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To Farzaneh, Darya and Soren, who make it all worthwhile.
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Contents

Preface

The subject of this dissertation is the field of continuous, or C^0, symplectic topology which studies continuous analogues of smooth symplectic objects (*e.g.* symplectic/Hamiltonian homeomorphisms and C^0 Lagrangians) and asks questions about persistence of various symplectic phenomena under C^0 limits and perturbations. Like the rest of symplectic topology, the field saw the light of day in the early 80s thanks to the C^0 rigidity theorem of Eliashberg and Gromov. In the past few years, the subject has undergone rapid expansion and has developed into an active and important area of research attracting wide interest in the symplectic and dynamics communities. Although C^0 symplectic topology is extremely young, it has been the subject of numerous articles, many of which address the interactions between C^0 symplectic topology and other subfields of symplectic topology and dynamical systems; see for example [AABZ15, AZ18, Arn15, Buh10, BEP12, BHS18b, BO16, BS13, CV08, EP10, EP09a, EP17, EPR10, EPZ07, HLS15b, HLS15a, HLS16, Ish16, MO17, Mül08, Mül19, MS15, MS14b, MS16, MS14a, MS13, Nak20, OM07, Oh10, Ops09, PR14, RZ18, Sey12, Sey13a, Sey13b, Sey15, She18, She19, Ush19, Ush20, Vit06, Vit07, Vit18, Zap07, Zap10].

This memoir presents an incomplete introduction to continuous symplectic topology with a focus on four of my contributions to the field:

- The results on C^0 continuity of spectral invariants and barcodes which are spread over several articles: this direction of research was initiated in [Sey13a] (see Theorem 1.8), as a part of my PhD thesis, with the simplicity conjecture, which at the time seemed completely out of reach, serving as an ultimate motivation; it was continued in the article [Sey12] and very recently culminated in the proof of C^0 continuity of barcodes (see Theorem 1.3) on surfaces [LRSV] and symplectically aspherical manifolds [BHS18a], written jointly with Le Roux-Viterbo and Buhovsky-Humilière.[1] We will also discuss the *displaced discs problem* [Sey13b] and other applications to dynamics.
- Rigidity of coisotropic submanifolds and their characteristic foliations which was obtained in collaboration with Humilière and Leclercq [HLS15a]; see Theorem 1.4. We will also briefly discuss the results from [HLS16] on reduction of symplectic homeomorphisms.
- The C^0 counter-example to the Arnold conjecture which was constructed jointly with Buhovsky and Humilière in [BHS18b]; see Theorem 1.15. I will also present a generalized version of the conjecture, proven in [BHS18a], which does hold in the topological setting; see Theorem 1.16.
- Proof of non-simplicity of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ the group of compactly supported area-preserving homeomorphisms of the disc (the simplicity conjecture) which was proven recently in collaboration with Cristofaro-Gardiner and Humilière [CGHS20].

The heart of the memoir is Chapter 1 in which I have attempted to introduce the above results while at the same time placing them into the broader context of symplectic topology and dynamics. The first chapter ends with a discussion on some of the open problems in C^0 symplectic topology. The

[1] As we will see, these results have been extended to other settings by Kawamoto [Kaw19] and Shelukhin [She18]

second chapter is meant to familiarize the reader with some of the technical tools of the field such as spectral invariants from Hamiltonian and Periodic Floer homologies. The third and the final chapter sketches some of the key ideas behind the proofs of the main results.

In the interest of keeping the length of this document under control, I have avoided going into an in-depth discussion of my articles [BS13, Sey12, HLS16, HLRS16], while my remaining articles [GHS09, Sey13c, Sey14, HLS15b, Sey15, BHS19, LRS20] are not discussed at all.

I want to emphasize that this document is a *mémoire d'habilatation*, and consequently the viewpoint on C^0 symplectic topology presented here is mostly limited to my own contributions to the field. Several interesting and important developments, such as the results on C^0 rigidity of the Poisson bracket and function theory on symplectic manifolds [Buh10, BEP12, CV08, EP10, EP09a, PR14], connections to weak KAM theory and the Hamilton-Jacobi equation [Vit07, MVZ12, Vic14, Arn10, AABZ15], and microlocal sheaf theory [Gui13] are not presented here.

Chapter 1

Overview of continuous symplectic topology

My goal in this memoir is to present an overview of C^0 symplectic topology from two different, but related, perspectives: One is a symplectic topological perspective which is informed by Gromov's **soft and hard** view of symplectic topology. The second perspective is motivated by the recent interactions between C^0 symplectic topology and dynamical systems. This latter viewpoint falls under the new field of **symplectic dynamics** and my goal here is to present dynamical aspects of C^0 symplectic topology. These two perspectives are intimately related and very often results in one direction inform our understanding of the other viewpoint and hence trigger progress in the other direction.

Continuous symplectic topology from the soft-hard viewpoint

Symplectomorphisms of a symplectic manifold (M, ω) preserve the volume form ω^n. Hence, the group of symplectomorphisms $\mathrm{Symp}(M, \omega)$ is a subgroup of the group of volume preserving diffemorphisms of M. In the early 1970s, Gromov proved his **soft vs. hard** alternative: With respect to the C^0 topology[1] $\mathrm{Symp}(M, \omega)$ is either dense (*softness or flexibility*) in the group of volume preserving diffeomorphisms or it forms a closed (*hardness or rigidity*) subset. Furthermore, the conclusion of the alternative is the same for all symplectic manifolds of a given dimension. The resolution of the alternative in favour of softness would indicate that symplectomorphisms possess no distinct qualitative properties in comparison to volume preserving diffeomorphisms, and so none of the major breakthroughs of symplectic topology, such as the Arnold conjecture or Gromov's non-squeezing theorem, would have a chance at being true. In the early 1980s Eliashberg [Eli, Eli87] resolved the alternative in favour of rigidity; this is now known as the rigidity theorem of Eliashberg & Gromov.[2]

The rigidity theorem constitutes a landmark in the development of symplectic topology. Indeed, Arnold [Ad86] refers to it as "the existence theorem of symplectic topology." This theorem also gave birth to the field of C^0 symplectic topology one of whose central goals is to further explore the mysterious boundary between C^0 rigidity and C^0 flexibility. From this viewpoint, one of the main goals of the field is to address the meta-mathematical question of whether symplectic topology belongs to the smooth world or to a larger universe of objects with less regularity. Throughout mathematics, one encounters smooth objects which under further examination admit interesting interpretations in non-smooth settings as well. An example of this is the notion of cohomology which is often first introduced via De Rham theory on smooth manifolds but, of course, it extends to more general topological

[1] That is the topology of uniform convergence on compact sets.

[2] In [Gro85], Gromov gave another proof of this theorem using his theory of pseudo-holomorphic curves. Yet another proof was given by Ekeland and Hofer via the theory of symplectic capacities [EH08, HZ94].

spaces. Another example is the notion of curvature which extends to metric spaces more general than Riemannian manifolds. Last, but not least, the example which concerns us the most is the very notion of a symplectic diffeomorphism: the rigidity theorem allows us to make sense of **symplectic homeomorphisms** and one of the main goals of C^0 symplectic topology is to understand the extent to which the underlying symplectic structure is preserved by these homeomorphisms.

In his 1986 ICM address [Gro87], Gromov presented his fundamental soft vs. hard philosophy: Symplectic topology is enriched by a beautiful interplay between rigidity and flexibility. Hardness, or rigidity, refers to a strong and rigid property of a certain class of objects, while softness, or flexibility, refers to a weak property which holds for a large class of objects. For example, the Darboux theorem, which says that all symplectic manifolds are locally equivalent, is a soft result, while the C^0 rigidity theorem of Eliashberg & Gromov is the quintessential hard result. Recent results, such as [Ops09, HLS15a, HLS16, BO16, BHS18b, BHS18a] have demonstrated that this contrast between rigidity and flexibility permeates in a rather surprising fashion to C^0 symplectic topology as well. Symplectic rigidity manifests itself when symplectic phenomena survive under C^0 limits. An example of such a phenomenon is the **coisotropic rigidity** theorem, proven by R. Leclercq, V. Humilière, and myself [HLS15a], which provides a generalization of the Eliashberg-Gromov theorem: Roughly speaking, it states that coisotropic submanifolds and their characteristic foliations are preserved by symplectic homeomorphisms. On the other hand, there exist instances when passage to C^0 limits results in spectacular loss of rigidity and prevalence of flexibility; examples of such phenomena include the exotic **area-shrinking** symplectic homeomorphism of Buhovsky-Opshtein [BO16] and the recent C^0 **counterexample** to the Arnold conjecture constructed by Buhovsky, Humilière, and myself [BHS18b].

Continuous symplectic topology from the viewpoint of symplectic dynamics

Through its interactions with dynamical systems, the field of C^0 symplectic topology has expanded beyond solely exploring Gromov's soft-hard philosophy; see for example [AABZ15, AZ18, Arn15, BHS18b, BHS18a, CGHS20, HLRS16, Sey13b, Vit07, Vit18]. Indeed, by now, these interactions constitute an equally important aspect of the field. Over the past three decades, symplectic methods, such as Floer theory, have had spectacular success in resolving seemingly intractable problems from dynamical systems. Examples of this include Floer's resolution of the Arnold conjecture on fixed points of Hamiltonian diffeomorphisms [Flo86a, Flo88a, Flo89a, Flo89c] (see also the articles of Conley-Zehnder [CZ83] and Hofer [Hof88a]), the impressive progress made towards the Weinstein conjecture [Vit87, Hof93, Tau10], and constructions of global surfaces of sections for three-dimensional Reeb flows [HWZ98, HWZ03]. These developments have prompted the genesis of **symplectic dynamics**: A new field which aims to develop the common core of symplectic topology and dynamical systems. It uses highly integrated ideas from both fields to study problems originating in dynamical systems; see [BH12, Bra15a, Bra15b, ABHSa18]. In recent years, continuous symplectic topology has played an important role in the advent of symplectic dynamics.

Continuous symplectic topology, and more generally symplectic topology, provide tools and methods which are not immediately available in dynamical systems and hence allow us to approach problems which might be out of reach if one were to solely rely on the tools of dynamics. In the case of C^0 symplectic topology, this is particularly applicable in dimension two where symplectic homeomorphisms coincide with area and orientation preserving homeomorphisms. Indeed, the study of area-preserving homeomorphisms is a domain where continuous symplectic topology has been quite successful with questions arising from dynamics. A first example of this is the **displaced discs problem** which was posed by the dynamists F. Béguin, S. Crovisier, and F. Le Roux and which asks, roughly speaking, if a

C^0 small Hamiltonian homeomorphism of a closed surface can displace a disc of large area.[3] I showed in the artciles [Sey13a, Sey13b] that the answer to this question is negative. A more recent example is my article with Cristofaro-Gardiner and Humilière [CGHS20] proving that the group of compactly supported area-preserving homeomorphisms of the two-disc is not simple, which settles what is known as the **simplicity conjecture** in the affirmative. The key symplectic tool used in the solutions to these problems is a class of invariants of area-preserving diffeomorphisms, called spectral invariants, which are constructed via various flavors[4] of Floer homology.

The soft-hard and dynamical viewpoints on continuous symplectic topology are intimately related. In fact, every single one of the results that I have mentioned above admits dynamical and soft-hard interpretations. For example, the theorem on C^0 rigidity of coisotropic submanifolds requires proving certain dynamical properties of continuous Hamiltonian flows. The solutions to the displaced discs problem and the simplicity conjecture require establishing C^0 rigidity of the spectral invariants from Hamiltonian Floer and periodic Floer homologies, respectively. Results on **continuity of barcodes**, proven in my articles with Buhovsky-Humilière and Le Roux-Viterbo [BHS18a, LRSV] constitute another manifestation of this interaction where we rely on C^0 rigidity of barcodes to draw dynamical conclusions about Hamiltonian homeomorphisms; for example, in [BHS18a] we show that, despite the C^0 counterexample to the Arnold conjecture, a version of the conjecture is true in the topological case; see Theorem 1.16 below.

One might naively wonder if there is any mathematical, or physical, necessity for studying objects of low regularity such as (symplectic) homeomorphisms. As a first response, one could argue that "natural" physical phenomena are generally non-smooth, and we study smooth approximations of what takes place in nature; for example, the article [Her19], in the astrophysics literature, speaks of the need for a notion of C^0 Hamiltonians in the study of the N–body problem. Moreover, even if we were to restrict our attention to smooth models of the physical world, in many situations we would still find oursleves face to face with non-smooth objects. Here is an example where exploring a smooth dynamical system necessitates the study of area-preserving homeomormphisms: Eversince the groundbreaking works of Poincaré [Poi87, Poi12] and Birkhoff [Bir13, Bir22, Bir66], an important method for studying a Hamiltonian flow on a hypersurface in a four dimensional symplectic manifold has been to construct a global surface of section and then study the so-called Poincaré return map which is an area-preserving map induced by the flow on the surface of section.[5] After compactification of the surface of section, one obtains an area-preserving *homeomorphism* of a closed surface. This is for example the case in the recent article of Cristofaro-Gardiner, Hutchings and Pomerleano [CGHP19] where they obtain an area-preserving homeomorphism of the sphere. This homeomorphism contains a wealth of information about the original smooth Hamiltonian flow. For example, by appealing to Franks' generalization of the Poincaré-Birkhoff theorem [Fra88, Fra92, Fra96a] one can deduce that the original Hamiltonian system possessed either two or infinitely many periodic orbits.

As is evident from the above, area-preserving homeomorphisms play a crucial role in dynamics and, as mentioned earlier, C^0 symplectic topology has already made important contributions in this direction [Sey13b, HLRS16, LRSV, CGHS20]. Here, one interesting feature of C^0 symplectic topology is that it provides a medium for generalising two-dimensional phenomena from surface dynamics to higher dimensions. An example of this is the article by Buhovsky, Humilière, and myself [BHS18a], in which we use notions from symplectic topology, namey the notion of spectral invariants, to generalise

[3] One should imagine here a *thin* disc which winds around the surface and could potentially be displaced by a C^0–small area-preserving homeomorphism.

[4] The solutions to the displaced discs problem and the simplicity conjecture rely, respectively, on *Hamiltonian Floer theory* and *periodic Floer homology.*

[5] In symplectic topology, following the pioneering work of Hofer-Wysocki-Zehnder [HWZ98], global genus zero surfaces of section are constructed via pseudo-holomorphic curve techniques. The authors of [CGHP19], use the theory of embedded contact homology.

non-trivial two-dimensional fixed point results of Matsumoto [Mat00] (see also Franks [Fra96b]) to higher dimensions. See also the recent work of Ginzburg and Gürel [GG18] who have generalized the notion of irrational pseudo-rotation from surfaces to higher dimensional projective spaces.

Invariant subsets, which are of great interest in smooth Hamiltonian and symplectic dynamics, are often very irregular and hence in the course of their study one is frequently obliged to deal with C^0 symplectic topology. This is, for instance, the case in Aubry-Mather theory, the study of solutions to the Hamilton-Jacobi equation and weak KAM theory: For example, in [AABZ15], Arcostanzo, Arnaud, Bolle and Zavidovique proved that a Tonelli Hamiltonian on $T^*\mathbb{T}^n$, with no conjugate points, is C^0 integrable in the sense that $T^*\mathbb{T}^n$ is foliated by invariant C^0 Lagrangians; see also [AZ18]. The notion of C^0 Lagrangians also appears in connection to generalizations of Birkhoff's second theorem on invariant Lagrangians for Tonelli Hamiltonians; see the articles [Bia89, BP92a, BP92b, Arn10, BdS12, AOdS18]. Moreover, in weak KAM theory one also comes across C^0 Hamiltonians as well as results concerning C^0 rigidity of the Poisson bracket; see the articles by Cardin-Viterbo [CV08], Davini-Zavidovique [DZ13], and Arnaud [Arn15].

In addition to the above, there are numerous other contexts, in symplectic topology and dynamical systems, in which one encounters irregular objects and notions from C^0 symplectic topology. This includes the Entov-Polterovich theory of (super)heavy subsets [EP06, EP09b], microsupports of sheaves [Gui13, GS14, GKS12], Viterbo's theory of symplectic homogenization [Vit07, MVZ12, Vic14, Vit18], the works of Artstein-Avidan and Ostrover concerning Hamiltonian dynamics on non-smooth convex hypersurfaces [AAO14], and the weak solutions of the *degenerate special Lagrangian (DSL)* equation arising in the work of Solomon-Rubinstein [RS17].

Lastly, we remark that studying objects with low regularity oftentimes leads to advancing our understanding of smooth objects themselves. This has, of course, occured very frequently in mathematics and there are several instances of it in symplectic topology as well. For example, the articles [HLS15b, HLS15a, HLS16] by Humilière, Lecelercq and myself, which were motivated by questions from C^0 symplectic topology, have led to the discovery of new properties of certain Floer theoretic invariants of Hamiltonian diffeomorphisms. Another example is the beautiful results on C^0 rigidity of the Poisson bracket by Cardin-Viterbo [CV08], Entov-Polterovich [EP10, EP09a], and Buhovsky [Buh10]. These articles have led to the discovery of Poisson bracket invariants of Buhovsky-Entov-Polterovich [BEP12] which have applications to smooth Hamiltonian dynamics [EP17] and have also partially motivated Polterovich's recent work on the connections between symplectic topology and quantum mechanics [Pol16, PR14].

1.1 Symplectic and Hamiltonian homeomorphisms

If ϕ is a homeomorphism of a symplectic manifold (M, ω), the form $\phi^*\omega$ does not exist because its definition involves the derivatives of ϕ and hence it does not make sense to ask if ϕ preserves the symplectic structure. However, as mentioned earlier, the C^0 rigidity theorem allows us to make sense of this notion.

Definition 1.1. *A homeomorphism of a symplectic manifold is said to be symplectic if it can be written as a uniform limit of symplecitc diffeomorphisms. We denote by* $\mathrm{Sympeo}(M, \omega)$ *the set of symplectic homeomorphisms of* (M, ω).

If a symplectic homeomorphism is smooth then it is symplectic, but if it is not smooth it is very mysterious how much of the symplectic topology is preserved by such transformations. Studying symplectic homeomorphisms is one of the cornerstones of C^0 symplectic topology and so not surprisingly several of my articles are dedicated to this cause; see [BS13, HLS15b, HLS15a, HLS16, BHS18b, BHS18a, Sey13c].

One may analogously define **Hamiltonian homeomorphisms** to be those homeomorphisms of M which may be written as uniform limits of Hamiltonian diffeomorphisms. A difficulty which arises immediately (in dimensions greater than 2) is whether a smooth Hamiltonian homeomorphism is a Hamiltonian diffeomorphism or not. This question is a (variation on a) longstanding and important open problem in symplectic topology that is referred to as the C^0 Flux conjecture; we will disucss this, and other open problems, in Section 1.4. Despite the difficulty posed by the C^0 Flux conjecture, this is indeed a good working definition and as we will see there has been some success in studying these objects. Furthermore, this definition is regularly used in the surface dynamics community.[6] In [OM07], S. Müller and Y.-G. Oh present an alternative definition for Hamiltonian homeomorphisms which we will encounter below in Section 3.2.

1.2 Flexibility and rigidity in C^0 symplectic topology

As mentioned above, the fascinating interplay between the rigid and flexible sides of symplectic topology runs in surprising ways through C^0 symplectic topology as well. I will briefly present in this section some of my results pertaining to this aspect of the domain.

One of the main challenges we have faced in C^0 symplectic topology is that the powerful tools of smooth symplectic topology are not available in the C^0 setting. For example, the tool which has been used very effectively in studying fixed and periodic points of Hamiltonian diffeomorphisms is the machinery of Hamiltonian Floer homology whose definition counts solutions to an elliptic PDE which simply cannot be written down for a Hamiltonian homeomorphism.

As early as a decade ago, the objective of developing viable tools and methods for studying symplectic homeomorphisms seemed out of reach. However, over the past few years, significant steps towards this objective have been taken. For example, starting with the techniques introduced in my article [Sey13a], and continuing with the articles [Sey12, HLS15b, HLS15a, HLS16], written by Humilière, Leclercq, and myself, a set of methods have been introduced to study various Floer theoretic invariants in C^0 settings. In my two recent collaborations with Le Roux-Viterbo [LRSV] and Buhovsky-Humilière [BHS18a], these methods culminated in a definition of filtered Floer homology for Hamiltonian *homeomorphisms* via the theory of **barcodes**. **Quantitative h-principles** form another set of important techniques which have resulted in surprising examples of symplectic homeomorphisms. These techniques were introduced by Buhovsky-Opshtein in [BO16] and were further developed in the paper by Buhovsky, Humilière, and myself [BHS18b]. Further details on these developments are provided below.

1.2.1 Barcodes and Hamiltonian homeomorphisms

In this section, we explain how Hamiltonian Floer homology, as seen through the lens of barcodes (or persistence homology), satisfies a strong C^0 rigidity property. To simplify the presentation we will work in the setting of a closed and symplectically aspherical[7] manifold (M, ω).

Definition 1.2. *A* **barcode** $\mathcal{B} = \{I_j\}_{1 \leqslant j \leqslant N}$ *is a finite collection of intervals (or bars) of the form* $I_j = (a_j, b_j]$, $a_j \in \mathbb{R}$, $b_j \in \mathbb{R} \cup \{+\infty\}$.

The space of barcodes can be equipped with the so-called **bottleneck distance** *(see e.g. [CdSGO16]) which is denoted by* d_{bottle} *and is defined as follows: We say that two barcodes* $\mathcal{B}$ *and* $\mathcal{C}$ *are* δ*-matched,* $\delta > 0$*, if after erasing some bars of length less than* 2δ *in* $\mathcal{B}$ *and* $\mathcal{C}$*, the remaining ones can be matched bijectively so that the endpoints of the corresponding intervals lie at a distance of less than* δ *from one another. The bottleneck distance* $d_{\text{bottle}}(\mathcal{B}, \mathcal{C})$ *is defined as the infimum of such* δ*.*

Barcodes have recently found several interesting applications in PDE, under the guise of Barannikov complexes (see [Bar94, LPNV13, LPNV20]), and also in symplectic topology, following the appearance

[6] On a symplectic surface, Hamiltonian homeomorphisms may be described as those area-preserving homeomorphisms of the surface which are in the path component of the identity and have vanishing mean rotation vector.

[7] M is said to be symplectically aspherical if ω and c_1, the first Chern class of M, both vanish on $\pi_2(M)$.

of the article of Polterovich and Shelukhin [PS16]. We refer the reader to the recent book [PRSZ19] of Polterovich et al for a thorough introduction to the subject and applications to geometry and analysis. We should mention that the notion of the bottleneck distance originated in the field of topological data analysis; see, for example, [ZC05, EH08, BL14, CdSGO16]

As explained in [PS16], using Hamiltonian Floer homology one can associate a canonical barcode $\mathcal{B}(H)$ to every Hamiltonian H.[8]The barcode $\mathcal{B}(H)$ encodes a significant amount of information about the Floer homology of H: it completely characterizes the filtered Floer complex of H up to quasi-isomorphism, and hence it subsumes all of the previously constructed filtered Floer theoretic invariants. For example, the spectral invariants of H, which will be reviewed in Chapter 2, correspond to the endpoints of the half-infinite bars in $\mathcal{B}(H)$.

