A (Really) Short Presentation of the Möbius Group

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Definition

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Note

$\forall \lambda \in \mathbb{C}^*, \forall A \in GL_2(\mathbb{C}),$

$$h_{\lambda A} = h_A$$

Fundamental Property

$\forall A, B \in GL_2(\mathbb{C})$,

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Calculus

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Calculus or projective interpretation. □

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Let add a point ∞ such as, $\forall z \in \mathbb{C}^*$,

$$z\infty = \infty \quad z + \infty = \infty \quad \frac{z}{\infty} = 0$$

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$h(z) = z$ is equivalent to an equation of degree less or equal to 2. For the translation case, take ∞. □

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Conclusion

If h fixes ∞, h is a similarity.

Decomposition into Simple Transformation

$z \mapsto \frac{az+b}{cz+d}$ could be decompose as:

$$z \mapsto z + \frac{d}{c}$$

$$z \mapsto \frac{1}{z}$$

$$z \mapsto -\frac{ad-bc}{c^2} z$$

$$z \mapsto z + \frac{a}{c}$$

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Proof.

Noticing equation of a line-circle could be written :

$$\mathcal{C} : az\bar{z} + bz + \bar{b}\bar{z} + c = 0 \quad \text{where } a, b, c \in \mathbb{C}$$

Let z be the inverse of a point of $\mathcal{C}$, so $\frac{1}{z} \in \mathcal{C}$ and

$$a\frac{1}{z\bar{z}} + b\frac{1}{z} + \bar{b}\frac{1}{\bar{z}} + c = 0$$

Thus

$$cz\bar{z} + \bar{b}z + b\bar{z} + a = 0$$

□

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Proof.

Complex analysis : $z \mapsto \frac{1}{z}$ is holomorphic, thus its differential is a direct similarity. Thus inversion is locally acting as a direct similarity, which is a conformal map, so inversion is also a conformal map. $\square$

Sending a Point at ∞

Let $M \in \mathcal{M}$ and $\xi_+, \xi_- \in \overline{\mathbb{C}}$ its fixed point(s). Let $F : \begin{cases} \xi_+ & \mapsto & 0 \\ \xi_- & \mapsto & \infty \end{cases}$.

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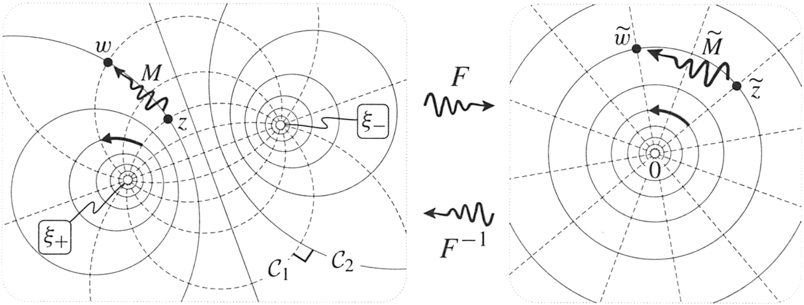
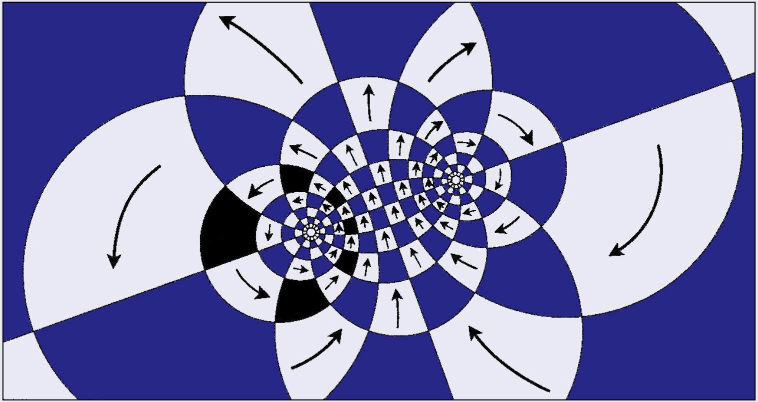


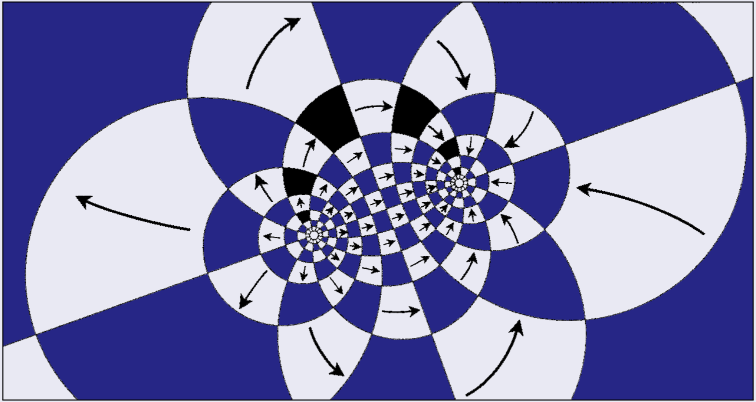
Elliptic Transformations

$$\widetilde{M} : z \mapsto e^{i\theta} z \quad (\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}) \quad \text{is a rotation}$$



Hyperbolic Transformations

$$\widetilde{M} : z \mapsto \rho z \quad (\rho \in \mathbb{R}^*) \quad \text{is an expansion}$$



Loxodromic Transformations

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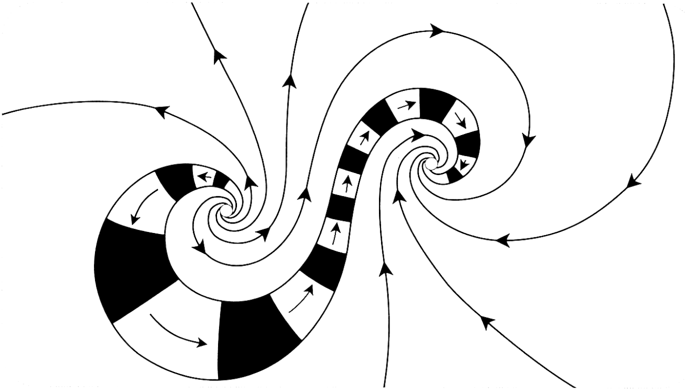

Parabolic Transformations

When M has got only one fixed point $\xi \in \mathbb{C}$, let $G : \xi \mapsto \infty$ given by $z \mapsto \frac{1}{z-\xi}$. Thus $\widetilde{M} = G \circ M \circ G^{-1}$ has an only fixed point ∞ and

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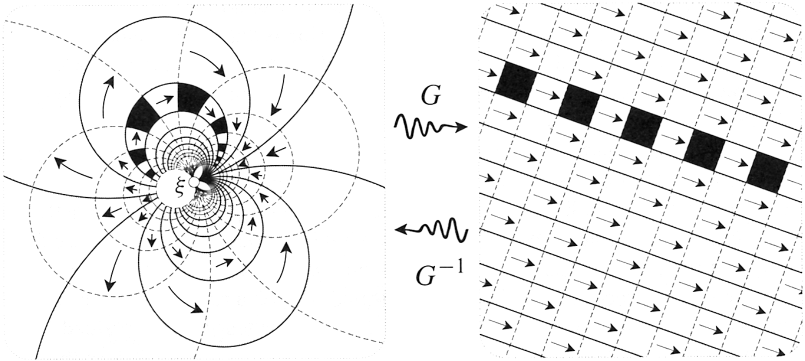

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