

QUANTUM NUMBERS AND q -DEFORMED CONWAY-COXETER FRIEZES

SOPHIE MORIER-GENOUD AND VALENTIN OVSIENKO

To the memory of John Conway

The notion of q -deformed real numbers was recently introduced in [13, 14]. The first steps of this theory were largely influenced by John Conway. In the fall of 2013, we had a chance to spend a week in his company, on the occasion of a conference that we organized in Luminy, where Conway gave two wonderful talks. Long conversations with John, who, between desperate attempts to teach us his famous doomsday algorithm, told us much about his work with Coxeter, is what we cannot forget. Our understanding of Conway and Coxeter's theorem have transformed into a certain combinatorial viewpoint on continued fractions [12], that eventually led us to q -deformations.

One goal of this article is to explain the origins of q -numbers and overview their main properties. We will also introduce q -deformed Conway-Coxeter friezes. We would have loved to talk of this with John.

1. q -DEFORMED CONWAY-COXETER FRIEZES

Friezes, or frieze patterns, is perhaps one of the most ingenious and the least known invention of Coxeter [4], who confessed [3] that friezes “caused [him] many restless nights”. The question of Coxeter about classification of friezes of positive integers was answered by Conway [2], and these friezes are often called Conway-Coxeter friezes. We will introduce q -deformations of Conway-Coxeter friezes.

1.1. q -analogues: why, what, where? In mathematics and theoretical physics, “ q -deformation” often means “quantization”, and vice-versa. Historically, q is the exponential of the Planck constant, $q = e^{\hbar}$. But, in quantization theory, both, q and \hbar , are parameters.

A number of q -deformations of algebraic, geometric, and analytic structures have been introduced and thoroughly studied in mathematics. Among the q -deformed structures, we encounter quantum groups, quantized Poisson structures, q -deformed special functions, just to mention the best known theories. Quantized sequences of integer numbers arise and play an important role in all of them.

Quantized quantities are functions in q , usually polynomials, or power series. A “good” q -analogue of a “classical” quantity must satisfy (at least) two requirements:

- (1) when $q \rightarrow 1$, we obtain the initial quantity;
- (2) coefficients of polynomials in the quantized object have a combinatorial meaning.

Whenever a combinatorial object counts something, its q -analogue also counts the same things, but with more precision. This is why the role of combinatorics in mathematical physics keeps increasing.

1.2. Euler and Gauss. Quantum integers have appeared in mathematics long before quantum physics made its first steps. For a positive integer n , the polynomial

$$(1) \quad [n]_q := 1 + q + q^2 + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}$$

is commonly called the q -analogue of n . It satisfies two recurrences:

$$(2) \quad [n + 1]_q = q [n]_q + 1, \quad [n + 1]_q = [n]_q + q^n,$$

both of them can be used as definition starting from the natural assumption $[0]_q = 0$. They were introduced by Euler, who studied the infinite product $\prod_{n \geq 1} (1 - q^n)$, now called the Euler function. Euler connected this function to permutations, thus founding simultaneously combinatorics and modular forms theory.

The q -factorial is defined by $[n]_q! := [1]_q [2]_q \cdots [n]_q$, and the Gaussian q -binomial coefficients by

$$\binom{n}{m}_q := \frac{[n]_q!}{[m]_q! [n-m]_q!}.$$

They are polynomials. The role of q -binomials in combinatorics is immense. They count points in Grassmannians over finite fields, Young diagrams, binary words, etc. Every coefficient of $\binom{n}{m}_q$ has a combinatorial meaning.

The first interesting example of q -binomial is $\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$ which is different from $[6]_q$. Moreover, $[3]_q! = 1 + 2q + 2q^2 + q^3$ gives yet another version of “quantum 6”. These and many other examples explain our viewpoint: we cannot quantize 6, or any other integer individually. What we quantize are not integers, but *sequences of integers*.

