Math. Z. (2009) 262:91–124
DOI 10.1007/s00209-008-0364-z

Mathematische Zeitschrift

L^2-singular dichotomy for orbital measures of classical simple Lie algebras

Sanjiv Kumar Gupta · Kathryn E. Hare · Sobhan Seyfaddini

Received: 8 May 2007 / Accepted: 25 February 2008 / Published online: 6 May 2008

Abstract We prove that the G-invariant orbital measures supported on adjoint orbits in the Lie algebra of a classical, compact, connected, simple Lie group satisfy a smoothness dichotomy: Either μ^k is singular to Lebesgue measure or $\mu^k \in L^2$. The minimum k for which $\mu^k \in L^2$ is specified and is also the minimum k such that the k-fold sum of the orbit has positive measure.

Keywords Classical Lie algebra · Compact Lie group · Adjoint orbit · Conjugacy class · Orbital measure

Mathematics Subject Classification (2000) Primary 43A80; Secondary 22E30 · 58C35

1 Introduction

Let G be a classical, compact, connected, simple Lie group and $\mathfrak{g}$ its Lie algebra. Given X in the torus of $\mathfrak{g}$, we let μ_X denote the G-invariant orbital measure supported on O_X, the orbit

S. K. Gupta appreciates the hospitality of the Department of Pure Mathematics at the University of Waterloo where some of this research was done. K. E. Hare was supported in part by NSERC.

S. K. Gupta
Department of Mathematics and Statistics, Sultan Qaboos University, P.O. Box 36,
Al Khodh 123, Sultanate of Oman
e-mail: gupta@squ.edu.om

K. E. Hare (✉)
Department of Pure Mathematics, University of Waterloo,
Waterloo, ON, Canada N2L 3G1
e-mail: kehare@uwaterloo.ca

S. Seyfaddini
Department of Mathematics, University of Toronto,
Toronto, ON, Canada M5S 2E4

of X under the adjoint action of G on $\mathfrak{g}$. In this paper, we prove that all the orbital measures (except the trivial case when $X = 0$) satisfy a surprising "L^2 versus singular" dichotomy:

$$\text{Either } \mu_X^k \in L^2(\mathfrak{g}) \bigcap L^1(\mathfrak{g}) \text{ or } \mu_X^k \text{ is singular to Lebesgue measure on } \mathfrak{g}.$$

We specify the minimum $k = k(X)$ for which $\mu_X^k \in L^2$; it depends only on the combinatorial structure of the annihilating roots of X. In fact, $\mu_X^{k(X)} \in L^p$ for some $p > 2$ which depends on the group, but need not be in L^p for all $p > 2$.

Adjoint orbits are submanifolds of proper dimension in the Lie algebra and hence have Lebesgue measure zero. Geometric properties of the algebra ensure that if a suitable number of (non-trivial) orbits are added together the resulting subset of $\mathfrak{g}$ has positive measure. Our results imply that this same value, $k(X)$, is also the least integer k such that the k-fold sum of O_X has positive measure.

Orbital measures on the group G, supported on conjugacy classes on which the exponential map is a diffeomorphism, are also shown to satisfy a similar L^2-singular dichotomy. In particular, this is true for all orbital measures on $SU(n)$.

The results in this paper improve upon [6,7] where the dichotomy was first observed for orbital measures supported on the minimal dimension conjugacy classes in the groups of Lie type $A_n, C_n, n > 3$ and D_n, and [8] where the dichotomy was shown for certain other orbital measures in $SU(n)$. These examples inspired us to conjecture that the dichotomy might hold for all orbital measures.

Our work generalizes, in spirit, the work of Ragozin [17] which built upon earlier work of Dunkl [4] and established that $\mu * \mu \in L^2(\mathbb{R}^n)$ when μ was the surface measure on a sphere in $\mathbb{R}^n$, $n \geq 3$. (Indeed, the surface measure on a sphere in $\mathbb{R}^3$ is the orbital measure on an adjoint orbit in the Lie algebra of $SU(2)$.) However, unlike spheres in $\mathbb{R}^n$ which are submanifolds of co-dimension one, adjoint orbits are typically manifolds of dimension much less than the dimension of the corresponding Lie algebra. Often their dimension is less than half the dimension of the algebra and in this case $k(X)$ will necessarily be greater than two.

Ricci and Stein [18,19] proved that if the surface measure μ on a compact submanifold of a Lie group satisfied $\mu^k \in L^1$, then the density function of μ^k satisfied a Lipschitz condition and thus $\mu^k \in L^{1+\varepsilon}$ for some $\varepsilon > 0$. But there was no suggestion in their proof that ε could be as large as 1. Other questions about the sum of orbits or convolutions of orbital measures have also been studied. For instance, in [2] and [22] a formula is given for the convolution of orbital measures. In [20] the L^p-improving behaviour of regular orbital measures is characterized; our results yield an extension of this. However, the L^2-singular dichotomy seems quite unexpected.

Our project was originally motivated by another classical result of Ragozin [16] that if μ was any central, continuous measure on G, then $\mu^{\dim G} \in L^1(G)$. Ragozin's result was improved in a series of papers by the authors, with various coauthors. Together the results in [6,7,10] imply that for the Lie groups of type A_n, C_n, $n > 3$ or D_n, it is the case that $\mu^k \in L^1(G)$ for all central, continuous measures μ if and only if k is at least $rank\, G + 1$ in type A_n and $rank\, G$ otherwise. The orbital measures supported on the conjugacy classes of minimal dimension were shown to be the sharp examples and, furthermore, these orbital measures were seen to satisfy the L^2-singular dichotomy. In [8] the dichotomy was also shown to hold for a class of orbital measures in $SU(n)$. All these examples follow as special cases of the current work. In [7] we also found the minimum k such that $(k)O_X$ has positive measure for all non-trivial orbits in all the classical Lie algebras and this too follows as a special case of the results here.

There are two main steps in obtaining our results. First, in Sects. 3, 4 and 5, we study the problem of showing that the measure of $(k)O$ is zero, for suitable choices of k and particular orbits, by proving that the dimension of the k-fold sum of tangent spaces to the orbit is less than the dimension of the algebra. Unlike [7], where the arguments varied considerably depending on the Lie type, in this paper, we develop a much more unified approach. For this part of the problem our methods are algebraic and geometric.

The second major task of the paper is to prove the sufficiency of the exponent $k(X)$ for the L^2 property and this we consider first for orbital measures supported on conjugacy classes in the Lie groups. In the previous papers the approach taken was to investigate the size of characters on Lie groups, a topic studied by many authors (c.f. [1,5]). Once the (uniform) rate of decay of the pointwise value of characters is known L^2 results can be easily derived for the orbital measures on the group by means of the Peter-Weyl theorem. This approach was already very complicated for the special orbital measures on $SU(n)$ that were studied in [8], where the structure of orbital measures/conjugacy classes is simpler than for general classical Lie groups. In this paper, we take a different approach, studying the l^2 norm of the Fourier transform of μ_x^k directly. This is a better strategy for our purposes since uniform decay estimates may not give the sharp L^2 results. However, we also obtain (new) estimates on the rate of decay of the characters as a consequence. One of the contributions of this research is to show that to verify the L^2 condition it is enough to check that the set of annihilating roots of the element x satisfy a combinatorial criterion. We then determine the optimal choice of k for which this criterion is satisfied. This analysis is carried out in Sect. 6.

To obtain results about orbital measures supported on adjoint orbits in the algebra from these results for orbital measures on the group we prove a general transference theorem that is based on ideas of Dooley and Wildberger [3]. This can be found in Sect. 7.

Our main result proved in Sect. 8. There the value of $k(X)$ is specified and we demonstrate how the work of the previous sections yield both the dichotomy and the sharp answer for the sum of orbits problem. Moreover, we show that $\mu_X^{k(X)} \in L^p$ for some $p > 2$. The L^2-singular dichotomy is also established for all orbital measures on the group that are supported on conjugacy classes of the same dimension as their preimage orbits (under the exponential map) and this includes all the orbital measures on $SU(n)$.

2 Basic facts about Lie algebras/groups and orbits

2.1 Introductory facts and notation

We begin by collecting required general facts and notation. In the next subsection we will specialize to the classical Lie groups/algebras that will be the focus of this paper. For further background details we refer the reader to [12,13] or [21], for example.

Let G denote a compact Lie group with (real) Lie algebra $\mathfrak{g}$. We let T denote the torus of the Lie group, $\mathfrak{t}$ the torus of the Lie algebra and W the Weyl group. We will denote by Φ the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and Φ^+ the positive roots.

The group G acts on the Lie algebra by the adjoint action $Ad(g)(X)$ for $g \in G$ and $X \in \mathfrak{g}$. The orbits in $\mathfrak{g}$ under this action always contain an element $H \in \mathfrak{t}$. We will denote by O_H the adjoint orbit of H and by $(k)O_H$ the k-fold sum of O_H, i.e., $O_H + \cdots + O_H$ (k times). Orbits are proper submanifolds of $\mathfrak{g}$. Similarly, a conjugacy class in G always contains some $h \in T$ and we denote by C_h the conjugacy class generated by h.[1] Given $Z \in O_H$, we will write $T_Z(O_H)$ for the tangent space to O_H at Z.

[1] We generally use capital letters to denote elements of the Lie algebra and small letters for the group.

By the orbital measure μ_H we mean the G-invariant measure supported on O_H given by:

$$\int_{\mathfrak{g}} f d\mu_H = \int_G f(Ad(g)(H))dm_G(g) \quad \text{for all } f \in C_c(\mathfrak{g})$$

(where m_G denotes the Haar measure on G.) The orbital measure μ_h supported on C_h is defined similarly. Orbital measures are continuous if $H \neq 0$ or h is not in the centre of G and they are central measures, meaning they commute with all other measures under convolution.[2]

Given a root $\alpha \in \Phi$, we let E_α denote a corresponding root vector and $\mathfrak{g}_\alpha$ the root space. We can express E_α as $RE_\alpha - \tau IE_\alpha$, with $RE_\alpha, IE_a \in \mathfrak{g}$ and $\tau^2 = -1$. (We reserve the usual symbol "i" for the quaternions, used later.) Then $E_{-\alpha} = RE_\alpha + \tau IE_\alpha$.

Given $H \in \mathfrak{t}$ we let

$$N_H = \{\alpha \in \Phi : \alpha(H) \neq 0\}$$

be the set of *non-annihilating roots* for H. The set of annihilating roots of H is a subroot system of Φ and it is of proper rank if $H \neq 0$. By the *type* of H we mean the Lie type of its annihilating roots. If $\alpha(H) \neq 0$ for all roots α, then H (or O_H) is called *regular*. These generate the orbits of maximal dimension.

The set of non-annihilating roots play an important role in the geometric structure of orbits. For example, it is known that the dimension of O_H is the cardinality of N_H [15, VI.4] and the following relationship was shown in [7].

Lemma 2.1 *If $H \in \mathfrak{t}$, then*

$$\begin{aligned} T_H(O_H) &= \text{span}\{RE_\alpha, IE_\alpha : \alpha \in N_H\} \\ &= \text{span}\{[H, RE_\alpha], [H, IE_\alpha] : \alpha \in N_H\}. \end{aligned}$$

Moreover, for a dense set, D_H, of $Z \in O_H$

$$T_Z(O_H) = \text{span}\{[Z, RE_\alpha], [Z, IE_\alpha] : \alpha \in N_H\}.$$

Since $T_{g^{-1}Hg}(O_H) = g^{-1}T_H(O_H)g$, this result implies:

Corollary 2.2 *If $X, Y \in \mathfrak{t}$ and $N_X \subseteq N_Y$, then for any $g \in G$,*

$$T_{g^{-1}Xg}(O_X) \subseteq T_{g^{-1}Yg}(O_Y).$$

Of course, $g^{-1}Xg$ and $g^{-1}Yg$ are typical elements in O_X and O_Y, respectively.

This corollary will be helpful for us (later in the paper) in deducing necessary lower bounds on the number of convolution powers needed for the L^1 problem.

2.2 Description of classical groups and properties of special elements

For the remainder of the paper, we will focus specifically on the classical, compact, connected, simple Lie groups G and their algebras $\mathfrak{g}$, those whose root systems are one of the four infinite families of type A_n, B_n, C_n or D_n. For the convenience of the reader, in this section we will summarize the set-up for each of these classical types.

Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ where $\mathbb{H} = \{\alpha + i\beta + j\gamma + k\phi : \alpha, \beta, \gamma, \phi \in \mathbb{R}\}$ is the set of quaternions. Note that $\mathbb{C}$ embeds in $\mathbb{H}$ by taking the quaternions with j, k coordinates 0 and

[2] Of course, the adjective "central" is only of interest in the group case when there exist non-central measures.

that for $q \in \mathbb{H}$, $\overline{q} = \alpha - i\beta - j\gamma - k\phi$. We should point out that the quaternions are not a field and the multiplication is not commutative. We denote by $M_n(\mathbb{F})$ the $n \times n$ matrices with entries in $\mathbb{F}$.

Our models of the classical compact connected Lie groups will be the special orthogonal, special unitary and sympletic groups:

$$SO(n) = \{g \in M_n(\mathbb{R}) : gg^t = g^t g = I, \det g = 1\}$$
$$SU(n) = \{g \in M_n(\mathbb{C}) : gg^* = g^* g = I, \det g = 1\}$$
$$Sp(n) = \{g \in M_n(\mathbb{H}) : gg^* = g^* g = I\}.$$

Note that $SO(n) \subseteq SU(n) \subseteq Sp(n)$.

Their Lie algebras are, respectively,

$$so(n) = \{A \in M_n(\mathbb{R}) : A = -A^t\}$$
$$su(n) = \{A \in M_n(\mathbb{C}) : A = -A^*, Tr(A) = 0\}$$
$$sp(n) = \{A \in M_n(\mathbb{H}) : A = -A^*\}.$$

The adjoint action here is given by $Ad(g)X = g^{-1}Xg$ for $g \in G$ and $X \in \mathfrak{g}$.

We will denote by E_{lm} the $n \times n$ matrix with 1 in entry (l, m) and 0 else, with the exception that in the case of $su(n)$, by E_{ll} we will mean the matrix with 1 in position (l, l), -1 in position (n, n) and 0 otherwise. As we will use them very frequently, we will put

$$R_{lm} = E_{lm} - E_{ml}; \quad I_{lm} = i(E_{lm} + E_{ml})$$
$$J_{lm} = j(E_{lm} + E_{ml}); \quad K_{lm} = k(E_{lm} + E_{ml}).$$

Some further notation: For $1 \leq r \leq n$, put

$$\Lambda_r(\mathfrak{g}) = \begin{matrix} \{n - r + 1, \ldots, n\} & \text{if } \mathfrak{g} = su(n) \text{ or } sp(n) \\ \{2n - 2r + 1, \ldots, 2n\} & \text{if } \mathfrak{g} = so(2n) \text{ or } so(2n + 1) \end{matrix}$$

and

$$\Omega_r(\mathfrak{g}) = \{(l, m) : 1 \leq l < m, \ m \in \Lambda_r(\mathfrak{g})\}$$

with the added condition that if $\mathfrak{g} = so(2n)$ or $so(2n + 1)$ and $l = m - 1$, then m is odd. When $\mathfrak{g}$ is clear we suppress the writing of it.

