$\mathrm{SL}_2(\mathbb{Z})$-TILINGS OF THE TORUS, COXETER-CONWAY FRIEZES AND FAREY TRIANGULATIONS

SOPHIE MORIER-GENOUD, VALENTIN OVSIENKO AND SERGE TABACHNIKOV

Abstract. The notion of SL_2-tiling is a generalization of that of classical Coxeter-Conway frieze pattern. We classify doubly antiperiodic SL_2-tilings that contain a rectangular domain of positive integers. Every such SL_2-tiling corresponds to a pair of frieze patterns and a unimodular 2×2-matrix with positive integer coefficients. We relate this notion to triangulated n-gons in the Farey graph.

1. Introduction

Frieze patterns were introduced and studied by Coxeter and Conway, [7, 6], in the 70's. A frieze pattern is an infinite array of numbers, bounding by two diagonals of 1's, such that every four adjacent numbers a, b, c, d forming a "small" square satisfy the relation $ad - bc = 1$ called *the unimodular rule*; for an example see Figure 1. The *width* of the frieze is the number of diagonals between the bounding diagonals of 1's.

The fundamental Conway-Coxeter theorem [6] offers the following classification: *frieze patterns with positive integer entries of width $n-3$, are in one-to-one correspondence with triangulations of a convex n-gon*; for a simple proof see [11]. More precisely, given a triangulated n-gon in the oriented plane, one constructs a frieze of width $n-3$ as follows. The diagonal next to the diagonal of 1's is formed by the numbers of triangles incident at each vertex (taken cyclically).

$$
\begin{array}{cccccccccccccccc}
\ddots &&&&&&&&&&&&&&& \\
& 1 & 2 & 3 & 1 & 1 & 1 &&&&&&&&& \\
&& 1 & 2 & 1 & 2 & 3 & 1 &&&&&&&& \\
&&& 1 & 1 & 3 & 5 & 2 & 1 &&&&&&& \\
&&&& 1 & 4 & 7 & 3 & 2 & 1 &&&&&& \\
&&&&& 1 & 2 & 1 & 1 & 1 & 1 &&&&& \\
&&&&&& 1 & 1 & 2 & 3 & 4 & 1 &&&& \\
&&&&&&& 1 & 3 & 5 & 7 & 2 & 1 &&& \\
&&&&&&&& 1 & 2 & 3 & 1 & 1 & 1 && \\
&&&&&&&&& 1 & 2 & 1 & 2 & 3 & 1 & \\
&&&&&&&&&&&&&&& \ddots
\end{array}
$$

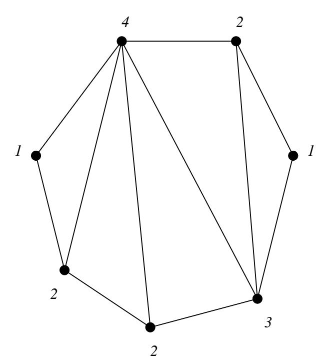

Figure 1. A 7-periodic frieze pattern and the corresponding triangulated heptagon.

Key words and phrases. Frieze pattern, SL_2-tiling, Farey graph, Modular group.

This, in particular, implies that every diagonal in a frieze of width $n-3$ is n-periodic. Throughout this paper, we will be considering frieze patterns with positive integer entries.

The following terminology is due to Conway and Coxeter [6]. A sequence of n positive integers $q = (q_0, \ldots, q_{n-1})$ is called a *quiddity* of order n, if there exists a triangulated n-gon such that every q_i is equal to the number of incident triangles at i-th vertex. For instance, the example in Figure 1 corresponds to the following quiddities of order 7: $(1,3,2,2,1,4,2)$, $(3,2,2,1,4,2,1), \ldots$ (cyclic permutation).

Every quiddity of order n determines a unique positive integer frieze pattern. Two quiddities correspond to the same positive integer frieze pattern if and only if they differ by a cyclic permutation. According to the Conway-Coxeter theorem, positive integer frieze patterns can be enumerated by the Catalan numbers.

Example 1.0.1. For each case $n = 3, 4$ and 5, there is a unique (up to cyclic permutation) quiddity: $(1,1,1)$, $(1,2,1,2)$ and $(1,3,1,2,2)$, respectively.

For $n = 6$, there are four different quiddities:

$$(1,3,1,3,1,3), \quad (1,4,1,2,2,2), \quad (1,2,3,1,2,3), \quad (1,3,2,1,3,2)$$

and their cyclic permutations.

We can also consider the "degenerate" case $n = 2$, where the corresponding "degenerate" quiddity is $(0,0)$.

Examples of frieze patterns can be constructed using the computer program [17].

Among many beautiful properties of Coxeter-Conway friezes, the property of periodicity and so-called Laurent phenomenon are particularly important. They relate frieze patterns to the theory of cluster algebras developed by Fomin and Zelevinsky, [8, 9].

Various generalizations of Coxeter-Conway friezes have been recently introduced and studied, see [5, 16, 2, 1, 13]. One of the generalizations, called SL_2-*tiling*, was first considered by Assem, Reutenauer and Smith [1], and further developed by Bergeron and Reutenauer [3]. An SL_2-tiling is an infinite array of numbers satisfying the above unimodular rule, without the condition of bounding diagonals of 1's. Unlike the frieze patterns, SL_2-tilings are not necessarily periodic. Nevertheless, correspondences between SL_2-tilings and triangulations can be established, [12, 4].

$$\begin{array}{cccccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 2 & 5 & 8 & 11 & 3 & -2 & -5 & -8 & -11 & -3 & \cdots \\
\cdots & 7 & 18 & 29 & 40 & 11 & -7 & -18 & -29 & -40 & -11 & \cdots \\
\cdots & 5 & 13 & 21 & 29 & 8 & -5 & -13 & -21 & -29 & -8 & \cdots \\
\cdots & 3 & 8 & 13 & 18 & 5 & -3 & -8 & -13 & -18 & -5 & \cdots \\
\cdots & -2 & -5 & -8 & -11 & -3 & 2 & 5 & 8 & 11 & 3 & \cdots \\
\cdots & -7 & -18 & -29 & -40 & -11 & 7 & 18 & 29 & 40 & 11 & \cdots \\
\cdots & -5 & -13 & -21 & -29 & -8 & 5 & 13 & 21 & 29 & 8 & \cdots \\
\cdots & -3 & -8 & -13 & -18 & -5 & 3 & 8 & 13 & 18 & 5 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

FIGURE 2. A $(4,5)$-antiperiodic SL_2-tiling with positive rectangular domain.