1.3. Friezes and Conway’s classification. A frieze is an array of n staggered infinite rows of positive integers, with the first and the last rows consisting of 1’s, that satisfies the local unimodular rule

$$\begin{array}{cccccccccccc}
 \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & \cdots \\
 & & \cdots & c_i & c_{i+1} & c_{i+2} & \cdots & c_{j-1} & c_j & \cdots \\
 & & & & & & & & & & \text{local rule} \\
 & & & & & & & & & & \begin{array}{ccc} b & & d \\ a & \diagdown & / \\ & c & \end{array} \\
 & & & & & & & & & & ad - bc = 1 \\
 & & & & & & \cdots & \cdots & \cdots & & \\
 \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & \cdots
 \end{array}$$

A frieze is determined by the second row $(c_i)_{i \in \mathbb{Z}}$. Every entry of the frieze is parametrized by two integers, i and j . The local rule allows then to calculate c_{ij} . For instance, $c_{ii} = c_i$, $c_{i,i+1} = c_i c_{i+1} - 1$, etc.

Example 1.1. The following 7-periodic frieze with 6 rows is the main example of [2] (and the favorite example of Coxeter who used it in all of his articles on the subject).

$$\begin{array}{cccccccccccc}
 \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & & 1 & 4 & 2 & 1 & 3 & 2 & 2 & 1 & \cdots \\
 \cdots & 1 & 3 & 7 & 1 & 2 & 5 & 3 & 1 & & \cdots \\
 & 1 & 2 & 5 & 3 & 1 & 3 & 7 & 1 & & \cdots \\
 \cdots & 1 & 3 & 2 & 2 & 1 & 4 & 2 & 1 & & \cdots \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots
 \end{array}$$

Coxeter proved [4] that every frieze is $(n+1)$ -periodic. The Conway and Coxeter theorem provides a classification of friezes, it states that friezes are in one-to-one correspondence with triangulations of a convex $(n+1)$ -gon.

Theorem 1 ([2]). *A sequence of positive integers (c_0, \dots, c_n) , is a cycle of the second row of an n -row frieze if and only if c_i is the number of triangles at the i -th vertex of a triangulated $(n+1)$ -gon.*

As acknowledged by Coxeter [3], this statement is actually due to John Conway.

Coxeter showed a connection of friezes to continued fractions, see Proposition 2.6 below. This connection is somewhat equivalent to existence of a natural embedding of friezes and triangulated $(n+1)$ -gons into the Farey graph; see Figure 1.

Interest to Conway-Coxeter’s friezes has much increased recently because of the connection to various areas of number theory, algebra, geometry and combinatorics. For a survey; see [11].

1.4. Quantum friezes. The following notion is new, although it was implicitly in [13].

where $a_i \geq 1$ and $c_j \geq 2$ (except for $a_1 \geq 0, c_1 \geq 1$). The notation used by Hirzebruch is $\frac{r}{s} = \llbracket c_1, \dots, c_k \rrbracket$; the coefficients a_i and c_j are connected by the Hirzebruch formula; see, e.g., [12].

Definition 2.1. The q -deformed regular continued fraction is defined by

$$(4) \quad [a_1, \dots, a_{2m}]_q := [a_1]_q + \frac{q^{a_1}}{[a_2]_{q^{-1}} + \frac{q^{-a_2}}{[a_3]_q + \frac{q^{a_3}}{[a_4]_{q^{-1}} + \frac{q^{-a_4}}{\ddots}}}}}$$

where $[a]_q$ is the Euler q -integer. The q -deformed Hirzebruch-Jung continued fraction is

$$(5) \quad \llbracket c_1, \dots, c_k \rrbracket_q := [c_1]_q - \frac{q^{c_1-1}}{[c_2]_q - \frac{q^{c_2-1}}{\ddots}}}$$

For $\frac{r}{s} = [a_1, \dots, a_{2m}] = \llbracket c_1, \dots, c_k \rrbracket$, the rational functions (4) and (5) coincide. This rational function is called the q -rational, and denoted by $[\frac{r}{s}]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$.

Both polynomials, \mathcal{R} and \mathcal{S} , depend on r and s .

Example 2.2. For instance, we obtain $[\frac{5}{2}]_q = \frac{1+2q+q^2+q^3}{1+q}$ and $[\frac{5}{3}]_q = \frac{1+q+2q^2+q^3}{1+q+q^2}$. Observe that ‘‘quantum 5’’ in the numerator depends on the denominator.

