# Solving cubic equation by paper folding

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### Constructible number using ruler and compass

- Axioms
- Basic constructions
- Theorems

### Constructible number using paper folding

- Axioms
- The link with cubic equation
- Solving a cubic equation



#### Axioms Basic constructions Theorems

## Axioms





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#### Axioms Basic constructions Theorems

## Axioms





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#### Axioms Basic constructions Theorems

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Axioms Basic constructions Theorems

# Axioms





Axioms Basic constructions Theorems

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Axioms Basic constructions Theorems

# Axioms





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Axioms Basic constructions Theorems

# Axioms





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Axioms Basic constructions Theorems

# Constructing an orthogonal frame





# Constructing $\mathbb{Z}$





# Constructing $\mathbb{Z}$





# Constructing $\mathbb{Z}$





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# Constructing $\mathbb{Z}$





# Constructing $\mathbb{Z}$





Axioms Basic constructions Theorems

# Constructing $\mathbb{Z}$





# Constructing $\mathbb{Z}$





# Constructing $\mathbb{Q}$







Axioms Basic constructions Theorems

# Constructing $\sqrt{a}$



# Pierre Wantzel theorem

#### Theorem

A number is constructible with compass and ruler if, and only if, it is expressed by sums, products, fractions and square roots of integers.



# Pierre Wantzel theorem

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A number is constructible with compass and ruler if, and only if, it is expressed by sums, products, fractions and square roots of integers.

#### Corollary

Every solution of a quadratic equation is a constructible number.



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# Pierre Wantzel theorem

#### Theorem

A number is constructible with compass and ruler if, and only if, it is expressed by sums, products, fractions and square roots of integers.

#### Corollary

Every solution of a quadratic equation is a constructible number.

#### Proof.

Solutions of 
$$ax^2 + bx + c = 0$$
 (with  $a \neq 0$ ) are

$$rac{-b\pm\sqrt{\Delta}}{2a}$$
 with  $\Delta=b^2-4ac.$ 

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### 2 Constructible number using paper folding

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#### Axioms

The link with cubic equation Solving a cubic equation

# New axioms





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## The link with cubic equation



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## The link with cubic equation



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## The link with cubic equation



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## The link with cubic equation



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## The link with cubic equation



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## The link with cubic equation



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## The link with cubic equation

$$A = (0, 1)$$

$$B = (x_B, y_B)$$

$$D:ax+by+c=0$$

$$C(t) = (t,0) \quad (t \in \mathbb{R})$$



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## The link with cubic equation

$$A = (0, 1)$$
  
 $B = (x_B, y_B)$   
 $D : ax + by + c = 0$   
 $C(t) = (t, 0) \quad (t \in \mathbb{R})$   
 $x_B, y_B, a, b, c?$ 



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## The link with cubic equation

When do we have BH = HI?



# The link with cubic equation

When do we have BH = HI? When

$$at^{3}+(b(y_{B}-1)+c-ax_{B})t^{2}+(2(bx_{B}+ay_{B})-a)t+ax_{B}+b(y_{B}+1)+c=0.$$



# Solving a cubic equation

We can obtain a *t* such that

 $at^{3} + (b(y_{B}-1)+c-ax_{B})t^{2} + (2(bx_{B}+ay_{B})-a)t + ax_{B}+b(y_{B}+1)+c = 0$ 

and we want that

$$t^3 + pt + q = 0$$
 (with  $p, q \in \mathbb{R}$ ).

# Solving a cubic equation

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Choosing b = 0, we can take

$$D: x=-rac{q}{2} \quad ext{and} \quad B=\left(rac{q}{2},rac{p+1}{2}
ight).$$

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Finding  $\sqrt[3]{2}$ 





Finding  $\sqrt[3]{2}$ 

p = 0 and q = -2



Finding 
$$\sqrt[3]{2}$$

$$\sqrt[3]{2}$$
 is the solution of  $X^3 - 2 = 0$ 

$$p = 0$$
 and  $q = -2$ 

$$B = \left(-1, \frac{1}{2}\right)$$

D: x = 1



Finding  $\sqrt[3]{2}$ 

Axioms The link with cubic equation Solving a cubic equation

1.5 D  $\sqrt[3]{2}$  is the solution of  $X^3 - 2 = 0$ 0.5 p = 0 and q = -20 -1.5 -1 -0.5 0.5 1.5  $B = (-1, \frac{1}{2})$ 0 D: x = 1-1

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# Trisecting the angle

$$\cos \frac{\theta}{3}$$
 is a solution of  $X^3 - \frac{3}{4}X - \cos \theta = 0$ 



# Trisecting the angle

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# Trisecting the angle

$$\cos \frac{\theta}{3}$$
 is a solution of  $X^3 - \frac{3}{4}X - \cos \theta = 0$ 

$$p = -\frac{3}{4}$$
 and  $q = -\frac{\cos\theta}{4}$   
 $B = \left(-\frac{\cos\theta}{8}, -\frac{1}{8}\right)$ 

$$D: x = \frac{\cos\theta}{8}$$



# Trisecting the angle



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#### Thank you for your attention!