The case of (n, m)-*antiperiodic, or "toric"* SL_2-tilings was suggested in [3]. In this paper, we study such tilings.

The main results of the paper are the following.

We classify doubly antiperiodic SL_2-tilings that contain a rectangular fundamental domain of positive integers. We show that every such SL_2-tiling is generated by a pair of quiddities and a unimodular 2×2-matrix with positive integer coefficients. Although there are infinitely many such SL_2-tilings, their description is very explicit.

Following the original idea of Coxeter [7], we also interpret the entries of a doubly periodic SL_2-tiling that contain a rectangular fundamental domain of positive integers in terms of the Farey graph of rational numbers. Every such SL_2-tiling corresponds to a triple: an n-gon, an m-gon in the Farey graph, and a totally positive matrix from $\mathrm{SL}_2(\mathbb{Z})$ relating them. We also obtain an explicit formula for the entries of the tiling.

2. Farey graph and the Conway-Coxeter theorem

In this section, we give an explanation of the relation between the Coxeter frieze patterns and triangulated n-gons.

It was already noticed by Coxeter [7] that a Farey series (of arbitrary order N) defines a frieze pattern. Moreover, every frieze pattern corresponds to an n-gon (i.e., an n-cycle) in the Farey graph. A Farey n-gon always carries a triangulation; we will prove that this triangulation is precisely that of Conway-Coxeter theorem. This statement seems to be new and extend the observation illustrated in [17].

2.1. **Farey graph, Farey series and Farey n-gons.** For two rational numbers, $v_1, v_2 \in \mathbb{Q}$, written as irreducible fractions $v_1 = \frac{a_1}{b_1}$ and $v_2 = \frac{a_2}{b_2}$, the *Farey "distance"* is defined by

$$d(v_1, v_2) := |a_1 b_2 - a_2 b_1|.$$

Note that the above "distance" does not satisfy the triangle inequality. Recall the definition of the *Farey graph.*

(1) The set of vertices of the Farey graph is $\mathbb{Q} \cup \{\infty\}$, with ∞ represented by $\frac{1}{0}$.
(2) Two vertices, v_1, v_2 are joined by a (non-oriented) edge (v_1, v_2) whenever $d(v_1, v_2) = 1$.

The Farey graph is often embedded into the hyperbolic half-plane, the edges being realized as geodesics joining rational points on the ideal boundary.

The following classical properties of the Farey graph can be found in [10] (the proof is elementary).

Proposition 2.1.1. *(i) Every* 3*-cycle of the Farey graph is of the form*

$$\left\{ \frac{a_1}{b_1}, \frac{a_1 + a_2}{b_1 + b_2}, \frac{a_2}{b_2} \right\}. \tag{2.1}$$

(ii) Every edge of the Farey graph belongs to a 3*-cycle.*

(iii) Edges in the Farey graph do not cross, i.e., for a quadruple $v_1 > v_2 > v_3 > v_4$ it is not possible to have edges (v_1, v_3) and (v_2, v_4).

Definition 2.1.2. The *Farey series* (also called *Farey sequence*) of order N is the sequence of irreducible fractions in $[0, 1]$ whose denominators do not exceed N.

We will write the sequences in the decreasing order; see Figure 3.

The following fundamental property of Farey series is also proved in [10]. It shows that every Farey series is a cycle in the Farey graph.

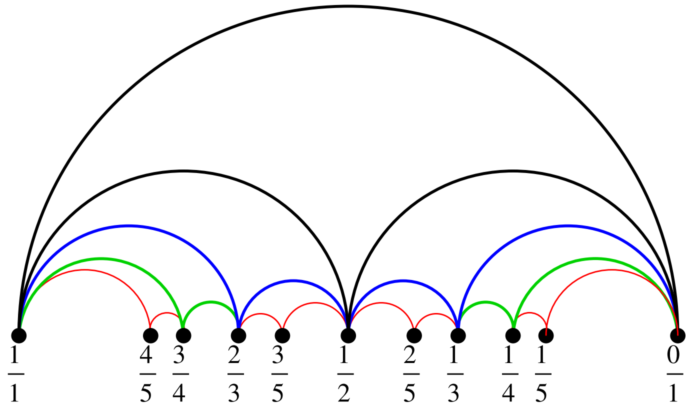

FIGURE 3. The Farey series of order 5 embedded in the Farey graph

Proposition 2.1.3. *Every two consecutive numbers in a Farey series are joined by an edge in the Farey graph.*

This is less elementary than Proposition 2.1.1, so we propose here a short proof. Our proof is different from the well-known one, it is based on the classical Pick formula.

Proof. Consider two consecutive numbers $\frac{a}{b} > \frac{c}{d}$, in a Farey series of some order N. Suppose that $ad - bc \geq 2$. The quantity $A = \frac{1}{2}(ad - bc)$ is the area of the Euclidean triangle spanned by the vertices $(0, 0)$, (a, b), (c, d). Pick's formula states:

$$A = I + \frac{B}{2} - 1,$$

where I is the number of integer points in the interior of the triangle, and B the number of integer points on the border. By assumption, $A \geq 1$, and therefore $I + \frac{B}{2} \geq 2$. It follows that there exists a point (x, y), which is either inside the triangle, or on the segment between (a, b) and (c, d) (since the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are irreducible). One then has:

$$y \leq \max(b, d) \leq N \qquad \text{and} \qquad \frac{a}{b} > \frac{x}{y} > \frac{c}{d}.$$

This contradicts the assumption that $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive numbers in the Farey series. □

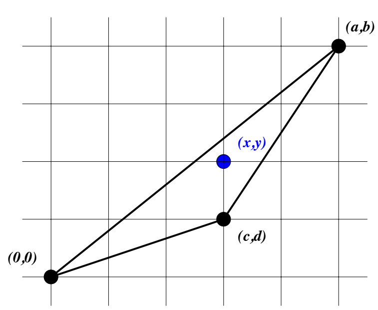

FIGURE 4. The case of interior point

Proposition 2.1.3 is used three times to prove the following.