2.2. The weighted Farey graph and Stern-Brocot tree. Let us give a recursive definition. The

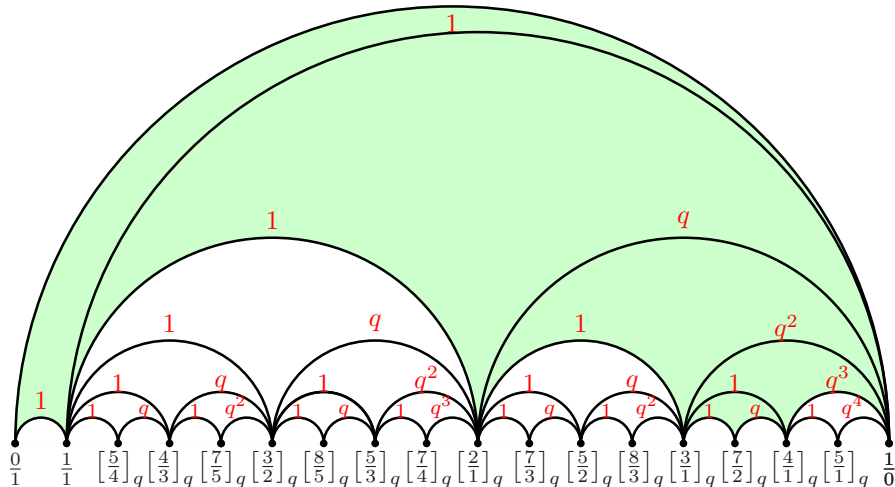
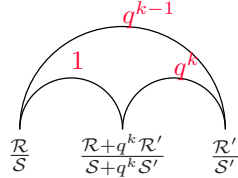


FIGURE 1. The weighted Farey graph between $\frac{0}{1}$ and $\frac{1}{0}$; the colored area corresponds to the frieze of Example 1.1.

set of rational numbers \mathbb{Q} , completed by $\infty := \frac{1}{0}$, are vertices of a graph called the *Farey graph*. Two rationals, $\frac{r}{s}$ and $\frac{r'}{s'}$ are connected by an edge if and only if $rs' - r's = \pm 1$. Edges of the Farey graph are often represented as geodesics of the hyperbolic plane which is triangulated.

Definition 2.3. The *weighted Farey graph*, see Figure 1, is just the classical Farey graph, in which the vertices are labeled by rational functions in q , and the edges are weighted by powers of q . The weights and the labels are defined recursively via the following local rule:

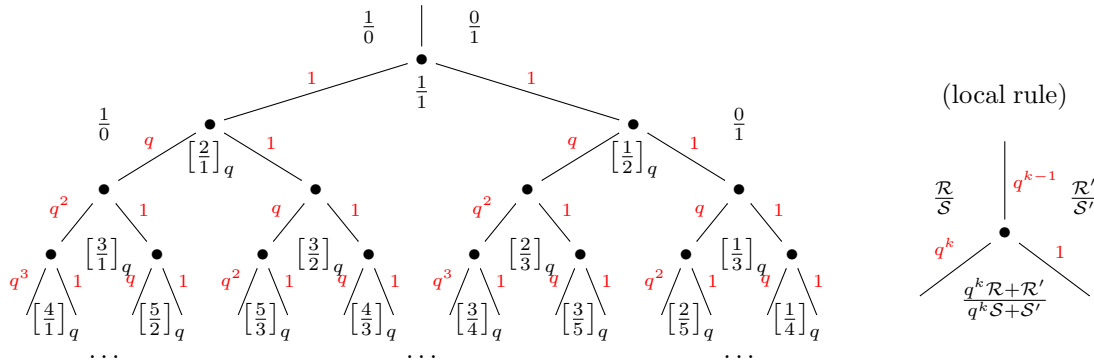


from the initial triangle $(\frac{0}{1}, \frac{1}{1}, \frac{1}{0})$ that remains undeformed.

Theorem 2 ([13]). *The vertices of the weighted Farey graph are, indeed, labeled by q -rationals.*

Remark 2.4. The Farey graph contains \mathbb{Z} as a subgraph (that form a sequence of triangles, $(\frac{n}{1}, \frac{n+1}{1}, \frac{1}{0})$). Our q -deformation restricted to \mathbb{Z} leads to Euler's formula (1).

Alternatively, one can use the Stern-Brocot tree instead of the Farey graph.