Given a barcode $\mathcal{B} = \{I_j\}_{1 \leqslant j \leqslant N}$ and $c \in \mathbb{R}$ define $\mathcal{B} + c = \{I_j + c\}_{1 \leqslant j \leqslant N}$, where $I_j + c$ is the interval obtained by adding c to the endpoints of I_j. Let $\sim$ denote the equivalence relation on the space of barcodes given by $\mathcal{B} \sim \mathcal{C}$ if $\mathcal{C} = \mathcal{B} + c$ for some $c \in \mathbb{R}$; we will denote the quotient space by $\widehat{Barcodes}$. Now the bottleneck distance descends to a distance on $\widehat{Barcodes}$ which we will continue to denote by d_{bottle}. If H, G are two Hamiltonians the time–1 maps of whose flows coincide, then it follows from results of Seidel [Sei97] and Schwarz [Sch00] that

$$\mathcal{B}(H) = \mathcal{B}(G)$$

in $\widehat{Barcodes}$. Hence, we obtain a well-defined map[9]

$$\mathcal{B} : (\text{Ham}(M, \omega), d_{C^0}) \to (\widehat{Barcodes}, d_{\text{bottle}}),$$

where $\text{Ham}(M, \omega)$ denotes the set of Hamiltonian diffeomorphisms of (M, ω).

The following is the key result connecting the theory of barcodes to C^0 symplectic topology.

Theorem 1.3. *Let (M, ω) be closed, connected, and symplectically aspherical. The mapping*

$$\mathcal{B} : (\text{Ham}(M, \omega), d_{C^0}) \to (\widehat{Barcodes}, d_{\text{bottle}})$$

is continuous and extends continuously to $\overline{\text{Ham}}(M, \omega)$.

For a diffeomorphism ϕ, the barcode $\mathcal{B}(\phi)$ is, as explained above, equivalent to the filtered Hamiltonian Floer homology of ϕ; if ϕ is a homeomorphism $\mathcal{B}(\phi)$ can be interpreted as its filtered Floer homology. This result provides us with a set of tools to study Hamiltonian homeomorphisms and allows us to prove various rigidity statements such as the topological version of the Arnold conjecture; see Theorem 1.15 below.

The above theorem was first proven in [LRSV], by Le Roux, Viterbo and myself, in the case of surfaces relying on certain fragmentation results which have only been proven in dimension two. In [BHS18a], Buhovsky, Humilière and I extended the above theorem to all dimensions via arguments which circumvent the two-dimensional fragmentation results. The proof in [BHS18a], which we will present in Section 3.1 relies crucially on a result of Kislev-Shelukhin [KS18, Cor. 6]. Lastly, we should mention that the above result has been generalized to projective spaces by Shelukhin [She18] (see also [Kaw19]) and to negative monotone manifolds by Kawamoto [Kaw19].

[8] See the article of Usher and Zhang [UZ16] for a construction of barcodes in settings more general than symplectically aspherical manifolds.

[9]The map $\mathcal{B}$ is often defined using mean-normalized Hamiltonians; see, for example, [PS16]. In this case, it takes values in the space of barcodes, as opposed to barcodes upto shift. However, this yields a discontinuous map (See Example 2.3 in [Sey13a]). This is why it is important to consider the space $\widehat{Barcodes}$, barcodes upto shift, in Theorem 1.3.

1.2.2 Submanifolds in C^0 symplectic topology

We now turn our attention to the interplay between softness and hardness of submanifolds in C^0 symplectic topology. The general form of the questions which we seek to answer is as follows: Consider a smooth submanifold V, of a symplectic manifold, satisfying a certain symplectic property P; this could be, for example, being Lagrangian, symplectic, (co)isotropic, having a certain symplectic volume, etc. Our goal is to understand whether P is preserved by symplectic homeorphisms in the following sense : Let h be a symplectic homeomorphism and suppose that $h(V)$ is smooth. Does $h(V)$ satisfy the property P?

What differentiates the C^0 world from the smooth one is that not every symplectic property is preserved by homeomorphisms. A general pattern which has emerged over the past few years is that those properties which are known to be rigid in the smooth world, such as being coisotropic, are preserved, and those which are flexible, i.e. satisfy an h-principle, such as being isotropic, are not preserved.

An example of hardness: rigidity of coisotropic submanifolds

A submanifold C of a symplectic manifold (M, ω) is called coisotropic if for all $p \in C$, $(T_pC)^\omega \subset T_pC$, where $(T_pC)^\omega$ denotes the symplectic orthogonal of T_pC. For instance, hypersurfaces and Lagrangians are coisotropic. A coisotropic submanifold carries a natural foliation $\mathcal{F}$ which integrates the distribution $(TC)^\omega$; $\mathcal{F}$ is called the characteristic foliation of C. Coisotropic submanifolds and their characteristic foliations have been studied extensively in symplectic topology. The various rigidity properties that they exhibit have been of particular interest; see for example [Gin07]. Here is the statement of the coisotropic rigidity theorem.

Theorem 1.4 (Humilière-Leclercq-S. [HLS15a])**.** *Let C be a smooth coisotropic submanifold of a symplectic manifold (M, ω) and suppose that θ is a symplectic homeomorphism. If $\theta(C)$ is smooth, then it is coisotropic. Furthermore, θ maps the characteristic foliation of C to that of $\theta(C)$.*

An important feature of the above theorem is its locality: C is not assumed to be necessarily closed. Moreover, as we explain in [HLS15a], θ need not be globally defined. An immediate, but surprising, consequence of Theorem 1.4 is that if the image of a coisotropic submanifold via a symplectic homeomorphism is smooth, then so is the image of its characteristic foliation.

Theorem 1.4 uncovers a link between two previous rigidity results and demonstrates that they are in fact extreme cases of a single rigidity phenomenon. One extreme case, where C is a hypersurface, was established by Opshtein [Ops09]. Clearly, in this case, the interesting part is the assertion on rigidity of characteristics, as the first assertion is trivially true. Lagrangians constitute the other extreme case. When C is Lagrangian, its characteristic foliation consists of one leaf, C itself. In this case the theorem reads: *If θ is a symplectic homeomorphism and $\theta(C)$ is smooth, then $\theta(C)$ is Lagrangian.* In [LS85], Laudenbach–Sikorav proved a similar result: *Let L be a closed manifold and ι_k denote a sequence of Lagrangian embeddings $L \to (M, \omega)$ which C^0–converges to an embedding ι. If $\iota(L)$ is smooth, then $\iota(L)$ is Lagrangian.* On one hand, their result only requires convergence of embeddings while Theorem 1.4 requires convergence of symplectomorphisms. On the other hand, Theorem 1.4 is local: It does not require the Lagrangian nor the symplectic manifold to be closed.

I should point out that Theorem 1.4 is a coisotropic generalization of the C^0 rigidity theorem. Indeed, it implies that if the graph of a symplectic homeomorphism is smooth, then it is Lagrangian.

Reduction of symplectic homeomorphisms

Let ϕ, C, C' be as in the hypothesis of the coisotropic rigidity theorem. Then, as a consequence of the theorem, ϕ induces a homeomorphism ϕ_R between the reductions[10] of C and C'. It is natural to ask whether ϕ_R is symplectic in any sense. For example, does it preserves any symplectic invariants? Some progress in this direction has been obtained recently in [HLS16, BO16, Bus19]. In [HLS16], Humilière, Leclercq and I prove that if ϕ_R is smooth then it is symplectic. Without the smoothness assumption on ϕ_R the problem becomes much more difficult. We show in [HLS16] that ϕ_R preserves a symplectic invariant, called the Viterbo capacity, in the specific case where the symplectic and coisotropic manifolds in question are all standard tori. In [BO16], Buhovsky and Opshtein prove that if the coisotropics are hypersurfaces then the reduced map ϕ_R satisfies certain non-squeezing properties. Similar results are obtained by Bustillo [Bus19]. It would be interesting to see if these results can be extended to more general settings.

An example of softness: an area-shrinking symplectic homeomorphism

A natural question which arises as a consequence of the coisotropic rigidity theorem is whether similar rigidity statements hold for symplectic and isotropic submanifolds. As we will now explain, the quantitative h-principles of Buhovsky and Opshtein provide strong evidence that this is in general not true.

In [BO16], Buhovsky and Opshtein proved that the notion of symplectic area, the most basic notion in symplectic topology, is not preserved by symplectic homeomorphisms.

Theorem 1.5 (Buhovsky-Opshtein [BO16])**.** *For $n \geq m+2$, there exists a symplectic homeomorphism $h : \mathbb{C}^n \to \mathbb{C}^n$ with support in an arbitrary neighborhood of $Q := \mathbb{D}^m \times 0_{n-m} \subset \mathbb{C}^n$, such that*

$$h|_Q = \frac{1}{2}\mathrm{Id}.$$

Here, $\mathbb{D}$ denotes the disc of radius 1 in $\mathbb{C}$ and 0_{n-m} stands for $0 \in \mathbb{C}^{n-m}$.

The key technical result needed in the proof of the above theorem is an h-principle-type statement which the authors refer to as a quantitaive h-principle for symplectic discs. Roughly speaking, it states that two symplectic discs of equal area which are C^0–close can be mapped to one another via a C^0–small Hamiltonian isotopy; see [BO16, Thm. 2]. The novelty of this quantitative h-principle is the fact it guarantees the existence of a C^0–small Hamiltonian isotopy; this is the crucial point which is needed in the construction of the homeomorphism h in Theorem 1.5.

Although it has not yet been proven, it is expected that quantitative h-principles could be used to construct symplectic a homeomorphism h mapping a (smooth) isotropic submanifold to a symplectic one.

Theorem 3 of [BO16] proves a rigidity statement for symplectic submanifolds of codimension 2 which implies in particular that the homeomorphism h, as in the above theorem, does not exist when $n = m + 1$.

Another example of softness is the C^0 counterexample to the Arnold conjecture which we present below because of its relevance to symplectic dynamics.

[10]The reduction of a coisotropic submanifold C is by definition the quotient space $\mathcal{R} := C/\mathcal{F}$, where $\mathcal{F}$ denotes the characteristic foliation of C. This space is at least locally smooth and it can be equipped with a natural symplectic structure induced by ω.

1.3 Homeomorphisms in symplectic dynamics

The remainder of this chapter addresses dynamical aspects of C^0 symplectic topology. There are numerous beautiful results and open problems at the intersection of continuous symplectic topology and dynamical systems and, as mentioned earlier, the two subjects have interacted frequently in recent years. As we will see below, these interactions has been particularly fruitful in the case of surfaces where symplectic homoemorphisms coincide with area and orientation preserving homeomorphisms.

1.3.1 Area-preserving homeomorphisms I : The displaced discs problem and related developments

My research in symplectic dynamics began with the article [Sey13b] which, relying on my results from [Sey13a], answers two beautiful questions, posed by the dynamists F. Béguin, S. Crovisier, and F. Le Roux, about area-preserving homeomorphisms. The first of these questions, the *displaced discs problem*, was mentioned earlier. Here is its precise formulation: Consider a closed and connected symplectic surface (Σ, ω). Recall that a homeomorphism ϕ is said to displace a set B if $\phi(B) \cap B = \emptyset$.

Question 1.6 (Displaced Discs). *For $a > 0$, define*

$$\mathcal{G}_a = \{\theta \in \overline{\text{Ham}}(\Sigma, \omega) : \theta \textit{ displaces a topological disc of area at least } a\}.$$

Does the C^0 closure of $\mathcal{G}_a$ contain the identity?

The second question of Béguin-Crovisier-Le Roux is intimately related to the first one and is formulated as follows.

Question 1.7 (Béguin-Crovisier-Le Roux). *Consider $\overline{\text{Ham}}(\mathbb{S}^2, \omega)$, the group of Hamiltonian homeomorphisms of the two-sphere,[11] equipped with the C^0 topology. Does this group possess a dense conjugacy class?[12]*

Interestingly enough, in posing the above questions Béguin, Crovisier, and Le Roux were partially motivated by attempts at better understanding questions on the algebraic structures of groups of area-preserving homeomorphisms, with the simplicity conjecture, which we discuss below, being the most prominent of such questions.

I will now explain how the results from [Sey13a] yield a negative answer to both of these questions. The main result of [Sey13a] establishes the C^0 continuity of a certain Floer theoretic invariant $\gamma : \text{Ham}(\Sigma, \omega) \to \mathbb{R}$ which is usually referred to as the spectral norm; it was introduced by Viterbo [Vit92], Schwarz [Sch00], and Oh [Oh99, Oh05].

Theorem 1.8 (S. [Sey13a]). *Let (Σ, ω) denote a closed symplectic surface. The spectral norm $\gamma : \text{Ham}(\Sigma, \omega) \to \mathbb{R}$ is continuous with respect to the C^0 topology on* $\text{Ham}(\Sigma, \omega)$. *Moreover, it extends continuously to* $\overline{\text{Ham}}(\Sigma, \omega)$.

It is a classical result that $\gamma|_{\mathcal{G}_a} \geq a$; see [Vit92, Sch00, Oh05]. Since $\gamma(\text{Id}) = 0$, it follow immediately that Id is not in the C^0 closure of $\mathcal{G}_a$. This answers Question 1.6. The answer to Question 1.7 is also negative because the conjugacy class of any $\phi \neq \text{Id}$ is contained in $\mathcal{G}_a$ for some $a > 0$.[13]

[11] In the case of $\mathbb{S}^2$, Hamiltonian homeomorphisms coincide with area and orientation persevering homeomorphisms.

[12] Suppose that $\Sigma \neq S^2$. It follows from the works of Gaumbado-Ghys [GG04] and Entov-Polterovich-Py [EPP12] that $\overline{\text{Ham}}(\Sigma, \omega)$ does not possess dense conjugacy classes. This follows from the fact the group carries C^0–continuous and homogeneous quasimorphisms; see Theorem 1.2 of [EPP12].

[13] I learned of Questions 1.6 & 1.7 after the publication of [Sey13a]. This is why the solutions to these questions appears in the separate note [Sey13b]. Working independently of me, Daniel N. Dore and Andrew D. Hanlon [DH13], obtained results similar to those of [Sey13b]; their arguments are closely related to those outlined here; in particular, they rely on Theorem 1.8.

I should add that Theorem 1.8 has recently been extended to some higher dimensional symplectic manifolds such as symplectically aspherical manifolds, in the article by Buhovsky-Humilière and myself [BHS18a], projective spaces, by Shelukhin [She18], and negatively monotone manifolds, by Kawamoto [Kaw19]. Consequently, one obtains negative answers to generalizations of Questions 1.6 & 1.7 in these settings.

In Section 3.1, we will present the proof of Theorem 1.8 given in [BHS18a].

Rokhlin groups and barcodes as weak conjugacy invariants

Question 1.7, which can be stated for any topological group, has been of interest in ergodic theory. Glasner-Weiss refer to a group with a dense conjugacy class as a *Rokhlin* group; see [GW01, GW08] and the references therein for further details on numerous example. An interesting example of a Rokhlin group is the group of homeomorphisms of the two-sphere which contrast the case of area-preserving homeomorphisms.

In the symplectic world, rigidity phenomena, such as Theorem 1.3, provide obstructions to $\overline{\mathrm{Ham}}(M,\omega)$ being Rokhlin. These obstructions appear in the form of conjugacy invariants which are C^0 continuous, such as barcodes and the spectral norm. Naturally, all classical dynamical invariants, like entropy or rotation vectors, are invariant under conjugation. However, what is very surprising is that when Béguin, Crovisier, and Le Roux posed their question there were no known dynamical invariants of Hamiltonian homeomorphisms which were continuous with respect to the C^0 topology. This is why their question is so difficult to approach from a purely dynamical angle.

The search for C^0 continuous conjugacy invariants naturally led Le Roux, Viterbo, and I [LRSV] to an equivalence relation on the group of Hamiltonian homeomorphisms that is better adapted to the C^0 topology and is called *weak conjugacy*. It is defined to be the strongest Hausdorff equivalence relation, on the group of Hamiltonian homeomorphisms, which is weaker than the conjugacy relation.[14] It has the following features: Its equivalence classes are closed and any C^0 continuous conjugacy invariant is automatically a weak conjugacy invariant. Furthermore, existence of a dense conjugacy class would imply the triviality of the weak conjugacy relation, *i.e.* any two homeomorphisms would be weakly conjugate.

It follows from our results on C^0 continuity of barcodes [LRSV, BHS18a], that *barcodes are weak conjugacy invariants*. In [LRSV], we use this fact to present a refinement of my solution to Question 1.7. Given a Hamiltonian homeomorphism f of a closed surface with finitely many contractible fixed points, define its absolute Lefschetz number to be the sum of the absolute values of the Lefschetz indices of its contractible fixed points. Now suppose that f, g are two Hamiltonian diffeomorphisms with finitely many fixed points. *If f and g are weakly conjugate, then the absolute Lefschetz number of f coincides with that of g.* Note that this immediately implies that the weak conjugacy relation on $\overline{\mathrm{Ham}}(\Sigma,\omega)$ is non-trivial and hence, $\overline{\mathrm{Ham}}(\Sigma,\omega)$ is not Rokhlin.

In the same article, we prove that the statement in the above paragraph extends to a large class of Hamiltonian homeomorphisms. We say an isolated fixed point of a Hamiltonian homeomorphism is *maximally degenerate* if it is accumulated by periodic orbits of every period. In [LRSV], using Le Calvez's theory of transverse foliations [LC05] and Le Roux's work on local dynamics [LR13], we prove the following generalization of the statement from the above paragraph:

Theorem 1.9 (Le Roux-S.-Viterbo [LRSV])**.** *Let (Σ,ω) denote a closed and connected symplectic surface. Suppose that f, g are two Hamiltonian homeomorphisms of (Σ,ω) with finitely many fixed*

[14] The weak conjugacy relation may be characterized by the following universal property: $\varphi \sim \psi$ if and only if $f(\varphi) = f(\psi)$ for any continuous function $f : \overline{\mathrm{Ham}}(M,\omega) \to Y$ such that f is invariant under conjugation and Y is a Hausdorff topological space. As a consequence, the notion of weak conjugacy arises naturally in settings where one needs to consider continuous conjugacy invariants. This is for example the case in the study of mapping class group actions on the circle; see the article by K. Mann and M. Wolff [MW] and the references therein.

points none of which are maximally degenerate. If f, g are weakly conjugate, then they have the same absolute Lefschetz number.

Idea of the proof. As mentioned earlier, we prove in the article that barcodes are weak conjugacy invariants. Furthermore, we prove that for a Hamiltonian homeomorphism as above the absolute Lefschetz number coincides with the total number of endpoints in its barcode. □

Towards a dynamical interpretation of spectral invariants

The spectral invariants of a Hamiltonian diffeomorphism are a collection of real numbers $\{c(a, \phi) \in \mathbb{R} : a \in H_*(M) \setminus \{0\}\}$ which are constructed via a min-max process on the Floer complex of ϕ. These invariants were introduced by Viterbo [Vit92], Schwarz [Sch00], and Oh [Oh99, Oh05] and, as the name suggests, they are intimately related to the spectral norm γ introduced above. Indeed, on aspherical manifolds, the spectral norm of a Hamiltonian diffeomorphism ϕ is given by the quantity $\gamma(\phi) := c([M], \phi) - c([pt], \phi)$. Spectral invariants, through various incarnations, have played an important role in symplectic topology and dynamics; see, for example, [Vit92, Sch00, Oh05, EP03, EP09b, Gin10, Iri15, AI16].

In the article [HLRS16], motivated by the solution to the displaced discs problem, Humilière, Le Roux and I used ideas from Le Calvez's theory of transverse foliations to give a purely dynamical construction of spectral invariants for autonomous Hamiltonian systems of closed surfaces other than the sphere. This dynamical interpretation allowed us to explicitly compute spectral invariants for a large class of examples. Furthermore, we answered open questions, raised in [Ent14], about Entov and Polterovich's theory of quasi-states and gave a purely topological characterization of super-heavy sets of closed surfaces.

1.3.2 Area-preserving homeomorphisms II: The simplicity conjecture and periodic Floer homology

Let $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ denote the group of area-preserving homeomorphisms of the two-disc which are the identity near the boundary. Recall that a group is *simple* if it does not have a non-trivial proper normal subgroup. The following fundamental question was raised[15] in the 1970s:

Question 1.10. *Is the group* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ *simple?*

Indeed, the algebraic structure of the group of volume-preserving homeomorphisms has been well-understood in dimension at least three since the work of Fathi [Fat80] from the 70s; but, the case of surfaces, and in particular Question 1.10, has long remained mysterious.

This question has been the subject of wide interest. For example, it is highlighted in the plenary ICM address of Ghys [Ghy07b, Sec. 2.2]; it appears on McDuff and Salamon's list of open problems [MS17, Sec. 14.7]; for other examples, see [Ban78, Fat80, Ghy07a, Bou08, LR10a, LR10b, EPP12]. It has generally been believed since the early 2000s that the group $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ is not simple: McDuff and Salamon refer to this as the simplicity conjecture.

The simplicity conjeture has been one of the main motivations behind the development of C^0-symplectic topology; for example, the articles [OM07, Oh10, Vit06, BS13, Hum11, LR10b, LR10a, EPP12] were, at least partially, motivated by this conjecture. My own work on C^0 continuity of spectral invariants [Sey12, Sey13a, Sey13b], which we discussed above, grew out of attempts to settle the conjecture via the use of spectral invariants from Hamiltonian Floer homology.

The main theorem of my article [CGHS20], written jointly with Cristofaro-Gardiner and Humilière, resolves this conjecture in the affirmative.

[15] The history of when this question was raised seems complicated. It is asked by Fathi in the paper [Fat80]. However, it has been suggested to us by Ghys that the question dates back to considerably earlier, possibly to Mather or Thurston.

Theorem 1.11 (Cristofaro-Gardiner-Humilière-S. [CGHS20]). *The group* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ *is not simple.*

In fact, we can obtain an a priori stronger result. Recall that a group G is called *perfect* if its commutator subgroup $[G, G]$ satisfies $[G, G] = G$. The commutator subgroup $[G, G]$ is a normal subgroup of G. Thus, every non-abelian simple group is perfect. However, in the case of certain transformation groups, such as $\mathrm{Homeo}_c(\mathbb{D}, \omega)$, a general argument due to Epstein and Higman [Eps70, Hig54] implies that perfectness and simplicity are equivalent. Hence, we obtain the following corollary.

Corollary 1.12 (Cristofaro-Gardiner-Humilière-S. [CGHS20]). *The group* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ *is not perfect.*

We remark that in higher dimensions, the analogue of Theorem 1.11 contrasts our main result: by [Fat80], the group $\mathrm{Homeo}_c(\mathbb{D}^n, \mathrm{Vol})$ of compactly supported volume-preserving homeomorphisms of the n-ball is simple for $n \geq 3$. It also seems that the structure of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ is radically different from that of the group $\mathrm{Diff}_c(\mathbb{D}, \omega)$ of compactly supported area-preserving diffeomorphisms, as we will review below.

Background

To place Theorem 1.11 in its appropriate context, we review here the long and interesting history surrounding the question of simplicity for various subgroups of homeomorphism groups of manifolds. Our focus will be on compactly supported homeomorphisms/diffeomorphisms of manifolds without boundary in the component of the identity[16].

In the 1930s, in the "Scottish Book", Ulam asked if the identity component of the group of homeomorphisms of the n–dimensional sphere is simple. In 1947, Ulam and von Neumann announced in an abstract [UvN47] a solution to the question in the Scottish Book in the case $n = 2$.

In the 50s, 60s, and 70s, there was a flurry of activity on this question and related ones.

First, the works of Anderson [And58], Fisher [Fis60], Chernavski, Edwards and Kirby [EK71] led to the proof of simplicity of the identity component in the group of compactly supported homeomorphisms of any manifold. These developments led Smale to ask if the identity component in the group of compactly supported diffeomorphisms of any manifold is simple [Eps70]. This question was answered affirmatively by Epstein [Eps70], Herman [Her73], Mather [Mat74a, Mat74b, Mat75], and Thurston [Thu74][17].