In the first half of this paper, we will be studying the orbits of torus elements $H = H_r \in \mathfrak{g} = su(n)$, $so(2n + 1)$, $sp(n)$ or $so(2n)$, whose annihilating roots are a subroot system of the same type as $\mathfrak{g}$, but with rank equal to $rank\, \mathfrak{g} - r$. We will also be interested in certain subspaces of $\mathfrak{g}$ of rank $n - r$ that we call $\mathfrak{g}_{(r)}(H_r) = \mathfrak{g}_{(r)}$. These are defined as the vector spaces generated by the annihilating roots of H_r, except if $\mathfrak{g} = su(n)$ when $\mathfrak{g}_{(r)}(H_r)$ has one more basis vector—see definition below. In all cases except $su(n)$, $\mathfrak{g}_{(r)}(H_r)$ is a Lie subalgebra.

In the chart that follows we summarize the specific details for the classical groups and algebras.

Type A_{n-1}, $n \geq 2$: (We will also say type $SU(n)$.) In this case $\tau = i$.

- $G = SU(n)$; $\mathfrak{g} = su(n)$; $\dim G = n^2 - 1$; $\mathfrak{t} =$ diagonal matrices in $su(n)$.
- $\Phi = \{\pm(e_l - e_m) : 1 \leq l < m \leq n\}$. The action of e_m on $\mathfrak{t}$ is given by $e_m(H) = \alpha_m$ when $H = \text{diag}(i\alpha_1, \ldots, i\alpha_n) \in \mathfrak{t}$.

– The root vectors satisfy the following conditions for $1 \le l < m \le n$:

$$RE_{e_l - e_m} = E_{lm} - E_{ml} = R_{lm}$$
$$IE_{e_l - e_m} = i(E_{lm} + E_{ml}) = I_{lm}.$$

Special element $H = H_r \in \mathfrak{t}$ of type A_{n-1-r} for some $r \ge 1$: Up to a Weyl conjugate these are the matrices

$$H = \mathrm{diag}(i\alpha, \ldots, i\alpha, i\alpha_{n-r+1}, \ldots, i\alpha_n) \tag{2.1}$$

with $\alpha, \alpha_{n-r+1}, \ldots, \alpha_n \in \mathbb{R}$, all distinct and $\sum_{l=n-r+1}^{n} \alpha_l = -(n-r)\alpha$. For these elements we have

– $N_H = \{\pm(e_l - e_m) : (l, m) \in \Omega_r\}$; $\dim O_H = (2n - r - 1)r$
– $T_H(O_H) = \mathrm{span}\{R_{lm}, I_{lm} : (l, m) \in \Omega_r\}$
– $\mathfrak{g}_{(r)}(H) = \mathrm{span}\{R_{lm}, I_{lm}, iE_{kk} : 1 \le l < m \le n - r, 1 \le k \le n - r\}$.

Type $B_n, n \ge 2$: $(\tau = i)$

– $G = SO(2n + 1)$; $\mathfrak{g} = so(2n + 1)$; $\dim G = n(2n + 1)$; $\mathfrak{t} = \{\sum_{l=1}^{n} \alpha_l(E_{2l-1,2l} - E_{2l,2l-1}) : \alpha_l \in \mathbb{R}\}$.
– $\Phi = \{\pm e_\lambda, \pm(e_l \pm e_m) : 1 \le \lambda \le n, 1 \le l < m \le n\}$ where $e_m(H) = \alpha_m$ for $H = \sum_{l=1}^{n} \alpha_l(E_{2l-1,2l} - E_{2l,2l-1})$.
– The root vectors are given by the formulas below for $1 \le l, m \le n$:

$$RE_{e_l} = R_{2l-1,2n+1}; \quad IE_{e_l} = R_{2l,2n+1},$$
$$RE_{e_l - e_m} = R_{2l-1,2m-1} + R_{2l,2m}; \quad IE_{e_l - e_m} = -R_{2l-1,2m} + R_{2l,2m-1}$$
$$RE_{e_l + e_m} = R_{2l-1,2m-1} - R_{2l,2m}; \quad IE_{e_l + e_m} = R_{2l-1,2m} + R_{2l,2m-1}.$$

Special element H of type B_{n-r} for $r \ge 1$: These are the matrices

$$H = \sum_{l=n-r+1}^{n} \alpha_l R_{2l-1,2l} \tag{2.2}$$

with $\alpha_{n-r+1}, \ldots, \alpha_n$ non-zero and distinct. Then

– $N_H = \{\pm e_m, \pm(e_l \pm e_m) : 1 \le l < m,\ n - r + 1 \le m \le n\}$; $\dim O_H = (2n - r)2r$
– $T_H(O_H) = \mathrm{span}\{R_{lm},\ R_{u,2n+1} : (l, m) \in \Omega_r, u \in \Lambda_r\}$
– $\mathfrak{g}_{(r)}(H) = \mathrm{span}\{R_{lm}, R_{u,2n+1} : 1 \le l < m \le 2n - 2r, 1 \le u \le 2n - 2r\}$.

Type $C_n, n \ge 3$:

– $G = Sp(n)$; $\mathfrak{g} = sp(n)$; $\dim G = n(2n + 1)$; $\mathfrak{t} = \{\sum_{l=1}^{n} i\alpha_l E_{ll} : \alpha_l \in \mathbb{R}\}$.
– $\Phi = \{\pm 2e_u, \pm(e_l \pm e_m) : 1 \le u \le n, 1 \le l < m \le n\}$ where $e_m(H) = \alpha_m$ for $H = \sum_{l=1}^{n} i\alpha_l E_{ll}$.
– For u, l, m as stated above, the root vectors are given by

$$RE_{2e_u} = jE_{uu}; \quad IE_{2e_u} = kE_{uu}$$
$$RE_{e_l - e_m} = R_{lm}; \quad IE_{e_l - e_m} = I_{lm}$$
$$RE_{e_l + e_m} = J_{lm}; \quad IE_{e_l + e_m} = K_{lm}.$$

Special element H of type C_{n-r} for $r \ge 1$: These are the $n \times n$ diagonal matrices

$$H = \mathrm{diag}(0, \ldots, 0, i\alpha_{n-r+1}, \ldots . i\alpha_n) \tag{2.3}$$

with $\alpha_{n-r+1}, \ldots, \alpha_n$ non-zero and distinct. Then
- $N_H = \{\pm 2e_m, \pm(e_l \pm e_m) : (l, m) \in \Omega_r\}$; $\dim O_H = (2n - r)2r$
- $T_H(O_H) = \mathrm{span}\{R_{lm}, I_{lm}, J_{lm}, K_{lm}, jE_{mm}, kE_{mm} : (l, m) \in \Omega_r\}$
- $\mathfrak{g}_{(r)}(H) = \mathrm{span}\{R_{lm}, I_{lm}, J_{lm}, K_{lm}, \beta E_{uu} : \beta = i, j, k, 1 \leq l < m \leq n - r, 1 \leq u \leq n - r\}$.

Type D_n, $n \geq 4$:

- $G = SO(2n)$; $\mathfrak{g} = so(2n)$; $\dim G = n(2n - 1)$; $\mathfrak{t}$ is the same as for $so(2n + 1)$, omitting the $2n + 1$ entry.
- $\Phi = \{\pm(e_l \pm e_m) : 1 \leq \lambda \leq n, 1 \leq l < m \leq n\}$ with the same action as in $SO(2n + 1)$.
- The root vectors are given by $RE_{e_l \pm e_m}$ and $IE_{e_l \pm e_m}$ as in $SO(2n + 1)$.

  Special element H of type D_{n-r} for $r \geq 1$: These are the matrices

$$H = \sum_{l=n-r+1}^{n} \alpha_l R_{2l-1,2l} \tag{2.4}$$

with $\alpha_{n-r+1}, \ldots, \alpha_n$ non-zero and distinct. Then
- $N_H = \{\pm(e_l \pm e_m) : 1 \leq l < m, n - r + 1 \leq m \leq n\}$; $\dim O_H = (2n - r - 1)2r$
- $T_H(O_H) = \mathrm{span}\{R_{lm} : (l, m) \in \Omega_r\}$
- $\mathfrak{g}_{(r)}(H) = \mathrm{span}\{R_{lm} : 1 \leq l < m \leq 2n - 2r\}$.

Terminology: When $\mathfrak{g} = su(n)$ (or $sp(n)$), a typical element $Z \in T_H(O_H)$ has the form

$$Z = \sum_{(l,m)\in\Omega_r} \alpha_{lm} R_{lm} + i\beta_{lm} I_{lm}$$

or

$$Z = \sum_{(l,m)\in\Omega_r} \alpha_{lm} R_{lm} + i\beta_{lm} I_{lm} + j\gamma_{lm} J_{lm} + k\phi_{lm} K_{lm} + \sum_{m\in\Lambda_r} (j\gamma_{mm} + k\phi_{mm}) E_{mm}$$

respectively. We will refer to $(z_{lm})_{(l,m)\in\Omega_r}$ as the *coefficients* of Z where z_{lm} is the complex (resp. quaternion) number

$$z_{lm} = \alpha_{lm} + i\beta_{lm} \ (+j\gamma_{lm} + k\phi_{lm}).$$

If $\mathfrak{g} = so(2n)$ (or $so(2n + 1)$) the typical element is

$$Z = \sum_{(l,m)\in\Omega_r} z_{lm} R_{lm} \left(+ \sum_{l\in\Lambda_r} z_{l,2n+1} R_{l,2n+1} \right).$$

where the coefficients $(z_{lm})_{(l,m)\in\Omega_r}$ are real.

3 Results for matrices

In this section and the next we will continue to assume that $H = H_r \in \mathfrak{t}$ and $\mathfrak{g}_{(r)} = \mathfrak{g}_{(r)}(H)$ are as described in the previous section. Let $P : \mathfrak{g} \to \mathfrak{g}_{(r)}$ denote the projection, say $P(X) = X_P$.

We will state and prove results either for all four Lie algebras, or for matrices in $sp(n)$ and $so(2n+1)$. In the latter case, the corresponding results for $su(n)$ and $so(2n)$ can be obtained by analogous methods.

In the next lemma $\mathfrak{g}$ could be any of the classical Lie algebras, $su(n)$, $so(2n+1)$, $sp(n)$ or $so(2n)$ and F_{lm} or F'_{lm} will denote any of R_{lm}, I_{lm}, J_{lm}, K_{lm}, depending upon which are defined for $\mathfrak{g}$.

Lemma 3.1 *Suppose* $(l, m), (u, v) \in \Omega_r(\mathfrak{g})$ *and* $u \in \Lambda_r(\mathfrak{g})$. *Then*

$$[F_{lm}, F'_{uv}]_P = 0.$$

Proof It can be seen that for any $1 \leq m_1, m_2, n_1, n_2 \leq n$,

$$[E_{m_1m_2}, E_{n_1n_2}] = \delta_{m_2n_1}E_{m_1n_2} - \delta_{m_1n_2}E_{n_1m_2}.$$

For any of the pairs, (l, u), (l, v), (m, u), (m, v) with indices as prescribed in the lemma, at most one index can lie in $\{1, \ldots, n-r\}$ if $\mathfrak{g} = su(n)$, $sp(n)$ or in $\{1, \ldots, 2n-2r\}$ if $\mathfrak{g} = so(2n)$, $so(2n+1)$, hence the lemma follows. □

Corollary 3.2 *Let* $Z \in T_H(O_H)$. *If* $(u, v) \in \Omega_r(\mathfrak{g})$ *and* $u \in \Lambda_r(\mathfrak{g})$, *then*

$$[Z, F_{uv}]_P = [Z, jE_{uu}]_P = [Z, kE_{uu}]_P = 0.$$

Notation. As they will arise frequently, for $Z \in \mathfrak{g}$ we will put

$$R_{uv}^{(Z)} = [Z, R_{uv}]_P$$

and similarly define $I_{uv}^{(Z)}$, $J_{uv}^{(Z)}$ and $K_{uv}^{(Z)}$ (as appropriate for the Lie algebra). In the case that $\mathfrak{g} = sp(n)$ we will put

$$V_{uv}^{(Z)} = \left(R_{uv}^{(Z)}, I_{uv}^{(Z)}, J_{uv}^{(Z)}, K_{uv}^{(Z)}\right).$$

We also introduce an "inner product" notation: Given a quaternion number $q = \alpha + i\beta + j\gamma + k\phi$ and 4-tuple $V = (A, B, C, D)$ (typically of elements in the Lie algebra) let

$$\langle z, V\rangle \equiv \alpha A + \beta B + \gamma C + \phi D.$$

Lemma 3.3 *Suppose* $Z, Z' \in T_H(O_H)$ *have coefficients* (z_{lm}), (z'_{lm}) *and that* $v, w \in \Lambda_r(\mathfrak{g})$.

(a) *For* $\mathfrak{g} = sp(n)$ *we have the following identities*:

- (i) $\sum_{u=1}^{n-r}\langle z_{uv}, V_{uv}^{(Z)}\rangle = 0.$
- (ii σ) $\sum_{u=1}^{n-r}\langle z'_{uv}\sigma, V_{uw}^{(Z)}\rangle + \sum_{u=1}^{n-r}\sigma^2\langle z_{uw}\sigma, V_{uv}^{(Z')}\rangle = 0$ *for any of* $\sigma = 1, i, j, k$. (*Here* $z_{uv}\sigma$ *is the quaternion product*).

(b) *For* $\mathfrak{g} = so(2n+1)$ *we have the following identities*:

- (i) $\sum_{u=1}^{2n-2r} z_{uv}R_{uv}^{(Z)} + z_{v,2n+1}R_{v,2n+1}^{(Z)} = 0.$
- (ii) $\sum_{u=1}^{2n-2r}\left(z'_{uv}R_{uw}^{(Z)} + z_{uw}R_{uv}^{(Z')}\right) + z'_{v,2n+1}R_{w,2n+1}^{(Z)} + z_{w,2n+1}R_{v,2n+1}^{(Z')} = 0.$

Proof (a) Let $m, v \in \Lambda_r$ and denote by F_{lm} any of R_{lm}, I_{lm}, J_{lm}, K_{lm}. One can easily check that

$$[F_{lm}, F_{uv}]_P = \delta_{mv}R_{ul};\ [R_{lm}, F_{uv}]_P = \delta_{mv}F_{ul}$$

$$[I_{lm}, K_{uv}]_P = -\delta_{mv} J_{ul}; [I_{lm}, J_{uv}]_P = \delta_{mv} K_{ul}$$
$$[J_{lm}, K_{uv}]_P = \delta_{mv} I_{ul}$$

with the understanding that $R_{uu} = 0$, $I_{uu} = 2iE_{uu}$, etc.