Corollary 2.1.4. *Every Farey series forms a triangulated polygon in the Farey graph.*

Proof. We prove this statement by induction on N (the order of Farey series). Assume that the series of order $N-1$ is triangulated. The series of order N is obtained from that of order $N-1$ by adding points of the form $\frac{k}{N}$.

First, we observe that two points, $\frac{k_1}{N}$ and $\frac{k_2}{N}$ cannot be consecutive. Indeed, $d(\frac{k_1}{N}, \frac{k_2}{N}) \neq 1$: that would contradict Proposition 2.1.3; therefore, every new point $\frac{k}{N}$ appears between two "old" points:

$$\frac{p_1}{q_1} > \frac{k}{N} > \frac{p_2}{q_2}. \tag{2.2}$$

Second, by Proposition 2.1.3, $\frac{k}{N}$ is joined by edges with $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$. Third, $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are joined by an edge, according to Proposition 2.1.3 applied to the series of order $N-1$. We conclude that (2.2) is a triangle. □

We will be interested in n-cycles (or "n-gons") in the Farey graph that are more general than Farey series.

Definition 2.1.5. (1) An *n-gon* in the Farey graph, or a *Farey n-gon* is a decreasing sequence of rationals $(v_0, \ldots, v_{n-1})$:

$$\infty \geq v_0 > v_1 > \ldots > v_{n-1} \geq 0,$$

such that every pair of consecutive numbers v_i, v_{i+1}, as well as v_{n-1}, v_0, are joined by an edge.

(2) The n-gon is called *normalized* if $v_0 = \infty$ and $v_{n-1} = 0$.

Since every n-gon can be embedded in a Farey series, Corollary 2.1.4 implies the following.

Corollary 2.1.6. *Every Farey n-gon is triangulated.*

We thus can speak of the *quiddity of a Farey n-gon*.

Proof. A Farey n-gon is obtained from a Farey series which is a triangulated polygon, by cutting along diagonals of the triangulation. □

We define the notion of *cyclic equivalence* of Farey n-gons. Given an n-gon $(v_0, \ldots, v_{n-1})$, consider the n-cycle $(v_1, \ldots, v_{n-1}, v_0)$, and renormalize it using the $\mathrm{SL}_2(\mathbb{Z})$-action so that $v_1 = \infty$ and $v_0 = 0$. The obtained n-gon is called cyclically equivalent to the given one. For an example, see Figure 5.

2.2. **Farey n-gons and Coxeter-Conway friezes.** Proposition 2.1.3 leads to the following observation due to Coxeter [7]: *every Farey series gives rise to a Coxeter-Conway frieze pattern of positive integers.* Along the same lines, we have the following strengthened statement.

Proposition 2.2.1. *The Coxeter-Conway frieze patterns of positive integers of width $n-3$ are in one-to-one correspondence with the normalized Farey n-gons, up to cyclic equivalence.*

Proof. The correspondence is given by considering the ratios of two consecutive rows of the frieze patterns. The sequence

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{a_1}{1}, \quad \ldots, \quad v_i = \frac{a_i}{b_i}, \quad \ldots, \quad v_{n-2} = \frac{1}{b_{n-2}}, \quad v_{n-1} = \frac{0}{1}$$

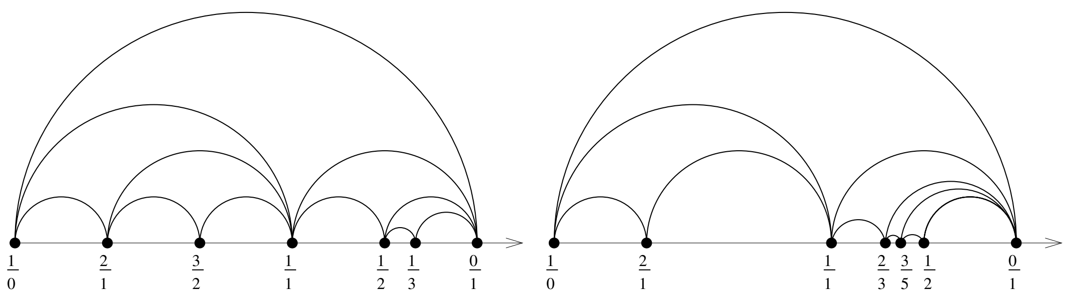

FIGURE 5. Two cyclically equivalent normalized heptagons in the Farey graph corresponding to the frieze of Figure 1

corresponds to the frieze determined by the rows

$$\begin{array}{cccccccc} 1 & a_1 & a_2 & \cdots & a_{n-3} & & 1 & 0 \\ 0 & 1 & b_2 & & & \cdots & b_{n-2} & 1 \end{array}$$

and *vice versa.* □

The Conway-Coxeter theorem mentioned in the introduction provides a relation between frieze patterns and triangulations. The following result somewhat "demystifies" this relation and provides an alternative proof of the Conway-Coxeter theorem.

Theorem 1. *The quiddity of a Farey n-gon coincides with the quiddity of the corresponding Coxeter-Conway frieze pattern.*

Proof. Consider a frieze pattern, and denote by $c_{i,j}$ its entries:

$$\begin{array}{cccccccc} 0 & 1 & c_{1,1} & c_{1,2} & \cdots & c_{1,n-3} & 1 & 0 \\ & 0 & 1 & c_{2,2} & & \cdots & c_{2,n-2} & 1 \\ & & \ddots & \ddots & & & & \ddots \end{array}$$

where

$$\begin{cases} c_{i,j} = 1, & i - j = 1 \text{ or } 3 - n, \\ c_{i,j} = 0, & i - j = 2 \text{ or } 2 - n. \end{cases}$$

The quiddity of the frieze pattern reads in the n-periodic line $(c_{i,i})$.

