Solving cubic equation by paper folding

Simon Allais

November 30, 2015

ENS DE LYON

Axioms

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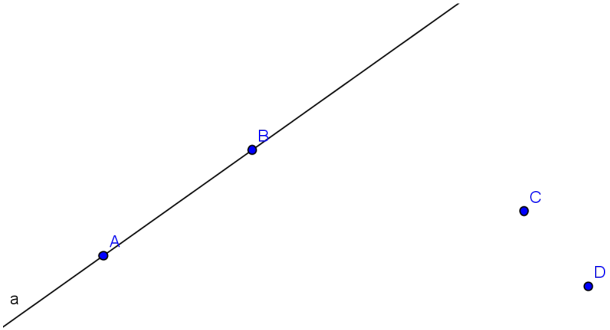

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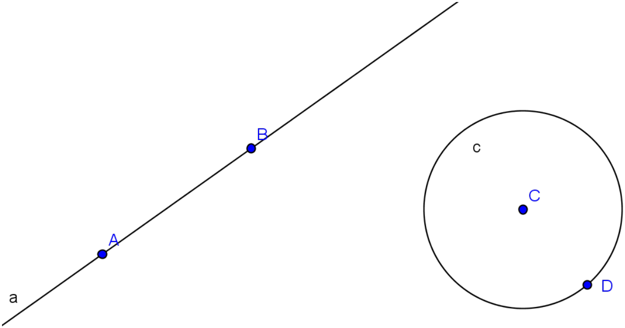

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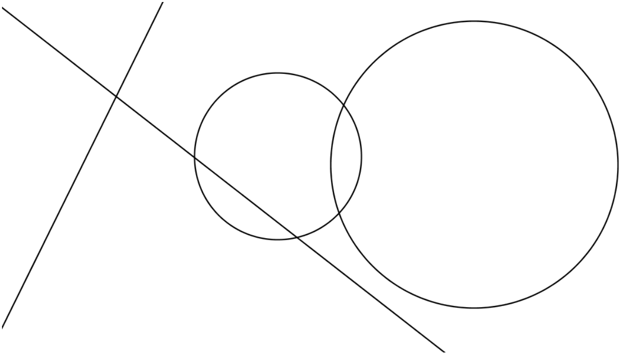

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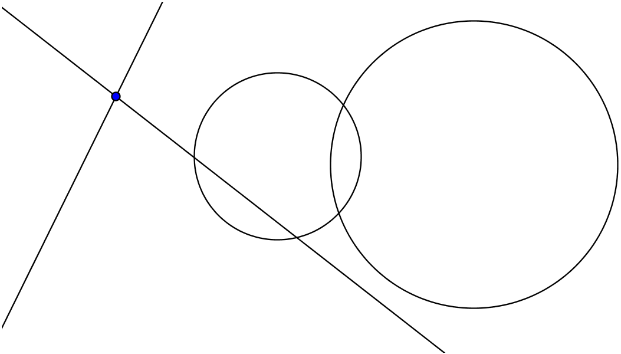

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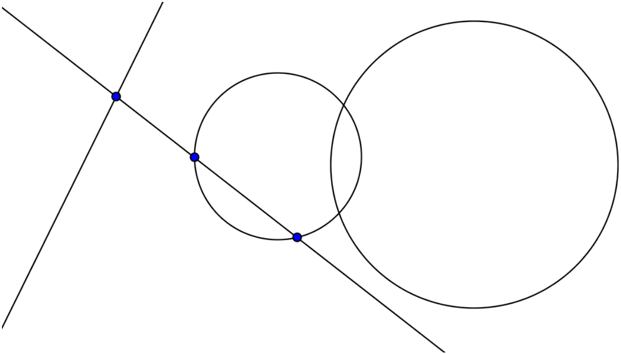