Using these facts it can be verified that for $1 \leq u \leq n-r < v \leq n$ and $V_{ul} = (R_{ul}, I_{ul}, J_{ul}, K_{ul})$,

$$\begin{aligned} R_{uv}^{(Z)} &= \left[\sum_{(l,m)\in\Omega_r} (\alpha_{lm} R_{lm} + i\beta_{lm} I_{lm} + j\gamma_{lm} J_{lm} + k\phi_{lm} K_{lm}), R_{uv} \right]_P \\ &\quad + \left[\sum_{m\in\Lambda_r} (j\gamma_{mm} + k\phi_{mm}) E_{mm}, R_{uv} \right]_P \\ &= \sum_{l=1}^{n-r} \alpha_{lv} R_{ul} - \beta_{lv} I_{ul} - \gamma_{lv} J_{ul} - \phi_{lv} K_{ul} = \sum_l \langle \overline{z_{lv}}, V_{ul} \rangle. \end{aligned}$$

Similarly,

$$I_{uv}^{(Z)} = \sum_{l=1}^{n-r} \beta_{lv} R_{ul} + \alpha_{lv} I_{ul} + \phi_{lv} J_{ul} - \gamma_{lv} K_{ul} = \sum_l \langle i\overline{z_{lv}}, V_{ul} \rangle,$$
$$J_{uv}^{(Z)} = \sum_{l=1}^{n-r} \gamma_{lv} R_{ul} - \phi_{lv} I_{ul} + \alpha_{lv} J_{ul} + \beta_{lv} K_{ul} = \sum_l \langle j\overline{z_{lv}}, V_{ul} \rangle,$$
$$K_{uv}^{(Z)} = \sum_{l=1}^{n-r} \phi_{lv} R_{ul} + \gamma_{lv} I_{ul} - \beta_{lv} J_{ul} + \alpha_{lv} K_{ul} = \sum_l \langle k\overline{z_{lv}}, V_{ul} \rangle.$$

The proofs of (i) and (ii σ) use these equations and the observation that $R_{ul} = -R_{lu}$ while I_{ul}, J_{ul} and K_{ul} are symmetric in u, l. For example, equation (ii σ) with $\sigma = 1$ simplifies to

$$\sum_{u,l=1}^{n-r} a_{ul} R_{ul} + b_{ul} I_{ul} + c_{ul} J_{ul} + d_{ul} K_{ul}$$

where, for example,

$$a_{ul} = \alpha'_{uv}\alpha_{lw} + \beta'_{uv}\beta_{lw} + \phi'_{uv}\phi_{lw} + \gamma'_{uv}\gamma_{lw} + \alpha'_{lv}\alpha_{uw} + \beta'_{lv}\beta_{uw} + \phi'_{lv}\phi_{uw} + \gamma'_{lv}\gamma_{uw}$$

and

$$c_{ul} = -\alpha'_{uv}\gamma_{lw} + \alpha'_{lv}\gamma_{uw} + \gamma'_{uv}\alpha_{lw} - \gamma'_{lv}\alpha_{uw} + \beta'_{uv}\phi_{lw} - \beta'_{lv}\phi_{uw} + \phi'_{lv}\beta_{uw} - \phi'_{uv}\beta_{lw}.$$

The antisymmetry of R_{ul} and symmetry of I_{ul} shows that

$$\sum_{u,l=1}^{n-r} a_{ul} R_{ul} = 0 = \sum_{u,l=1}^{n-r} c_{ul} J_{ul}.$$

The other two terms are similar.

(b) One can similarly check that for $1 \leq u \leq 2n-2r < v \leq 2n$ we have

$$R^{(Z)}_{uv} = \sum_{l=1}^{2n-2r} z_{lv} R_{ul} - z_{v,2n+1} R_{u,2n+1} \quad \text{and}$$

$$R^{(Z)}_{v,2n+1} = \sum_{l=1}^{2n-2r} z_{lv} R_{l,2n+1}.$$

The two identities follow from this. □

Analogous results hold for $su(n)$ and $so(2n)$.

4 Sums of tangent spaces

In this section we will prove that $(q+1)O_H$ has measure zero for $H = H_r$ as specified in (2.1)–(2.4) and $q = [\frac{n-1}{r}] - 1$. To do this, we first describe a spanning set for the projection of $T_X(O_H)$ into $\mathfrak{g}_{(r)}$, the Lie algebra of rank r less than $rank\, \mathfrak{g}$, spanned by the annihilating roots of H for any Z in the dense set D_H where we know a basis for the tangent space (Lemma 4.2). The results of the previous section on matrices in $\mathfrak{g}$ will allow us to show that there are many dependencies between these spanning sets for arbitrary $X^{(1)}, \ldots, X^{(q)} \in D_H$. This enables us to prove that the dimension of the projected space,

$$\left(T_{X^{(1)}}(O_H) + \cdots + T_{X^{(q)}}(O_H)\right)_P,$$

is less than the dimension of $\mathfrak{g}_{(r)}$ for all $X^{(1)}, \ldots, X^{(q)} \in D_H$. A continuity argument ensures that the same is true for all $X^{(1)}, \ldots, X^{(q)} \in O_H$. Since $T_H(O_H)$ lies in the orthogonal complement of $\mathfrak{g}_{(r)}$ this will prove that

$$\dim \left(T_H(O_H) + T_{X^{(1)}}(O_H) + \cdots + T_{X^{(q)}}(O_H)\right) < \dim \mathfrak{g}$$

giving Theorem 4.1. As we explain in Corollary 4.5, Sard's theorem then implies that the measure of $(q+1)O_H$ is zero.

Theorem 4.1 *Let $\mathfrak{g}$ be any of the classical compact Lie algebras $su(n)$, $so(2n+1)$, $sp(n)$ or $so(2n)$. Let $H \in \mathfrak{t}$ be as specified in Sect. 2.2 and let $q = \left[\frac{n-1}{r}\right] - 1$. Then for a dense set of $X^{(1)}, \ldots, X^{(q)} \in O_H$,*

$$\dim \left(T_H(O_H) + T_{X^{(1)}}(O_H) + \cdots + T_{X^{(q)}}(O_H)\right) < \dim \mathfrak{g}.$$

As indicated in the introduction to this section, to prove the theorem we need several further lemmas.

Our first lemma is a reformulation of [8, 3.1]. We remind the reader that D_H is the dense set of points $X \in O_H$ satisfying

$$T_X(O_H) = \operatorname{span}\{[X, RE_\alpha], [X, IE_\alpha] : \alpha \in N_H\}$$

and that P is the projection onto $\mathfrak{g}_{(r)}$. We also introduce another projection: the map $X \to X_H$ from the Lie algebra $\mathfrak{g}$ onto $T_H(O_H)$.

Lemma 4.2 *For $X \in D_H$,*

$$(T_X(O_H))_P = \operatorname{span}\{[X_H, RE_\alpha]_P, [X_H, IE_\alpha]_P : \alpha \in N_H\}.$$

Proof Since $\mathfrak{g} = \mathfrak{t} \bigoplus \sum_{\alpha\in\Phi} \text{span}\{RE_\alpha, IE_\alpha\}$, we may assume

$$X = \sum_{\alpha\in\Phi\diagdown N_H} c_\alpha RE_\alpha + d_\alpha IE_\alpha + \sum_{\alpha\in N_H} c_\alpha RE_\alpha + d_\alpha IE_\alpha + X_0$$

for $X_0 \in \mathfrak{t}$. Thus

$$X - X_H = \sum_{\alpha\in\Phi\diagdown N_H} c_\alpha RE_\alpha + d_\alpha IE_\alpha + X_0.$$

If β is a root in N_H, then $[X - X_H, RE_\beta]$ and $[X - X_H, IE_\beta]$ lie in the span of

$$\{RE_\beta, IE_\beta, RE_{\alpha+\beta}, IE_{\alpha+\beta}, RE_{\alpha-\beta}, IE_{\alpha-\beta} : \alpha \in \Phi\diagdown N_H, \beta \in N_H\}.$$

But $\beta, \alpha \pm \beta \in N_H$ for $\alpha \in \Phi\diagdown N_H$ and $\beta \in N_H$. Hence

$$[X - X_H, RE_\beta]_P = [X - X_H, IE_\beta]_P = 0$$

and the lemma follows. □

For any $Z \in T_H(O_H)$, formally define T_Z to be the subspace of $\mathfrak{g}_{(r)}$,

$$T_Z = \text{span}\{[Z, RE_\alpha]_P, [Z, IE_\alpha]_P : \alpha \in N_H\}.$$

For $X \in O_H$, $T_{X_H} = (T_X(O_H))_P$.

Lemma 4.3 *For any* $Z \in T_H(O_H)$

$$\dim T_Z \le d = \begin{cases} 2r(n-r) & \text{if } \mathfrak{g} = su(n) \\ 4r(n-r) + 2r & \text{if } \mathfrak{g} = so(2n+1) \\ 4r(n-r) & \text{if } \mathfrak{g} = sp(n) \\ 4r(n-r) & \text{if } \mathfrak{g} = so(2n) \end{cases}$$

Remark 4.4 Together with Lemma 4.2 this implies $\dim\,(T_X(O_H))_P \le d$ for all $X \in O_H$.

Proof The result follows from Corollary 3.2. For instance, when $\mathfrak{g} = su(n)$ (or $sp(n)$), T_Z is spanned by the $2r(n-r)$ (respectively, $4r(n-r)$) vectors

$$\{R_{uv}^{(Z)}, I_{uv}^{(Z)}, \text{ (and } J_{uv}^{(Z)}, K_{uv}^{(Z)}) : 1 \le u \le n-r < v \le n\}. \tag{4.1}$$

When $\mathfrak{g} = so(2n)$ (or $so(2n+1)$), T_Z is spanned by the $4r(n-r)$ $(+2r)$ vectors

$$\{R_{uv}^{(Z)}, \text{ (and } R_{v,2n+1}^{(Z)}) : 1 \le u \le 2n-2r < v \le 2n\}. \tag{4.2}$$

□

Proof of Theorem We claim it is enough to prove that

$$\dim V_q < \dim \mathfrak{g} - \dim O_H - r \ (+1 \text{ for } su(n)) \tag{4.3}$$

for a dense set of $X_1, \ldots, X_q \in O_H$, where

$$V_q = \left(T_{X^{(1)}}(O_H) + \cdots + T_{X^{(q)}}(O_H)\right)_P.$$

The claim holds since $T_H(O_H) \subseteq \mathfrak{g}_{(r)}^\perp$, $\dim \mathfrak{g}_{(r)}^\perp = \dim O_H + r$ $(-1$ for $su(n))$ and

$$\left(T_H(O_H) + T_{X^{(1)}}(O_H) + \cdots + T_{X^{(q)}}(O_H)\right)_P = V_q \subseteq \mathfrak{g}_{(r)}.$$

For $Z^{(1)}, \ldots, Z^{(q)} \in T_H(O_H)$, let

$$W_q = T_{Z^{(1)}} + \cdots + T_{Z^{(q)}}.$$

We will prove that $\dim W_q < \dim \mathfrak{g} - \dim O_H - r$ ($+1$ for $su(n)$) for all $Z^{(1)}, \ldots, Z^{(q)}$ and this will certainly establish (4.3) for all $(X^{(1)}, \ldots, X^{(q)}) \in D_H^q$.

Let $\vec{T}_Z$ be the vector space of dimension d, with basis labelled by the collections of vectors spanning T_Z that are specified in (4.1) or (4.2), depending on the Lie algebra. We will denote these basis vectors by $\vec{R}_{uv}^{(Z)}$ etc. There is a natural map from $\vec{T}_Z$ onto T_Z and this extends to a map from $\prod_{l=1}^q \vec{T}_{Z^{(l)}} \to \prod_{l=1}^q T_{Z^{(l)}}$. Let L be the composition of this map with the addition map $L_1 : \prod_{l=1}^q T_{Z^{(l)}} \to \mathfrak{g}_{(r)}$ given by $L_1(Y_1, \ldots, Y_q) = Y_1 + \cdots + Y_q$. Clearly,

$$\dim W_q = Rank\ L \leq qd - Nullity\ L.$$

Our strategy is to find a good lower bound for $Nullity\ L$.

We will do this first for $\mathfrak{g} = sp(n)$. Suppose that for $l = 1, \ldots, q$, $Z^{(l)} \in T_H(O_H)$ has coefficients

$$z_{uv}^{(l)} = \alpha_{uv}^{(l)} + i\beta_{uv}^{(l)} + j\gamma_{uv}^{(l)} + k\phi_{uv}^{(l)}.$$

Put

$$\vec{V}_{uv}^{(l)} = \left(\vec{R}_{uv}^{(l)}, \vec{I}_{uv}^{(l)}, \vec{J}_{uv}^{(l)}, \vec{K}_{uv}^{(l)}\right)$$

(where we write $\vec{R}_{uv}^{(l)}$ in place of $\vec{R}_{uv}^{(Z^{(l)})}$ etc.) We will again use the $\langle\,,\rangle$ notation defined immediately prior to Lemma 3.3.

If $A \in \prod_{l=1}^q \vec{T}_{Z^{(l)}}$, we denote the s-coordinate of A by $A(s)$. For $\sigma = 1, i, j, k$; $1 \leq l \leq m \leq q$; $w, v \in \Lambda_r$ and $w \neq v$ if $l = m$, define A_{lv} and $A_{lmwv}^{(\sigma)} \in \prod_{l=1}^q \vec{T}_{Z^{(l)}}$ by

$$A_{lv}(s) = \sum_{u=1}^{n-r} \langle z_{uv}^{(l)}, \vec{V}_{uv}^{(l)} \rangle \delta_{ls}$$

and

$$A_{lmwv}^{(\sigma)}(s) = \sum_{u=1}^{n-r} \langle z_{uv}^{(m)}\sigma, \vec{V}_{uw}^{(l)} \rangle \delta_{sl} + \sigma^2 \sum_{u=1}^{n-r} \langle z_{uw}^{(l)}\sigma, \vec{V}_{uv}^{(m)} \rangle \delta_{sm}.$$

(Of course, A_{lv} and $A_{lmwv}^{(\sigma)}$ also depend on $Z = (Z^{(1)}, \ldots, Z^{(q)})$ but we suppress this dependence in the notation).

It is clear from Lemma 3.3(a) that all of these vectors belong to the nullspace of L. We will show that for almost all choices of $Z = (Z^{(1)}, \ldots, Z^{(q)})$ this collection of vectors is linearly independent and therefore for a dense set of Z,

$$Nullity\ L \geq 2qr(qr - 1) + rq.$$

Hence we will have

$$Rank\ L < \dim \mathfrak{g} - \dim O_H - r$$

for a dense set of Z provided

$$4qr(n - r) - 2qr(qr - 1) - qr < n(2n + 1) - 2r(2n - r) - r,$$

and this is true for the choice of q. But then a continuity argument implies $Rank\ L < \dim \mathfrak{g} - \dim O_H - r$ for all Z, giving the desired result.

The proof for $\mathfrak{g} = su(n)$ follows by the natural embedding into $sp(n)$. Take Z with all j, k coefficients equal to zero. One can easily see that the null space of L contains the set

$$\left\{A_{lv}, A_{lmwv}^{(\sigma)} \text{with } \sigma = 1, i,\ 1 \leq l \leq m \leq q, w, v \in \Lambda_r,\ w \neq v \text{ if } l = m\right\}.$$

Again assuming linear independence it follows that $Nullity\, L \geq qr(qr-1) + rq$ and since

$$qr(2n-2r) - qr(qr-1) - rq < n^2 - 1 - r(2n-r-1) - r + 1$$

we see that $\dim W_q < \dim \mathfrak{g} - \dim O_H - r + 1$, as we desired to show.