The connected component of the identity in volume-preserving, and symplectic, diffeomorphisms admits a homomorphism, called **flux**, to a certain abelian group. Hence, these groups are not simple when this homomorphism is non-trivial. Thurston proved, however, that the kernel of flux is simple in the volume-preserving setting for any manifold of dimension at least three; see [Ban97, Chapter 5]. In the symplectic setting, Banyaga [Ban78] then proved that this group is simple when the symplectic manifold is closed; otherwise, it is not simple as it admits a non-trivial homomorphism, called **Calabi**, and Banyaga showed that the kernel of Calabi is always simple. We will recall the definition of Calabi in the case of the disc in Section 3.3.

As mentioned above, the simplicity of the identity component in compactly supported volume-preserving homeomorphisms is well-understood in dimensions greater than two,[18] thanks to the article [Fat80], in which Fathi shows that, in all dimensions, the group admits a homomorphism, called "mass-flow"; moreover, the kernel of mass-flow is simple in dimensions greater than two. On simply connected manifolds, the mass-flow homomorphism is trivial, and so the group is indeed simple in dimensions

[16] The simplicity question is interesting only for compactly supported maps in the identity component, because this is a normal subgroup of the larger group. The group $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ coincides with its identity component.

[17] More precisely, Epstein, Herman and Thurston settled the question in the case of smooth diffeomorphisms, while Mather resolved the case of C^r diffeomorphisms for $r < \infty$ and $r \neq \dim(\mathrm{M}) + 1$. The case of $r = \dim(\mathrm{M}) + 1$ remains open.

[18] The group is trivial in dimension 1.

greater than two — in particular, $\text{Homeo}_c(\mathbb{D}^n, \text{Vol})$, the higher dimensional analogue of the group in Question 1.10, is simple for $n > 2$, as stated above.

Thus, the following rather simple picture emerges from the above cases of the simplicity question. In the non-conservative setting, the connected component of the identity is simple. In the conservative setting, there always exists a natural homomorphism (flux, Calabi, mass-flow) which obstructs the simplicity of the group. However, the kernel of the homomorphism is always simple.

The case of surfaces and our case of the disc

Despite the clear picture above, established by the end of the 70s, the case of area-preserving homeomorphisms of surfaces has remained unsettled — the simplicity question has remained open for the disc and more generally for the kernel of the mass-flow homomorphism— underscoring the importance of answering Question 1.10.

In fact, the case of area-preserving homeomorphisms of the disc does seem drastically different. For example, the natural homomorphisms flux, Calabi, and mass-flow mentioned above that obstruct simplicity are all continuous with respect to a natural topology on the group; however, there can not exist a continuous homomorphism out of $\text{Homeo}_c(\mathbb{D}, \omega)$ with a proper non-trivial kernel, when $\text{Homeo}_c(\mathbb{D}, \omega)$ is equipped with the C^0-topology; see, for example, [CGHS20, Cor. 2.5].

Of course, $\text{Homeo}_c(\mathbb{D}, \omega)$ could still admit a discontinuous homomorphism. For example, Ghys asks, in his ICM address [Ghy07b], whether the Calabi invariant extends to $\text{Homeo}_c(\mathbb{D}, \omega)$. However, as far as we know, even if one does not demand that the homomorphism is continuous, there is still no natural geometrically constructed homomorphism out of $\text{Homeo}_c(\mathbb{D}, \omega)$ that would in any sense be an analogue of the flux, Calabi, or mass-flow; rather, as we will see, our proof of non-simplicity goes by explicitly constructing a non-trivial normal subgroup and showing that it is proper. It seems likely to us that $\text{Homeo}_c(\mathbb{D}, \omega)$ indeed has a more complicated algebraic structure.

We will now see in the discussion of Le Roux's work below another way in which the case of area-preserving homeomorphisms of the disc is quite different.

"Lots" of normal subgroups and the failure of fragmentation

Le Roux [LR10a] has previously studied the simplicity question for $\text{Homeo}_c(\mathbb{D}, \omega)$, and this provides useful context for our work; it is also valuable to combine his conclusions with our Theorem 1.11.

As Le Roux explains, Fathi's proof of simplicity in higher-dimensions does not work in dimension 2, because it relies on the following "fragmentation" result: any element of $\text{Homeo}_c(\mathbb{D}^n, \text{Vol})$ can be written as the product of two others, each of which are supported on topological balls of three-fourths of the total volume; this is not true on the disc.

Building on this, Le Roux constructs a whole family P_ρ, for $0 < \rho \leq 1$, of quantitative fragmentation properties for $\text{Homeo}_c(\mathbb{D}, \omega)$: the property P_ρ asserts that there is some $\rho' < \rho$ and a positive integer N, such that any group element supported on a topological disc of area ρ can be written as a product of N elements, each supported on discs of area at most ρ'; we refer the reader to [LR10a] for more details. For example, if Fathi's fragmentation result held in dimension 2, then P_1 would hold.

Le Roux then establishes the following alternative: if any one of these fragmentation properties holds, then $\text{Homeo}_c(\mathbb{D}, \omega)$ is simple; otherwise, there is a huge number of proper normal subgroups, constructed in terms of "fragmentation norms". Thus, in view of our Theorem 1.11, we have not just one proper normal subgroup but "lots" of them; for example, combining Le Roux's work [LR10a, Cor. 7.1] with our Theorem 1.11 yields the following.

Corollary 1.13. *Every compact*[19] *subset of* $\text{Homeo}_c(\mathbb{D}, \omega)$ *is contained in a proper normal subgroup.*

[19] As above, we are working in the C^0-topology.

As Le Roux explains [LR10a, Sec. 7], this is “radically” different from the situation for the group $\mathrm{Diff}_c(\mathbb{D}, \omega)$ of compactly supported area-preserving diffeomorphisms of the disc with its usual topology, where the normal closure of many one-parameter subgroups is the entire group; a similar argument can be used to show that Corollary 1.13 is false for the other cases of the simplicity question reviewed above.

In view of Theorem 1.11, we also have the following result about the failure[20] of fragmentation in dimension 2, which follows by invoking [LR10a, Thm. 0.2].

Corollary 1.14. *None of the fragmentation properties P_ρ for $0 < \rho \leq 1$ hold in* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$.

This generalizes a result of Entov-Polterovich-Py [EPP12, Sec. 5.1], who established Corollary 1.14 for $1/2 \leq \rho \leq 1$.

1.3.3 The Arnold conjecture in the topological setting: a counterexample and a proof

The Arnold conjecture, in its original formulation, states that a Hamiltonian diffeomorphism of a closed and connected symplectic manifold (M, ω) must have at least as many fixed points as the minimal number of the critical points of a smooth function on M. There exist other formulations of the conjecture some of which we will discuss below.

What makes this conjecture so remarkable is the large number of fixed points predicted by it, which is often interpreted as a manifestation of symplectic rigidity. In contrast to Arnold's conjecture, the classical Lefschetz fixed-point theorem cannot predict the existence of more than one fixed point for a general diffeomorphism. Ever since its inception, this simple and beautiful conjecture has been a powerful driving force in the development of symplectic topology. The most important breakthrough towards a solution of this conjecture came with Floer's invention of what is now called *Hamiltonian Floer homology* which established a variant of the Arnold conjecture on a large class of symplectic manifolds [Flo86b, Flo88b, Flo89d]. The above version of the Arnold conjecture has been established on symplectically aspherical manifolds by Rudyak and Oprea in [RO99] who built on earlier works of Floer [Flo89b] and Hofer [Hof88b]. We should mention that prior to the discovery of Floer homology, the Arnold conjecture was proven by Eliashberg [Eli79] on closed surfaces (see also Sikorav [Sik85]), by Conley and Zehnder [CZ83] on higher dimensional tori, and by Fortune and Weinstein [For85, FW85] on complex projective spaces.

Fixed, and periodic, points of Hamiltonian homeomorphisms have been studied extensively in dimension two: Matsumoto [Mat00], building on an earlier paper of Franks [Fra96b], has proven that Hamiltonian homeomorphisms of surfaces satisfy the Arnold conjecture. Le Calvez's theory of transverse foliations [LC05] not only proves the Arnold conjecture but also the Conley conjecture on existence of infintely many periodic points for these homeomorphisms [LC06].

In striking contrast to the rich theory in dimension two, there are very few results on fixed point theory of Hamiltonian homeomorphisms in higher dimensions. Indeed, none of the powerful tools of surface dynamics seem to generalize in an obvious manner to dimensions higher than two. The main theorem of [BHS18b] proves that the Arnold conjecture fails rather spectacularly for Hamiltonian homeomorphisms in dimensions greater than two. As I mentioned earlier, there exist alternate definitions for Hamiltonian homeomorphisms and so one might wonder whether any of those definitions could satisfy the conjecture. In fact, as we will explain below, there is no hope for proving the Arnold conjecture, as formulated above, for any alternate definition of Hamiltonian homeomorphisms as long as a minimal set of requirements is satisfied. Here is the main result of [BHS18b].

[20] One should stress that a weaker version of fragmentation does hold, even in dimension 2, for example if we remove the requirement in the definition of P_ρ that the product has N elements then the corresponding property holds for all ρ by [Fat80].

Theorem 1.15 (Buhovsky-Humilière-S. [BHS18b])**.** *Let (M,ω) denote a closed and connected symplectic manifold of dimension at least 4. There exists $f \in \overline{\mathrm{Ham}}(M,\omega)$ with a single fixed point. Furthermore, f can be chosen to satisfy either of the following additional properties.*

1. *Let $\mathcal{H}$ be a normal subgroup of* $\mathrm{Sympeo}(M,\omega)$ *which contains* $\mathrm{Ham}(M,\omega)$ *as a proper subset. Then, $f \in \mathcal{H}$.*

2. *Let p denote the unique fixed point of f. Then, f is a symplectic diffeomorphism of $M \setminus \{p\}$.*

A few remarks are in order. First, we should point out that every Hamiltonian homeomorphism possesses at least one fixed point. This is because a Hamiltonian homeomorphism is by definition a uniform limit of Hamiltonian diffeomorphisms and it is a non-trivial fact that a Hamiltonian diffeomorphism has at least one fixed point.[21]

With regards to the second property, we point out that it is natural to expect f to have at least one non-smooth point. Indeed, since Hamiltonian Floer homology predicts that a Hamiltonian diffeomorphism can never have as few as one fixed point, our homeomorphism f must necessarily be non-smooth on any symplectic manifold (M,ω) with the property [22] that $\overline{\mathrm{Ham}}(M,\omega) \cap \mathrm{Diff}(M) = \mathrm{Ham}(M,\omega)$.

Next, we remark that $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Symp}(M,\omega)$. Hence, it is reasonable to expect that any alternative candidate, say $\mathcal{H}$, for the group of Hamiltonian homeomorphisms should contain $\mathrm{Ham}(M,\omega)$ and be a normal subgroup of $\mathrm{Sympeo}(M,\omega)$. It is indeed the case that $\overline{\mathrm{Ham}}(M,\omega) \trianglelefteq \mathrm{Sympeo}(M,\omega)$. Therefore, the first property in the above theorem states that there is no hope of proving the Arnold conjecture for any alternate definition of Hamiltonian homeomorphisms.

Lastly, I should mention that, according to a result of G. Lu, the Arnold conjecture does hold if one requires some minimal level of regularity; see [Lu05] for further details.

A topological version of the Arnold conjecture: barcodes to the rescue

We will now explain how Theorem 1.3 allows us to present a generalization of the Arnold conjecture which does hold for Hamiltonian homeomorphisms. An important difference between the smooth and topological settings which should be kept in mind is that, C^0 generically, Hamiltonian homeomorphisms, even diffeomorphisms, have infinitely many fixed points; see [BHS18b, App. A]. Hence, our goal here will be to address the conjecture for all elements of $\overline{\mathrm{Ham}}(M,\omega)$ and not a generic subset of it.

The homological version of the Arnold conjecture states that a Hamiltonian *diffeomorphism* of a closed and connected symplectic manifold (M,ω) must have at least as many fixed points as the *cup length* of M. Cup length, denoted by $\mathrm{cl}(M)$, is a topological invariant of M which is defined as follows:[23]

$$\mathrm{cl}(M) := \max\{k+1 \,:\, \exists\, a_1,\dots,a_k \in H_*(M),\ \forall i, \deg(a_i) \neq \dim(M) \\ \text{and } a_1 \cap \dots \cap a_k \neq 0\}.$$

This version of the Arnold conjecture was proven, for Hamiltonian diffeomorphisms, on $\mathbb{C}P^n$ [For85, FW85] and on symplectically aspherical manifolds [Flo89d, Hof88b, RO99].[24]

Recall that Theorem 1.3 allows us to define the barcode of a Hamiltonian homeomorphism (upto shift). In particular, we can now make sense of *the total number of spectral invariants of a Hamiltonian*

[21] This fact is an immediate consequence of the Arnold conjecture.

[22] It can be shown that this property holds for closed symplectic surfaces, as well as for the standard $\mathbb{C}P^2$ and monotone $S^2 \times S^2$.

[23] Here, $\cap$ refers to the intersection product in homology. Cup length can be equivalently defined in terms of the cup product in cohomology.

[24] I should emphasize that cuplength estimates have not been established for general symplectic manifolds and, in fact, Floer himself tended to "believe that there are more than technical reasons for this"; see [Flo89d, Page 577].

homeomorphism ϕ: For example, we can simply define this quantity to be the total number of half-infinite rays in the barcode $\mathcal{B}(\phi)$. The theorem below shows that, in spite of the counter-example from [BHS18b], the cup length estimate from the homological Arnold conjecture survives if we include in the count the total number of spectral invariants.

We need the following notion before stating the result: A subset $A \subset M$ is homologically non-trivial if for every open neighborhood U of A the map $i_* : H_j(U) \to H_j(M)$, induced by the inclusion $i : U \hookrightarrow M$, is non-trivial for some $j > 0$. Clearly, homologically non-trivial sets are infinite.

Theorem 1.16 (Buhovsky-Humilière-S. [BHS18a])**.** *Let (M, ω) denote a closed, connected and symplectically aspherical manifold. Let $\phi \in \overline{\mathrm{Ham}}(M, \omega)$ be a Hamiltonian homeomorphism. If the total number of spectral invariants of ϕ is smaller than* $\mathrm{cl}(M)$*, then the set of fixed points of ϕ is homologically non-trivial; hence, it is infinite.*

In the smooth case, Theorem 1.16 was established by Howard [How12], and our proof is inspired by his. For a smooth Hamiltonian diffeomorphism, spectral invariants correspond to actions of certain fixed points. Therefore, Theorem 1.16 is a generalization of the Arnold conjecture in the smooth setting. However, when it comes to Hamiltonian homeomorphisms, there is a total breakdown in the correspondence between spectral invariants and actions of fixed points: Indeed, the Hamiltonian homeomorphism we construct in [BHS18b] has a single fixed point and many[25] distinct spectral invariants.

1.4 Open problems

I will end this chapter with a brief discussion of several open questions in continuous symplectic topology. Some of these have naturally emerged as the field has grown while others have been long-open problems which have served as guiding lights for the field.

The symplectic four-sphere problem

Having the notion of symplectic homeomorphisms at hand, one can define a **topological symplectic manifold** to be a topological manifold equipped with an atlas whose transition maps consist of symplectic homeomorphisms. Here is one of the most fascinating and longstanding open problems in C^0 symplectic topology.[26]

Problem 1. *Could a non-symplectic manifold, such as the four-sphere $\mathbb{S}^4$, admit the structure of a topological symplectic manifold?*

The most elementary necessary condition for the existence of a smooth symplectic structure on a manifold of dimension $2n$ is the existence of a non-zero degree–2 cohomology class. The four sphere $\mathbb{S}^4$ is the most conspicuous manifold not meeting this condition. For this reason, this question is often referred to as the **symplectic four-sphere problem.** Of course, the spheres $\mathbb{S}^{2n}$, for $n > 1$, are non-symplectic (in the smooth sense) for the same reason. This discussion leads to the following question:

Problem 2. *Suppose that M admits the structure of a topological symplectic manifold. Could the cohomology of M vanish in degree 2?*

[25] The spectral invariants of this Hamiltonian homeomorphism coincide with those of a C^2–small Morse function. Hence, their count is at least the cup length of the manifold. This is perhaps an indication that, on symplectic manifolds of dimension at least four, one cannot define the notion of action for fixed points of an arbitrary Hamiltonian homeomorphism. See Remark 20 in [BHS18b].

[26] As far as I know, this question was first proposed by Hofer in the late 1980s following the discovery of symplectic capacities.

Clearly, one can view these questions in light of Gromov's soft versus hard philosophy with the negative/positive answer indicating the prevalence of rigidity/flexibility. Interestingly enough, it is even plausible that a topological manifold which does not admit a smooth structure could be symplectic in the above sense.

The C^0 flux conjecture

Consider a closed symplectic manifold (M, ω). Denote by $\mathrm{Symp}_0(M, \omega)$ the connected component of the identity in $\mathrm{Symp}(M, \omega)$. We know that $\mathrm{Ham}(M, \omega)$ is a subset of $\mathrm{Symp}_0(M, \omega)$; in fact, it is a normal subgroup of it. Here is the statement of the C^0 flux conjecture.

Conjecture. $\mathrm{Ham}(M, \omega)$ *is* C^0 *closed in* $\mathrm{Symp}_0(M, \omega)$.

The article [LMP98], by Lalonde, McDuff, and Polterovich, contains a full discussion of this conjecture and a proof of it in special cases. More recently, Buhovsky [Buh14] has established the conjecture on a larger class of manifolds. As is noted in [LMP98], the C^0-Flux conjecture is a fundamental question lying at the boundary between hard and soft symplectic topology; very little is known about the C^0-Flux conjecture, beyond what appears in the above articles, and today's symplectic technology seems to be insufficient for approaching this question in full generality.

I should point out that the analogous statement in the C^1 topology was known as the flux conjecture and was settled by Ono [Ono06]. For more detailed discussions on these conjectures, see [LMP98, MS17].

Coisotropic rigidity

According to Theorem 1.4, if C' is the smooth image of a coisotroic submanifold under a symplectic homeomorphism, then it is coisotropic. Now, observe that C' can be written as a limit of coisotropic embeddings. Hence, one is naturally led to ask if the coisotropic rigidity theorem is a manifestation of more general results about coisotropic embeddings.

Problem 3. *Let C_i denote a sequence of closed coisotropic submanifolds which converge uniformly, as embeddings, to a submanifold C. Is C coisotropic?*

As mentioned earlier, it is proven in [LS85] that if the C_is are closed Lagrangians, then C is indeed Lagrangian. Although we know of no counterexamples to the above question, it seems reasonable to require the C_is to be Hamiltonian diffeomorphic and satisfy a technical condition called *stability* which implies that a neighborhood of the coisotropic in question is symplectically trivial. The reader might wonder why we do not raise any questions about the charecteristic foliations of the C_is. The reason is that in [GG15] Ginzburg and Gürel give an example, where the C_is converge to a coisotropic C, but the characteristic foliations of the C_is do not converge to that of C. However, in their construction the coisotropics C_i are not stable. Lastly, we should mention that in contrast to the coisotropic rigidity theorem, where C is not assumed to be closed, it is important here to suppose that C_i, C are closed. Indeed, there exist folkloric examples indicating that the answer is negative without this assumption.

Another interesting direction to pursue is to see if the aforementioned results on reduction of symplectic homeomorphisms [HLS16, BO16, Bus19] extend to more general settings.

Contact homeomorphisms and C^0 contact geometry

There exists a version of the Eliashberg-Gromov theorem in contact geometry stating that a smooth C^0 limit of contact diffeomorphisms is again a contact diffeomorphism. As before, this gives rise to the notion of contact homeomorphisms and C^0 contact geometry. One is then immediately led to ask how much of the underlying contact geometry is preserved by contact homeomorphisms. Although the topic has been studied recently, see for example [BS17, MS15, MS14b, MS16, MS14a, Mül19, RZ18, Ush20, Nak20], it remains relatively unexplored and there are many interesting open problems. For example, we do not know if the coisotropic rigidity theorem of [HLS15b] has an analogue in C^0 contact geometry.

Problem 4. *Suppose that the image of a coisotropic submanifold under a contact homeomorphism is smooth. Is the image necessarily coisotropic?*

The standard approach of working in symplectizations is not well suited to the above question since contact homeomorphisms do not lift to symplectic homeomorphisms of the symplectization. There has been some progress in recent years towards this problem. Usher [Ush20], has shown that the answer is affirmative under certain assumptions on the contact homeomorphism; see also [RZ18].

The situation here seems to be genuinely different than in the symplectic world, and we might in fact be facing a mix of flexibility and rigidity: Recall that, by [LS85], a smooth C^0–limit of Lagrangian embeddings is itself Lagrangian. The analogous statement does not hold for Legendrians as it is well-known that a sequence of closed Legendrian embeddings could converge to a non-Legendrian, *e.g.* any knot can be approximated by Legendrian knots. However, recently, Nakamura has shown that under certain assumptions the C^0-limit of a sequence of Legendrian submanifolds with uniformly bounded Reeb chords is again Legendrian [Nak20].

Barcodes and Hamiltonian homeomorphisms

The construction of barcodes for Hamiltonian homeomorphisms is very technical and abstract. They are first constructed via Hamiltonian Floer theory, which is complicated enough, for generic Hamiltonian diffeomorphisms. The definition is then extended to Hamiltonian homeomorphisms via a limiting process whose well-definedness rests on non-trivial continuity results from [LRSV, BHS18a]. As a consequence, barcodes of Hamiltonian homeomorphisms are dynamically mysterious and it is non-trivial to extract dynamical information from them. Hence, it would be beneficial to find a dynamical interpretation of the theory of barcodes and further develop this theory for Hamiltonian homeomorphisms.

This direction of research seems to be particularly promising in dimension two where one can combine the methods of C^0 symplectic geometry and those of surface dynamics such as Le Calvez's transverse foliation theory [LC05] or Le Roux's work on local dynamics [LR13]. I will list a few concrete questions here.

In the smooth setting, the endpoints of bars (of the barcode of a Hamiltonian diffeomorphism) correspond to actions of fixed points. Recently, J. Wang [Wan17, Wan16] has defined the notion of action for fixed points of Hamiltonian homeomorphisms of surfaces.[27]

Problem 5. *Do the endpoints of the bars, in the barcode of a Hamiltonian homeomorphism of a closed surface, correspond to actions of fixed points?*

One can associate to an isolated fixed point of a Hamiltonian diffeomorphism a vector space called the **local Floer homology** of the fixed point; it depends only on a germ of the Hamiltonian diffeomorphism near the fixed point. Local Floer homology is a crucial tool for studying local dynamics of fixed/periodic points of Hamiltonian diffeomorphisms and it has played an extremely important role in Ginzburg's proof of the Conley conjecture [Gin10] and the subject of periodic points of Hamiltonian diffeomorphisms. An affirmative answer to the question below would provide us with a definition of local Floer homology for fixed points of Hamiltonian homeomorphisms.

Problem 6. *Is it possible to associate a local barcode to an isolated fixed point of a Hamiltonian homeomorphism?*

[27] To be more precise, Wang defines the notion of action for a certain class of fixed points which, by an unpublished result of Le Roux, form a dense subset of all fixed points.