Clearly, two consecutive rows determine the rest of the frieze; the following formula was proved in [7], formula (5.6):

$$c_{i,j} = c_{1,i-2}c_{2,j} - c_{1,j}c_{2,i-2}.$$

In particular, we have:

$$c_{i,i} = c_{1,i-2}c_{2,i} - c_{1,i}c_{2,i-2}. \tag{2.3}$$

The corresponding Farey n-gon has the following vertices

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{c_{1,1}}{1}, \quad \ldots \quad v_i = \frac{c_{1,i}}{c_{2,i}}, \quad \ldots \quad v_{n-2} = \frac{1}{c_{2,n-2}}, \quad v_{n-1} = \frac{0}{1}.$$

Therefore, the expression (2.3) reads: $c_{i,i} = d(v_{i-2}, v_i)$. It remains to calculate the Farey distance between pairs of vertices v_{i-2} and v_i in a Farey n-gon.

Lemma 2.2.2. *Given a (triangulated) Farey n-gon*

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{a_1}{1}, \quad \ldots, \quad v_i = \frac{a_i}{b_i}, \quad \ldots, \quad v_{n-2} = \frac{1}{b_{n-2}}, \quad v_{n-1} = \frac{0}{1},$$

the Farey distance $d(v_{i-1}, v_{i+1})$ coincides with the number of triangles incident at v_i.

Proof. Among all the vertices of the n-gon (v_i), let us select those connected to v_i by edges of the Farey graph. Denote by $\{v_{i_1}, \ldots, v_{i_k}\}$, resp. $\{v_{i_{k+1}}, \ldots, v_{i_{k+\ell}}\}$ the vertices at the left, resp. right, of v_i, so that

$$v_{i_1} > \ldots > v_{i_k} > v_i > v_{i_{k+1}} > \ldots > v_{i_{k+\ell}},$$

(note that $v_{i_k} = v_{i-1}$ and $v_{i_{k+1}} = v_{i+1}$). The number of triangles incident at v_i is then equal to $k + \ell - 1$.

Two consecutive selected vertices, v_{i_j} and $v_{i_{j+1}}$ are connected by an edge. Indeed, this follows from the fact that every Farey polygon is triangulated. Therefore, the vertices $(v_{i_j}, v_{i_{j+1}}, v_i)$ form a triangle (a 3-cycle) in the Farey graph. Using Eq. (2.1), we obtain by induction:

$$v_{i-1}(= v_{i_k}) = \frac{a_{i_1} + (k-1)a_i}{b_{i_1} + (k-1)b_i}, \qquad v_{i+1}(= v_{i_{k+1}}) = \frac{a_{i_{k+\ell}} + (\ell-1)a_i}{b_{i_{k+\ell}} + (\ell-1)b_i}.$$

We have:

$$d(v_{i-1}, v_{i+1}) = a_{i_1} b_{i_{k+\ell}} - b_{i_1} a_{i_{k+\ell}} + (k-1)(a_i b_{i_{k+\ell}} - b_i a_{i_{k+\ell}}) + (\ell-1)(a_{i_1} b_i - b_{i_1} a_i).$$

By assumption, v_i is joined by edges with v_{i_1} and $v_{i_{k+\ell}}$, hence $a_i b_{i_{k+\ell}} - b_i a_{i_{k+\ell}} = 1$, and $a_{i_1} b_i - b_{i_1} a_i = 1$. Furthermore, $(v_{i_1}, v_i, v_{i_{k+\ell}})$ is also a triangle, therefore $a_{i_1} b_{i_{k+\ell}} - b_{i_1} a_{i_{k+\ell}} = 1$. We have finally:

$$d(v_{i-1}, v_{i+1}) = k + \ell - 1. \tag{2.4}$$

Hence the lemma. □

Theorem 1 is proved. □

2.3. **Entries of the frieze pattern.** Coxeter's formula (5.6) in [7] for the entries of the frieze pattern translates into our language as the following general expression:

$$c_{i,j} = d(v_{i-2}, v_j), \tag{2.5}$$

where, as above, (v_i) is the Farey n-gon corresponding to the frieze pattern.

3. SL_2-tilings

In this section, we introduce the main notions studied in this paper.

3.1. **Tame SL_2-tilings.** Let us first recall the notion of SL_2-tiling introduced in [3].

(1) An SL_2-tiling, is an infinite matrix $\mathcal{A} = (a_{i,j})_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$, such that every adjacent 2×2-minor equals 1:

$$\begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} = 1,$$

for all $(i,j) \in \mathbb{Z} \times \mathbb{Z}$.

(2) The tiling is called *tame* if every adjacent 3×3-minor equals 0:

$$\begin{vmatrix} a_{i,j} & a_{i,j+1} & a_{i,j+2} \\ a_{i+1,j} & a_{i+1,j+1} & a_{i+1,j+2} \\ a_{i+2,j} & a_{i+2,j+1} & a_{i+2,j+2} \end{vmatrix} = 0,$$

for all $(i,j) \in \mathbb{Z} \times \mathbb{Z}$.

Let us stress on the fact that a *generic* SL_2-tiling is tame.

3.2. **Antiperiodicity.** The following condition was also suggested in [3].

An SL_2-tiling is called (n, m)-*antiperiodic* if every row is n-antiperiodic, and every column is m-antiperiodic:

$$\begin{array}{rcl} a_{i,j+n} & = & -a_{i,j}\ , \\ a_{i+m,j} & = & -a_{i,j}\ , \end{array}$$

for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.

The following relation between (n, m)-antiperiodic SL_2-tilings and the classical Coxeter-Conway frieze patterns shows that the antiperiodicity condition for the SL_2-tilings is natural and interesting.