The arguments for $so(2n)$ and $so(2n+1)$ follow similarly from Lemma 3.3(b). For $Z^{(l)} \in T_H(O_H)$ with coefficients $(z_{uv}^{(l)})$ we put

$$A_{lv}(s) = \left(\sum_{u=1}^{2n-2r} z_{uv}^{(l)} \vec{R}_{uv}^{(l)} + z_{v,2n+1}^{(l)} \vec{R}_{v,2n+1}^{(l)}\right) \delta_{ls}$$

and

$$A_{lmwv}(s) = \left(\sum_{u=1}^{2n-2r} z_{uv}^{(m)} \vec{R}_{uw}^{(Z^{(l)})} + z_{v,2n+1}^{(m)} \vec{R}_{w,2n+1}^{(Z^{(l)})}\right) \delta_{sl} + \left(\sum_{u=1}^{2n-2r} z_{uw}^{(l)} \vec{R}_{uv}^{(Z^{(m)})} + z_{w,2n+1}^{(l)} \vec{R}_{v,2n+1}^{(Z^{(m)})}\right) \delta_{sm}$$

for $w, v \in \Lambda_r$ and $v \neq w$ if $l = m$. Assuming linear independence one can again see there are enough vectors here to argue that $\dim W_q < \dim \mathfrak{g} - \dim O_H - r$.

The result for $so(2n)$ follows by simply deleting the terms having $2n+1$ as a subscript.

It only remains to prove the linear independence for a dense set of Z. In fact, an analyticity argument implies that either the vectors are linearly independent for almost all Z, or they are linearly dependent for every choice. Thus it suffices to prove linear independence for a single choice of $Z = (Z^{(1)}, \ldots, Z^{(q)})$.

In the case of $sp(n)$ we take

$$Z^{(l)} = \sum_{t=1}^{r} R_{(l-1)r+t,n-r+t} \quad \text{for } l = 1, \ldots, q.$$

Thus the coefficients of $Z^{(l)}$ are zero except if $v = n - r + t$ for some $t = 1, \ldots, r$ and $u = (l-1)r + t \equiv \pi_l(v)$, in which case $z_{uv}^{(l)} = 1$.

Now suppose

$$\sum_{1 \leq l \leq m}^{q} \sum_{\sigma=1,i,j,k} \sum_{v \in \Lambda_r} a_{lmwv}^{(\sigma)} A_{lmwv}^{(\sigma)} + \sum_{l=1}^{q} \sum_{v \in \Lambda_r} a_{lv} A_{lv} = 0$$

(where we understand $v \neq w$ if $l = m$).

Evaluating at the qth coordinate we obtain

$$0 = \sum_{v \in \Lambda_r} a_{qv} \sum_{u=1}^{n-r} \langle z_{uv}^{(q)}, \vec{V}_{uv}^{(q)} \rangle + \sum_{l=1}^{q-1} \sum_{\sigma=1,i,j,k} \sum_{v,w \in \Lambda_r} a_{lqwv}^{(\sigma)} \sigma^2 \sum_{u=1}^{(q-1)r} \langle z_{uw}^{(l)} \sigma, \vec{V}_{uv}^{(q)} \rangle + \sum_{\sigma=1,i,j,k} \sum_{v<w \in \Lambda_r} a_{qqwv}^{(\sigma)} \left(\sum_{u=(q-1)r+1}^{qr} \langle z_{uv}^{(q)} \sigma, \vec{V}_{uw}^{(q)} \rangle + \sigma^2 \langle z_{uw}^{(q)} \sigma, \vec{V}_{uv}^{(q)} \rangle \right).$$

Since $z_{uv}^{(q)} = 0$ unless $u = \pi_q(v)$,

$$\sum_{v\in\Lambda_r} a_{qv} \sum_{u=1}^{n-r} \langle z_{uv}^{(q)}, \vec{V}_{uv}^{(q)}\rangle = \sum_{v\in\Lambda_r} a_{qv} \vec{R}_{\pi_q(v),v}^{(q)}.$$

Similarly, for each fixed $u \le (q-1)r$, there is a unique choice of $l = l(u) \in \{1,\ldots,q-1\}$ and $w = w(u)$ (namely, chosen such that $u = \pi_l(w)$) with $z_{uw}^{(l)} \ne 0$. Consequently, the second line in the evaluation of the q-th coordinate above simplifies to

$$\sum_{v\in\Lambda_r} \sum_{u=1}^{(q-1)r} a_{lqwv}^{(1)} \vec{R}_{uv}^{(q)} - a_{lqwv}^{(i)} \vec{I}_{uv}^{(q)} - a_{lqwv}^{(j)} \vec{J}_{uv}^{(q)} - a_{lqwv}^{(k)} \vec{K}_{uv}^{(q)}.$$

We remark that $u \ne \pi_q(v)$ as $u \le (q-1)r$.

The third line can be rewritten as

$$\sum_{p\in\Lambda_r} \sum_{u=(q-1)r+1}^{qr} \sum_{\sigma} \left(\sum_{v<p} a_{qqpv}^{(\sigma)} \langle z_{uv}^{(q)}\sigma, \vec{V}_{up}^{(q)}\rangle + \sum_{w>p} a_{qqwp}^{(\sigma)} \sigma^2 \langle z_{uw}^{(q)}\sigma, \vec{V}_{up}^{(q)}\rangle \right).$$

For $v < p$, $z_{uv}^{(q)} \ne 0$ only if $u = \pi_q(v) < \pi_q(p)$, while for $w > p$, $z_{uw}^{(q)} \ne 0$ only if $u = \pi_q(w) > \pi_q(p)$. Thus the third line is a linear combination of the basis vectors $\vec{R}_{up}^{(q)}$, $\vec{I}_{up}^{(q)}$, $\vec{J}_{up}^{(q)}$, $\vec{K}_{up}^{(q)}$ but with $u > (q-1)r$ and $u \ne \pi_q(p)$, having coefficients $\pm a_{qqpv}^{(\sigma)}$ or $\pm a_{qqwp}^{(\sigma)}$ for a suitable choice of signs.

The linear independence of the basis vectors implies that all $a_{lqwv}^{(\sigma)} = a_{qv} = 0$. Repeating this argument on each coordinate $q-1,\ldots,1$ gives the result.

The linear independence arguments for the nullity for the other Lie algebras are similar, with the appropriate change of notation. This completes the proof of the theorem. □

Corollary 4.5 *For* $q = \left[\frac{n-1}{q}\right] - 1$, *the measure of* $(q+1)O_H$ *is zero and* μ_H^{q+1} *is singular to Lebesgue measure on* $\mathfrak{g}$.

Proof Since $T_{Ad(g)X}(O_H) = Ad(g)\,(T_X(O_H))$, it follows from the theorem that

$$\dim\left(\sum_{i=0}^{q} T_{X_i}(O_H)\right) < \dim \mathfrak{g}$$

for a dense set of $X_0,\ldots,X_q \in O_H$.

Consider the addition map

$$F : O_H^{q+1} \to (q+1)O_H \subseteq \mathfrak{g}$$

given by

$$F(X_0,\ldots,X_q) = \sum_{i=0}^{q} X_i.$$

The differential of F at $X = (X_0,\ldots,X_q)$ maps onto the sum of the tangent spaces to O_H at X_i, $\sum_{i=1}^{q+1} T_{X_i}(O_H)$. A continuity argument shows that if the differential map had rank equal to $\dim \mathfrak{g}$ at some X, then it would have rank equal to the $\dim \mathfrak{g}$ on a neighbourhood of X. Consequently, the differential map has rank less than $\dim \mathfrak{g}$ at every point of O_H^{q+1} and therefore Sard's theorem [14, p. 286] implies that the measure of the image of F is zero. But the image of F is $(q+1)O_H$.

Of course, μ_H^{q+1} is singular being supported on $(q+1)O_H$. □

5 Special case $SU(n/2) \times SU(n/2)$

There is one other orbit we need to consider, the orbit $O_H \subseteq su(n)$ with n even and

$$H = a\sum_{l=1}^{n/2} E_{ll} + b\sum_{l=n/2+1}^{n} E_{ll} \text{ where } a+b=0.$$

The results of the previous section are not helpful as they simply imply that $m(O_H)=0$. However, with additional work we can prove the two-fold sum has measure zero.

Proposition 5.1 *For H as above,*

$$\dim\left(T_H(O_H)+T_Z(O_H)\right) < \dim su(n)$$

for all $Z \in O_H$. Hence $O_H + O_H$ has measure zero and $\mu_H * \mu_H$ is singular to Lebesgue measure on $su(n)$.

Proof We will continue to use the notation introduced previously.

Notice the set of non-annihilating roots for H is

$$N_H = \{\pm(e_l - e_m) : l = 1, \ldots, n/2; m = n/2+1, \ldots, n\}.$$

Thus if $L = \{1, \ldots, n/2\}$ and $M = \{n/2+1, \ldots, n\}$, then

$$T_H(O_H) = \text{span}\{R_{lm}, I_{lm} : l \in L, m \in M\}.$$

If we let P be the projection of $su(n)$ onto $(T_H(O_H))^{\perp}$, then the proposition will be proven once we establish that $\dim(T_Z(O_H))_P < n^2/2 - 1$.

For $l \in L$ and $m \in M$, $[Z_H, R_{lm}]_P = [Z, R_{lm}]$ and $[Z_H, I_{lm}]_P = [Z, I_{lm}]$. Thus for Z in a dense subset,

$$(T_Z(O_H))_P = \text{span}\{[Z_H, R_{lm}],\ [Z_H, I_{lm}] : l \in L, m \in M\}.$$

Consequently, it suffices to prove that if $Z \in T_H(O_H)$ and

$$T_Z = \text{span}\{[Z, R_{lm}],\ [Z, I_{lm}] : l \in L, m \in M\},$$

then $\dim(T_Z) < n^2/2 - 1$.

Let $Z \in T_H(O_H)$ be given by $Z = \sum_{l\in L, m\in M} \lambda_{lm} R_{lm} + \beta_{lm} I_{lm}$. A straightforward computation shows that

$$R_{uv}^{(Z)} \equiv [Z, R_{uv}] = \sum_{l\in L}(\alpha_{lv} R_{ul} - \beta_{lv} I_{ul}) + \sum_{m\in M}(\alpha_{um} R_{vm} + \beta_{um} I_{vm})$$
$$I_{uv}^{(Z)} \equiv [Z, I_{uv}] = \sum_{l\in L}(\beta_{lv} R_{ul} + \alpha_{lv} I_{ul}) + \sum_{m\in M}(\beta_{um} R_{vm} - \alpha_{um} I_{vm}).$$

We introduce further notation (which will be used only in this proof): For $m, v, s, t \in M$ and $s < t$ put

$$R_v = \sum_{u \in L} \alpha_{uv} R_{uv}^{(Z)} + \beta_{uv} I_{uv}^{(Z)}$$
$$R(s,t) = \sum_{u \in L} \alpha_{us} R_{ut}^{(Z)} + \beta_{us} I_{ut}^{(Z)} + \alpha_{ut} R_{us}^{(Z)} + \beta_{ut} I_{us}^{(Z)}$$
$$I(s,t) = \sum_{u \in L} \beta_{us} R_{ut}^{(Z)} - \alpha_{us} I_{ut}^{(Z)} - \beta_{ut} R_{us}^{(Z)} + \alpha_{ut} I_{us}^{(Z)}$$

and

$$\alpha'_{vm} = \sum_{u \in L} \alpha_{uv} \alpha_{um} + \beta_{uv} \beta_{um},$$
$$\beta'_{vm} = \sum_{u \in L} \beta_{um} \alpha_{uv} - \alpha_{um} \beta_{uv}.$$

Note that $\alpha'_{vm} = \alpha'_{mv}$, $\beta'_{vm} = -\beta'_{mv}$ and $\beta'_{vv} = 0$. With this notation we can write

$$R(s,t) = \sum_{m \in M} \alpha'_{ms} R_{tm} + \beta'_{sm} I_{tm} + \alpha'_{mt} R_{sm} + \beta'_{tm} I_{sm}$$
$$I(s,t) = \sum_{m \in M} \beta'_{ms} R_{tm} + \alpha'_{sm} I_{tm} - \beta'_{mt} R_{sm} - \alpha'_{tm} I_{sm}.$$

We claim

(i) $\sum_{v \in M} R_v = 0$;
(ii) $\sum_{v \in M} \alpha'_{vv} R_v + \sum_{s<t \in M} \alpha'_{st} R(s,t) + \beta'_{st} I(s,t) = 0$.

Claim (i) is simply the observation that $\sum_{v \in M} R_v = [Z, Z]$. To prove (ii), we first note that the symmetry of I_{ul} and antisymmetry of R_{ul} ensures that

$$0 = \sum_{u,l \in L} \alpha_{uv} \alpha_{lv} R_{ul} = \sum_{u,l \in L} \beta_{uv} \beta_{lv} R_{ul} = \sum_{u,l \in L} (-\beta_{lv} \alpha_{uv} + \beta_{uv} \alpha_{lv}) I_{ul}.$$

Thus the left side of (ii) simplifies to

$$= \sum_{m,t \in M} \alpha'_{tt} \left(\alpha'_{mt} R_{mt} + \beta'_{tm} I_{tm} \right) + \sum_{m,s<t \in M} \alpha'_{st} \left(\alpha'_{ms} R_{tm} + \beta'_{sm} I_{tm} + \alpha'_{mt} R_{sm} + \beta'_{tm} I_{sm} \right)$$
$$+ \sum_{m,s<t \in M} \beta'_{st} \left(\beta'_{ms} R_{tm} + \alpha'_{sm} I_{tm} - \beta'_{mt} R_{sm} - \alpha'_{tm} I_{sm} \right).$$

Fix $a, b \in M$, $a < b$. We will show that the coefficient of R_{ab} in the left side of (ii) is zero. Similarly, the coefficient of $I_{ab} = 0$ and hence (ii) will be proved. Reading off, we see that the coefficient of R_{ab} is

$$= -\alpha'_{bb} \alpha'_{ab} + \alpha'_{aa} \alpha'_{ba} + \sum_{s<a} \left(\alpha'_{sa} \alpha'_{bs} + \beta'_{sa} \beta'_{bs} \right) + \sum_{t>a} \left(\alpha'_{at} \alpha'_{bt} - \beta'_{at} \beta'_{bt} \right)$$
$$- \sum_{s<b} \left(\alpha'_{sb} \alpha'_{as} + \beta'_{sb} \beta'_{as} \right) - \sum_{t>b} \left(\alpha'_{bt} \alpha'_{at} - \beta'_{bt} \beta'_{at} \right)$$
$$= -\alpha'_{bb} \alpha'_{ab} + \alpha'_{aa} \alpha'_{ba} - \sum_{a \le s < b} \left(\alpha'_{sa} \alpha'_{bs} + \beta'_{sa} \beta'_{bs} \right) + \sum_{a < t \le b} \left(\alpha'_{at} \alpha'_{bt} - \beta'_{at} \beta'_{bt} \right).$$

As $\beta'_{bb} = \beta'_{aa} = 0$, this further simplifies to

$$-\sum_{a<s<b} \left(\alpha'_{sa}\alpha'_{bs} + \beta'_{sa}\beta'_{bs}\right) + \sum_{a<t<b} \left(\alpha'_{at}\alpha'_{bt} - \beta'_{at}\beta'_{bt}\right) = 0.$$

The two vectors $\sum_{v\in M} R_v$ and $\sum_{v\in M} \alpha'_{vv} R_v + \sum_{s<t\in M} \alpha'_{st} R(s,t) + \beta'_{st} I(s,t)$ are linear combinations of $R^{(Z)}_{lm}, I^{(Z)}_{lm}$ for $l \in L$, $m \in M$ and thus belong to T_Z. Moreover, they are linearly independent combinations when $Z = R_{1,n} + \frac{1}{2} R_{2,n-1}$, for example, and hence for almost all $Z \in T_H(O_H)$. Consequently, $\dim T_Z \leq 2(n/2)^2 - 2 < n^2/2 - 1$ and this completes the proof. □

6 Combinatorial criterion for L^2

In this section we will develop a combinatorial approach to studying the L^2 side of the problem.