A positive answer to the above question, even if it is limited to the case of surfaces, would be of great interest from a dynamical viewpoint. Indeed, defining local Floer homology would lead to a new set of tools for studying local dynamics of fixed and periodic points of area-preserving homeomorphisms. This is a subject which has received much interest from surface dynamists; see for example the works of Le Calvez, Le Roux, and Yoccoz [LC03, LR13, LCY97].

Another interesting problem in this direction is to see whether the hypothesis on non-existence of maximally degenerate fixed points can be removed from Theorem 1.9.

Problem 7. *Does the statement of Theorem 1.9 hold for homeomorphisms which possess maximally degenerate fixed points?*

It seems likely that affirmative answers to the previous two problems could be of help with the removal of the said hypothesis.

The Conley conjecture

Although the Arnold conjecture is fairly well-understood in topological settings, the Conley conjecture remains completely mysterious. To keep the discussion simple we will restrict our attention to symplectically aspherical manifolds. In its simplest form, this conjecture asks if every Hamiltonian diffeomorphism has infinitely many periodic points. This was proven to be true by Franks-Handel in dimension 2 [FH03], by Hingston on tori [Hin09], and by Ginzburg [Gin10] on all aspherical manifolds.

Problem 8. *Suppose that* (M, ω) *is symplectically aspherical. Does every* $\phi \in \overline{Ham}(M)$ *have infinitely many periodic points?*

In the case of closed aspherical surfaces, Le Calvez [LC06] has proven that every homeomorphism $\phi \in \overline{\mathrm{Ham}}(\Sigma)$ has infinitely many periodic points.

The simplicity conjecture

Despite the recent progress towards the simplicity conjecture, numerous related questions remain open. In this section, I will present a few of these and refer the reader to [CGHS20, Sec. 7] for a more detailed discussion. Let us begin with an immediate generalization of the simplicity conjecture to higher dimensions.

Problem 9. *Is* $\overline{\mathrm{Symp}}_c(\mathbb{D}^{2n}, \omega)$*, the group of compactly supported symplectic homeomorphisms of the standard ball, simple?*

In the case of the disc, area-preserving homeomorphisms coincide with (compactly supported) Hamiltonian homeomorphisms of the disc. This fact motivates the following question.

Problem 10. *Let* (M, ω) *be a closed symplectic manifold. Is the group of Hamiltonian homeomorphisms* $\overline{\mathrm{Ham}}(M, \omega)$ *a simple group?*

In comparison, Banyaga's theorem states that the group of Hamiltonian diffeomorphisms is simple for closed M. The question can also be stated for non-closed manifolds but I will not discuss that here for the sake of brevity. I should add that above question remains open even in dimension two. Indeed, the two-dimensional case of the question was raised in Fathi's article; see [Fat80, App. A.6].

As I will explain in Section 3.3, we prove the simplicity conjecture by constructing an explicit proper normal subgroup which we call **finite energy homemorphisms** and denote by $\mathrm{FHomeo}_c(\mathbb{D}, \omega)$. It is rather easy to see that the group of finite energy homeomorphisms, $\mathrm{FHomeo}(M, \omega)$, can be defined on arbitrary symplectic manifolds; the construction is analogous to what is done in Section 3.3. As before, it forms a normal subgroup of $\overline{\mathrm{Ham}}(M, \omega)$. However, we do not know if it is proper.

Problem 11. *Is* $\mathrm{FHomeo}(M,\omega)$ *a proper subset of* $\overline{\mathrm{Ham}}(M,\omega)$*?*

Clearly, a positive answer to this question would settle the above simplicity questions. In the case of surfaces, one could hope to apply the machinery of PFH to approach this question. However, a serious obstacle arises in higher dimensions where PFH, and the related Seiberg-Witten theory, have no known generalization.

We now return to the case of the disc, where we know that $\mathrm{FHomeo}_c(\mathbb{D},\omega)$ is a proper normal subgroup of $\mathrm{Homeo}_c(\mathbb{D},\omega)$. This immediately gives rise to several interesting questions about $\mathrm{FHomeo}_c(\mathbb{D},\omega)$.

Problem 12. *Is* $\mathrm{FHomeo}_c(\mathbb{D},\omega)$ *simple?*

Another interesting direction to explore is the algebraic structure of the quotient $\mathrm{Homeo}_c(\mathbb{D},\omega)/\mathrm{FHomeo}_c(\mathbb{D},\omega)$. At present we are not able to say much beyond the fact that this quotient is abelian. Here are two sample questions.

Problem 13. *Is the quotient* $\mathrm{Homeo}_c(\mathbb{D},\omega)/\mathrm{FHomeo}_c(\mathbb{D},\omega)$ *torsion-free? Is it divisible?*

It would also be very interesting to know if the quotient $\mathrm{Homeo}_c(\mathbb{D},\omega)/\mathrm{FHomeo}_c(\mathbb{D},\omega)$ admits a geometric interpretation.

Chapter 2

Preliminaries

The goal of this chapter is to review some basics of symplectic geometry and to familiarize the reader with some of the technical tools needed in the proofs of the results in the previous chapter. Since the next chapter provides only the main ideas of the proofs, and not the details, a careful reading of the current chapter is not necessary for the reader who has some basic familiarity with the keywords appearing in the titles of the sections.

2.1 Basic notions and notations

Throughout chapter (M, ω) will denote a closed and connected symplectic manifold. Recall that a symplectic diffeomorphism is a diffeomorphism $\theta : M \to M$ such that $\theta^* \omega = \omega$. The set of all symplectic diffeomorphisms of M is denoted by $\mathrm{Symp}(M, \omega)$.

Hamiltonian diffeomorphisms, which constitute an important class of examples of symplectic diffeomorphisms, are defined as follows: A smooth Hamiltonian $H \in C^\infty([0,1] \times M)$ gives rise to a time-dependent vector field X_H which is defined via the equation: $\omega(X_{H_t}, \cdot) = dH_t$. The Hamiltonian flow of H, denoted by ϕ^t_H, is by definition the flow of X_H. A Hamiltonian diffeomorphism is a diffeomorphism which arises as the time-one map of a Hamiltonian flow. The set of all Hamiltonian diffeomorphisms is denoted by $\mathrm{Ham}(M, \omega)$.

It is clear that the set of all symplectomorphisms $\mathrm{Symp}(M, \omega)$ is a group. Moreover, $\mathrm{Ham}(M, \omega)$ forms a normal subgroup of $\mathrm{Symp}(M, \omega)$. Indeed, it is well-known (and proved for example in [HZ94, Sec. 5.1, Prop. 1]) that the Hamiltonians

$$H \# G(t,x) := H(t,x) + G(t, (\phi^t_H)^{-1}(x)), \quad \bar{H}(t,x) := -H(t, \phi^t_H(x)), \tag{2.1}$$

generate the Hamiltonian flows $\phi^t_H \phi^t_G$ and $(\phi^t_H)^{-1}$ respectively. Furthermore, given $\psi \in \mathrm{Diff}_c(\mathbb{D}, \omega)$, the Hamiltonian

$$H \circ \psi(t,x) := H(t, \psi(x)), \tag{2.2}$$

generates the flow $\psi^{-1} \phi^t_H \psi$.

We equip M with a Riemannian distance d. Given two maps $\phi, \psi : M \to M$, we denote

$$d_{C^0}(\phi, \psi) = \max_{x \in M} d(\phi(x), \psi(x)).$$

We will say that a sequence of maps $\phi_i : M \to M$, converges uniformly, or C^0–converges, to ϕ, if $d_{C^0}(\phi_i, \phi) \to 0$ as $i \to \infty$. Of course, the notion of C^0–convergence does not depend on the choice of the Riemannian metric.

As mentioned earlier, a homeomorphism $\theta : M \to M$ is said to be symplectic/Hamiltonian if it is the C^0–limit of a sequence of symplectic/Hamiltonian diffeomorphisms. We will denote the sets of symplectic and Hamiltonian homeomorphisms by $\mathrm{Sympeo}(M, \omega)$ and $\overline{\mathrm{Ham}}(M, \omega)$, respectively. It is not difficult to see that $\overline{\mathrm{Ham}}(M, \omega)$ forms a normal subgroup of $\mathrm{Sympeo}(M, \omega)$.

As mentioned earlier, alternative definitions for Hamiltonian homeomorphisms do exist within the literature of C^0 symplectic topology. Most notable of these is a definition given by Müller and Oh in [OM07] which has received much attention. A homeomorphism which is Hamiltonian in the sense of [OM07] is necessarily Hamiltonian in the sense defined above.

2.1.1 Hofer's distance

We will denote the Hofer norm on $C^\infty([0,1] \times M)$ by

$$\|H\| = \int_0^1 \left(\max_{x \in M} H(t, \cdot) - \min_{x \in M} H(t, \cdot) \right) dt.$$

The Hofer distance on $\mathrm{Ham}(M, \omega)$ is defined via

$$d_{\mathrm{Hofer}}(\phi, \psi) = \inf \|H - G\|,$$

where the infimum is taken over all H, G such that $\phi^1_H = \phi$ and $\phi^1_G = \psi$. This defines a bi-invariant distance on $\mathrm{Ham}(M, \omega)$.

Given $B \subset M$, we define its *displacement energy* to be

$$e(B) := \inf\{d_{\mathrm{Hofer}}(\phi, \mathrm{Id}) : \phi(B) \cap B = \emptyset\}.$$

Non-degeneracy of the Hofer distance is a consequence of the fact that $e(B) > 0$ when B is an open set. This was proven in [Hof90, Pol93, LM95].

2.1.2 Rotation numbers

We need to recall some basic facts about rotation numbers of fixed points of area-preserving diffeomorphisms of the two-sphere $\mathbb{S}^2$. The contents of this section will be used later to give a precise definition of the spectral invariants of periodic Floer homology.

Let p be a fixed point of $\varphi \in \mathrm{Diff}(\mathbb{S}^2, \omega)$. One can find a Hamiltonian isotopy $\{\varphi^t\}_{t \in [0,1]}$ such that $\varphi^0 = \mathrm{Id}, \varphi^1 = \varphi$ and $\varphi^t(p) = p$ for all $t \in [0,1]$. In this section, we briefly review the definitions and some of the properties of $\mathrm{rot}(\{\varphi^t\}, p)$, the rotation number of the isotopy $\{\varphi^t\}_{t \in [0,1]}$ at p, and $\mathrm{rot}(\varphi, p)$, the rotation number of φ at p. Our conventions are such that $\mathrm{rot}(\varphi, p)$ will be a real number in the interval $(-\frac{1}{2}, \frac{1}{2}]$. For further details on this subject, we refer the reader to [KH95].

The derivative of the isotopy $D_p\varphi^t : T_p\mathbb{S}^2 \to T_p\mathbb{S}^2$, viewed as a linear isotopy of $\mathbb{R}^2$, induces an isotopy of the circle $\{f_t\}_{t \in [0,1]}$ with $f_0 = \mathrm{Id}$ and $f_1 = D_p\varphi$. We define $\mathrm{rot}(\{\varphi^t\}, p)$, the **rotation number of the isotopy** $(\varphi^t)_{t \in [0,1]}$ **at** p, to be the Poincaré rotation number of the circle isotopy $\{f_t\}$.

The number $\mathrm{rot}(\{\varphi^t\}, p)$ depends only on $\{D_p\varphi^t\}_{t \in [0,1]}$ and it satisfies the following properties. Let $\{\psi^t\}_{t \in [0,1]}$ be another Hamiltonian isotopy such that $\psi^0 = \mathrm{Id}, \psi^1 = \varphi^1$, and $\psi^t(p) = p$ for $t \in [0,1]$. Then,

1. $\mathrm{rot}(\{\varphi^t\}, p) - \mathrm{rot}(\{\psi^t\}, p) \in \mathbb{Z}$,
2. $\mathrm{rot}(\{\varphi^t\}, p) = \mathrm{rot}(\{\psi^t\}, p)$ if the two isotopies are homotopic rel. endpoints among isotopies fixing the point p.

The above facts may be deduced from the standard properties of the Poincaré rotation number; see, for example, [KH95].

Lastly, we define $\mathrm{rot}(\varphi, p)$, the **rotation number of** φ **at** p, to be the unique number in $(-\frac{1}{2}, \frac{1}{2}]$ which coincides with $\mathrm{rot}(\{\varphi^t\}, p) \bmod \mathbb{Z}$. It follows from the discussion in the previous paragraph that it depends only on $D_p\varphi$.

2.2 Preliminaries on Hamiltonian Floer theory, spectral invariants, and barcodes

In this section, (M, ω) will denote a closed, connected and symplectically aspherical manifold of dimension $2n$. We fix a ground field $\mathbb{F}$, e.g. $\mathbb{Z}_2, \mathbb{Q}$, or $\mathbb{C}$. Singular homology, Floer homology and all notions relying on these theories depend on the field $\mathbb{F}$.

The action functional and its spectrum. Let $\Omega(M)$ denote the space of smooth contractible loops in M, viewed as maps $\mathbb{R}/\mathbb{Z} \to M$. Let $H : [0,1] \times M$ denote a smooth Hamiltonian. The associated action functional $\mathcal{A}_H : \Omega(M) \to \mathbb{R}$ is defined by

$$\mathcal{A}_H(z) := \int_0^1 H(t, z(t))dt \ + \int_{D^2} u^*\omega,$$

where $u : D^2 \to M$ is a capping disk for z. Note that because $\omega|_{\pi_2(M)} = 0$, the value of $\mathcal{A}_H(z)$ does not depend on the choice of u.

It is a well-known fact that the set of critical points of $\mathcal{A}_H$, denoted by $\mathrm{Crit}(\mathcal{A}_H)$, consists of 1–periodic orbits of the Hamiltonian flow ϕ_H^t. The action spectrum of H, denoted by $\mathrm{Spec}(H)$, is the set of critical values of $\mathcal{A}_H$. The set $\mathrm{Spec}(H)$ has Lebesgue measure zero.

Suppose that H and G are two Hamiltonians such that $\phi_H^1 = \phi_G^1$. Then, there exists a constant $C \in \mathbb{R}$ such that

$$\mathrm{Spec}(H) = \mathrm{Spec}(G) + C, \tag{2.3}$$

where $\mathrm{Spec}(G) + C$ is the set obtained from $\mathrm{Spec}(G)$ by adding C to each of its elements. It follows that, given a Hamiltonian diffeomorphism ϕ, its spectrum $\mathrm{Spec}(\phi)$ is a subset of $\mathbb{R}$ which is well-defined upto a shift.

Hamiltonian Floer theory. We say that a Hamiltonian H is non-degenerate if the graph of ϕ_H^1 intersects the diagonal in $M \times M$ transversally. The Floer chain complex of (non-degenerate) H, $CF_*(H)$, is the vector space spanned by $\mathrm{Crit}(\mathcal{A}_H)$ over the ground field $\mathbb{F}$. The boundary map of $CF_*(H)$ counts certain solutions of a perturbed Cauchy-Riemann equation for a chosen ω-compatible almost complex structure J on TM, which can be viewed as isolated negative gradient flow lines of $\mathcal{A}_H$. There exists a canonical isomorphism, $\Phi : H_*(M) \to HF_*(H)$, between the homology of Floer's chain complex and the singular homology of M; [Flo89d, PSS96]. We will denote this isomorphism by $\Phi : H_*(M) \to HF_*(H)$.

For any $a \in \mathbb{R}$, we will define $CF_*^a(H) := \{\sum a_z z \in CF_*(H) : \mathcal{A}_H(z) < a\}$. It turns out that the Floer boundary map preserves $CF_*^a(H)$ and hence one can define its homology $HF_*^a(H)$. The homology groups $HF_*^a(H)$ are referred to as the filtered Floer homology groups of H.

More generally, filtered Floer homology groups may be defined for any interval of the form $(a, b) \subset \mathbb{R}$: $HF_*^{(a,b)}(H)$ is defined to be the homology of the quotient complex $CF_*^{(a,b)}(H) = CF_*^b(H)/CF_*^a(H)$. Let us remark that the filtered Floer homology groups do not depend on the choice of the almost complex structure J.

We should also add that one can define filtered Floer homology even when H is degenerate. Consider $(a, b) \subset \mathbb{R}$ such that $-\infty \leqslant a, b \leqslant \infty$ are not in $\mathrm{Spec}(H)$ and define $HF_*^{(a,b)}(H)$ to be $HF_*^{(a,b)}(\tilde{H})$, where $\tilde{H}$ is non-degenerate and sufficiently C^2–close to H. It can be shown that $HF_*^{(a,b)}(H)$ does not depend on the choice of $\tilde{H}$.

It turns out that filtered Floer homology groups are in fact invariants of the time-1 map ϕ_H^1 in the following sense: Suppose that H and G are two Hamiltonians such that $\phi_H^1 = \phi_G^1$. Then, there exists a constant $C \in \mathbb{R}$ such that we have a canonical isomorphism

$$HF_*^a(H) \cong HF_*^{a+C}(G), \quad \forall a \in \mathbb{R}. \tag{2.4}$$

As explained in Remark 2.10 of [PS16], the above is a consequence of results from [Sei97, Sch00].

2.2.1 Spectral invariants

Spectral invariants of Hamiltonian diffeomorphisms were first introduced by Viterbo in [Vit92] in the case of $\mathbb{R}^{2n}$ and were later generalized in the works of Schwarz [Sch00] and Oh [Oh05]. Here, we will be closely following Schwarz [Sch00] which treats the case of closed and symplectically aspherical manifolds.[1]

Denote by $i_a^* : HF_*^a(H) \to HF_*(H)$ the map induced by the inclusion $i_a : CF_*^a(H) \to CF_*(H)$ and let α be a non-zero homology class. The spectral invariant $c(\alpha, H)$ is defined by

$$c(\alpha, H) := \inf\{a \in \mathbb{R} : \Phi(\alpha) \in \mathrm{Im}(i_a^*)\},$$

where $\mathrm{Im}(i_a^*)$ denotes the image of $i_a^* : HF_*^a(H) \to HF_*(H)$. (Recall that Φ is the canonical isomorphism between $H_*(M)$ and $HF_*(H)$).

It is well-known that (see [Sch00]) that $|c(\alpha, H) - c(\alpha, G)| \leqslant \|H - G\|$, where $\|H\| = \int_0^1 (\max_{x\in M} H(t,\cdot) - \min_{x\in M} H(t,\cdot))\,dt$ denotes the Hofer norm of H. This allows us to define $c(\alpha, H)$ for any smooth (or even continuous) Hamiltonian: we set $c(\alpha, H) := \lim c(\alpha, H_i)$, where H_i is a sequence of smooth, non-degenerate Hamiltonians such that $\|H - H_i\| \to 0$.

Spectral invariants satisfy the following properties whose proofs can be found in [Sch00] as well as [Oh05, Oh06, Ush10].

Proposition 2.1. *The function $c : (H_*(M) \setminus \{0\}) \times C^\infty([0,1] \times M) \to \mathbb{R}$ has the following properties:*

1. $c(\alpha, H) \in \mathrm{Spec}(H)$,
2. $c(\alpha \cap \beta, H\#G) \leqslant c(\alpha, H) + c(\beta, G)$,
3. $|c(\alpha, H) - c(\alpha, G)| \leqslant \|H - G\|$,
4. $c([M], H) = -c([pt], \bar{H})$,
5. *Let $f \in C^\infty(M)$ denote an autonomous Hamiltonian and suppose that $\alpha \in H_*(M)$ is a non-zero homology class. Then, for $\varepsilon > 0$ sufficiently small,*

$$c(\alpha, \varepsilon f) = c_{LS}(\alpha, \varepsilon f) = \varepsilon\, c_{LS}(\alpha, f),$$

where $c_{LS}(\alpha, f)$ is the topological quantity[2] *defined by*

$$c_{LS}(\alpha, f) = \inf\{a \in \mathbb{R} : \alpha \in \mathrm{Im}\,(H_*(\{f < a\}) \to H_*(M))\}.$$

As a consequence of Equation (2.4), spectral invariants are invariants of the time-1 map ϕ_H^1 in the following sense: If H and G are two Hamiltonians such that $\phi_H^1 = \phi_G^1$, then there exists a constant $C \in \mathbb{R}$ such that

$$c(\alpha, H) - c(\alpha, G) = C, \quad \forall \alpha \in H_*(M) \setminus \{0\}. \tag{2.5}$$

Hence, we see that the difference of two spectral invariants defined via

$$\gamma(\alpha, \beta; \phi_H^1) := c(\alpha, H) - c(\beta, H) \tag{2.6}$$

depends only on ϕ_H^1. The so-called spectral norm

$$\gamma : \mathrm{Ham}(M, \omega) \to \mathbb{R}$$

is defined via

$$\gamma(\cdot) := \gamma([M], [pt]; \cdot).$$

It satisfies the following list of properties:

[1] See [Oh05] for the construction of these invariants on general symplectic manifolds.

[2] Here, the subscript "LS" refers to "Lusternik-Schnirelman" since this quantity is related to Lusternick-Schnirelman theory.

1. Non-degeneracy: $\gamma(\phi) \geqslant 0$ with equality if and only $\phi = \mathrm{Id}$,
2. Hofer boundedness: $\gamma(\phi) \leqslant d_{\mathrm{Hofer}}(\phi, \mathrm{Id})$,
3. Conjugacy invariance: $\gamma(\psi\phi\psi^{-1}) = \gamma(\phi)$ for any $\phi \in \mathrm{Ham}(M, \omega)$ and any $\psi \in \mathrm{Symp}(M, \omega)$,
4. Triangle inequality: $\gamma(\phi\psi) \leqslant \gamma(\phi) + \gamma(\psi)$,
5. Duality: $\gamma(\phi^{-1}) = \gamma(\phi)$,
6. Energy-Capacity inequality: $\gamma(\phi) \leqslant 2e(\mathrm{Supp}(\phi))$, where $e(\mathrm{Supp}(\phi))$ denotes the displacement energy of the support of ϕ.

Let us point out that the non-degeneracy property is an immediate consequence of the following: Let $B \subset M$ denote a symplectically embedded ball of radius r. If $\phi(B) \cap B = \emptyset$, then,

$$\pi r^2 \leqslant \gamma(\phi). \tag{2.7}$$

The above inequality, which is also referred to as the energy-capacity inequality, follows from the results in [Ush10].

2.2.2 Barcodes

A *finite barcode* $\mathcal{B} = \{(I_j, m_j)\}_{1 \leqslant j \leqslant N}$ is a finite set of intervals (or bars) $I_j = (a_j, b_j], a_j \in \mathbb{R}, b_j \in \mathbb{R} \cup \{\infty\}$ with multiplicities $m_j \in \mathbb{N}$. Two barcodes $\mathcal{B}_1, \mathcal{B}_2$ are said to be δ-matched if, upto adding/deleting some intervals of length less than 2δ, there exists a bijective matching between the bars of $\mathcal{B}_1$ and $\mathcal{B}_2$ such that the endpoints of the matched intervals are placed within distance at most δ of each other[3]. The bottleneck distance $d_{\mathrm{bottle}}(\mathcal{B}_1, \mathcal{B}_2)$ is defined to be the infimum of such δ.

The space of all finite barcodes, equipped with the bottleneck distance, is not a complete metric space. In order to form its completion, we will need to allow certain non-finite barcodes. We define a barcode $\mathcal{B} = \{(I_j, m_j)\}_{j \in \mathbb{N}}$ to be a collection of intervals (or bars) $I_j = (a_j, b_j], a_j \in \mathbb{R}, b_j \in \mathbb{R} \cup \{\infty\}$, with multiplicities $m_j \in \mathbb{N}$, such that for any $\epsilon > 0$ only finitely many of the intervals I_j are of length greater than ϵ.