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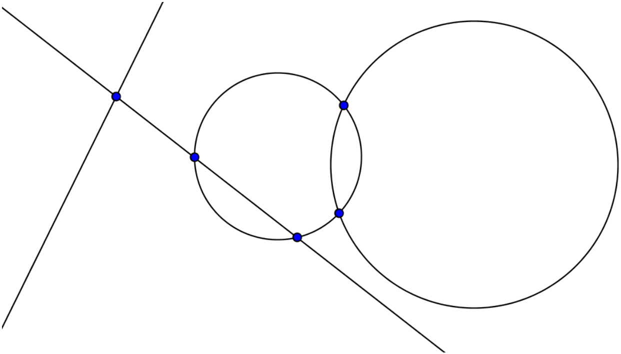
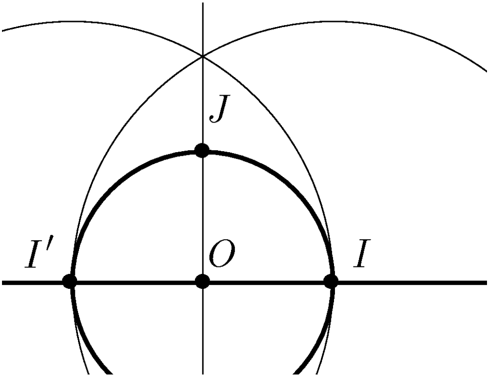
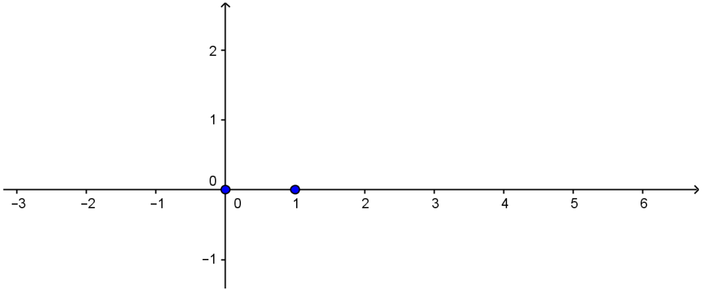


Constructing an orthogonal frame



Constructing $\mathbb{Z}$



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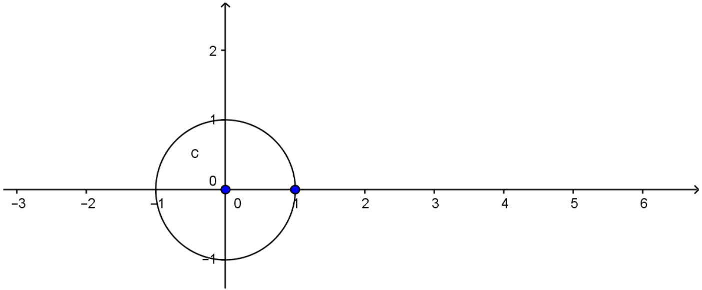

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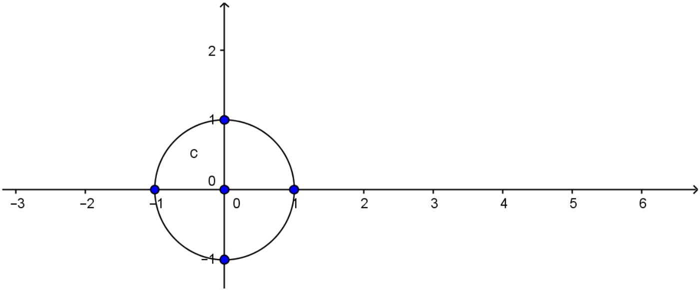

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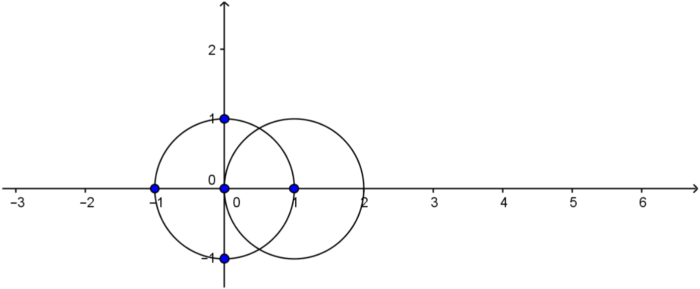

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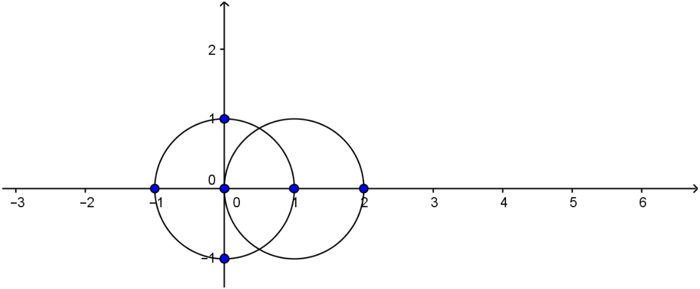