3.3. **Frieze patterns and** (n, n)**-antiperiodic** SL_2**-tilings.** As explained in [3], every Coxeter-Conway frieze pattern of width $n-3$ can be extended to a tame (n, n)-antiperiodic SL_2-tiling, in a unique way.

The construction is as follows. One adds two diagonals of 0's next to the diagonals of 1's, and then continues by antiperiodicity.

Example 3.3.1. The frieze pattern in Figure 1 corresponds to the following $(7, 7)$-antiperiodic tame SL_2-tiling.

$$\begin{array}{cccccccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & 2 & 3 & 1 & 1 & 1 & 0 & -1 & -2 & -3 & -1 & -1 & \cdots \\
\cdots & 0 & 1 & 2 & 1 & 2 & 3 & 1 & 0 & -1 & -2 & -1 & -2 & \cdots \\
\cdots & -1 & 0 & 1 & 1 & 3 & 5 & 2 & 1 & 0 & -1 & -1 & -3 & \cdots \\
\cdots & -2 & -1 & 0 & 1 & 4 & 7 & 3 & 2 & 1 & 0 & -1 & -4 & \cdots \\
\cdots & -1 & -1 & -1 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & -1 & \cdots \\
\cdots & -2 & -3 & -4 & -1 & 0 & 1 & 1 & 2 & 3 & 4 & 1 & 0 & \cdots \\
\cdots & -3 & -5 & -7 & -2 & -1 & 0 & 1 & 3 & 5 & 7 & 2 & 1 & \cdots \\
\cdots & -1 & -2 & -3 & -1 & -1 & -1 & 0 & 1 & 2 & 3 & 1 & 1 & \cdots \\
\cdots & 0 & -1 & -2 & -1 & -2 & -1 & -1 & 0 & 1 & 2 & 1 & 2 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

For the details of the above construction and the "antiperiodic nature" of Conway-Coxeter's friezes; see [3, 14].

3.4. **Positive rectangular domain.** In this paper, we are considering (n, m)-antiperiodic SL_2-tilings that contain an $m \times n$-rectangular domain of positive integers.

More precisely, we are interested in SL_2-tilings of the following form:

$$\begin{array}{c|c|c|c|c}
 & \vdots & \vdots & \vdots & \\
\hline
\cdots & P & -P & P & \cdots \\
\hline
\cdots & -P & P & -P & \cdots \\
\hline
 & \vdots & \vdots & \vdots &
\end{array} \tag{3.1}$$

where P is an $m \times n$-matrix with entries in $\mathbb{Z}_{>0}$. An example of such an SL_2-tilling is presented in Figure 2.

The following property is important for us.

Proposition 3.4.1. *An (n,m)-antiperiodic SL_2-tiling that contains a positive $m \times n$-rectangular domain is tame.*

Proof. This is a consequence of the Jacobi identity or Dodgson formula on determinants:

$$\begin{vmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{vmatrix} \begin{vmatrix} \circ & \circ & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \circ \end{vmatrix} = \begin{vmatrix} \bullet & \bullet & \circ \\ \bullet & \bullet & \circ \\ \circ & \circ & \circ \end{vmatrix} \begin{vmatrix} \circ & \circ & \circ \\ \circ & \bullet & \bullet \\ \circ & \bullet & \bullet \end{vmatrix} - \begin{vmatrix} \circ & \circ & \circ \\ \bullet & \bullet & \circ \\ \bullet & \bullet & \circ \end{vmatrix} \begin{vmatrix} \circ & \bullet & \bullet \\ \circ & \bullet & \bullet \\ \circ & \circ & \circ \end{vmatrix}$$

where the white dots represent deleted entries, and the black dots initial entries.

Since the values are non zero and the 2×2-minors all equal to 1, the above identity implies that all the 3×3-minors vanish. □

4. The main theorem

In this section, we formulate our main result. The proof will be given in Section 6.

4.1. **Classification.**

It turns out that every SL_2-tiling corresponds to a pair of frieze patterns and a positive integer 2×2-matrix M satisfying some conditions.

Theorem 2. *The set of (n,m)-antiperiodic SL_2-tilings containing a fundamental rectangular domain of positive integers is in a one-to-one correspondence with the set of triples (q, q', M), where*

$$q = (q_0, \dots, q_{n-1}), \qquad q' = (q'_0, \dots, q'_{m-1})$$

are quiddities of order n and m, respectively, and where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a unimodular 2×2-matrix with positive integer coefficients, such that the inequalities

$$q_0 < \frac{b}{a}, \qquad q'_0 < \frac{c}{a} \tag{4.1}$$

are satisfied.

Remark 4.1.1. It is important to notice that inequalities (4.1) also imply

$$q_0 < \frac{d}{c}, \qquad q'_0 < \frac{d}{b}. \tag{4.2}$$

Indeed, the unimodular condition $ad - bc = 1$ and the assumption that a, b, c, d are positive integers imply that $\frac{b}{a} < \frac{d}{c}$ and $\frac{c}{a} < \frac{d}{b}$.

Corollary 4.1.2. *For every pair of quiddities q, q', there exist infinitely many (n,m)-antiperiodic SL_2-tilings containing a fundamental rectangular domain of positive integers.*

Proof. Given arbitrary pair of quiddities q and q', the matrices:

$$\begin{pmatrix} 1 & b \\ c & bc+1 \end{pmatrix}$$

satisfy (4.1) for sufficiently large b, c. □

4.2. **The semigroup $\mathcal{S}$.** Consider the set of 2×2-matrices with positive integral entries satisfying the following conditions of positivity:

$$\mathcal{S} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \begin{array}{l} 0 < a < b < d, \\ 0 < a < c < d \end{array} \right\}. \tag{4.3}$$

Note that the inequalities $b < d$ and $c < d$ are included for the sake of completeness. These inequalities actually follow from $a < b,\ a < c$ together with $ad - bc = 1$ and the assumption that a, b, c, d are positive.

We have the following property.