The weight of every edge of the weighted Stern-Brocot tree is determined by the local rule along the tree: the weight of the right branch is multiplied by q , the weight of the left branch is 1. This allows us to calculate q -rationals inductively. For instance, $[\frac{2}{1}]_q = \frac{q+1}{1} = [2]_q$, $[\frac{1}{2}]_q = \frac{q}{q+1}$. The left branch of the tree consists in the classical q -integers (1).

2.3. q -deformed $SL(2, \mathbb{Z})$. The *modular group* $SL(2, \mathbb{Z})$ is a useful tool to work with continued fractions. In our case, we need to q -deform $SL(2, \mathbb{Z})$, so we consider the following three matrices:

$$(6) \quad R_q := \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \quad L_q := \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}, \quad S_q := \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}.$$

For $q = 1$, we obtain the standard matrices any two of which can be chosen as generators of $SL(2, \mathbb{Z})$.

Proposition 2.5 ([13]). *Given a rational, $\frac{r}{s} = [a_1, \dots, a_{2m}] = \llbracket c_1, \dots, c_k \rrbracket$, the polynomials $\mathcal{R}(q)$ and $\mathcal{S}(q)$ of $[\frac{r}{s}]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$ can be calculated as the entries of first column of the following matrix products*

$$R_q^{a_1} L_q^{a_2} \dots R_q^{a_{2m-1}} L_q^{a_{2m}} = \begin{pmatrix} q\mathcal{R} & \mathcal{R}' \\ q\mathcal{S} & \mathcal{S}' \end{pmatrix}, \quad R_q^{c_1} S_q R_q^{c_2} S_q \dots R_q^{c_k} S_q = \begin{pmatrix} \mathcal{R} & -q^{c_k-1} \mathcal{R}'' \\ \mathcal{S} & -q^{c_k-1} \mathcal{S}'' \end{pmatrix},$$

where \mathcal{R}' , \mathcal{S}' and \mathcal{R}'' , \mathcal{S}'' are lower degree polynomials corresponding to the previous convergents of the continued fractions.

This way of understanding q -rationals clarify Definition 1.2. Indeed, (3) is the determinant of the second matrix. Note that the matrices (6) are proportional to the matrices arising in the quantum Teichmüller theory; see [5]. This interesting connection is yet to be investigated.

2.4. Connection to q -friezes. It turns out that q -rationals appear everywhere in q -deformed Conway-Coxeter friezes, as quotients of the neighboring entries, so that we can consider q -friezes as yet another way to describe q -deformations of rationals.

Let us first recall the connection between the Conway-Coxeter friezes and continued fractions.

Proposition 2.6 ([4]). *If (c_{ij}) are the entries of a frieze, then $\frac{c_{ij}}{c_{i+1,j}} = \llbracket c_i, \dots, c_j \rrbracket$.*

This statement has a straightforward q -analogue.

Proposition 2.7. *If (c_{ij}) are the entries of a frieze and $(C_{ij}(q))$ the entries of the q -deformed frieze, then $\frac{C_{ij}(q)}{C_{i+1,j}(q)} = \left[\frac{c_{ij}}{c_{i+1,j}} \right]_q = \llbracket c_i, \dots, c_j \rrbracket_q$.*

The q -frieze of Example 1.3 contains many examples of q -rationals: $\frac{\{5\}}{\{2\}} = \left[\frac{5}{2} \right]_q$, $\frac{\{7\}}{\{3\}} = \left[\frac{7}{3} \right]_q$, etc.

2.5. q -deformed irrational numbers. Let $x \geq 0$ be an irrational number, and $(x_n)_{n \geq 1}$ a sequence of rationals converging to x . Our definition is the following.

Take the sequence of rational functions $[x_1]_q, [x_2]_q, \dots$. Consider their Taylor expansions at $q = 0$, for which we will use the notation: $[x_n]_q = \sum_{k \geq 0} \varkappa_{n,k} q^k$.