We will let *Barcodes* denote the set of all barcodes. Observe that the bottleneck distance extends to *Barcodes*. The space $(\mathit{Barcodes}, d_{\mathrm{bottle}})$ is indeed the completion of the space of finite barcodes. A related notion referred to as the space of q-tame barcodes was introduced in [CdSGO16].

Given a barcode $\mathcal{B} = \{(I_j, m_j)\}_{j \in \mathbb{N}}$, we will define its spectrum, $\mathrm{Spec}(\mathcal{B})$, to be the set of endpoints of the intervals I_j.

Barcodes for Hamiltonians

We will now give a brief description of how one may associate a collection of canonical barcodes to every Hamiltonian. To do so, we will pass through the theory of persistence modules, following the exposition of [PS16]. Alternatively, one could use the theory Barannikov complexes, as explained in [LRSV]. The two methods are equivalent; see [LRSV].[4]

Definition 2.2 (Persistence module)**.** *Given a field $\mathbb{F}$, a persistence module is a family of $\mathbb{F}$-vector spaces $Q = (Q^t)_{t \in \mathbb{R}}$ endowed with maps $\iota_s^t(Q) : Q^s \to Q^t$ for all $s \leqslant t \in \mathbb{R}$ (we will write ι_s^t when there is no possible confusion), satisfying:*

- *there exists $t_0 \in \mathbb{R}$, such that $Q^t = 0$ for all $t < t_0$,*

[3]For instance, the barcodes $\{([0, 1], 1), ([0, 3], 2), ([10, 20], 1)\}$ and $\{([0, 4], 1), ([1, 3], 1), ([11, 18], 1)\}$ can be 2-matched.
[4]Barcodes may be defined on symplectic manifolds which are not necessarily aspherical; see [UZ16].

- *For all* $r, s, t \in \mathbb{R}$*, such that* $r \leqslant s \leqslant t$*, one has* $\iota_s^t \circ \iota_r^s = \iota_r^t$*,*
- *For all* $t \in \mathbb{R}$*,* ι_t^t *is the identity map* $Q^t \to Q^t$*,*
- *There exists a finite set* $\mathrm{Spec}(Q)$*, called the spectrum of* Q*, such that if* s, t *are in the same connected component of* $\mathbb{R} \setminus S(Q)$*, then* ι_s^t *is an isomorphism.*
- *For all* $t \in \mathbb{R}$*,* $\varinjlim_{s<t} Q^s = Q^t$*.*

Observe that given two persistence modules Q_1, Q_2 one can form the direct sum $Q := (Q_1^t \oplus Q_2^t)_{t \in \mathbb{R}}$. The morphisms ι_s^t of Q are obtained by taking the direct sums of the morphisms of to Q_1 and Q_2. Given an interval $I = (a, b]$, where $a \in \mathbb{R}, b \in \mathbb{R} \cup \infty$, we define a persistence module by $Q(I)^t = \mathbb{F}$ if $a < t \leqslant b$ and $Q(I)^t = 0$, otherwise. We set the morphisms $\iota_s^t(Q(I)) : Q(I)^s \to Q(I)^t$ to be the identity if $a < s \leqslant t \leqslant b$ and zero otherwise.

The *structure theorem* for persistence modules [CB15] states that for every persistence module Q^t there exists a unique finite barcode $\mathcal{B}(Q) = \{(I_j, m_j)\}_{1 \leqslant j \leqslant N}$ such that Q is isomorphic to $\oplus_{j=1}^N Q(I_j)^{m_j}$. Here, $Q(I_j)^{m_j}$ denotes the direct sum of m_j copies of $Q(I_j)$. A version of this theorem for Morse persistence modules was proven, using a different terminology, by Barannikov in [Bar94].

Suppose that H is non-degenerate and consider the family of vector spaces $Q^t := HF_*^t(H)$, equipped with the maps $\iota_s^t : HF_*^s(H) \to HF_*^t(H)$ induced by inclusions of chain complexes; this has the algebraic structure of a persistence module (over the ground field $\mathbb{F}$). Similarly, if we fix a Conley–Zehnder index j and consider the family of vector spaces $Q_j^t := HF_j^t(H)$, equipped with the same maps as above, we obtain persistence modules Q_j^t. Clearly, $Q^t = \oplus_j Q_j^t$.

It follows from the structure theorem for persistence modules that to every non-degenerate Hamiltonian H, we can associate barcodes $\mathcal{B}(H)$, corresponding to Q^t, and $\mathcal{B}_j(H)$ corresponding to Q_j^t. It is easy to see that

$$\mathcal{B}(H) = \sqcup_j \mathcal{B}_j(H).$$

As mentioned in the introduction, the barcode $\mathcal{B}(H)$ determines the filtered Floer complex $CF_*^t(H)$ upto quasi-isomorphism. Hence, it subsumes all the filtered Floer theoretic invariants of H. For example, the spectral invariants of H correspond to the endpoints of the half-infinite bars of $\mathcal{B}(H)$. The rest of this section is dedicated to describing some of the properties of the barcodes which arise in the above manner.

Continuity: It can be shown, via standard Floer theoretic arguments (see e.g. Equation (4) in [PS16]), that

$$d_{\mathrm{bottle}}(\mathcal{B}_j(H), \mathcal{B}_j(G)) \leqslant \|H - G\|. \tag{2.8}$$

The above inequality allows us to define the barcode of any smooth, or even continuous, $H : [0,1] \times M \to \mathbb{R}$. Indeed, for an arbitrary H, take H_i to be a sequence of non-degenerate Hamiltonians such that $\|H - H_i\| \to 0$, and define $\mathcal{B}_j(H) := \lim \mathcal{B}_j(H_i)$ where the limit is taken with respect to the bottleneck distance. We have now obtained a map

$$\mathcal{B}_j : C^\infty([0,1] \times M) \to \textit{Barcodes},$$

which continues to satisfy Equation (2.8).

It is clear that the continuity property holds, as stated above, for the barcode $\mathcal{B}(H)$ as well.

Spectrality: If H is non-degenerate, then it can easily be seen that the set of endpoints of the bars of $\mathcal{B}(H)$ is exactly the action spectrum of H, i.e. $\mathrm{Spec}(\mathcal{B}(H)) = \mathrm{Spec}(H)$.[5] If H is an arbitrary smooth Hamiltonian, then

$$\mathrm{Spec}(\mathcal{B}(H)) \subset \mathrm{Spec}(H). \tag{2.9}$$

[5] As for the set of endpoints of $\mathcal{B}_j(H)$, it can be shown that the lower and upper ends of its bars are actions of periodic orbits with Conley–Zenhnder indices j and $j+1$, respectively.

This latter statement can be proven by writing H as the limit, in C^2 topology, of a sequence of non-degenerate Hamiltonians H_i and applying the continuity and spectrality properties to the H_i's. It is clear that we also have

$$\mathrm{Spec}(\mathcal{B}_j(H)) \subset \mathrm{Spec}(H).$$

Conjugacy Invariance: Suppose that $\psi \in \mathrm{Symp}(M, \omega)$. Then, for any smooth H,

$$\mathcal{B}_j(H \circ \psi) = \mathcal{B}_j(H). \tag{2.10}$$

The above follows from the fact that, for non-degenerate H, the filtered Floer complexes $CH^t_*(H)$ and $CF^t_*(H \circ \psi)$ are isomorphic. See [PS16] for further details. It is clear that we also have

$$\mathcal{B}(H \circ \psi) = \mathcal{B}(H).$$

Finally, we should mention that the barcode $\mathcal{B}_j(H)$ for the Hamiltonian $H = c$, where c is a constant, consists of the single interval (c, ∞) with multiplicity $\mathrm{rank}(H_j(M))$.

Barcodes for Hamiltonian diffeomorphisms

Recall that given a barcode $\mathcal{B} = \{(I_j, m_j)\}_{j\in\mathbb{N}}$ and $c \in \mathbb{R}$, we have defined $\mathcal{B} + c = \{(I_j + c, m_j)\}_{j\in\mathbb{N}}$, where $(a_j, b_j] + c = (a_j + c, b_j + c]$. Let $\sim$ denote the equivalence relation on the space of barcodes given by $\mathcal{B} \sim \mathcal{C}$ if $\mathcal{C} = \mathcal{B} + c$ for some $c \in \mathbb{R}$; we will denote the quotient space by $\widehat{Barcodes}$. The bottleneck distance defines a distance on $\widehat{Barcodes}$ which we will continue to denote by d_{bottle}.

Now, suppose that H, H' are two Hamiltonians such that $\phi^1_H = \phi^1_{H'}$. Then, as a consequence of Equation (2.4), there exists a constant $c \in \mathbb{R}$ such that $\mathcal{B}_j(H) = \mathcal{B}_j(H') + c$. Of course, it is also true that $\mathcal{B}(H) = \mathcal{B}(H') + c$. We conclude that the maps $\mathcal{B}$, $\mathcal{B}_j : C^\infty(\mathbb{S}^1 \times M) \to Barcodes$, introduced above, induces a map which we will continue to denote by $\mathcal{B}$, $\mathcal{B}_j$:

$$\mathcal{B},\ \mathcal{B}_j : \mathrm{Ham}(M) \to \widehat{Barcodes}.$$

Remark 2.3. An alternative to our approach in this article, is to define $\mathcal{B}_j(\tilde\varphi), \mathcal{B}(\tilde\varphi)$ to be $\mathcal{B}_j(H), \mathcal{B}(H)$ where H is a *mean-normalized* Hamiltonian whose flow is a representative of $\tilde\varphi$. Being mean-normalized means $\int_0^1 \int_M H\omega^n = 0$. This defines $\mathcal{B}_j(\varphi), \mathcal{B}(\varphi)$ without any ambiguity as a barcode, as opposed to a barcode upto shift, and so one obtains maps $\mathcal{B}_j, \mathcal{B} : \mathrm{Ham}(M, \omega) \to \widehat{Barcodes}$. This is the approach taken in [PS16], and indeed, it is a more natural approach from the point of view of Hofer geometry. However, as pointed out in [LRSV], this approach yields barcodes which are discontinuous in the C^0-topology on $\mathrm{Ham}(M, \omega)$. ◀

The barcodes $\mathcal{B}(\phi)$, $\mathcal{B}_j(\phi)$, where $\phi \in \mathrm{Ham}(M, \omega)$, inherit appropriately restated versions of the properties listed above. We will list them here for the record.

Continuity: It is easy to see that Equation (2.8) translates to: $d_{\mathrm{bottle}}(\mathcal{B}_j(\phi), \mathcal{B}_j(\psi)) \leqslant d_{\mathrm{Hofer}}(\phi, \psi)$, for any $\phi, \psi \in \mathrm{Ham}(M, \omega)$. Similarly, we have $d_{\mathrm{bottle}}(\mathcal{B}(\phi), \mathcal{B}(\psi)) \leqslant d_{\mathrm{Hofer}}(\phi, \psi)$.

Spectrality: Let $\mathcal{B}$ be (the equivalence class of a) barcode in $\widehat{Barcodes}$. Then, the set of endpoints of the bars of $\mathcal{B}$, $\mathrm{Spec}(\mathcal{B})$, is well-defined upto a shift by a constant. As mentioned earlier (see Equation (2.5)), the same is true for the spectrum of a Hamiltonian diffeomorphism. Hence, the spectrality property from above translates to

$$Spec(\mathcal{B}(\phi)) \subset \mathrm{Spec}(\phi), \tag{2.11}$$

which means that $\mathrm{Spec}(\mathcal{B}(\phi))$ is a subset of $\mathrm{Spec}(\phi)$ upto a shift. Clearly, it is also true that

$$\mathrm{Spec}(\mathcal{B}_j(\phi)) \subset \mathrm{Spec}(\phi).$$

Clearly the total number of endpoints of $\mathcal{B}(\phi)$ is independent of the choice of the representative of the equivalence class of $\mathcal{B}(\phi)$ in $\widehat{Barcodes}$. It follows immediately that the total number of the endpoints of the bars in $\mathcal{B}(\phi)$ gives a lower bound for the total number of fixed points of ϕ. The two numbers coincide when ϕ is non-degenerate.

Conjugacy invariance: Suppose that $\psi \in \mathrm{Symp}(M, \omega)$. Then, for any $\phi \in \mathrm{Ham}(M, \omega)$,

$$\mathcal{B}_j(\psi^{-1}\phi\psi) = \mathcal{B}_j(\phi). \tag{2.12}$$

This follows from Equation (2.10) and the fact that $\phi^t_{H\circ\psi} = \psi^{-1}\phi^t_H\psi$ for any Hamiltonian H. Clearly, we also have

$$\mathcal{B}(\psi^{-1}\phi\psi) = \mathcal{B}(\phi).$$

Finally, note that $\mathcal{B}_j(\mathrm{Id})$, can be represented by any barcode given by a single interval of the form (c, ∞) with multiplicity $\mathrm{rank}(H_j(M))$.

2.3 Periodic Floer Homology and basic properties of the PFH spectral invariants

Our solution to the simplicity conjecture uses the theory of **periodic Floer homology** (PFH), a version of Floer homology for area-preserving diffeomorphisms which was introduced by Hutchings [Hut02, HS05]. In this section, I recall the definition of Periodic Floer Homology and sketch the construction of the spectral invariants which arise from this theory, also due to Hutchings [Hut17].

As with ordinary Floer homology, PFH can be used to define a sequence of functions

$$c_d : \mathrm{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R},$$

where $d \in \mathbb{N}$, called **PFH spectral invariants**, which satisfy various useful properties; here $\mathrm{Diff}_c(\mathbb{D}, \omega)$ stands for the set of compactly supported area-preserving diffeomorphisms of the (open) disc.

The definition of PFH spectral invariants is due to Michael Hutchings [Hut17], but very few properties have been established about these. We prove in [CGHS20, Thm. 3.6] that the PFH spectral invariants satisfy the following properties:

1. Normalization: $c_d(\mathrm{Id}) = 0$,
2. Monotonicity: Suppose that $H \leqslant G$ where $H, G \in C^\infty_c([0,1] \times \mathbb{D})$. Then, $c_d(\phi^1_H) \leqslant c_d(\phi^1_G)$ for all $d \in \mathbb{N}$,
3. Hofer Continuity: $|c_d(\phi^1_H) - c_d(\phi^1_G)| \leqslant d\|H - G\|$,
4. Spectrality: $c_d(\phi^1_H) \in \mathrm{Spec}_d(H)$ for any $H \in \mathcal{H}$, where $\mathrm{Spec}_d(H)$ is the **order** d **action spectrum** of H and is defined below.

I will now elaborate on the spectrality property and the order d action spectrum of H. We view $\phi \in \mathrm{Diff}_c(\mathbb{D}, \omega)$ as an area-preserving diffeomorphism of the sphere $\mathbb{S}^2$; we will do so by embedding the disc $\mathbb{D}$ into $\mathbb{S}^2$ such that it coincides with the northern hemisphere. Now, pick a Hamiltonian H supported in $\mathbb{D}$, i.e. the northern hemisphere of $\mathbb{S}^2$, such that $\phi = \phi^1_H$. Denote by H^k the k-times composition of H with itself, where composition is the # operation defined in Equation (2.1). For any $d > 0$, we now define the **order d spectrum** of H by

$$\mathrm{Spec}_d(H) := \cup_{k_1+\ldots+k_j=d}\ \mathrm{Spec}(H^{k_1}) + \ldots + \mathrm{Spec}(H^{k_j}),$$

where $\mathrm{Spec}(\cdot)$ is the usual action spectrum defined in Section 2.2.

Note that $\mathrm{Spec}_d(H)$ may equivalently be described as follows: For every value $a \in \mathrm{Spec}_d(H)$ there exist capped periodic orbits $(z_1, u_1), \ldots, (z_k, u_k)$ of H the sum of whose periods is d and such that

$$a = \sum \mathcal{A}_{H^{k_i}}(u_i, z_i).$$

The set $\mathrm{Spec}_d(H)$ depends only on ϕ: Indeed, as we explain in [CGHS20, Sec. 2.5], if H, H' are supported in the northern hemisphere and generate the same time-1 map ϕ, then we in fact have $\mathrm{Spec}_d(H) = \mathrm{Spec}_d(H')$ for all $d > 0$.

My goal in this section is to only review the definition of PFH, and its spectral invariants, and so I will not prove any of the above properties here; for proofs, I refer the interested reader to [CGHS20, Sec. 3].

2.3.1 Preliminaries on J–holomorphic curves and stable Hamiltonian structures

A stable Hamiltonian structure (SHS) on a closed three-manifold Y is a pair (α, Ω), consisting of a 1–form α and a closed two-form Ω, such that

1. $\alpha \wedge \Omega$ is a volume form on Y,
2. $\ker(\Omega) \subset \ker(d\alpha)$.

Observe that the first condition implies that Ω is non-vanishing, and as a consequence, the second condition is equivalent to $d\alpha = g\Omega$, where $g : Y \to \mathbb{R}$ is a smooth function.

A stable Hamiltonian structure determines a plane field $\xi := \ker(\alpha)$ and a **Reeb** vector field R on Y given by

$$R \in \ker(\Omega),\ \alpha(R) = 1.$$

Closed integrals curves of R are called **Reeb orbits**; we regard Reeb orbits as equivalent if they are equivalent as currents.

Stable Hamiltonian structures were introduced in [BEH+03, CM05] as a setting in which one can obtain general Gromov-type compactness results, such as the SFT compactness theorem, for pseudo-holomorphic curves in $\mathbb{R} \times Y$. Here are two examples of stable Hamiltonian structures which are relevant to our story.

Example 2.4. A **contact** form on Y is a 1-form λ such that $\lambda \wedge d\lambda$ is a volume form. The pair $(\alpha, \Omega) := (\lambda, d\lambda)$ gives a stable Hamiltonian structure with $g \equiv 1$. The plane field ξ is the associated contact structure and the Reeb vector field as defined above gives the usual Reeb vector field of a contact form.

The **contact symplectization** of Y is

$$X := \mathbb{R} \times Y_{\varphi},$$

which has a standard symplectic form, defined by

$$\Gamma = d(e^s \lambda), \tag{2.13}$$

where s denotes the coordinate on $\mathbb{R}$. ◀

Example 2.5. Let (S, ω_S) be a closed surface and denote by φ a smooth area-preserving diffeomorphism of S. Define the mapping torus

$$Y_\varphi := \frac{S \times [0,1]}{(x,1) \sim (\varphi(x), 0)}.$$

Let r be the coordinate on $[0,1]$. Now, Y_φ carries a stable Hamiltonian structure $(\alpha, \Omega) := (dr, \omega_\varphi)$, where ω_φ is the canonical closed two form on Y_φ induced by ω_S. Note that the plane field ξ is given by the vertical tangent space of the fibration $\pi : Y_\varphi \to \mathbb{S}^1$ and the Reeb vector field is given by $R = \partial_r$. Here, $g \equiv 0$. Observe that the Reeb orbits here are in correspondence with the periodic orbits of φ.

We define the **symplectization** of Y_φ

$$X := \mathbb{R} \times Y_\varphi,$$

which has a standard symplectic form, defined by

$$\Gamma = ds \wedge dr + \omega_\varphi, \tag{2.14}$$

where s denotes the coordinate on $\mathbb{R}$. ◀

We say an almost complex structure J on $X = \mathbb{R} \times Y$ is **admissible**, for a given SHS (α, Ω), if the following conditions are satisfied:

1. J is invariant under translation in the $\mathbb{R}$-direction of $\mathbb{R} \times Y$,
2. $J\partial_s = R$, where s denotes the coordinate on the $\mathbb{R}$-factor of $\mathbb{R} \times Y$,
3. $J\xi = \xi$, where $\xi := \ker(\alpha)$, and $\Omega(v, Jv) > 0$ for all nonzero $v \in \xi$.

We will denote by $\mathcal{J}(\alpha, \Omega)$ the set of almost complex structures which are admissible for (α, Ω). The space $\mathcal{J}(\alpha, \Omega)$ equipped with the C^∞ topology is path connected, and even contractible.

Define a ***J*-holomorphic map** to be a smooth map

$$u : (\Sigma, j) \to (X, J),$$

satisfying the equation

$$du \circ j = J \circ du, \tag{2.15}$$

where (Σ, j) is a closed Riemann surface (possibly disconnected), minus a finite number of punctures. As is common in the literature on ECH, we will sometimes have to consider J-holomorphic maps up to equivalence of currents, and we call such an equivalence class a ***J*-holomorphic current**; see [Hut14] for the precise definition of this equivalence relation. An equivalence class of J-holomorphic maps under the relation of biholomorphisms of the domain will be called a ***J*-holomorphic curve**; this relation might be more familiar to the reader, but is not sufficient for our needs.

2.3.2 Definition of periodic Floer homology

Periodic Floer homology (PFH) is a version of Floer homology, defined by Hutchings [Hut02, HS05], for area-preserving maps of surfaces. The construction of PFH is closely related to the better-known embedded contact homology (ECH) and, in fact, predates the construction of ECH. We now review the definition of PFH; for further details on the subject we refer the reader to [Hut02, HS05].

Let (S, ω_S) be a closed[6] surface with an area form, and φ a **nondegenerate** smooth area-preserving diffeomorphism. Non-degeneracy is defined as follows: A periodic point p of ϕ, with period k, is said

[6]PFH can still be defined if S is not closed, but we will not need this here.

to be non-degenerate if the derivative of φ^k at the point p does not have 1 as an eigenvalue. We say φ is d–**nondegenerate** if all of its periodic points of period at most d are nondegenerate; if φ is d–nondegenerate for all d, then we say it is non-degenerate. A C^∞-generic area-preserving diffeomorphism is nondegenerate.

Recall the definition of Y_φ from Example 2.5 and take $0 \neq h \in H_1(Y_\varphi)$. If φ is nondegenerate and satisfies a certain "monotonicity" assumption,[7] which we do not need to discuss here as it automatically holds when $S = \mathbb{S}^2$, the **periodic Floer homology** $PFH(\varphi, h)$ is defined; it is the homology of a chain complex $PFC(\varphi, h)$ which we define below.

REMARK 2.6. If we carry out the construction outlined below, nearly verbatim, for a contact SHS $(\lambda, d\lambda)$, rather than the SHS (dr, ω_φ), then we would obtain the **embedded contact homology** ECH; see [Hut14] for further details. ◀

PFH generators

The chain complex PFC is freely generated over[8] $\mathbb{Z}_2$, by certain finite orbit sets $\alpha = \{(\alpha_i, m_i)\}$ called **PFH generators**. Specifically, we require that each α_i is an embedded Reeb orbit, the α_i are distinct, the m_i are positive integers, $m_i = 1$ whenever α_i is *hyperbolic*,[9] and $\sum m_i[\alpha_i] = h$.

The ECH index

The $\mathbb{Z}_2$ vector space $PFC(\varphi, h)$ has a relative $\mathbb{Z}$ grading which we now explain. Let $\alpha = \{(\alpha_i, m_i)\}, \beta = \{(\beta_j, n_j)\}$ be two PFH generators in $PFC(\varphi, h)$. Define $H_2(Y_\varphi, \alpha, \beta)$ to be the set of equivalence classes of 2–chains Z in Y_φ satisfying $\partial Z = \sum m_i \alpha_i - \sum n_i \beta_i$. Note that $H_2(Y_\varphi, \alpha, \beta)$ is an affine space over $H_2(Y_\varphi)$.