Proposition 4.2.1. *The set $\mathcal{S} \subset \mathrm{SL}_2(\mathbb{Z})$ is a semigroup, i.e., it is stable by multiplication.*

Proof. Straightforward. □

The semigroup $\mathcal{S}$ naturally appears in our context. Indeed, if $n, m \geq 3$, then the inequalities (4.1) imply $M \in \mathcal{S}$. Moreover every quiddity q contains a unit entry, so that after a cyclic permutation of any quiddity one can obtain $q_0 = 1$. The inequalities (4.1) then coincide with the conditions (4.3).

4.3. **Examples.** Let us give two simple examples of SL_2-tilings.

Example 4.3.1. There is a one-to-one correspondence between $(3, 3)$-antiperiodic SL_2-tilings containing a fundamental domain of positive integers and elements of the semigroup $\mathcal{S}$. Indeed, the only quiddity of order 3 is $q = (1, 1, 1)$. To every matrix (4.3) there corresponds the following SL_2-tiling:

$$\begin{array}{ccccc} & \vdots & \vdots & \vdots & \\ \cdots & a & b & b-a & \cdots \\ \cdots & c & d & d-c & \cdots \\ \cdots & c-a & d-b & d-b-c+a & \cdots \\ & \vdots & \vdots & \vdots & \end{array}$$

It is a good exercise to check that the positivity condition $d - b - c + a > 0$ follows from (4.3) together with $ad - bc = 1$.

Example 4.3.2. In the case $n = 2$ or $m = 2$, the conditions (4.1) become trivial.

Consider also the simplest (degenerate) case of $(2, 2)$-antiperiodic SL_2-tilings. A $(2, 2)$-antiperiodic SL_2-tiling containing a fundamental domain of positive integers is of the form:

$$\begin{array}{cccccc} & \vdots & \vdots & \vdots & \vdots & \\ \cdots & a & b & -a & -b & \cdots \\ \cdots & c & d & -c & -d & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{array}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an arbitrary unimodular matrix with positive integer coefficients. Note that this case corresponds to the "degenerate quiddity" of order 2, namely $q = (0, 0)$.

5. Frieze patterns and linear recurrence equations

We will recall here a remarkable and well-known property of Coxeter-Conway frieze patterns. It concerns a relation of frieze patterns and linear recurrence equations. The statement presented in this subsection was implicitly obtained in [6]; for details see [14]. We recall this statement without proof.

5.1. **Discrete non-oscillating Hill equations.**

Definition 5.1.1. Let $(c_i)_{i\in\mathbb{Z}}$ be an arbitrary n-periodic sequence of numbers.

(a) A linear difference equation

$$V_{i+1} = c_i V_i - V_{i-1}, \tag{5.1}$$

where the sequence (c_i) is given (the coefficients) and where (V_i) is unknown (the solution), is called a discrete Hill, or Sturm-Liouville, or one-dimensional Schrödinger equation.

(b) The equation (5.1) is called *non-oscillating* if every solution (V_i) is antiperiodic:

$$V_{i+n} = -V_i,$$

for all i, and has exactly one sign change in any sequence $(V_i, V_{i+1}, \dots V_{i+n})$.

In other words, every solution of a non-oscillating equation must have non-negative intervals of length n, that is, n consecutive non-negative values: $(V_k, \dots, V_{k+n-1})$.

Moreover, for *generic* solution of (5.1), all the elements V_j of a non-negative interval are *strictly positive*. Zero values can only occur at the endpoints: $V_k = 0$, or $V_{k+n-1} = 0$.

Note also that the coefficients in a non-oscillating equation are necessarily positive.

5.2. **Frieze patterns and difference equations.**

The relation between the equations (5.1) and Coxeter-Conway frieze patterns is as follows.

Proposition 5.2.1. *Given an equation* (5.1) *with integer coefficients, it is a non-oscillating equation if and only if the coefficients* $(c_0, c_1, \dots, c_{n-1})$ *form a quiddity.*

Proof. This is an immediate consequence of properties established by Coxeter and Conway. Indeed, it was proved in [7] (see also [6] property (17)) that the entries in any row of the pattern (extended by antiperiodicity) form a solution of an equation (5.1), where the coefficients c_i are given by the sequence on the first non-trivial diagonal. Thus, from an non-oscillating equation one can write down a frieze, and vice versa.

$$\begin{array}{cccccccccc}
\ddots & \ddots & & \ddots & \ddots & \ddots & & & & \\
 & 1 & c_0 & \cdots & 1 & 0 & -1 & \cdots & & \\
 & & 1 & c_1 & \cdots & 1 & 0 & -1 & \cdots & \\
 & & & 1 & c_2 & \cdots & 1 & 0 & -1 & \cdots \\
 & & & & \ddots & \ddots & & \ddots & \ddots & \ddots
\end{array}$$

Finally, the integer condition establish the correspondence with quiddities. □

Of course, for an arbitrary non-oscillating equation (5.1), the corresponding frieze pattern does not necessarily have integer entries. In [14], the space of frieze patterns and the space of non-oscillating equation (5.1) are identified in a more general setting.

Example 5.2.2. (a) The simplest quiddity $q = (1, 1, 1)$ corresponds to the non-oscillating equation with all $c_i = 1$. Every solution of this equation is 3-antiperiodic and can be obtained as a linear combination of the following two solutions:

$$(V_i^{(1)}) = (\ldots, 0, 1, 1, 0, -1, -1, \ldots), \qquad (V_i^{(2)}) = (\ldots, 1, 1, 0, -1, -1, 0 \ldots).$$

This corresponds to a degenerate frieze of Coxeter-Conway of width 0. (b) The frieze from Figure 1 corresponds to the non-oscillating equation with 7-antiperiodic solutions that are linear combinations of the following two:

$$(V_i^{(1)}) = (\ldots, 1, 2, 3, 1, 1, 1, 0, \ldots), \qquad (V_i^{(2)}) = (\ldots, 0, 1, 2, 1, 2, 3, 1, \ldots).$$

The above two solutions are exactly the first two rows of the frieze in Figure 1. One can of course choose different rows for a basis.