6.1 Criterion for belonging to L^2

In [6–10] the L^2 problem for convolution powers of orbital measures supported on conjugacy classes in the group G was studied. The approach taken there was to compare the rate of decay of the pointwise value of the characters of the group with their degree. To be more precise, we proved bounds such as $Tr\lambda(x)/d_\lambda \leq C d_\lambda^{-s}$ for all representations λ, where C, s were positive constants that did not depend on λ and $d_\lambda = \deg \lambda$. Estimates on the pointwise values of characters are relevant for an orbital measure μ_x on G because $\widehat{\mu_x(\lambda)} = Tr\lambda(x)/d_\lambda$. It was established in [9] that $\sum_{\lambda\in\widehat{G}} d_\lambda^t < \infty$ for $t < -rank\, G/\left|\Phi^+\right|$, thus L^2 results follow easily from these estimates using the Peter–Weyl theorem.

This approach does not seem to adapt well to some of the more complicated orbits. Instead, in this paper, we directly bound the l^2 norm of the Fourier transform of $\widehat{\mu^k}$. The novelty of our new approach is that we show that the problem of finding an integer k such that $\widehat{\mu^k} \in l^2$ can be reduced to a purely combinatorial problem, as we will now explain.

Notation. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis for a root system Φ of rank n and $\{\lambda_1, \ldots, \lambda_n\}$ be the fundamental dominant weights, i.e., the dual basis vectors which satisfy $(\alpha_i, \lambda_j) = \delta_{ij}$.[3] Given a set of l integers $i_1, \ldots, i_l$ satisfying

$$n \geq i_1 > i_2 > \cdots > i_l \geq 1$$

and a subroot system Ψ of Φ, we will let

$$X_j = \left\{\alpha \in \Phi^+ \setminus \bigcup_{k=1}^{j-1} X_k : (\alpha, \lambda_{i_j}) \neq 0\right\} \quad \text{for } j = 1, \ldots, l,$$

$$B_j = B_j(\Psi) = \left\{\alpha \in \Psi^+ \setminus \bigcup_{k=1}^{j-1} B_k : (\alpha, \lambda_{i_j}) \neq 0\right\} \text{ for } j = 1, \ldots, l$$

and $G_j = X_j \setminus B_j$. (We call G_j and B_j the "good" and "bad" roots, respectively, arising at step j relative to the given set of indices).

[3] Note that in this section i, j, k will no longer be quaternions.

Let $k(i_1, \dots, i_l, \Psi)$ be the minimum integer k such that

$$\sum_{j=1}^{l}(k-1)\left|X_j\right| - k\left|B_j\right| = \sum_{j=1}^{l}(k-1)\left|G_j\right| - \left|B_j\right| > l/2.$$

Given $x \in T$, we say $\alpha \in \Phi$ is an *annihilating root* of x if $\alpha(x) \equiv 0 \mod 2\pi$. The set of annihilating roots is a subroot system of Φ and by the *type* of x we mean the Lie type of its set of annihilating roots. This set is proper if x does not belong to the centre of G.

With this notation we can state our combinatorial criterion for μ_x^k to belong to $L^2(G)$.

Theorem 6.1 *Suppose $x \in T$ and $\Phi(x)$ is the set of annihilating roots of x. Let*

$$k_0(x) = \max\left(k(i_1, \dots, i_l, \Psi)\right)$$

where the maximum is taken over all $l = 1, \dots, n = \operatorname{rank} G$, all sets of indices $n \geq i_1 > i_2 > \cdots > i_l \geq 1$ and all subroot systems, Ψ, Weyl conjugate to $\Phi(x)$. Then $\mu_x^{k_0(x)} \in L^2(G)$.

Proof Throughout the proof the constant C may depend on x and G, but not λ, and may vary from one occurrence to another.

The Peter–Weyl theorem implies that $\mu_x^k \in L^2(G)$ if and only if

$$\left\|\widehat{\mu_x}^k\right\|_2^2 = \sum_{\lambda \in \widehat{G}} d_\lambda^{2-2k} \left|Tr\lambda(x)\right|^{2k} < \infty.$$

It was proven in [9] that

$$\left|Tr\lambda(x)\right| \leq C \max_{w \in W} \prod_{\alpha \in w(\Phi^+(x))} \left|(\rho + \lambda, \alpha)\right|$$

and since the Weyl-dimension formula states that

$$d_\lambda = C \prod_{\alpha \in \Phi^+} (\rho + \lambda, \alpha),$$

this gives the bound

$$\left\|\widehat{\mu_x}^k\right\|_2^2 \leq C \max_{w \in W} \sum_{\lambda \in \widehat{G}} \prod_{\alpha \in \Phi^+} \left|(\rho + \lambda, \alpha)\right|^{2-2k} \prod_{\alpha \in w(\Phi^+(x))} \left|(\rho + \lambda, \alpha)\right|^{2k}.$$

Temporarily fix $\lambda = \sum_{i=1}^{n} m_i \lambda_i \in \widehat{G}$, where $\lambda_1, \dots, \lambda_n$ are the fundamental dominant weights, and order the coefficients m_i as $m_{i_1} \geq m_{i_2} \geq \cdots \geq m_{i_n}$. Let $w \in W$. For $j = 1, \dots, l$ put

$$X_j = \left\{\alpha \in \Phi^+ \diagdown \bigcup_{k=1}^{j-1} X_k : (\alpha, \lambda_{i_j}) \neq 0\right\}$$

and

$$B_j = B_j(w(\Phi(x))) = \left\{\alpha \in w(\Phi^+(x)) \diagdown \bigcup_{k=1}^{j-1} B_k : (\alpha, \lambda_{i_j}) \neq 0\right\}.$$

The sets X_j, $j = 1, \dots, n$ are a disjoint partitioning of Φ^+ and the sets B_j, $j = 1, \dots, n$ a disjoint partitioning of $w(\Phi^+(x))$, the positive roots of a subroot system conjugate to $\Phi(x)$.

If $\alpha \in X_j$ or B_j, then

$$1 + m_{i_j} \le |(\rho + \lambda, \alpha)| \le C(1 + m_{i_j}).$$

With this notation

$$\prod_{\alpha \in \Phi^+} |(\rho + \lambda, \alpha)|^{2-2k} \prod_{\alpha \in w(\Phi^+(x))} |(\rho + \lambda, \alpha)|^{2k} \le C \prod_{j=1}^{n} (1 + m_{i_j})^{|X_j|(2-2k)+|B_j|2k}.$$

It is easy to see that there is no loss of generality in assuming the indices i_j are in (strictly) decreasing order for calculating the sums, $\sum_{j=1}^{l} |X_j|$ and $\sum_{j=1}^{l} |B_j|$. Thus taking $k = k_0(x)$ ensures that

$$\sum_{j=1}^{l} |X_j| (2 - 2k) + |B_j| 2k \le -l - 1.$$

Simplifying, it follows that

$$\begin{aligned}
&\prod_{j=1}^{n} (1 + m_{i_j})^{|X_j|(2-2k)+|B_j|2k} \\
&\quad \le (1 + m_{i_1})^{-1-\frac{1}{n}} (1 + m_{i_2})^{\sum_{j=1}^{2}|X_j|(2-2k)+|B_j|2k+1+\frac{1}{n})} \prod_{j=3}^{n} (1 + m_{i_j})^{|X_j|(2-2k)+|B_j|2k} \\
&\quad \le \prod_{j=1}^{n} (1 + m_j)^{-1-\frac{1}{n}}
\end{aligned}$$

and hence

$$\begin{aligned}
\left\| \widehat{\mu_x}^k \right\|_2^2 &\le C |W| \sum_{\lambda \in \widehat{G}} \prod_{j=1}^{n} (1 + m_j)^{-1-1/n} \\
&\le C \left(\sum_{m=1}^{\infty} m^{-1-1/n} \right)^n < \infty
\end{aligned}$$

as we desired to prove. □

Remark 6.2 Similar reasoning shows that if $\sum_{j=1}^{l} |G_j| (s - 1) + |B_j| s \le 0$, then

$$\frac{|Tr\lambda(x)|}{d_\lambda} \le c d_\lambda^{-s} \quad \text{for all } \lambda.$$

Thus for all λ,

$$\frac{|Tr\lambda(x)|}{d_\lambda} \le c d_\lambda^{-1/k_0(x)}.$$

6.2 Counting arguments

The previous theorem shows it is important to determine the maximum value of $k(i_1, \ldots, i_l, \Psi)$ taken over all $l = 1, \ldots, rank\, G$, indices $i_1 > i_2 \cdots > i_l$ and subroot systems Ψ of a given type. As we will see later, it will be enough to determine this number for a relatively small set of subroot systems, which we analyze in this section. Our typical strategy will involve an induction argument and counting.

In what follows, C_1 will mean a single long root $\{2e_J\}$ and when we write $SU(J) \times SU(1)$ in $SU(J+1)$ we simply mean the subroot system $SU(J)$ in $SU(J+1)$.

6.2.1 $C_J \times C_K$ in C_{J+K} and $SU(J) \times SU(K)$ in $SU(J+K)$

Lemma 6.3 *Suppose G is either of type C_{J+K} or $SU(J+K)$ and that Ψ is a subroot system of type $C_J \times C_K$ in the first case and type $SU(J) \times SU(K)$ in the second. Assume $J \geq \max(K, 2)$, $K \geq 1$ and Ψ is not type $C_2 \times C_1$. Then for any $l = 1, \ldots, rank\, G$ and any set of indices satisfying $rank\, G \geq i_1 > i_2 > \cdots > i_l \geq 1$ we have*

$$\sum_{j=1}^{l} \max\left(\frac{J}{K}, 2\right) G_j - B_j > \frac{l}{2}$$

(*where we write G_j, B_j for the cardinality of the sets G_j, B_j*).

Remark 6.4 In [10] it was shown that for x of type $C_2 \times C_1$ in C_3, $\mu_x^k \in L^2(G)$ if and only if $k \geq 4$, so the inequality above cannot hold in this case. However, one can check that if Ψ is type C_2 in C_3, then $\sum_{j=1}^{l} 2G_j - B_j > l/2$.

Proof First, suppose G is type C_{J+K}. The roots of a subroot system Ψ of type $C_J \times C_K$ are of the form

$$\left\{\pm(e_i \pm e_j), \pm 2e_j : i, j \in J\right\} \bigcup \left\{\pm(e_i \pm e_j), \pm 2e_j : i, j \in K\right\}$$

where J, K are disjoint subsets of $\{1, \ldots, n\}$ for $n = J + K$. (As in the statement of the lemma, we abuse notation and write J, K both for sets of indices and their cardinalities. Which is meant should be clear from the context). When $K = 1$, we simply mean the long root, $2e_k$, on the single letter $k \in K$.

Our proof will proceed by induction on J, assuming the results holds whenever $K \leq J-1$. The base cases, $C_3 \times C_1$ and $C_2 \times C_2$, will be left for the reader.

There are two cases for the induction step.

Case 1: $C_J \times C_K$ in C_{J+K} with $K < J$.

Fix l and a set of indices $J + K \geq i_1 > i_2 > \cdots > i_l \geq 1$. For $m = 0, \ldots, l$, let

$$K_m = (i_{m+1}, i_m] \bigcap K \quad \text{and} \quad J_m = (i_{m+1}, i_m] \bigcap J$$

where we put $i_0 = J + K$ and $i_{l+1} = 0$. We will also write K_m and J_m for the cardinality of these sets.

As

$$\sum_{m=0}^{l} J_m = J = \frac{J}{K} \sum_{m=0}^{l} K_m,$$

there must be some index $0 \leq s \leq l$ such that $J_s \geq \frac{J}{K} K_s$. Choose any letter $j_0 \in J_s$ and consider the combinatorial criterion problem with the root system Φ' consisting of all the roots in Φ on the letters $\{1, \ldots, n\} \backslash \{j_0\}$ and Ψ' the subroot system $\Psi \bigcap \Phi'$. This is a problem of type $C_{J \backslash \{j_0\}} \times C_K$ in type C_{J+K-1} and so the induction assumption will apply to it. We will explain how to use this fact later in the proof.

But first, we count the total number of good and bad roots arising at steps 1 through l that are not in Φ'. Recall that the roots can be expressed in terms of the base $\{\alpha_1, \ldots, \alpha_n\}$ by the rules

$$\begin{aligned} e_i - e_j &= \alpha_i + \cdots + \alpha_{j-1}, \\ e_i + e_j &= \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n, \\ 2e_i &= 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n, \\ 2e_n &= \alpha_n \end{aligned}$$

when $i < j$. Thus if $s \neq 0$, the good roots not in Φ' are those of the form $e_{j_0} \pm e_k$ for all $k \in K \backslash K_s$ and $e_{j_0} + e_k$ for all $k \in K_s$, for a total of $2K - K_s \equiv g$ good roots. The bad roots are $e_{j_0} \pm e_j$ for all $j \in J \backslash J_s$, $e_{j_0} + e_j$ for $j \in J_s$ and the long root $2e_{j_0}$, for a total of $2J - J_s \equiv b$ bad roots. Note that

$$\frac{J}{K} g - b \geq J_s - \frac{J}{K} K_s \geq 0.$$

The analysis is similar and the conclusion the same if $s = 0$. Moreover, observe that $\frac{J}{K} g - b = 0$ if and only if $J_s = \frac{J}{K} K_s$.

If $j_0 \notin \{i_1, \ldots, i_l\}$, then consider the combinatorial problem on the subroot system Ψ' in Φ', with the indices $i_1 > i_2 > \cdots > i_l$ partitioning the letters $\{1, \ldots, n\} \backslash \{j_0\}$. If we denote by G'_j and B'_j the good and bad roots arising at step j of this problem, then it is clear that

$$\sum_{j=1}^{l} G_j = \sum_{j=1}^{l} G'_j + g \quad \text{and} \quad B_j = \sum_{j=1}^{l} B'_j + b.$$

The induction assumption implies that

$$\sum_{j=1}^{l} \max\left(\frac{J-1}{K}, 2\right) G'_j - B'_j > \frac{l}{2}.$$

Thus

$$\sum_{j=1}^{l} \max\left(\frac{J}{K}, 2\right) G_j - B_j = \max\left(\frac{J}{K}, 2\right) \left(\sum_{j=1}^{l} G'_j + g\right) - \left(\sum_{j=1}^{l} B'_j + b\right) > \frac{l}{2}.$$

If $j_0 = i_s$, but $j_0 > 1$ and $j_0 - 1 \notin \{i_1, \ldots, i_l\}$, then consider the combinatorial problem (Ψ', Φ') with the indices $i_1 > \cdots > i_{s-1} > j_0 - 1 > i_{s+1} > \cdots > i_l$ partitioning the letters $\{1, \ldots, n\} \backslash \{j_0\}$. The reasoning is similar.