We define the **ECH index**

$$I(\alpha, \beta, Z) = c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{n_j} CZ_\tau(\beta_j^k), \tag{2.16}$$

where τ is (the homotopy class of a trivialization) of the plane field ξ over all Reeb orbits, $c_\tau(Z)$ denotes the relative first Chern class of ξ over Z, $Q_\tau(Z)$ denotes the relative intersection pairing, and $CZ_\tau(\gamma^k)$ denotes the Conley-Zehnder index of the k^{th} iterate of γ; all of these quantities are computed using the trivialization τ. For the definitions of c_τ, CZ_τ, and Q_τ we refer the reader to [Hut02].

It is proven in [Hut02] that although the individual terms in the above definition do depend on the choice of τ, the ECH index itself does not depend on τ. According to [Hut02, Prop 1.6], the change in index caused by changing the relative homology class Z to another $Z' \in H_2(Y_\varphi, \alpha, \beta)$ is given by the formula

$$I(\alpha, \beta, Z) - I(\alpha, \beta, Z') = \langle c_1(\xi) + 2PD(h), Z - Z' \rangle. \tag{2.17}$$

The differential

Let $J \in \mathcal{J}(dr, \omega_\varphi)$ be an almost complex structure on $X = \mathbb{R} \times Y_\varphi$ which is admissible for the SHS (dr, ω_φ) and define

$$\mathcal{M}_J^{I=1}(\alpha, \beta)$$

[7]If the monotonicity assumption does not hold, we can still define PFH, but we need a different choice of coefficients; this is beyond the scope of the present work.

[8]We could also define PFH over $\mathbb{Z}$, but we do not need this here.

[9]Being hyperbolic means that the eigenvalues at the corresponding periodic point of φ are real. Otherwise, the orbit is called *elliptic*.

to be the space of J-holomorphic currents C in X, modulo translation in the $\mathbb{R}$ direction, with ECH index $I(\alpha, \beta, [C]) = 1$, which are asymptotic to α as $s \to +\infty$ and β as $s \to -\infty$; we refer the reader to [HS05], page 307, for the precise definition of asymptotic in this context.

Assume now and below for simplicity that $S = \mathbb{S}^2$. (For other surfaces, a similar story holds, but we will not need this.) Then, for generic J, $\mathcal{M}_J^{I=1}(\alpha, \beta)$ is a compact 0-dimensional manifold and we can define the PFH differential by the rule

$$\langle \partial\alpha, \beta \rangle = \#\mathcal{M}_J^{I=1}(\alpha, \beta), \tag{2.18}$$

where # denotes mod 2 cardinality. It is shown in [HT07, HT09] that[10] $\partial^2 = 0$, hence the homology PFH is defined.

Lee and Taubes [LT12] proved that the homology does not depend on the choice of J; in fact, [LT12, Corollary 1.1] states that for any surface, the homology depends only on the Hamiltonian isotopy class of φ and the choice of $h \in H_1(Y_\varphi)$. In our case, where $S = \mathbb{S}^2$, all orientation preserving area-preserving diffeomorphisms are Hamiltonian isotopic, and so we obtain a well-defined invariant which we denote by $PFH(Y_\varphi, h)$. For future motivation, we note that the Lee-Taubes invariance results discussed here come from an isomorphism of PFH and a version of the Seiberg-Witten Floer theory from [KM07].

Importantly, for the applications to this paper, we can relax the assumption that φ is nondegenerate to requiring only that φ is d-nondegenerate, where d, called the **degree**, is the positive integer determined by the intersection of h with the fiber class of the map $\pi : Y_\varphi \to \mathbb{S}^1$; note that any orbit set α with $[\alpha] = h$ must correspond to periodic points with period no more than d.

2.3.3 Twisted PFH and the action filtration

One can define a twisted version of PFH, where we keep track of the relative homology classes of J–holomorphic currents, that we will need to define spectral invariants. It has the same invariance properties of ordinary PFH, e.g. by [LT12, Corollary 6.7], where it is shown to agree with an appropriate version of Seiberg-Witten Floer cohomology.

The main reason we want to use twisted PFH is because while PFH does not have a natural action filtration, the twisted PFH does. As above, we are continuing to assume $S = \mathbb{S}^2$.

First, note that in this case, Y_φ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ and so $H_1(Y_\varphi) = \mathbb{Z}$. A class $h \in H_1(Y_\varphi)$ is then determined by its intersection with the homology class of a fiber of the map $\pi : Y_\varphi \to \mathbb{S}^1$, which we defined above to be the **degree** and denote by the integer d; from now on we will write the integer d in place of h, since these two quantities determine each other.

Choose a reference cycle γ_0 in Y_φ such that $\pi|_{\gamma_0} : \gamma_0 \to \mathbb{S}^1$ is an orientation preserving diffeomorphism and fix a trivialization τ_0 of ξ over γ_0. We can now define the $\widetilde{PFH}$ chain complex $\widetilde{PFC}(\varphi, d)$. A generator of $\widetilde{PFC}(\varphi, d)$ is a pair (α, Z), where α is a PFH generator of degree d, and Z is a relative homology class in $H_2(Y_\varphi, \alpha, d\gamma_0)$. The $\mathbb{Z}_2$ vector space $\widetilde{PFC}(\varphi, d)$ has a canonical $\mathbb{Z}$-grading I given by

$$I(\alpha, Z) = c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k). \tag{2.19}$$

The terms in the above equation are defined as in the definition of the ECH index given by Equation (2.16). Note that the above index depends on the choice of the reference cycle γ_0 and the trivialization τ_0 of ξ over γ_0.

The index defined here is closely related to the ECH index of Equation (2.16): Let (α, Z) and (β, Z') be two generators of $\widetilde{PFC}(\varphi, d)$. Note that $Z - Z'$ is a relative homology class in $H_2(Y_\varphi, \alpha, \beta)$. Then,

[10] More precisely, [HT09] proves that the differential in embedded contact homology squares to zero. As pointed out in [HT07] and [LT12] this proof carries over, nearly verbatim, to our setting.

it follows from[Hut02, Prop 1.6] that

$$I(\alpha, \beta, Z - Z') = I(\alpha, Z) - I(\beta, Z').$$

As a consequence, we see that the index difference $I(\alpha, Z) - I(\beta, Z')$ does not depend on the choices involved in the definition of the index.

We now define the differential on $\widetilde{PFC}(\varphi, d)$. We say that C is a J–holomorphic curve, or current, from (α, Z) to (β, Z') if it is asymptotic to α as $s \to +\infty$ and β as $s \to -\infty$, and moreover satisfies

$$Z' + [C] = Z, \tag{2.20}$$

as elements of $H_2(Y_\varphi, \alpha, d\gamma_0)$.

Suppose that $I(\alpha, Z) - I(\beta, Z') = 1$ and let $J \in \mathcal{J}(dr, \omega_\varphi)$. We define

$$\mathcal{M}_J((\alpha, Z), (\beta, Z'))$$

to be the moduli space of J-holomorphic currents in $X = \mathbb{R} \times Y_\varphi$, modulo translation in the $\mathbb{R}$ direction, from (α, Z) to (β, Z'). As before, for generic $J \in \mathcal{J}(dr, \omega_\varphi)$, the above moduli space is a compact 0–dimensional manifold and we define the differential by the rule

$$\langle \partial(\alpha, Z), (\beta, Z') \rangle = \#\mathcal{M}_J((\alpha, Z), (\beta, Z')),$$

where # denotes mod 2 cardinality. As before, $\partial^2 = 0$ by [HT07, HT09], and so the homology $\widetilde{PFH}$ is well-defined; by [LT12] it depends only on the degree d.

By a direct and relatively simple computation, whose details we will not provide here, in the case where φ is an irrational rotation of the sphere, i.e. $\varphi(z, \theta) = (z, \theta + \alpha)$ with α being irrational, one obtains

$$\widetilde{PFH}_*(Y_\varphi, d) = \begin{cases} \mathbb{Z}_2, & \text{if } * = d \bmod 2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.21}$$

The vector space $\widetilde{PFC}(\varphi, d)$ carries a filtration, called the **action filtration**, defined by

$$\mathcal{A}(\alpha, Z) = \int_Z \omega_\varphi.$$

The quantity $\mathcal{A}(\alpha, Z)$ is called the action of (α, Z) and is related to the Hamiltonian action functional discussed in Section 2.2: We show in [CGHS20, Lem. 3.9] that $\mathcal{A}(\alpha, Z) \in \mathrm{Spec}_d(H)$, for any Hamiltonian H supported in the northern hemishpere such that $\varphi = \phi^1_H$.

We define $\widetilde{PFC}^L(\varphi, d)$ to be the $\mathbb{Z}_2$ vector space spanned by generators (α, Z) with $\mathcal{A}(\alpha, Z) \leqslant L$.

It can be shown that ω_φ is pointwise nonnegative along any J-holomorphic curve C, and so $\int_C \omega_\varphi \geqslant 0$; see [CGHS20, Lem. 3.3]. This implies that the differential does not increase the action filtration, i.e.

$$\partial(\widetilde{PFC}^L(\varphi, d)) \subset \widetilde{PFC}^L(\varphi, d).$$

Hence, it makes sense to define $\widetilde{PFH}^L(\varphi, d)$ to be the homology of the subcomplex $\widetilde{PFC}^L(\varphi, d)$.

We are now in position to define the PFH spectral invariants. There is an inclusion induced map

$$\widetilde{PFH}^L(\varphi, d) \to \widetilde{PFH}(Y_\varphi, d). \tag{2.22}$$

If $0 \neq \sigma \in \widetilde{PFH}(\varphi, d)$ is any nonzero class, then we define the **PFH spectral invariant**

$$c_\sigma(\varphi)$$

to be the infimum, over L, such that σ is in the image of the inclusion induced map (2.22) above. The number $c_\sigma(\varphi)$ is finite, because φ is non-degenerate and so there are only finitely many Reeb orbit sets of degree d, and hence only finitely many pairs (α, Z) of a fixed grading. We remark that $c_\sigma(\varphi)$ is given by the action of some (α, Z). Indeed, this can be deduced from the following two observations:

1. If $L < L'$ are such that there exists no (α, Z) with $L \leqslant \mathcal{A}(\alpha, Z) \leqslant L'$, then the two vector spaces $\widetilde{PFC}^L(\varphi, d)$ and $\widetilde{PFC}^{L'}(\varphi, d)$ coincide and so $\widetilde{PFH}^L(\varphi, d) \to \widetilde{PFH}(Y_\varphi, d)$ and $\widetilde{PFH}^{L'}(\varphi, d) \to \widetilde{PFH}(Y_\varphi, d)$ have the same image.
2. The set of action values $\{\mathcal{A}(\alpha, Z) : (\alpha, Z) \in \widetilde{PFC}^L(\varphi, d)\}$ forms a discrete subset of $\mathbb{R}$. This is a consequence of the fact that, as stated above, there are only finitely many Reeb orbit sets of degree d.

As explained in [CGHS20, Rem. 3.10], the value of the spectral invariant does not depend on the choice of the admissible almost complex structure J. Note, however, that $c_\sigma(\varphi)$ does depend on the choice of the reference cycle γ_0.

2.3.4 Initial properties of PFH spectral invariants

We now wish to define the PFH spectral invariant $c_d(\varphi)$, for $\varphi \in \mathrm{Diff}_c(\mathbb{D}, \omega)$, without the aforementioned ambiguity. To do so, we will restrict out attention to the subset $\mathcal{S} \subset \mathrm{Diff}(\mathbb{S}^2, \omega)$ which contains $\mathrm{Diff}_c(\mathbb{D}, \omega)$.

Let $p_- = (0, 0, -1) \in \mathbb{S}^2$, where we are viewing $\mathbb{S}^2$ as the unit sphere in $\mathbb{R}^3$. We denote

$$\mathcal{S} := \{\varphi \in \mathrm{Diff}(\mathbb{S}^2, \omega) : \varphi(p_-) = p_-, \, -\frac{1}{4} < \mathrm{rot}(\varphi, p_-) < \frac{1}{4}\},$$

where $\mathrm{rot}(\varphi, p_-)$ denotes the rotation number of φ at p_- as defined in Section 2.1.2. We remark that our choice of the constant $\frac{1}{4}$ is arbitrary; any other constant in $(0, \frac{1}{2})$ would be suitable for us. As we explain below, we can define the promised PFH spectral invariant c_d without any ambiguity for non-degenerate elements of $\mathcal{S}$.

Recall from the previous section that the spectral invariant c_σ depends on the choice of reference cycle $\gamma_0 \in Y_\varphi$. For $\varphi \in \mathcal{S}$, there is a unique embedded Reeb orbit through p_-, and we set this to be the reference cycle γ_0.

The grading on $\widetilde{PFH}$ depends on the choice of trivialization τ_0 over γ_0; our convention in this paper is that we always choose τ_0 such that the rotation number θ of the linearized Reeb[11] flow along γ_0 with respect to τ_0 satisfies $-\frac{1}{4} < \theta < \frac{1}{4}$: this determines τ_0 uniquely.

We will want to single out some particular spectral invariants for $\varphi \in \mathcal{S}$, and show that they have various convenient properties; we will use these to define the spectral invariants for $\varphi \in \mathrm{Diff}_c(\mathbb{D}, \omega)$.

Having set the above conventions, we do this as follows. Suppose that $\varphi \in \mathcal{S}$ is non-degenerate. According to Equation (2.21), for every pair (d, k) with $k = d \bmod 2$, we have a distinguished nonzero class $\sigma_{d,k}$ with degree d and grading k, and so we can define

$$c_{d,k}(\varphi) := c_{\sigma_{d,k}}(\varphi).$$

Lastly, we also define[12]

$$c_d(\varphi) := c_{d,-d}(\varphi). \tag{2.23}$$

[11] Following [Hut14, Section 3.2], we define the rotation number θ as follows: Let $\{\psi_t\}_{t\in\mathbb{R}}$ denote the 1–parameter group of diffeomorphisms of Y_φ given by the flow of the Reeb vector field. Then, $D\psi_t : T_{\gamma_0(0)}Y_\varphi \to T_{\gamma_0(t)}Y_\varphi$ induces a symplectic linear map $\phi_t : \xi_{\gamma_0(0)} \to \xi_{\gamma_0(t)}$, which using the trivialization τ_0 we regard as a symplectic linear transformation of $\mathbb{R}^2$. We define θ to be the rotation number of the isotopy $\{\phi_t\}_{t\in[0,1]}$ as defined in Section 2.1.2.

[12] Alternatively, one may define $c_d(\varphi) := c_{d,k}(\varphi)$ for any $-d \leq k \leq d$ satisfying $k = d \bmod 2$. These alternative definitions are all suitable for our purposes.

Theorem 2.7, stated below and proven in [CGHS20, Sec. 3], allows us to defined $c_{d,k}(\varphi)$ for all $\varphi \in \mathcal{S}$ by Hofer continuity.

To state the next theorem, we identify a class of Hamiltonians $\mathcal{H}$ with the key property, among others, that $\mathcal{S} = \{\varphi_H^1 : H \in \mathcal{H}\}$. Denote

$$\mathcal{H} := \{H \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^2) : \varphi_H^t(p_-) = p_-, H(t, p_-) = 0, \forall t \in [0,1], \\ -\frac{1}{4} < \operatorname{rot}(\{\varphi_H^t\}, p_-) < \frac{1}{4}\},$$

where $\operatorname{rot}(\{\varphi_H^t\}, p_-)$ is the rotation number of the isotopy $\{\varphi_H^t\}_{t\in[0,1]}$ at p_-; see Section 2.1.2. Observe that $\mathcal{S} = \{\varphi_H^1 : H \in \mathcal{H}\}$.

We end this section with the theorem below establishes some of the key properties of the PFH spectral invariants and furthermore, as mentioned above, allows us to extend the definition of these invariants to all, possibly degenerate, $\varphi \in \mathcal{S}$.

Theorem 2.7. *The PFH spectral invariants $c_{d,k}(\varphi)$ admit a unique extension to all $\varphi \in \mathcal{S}$ satisfying the following properties:*

1. *Monotonicity: Suppose that $H \leqslant G$, where $H, G \in \mathcal{H}$. Then,*

$$c_{d,k}(\varphi_H^1) \leqslant c_{d,k}(\varphi_G^1).$$

2. *Hofer Continuity: For any $H, G \in \mathcal{H}$, we have*

$$|c_{d,k}(\varphi_H^1) - c_{d,k}(\varphi_G^1)| \leqslant d||H - G||_{(1,\infty)}.$$

3. *Spectrality: $c_{d,k}(\varphi_H^1) \in \operatorname{Spec}_d(H)$ for any $H \in \mathcal{H}$.*

4. *Normalization: $c_{d,-d}(\mathrm{Id}) = 0$.*

The proof of this is given in [CGHS20, Sec. 3].

Chapter 3

Main ideas of proofs

This chapter sketches the main ideas behind the proofs of the results announced in Chapter 1.

3.1 C^0 continuity of barcodes and the spectral norm

In this section I will outline the proofs of C^0 continuity of barcodes and the spectral norm γ, Theorems 1.3 and 1.8, respectively.

As mentioned earlier, Theorem 1.3 was initially proven in the case of surfaces by Le Roux, Viterbo [LRSV] relying on certain fragmentation results which have only been proven in dimension two. We will be presenting here the proofs given in [BHS18a] which applies more generally to all symplectically aspherical manifolds.

It has recently been proven by Kislev and Shelukhin [KS18] that the following inequality holds for all Hamiltonian diffeomorphisms ϕ, ψ:

$$d_{\text{bottle}}(\mathcal{B}(\phi), \mathcal{B}(\psi)) \leqslant \tfrac{1}{2}\gamma(\psi^{-1} \circ \phi). \tag{3.1}$$

The result follows immediately from the above inequality and the next theorem establishing C^0 continuity of the spectral norm γ on symplectically aspherical manifolds.

Theorem 3.1. *Let (M, ω) denote a closed and symplectically aspherical manifold. The spectral norm $\gamma : \mathrm{Ham}(M, \omega) \to \mathbb{R}$ is continuous with respect to the C^0 topology on $\mathrm{Ham}(M, \omega)$. Moreover, it extends continuous to $\overline{\mathrm{Ham}}(M, \omega)$.*

As mentioned earlier, the above result was initially proven for surfaces in [Sey13a] as a part of my PhD thesis. The proof in [Sey13a] required fragmentation techniques which are not available in higher dimensions. In [BHS18a], my collaborators and I presented an argument eliminating the need for the the two-dimensional fragmentation techniques; this is what yields the above result. More recently, C^0 continuity of γ has been generalized to projective spaces by Shelukhin [She18] (see also [Kaw19]) and to negative monotone manifolds by Kawamoto [Kaw19].

Proof of Theorem 3.1. It was proved in [Sey13a, Thm. 1] that the spectral norm γ is C^0-continuous on the subset of diffeomorphisms of $\mathrm{Ham}(M, \omega)$ generated by Hamiltonians supported in the complement of a given open subset. Our strategy involves reducing Theorem 3.1 to a slight variant of [Sey13a, Thm. 1] this as formulated in the next lemma.

Lemma 3.2. *Let (M, ω) be a closed symplectically aspherical manifold, and let U be a connected open subset in M. Then, for every $\varepsilon > 0$, there exists $\Delta > 0$ such that for any $\phi \in \mathrm{Ham}(M, \omega)$ satisfying $\phi(x) = x$ for all $x \in U$, and $d_{C^0}(\phi, \mathrm{Id}) < \Delta$, we have $\gamma(\phi) < \varepsilon$.*

The proof is very similar to the one provided in [Sey13a], with a small modification due to the fact that our map ϕ is not supposed to be generated by a Hamiltonian supported in $M \setminus U$.

Proof. By assumption, the points of U are all fixed points of ϕ. The value of their action depends on the choice of the Hamiltonian which generates ϕ. However, since U is assumed to be connected, this value is constant on U, and we will denote it by A.

Let F be a Morse function on M all of whose critical points are located in U. We assume that F is so small that its Hamiltonian flow does not admit any other periodic orbits of length $\leqslant 1$ than its critical points, and that $\max F - \min F < \varepsilon$. Thus, the spectrum of F is the set of critical values of F. This also implies that ϕ_F^1 has no fixed points in $M \setminus U$. Thus, there exists $\Delta > 0$ such that for all $x \in M \setminus U$, we have $d(\phi_F^1(x), x) > \Delta$ (in the terminology of [Sey13a], the map ϕ_F^1 "Δ-shifts" $M \setminus U$).

As a consequence, if $d_{C^0}(\phi, \mathrm{Id}) < \Delta$, then $\phi_F^1 \circ \phi$ does not have any fixed point in $M \setminus U$. Since ϕ acts as the identity on U, we get that $\phi_F^1 \circ \phi$ has the same set of fixed points as ϕ_F^1, which is in turn the set of critical points of F. Moreover, the action of point x is $F(x)$ if we think of x as a fixed point of ϕ_F^1, and is $A + F(x)$ if we see it as fixed point of $\phi_F^1 \circ \phi$.

Therefore, each spectral invariant $c(\alpha, \phi)$ of ϕ takes the form $A + F(x)$ for some critical point x of F. In particular,

$$\gamma(\phi_F^1 \circ \phi) \leqslant A + \max F - A - \min F < \tfrac{\varepsilon}{2}.$$

Using the triangle inequality, we deduce that under the condition $d_{C^0}(\phi, \mathrm{Id}) < \Delta$, we have:

$$\gamma(\phi) \leqslant \gamma(\phi_F^{-1}) + \gamma(\phi_F^1 \circ \phi) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

□

In order to reduce to Lemma 3.2, we will use a trick which consists in doubling coordinates by introducing the auxiliary map:

$$\Phi = \phi \times \phi^{-1} : \ M \times M \to M \times M, \qquad (x, y) \mapsto (\phi(x), \phi^{-1}(y)),$$

where we endow $M \times M$ with the symplectic form $\omega \oplus \omega$. The map Φ is a Hamiltonian diffeomorphism. More precisely, if ϕ is the time-1 map of a Hamiltonian H, then Φ is the time-1 map of the Hamiltonian $K : [0,1] \times M \times M \to \mathbb{R}$, $(t, x, y) \mapsto H(t, x) - H(t, \phi_H^t(y))$. Moreover, if ϕ is C^0 close to the identity, so is Φ. According to the product formula for spectral invariants (Theorem 5.1 in [EP09c]), we have $c(K) = c(H) + c(\bar{H}) = \gamma(\phi)$ and similarly, $c(\bar{K}) = c(\bar{H}) + c(H) = \gamma(\phi)$. Thus,

$$\gamma(\Phi) = 2\gamma(\phi). \tag{3.2}$$

We will prove the following Lemma.

Lemma 3.3. *For any ball B in M there exists a smaller ball $B' \subset B$ with the following property. For any $\Delta > 0$ there exists $\delta > 0$ such that if $\phi \in \mathrm{Ham}(M, \omega)$ satisfies $d_{C^0}(\phi, \mathrm{Id}_M) < \delta$, then one can find a Hamiltonian diffeomorphism $\Psi \in \mathrm{Ham}(M \times M, \omega \oplus \omega)$ satisfying the following properties:*

(i) $\mathrm{supp}\,(\Psi) \subset B \times B$ *and* $\mathrm{supp}\,(\Phi \circ \Psi) \subset M \times M \setminus B' \times B'$,

(ii) $d_{C^0}(\Psi, \mathrm{Id}_{M \times M}) < \Delta$ *and* $d_{C^0}(\Phi \circ \Psi, \mathrm{Id}_{M \times M}) < \Delta$.