Note that, in the both cases, the basis solutions $(V_i^{(1)}), (V_i^{(2)})$ are not generic since they contain zeros.

6. Proof of Theorem 2

6.1. **The construction.** Given a triple (q, q', M) as in Theorem 2, we will construct an SL_2-tiling satisfying the above conditions. Define $T = (a_{i,j})$ using the following recurrence relations:

$$\begin{array}{lcl} a_{i,j+1} & := & q_j a_{i,j} - a_{i,j-1}, \\ a_{i+1,j} & := & q'_i a_{i,j} - a_{i-1,j}, \end{array} \tag{6.1}$$

for all $i, j \in \mathbb{Z}$, where the quiddities are periodically extended, i.e $q_i = q_{i+n}, q'_i = q'_{i+m}$, and taking the initial conditions

$$\begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.2}$$

It is very easy to check that the tiling T is well-defined, i.e., the two recurrences commute and the calculations along the rows and columns give the same result. We show that the defined tiling T contains a fundamental rectangular domain of positive integers.

By Proposition 5.2.1, the defined tiling T is (n, m)-antiperiodic. Consider the following $m \times n$-subarray of T

$$P = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ \cdots & & & \\ a_{m-1,0} & a_{m-1,1} & \cdots & a_{m-1,n-1} \end{pmatrix}. \tag{6.3}$$

The main step of the proof of Theorem 2 is the following lemma.

Lemma 6.1.1. *The entries of P are positive integers.*

Proof. It turns out that thanks to Proposition 5.2.1 we will only need to perform "local" calculation of the elements neighboring to the initial ones:

$$\begin{array}{c|cc} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ \hline a_{0,-1} & a & b \\ a_{1,-1} & c & d \end{array}$$

The conditions (4.1) imply: $a_{0,-1} < 0$ and $a_{-1,0} < 0$. Indeed, from (6.1) and (6.2), one has

$$a_{0,-1} = q_0 a - b, \qquad a_{-1,0} = q'_0 a - c.$$

Since the rows and the columns of P are solutions of non-oscillating equations, and a is positive, this implies that all the values of the first row and the first column of P are positive.

Furthermore, again from the recurrence (6.1), one has

$$a_{-1,-1} = q_0 q_0' a - q_0 c - q_0' b + d.$$

The condition (4.1) then implies $a_{-1,-1} > 0$. Indeed, one establishes

$$0 < q_0 = a q_0 (d - q_0' b) - b q_0 (c - q_0' a) < b(d - q_0' b) - b q_0 (c - q_0' a) = b(q_0 q_0' a - q_0 c - q_0' b + d).$$

Proposition 5.2.1 then guarantees that

$$\begin{array}{lll} a_{0,-1} < 0, & \ldots, & a_{m-1,-1} < 0, \\ a_{-1,0} < 0, & \ldots, & a_{-1,n-1} < 0, \end{array}$$

and applying again Proposition 5.2.1, we deduce that all the entries in P are positive. □

6.2. **From tilings to triples.** Conversely, consider a (n, m)-periodic SL$_2$-tiling $T = (a_{i,j})_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}$ such that the $m \times n$-subarray P given by (6.3) consists in positive integers. We claim that T can be obtained by the above construction.

Lemma 6.2.1. *The ratios of the first two rows of P form a decreasing sequence:*

$$\frac{a_{0,0}}{a_{1,0}} > \frac{a_{0,1}}{a_{1,1}} > \ldots > \frac{a_{0,n-1}}{a_{1,n-1}},$$

and similarly for the ratios of the first two columns of P:

$$\frac{a_{0,1}}{a_{0,0}} > \frac{a_{1,1}}{a_{1,0}} > \ldots > \frac{a_{m-1,1}}{a_{m-1,0}}.$$

Proof. This follows from the unimodular conditions $a_{0,j} a_{1,j+1} - a_{0,j+1} a_{1,j} = 1$ and the assumption that all the entries of P are positive. □

Lemma 6.2.2. *The entries of T satisfy the recurrence relations (6.1) where $q = (q_j)$ and $q' = (q_i')$ are n-periodic and m-periodic sequences of positive integers, respectively.*

Proof. Given (i, j), there is a linear relation

$$\begin{pmatrix} a_{i,j+1} \\ a_{i+1,j+1} \end{pmatrix} = \lambda_{i,j} \begin{pmatrix} a_{i,j} \\ a_{i+1,j} \end{pmatrix} + \mu_{i,j} \begin{pmatrix} a_{i,j-1} \\ a_{i+1,j-1} \end{pmatrix}.$$

Using the SL$_2$ conditions one immediately obtains the values

$$\lambda_{i,j} = a_{i,j-1} a_{i+1,j+1} - a_{i,j+1} a_{i+1,j-1}, \qquad \mu_{i,j} = -1.$$

From Lemma 6.2.1, one has $\lambda_{i,j} > 0$. Furthermore, it readily follows from the tameness property (see Proposition 3.4.1) that $\lambda_{i,j}$ actually does not depend on i, so we use the notation $q_j := \lambda_{i,j}$.

The arguments for the rows are similar. □

Lemma 6.2.3. *The above sequences $(q_0, \ldots, q_{m-1})$ and $(q_0', \ldots, q_{n-1})$ are quiddities.*

Proof. The rows, resp. columns, of T are antiperiodic solutions of an equation (5.1) with $c_i = c_{i+n} = q_i$, resp. $c_i = c_{i+m} = q_i'$. It follows from Proposition 5.2.1 that the coefficients are quiddities. □

Lemma 6.2.4. *The 2×2 left upper block of P, satisfies*

$$\begin{array}{rcl} q_0\, a_{0,0} & < & a_{0,1}, \\ q_0'\, a_{0,0} & < & a_{1,0}. \end{array}$$

Proof. By antiperiodicity, $a_{0,-1} < 0$. One has from (6.1): $a_{0,1} = q_0\, a_{0,0} - a_{0,-1}$, and similarly for q'_0. Hence the result. □

In other words, the elements of the matrix

$$\begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfy (4.1).