Theorem 3 ([14]). *(i) For every $k \geq 0$, the coefficients of the Taylor series of $[x_n]_q$ stabilize as n grows. (ii) The limit coefficients $\varkappa_k = \lim_{n \rightarrow \infty} \varkappa_{n,k}$ do not depend on the sequence $(x_n)_{n \geq 1}$, but only on x .*

This stabilization phenomenon allows us to define the q -deformation of $x \geq 0$ as a power series in q :

$$[x]_q = \varkappa_0 + \varkappa_1 q + \varkappa_2 q^2 + \varkappa_3 q^3 + \dots$$

In the case of $x < 0$, the definition of q -deformation is based on the recurrence $[x-1]_q := q^{-1}[x]_q - q^{-1}$, cf. Section 4.1. It turns out that the resulting series in q is a *Laurent series* (with integer coefficients):

$$[x]_q = -q^{-N} + \varkappa_{1-N} q^{1-N} + \varkappa_{2-N} q^{2-N} + \dots$$

where $N \in \mathbb{Z}_{>0}$ such that $-N \leq x < 1 - N$.

3. EXAMPLES: q -FIBONACCI AND q -PELL NUMBERS, q -GOLDEN RATIO AND $[\sqrt{2}]_q$

Let us consider two remarkable sequences of rationals,

$$\frac{F_{n+1}}{F_n} = \underbrace{[1, 1, \dots, 1]}_n \quad \text{and} \quad \frac{P_{n+1}}{P_n} = \underbrace{[2, 2, \dots, 2]}_n,$$

where F_n is the n^{th} Fibonacci number, and P_n is the n^{th} Pell number. Quantizing them, we obtain sequences of polynomials with a very particular ‘‘swivel’’ property: the polynomials corresponding to F_n (resp. P_n) in the numerator and denominator are mirrors of each other. The stabilized Taylor series of $\left[\frac{F_{n+1}}{F_n} \right]_q$ and $\left[\frac{P_{n+1}}{P_n} \right]_q$ give rise of q -analogues of $\frac{1+\sqrt{5}}{2}$ and $1 + \sqrt{2}$, respectively.

3.1. q -Fibonacci numbers. Let $\mathcal{F}_n(q)$ be a sequences of polynomials defined by the recurrence

$$\mathcal{F}_{n+2} = [3]_q \mathcal{F}_n - q^2 \mathcal{F}_{n-2},$$

where $[3]_q = 1 + q + q^2$ is Euler’s quantum 3, and the initial conditions $(\mathcal{F}_0(q) = 0, \mathcal{F}_2(q) = 1)$ and $(\mathcal{F}_1(q) = 1, \mathcal{F}_3(q) = 1 + q)$. The sequence of polynomials $\mathcal{F}_n(q)$ is a q -deformation of the Fibonacci sequence, i.e., $\mathcal{F}_n(1) = F_n$. Consider also the *mirror polynomials* $\tilde{\mathcal{F}}_n(q) := q^{n-1} \mathcal{F}_n(\frac{1}{q})$. The triangles of

More and more coefficients repeat in every next series, the series eventually stabilize to

$$[\varphi]_q = 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 37q^8 - 82q^9 + 185q^{10} - 423q^{11} + 978q^{12} - 2283q^{13} \\ + 5373q^{14} - 12735q^{15} + 30372q^{16} - 72832q^{17} + 175502q^{18} - 424748q^{19} + 1032004q^{20} \dots$$

This power series is our quantized golden ratio $\left[\frac{1+\sqrt{5}}{2}\right]_q := [\varphi]_q$.

The series $[\varphi]_q$ satisfies the equation $q[\varphi]_q^2 - (q^2 + q - 1)[\varphi]_q - 1 = 0$, which is a q -analogue of $x^2 - x + 1 = 0$. Therefore, the generating function of the series $[\varphi]_q$ is

$$(7) \quad [\varphi]_q = \frac{q^2 + q - 1 + \sqrt{(q^2 + 3q + 1)(q^2 - q + 1)}}{2q}.$$

Let us mention that the coefficients of the series $[\varphi]_q$ remarkably coincide with Sequence A004148 of [15] (with alternating signs), called the Generalized Catalan numbers.