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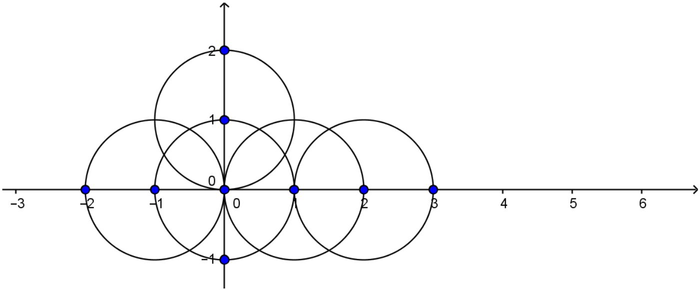

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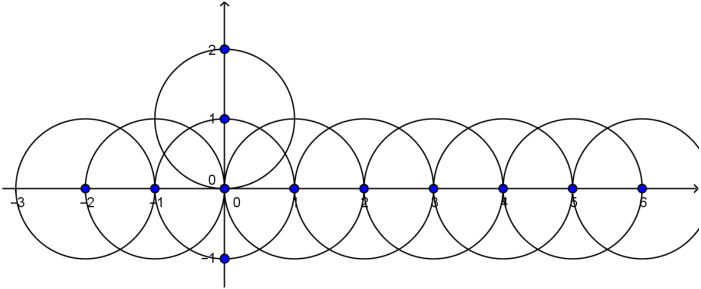
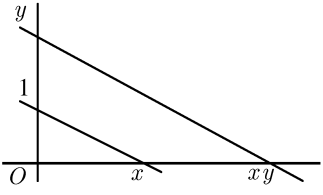
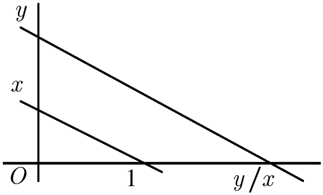


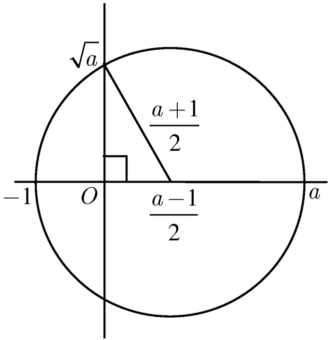
Constructing $\mathbb{Q}$





Constructing $\sqrt{a}$

$$\sqrt{a} = \sqrt{\left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2}$$



Pierre Wantzel theorem

Theorem

A number is constructible with compass and ruler if, and only if, it is expressed by sums, products, fractions and square roots of integers.

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Every solution of a quadratic equation is a constructible number.

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Corollary

Every solution of a quadratic equation is a constructible number.

Proof.

Solutions of $ax^2 + bx + c = 0$ (with $a \neq 0$) are

$$\frac{-b \pm \sqrt{\Delta}}{2a} \quad \text{with} \quad \Delta = b^2 - 4ac.$$

□

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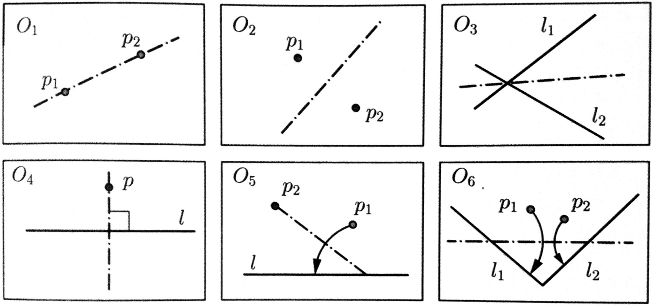
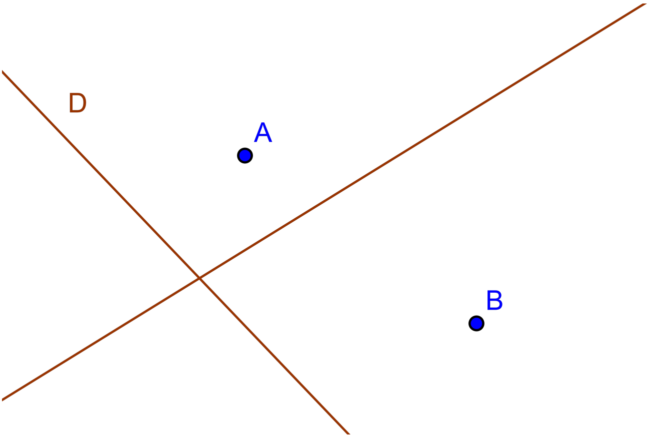
New axioms