We now explain why this lemma implies C^0 continuity of γ. Pick any ball $B \subset M$ such that $M \setminus B$ has a non-empty interior, and let $B' \subset B$ be a ball as provided by Lemma 3.3. Let $\varepsilon > 0$, and pick Δ as provided by Lemma 3.2 for the cases of $(M \times M, \omega \oplus \omega)$, $U = B' \times B'$ and $U = \mathrm{int}(M \times M \setminus B \times B)$. Moreover, let δ as also provided by Lemma 3.3. Finally let ϕ be such that $d_{C^0}(\phi, \mathrm{Id}_M) < \delta$ and pick Ψ as given by Lemma 3.3. By the conclusion (ii) of Lemma 3.3 and by Lemma 3.2 we have $\gamma(\Psi), \gamma(\Phi \circ \Psi) < \varepsilon$. Therefore, using (3.2), the triangle inequality and the duality property, we get

$$\gamma(\phi) = \tfrac{1}{2}\gamma(\Phi) \leqslant \tfrac{1}{2}\gamma(\Phi \circ \Psi) + \tfrac{1}{2}\gamma(\Psi^{-1}) \leqslant \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

This shows continuity of γ at identity. This, in fact, implies that γ is continuous everywhere because γ, being a norm, satisfies the triangle inequality.

We now turn our attention to the proof of Lemma 3.3.

Proof of Lemma 3.3. Let $\varepsilon > 0$, and let B be a non empty open ball in M.

The following claim asserts the existence of a convenient Hamiltonian diffeomorphism which switches coordinates on a small open set.

Claim 3.4. *There exists a non empty open ball $B'' \subset B$ and a Hamiltonian diffeomorphism f on $M \times M$, such that:*

- *f is the time-1 map of a Hamiltonian supported in $B \times B$,*
- *for all $(x, y) \in B'' \times B''$, we have $f(x, y) = (y, x)$.*

Proof. Using a Darboux chart and shrinking B if needed, we may assume without loss of generality that B is a neighborhood of 0 in $\mathbb{R}^{2n}$. Since the space $\mathrm{Sp}(4n, \mathbb{R})$ of symplectic matrices of $\mathbb{R}^{4n} \simeq \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is connected, we can choose a path $(A^t)_{t\in[0,1]}$ of such matrices such that $A^0 = \mathrm{Id}$ and A^1 is the linear map $(x, y) \mapsto (y, x)$. Let B'' be a small ball containing 0, such that for all $t \in [0, 1]$, the closure of $A^t(B'' \times B'')$ is included in $B \times B$. Let $Q_t(x)$ be a generating (quadratic) Hamiltonian for A^t and let ρ be a cut-off function supported in $B \times B$ and taking value 1 on $\bigcup_{t\in[0,1]} A^t(B'' \times B'')$. The Hamiltonian $F_t(x) = \rho(x)Q_t(x)$ generates a flow which coincides with A^t on $B'' \times B''$. Thus, its time-one map $f = \phi^1_F$ suits our needs. □

For the rest of the proof of Lemma 3.3, we pick a ball B'' and a Hamiltonian diffeomorphism f as provided by Claim 3.4. Let B' be a ball whose closure is included in B'', let $\Upsilon = \phi \times \mathrm{Id}_M$ and let

$$\Psi = \Upsilon^{-1} \circ f^{-1} \circ \Upsilon \circ f.$$

If ϕ tends to Id_M, then Φ and Ψ converge to $\mathrm{Id}_{M\times M}$, which shows property (ii).

Now, assume that ϕ is close enough to Id_M so that $\Upsilon^{-1}(\operatorname{supp} f) \subset B \times B$. Then we have $\operatorname{supp} \Upsilon^{-1} \circ f^{-1} \circ \Upsilon \subset B \times B$, and since we moreover have $\operatorname{supp} f \subset B \times B$, we conclude that $\operatorname{supp} \Psi \subset B \times B$. Assume now that ϕ is close enough to Id_M so that in addition we have $\phi(B') \subset B''$. Then for all $(x, y) \in B \times B$, we have

$$\begin{aligned}
\Phi \circ \Psi(x, y) &= \Phi \circ \Upsilon^{-1} \circ f^{-1} \circ \Upsilon \circ f(x, y) \\
&= \Phi \circ \Upsilon^{-1} \circ f^{-1} \circ \Upsilon(y, x) \\
&= \Phi \circ \Upsilon^{-1} \circ f^{-1}(\phi(y), x) \\
&= \Phi \circ \Upsilon^{-1}(x, \phi(y)) \\
&= \Phi(\phi^{-1}(x), \phi(y)) = (x, y).
\end{aligned}$$

Thus, $\Phi \circ \Psi$ coincides with the identity on $B \times B$. This establishes Property (i). □

We have completed the proof of Theorem 3.1. □

Remark 3.5. A slight modification of the proofs of Lemmas 3.2 and 3.3 would show that, in fact, γ is locally Lipschitz continuous with respect to the C^0 metric on $\mathrm{Ham}(M,\omega)$, that is, we have $\gamma(\phi) \leqslant C d_{C^0}(\phi, \mathrm{Id}_M)$ for every ϕ which is C^0-close enough to the identity, where $C = C(M,\omega)$.

Indeed, in Lemma 3.2, by picking a Morse function $H : M \to \mathbb{R}$ with $\max H - \min H < 1$, whose critical points all lie in U, we can always take the function F in the proof of the lemma to be of the form $F = \varepsilon H$. This shows that in the lemma, for small enough ε we can take $\Delta = c\varepsilon$ where $c = c(M,\omega)$. Secondly, it is easy to see that also in Lemma 3.3, for small enough Δ we can take $\delta = C\Delta$, where $C = C(M,\omega,B)$. Indeed, in the proof of the lemma we have defined $\Psi = \Upsilon^{-1} \circ f^{-1} \circ \Upsilon \circ f$ which means that $d_{C^0}(\Psi, \mathrm{Id}_M) \leqslant d_{C^0}(\Upsilon^{-1}, \mathrm{Id}_M) + d_{C^0}(f^{-1} \circ \Upsilon \circ f, \mathrm{Id}_M) \leqslant C d_{C^0}(\phi, \mathrm{Id}_M)$ where C depends only on the Lipschitz constants of the diffeomorphism f and of its inverse. ◀

3.2 Coisotropic rigidity

In this section, I will outline some of the key ideas involved in the proof of Theorem 1.4 without going into the details of the proof. Very recently, Usher has discovered a new, and shorter, proof of this theorem; see [Ush19].

The proof of Theorem 1.4 given in [HLS15a] relies on dynamical properties of coisotropic submanifolds. In particular, we use C^0–Hamiltonian dynamics as defined by Müller and Oh [OM07].

Following [OM07], we call a path of homeomorphisms ϕ^t a **hameotopy** if there exists a sequence of smooth Hamiltonian functions H_k such that the isotopies $\phi^t_{H_k}$ C^0–converge to ϕ^t and the Hamiltonians H_k C^0–converge to a continuous function H. Then, H is said to generate the hameotopy ϕ^t, and to emphasize this we write ϕ^t_H; the set of such generators will be denoted C^0_{Ham}. An important result of C^0–Hamiltonian dynamics is the uniqueness of generators theorem, proven by Viterbo [Vit06], and later generalized by Buhovsky and myself [BS13], which states that the trivial hameotopy, $\phi^t = \mathrm{Id}$, can only be generated by those functions in C^0_{Ham} which solely depend on time.

Here is an important example of a hameotopy which is needed in the proof of Theorem 1.4. Let H be a smooth Hamiltonian and θ a symplectic homeomorphism. Then, $H \circ \theta \in C^0_{\mathrm{Ham}}$ and its associated hameotopy is $\theta^{-1} \circ \phi^t_H \circ \theta$.

Recall the following dynamical property of coisotropic submanifolds C: Assume that C is a closed submanifold of (M,ω) and let $H \in C^\infty([0,1] \times M)$ be a smooth Hamiltonian. Then,

$H|_C$ is a function of time if and only if ϕ_H (preserves C and) flows along the characteristic foliation of C. By flowing along characteristics we mean that for any point $p \in C$ and any time $t \geqslant 0$, $\phi^t_H(p) \in \mathcal{F}(p)$, where $\mathcal{F}(p)$ stands for the characteristic leaf through p.

The C^0–analogue of the above property, stated below, plays an important role in the proof of Theorem 1.4.

Theorem 3.6 (Humilière-Leclercq-S. [HLS15a])**.** *Denote by C a connected coisotropic submanifold of a symplectic manifold (M,ω) which is closed as a subset[1] of M. Let $H \in C^0_{\mathrm{Ham}}$ with induced hameotopy ϕ^t_H. The restriction of H to C is a function of time if and only if ϕ_H preserves C and flows along the leaves of its characteristic foliation.*

Main idea of the proof of Theorem 1.4. The proof presented in [HLS15a] exploits the following dynamical characterization of coisotropic submanifolds: A submanifold C is coisotropic if and only if for every autonomous Hamiltonian H which is constant on C, the Hamiltonian flow of H preserves C.

Now, let C be a smooth closed coisotropic submanifold and θ a symplectic homeomorphisms. Suppose that $C' := \theta(C)$ is smooth. Let H be an autonomous Hamiltonian which is constant on C'. Then, as mentioned above, $H \circ \theta \in C^0_{\mathrm{Ham}}$ and, moreover, it is constant on the coisotropic C. Thus, by Theorem 3.6 the induced hameotopy $\phi_{H\circ\theta}$ preserves C. Since $\phi^t_{H\circ\theta} = \theta^{-1}\phi_H\theta$, we conclude that

[1] It is our convention that submanifolds have no boundary. Note that a submanifold is closed as a subset if and only if it is properly embedded.

ϕ_H^t preserves C'. It follows from the above characterization of coisotropic submanifolds that C' is coisotropic.

To prove the satement about preservation of characteristic foliations, we apply a similar argument using the following characterization of characteristic leaves: The leaf $\mathcal{F}$ through a point p is the union of the orbits of p under all Hamiltonians which are constant on C. □

3.3 The simplicity conjecture

My goal in this section is to outline the proof of Theorem 1.11 as presented in [CGHS20]. Along the way, I will state several other results of potentially independent interest, including a resolution of the "Infinite Twist Conjecture".

A proper normal subgroup of $\mathrm{Homeo}_c(\mathbb{D},\omega)$

To prove Theorem 1.11, we will define below a normal subgroup of $\mathrm{Homeo}_c(\mathbb{D},\omega)$ which is a variation on the construction of Oh-Müller [OM07]. We will show that this normal subgroup is proper.

Recall that every $\phi \in \mathrm{Diff}_c(\mathbb{D},\omega)$ is a Hamiltonian diffeomorphism, i.e. there exists $H \in C_c^\infty([0,1]\times\mathbb{D})$ such that $\phi = \phi_H^1$, where $\mathrm{Diff}_c(\mathbb{D},\omega)$ and $C_c^\infty([0,1]\times\mathbb{D})$ denote, respectively, the sets of area-preserving diffeomorphisms and Hamiltonians of the disc whose supports are contained in the interior of $\mathbb{D}$. Recall also, from Section 2.1.1, that $\|H\|$ denotes the Hofer norm of the Hamiltonian H.

Definition 3.7. *An element $\phi \in \mathrm{Homeo}_c(\mathbb{D},\omega)$ is a* **finite-energy homeomorphism** *if there exists a sequence of smooth Hamiltonians $H_i \in C_c^\infty([0,1]\times\mathbb{D})$ such that the sequence $\|H_i\|$ is bounded, i.e. $\exists\, C \in \mathbb{R}$ such that $\|H_i\| \leqslant C$, and the Hamiltonian diffeomorphisms $\phi_{H_i}^1$ converge uniformly to ϕ. We will denote the set of all finite-energy homeomorphisms by* $\mathrm{FHomeo}_c(\mathbb{D},\omega)$.

We will now show that $\mathrm{FHomeo}_c(\mathbb{D},\omega)$ is a normal subgroup.

Proposition 3.8. $\mathrm{FHomeo}_c(\mathbb{D},\omega)$ *is a normal subgroup of* $\mathrm{Homeo}_c(\mathbb{D},\omega)$.

Proof. Consider smooth Hamiltonians $H, G \in C_c^\infty([0,1]\times\mathbb{D})$. According to Equations (2.1) and (2.2) the Hamiltonians

$$H\#G(t,x) := H(t,x) + G(t,(\phi_H^t)^{-1}(x)), \quad \bar{H}(t,x) := -H(t,\phi_H^t(x)), \tag{3.3}$$

generate the Hamiltonian flows $\phi_H^t\phi_G^t$ and $(\phi_H^t)^{-1}$ respectively. Furthermore, given $\psi \in \mathrm{Diff}_c(\mathbb{D},\omega)$, the Hamiltonian

$$H\circ\psi(t,x) := H(t,\psi(x))$$

generates the flow $\psi^{-1}\phi_H^t\psi$.

We now show that FHomeo_c is closed under conjugation. Take $\phi \in \mathrm{FHomeo}_c(\mathbb{D},\omega)$ and let H_i and C be as in Definition 3.7. Let $\psi \in \mathrm{Homeo}_c(\mathbb{D},\omega)$ and take a sequence $\psi_i \in \mathrm{Diff}_c(\mathbb{D},\omega)$ which converges uniformly to ψ. Consider the Hamiltonians $H_i\circ\psi_i$. The corresponding Hamiltonian diffeomorphisms are the conjugations $\psi_i^{-1}\phi_{H_i}^1\psi_i$ which converge uniformly to $\psi^{-1}\phi\psi$. Furthermore,

$$\|H_i\circ\psi_i\| = \|H_i\| \leqslant C,$$

where the inequality follows from the definition of $\mathrm{FHomeo}_c(\mathbb{D},\omega)$.

We will next check that FHomeo_c is a group. Take $\phi,\psi \in \mathrm{FHomeo}_c$ and let $H_i, G_i \in C_c^\infty([0,1]\times\mathbb{D})$ be two sequences of Hamiltonians such that $\phi_{H_i}^1, \phi_{G_i}^1$ converge uniformly to ϕ,ψ, respectively, and $\|H_i\|, \|G_i\| \leqslant C$ for some constant C. Then, the sequence $\phi_{H_i}^{-1}\circ\phi_{G_i}^1$ converges uniformly to $\phi^{-1}\circ\psi$. Moreover, by the above formulas, we have $\phi_{H_i}^{-1}\circ\phi_{G_i}^1 = \phi_{\overline{H}_i\#G_i}^1$. Since $\|\overline{H}_i\#G_i\| \leqslant \|H_i\| + \|G_i\| \leqslant 2C$, this proves that $\phi^{-1}\circ\psi \in \mathrm{FHomeo}_c$ which completes the proof that FHomeo_c is a group. □

Theorem 1.11 will follow from the following result:

Theorem 3.9 (Thm. 1.7 in [CGHS20])**.** $\mathrm{FHomeo}_c(\mathbb{D}, \omega)$ *is a proper normal subgroup of* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$.

Remark 3.10. In defining $\mathrm{FHomeo}_c(\mathbb{D}, \omega)$ as above, we were inspired by the article of Oh and Müller [OM07], who defined the normal subgroup of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$, denoted by $\mathrm{Hameo}_c(\mathbb{D}, \omega)$, which is usually referred to as the group of **hameomorphisms**; these are, by definition, the time–1 maps of hameotopies which were introduced above in Section 3.2. It has been conjectured that $\mathrm{Hameo}_c(\mathbb{D}, \omega)$ is a proper normal subgroup of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$; see for example [OM07, Question 4.3].

It can easily be verified that $\mathrm{Hameo}_c(\mathbb{D}, \omega) \subset \mathrm{FHomeo}_c(\mathbb{D}, \omega)$. Hence, it follows from the above theorem that $\mathrm{Hameo}_c(\mathbb{D}, \omega)$ is a proper normal subgroup of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$. ◀

Remark 3.11. Theorem 3.9 gives an affirmative answer to [LR10b, Question 1] where Le Roux asks if there exist area-preserving homeomorphisms of the disc which are "infinitely far in Hofer's distance" from area-preserving diffeomorphisms.

To elaborate, for any $\phi \in \mathrm{Homeo}_c(\mathbb{D}, \omega) \setminus \mathrm{FHomeo}_c(\mathbb{D}, \omega)$ and any sequence $H_i \in C_c^\infty([0,1] \times \mathbb{D})$ of smooth Hamiltonians such that the Hamiltonian diffeomorphisms $\phi^1_{H_i} \to \phi$ uniformly, then, $\|H_i\| \to \infty$. This implies in particular that for any sequence of diffeomorphisms $\phi_i \in \mathrm{Diff}_c(\mathbb{D}, \omega)$ converging to ϕ, the ϕ_is become arbitrarily far from the identity in the Hofer metric on $\mathrm{Diff}_c(\mathbb{D}, \omega)$, hence Le Roux's formulation above.

We remark that we can also regard such a ϕ as having "infinite Hofer energy". Prior to our work, it was not known whether or not such infinite energy maps existed.

In the next section, we will see explicit examples of ϕ that we will show are in $\mathrm{Homeo}_c(\mathbb{D}, \omega) \setminus \mathrm{FHomeo}_c(\mathbb{D}, \omega)$. ◀

The Calabi invariant and the infinite twist

As we saw above, proving that $\mathrm{FHomeo}_c(\mathbb{D}, \omega)$ forms a normal subgroup of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ is not that hard. The hard part is to show properness. Here we describe the key example of an area-preserving homeomorphism that we later show is not in $\mathrm{FHomeo}_c(\mathbb{D}, \omega)$.

We first summarize some background that will motivate what follows. As mentioned earlier, for smooth, area-preserving compactly supported two-disc diffeomorphisms, non-simplicity is known, via the Calabi invariant. More precisely, the **Calabi invariant** of $\theta \in \mathrm{Diff}_c(\mathbb{D}, \omega)$ is defined as follows. Pick any Hamiltonian $H \in C_c^\infty([0,1] \times \mathbb{D})$ such that $\theta = \phi^1_H$. Then,

$$\mathrm{Cal}(\theta) := \int_{\mathbb{S}^1} \int_{\mathbb{D}} H \, \omega \, dt.$$

It is well-known that the above integral does not depend on the choice of H and so $\mathrm{Cal}(\theta)$ is well-defined; it is also known that $\mathrm{Cal} : \mathrm{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R}$ is a non-trivial group homomorphism, *i.e.* $\mathrm{Cal}(\theta_1 \theta_2) = \mathrm{Cal}(\theta_1) + \mathrm{Cal}(\theta_2)$. For further details on the Calabi homomorphism see [MS17].

We will need to know the value of the Calabi invariant for the following class of area-preserving diffeomorphisms. Let $f : [0,1] \to \mathbb{R}$ be a smooth function vanishing near 1 and define $\phi_f \in \mathrm{Diff}_c(\mathbb{D}, \omega)$ by $\phi_f(0) := 0$ and $\phi_f(r, \theta) := (r, \theta + 2\pi f(r))$. If the function f is taken to be positive monotone, then the map ϕ_f is referred to as a monotone twist.

Now suppose that $\omega = \frac{1}{2\pi} r dr \wedge d\theta$. A simple computation shows that ϕ_f is the time–1 map of the flow of the Hamiltonian defined by

$$F(r, \theta) = \int_r^1 s f(s) ds. \tag{3.4}$$

From this we compute:

$$\mathrm{Cal}(\phi_f) = \int_0^1 \int_r^1 s f(s) ds \, r dr. \tag{3.5}$$

We can now introduce the element that will not be in $\mathrm{FHomeo}_c(\mathbb{D},\omega)$. Let $f:(0,1]\to\mathbb{R}$ be a smooth function which vanishes near 1, is decreasing, and satisfies $\lim_{r\to 0} f(r)=\infty$. Define $\phi_f\in\mathrm{Homeo}_c(\mathbb{D},\omega)$ by $\phi_f(0):=0$ and

$$\phi_f(r,\theta):=(r,\theta+2\pi f(r)). \tag{3.6}$$

It is not difficult to see that ϕ_f is indeed an element of $\mathrm{Homeo}_c(\mathbb{D},\omega)$ which is in fact smooth away from the origin. We will refer to ϕ_f as an **infinite twist.**

We will show that if

$$\int_0^1\int_r^1 sf(s)ds\, rdr=\infty, \tag{3.7}$$

then

$$\phi_f\notin \mathrm{FHomeo}_c(\mathbb{D},\omega). \tag{3.8}$$

The infinite twist conjecture

The idea outlined in the above section is inspired by Fathi's suggestion that the Calabi homomorphism could extend to the subgroup $\mathrm{Hameo}_c(\mathbb{D},\omega)$, the normal subgroup constructed by Oh and Müller mentioned in Remark 3.10; see Conjecture 6.8 in Oh's article [Oh10]. Moreover, [Oh10, Thm. 7.2] shows that if the Calabi invariant extended to $\mathrm{Hameo}_c(\mathbb{D},\omega)$, then any infinite twist ϕ_f, satisfying (3.7), would not be in $\mathrm{Hameo}_c(\mathbb{D},\omega)$.

Hence, it seemed reasonable to conjecture that $\phi_f\notin\mathrm{Hameo}_c(\mathbb{D},\omega)$ if (3.7) holds. Indeed, McDuff and Salamon refer to this[2] as the **Infinite Twist Conjecture** and it is Problem 43 on their list of open problems; see [MS17, Section 14.7]. Since, as stated in Remark 3.10, $\mathrm{Hameo}_c(\mathbb{D},\omega)\subset\mathrm{FHomeo}_c(\mathbb{D},\omega)$, we obtain the following corollary from (3.8).

Corollary 3.12 ("Infinite Twist Conjecture")**.** *Any infinite twist ϕ_f satisfying* (3.7) *is not in* $\mathrm{Hameo}_c(\mathbb{D},\omega)$.

Spectral invariants from periodic Floer homology

To show that the infinite twist is not in $\mathrm{FHomeo}_c(\mathbb{D},\omega)$, we use the theory of **periodic Floer homology** (PFH) which was introduced by Hutchings [Hut02, HS05]. Recall that PFH can be used to define a sequence of functions $c_d:\mathrm{Diff}_c(\mathbb{D},\omega)\to\mathbb{R}$, where $d\in\mathbb{N}$, called **spectral invariants**, which satisfy various useful properties. These spectral invariants were defined by Michael Hutchings [Hut17] and we saw in Theorem 2.7 that they satisfy the following properties:

1. Normalization: $c_d(\mathrm{Id})=0$,
2. Monotonicity: Suppose that $H\leqslant G$ where $H,G\in C_c^\infty([0,1]\times\mathbb{D})$. Then, $c_d(\phi_H^1)\leqslant c_d(\phi_G^1)$ for all $d\in\mathbb{N}$,
3. Hofer Continuity: $|c_d(\phi_H^1)-c_d(\phi_G^1)|\leqslant d\|H-G\|$,
4. Spectrality: $c_d(\phi_H^1)\in\mathrm{Spec}_d(H)$ for any $H\in\mathcal{H}$, where $\mathrm{Spec}_d(H)$ is the **order d spectrum** of H and is defined in Section 2.3.

[2]The actual formulation in [MS17] is slightly different than this, because it does not include the condition (3.7). However, without this condition, one can produce infinite twists that lie in $\mathrm{Hameo}_c(\mathbb{D},\omega)$, by the following argument. Pick a function f with $\lim_{r\to 0} f(r)=\infty$ such that the function F from (3.4) is bounded. For such a function f, the map ϕ_f satisfies the assumptions given in [MS17], but it can be approximated by smooth monotone twist maps in a standard way, showing that ϕ_f actually belongs to $\mathrm{Hameo}_c(\mathbb{D},\omega)$. The authors of [MS17] have confirmed in private communication with us that imposing condition (3.7) is consistent with what they intended.