3.4. **q -deformed $\sqrt{2}$.** Quotients of the Pell polynomials stabilize to the series $[1 + \sqrt{2}]_q$, from which we deduce

$$[\sqrt{2}]_q = 1 + q^3 - 2q^5 + q^6 + 4q^7 - 5q^8 - 7q^9 + 18q^{10} + 7q^{11} - 55q^{12} + 18q^{13} \\ + 146q^{14} - 155q^{15} - 322q^{16} + 692q^{17} + 476q^{18} - 2446q^{19} + 307q^{20} \dots$$

This series is a solution of $q^2[\sqrt{2}]_q^2 - (q^3 - 1)[\sqrt{2}]_q = q^2 + 1$, which is our version of q -analogue of $x^2 = 2$. The generating function is then equal to

$$(8) \quad [\sqrt{2}]_q = \frac{q^3 - 1 + \sqrt{(q^4 + q^3 + 4q^2 + q + 1)(q^2 - q + 1)}}{2q^2}.$$

Note that the coefficients of $[\sqrt{2}]_q$ grow much slower than those of $[\varphi]_q$, and does not match with any known sequence.

3.5. **Quadratic q -irrationals.** Several observations can be done analyzing (7) and (8). The polynomial under the radical of these q -numbers is a *palindrome*. This remarkable property remain true for arbitrary *quadratic irrationals*, i.e., numbers of the form $x = \frac{a+\sqrt{b}}{c}$, where $a, b > 0, c$ are integers.

Theorem 4 ([7]). *For every quadratic irrational, $\left[\frac{a+\sqrt{b}}{c}\right]_q = \frac{A(q)+\sqrt{B(q)}}{C(q)}$, where A, B and C are polynomials in q . Furthermore, B is a monic polynomial whose coefficients form a palindrome.*

A quadratic irrational can also be characterized as a fixed point of an element of $\text{PSL}(2, \mathbb{Z})$. A q -analogue of this property will be provided by Theorem 5 in the next section.

3.6. **Radius of convergence.** The modulus of the smallest (i.e., closest to 0) root of the polynomials under the radical in (7) and (8) is equal to

$$R_\varphi^- = \frac{3 - \sqrt{5}}{2} \quad \text{and} \quad R_{\sqrt{2}}^- = \frac{1 + \sqrt{2} - \sqrt{2\sqrt{2} - 1}}{2},$$

respectively. This are the *radius of convergence* of the Taylor series (7) and (8). The modulus of the largest roots are $R_\varphi^+ = 1/R_\varphi^-$ and $R_{\sqrt{2}}^+ = 1/R_{\sqrt{2}}^-$, viz

$$R_\varphi^+ = \frac{3 + \sqrt{5}}{2} \quad \text{and} \quad R_{\sqrt{2}}^+ = \frac{1 + \sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2}.$$

Aesthetic aspect of these formulas pushed us to analyze several more examples. One of them is another remarkable number, $\frac{9+\sqrt{221}}{14}$, sometimes called the “bronze ratio”. This is the third, after φ

and $\sqrt{2}$, badly approximated number in the Markov theory. The modulus of the minimal and maximal roots of the polynomial under the radical in $\left[\frac{9+\sqrt{221}}{14}\right]_q$ are

$$R^- = \frac{1 + \sqrt{13} - \sqrt{2(\sqrt{13} - 1)}}{4} \quad \text{and} \quad R^+ = \frac{1 + \sqrt{13} + \sqrt{2(\sqrt{13} - 1)}}{4}.$$

Note that $221 = 13 \cdot 17$. We do not know if the striking resemblance with the case of $\sqrt{2}$ is a coincidence. A work on the analytic properties of q -numbers is in progress [8].

4. SOME PROPERTIES OF q -RATIONALS

The notion of q -rationals arose from an attempt to understand the connection between several different theories, such as continued fractions, Jones polynomial of (rational) knots, quantum Teichmüller theory, and cluster algebra. These connections were discovered by many authors; see [5, 9]. Our definition is a specialization of such notions as F -polynomials, quantum geodesic length, snake graphs. We present some concrete properties of q -rationals that we consider as most important, many of them reflect this deep connection. For more details; see [13, 14].

4.1. $\text{PSL}(2, \mathbb{Z})$ -action. Our first important property is the following.