If both j_0 and $j_0 - 1$ are in the set $\{i_1, \ldots, i_l\}$ then, instead, we consider the combinatorial problem (Ψ', Φ') but with the $l-1$ indices $i_1 > \cdots > i_{s-1} > i_{s+1} > \cdots > i_l$. We still have $\sum G_j = \sum G'_j + g$ and $\sum B_j = \sum B'_j + b$, but now the sum $\sum G'_j$ (or $\sum B'_j$) is over $l-1$ indices rather than l. Thus the induction argument yields

$$\sum_{j=1}^{l-1} \max\left(\frac{J-1}{K}, 2\right) G'_j - B'_j > \frac{l-1}{2}.$$

However, as both $j_0, j_0 - 1 \in \{i_1, \ldots, i_l\}$, the set J_s is the singleton j_0 and K_s is empty, so $\frac{J}{K} g - b \geq J_s = 1$. Therefore

$$\sum_{j=1}^{l} \max\left(\frac{J}{K}, 2\right) G_j - B_j = \sum_{j=1}^{l-1} \max\left(\frac{J-1}{K}, 2\right) G'_j - B'_j + \frac{J}{K} g - b$$
$$> \frac{l-1}{2} + 1 > \frac{l}{2}. \tag{6.1}$$

If $j_0 = 1 = i_l$ and $l \neq 1$ we argue similarly. Finally, if $j_0 = 1 = i_1$ $(l = 1)$ we do not eliminate j_0, but rather directly count that $G_1 = 2K$ and $B_1 = 2J - 1$, which gives the desired inequality, $\frac{J}{K} G_1 - B_1 > \frac{1}{2}$.

This completes the argument for $C_J \times C_K$ with $K < J$.

Case 2: $C_J \times C_K$ with K, J the same cardinality.

Here we can eliminate one of the letters in K and reduce the problem to one of type $C_J \times C_{K-1}$. To do this, just choose $K_s \geq J_s$ (with the notation as above) and pick any $k_0 \in K_s$. Count the number of good and bad roots involving the letter k_0 and argue in a similar fashion to case 1, noting that $\max(J/K, 2) = 2 = \max(J/(K-1), 2)$.

Now assume G is $SU(J + K)$. There are similarities between this counting problem and that for C_{J+K}, but differences, as well. One obvious difference is that there are no roots of the form $e_i + e_j$. But this does not affect the conclusion of the counting argument as these roots contribute proportionally to both the count of the good and bad roots. More significantly, $SU(J + K)$ has rank $J + K - 1$, so the indices $i_1 > i_2 > \cdots > i_l \geq 1$ must satisfy $i_1 \leq J + K - 1$, and yet we should still view them as defining a partition of $\{1, \ldots, J + K\}$. Indeed, we define K_m and J_m just as before, with $i_0 = J + K$. We consider the two cases, as before, and choose a letter $j_0 \in J_s$ (or $k_0 \in K_s$, in the second case) to eliminate. If the letter to eliminate is n and if $i_1 = n - 1$, then it would not be valid to apply the induction hypothesis to the root system Φ' consisting of the roots on the letters $\{1, \ldots, n - 1\}$ with the indices $n - 1 = i_1 > \cdots > i_l \geq 1$. Instead we discard i_1 and work with only $l - 1$ indices. If G'_j and B'_j are the good and bad roots arising at step j of this problem (beginning the indexing at step 2), then we still have $\sum_{j=1}^{l} G_j = \sum_{j=2}^{l-1} G'_j + g$, $\sum B_j = \sum B'_j + b$. But in this case, J_0 is a singleton and K_0 is empty, hence $g - b \geq 1$. Thus, after applying the induction step we get the same inequality as in (6.1). □

6.2.2 Counting in D_n

It will frequently be convenient to identify the root system D_n with the subset B_n^0 of B_n consisting of all the roots except the short ones, $\{e_i\}_{i=1}^{n}$, i.e.,

$$B_n^0 = \{\pm(e_i \pm e_j) : 1 \leq i < j \leq n\}.$$

There is no harm in making this identification as far as the counting for the L^2 combinatorial criterion is concerned because if Ψ is a subroot system of D_n and $i_1 > \cdots > i_l$ is a partition of $\{1, \ldots, n\}$, then if $i_1 \neq n - 1$ the counting of the good and bad roots in D_n is the same as counting in B_n^0, while if $i_1 = n - 1$, counting in D_n is the same as counting the good and bad roots corresponding to the subroot system $w(\Psi)$ in B_n^0 where the Weyl element, w, is the simple sign change, $w(e_n) = -e_n$ and we take the partition $n = i'_1 > i_2 > \cdots > i_l$.

6.2.3 $SU(J)$ in B_n or D_n

Lemma 6.5 *If Ψ is either the subroot system $SU(J) \times D_K$ in $\Phi = D_{J+K}$ or $SU(J) \times B_K$ in B_{J+K}, with $J \geq K \geq 1$ (or $K \geq 2$ in D_{J+K}), then for any $l = 1, \dots, J+K$,*

$$\sum_{j=1}^{l} G_j - B_j > \frac{l}{2}.$$

Proof Note that by type $SU(J)$ we mean the subroot system

$$\Psi = \{s_i e_i - s_j e_j : i \neq j \in J\}$$

where $s_i = \pm 1$ is a fixed choice of signs.

We proceed, again, by induction on J assuming the results holds whenever $K \leq J - 1$ and for the induction step we consider the two cases, $J > K$ and $J = K$.

First we study the problem of $SU(J) \times D_K$ in D_{J+K} with $J > K$, viewing D_n as B_n^0, as remarked above. Take a partition $i_1 > \cdots > i_l$ of $\{1, \dots, J+K\}$. The counting here is slightly different from the $C_J \times C_K$ problem because we must take into account the choice of signs s_i. So we will define K_m and J_m, as before but, in addition, we will put

$$J_m^+ = \{j \in J_m : s_j = +1\},$$
$$J_m^- = \{j \in J_m : s_j = -1\}.$$

Pick an index $0 \leq s \leq l$ such that $J_s \neq 0$ and without loss of generality assume $J_s^+ \geq J_s^-$. Choose $j_0 \in J_s^+$. The strategy will be to eliminate this letter and appeal to induction.

As before, we count the good and bad roots associated with j_0: If $s \neq 0$, then $e_{j_0} \pm e_k$ for all $k \notin K_s$ and $e_{j_0} + e_k$ for $k \in K_s$ are good roots. As well, for each $j \notin J_s$, one of $e_{j_0} + e_j$ or $e_{j_0} - e_j$ is good and the other is bad. Finally, $e_{j_0} + s_j e_j \notin SU(J)$ and thus $e_{j_0} + e_j$ is a good root for each $j \in J_s^+ \setminus \{j_0\}$. This gives a total of

$$g = J - J_s^- + 2K - K_s - 1$$

good roots associated with the letter j_0. Similarly, there are

$$b = J - J_s^+$$

bad roots associated with j_0. Thus $g - b \geq K - 1 \geq 1$.

We will pick the index $s = 0$ only if $J_i = 0$ for all $i \neq 0$. This ensures $K_0 \neq K$ and hence if $s = 0$, then $g = 2(K - K_0)$ and $b = 0$, so $g - b \geq 2$.

If $j_0 = 1 = i_1$ a direct count gives $G_1 - B_1 > 1/2$. Otherwise, if G_j' and B_j' denote the remaining good and bad roots after eliminating the letter j_0, the same logic as in the $C_J \times C_K$ problem shows that

$$\sum_{j=1}^{l} G_j' - B_j' > \frac{l-1}{2}$$

so that $\sum_{j=1}^{l} G_j - B_j > \frac{l}{2}$, as we needed to show.

The argument for type $SU(J) \times B_K$ in B_{J+K} with $J > K$ is similar. We get one extra good root from the short root e_{j_0} and no additional bad roots. Thus, even if $K = 1$ we have $g - b \geq 1$, which shows the induction step holds.

When $J = K$ we pick a letter from K to eliminate and reduce the problem to $SU(J) \times D_{K-1}$ in D_{J+K-1} (resp., $SU(J) \times B_{K-1}$ in B_{J+K-1}). We pick $k_0 \in K_s$ where $K_s \geq J_s$, with the strict inequality holding if possible. In type B_{J+K}, the short root e_{k_0} is a bad root when $s \neq 0$, so that $g - b = K_s - J_s$, while in type D_{J+K}, $g - b \geq K_s - J_s + 1$. In either case, $g - b = K_0 - J_0$ if $s = 0$. Thus it is possible to have $g - b = 0$, but only in type B_{J+K} and only if all $J_s = K_s$. But in this situation there is no loss of an index in passing to the induction assumption, hence the proof will again follow. □

Lemma 6.6 (i) *If either Ψ is a subroot system of type $SU(n)$ in B_n or Ψ is type $SU(n-1)$ in D_n, then*

$$\sum_{j=1}^{l} G_j - B_j > \frac{l}{2}.$$

(ii) *If Ψ is a subroot system of type $SU(n)$ in D_n and $n \geq 5$, then*

$$\sum_{j=1}^{l} 2G_j - B_j > \frac{l}{2}.$$

Proof (i) $SU(n)$ in B_n. Rather than an induction argument it is simple enough to do a direct count of good and bad roots. We suppose

$$\Psi = \{s_i e_i - s_j e_j : 1 \leq i < j \leq n\}$$

and given a partition $i_1 > \cdots > i_l$ we let

$$S_m^+ = \{j \in (i_{m+1}, i_m] : s_j = +1\}$$

and

$$S_m^- = \{j \in (i_{m+1}, i_m] : s_j = -1\}.$$

Of course, bad roots can only be of the form $e_j \pm e_k$, $j < k$, and we split these into two groups.
Group 1: $j, k \in (i_{m+1}, i_m]$ for some $1 \leq m \leq l$. From these roots the bads ones are of the form $s_j e_j - s_k e_k$ where $s_j s_k = -1$ and the total number of these is $\sum_{i=1}^{l} S_m^+ S_m^-$.
Group 2: These are the roots for which there exists t with $j \leq i_t < k$. But then one of each pair $e_j \pm e_k$ will be a good root and the other bad.

Consequently, to prove $\sum_{i=1}^{l} G_i - B_i > l/2$ it is enough to prove that the number of remaining good roots exceeds the number of group 1 bad roots by at least $l/2$. These additional good roots are those of the form $s_j e_j + s_k e_k$ with $j, k \in (i_{m+1}, i_m]$ for some $m \geq 1$ and satisfying $s_j s_k = +1$, as well as the short roots e_j with $j \leq i_1$, for a total of

$$\sum_{i=1}^{l} \binom{S_i^+}{2} + \binom{S_i^-}{2} + S_i^+ + S_i^-.$$

Thus the number of additional good roots less the group 1 bad roots is equal to $\frac{1}{2} \sum_{i=1}^{l} (S_i^+ - S_i^-)^2 + (S_i^+ + S_i^-)$. Since $S_i^+ + S_i^- \geq 1$ it is easy to see the sum above is greater than $l/2$, as desired.

$SU(n-1)$ in D_n. This problem is similar to type $SU(n)$ in B_n with the good roots involving the free letter effectively replacing the good short roots in the B_n problem. We leave the details for the reader.

(ii) Type $SU(n)$ in D_n has the same good and bad roots as $SU(n)$ in B_n, except it is missing the good short roots. Thus

$$\sum_{j=1}^{l} G_j - B_j \geq \sum_{i=1}^{l} \frac{(S_i^+ - S_i^-)^2 - (S_i^+ + S_i^-)}{2}$$

and obviously $\sum_{j=1}^{l} 2G_j - B_j \geq \sum_{j=1}^{l} G_j - B_j + G_1$. Similar analysis to that above establishes

$$G_1 = \sum_{i=1}^{l} (S_i^+ + S_i^-)(S_0^+ + S_0^-) + \binom{\sum_{i=1}^{l} S_i^+}{2} + \binom{\sum_{i=1}^{l} S_i^-}{2}$$

and a routine calculation shows that for $n \geq 5$ this gives the desired conclusion. □

6.3 $SU(J) \times SU(K) \times SU(L)$ in $SU(J + K + L)$

Lemma 6.7 *Suppose Ψ is a subroot system of type $SU(J) \times SU(K) \times SU(L)$ in $SU(J + K + L)$ with $J \geq K \geq L$ and $J \leq K + L$. Then*

$$\sum_{j=1}^{l} G_j - B_j > \frac{l}{2}.$$

Proof The idea here is to proceed by induction on $J + K + L$. We will allow $J, K, L = 1$ with the understanding that type $SU(1)$ means one free letter. The result is trivially true for the base case, $SU(1) \times SU(1) \times SU(1)$.

We proceed by eliminating a suitable letter j_0 from J. The reduced problem on the sets $J \backslash \{j_0\}$, K and L (upon reordering if necessary) is still of the same structure, so the induction hypothesis applies to it. To pick the suitable letter, we define J_m and K_m as before, and L_m similarly. Pick an index s so that J_s is not empty and $J_s - (K_s + L_s) \geq J - (K + L)$. Such an index must exist because otherwise

$$\begin{aligned} J - (K + L) &= \sum_{m=1}^{l} J_m - (K_m + L_m) \\ &= \sum_{m \text{ with } J_m \neq \phi} J_m - (K_m + L_m) + \sum_{m \text{ with } J_m = \phi} -(K_m + L_m) \\ &< c(J - (K + L)) \end{aligned}$$

where c is number of m such that J_m is not empty. As $J - (K + L) \leq 0$ this is a contradiction.

Counting good and bad roots associated with j_0, one sees that $g - b = J_s - (K_s + L_s) - J + K + L \geq 0$ with equality avoidable if $J_s \cup K_s \cup L_s$ is a singleton. Thus an appeal to the induction assumption gives the desired result whether there is a loss of an index in passing to the reduced problem or not. □

7 Transference theorem

To obtain L^2 results for orbital measures on the Lie algebras we will prove a transference theorem.

Given any compact Lie group G, we let Ψ be the wrapping map of G introduced by Dooley and Wildberger in [3], our main reference for this section. This is the map $\Psi : M(\mathfrak{g}) \to M(G)$ given by

$$\int_G f d\Psi(\mu) = \int_{\mathfrak{g}} j(X)\widetilde{f}(X)d\mu(X) \quad \text{for } f \text{ continuous}$$

where $\widetilde{f} = f \circ \exp$ and j is an analytic square root of the determinant of the exponential map such that $j(0) = 1$.

It is known that if μ, ν are G-invariant measures on $\mathfrak{g}$, i.e., $\mu(Ad(g)E) = \mu(E)$ for all $g \in G$ and Borel sets $E \subseteq \mathfrak{g}$, then

$$\Psi(\mu * \nu) = \Psi(\mu) * \Psi(\nu)$$

where the convolution on the left is in $\mathfrak{g}$ and the convolution on the right is in G.

Fix an open set $U \subseteq \mathfrak{g}$ containing 0, having compact closure and which has the property that

$$\exp : U \to \exp(U)$$

is a diffeomorphism on $closure(U)$. We state a lemma whose proof is similar to Theorem 1 in [3].