The link with cubic equation



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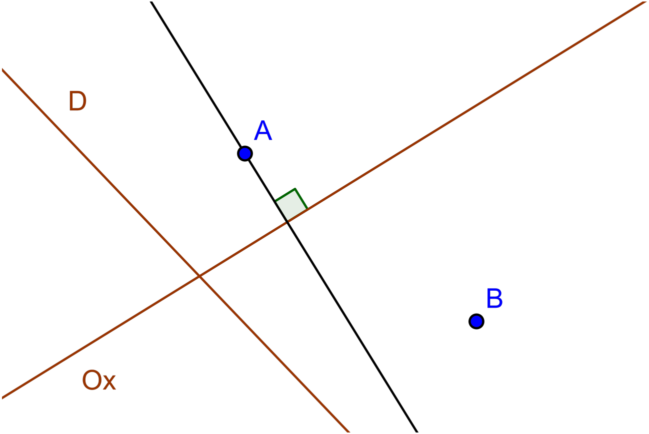

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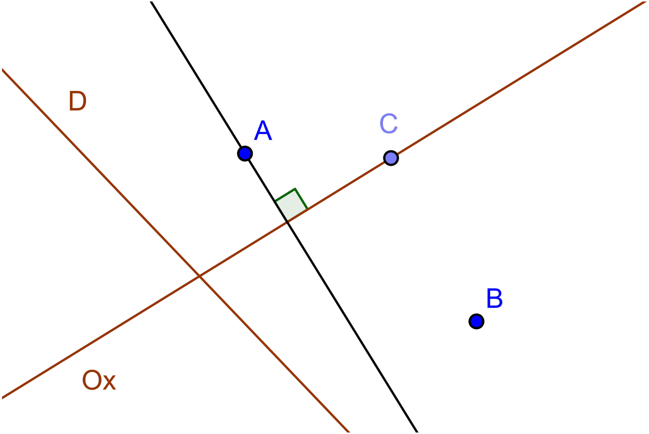

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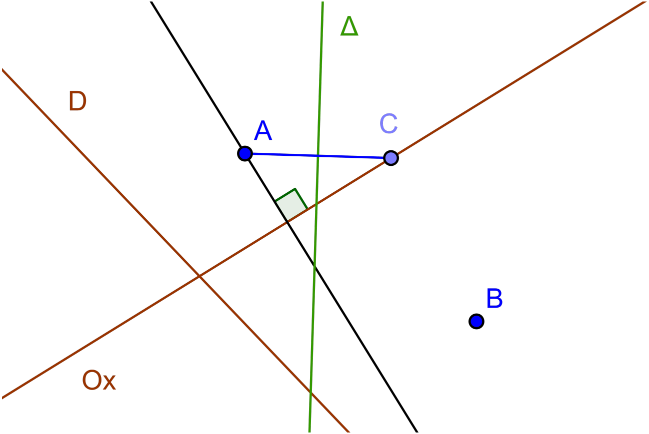

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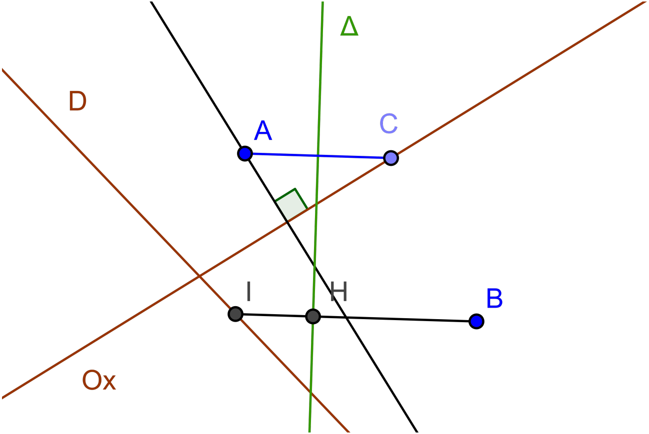