Theorem 5 ([7]). *The procedure of q -deformation commutes with the $\text{PSL}(2, \mathbb{Z})$ -action.*

Indeed, the first recurrence (2) remains true for q -rationals. For every $x \in \mathbb{Q}$, we have

$$(9) \quad [x + 1]_q = q[x]_q + 1.$$

Recurrence (9) readily follows from (4). The recurrence (9) is very useful for us, since it allows us to define q -deformations of $x < 0$. Furthermore, we have

$$(10) \quad \left[-\frac{1}{x}\right]_q = -\frac{1}{q[x]_q}, \quad [-x]_q = -q^{-1}[x]_{q^{-1}}.$$

Together, (9) and (10), define an action of $\text{PSL}(2, \mathbb{Z})$ on q -rationals generated by the matrices R_q and qS_q in (6). This was (implicitly) checked in [13]; see Lemma 4.6, and will be further developed in [7].

Remark 4.1. Recurrence (9) appear, e.g., for q -integers, and plays a crucial role in quantum algebra. For instance this recurrence is necessary for the quantum binomial formula. Identities (10) look more intriguing and need to be better understood.

4.2. Total positivity. The polynomials in the numerator and denominator of a positive q -rational have positive integer coefficients, with 1 at lowest and higher orders. Moreover, the set of all q -rationals has a much stronger property of *total positivity*. Consider two q -rationals, $\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$ and $\left[\frac{r'}{s'}\right]_q = \frac{\mathcal{R}'(q)}{\mathcal{S}'(q)}$.

Theorem 6 ([13]). *If $\frac{r}{s} > \frac{r'}{s'} > 0$, then the polynomial*

$$(11) \quad \mathcal{X}_{\frac{r}{s}, \frac{r'}{s'}}(q) := \mathcal{R}(q)\mathcal{S}'(q) - \mathcal{S}(q)\mathcal{R}'(q)$$

has positive integer coefficients.

The main ingredient of the proof of this theorem is the fact that (11) is a monomial, i.e., proportional to a power of q , if and only if $\frac{r}{s}$ and $\frac{r'}{s'}$ are connected in the Farey graph.

Theorem 6 means that the “quantization preserves the order”, in the sense that it is a homeomorphism of \mathbb{Q} into an ordered subset of partially ordered set of rational functions. The notion of total positivity has a long history in mathematics and manifests in every area of it.

4.3. Relation to the Jones polynomial. One of the ingenious inventions of Conway [1] was to encode a certain large class of knots, called “rational”, or “two-bridge” knots, by continued fractions. Every notion of the knot theory, such as knot invariants, can then be directly associated to continued fractions.

The Jones polynomial is a powerful invariant in the knot theory. It turns out that one can express the Jones polynomial $J_{\frac{r}{s}}(q)$ of the two-bridge knot associated to a rational $\frac{r}{s}$, as a combination of the polynomials \mathcal{R} and \mathcal{S} of the q -rational $[\frac{r}{s}]_q$.

Theorem 7 ([13]). *The Jones polynomial of a two-bridge knot is $J_{\frac{r}{s}}(q) = q\mathcal{R}(q) + (1 - q)\mathcal{S}(q)$.*

The proof is based on connection to cluster algebra; see [9].

4.4. Unimodality conjecture. Long computer experimentation and evidence in the simplest cases convinced us in yet another property of q -rationals.

Conjecture 1 ([13]). *For every $[\frac{r}{s}]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$, the coefficients of the polynomials $\mathcal{R}(q)$ and $\mathcal{S}(q)$ form unimodal sequences.*

This means that the coefficients of the polynomials increase from 1 to the maximal value (that can be taken by one or more consecutive coefficients) and then decrease to 1. Unimodal sequences appear in mathematics, and this property is interesting because it often hides some combinatorial or geometric structure.

A proof of several particular cases of the conjecture, and connection to old problems of combinatorics, was obtained in [10]. However, the general problem is open.

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SOPHIE MORIER-GENOUD, SORBONNE UNIVERSITÉ AND UNIVERSITÉ DE PARIS, CNRS, IMJ-PRG, F-75006 PARIS, FRANCE

VALENTIN OVSIENKO, CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE, LABORATOIRE DE MATHÉMATIQUES U.F.R. SCIENCES EXACTES ET NATURELLES MOULIN DE LA HOUSSE - BP 1039 51687 REIMS CEDEX 2, FRANCE

Email address: sophie.morier-genoud@imj-prg.fr, valentin.ovsienko@univ-reims.fr