Lemma 7.1 *Let $\phi \in L^1(\mathfrak{g})$ be G-invariant and assume support $\phi \subseteq U$. Then $\Psi(\phi j)$ is a G-invariant function whose value on T is given by*

$$\Psi(\phi j)(\exp H) = \begin{cases} \phi(H) & \text{for } H \in U \bigcap \mathfrak{t} \\ 0 & \text{for } H \in \mathfrak{t} \setminus U \end{cases}.$$

Theorem 7.2 *Let $1 \le p < \infty$. There are constants $A, B > 0$ such that if f is a G-invariant function supported on $U \subseteq \mathfrak{g}$, then*

$$A \|f\|_p \le \|\Psi(f)\|_p \le B \|f\|_p$$

and

$$\|\Psi(f)\|_2 = \|f\|_2 .$$

Proof Since $\Psi(f)$ is concentrated on $\exp U$, a change of variables argument and the Weyl integration formula gives

$$\begin{aligned} \|\Psi(f)\|_p^p &= \int |\Psi(f)(g)|^p \, dg \\ &= \int_U |\Psi(f)(\exp H)|^p \, |j(H)|^2 \, dH \\ &= \int_{\mathfrak{t}^+ \bigcap U} |\Psi(f)(H)|^p \, |j(H)|^2 \left(\prod_{\alpha \in \Phi^+} \alpha(H) \right)^2 dH. \end{aligned}$$

Applying the lemma to $\phi = f/j$, we see that

$$\|\Psi(f)\|_p^p = \int\limits_{\mathfrak{t}^+\bigcap U} \left|\frac{f}{j}(H)\right|^p |j(H)|^2 \left(\prod_\alpha \alpha(H)\right)^2 dH$$
$$= \int\limits_{\mathfrak{t}^+\bigcap U} |f(H)|^p \, |j(H)|^{2-p} \left(\prod_\alpha \alpha(H)\right)^2 dH.$$

Similarly,

$$\|f\|_p^p = \int\limits_{\mathfrak{t}^+\bigcap U} |f(H)|^p \left(\prod_\alpha \alpha(H)\right)^2 dH.$$

Since exp is a diffeomorphism on $closure(U)$, there are constants $a, b > 0$ such that

$$a \leq |j(H)| \leq b \text{ on } U.$$

Thus

$$a^{(2-p)/p} \, \|f\|_p \leq \|\Psi(f)\|_p \leq b^{(2-p)/p} \, \|f\|_p$$

and

$$\|\Psi(f)\|_2 = \|f\|_2 .$$

□

Corollary 7.3 *Let $H \in U$ be such that $O_H + \cdots + O_H \subseteq U$. Assume $h = \exp H$. Then for $1 \leq p < \infty$, $\mu_H^k \in L^p(\mathfrak{g})$ if and only if $\mu_h^k \in L^p(G)$.*

Proof This follows from the theorem and the boundedness of j on U, together with the observation that $\Psi(\mu_H) = j(H)\mu_h$. □

8 Dichotomy result

8.1 Types

We recall that by the type of an element $X \in \mathfrak{t}$ or $x \in T$ we mean the Lie type of its annihilating root system. We define the *type* of an orbit (or conjugacy class) to be the type of any torus element generating the orbit (or conjugacy class). This is well defined as all such elements are Weyl conjugate and hence have the same type. The zero element in $\mathfrak{g}$ and the central elements in G are characterized as being annihilated by all the roots.

Before stating and proving the main theorem of the paper, we need to identify the types of subroot systems that are the sets of annihilating roots of elements of $\mathfrak{t}$. Not all subroot systems can arise this way since any set of annihilating roots is closed under all linear combinations (that are still roots). One example of a subroot system that is not the set of annihilating roots is $C_J \times C_K$ in C_{J+K}. This set is not closed under linear combinations as every root in C_{J+K} is a linear combination of roots in $C_J \times C_K$.

To determine which subroot systems are sets of annihilating roots consider a typical non-zero element $X \in \mathfrak{t}$. For $\mathfrak{g}$ of type A_{n-1}, B_n, C_n or D_n it is convenient to identify $\mathfrak{t}$ with

$\mathbb{R}^n$ by the identification $X \sim (\alpha_1, \ldots, \alpha_n)$ if $X = \sum_{l=1}^n i\alpha_l E_{ll}$ in type A_{n-1} or C_n, or $X = \sum_{l=1}^n \alpha_l (E_{2l-1,2l} - E_{2l,2l-1})$ in type B_n or D_n.

If $\mathfrak{g}$ is type A_{n-1}, then up to a Weyl conjugate

$$X \sim (a_1 \ (J_1 \text{ times}), a_2 \ (J_2 \text{ times}), \ldots, a_t \ (J_t \text{ times}), b_1, \ldots, b_r)$$

where a_i, b_j are distinct and $J_l \geq 2$. The set of annihilating roots is Lie type $A_{J_1-1} \times \cdots \times A_{J_t-1}$, but we prefer to say that X is

$$\text{type } SU(J_1) \times \cdots \times SU(J_t) + r \text{ free}$$

(and that $\mathfrak{g}$ is type $SU(n)$). We simply say X is type $SU(n-r)$ if $t = 1$. If $t = 0$, X is regular.

For types B_n, C_n or D_n we can suppose

$$X \sim (0 \ (J \text{ times}), A_1, \ldots, A_t, b_1, \ldots, b_r)$$

where $A_l = (a_l \ (K_l^+ \text{ times}), -a_l \ (K_l^- \text{ times}))$, with $K_l^+ + K_l^- \geq 2$ and $\pm a_l, \pm b_j$ real, non-zero and distinct. If $\mathfrak{g}$ is type B_n the annihilating roots of X are

$$\pm\{e_i, e_i \pm e_j : i, j \in J\} \bigcup \bigcup_{l=1}^{t} X_l$$

where

$$X_l = \{s_i e_i - s_j e_j : i, j \in K_l^+ \cup K_l^- \text{ with } s_i = 1 \text{ if } e_i \in K_l^+, s_i = -1 \text{ if } e_i \in K_l^-\}$$

and we write $J, K_1, \ldots, K_t$ to denote disjoint sets of cardinalities $J, K_1, \ldots, K_t$, respectively, whose union is $\{1, \ldots, n-r\}$. The annihilating roots of X are of

$$\text{type } B_J \times SU(K_1) \times \cdots \times SU(K_t) + r \text{ free},$$

or type B_{n-r} if $t = 0$. Note that we write B_1 for the single short root $\{e_J\}$.

The annihilating subroot systems are similar in C_n and D_n, but in the latter case, if $J = 1$ the root system is type $SU(K_1) \times \cdots \times SU(K_t) + r + 1$ free and D_2 and D_3 are understood as meaning the sets $\{\pm(e_i \pm e_j) : i, j \in J\}$ where the cardinality of J is two or three respectively.

The annihilating roots of a Weyl conjugate of X will be of the same type, although with possibly different disjoint sets of letters and signs changed, according to the action of the Weyl element.

In this terminology, Theorem 4.1 and Corollary 4.5 could be restated as

Corollary 8.1 *Suppose X is type $SU(n-r)$ in $su(n)$, type B_{n-r} in B_n, type C_{n-r} in C_n or type D_{n-r} in D_n. If $k \leq \left[\frac{n-1}{r}\right]$, then*

$$\dim\left(\sum_{i=1}^{k} T_{g_i^{-1} X g_i}(O_X)\right) < \dim \mathfrak{g}$$

for all $g_1, \ldots, g_k \in G$, the measure of $(k)\ O_X$ is zero and μ_X^k is singular.

Proof This was actually proved for specific choices of $H \in \mathfrak{t}$ of these types, however any other X of the same type is Weyl conjugate to one of those choices of H and hence generates the same orbit. □

8.2 Main theorem

Theorem 8.2 *Suppose $X \neq 0$ belongs to the torus of any of the classical compact simple Lie algebras. Then either $\mu_X^k \in L^2(\mathfrak{g}) \bigcap L^1(\mathfrak{g})$ or μ_X^k is singular to Lebesgue measure on $\mathfrak{g}$. Moreover, if we let*

$$k(X) = \min\{k : \mu_X^k \in L^2\}$$

then $m((k)O_X) > 0$ if and only if $k \geq k(X)$.

If X is a regular element, then $k(X) = 2$. For all other X the value of $k(X)$ depends only on the type of X and is specified below.

Type of $\mathfrak{g}$	*Type of X*	*Value of* $k(X)$
$SU(n)$	$SU(n-r),\ n-r \geq 2$	$\left[\frac{n-1}{r}\right]+1$
	$SU(k_1) \times \cdots \times SU(k_t) + r$ *free with* $k_1 = \max\{k_i\}$ *and* $t \geq 2$	$\begin{cases} \left[\frac{n-1}{n-k_1}\right]+1 & \text{if } k_1 > \sum_{i=2}^t k_i + r \\ 3 & \text{if } t=2,\ r=0 \text{ and } k_1 = k_2 \\ 2 & \text{else} \end{cases}$
B_n	B_{n-r}	$\left[\frac{n-1}{r}\right]+1$
	$B_J \times SU(k_1) \times \cdots \times SU(k_t) + r$ *free with* $t \geq 1$	$\begin{cases} \left[\frac{n-1}{n-J}\right]+1 & \text{if } J > \sum_{i=1}^t k_i + r \\ 2 & \text{if } J \leq \sum_{i=1}^t k_i + r \end{cases}$
	$SU(k_1) \times \cdots \times SU(k_t) + r$ *free with* $t \geq 1$	2
D_n	$D_{n-r},\ n-r \geq 2$	$\left[\frac{n-1}{r}\right]+1$
	$D_J \times SU(k_1) \times \cdots \times SU(k_t) + r$ *free with* $t \geq 1$	$\begin{cases} \left[\frac{n-1}{n-J}\right]+1 & \text{if } J > \sum_{i=1}^t k_i + r \\ 2 & \text{if } J \leq \sum_{i=1}^t k_i + r \end{cases}$
	$SU(n)$	$\begin{cases} 3 & \text{if } n \geq 5 \\ 4 & \text{if } n = 4 \end{cases}$
	$SU(k_1) \times \cdots \times SU(k_t) + r$ *free with* $t \geq 2$ *or* $t = 1$ *and* $r \geq 1$	2

The results for Lie type C_n are the same as for B_n with the obvious changes in notation.

Proof We will begin by proving that $\mu_X^{k(X)} \in L^2(\mathfrak{g})$. As $(k)O_X$ is compact, this also implies $\mu_X^k \in L^1(\mathfrak{g})$. Our approach will be to solve the L^2 problem first for certain orbital measures on the group G and then use our transference theorem to derive the appropriate result for $L^2(\mathfrak{g})$.

To begin, fix a 0-neighbourhood $U \subseteq \mathfrak{g}$, with compact closure, on which the exponential map is a diffeomorphism. Given non-zero $X \in \mathfrak{t}$, choose $\lambda_0 > 0$ such that

$$(k(X))O_{\lambda X} = \lambda k(X) O_X \subseteq U \quad \text{for all } \lambda \leq \lambda_0.$$

Since the Fourier transform of μ_X and $\mu_{\lambda X}$ are dilates, it follows that μ_X^k belongs to $L^2(\mathfrak{g})$ for the same choices of k as $\mu_{\lambda X}^k$. This shows there is no loss of generality in assuming

$(k(X))O_X \subseteq U$ and hence by the transference argument, Corollary 7.3, it suffices to prove that $\mu_{\exp X}^{k(X)} \in L^2(G)$.

The annihilating roots for $\exp X$ will always contain those of X, but strict containment is possible. But for almost all non-zero scalars λ, the matrices λX, X and $\exp \lambda X$ have precisely the same annihilating roots. As explained above, there is no harm in replacing X by such a choice of λX (for λ suitably small), thus in order to prove that $\mu_X^{k(X)} \in L^2(\mathfrak{g})$ for all $X \neq 0$ it will be enough to prove that $\mu_{\exp X}^{k(X)} \in L^2(G)$ whenever X and $\exp X$ have the same type.

In some special cases the L^2 result for orbital measures on G is already known. For example, in [10] it was shown that for a regular element $x \in G$, $\mu_x^2 \in L^2(G)$. Hence $k(X) = 2$ for X regular. This can also easily be seen from our combinatorial approach since none of the roots are "bad". In [8] it was shown that when $G = SU(n)$ and x is type $SU(n-r)$, then $\mu_x^k \in L^2(G)$ for $k = \left[\frac{n-1}{r}\right] + 1$ and this is sharp. The elements X of type B_{n-1} in B_n, C_{n-1} in C_n and D_{n-1} in D_n generate orbits of minimum dimension. These were the types of orbits studied in [7] and the proof that $\mu_{\exp X}^n \in L^2(G)$ can be found there. The results of both [7,8] will be recovered as special cases of our theorem.

We will give the details of the proof of the L^2 result for the orbital measures in groups of type $SU(n)$ or D_n. Our approach will be to check that the value of $k(X)$ is at least as great as the value of $k_0(\exp X)$ specified in the L^2 combinatorial criterion, Theorem 6.1, so that $\mu_{\exp X}^{k(X)}$ will belong to $L^2(G)$.

The proof for G of type B_n is similar to D_n and will be left for the reader. The results for C_n are identical to B_n since their root systems have the same combinatorial structure.

Case: G of type $SU(n)$. We consider the possible types of X.

(i) X of Type $SU(n-r)$ with $n-r > r$: The subroot system of type $SU(n-r)$ is contained in one of type $SU(n-r) \times SU(r)$ and it clearly suffices to check the L^2 combinatorial criterion for the larger system. (This is an observation we will use repeatedly). Lemma 6.3 implies (with the notation of that lemma) that for all choices of indices,

$$k(i_1, \ldots, i_l, SU(n-r) \times SU(r)) \leq \max\left(\frac{n}{r}, 3\right),$$

thus we may take $k_0(\exp X) = \max\left(\frac{n}{r}, 3\right)$ in Theorem 6.1. Hence $\mu_{\exp X}^k \in L^2(G)$ for the least integer $k \geq n/r$, namely, $k = \left[\frac{n-1}{r}\right] + 1$.

(ii) Type $SU(n-r)$ with $n-r \leq r$: In this case $SU(n-r)$ is contained in a subroot system of type $SU(n-r) \times SU(r-1) \times SU(1)$ and Lemma 6.7 implies we may take $k_0(X) = 2$.

(iii) Type $SU(n/2) \times SU(n/2)$ follows directly from Lemma 6.3.

(iv) Type $SU(k_1) \times \cdots \times SU(k_t) + r$ free with $k_1 > \sum_{i=2}^t k_i + r$ is contained in one of type $SU(k_1) \times SU(\sum_{i=2}^t k_i + r)$. Again the result is immediate from Lemma 6.3.

(v) Type $SU(J) \times SU(K) + r$ free with $J \leq K + r$ and $r \geq 1$: If $r \leq J$, then this subroot system is contained in one of type $SU(J) \times SU(K) + SU(r)$ and Lemma 6.7 gives the result $k_0(X) = 2$. Otherwise, by distributing the "free" letters suitably between the sets J, K we can assume the subroot system is contained in one of type $SU(J') \times SU(K') \times SU(r')$ where $J \subseteq J'$, $K \subseteq K'$, $1 \leq r' \leq r$ and $J' \leq K' + r'$. Again, we call upon Lemma 6.7.