A key property, which allows us to use the PFH spectral invariants for studying homeomorphisms (as opposed to diffeomorphisms) is the following theorem, which we prove in [CGHS20] via the methods of continuous symplectic topology.

Theorem 3.13 (Thm. 1.10 in [CGHS20])**.** *The spectral invariant* $c_d : \mathrm{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R}$ *is continuous with respect to the* C^0 *topology on* $\mathrm{Diff}_c(\mathbb{D}, \omega)$*. Furthermore, it extends continuously to* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$*.*

Another key property is the following which was originally conjectured in greater generality by Hutchings [Hut17]:

Theorem 3.14 (Thm. 1.11 in [CGHS20])**.** *The PFH spectral invariants* $c_d : \mathrm{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R}$ *satisfy the* **Calabi property**

$$\lim_{d \to \infty} \frac{c_d(\varphi)}{d} = \mathrm{Cal}(\varphi) \tag{3.9}$$

if φ *is a monotone twist map of the disc.*

The property (3.9) can be thought of as a kind of analogue of the "Volume Property" for ECH spectral invariants proved in [CGHR15]. ECH has many similarities to PFH, which is part of the motivation for conjecturing that something like (3.9) might be possible.

Remark 3.15. In fact, Hutchings [Hut17] has conjectured that the Calabi property in Theorem 3.14 holds more generally for all $\varphi \in \mathrm{Diff}_c(\mathbb{D}, \omega)$. The point is that we verify this conjecture for monotone twists; and, this is sufficient for our purposes. ◀

Proof of Theorem 3.9

We will now give the proof of Theorem 3.9, assuming the results stated above on properties of PFH spectral invariants.

We begin by explaining the basic idea. As was already explained above, the challenge with our approach is to show that the infinite twist is not in $\mathrm{FHomeo}_c(\mathbb{D}, \omega)$. Here is how we do this. Theorem 3.13 allows to define the PFH spectral invariants for any $\psi \in \mathrm{Homeo}_c(\mathbb{D}, \omega)$. We will show, by using the Hofer Continuity property, that if ψ is a finite-energy homeomorphism then the sequence of PFH spectral invariants $\{c_d(\psi)\}_{d \in \mathbb{N}}$ grows at most linearly. On the other hand, in the case of an infinite twist ϕ_f, satisfying the condition in Equation (3.7), the sequence $\{c_d(\phi_f)\}_{d \in \mathbb{N}}$ has super-linear growth, as a consequence of the Calabi property (3.9). From this we can conclude that $\phi_f \notin \mathrm{FHomeo}_c(\mathbb{D}, \omega)$, as desired.

The details are as follows. We begin with the following lemma which tells us that for a finite-energy homeomorphism ψ the sequence of PFH spectral invariants $\{c_d(\psi)\}_{d \in \mathbb{N}}$ grows at most linearly.

Lemma 3.16. *Let* $\psi \in \mathrm{FHomeo}_c(\mathbb{D}, \omega)$ *be a finite-energy homeomorphism. Then, there exists a constant* C*, depending on* ψ*, such that*

$$\frac{c_d(\psi)}{d} \leqslant C, \ \forall d \in \mathbb{N}.$$

Proof. By definition, ψ being a finite-energy homeomorphism implies that there exist smooth Hamiltonians $H_i \in C_c^\infty([0,1] \times \mathbb{D})$ such that the sequence $\|H_i\|$ is bounded, *i.e.* $\exists\, C \in \mathbb{R}$ such that $\|H_i\| \leqslant C$, and the Hamiltonian diffeomorphisms $\phi^1_{H_i}$ converge uniformly to ψ.

The Hofer continuity property and the fact that $c_d(\mathrm{Id}) = 0$ imply that

$$c_d(\phi^1_{H_i}) \leqslant d\|H_i\| \leqslant dC,$$

for each $d \in \mathbb{N}$.

On the other hand, by Theorem 3.13 $c_d(\psi) = \lim_{i \to \infty} c_d(\varphi^1_{H_i})$. We conclude from the above inequality that $c_d(\psi) \leqslant dC$ for each $d \in \mathbb{N}$. □

We now turn our attention to showing that the PFH spectral invariants of an infinite twist ϕ_f, which satisfies Equation (3.7), violate the inequality from the above lemma. We will need the following.

Lemma 3.17. *There exists a sequence of smooth monotone twists $\phi_{f_i} \in \mathrm{Diff}_c(\mathbb{D}, \omega)$ satisfying the following properties:*

1. *The sequence ϕ_{f_i} converges in the C^0 topology to ϕ_f,*
2. *There exist Hamiltonians F_i, compactly supported in the interior of the disc $\mathbb{D}$, such that $\varphi^1_{F_i} = \phi_{f_i}$ and $F_i \leqslant F_{i+1}$,*
3. $\lim_{i\to\infty} \mathrm{Cal}(\phi_{f_i}) = \infty$.

Proof. Recall that f is a decreasing function of r which vanishes near 1 and satisfies $\lim_{r\to 0} f(r) = \infty$. It is not difficult to see that we can pick smooth functions $f_i : [0,1] \to \mathbb{R}$ satisfying the following properties:

1. $f_i = f$ on $[\frac{1}{i}, 1]$,
2. $f_i \leqslant f_{i+1}$.

Let us check that the monotone twists ϕ_{f_i} satisfy the requirements of the lemma. To see that they converge to ϕ_f, observe that ϕ_f and ϕ_{f_i} coincide outside the disc of radius $\frac{1}{i}$. Hence, $\phi_f^{-1}\phi_{f_i}$ converges uniformly to Id because it is supported in the disc of radius $\frac{1}{i}$. Next, note that by Formula (3.4), ϕ_{f_i} is the time–1 map of the Hamiltonian flow of $F_i(r,\theta) = \int_r^1 s f_i(s)ds$. Clearly, $F_i \leqslant F_{i+1}$ because $f_i \leqslant f_{i+1}$. Finally, by Formula (3.5) we have

$$\mathrm{Cal}(\phi_{f_i}) = \int_0^1 \int_r^1 s f_i(s)ds\, rdr \geqslant \int_{\frac{1}{i}}^1 \int_r^1 s f_i(s)ds\, rdr = \int_{\frac{1}{i}}^1 \int_r^1 s f(s)ds\, rdr.$$

Recall that f has been picked such that $\int_0^1 \int_r^1 sf(s)ds\, rdr = \infty$; see Equation (3.7). We conclude that $\lim_{i\to\infty} \mathrm{Cal}(\phi_{f_i}) = \infty$. ☐

We will now use Lemma 3.17 to complete the proof of Theorem 3.9. By Proposition 3.8, FHomeo_c is a normal subgroup; it is certainly non-trivial, so it remains to show it is proper.

By the Monotonicity property, we have $c_d(\phi_{f_i}) \leqslant c_d(\phi_{f_{i+1}})$ for each $d \in \mathbb{N}$. Since ϕ_{f_i} converges in C^0 topology to ϕ_f, we conclude from Theorem 3.13 that $c_d(\phi_f) = \lim_{i\to\infty} c_d(\phi_{f_i})$. Combining the previous two lines we obtain the following inequality:

$$c_d(\phi_{f_i}) \leqslant c_d(\phi_f), \forall d, i \in \mathbb{N}.$$

Now the Calabi property of Theorem 3.14 tells us that $\lim_{d\to\infty} \frac{c_d(\phi_{f_i})}{d} = \mathrm{Cal}(\phi_{f_i})$. Combining this with the previous inequality we get $\mathrm{Cal}(\phi_{f_i}) \leqslant \lim_{d\to\infty} \frac{c_d(\phi_f)}{d}$ for all i. Hence, by the third item in Lemma 3.17

$$\lim_{d\to\infty} \frac{c_d(\phi_f)}{d} = \infty,$$

and so by Lemma 3.16 ϕ_f is not in $\mathrm{FHomeo}_c(\mathbb{D}, \omega)$.

Remark 3.18. The proof outlined above does not use the full force of Theorem 3.14; it only uses the fact that $\lim_{d\to\infty} \frac{c_d(\varphi)}{d} \geqslant \mathrm{Cal}(\varphi)$. ◀

3.4 The Arnold conjecture

In this section, I will outline the construction of the C^0 counterexample to the Arnold conjecture, Theorem 1.15, and the proof of Theorem 1.16.

3.4.1 C^0 counterexample to the Arnold conjecture: A brief outline of the construction

Construction of the homeomorphism f, as prescribed in Theorem 1.15, takes place in two main steps. The first step, which is the more difficult of the two, can be summarized in the following theorem.

Theorem 3.19. *Let (M, ω) denote a closed and connected symplectic manifold of dimension at least 4. There exists $\psi \in \overline{Ham}(M, \omega)^3$ and an embedded tree $T \subset M$ such that*

1. *T is invariant under ψ, i.e. $\psi(T) = T$,*
2. *All of the fixed points of ψ are contained in T,*
3. *ψ is smooth in the complement of T.*

The proof of this theorem forms the technical heart of our paper [BHS18b]. An important ingredient used in the construction of the invariant tree T is the following quantitative h-principle for curves:

Proposition 3.20 (Quantitative h-principle for curves)**.** *Denote by (M, ω) a symplectic manifold of dimension at least 4. Let $\varepsilon > 0$. Suppose that $\gamma_0, \gamma_1 : [0, 1] \to M$ are two smooth embedded curves such that*

i. *γ_0 and γ_1 coincide near $t = 0$ and $t = 1$,*

ii. *there exists a homotopy, rel.end points, from γ_0 to γ_1 under which the trajectory of any point of γ_0 has diameter less than ε, and the symplectic area of the element of $\pi_2(M, \gamma_1 \sharp \overline{\gamma_0})$ defined by this homotopy has area 0.*

Then, for any $\rho > 0$, there exists a compactly supported Hamiltonian F, generating a Hamiltonian isotopy $\varphi^s : M \to M,\ s \in [0, 1]$ such that

1. *F vanishes near $\gamma_0(0)$ and $\gamma_0(1)$ (in particular, φ^s fixes γ_0 and γ_1 near the extremities),*
2. *$\varphi^1 \circ \gamma_0 = \gamma_1$,*
3. *$d_{C^0}(\varphi^s, Id) < 2\varepsilon$ for each $s \in [0, 1]$, and $\|F\| \leqslant \rho$,*
4. *F is supported in a 2ε-neighborhood of the image of γ_0.*

The existence of a Hamiltonian F satisfying only properties 1 and 2 is well known. The aspect of the above proposition which is non-standard is the fact that F can be picked such that properties 3 and 4 are satisfied as well. I should emphasize that M having dimension at least four is used in a crucial way in the proof of this theorem. In fact, the second step of the construction, which is outlined below, can be carried out on surfaces as well.

The second major step of our construction consists of "collapsing" the invariant tree T to a single point which will be the fixed point of our homeomorphism f. Here is a brief outline of how this is done. Fix a point $p \in M$. We construct a sequence $\varphi_i \in \mathrm{Symp}(M, \omega)$ such that φ_i converges uniformly to a map $\varphi : M \to M$ with the following two properties:

[3] In fact, ψ can be picked in the Oh-Müller group of hameomorphisms $\mathrm{Hameo}(M, \omega)$ which is contained in $\mathrm{Ham}(M, \omega)$.

1. $\varphi(T) = p$,
2. φ is a symplectic diffeomorphism from $M \setminus T$ to $M \setminus \{p\}$.

Note that the first property implies that φ is not a 1-1 map and hence, the sequence φ_i^{-1} is not convergent. Define $f : M \to M$ as follows: $f(p) = p$ and

$$\forall x \in M \setminus \{p\},\ f(x) = \varphi \circ \psi \circ \varphi^{-1}(x).$$

It is not difficult to see that p is the unique fixed point of f. Indeed, on $M \setminus \{p\}$, the map f is conjugate to $\psi : M \setminus T \to M \setminus T$ which is fixed point free by construction.

By picking the above sequence of symplectomorphisms φ_i carefully, it is possible to ensure that the sequence of conjugations $\varphi_i \psi \varphi_i^{-1}$ converges uniformly to f. The uniform convergence of $\varphi_i \psi \varphi_i^{-1}$ to f relies heavily on the invariance of the tree T and it occurs despite the fact that the sequence φ_i^{-1} diverges. The details of this are carried out in [BHS18b, Sec. 3.1]. It follows that f can be written as the uniform limit of a sequence of Hamiltonian diffeomorphisms.

It is not difficult to see that f is smooth on the complement of its unique fixed point. However, proving that f satisfies the first property listed in Theorem 1.15 requires some more work; see [BHS18b, Sec. 3.1].

3.4.2 Proof of Theorem 1.16: the generalized Arnold conjecture

Let $a, b \in H_*(M)$ be non-zero homology classes. Recall that for any $\phi \in \mathrm{Ham}(M, \omega)$ we have defined, in Equation (2.6), the difference of spectral invariants

$$\gamma(a, b; \phi) := c(a, H) - c(b, H)$$

where H is any Hamiltonian the time–1 map of whose flow is ϕ. As mentioned in Section 2.2.1, when (M, ω) is symplectically aspherical, which is our standing assumption throughout this section, the quantity $\gamma(a, b; \phi)$ does not depend on the choice of H and so it is well-defined.

In the main theorem of [BHS18a], we prove the following generalization of Theorem 3.1.

Theorem 3.21. *Let (M, ω) be closed, connected, and symplectically aspherical. For any $a, b \in H_*(M) \setminus \{0\}$, the difference of spectral invariants $\gamma(a, b; \cdot) : \mathrm{Ham}(M, \omega) \to \mathbb{R}$ is continuous with respect to the C^0 topology on $\mathrm{Ham}(M, \omega)$ and extends continuously to $\overline{\mathrm{Ham}}(M, \omega)$.*

Note that the above does imply Theorem 3.1. Moreover, it implies that we can define the quantity $\gamma(a, b; \phi)$ for any Hamiltonian homeomorphism $\phi \in \overline{\mathrm{Ham}}(M, \omega)$. This allows us to state the following result, which immediately implies Theorem 1.16.

Theorem 3.22. *Let (M, ω) denote a closed, connected and symplectically aspherical manifold, and let $\phi \in \overline{Ham}(M, \omega)$. If there exist $\alpha, \beta \in H_*(M) \setminus \{0\}$ with $\deg(\beta) < \dim(M)$, such that $\gamma(\alpha, \alpha \cap \beta; \phi) = 0$, then the set of fixed points of ϕ is homologically non-trivial.*

Our proof of the above is a generalization of the one presented in the smooth case in [How12]. To present the proof we will need to recall certain aspects of Lusternik–Schnirelmann theory, which will be done in the next section.

Preparation for the proof: min-max critical values

Let M be a closed and connected smooth manifold. Denote by $f \in C^\infty(M)$ a smooth function on M and for any $a \in \mathbb{R}$, let $M^a = \{x \in M : f(x) < a\}$. Recall that the inclusion $i_a : M^a \hookrightarrow M$ induces a map $i_a^* : H_*(M^a) \to H_*(M)$. Let $\alpha \in H_*(M)$ be a non-zero singular homology class and define

$$c_{\mathrm{LS}}(\alpha, f) := \inf\{a \in \mathbb{R} : \alpha \in \mathrm{Im}(i_a^*)\}.$$

Note that the numbers $c_{\text{LS}}(\alpha, f)$ already appeared in 2.1. They are critical values of f and such critical values are often referred to as *homologically essential* critical values. The function $c_{\text{LS}} : H_*(M) \setminus \{0\} \times C^\infty(M) \to \mathbb{R}$ is often called a *min-max* critical value selector. In the following proposition $[M]$ denotes the fundamental class of M and $[pt]$ denotes the class of a point.

Proposition 3.23. *The min-max critical value selector c_{LS} possesses the following properties:*

1. $c_{\text{LS}}(\alpha, f)$ *is a critical value of f,*
2. $\min(f) = c_{\text{LS}}([pt], f) \leqslant c_{\text{LS}}(\alpha, f) \leqslant c_{\text{LS}}([M], f) = \max(f)$,
3. $c_{\text{LS}}(\alpha \cap \beta, f) \leqslant c_{\text{LS}}(\alpha, f)$, *for any $\beta \in H_*(M)$ such that $\alpha \cap \beta \neq 0$,*
4. *Suppose that $\deg(\beta) < \dim(M)$ and $c_{\text{LS}}(\alpha \cap \beta, f) = c_{\text{LS}}(\alpha, f)$. Then, the set of critical points of f with critical value $c_{\text{LS}}(\alpha, f)$ is homologically non-trivial.*

The above are well-known results from Lusternik-Schnirelmann theory and hence we will not present a proof here. For details, we refer the reader to [LRS20, CLOT03, Vit92].

However, for the reader's convenience, we briefly sketch below the proof of the fourth property, in the case $\alpha = [M]$ (which is the only case that we will be using).

Proof of Prop 3.23, point 4. (in the case where α is the fundamental class). Since $c_{\text{LS}}([M], f) = \max(f)$, we want to prove that, if $c_{\text{LS}}(\beta, f) = \max(f)$, then the set of points where f reaches its maximum is homologically non-trivial. Let σ be a cycle which represents β. By definition, the maximum of f on the support of σ is at least $c_{LS}(\beta, f)$, which is nothing but $\max(f)$. Thus, f attains its maximum on σ. We deduce that there is no cycle representing β and supported in $M \setminus f^{-1}(\max(f))$. For every neighborhood U of $f^{-1}(\max(f))$, the homology $H_*(M)$ is generated by the homologies $H_*(U)$ and $H_*(M \setminus f^{-1}(\max(f)))$. Since β cannot be represented in $M \setminus f^{-1}(\max(f))$, this implies that U has non trivial homology. □

The proof

I will now provide the proof of Theorem 3.22, which immediately implies Theorem 1.16.

Proof of Theorem 3.22. Let U be any open neighborhood of the fixed-point set of ϕ. We will show that $\bar{U}$, the closure of U, is homologically non-trivial. This clearly implies the theorem.

Let ϕ_i be any sequence of Hamiltonian diffeomorphisms C^0-converging to ϕ, and for each i, we pick a Hamiltonian H_i whose time-one map is ϕ_i. Theorem 3.21 implies that $\gamma(\alpha \cap \beta, \alpha; \phi_i)$ converges to 0 as i goes to ∞. Denote by $f : M \to \mathbb{R}$ a smooth function such that $f = 0$ on $\bar{U}$ and $f < 0$ on $M \setminus \bar{U}$.

Claim 3.24. *For any $a \in H_*(M) \setminus \{0\}$, there exists $\varepsilon_0 > 0$ and an integer i_0 such that for any $0 < \varepsilon \leqslant \varepsilon_0$ and any $i \geqslant i_0$,*

$$c(a, H_i \# \varepsilon f) = c(a, H_i).$$

Proof. Let $\delta > 0$ be such that $d(\phi(x), x) > \delta$ for all $x \notin U$. The map ϕ is the C^0-limit of the sequence $\phi_i = \phi^1_{H_i}$, hence there exists some large integer i_0 such that:

$$d(\phi^1_{H_i}(x), x) > \tfrac{\delta}{2}, \quad \text{for all } x \notin U \text{ and } i \geqslant i_0.$$

Now take $\varepsilon > 0$ so small that $d_{C^0}(\phi^{s\varepsilon}_f, \mathrm{Id}) < \frac{\delta}{2}$, for all $s \in [0, 1]$. If x does not belong to U, neither does $\phi^{s\varepsilon}_f(x)$. Thus, for all $x \notin U$,

$$d(\phi^1_{H_i} \circ \phi^{s\varepsilon}_f(x), x) \geqslant d(\phi^1_{H_i} \circ \phi^{s\varepsilon}_f(x), \phi^{s\varepsilon}_f(x)) - d(\phi^{s\varepsilon}_f(x), x) > \tfrac{\delta}{2} - \tfrac{\delta}{2} = 0.$$

In words, $\phi^1_{H_i} \circ \phi^{s\varepsilon}_f$ has no fixed point in $M \setminus U$. Since $f = 0$ on U, we deduce that the fixed points of $\phi^1_{H_i} \circ \phi^{s\varepsilon}_f$ are the same as those of $\phi^1_{H_i}$.

Moreover, the actions of the corresponding orbits coincide. Indeed, to see this fact, note that $\phi^1_{H_i} \circ \phi^{s\varepsilon}_f$ can also be generated by the "concatenated" Hamiltonian:

$$K_{i,\varepsilon}(t,x) = \begin{cases} \rho'(t)s\varepsilon f(x) & \text{if } t \in [0, \frac{1}{2}] \\ \rho'(t-\frac{1}{2})H_i(\rho(t),x) & \text{if } t \in [\frac{1}{2}, 1], \end{cases} \tag{3.10}$$

where $\rho : [0, \frac{1}{2}] \to [0,1]$ is any smooth non-decreasing function which is 0 near 0 and 1 near $\frac{1}{2}$. It is a standard fact that the paths in $\mathrm{Ham}(M,\omega)$ generated by $H_i \# s\varepsilon f$ and $K_{i,\varepsilon}$ are homotopic with fixed end-points. Since the mean values of these two Hamiltonians are the same, this implies[4] that, given a fixed point x of $\phi^1_{H_i} \circ \phi^{s\varepsilon}_f$, the action of the associated 1-periodic orbits will be the same for $H_i \# s\varepsilon f$ and $K_{i,\varepsilon}$. Now since x does not belong to the support of $s\varepsilon f$, we easily deduce from (3.10) that this action is exactly that of H_i.

It follows that the spectrum of $H_i \# s\varepsilon f$ (which generates $\phi^1_{H_i} \circ \phi^{s\varepsilon}_f$) remains constant for $s \in [0,1]$. Now continuity of spectral invariants and the fact that the spectrum has measure zero, imply that the number $c(a, H_i \# s\varepsilon f)$ remains constant for $s \in [0,1]$. This proves the Claim. □

It follows from the above claim that for i large enough and ε small enough, $c(\alpha \cap \beta, H_i \# \varepsilon f) = c(\alpha \cap \beta, H_i)$. On the other hand, the triangle inequality of Proposition 2.1 implies that $c(\alpha \cap \beta, H_i \# \varepsilon f) \leqslant c(\alpha, H_i) + c(\beta, \varepsilon f)$. Thus, for all i, $\gamma(\alpha \cap \beta, \alpha; \phi_i) \leqslant c(\beta, \varepsilon f)$. Taking limit $i \to \infty$, we obtain $c(\beta, \varepsilon f) \geqslant 0$.

We can now conclude our proof as follows. On one hand Proposition 2.1.5 implies that for sufficiently small $\varepsilon > 0$, one has

$$c(\beta, \varepsilon f) = c_{LS}(\beta, \varepsilon f) = c_{LS}([M] \cap \beta, \varepsilon f).$$

On the other hand, Proposition 3.23.2 implies

$$c_{LS}(\beta, \varepsilon f) \leqslant c_{LS}([M], \varepsilon f) = 0.$$

Recalling that $c(\beta, \varepsilon f) \geqslant 0$, we conclude

$$0 \leqslant c(\beta, \varepsilon f) = c_{LS}(\beta, \varepsilon f) = c_{LS}([M] \cap \beta, \varepsilon f) \leqslant c_{LS}([M], \varepsilon f) = 0,$$

and in particular we obtain the equality $c_{LS}([M] \cap \beta, \varepsilon f) = c_{LS}([M], \varepsilon f)$. By Proposition 3.23.4 it follows that the zero level set of f, that is $\bar{U}$, is homologically non-trivial. This concludes the proof of Theorem 3.22. □

[4]By using the well-known fact [Sch00] that on a closed symplectically aspherical manifold, the action spectrum of a contractible normalized Hamiltonian loop is $\{0\}$.

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