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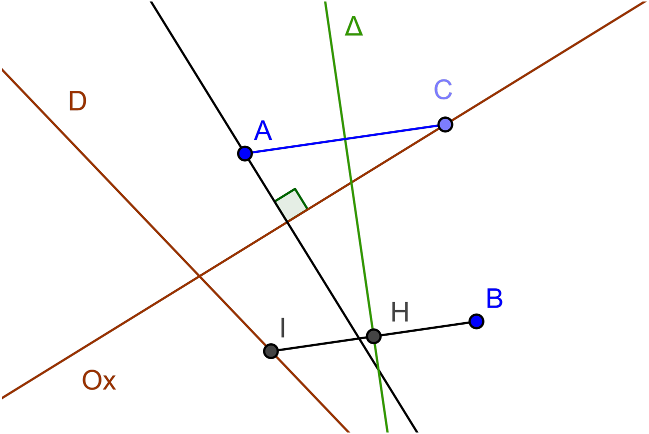


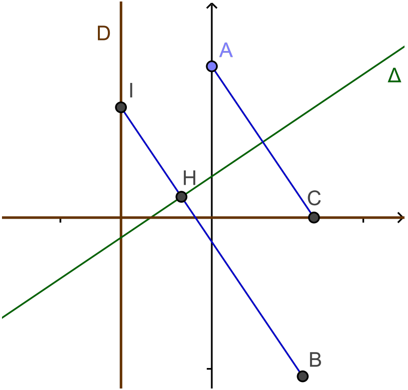
The link with cubic equation

$A = (0, 1)$

$B = (x_B, y_B)$

$D : ax + by + c = 0$

$C(t) = (t, 0) \quad (t \in \mathbb{R})$



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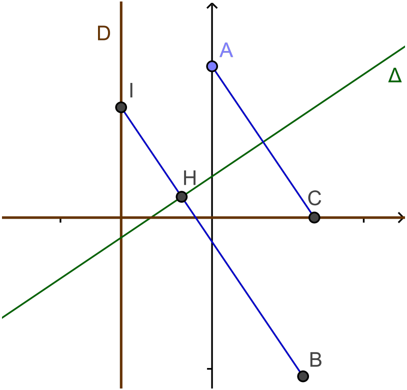
$x_B, y_B, a, b, c?$



The link with cubic equation

When do we have $BH = HI$?

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When

$$at^3+(b(y_B-1)+c-ax_B)t^2+(2(bx_B+ay_B)-a)t+ax_B+b(y_B+1)+c=0.$$

Solving a cubic equation

We can obtain a t such that

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and we want that

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Choosing $b=0$, we can take

$$D: x=-\frac{q}{2} \quad \text{and} \quad B=\left(\frac{q}{2},\frac{p+1}{2}\right).$$

Finding $\sqrt[3]{2}$

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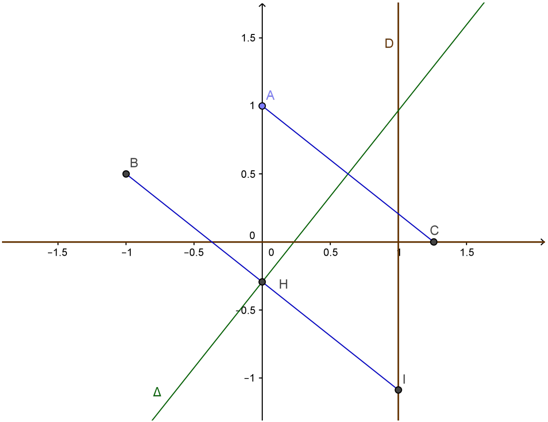

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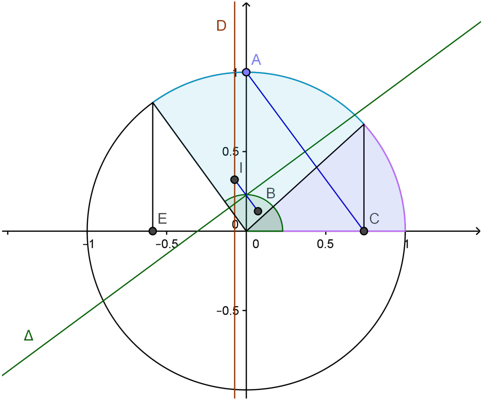

Thank you for your attention!