(vi) Type $SU(k_1) \times \cdots \times SU(k_t) + r$ free with $k_1 \leq \sum_{i=2}^t k_i + r$ and $t \geq 3$: Again, by distributing the free letters across the sets $k_1, \ldots, k_t$, suitably, we can assume, without loss of generality, that $r = 0$. If $t \geq 4$, we note that $SU(k_1) \times \cdots \times SU(k_t)$ is contained in one of type $SU(k_1) \times \cdots \times SU(k_{t-2}) \times SU(k_{t-1} + k_t)$. Reorder $k_1, \ldots, k_{t-2}, k_{t-1} + k_t$ if necessary. Since $k_{t-1} + k_t \leq k_1 + k_2$, the condition that the cardinality of the largest set is at most the sum of the cardinalities of the remainder continues to be satisfied. Thus without loss of generality $t = 3$ and Lemma 6.7 applies.

Case: G of type D_n. Again, consider the possible types of X.

(i) X of Type D_{n-r} with $n - r > r$: Again we observe it suffices to check the L^2 combinatorial criterion for the larger system, $D_{n-r} \times D_r$. But compare the appropriate counting of good and bad roots in $D_{n-r} \times D_r$ with counting in $C_{n-r} \times C_r$. Since the long roots of C_n can only contribute to the count as bad roots, it follows from Lemma 6.3 that for all choices of indices,

$$k(i_1, \ldots, i_l, D_{n-r} \times D_r) \leq k(i_1, \ldots, i_l, C_{n-r} \times C_r) \leq \max\left(\frac{n}{r}, 3\right).$$

Hence $\mu^k_{\exp X} \in L^2$ for $k = \left[\frac{n-1}{r}\right] + 1$.

(ii) Type D_{n-r} with $n - r \leq r$: A subroot system of this type is contained in one of type $SU(r) \times D_{n-r}$ and by Lemma 6.5 it follows that $\mu^2_{\exp X} \in L^2$.

(iii) Type $D_J \times SU(k_1) \times \cdots \times SU(k_t) + r$ free is contained in both $SU(K) \times D_J$ and $D_J \times D_K$ for $K = \sum_{i=1}^t k_i + r$. If $K \geq J$, then the first containment and Lemma 6.5 implies $\mu^2_X \in L^2$. Otherwise, as in the explanation of (i), $k(X) = \left[\frac{n-1}{K}\right] + 1$ suffices.

(iv) Type $SU(k_1) \times \cdots \times SU(k_t) + r$ free for $r \geq 1$ is contained in one of type $SU(n-1)$ and thus Lemma 6.6(i) implies $\mu^2_{\exp X} \in L^2$. If $r = 0$ but $t \geq 2$, then the subroot system is contained in $SU(\sum_{i=1}^{t-1} k_i) \times D_{k_t}$ where $k_t = \min(k_1, \cdots, k_t)$. Lemma 6.5 ensures $\mu^2_{\exp X} \in L^2$.

(v) Type $SU(n)$ in D_n is dealt with in Lemma 6.6(ii) for $n \geq 5$. Lastly, we remark that the fact that $\mu^4_{\exp X} \in L^2(G)$ for X of type $SU(4)$ in D_4 was shown in [10].

As $\mu_X^{k(X)}$ is a non-zero, absolutely continuous measure supported on $(k(X))O_X$ the measure of this set is positive.

We now turn to the "singularity" side of the problem. Of course, the singularity of $\mu_X^{k(X)-1}$ will follow once we establish that $m((k(X) - 1)O_X) = 0$. This is obvious if $k(X) = 2$ since any orbit is a submanifold of proper dimension and hence has measure zero. The desired result is proven in Corollary 8.1 for the orbits of types $SU(n-r)$ in $SU(n)$, B_{n-r} in B_n, C_{n-r} in C_n and D_{n-r} in D_n and for the orbits of type $SU(n/2) \times SU(n/2)$ in $SU(n)$ in Proposition 5.1.

For most of the other types we rely upon the following observation made in Corollary 2.2: If the set of annihilating roots of X contains those of Y, then for all $g \in G$, $T_{g^{-1}Xg}(O_X) \subseteq T_{g^{-1}Yg}(O_Y)$. Thus if

$$\dim\left(\sum_{i=1}^{k} T_{g_i^{-1}Yg_i}(O_Y)\right) < \dim \mathfrak{g}$$

for all choices $g_1, \ldots, g_k \in G$, then the same must be true for the dimension of $\sum_{i=1}^k T_{g_i^{-1}Xg_i}(O_X)$ and therefore $(k)O_X$ will have measure zero as in the reasoning of Corollary 4.5.

For example, if X is type $SU(J) \times SU(k_1) \times \cdots \times SU(k_t) + r$ free in $SU(n)$ with $J > \sum_{i=1}^t k_i + r$, then the annihilating roots of X contain the annihilating roots of some Y of type $SU(J)$ in $SU(n)$. By Corollary 8.1

$$\dim\left(\sum_{i=1}^{k} T_{g_i^{-1}Yg_i}(O_Y)\right) < \dim \mathfrak{g}$$

if $k \leq \left[\frac{n-1}{n-J}\right]$ and thus $m((k)O_X) = 0$.

The same reasoning applies to type B_J (or C_J, D_J) $\times SU(k_1) \times \cdots \times SU(k_t) + r$ free in B_n (or C_n, D_n).

This leaves only type $SU(n)$ in D_n. For $n \geq 5$, a simple dimension argument suffices: Since $\dim O_X = n(n-1)$ and $\dim D_n = 2n^2 - n$,

$$\dim\left(T_X(O_X) + T_{g^{-1}Xg}(O_X)\right) \leq 2n(n-1) < \dim D_n$$

for all $g \in G$ and therefore $m((2)O_X) = 0$.

Lastly, we need to prove that if X is type $SU(4)$ in D_4, then $m((3)O_X) = 0$. There is an automorphism π of the root system of D_4 that maps the set of annihilating roots of X onto a subroot system of type D_3. This automorphism extends to an isomorphism on the torus which maps X to the element $\pi(X)$ whose set of annihilating roots is the type D_3 subroot system. Furthermore, π induces a Lie algebra isomorphism, also called π. One can check that $\pi(O_X) = O_{\pi(X)}$ and that

$$\pi(T_{g^{-1}Xg}(O_X)) = T_{g'^{-1}\pi(X)g'}(O_{\pi(X)})$$

where if $g = \exp H$, then $g' = \exp \pi(H)$. Theorem 4.1 shows that if $\pi(X)$ is type D_3, then

$$\dim\left(\sum_{i=1}^{3} T_{g_i^{-1}\pi(X)g_i}(O_{\pi(X)})\right) < \dim D_4 \quad \text{for all } g_1, g_2, g_3 \in G.$$

As $\sum_{i=1}^{3} T_{g_i^{-1}Xg_i}(O_X)$ has the same dimension we see that $m((3)O_X) = 0$.

This completes the proof of the theorem. □

Corollary 8.3 *Suppose $x \in G$ and there is some $X \in \mathfrak{g}$ of the same type as x and satisfying $\exp X = x$. Then $\mu_x^{k(X)} \in L^2(G)$, $\mu_x^{k(X)-1}$ is singular to the Haar measure on G, and $m_G(C_x^k) > 0$ if and only if $k \geq k(X)$.*

Proof The L^2 results were already established in the proof of the theorem.

It is known that $C_x^k \subseteq \exp((k)O_X)$ [3]. Thus $m((k(X)-1)O_X) = 0$ implies $m_G(C_x^{k(X)-1}) = 0$ and therefore $\mu_x^{k(X)-1}$ is singular to m_G. The measure of $C_x^{k(X)}$ is positive since the set supports a non-zero, absolutely continuous measure. □

The dimensions of O_X and $C_{\exp X}$ coincide precisely when X and $\exp X$ are the same type. This is always the case when G is type A_n, consequently we also have the dichotomy holding on these groups.

Corollary 8.4 *All orbital measures on $SU(n)$ satisfy the L^2-singular dichotomy.*

Remark 8.5 In Lie types other than A_n it is not always true that X and $\exp X$ are the same type, although always $\dim O_X \geq \dim C_{\exp X}$. One example where the inequality is strict is when G is of type B_n. The torus element $x \sim (\pi, \pi, \ldots, \pi)$ in G is of type D_n and generates a conjugacy class of dimension $2n$, but any preimage orbit has dimension at least $n(n+1)/2$. Indeed, the orbit of minimal dimension in the algebra has dimension $4n-2$ and there is no orbit in the Lie algebra B_n of type D_n.

It was shown in [10] that $\mu_x^{2n} \in L^2(G)$ and this is sharp. In [7] we verified that $m((n)O_X) > 0$ for all non-trivial orbits in B_n, but we cannot deduce that C_x^n has positive Haar measure from this fact since the exponential map is not a local diffeomorphism near x and need not preserve positive measure. Thus new ideas will be needed to resolve the L^2-singular dichotomy question for such orbital measures on the Lie groups.

The main result of [7] is a special case of our theorem.

Corollary 8.6 *Let $\mathfrak{g}$ be one of the classical simple Lie algebras. The Lebesgue measure of $(k)O_X$ is non-zero for all $X \neq 0$ if and only if $k \geq rank\,\mathfrak{g} + 1$ if $\mathfrak{g}$ is type A_n and $rank\,\mathfrak{g}$ otherwise.*

Proof One can see from the chart that $\mu_X^k \in L^2$ for all non-trivial X for the choice of k specified. The elements of type A_{n-1}, B_{n-1}, C_{n-1} or D_{n-1} in the Lie algebras of type A_n, B_n, C_n or D_n, respectively, generate the orbits of minimal dimension and for these orbits the choice of k is sharp. □

Ricci and Stein [18] proved that if $\mu_X^k \in L^1$, then $\mu_X^k \in L^p$ for some $p > 1$. In fact, our methods show this is true for some $p > 2$.

Corollary 8.7 *If $X \neq 0$, then $\mu_X^{k(X)} \in L^p(\mathfrak{g})$ for some $p = p(X) > 2$.*

Proof As explained in the proof of the theorem it is enough to check that $\mu_x^{k(X)} \in L^p(G)$ for $x \in G$ with $\exp X = x$ and of the same type as X.

We note that we do not need k to be an integer in the combinatorial criterion. Indeed, the proof of Theorem 6.1 shows that provided

$$\sum_{j=1}^{l}(k-1)\left|G_j\right| - \left|B_j\right| > l/2 \tag{8.1}$$

for all indices satisfying $rank\,G \geq i_1 > \cdots > i_l \geq 1$ and all subroot systems of a given type, we can conclude that the fractional Fourier transform, $\widehat{\mu^k_{\exp X}} \in l^2$. Since the strict inequality is satisfied by $k(X)$, it must be the case that for some $k < k(X)$ (non-integer valued) inequality (8.1) holds for all of the finitely many choices of indices and subroot systems of the given type. (In fact, we can take $k = k(X) - \varepsilon$ where $\varepsilon < 1/\dim O_X$).

Thus $\widehat{\mu^{k(X)}_{\exp X}} \in l^q$ for $q = 2k/k(X) < 2$ and by the Hausdorff-Young inequality $\mu^{k(X)}_{\exp X} \in L^p(G)$ where $p > 2$ is the conjugate index to q. □

Remark 8.8 It is not the case, however, that $\mu_X^{k(X)} \in L^p$ for all p. Our reasoning above implies that $\mu_X^2 \in L^{3-\varepsilon}(su(2))$ for all $\varepsilon > 0$, but it follows from [17] that $\mu_X^2 \notin L^3$.

Ricci and Travaglini [20] proved that if X is a regular element (in the Lie group or algebra) then $\mu_X * L^p \subseteq L^2$ if and only if $p \geq 1 + rank\,G/(2\dim G - rank\,G)$. Since $\mu_X^{k(X)} * L^1 \subseteq L^2$ for any $X \neq 0$, an application of Stein's interpolation theorem (c.f. [9]) yields the following extension of their result.

Corollary 8.9 *For any $X \neq 0$, $\mu_X * L^p \subseteq L^2$ if $p > 2 - 2/(k(X) + 1)$.*

Remark 8.10 The dichotomy conjecture is open for the exceptional Lie groups and algebras. There are sufficient conditions known for the L^2 property [11], but it is unknown if these are sharp. The more complicated combinatorial structure of the root systems of the exceptional groups (especially, E_6, E_7 and E_8) make these algebras much more difficult to work with.

References

1. Dooley, A.: Norms of characters and lacunarity for compact Lie groups. J. Funct. Anal. **32**, 254–267 (1979)
2. Dooley, A., Repka, J., Wildberger, N.: Sums of adjoint orbits. Linear Multilinear Algebra **36**, 79–101 (1993)

3. Dooley, A., Wildberger, N.: Harmonic analysis and the global exponential map for compact Lie groups. Funct. Anal. Appl. **27**, 21–27 (1993)
4. Dunkl, C.: Operators and harmonic analysis on the sphere. Trans. Am. Math. Soc. **125**, 250–263 (1966)
5. Giulini, S., Soardi, P., Travaglini, G.: Norms of characters and Fourier series on compact Lie groups. J. Funct. Anal. **46**, 88–101 (1982)
6. Gupta, S., Hare, K.: Singularity of orbits in $SU(n)$. Israel J. Math. **130**, 93–107 (2002)
7. Gupta, S., Hare, K.: Singularity of orbits in classical Lie algebras. Geom. Funct. Anal. **13**, 815–844 (2003)
8. Gupta, S., Hare, K.: The dichotomy problem for orbital measures of $SU(n)$. Monatsch. Math. **146**, 227–238 (2005)
9. Hare, K.: The size of characters of compact Lie groups. Stud. Math. **129**, 1–18 (1998)
10. Hare, K., Wilson, D., Yee, W.L.: Pointwise estimates of the size of characters of compact Lie groups. J. Aust. Math. Soc. (Series A) **69**, 61–84 (2000)
11. Hare, K., Yeats, K.: Size of characters of exceptional Lie groups. J. Aust. Math. Soc. **77**, 1–16 (2004)
12. Humphreys, J.: Introduction to Lie Algebras and Representation Theory. Springer, New York (1994)
13. Knapp, A.: Representation Theory of Semisimple Groups. An Overview Based on Examples. University Press, Princeton (1986)
14. Lang, S.: Differential and Riemannian Manifolds. Springer, New York (1995)
15. Mimura, M., Toda, H.: Topology of Lie Groups. Trans. Math. Monographs, vol. 91. American Mathematical Society, Providence (1991)
16. Ragozin, D.: Central measures on compact simple Lie groups. J. Funct. Anal. **10**, 212–229 (1972)
17. Ragozin, D.: Rotation invariant measure algebras on Euclidean space. Ind. Univ. J. **23**, 1139–1154 (1974)
18. Ricci, F., Stein, E.: Harmonic analysis on nilpotent groups and singular integrals. II. Singular kernels supported on submanifolds. J. Funct. Anal. **78**, 56–84 (1988)
19. Ricci, F., Stein, E.: Harmonic analysis on nilpotent groups and singular integrals. III Fractional integration along manifolds. J. Funct. Anal. **86**, 360–389 (1989)
20. Ricci, F., Travaglini, G.: $L^p - L^q$ estimates for orbital measures and Radon transforms on compact Lie groups and Lie algebras. J. Funct. Anal. **129**, 132–147 (1995)
21. Varadarajan, V.S.: Lie Groups and Lie Algebras and their Representations. Springer, New York (1984)
22. Wildberger, N.: On a relationship between adjoint orbits and conjugacy classes of a Lie group. Can. Math. Bull. **33**, 297–304 (1990)