Theorem 2 is proved.

7. SL_2-tilings and the Farey graph

In this section, we give an interpretation of the entries $a_{i,j}$ of a doubly periodic SL_2-tiling. We follow the idea of Coxeter [7] and consider n-gons in the classical Farey graph.

7.1. **The distance between two n-gons.**

Consider a doubly periodic SL_2-tiling $T = (a_{i,j})$ and the corresponding triple (q, q', M) (see Theorem 2). Our next goal is to give an explicit expression for the numbers $a_{i,j}$ similar to (2.5).

From the triple (q, q', M) we construct the unique n-gon $(v_0, v_1, \ldots, v_{n-1})$ and the unique m-gon $(v'_0, v'_1, \ldots, v'_{m-1})$ with the "initial" conditions:

$$(v_0, v_1) := \left(\tfrac{a}{c},\ \tfrac{b}{d}\right), \qquad \left(v'_0, v'_{m-1}\right) := \left(\tfrac{1}{0},\ \tfrac{0}{1}\right),$$

and with the quiddities $(q_0, \ldots, q_{n-1})$ and $(q'_1, \ldots, q'_m)$, respectively. Notice that the quiddity q' is shifted cyclically.

Theorem 3. *The entries of the* SL_2*-tiling* $T = (a_{i,j})$ *are given by*

$$a_{i,j} = d(v'_{i-1}, v_j),$$

for all $0 \leq i \leq m-1,\ 0 \leq j \leq n-1$.

Proof. The main idea of the proof is to include the n-gon v and the m-gon v' into a bigger N-gon in a Farey graph, and then apply Eq. (2.5). In other words, we will include the fundamental domain P into a (bigger) frieze pattern.

First, let us show that

$$v'_{m-2} > v_0 > v_1 > \ldots > v_{n-1} > v'_{m-1}.$$

Indeed, the vertices v'_{m-2}, v'_{m-1}, v'_0 are consecutive vertices of the m-gon v'. By assumption, $v'_{m-1} = \frac{0}{1}$, so that the condition

$$d(v'_{m-2}, v'_{m-1}) = 1$$

implies $v'_{m-2} = \frac{1}{\ell}$ for some ℓ. By Lemma 2.2.2, the distance $d(v'_0, v'_{m-2})$ coincides with the number of triangles at the vertex v'_{m-1} which is, by construction, equal to q'_0. We finally have:

$$d(v'_0, v'_{m-2}) = \ell = q'_0,$$

so that $v'_{m-2} = \frac{1}{q'_0}$. The inequality $v'_{m-1} > v_0$ then follows from the second inequality (4.1).

It is well-known that the Farey graph is connected; see [10]. Therefore, two disjoint polygons, v and v', belong to some N-gon that contain the n-gon v and the m-gon v'.

Theorem 3 then follows from formula (2.5). □

Example 7.1.1. Consider the tiling given in Figure 2. It corresponds to the following data

$$q = (1,2,2,1,3), \qquad q' = (2,1,2,1), \qquad M = \begin{pmatrix} 2 & 5 \\ 7 & 18 \end{pmatrix}.$$

The associated 5-gon and 4-gon in the Farey graph are as follows:

$$v = \left(\frac{2}{7}, \frac{5}{18}, \frac{8}{29}, \frac{11}{40}, \frac{3}{11}\right), \quad \text{and} \quad v' = \left(\frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{0}{1}\right),$$

respectively. They can be included in an 11-gon; see Figure 6.

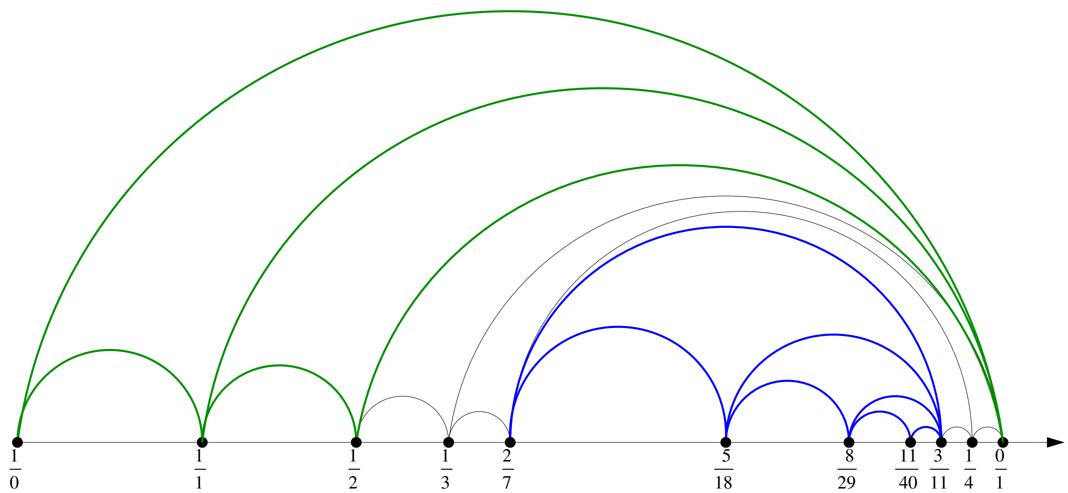

FIGURE 6. The subgraph associated with the tiling in Figure 2

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Sophie Morier-Genoud, Sorbonne Universités, UPMC Univ Paris 06, UMR 7586, Institut de Mathématiques de Jussieu- Paris Rive Gauche, Case 247, 4 place Jussieu, F-75005, Paris, France

Valentin Ovsienko, CNRS, Laboratoire de Mathématiques U.F.R. Sciences Exactes et Naturelles Moulin de la Housse - BP 1039 51687 REIMS cedex 2, France

Serge Tabachnikov, Pennsylvania State University, Department of Mathematics, University Park, PA 16802, USA

E-mail address: sophie.morier-genoud@imj-prg.fr, valentin.ovsienko@univ-reims.fr, tabachni@math.psu.edu