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Representations of asl_2

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We study representations of the algebra asl_2 introduced in [5]. We show that representations of asl_2 form a subclass of representations of the Lie superalgebra $\mathrm{osp}(1|2)$ characterized by the property $C = 0$, where C is the Casimir element of the universal enveloping algebra $\mathcal{U}(\mathrm{osp}(1|2))$. We deduce that asl_2 has no nontrivial finite-dimensional representations and classify infinite-dimensional weighted (Harish-Chandra) representations. It turns out that asl_2 has exactly one highest (and one lowest) weight representation.

1 Introduction and Main Results

Lie antialgebras form a new class of algebras introduced by V. Ovsienko [5]. These algebras appear naturally in the context of symplectic and contact geometry of $\mathbb{Z}_2$-graded spaces, but their algebraic properties are not yet well understood. They exhibit a surprising mixture of the properties of commutative algebras and Lie algebras.

One of the first example of Ovsienko's algebras is the simple Lie antialgebra $\mathrm{asl}_2(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. This algebra is of dimension 3 and has linear basis $\{\varepsilon; a, b\}$ subject to the following relations:

$$
\begin{aligned}
&\varepsilon\cdot\varepsilon = \varepsilon,\\
&\varepsilon\cdot a = a\cdot\varepsilon = \tfrac{1}{2}\,a, \quad \varepsilon\cdot b = b\cdot\varepsilon = \tfrac{1}{2}\,b,\\
&a\cdot b = -b\cdot a = \tfrac{1}{2}\,\varepsilon, \quad a\cdot a = b\cdot b = 0.
\end{aligned}
\tag{1.1}
$$

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Representations of a Lie antialgebras were also defined in [5], where the problem of classification of representations of simple Lie antialgebras was formulated. In this article, we study representations of $\mathrm{asl}_2(\mathbb{K})$.

Any representation of asl_2 is given by three operators: $\mathcal{E}$, A, and B, satisfying

$$\begin{aligned} AB - BA &= \mathcal{E}, \\ A\mathcal{E} + \mathcal{E}A &= A, \\ B\mathcal{E} + \mathcal{E}B &= B, \\ \mathcal{E}^2 &= \mathcal{E}. \end{aligned} \tag{1.2}$$

These relations are very similar to the canonical Heisenberg relations $AB - BA = \mathrm{Id}$, but they are different. In particular, the operator $\mathcal{E}$ is not the identity. We will prove the following.

Proposition 1.1. Any irreducible representation of the relations (1.2) is equivalent to a representation $V = V_0 \oplus V_1$ such that

$$\mathcal{E}|_{V_0} = 0, \quad \mathcal{E}|_{V_1} = \mathrm{Id}. \qquad \square$$

This provides a natural $\mathbb{Z}_2$-grading in which the operators A and B are odd. We then obtain a result similar to the classical result for the Heisenberg algebra.

Theorem 1. The Lie antialgebra $\mathrm{asl}_2(\mathbb{K})$ has no nontrivial finite-dimensional representations. $\square$

We introduce and classify infinite-dimensional representations of $\mathrm{asl}_2(\mathbb{K})$, which we call *weighted representations*. We construct a family of weighted representations $V(\ell)$, for $\ell \in \mathbb{K}$ (see Section 4.2). Considering the set of parameters $\mathcal{P} = [-1, 1]$ in the real case, or $\mathcal{P} = [-1, 1] \cup \{\ell \in \mathbb{C} \mid -1 \leq \mathrm{Re}(\ell) < 1\}$ in the complex case, we obtain the complete classification of irreducible weighted representations.

Theorem 2. Any irreducible weighted representation is isomorphic to $V(\ell)$ for a unique $\ell \in \mathcal{P}$. $\square$

In particular, we obtain the following statement.

Corollary 1.2. The algebra asl_2 has exactly one nontrivial highest weight representation and exactly one nontrivial lowest weight representation. □

Indeed, the highest and lowest weight representations are the representations $V(-1)$ and $V(1)$, respectively.

The Lie antialgebra $\mathrm{asl}_2(\mathbb{K})$ is closely related to the simple classical Lie superalgebra $\mathrm{osp}(1|2)$. One has

$$\mathrm{osp}(1|2) = \mathrm{Der}(\mathrm{asl}_2(\mathbb{K})).$$

It was shown in [5] that every representation of $\mathrm{asl}_2(\mathbb{K})$ is naturally a representation of $\mathrm{osp}(1|2)$. The first result of this article determines the class of $\mathrm{osp}(1|2)$-representations corresponding to $\mathrm{asl}_2(\mathbb{K})$-representations.

Theorem 3. There is a one-to-one correspondence between representations of $\mathrm{asl}_2(\mathbb{K})$ and representations of $\mathrm{osp}(1|2)$ satisfying one of the following equivalent conditions:

$$C = 0, \tag{1.3}$$

where C is the Casimir element of $\mathcal{U}(\mathrm{osp}(1|2))$;

$$\Gamma^2 = \tfrac{1}{4}\,\mathrm{Id}, \tag{1.4}$$

where Γ is the ghost Casimir element. □

Recall that the ghost Casimir element of $\mathcal{U}(\mathrm{osp}(1|2))$ is an invariant of so-called *twisted adjoint action* (see [1, 2, 4, 6]).

Let us mention that [3] contains a classification of the irreducible Harish-Chandra modules of $\mathrm{osp}(1|2)$. This classification will be relevant for this article (cf. Section 4.8) as application of Theorem 3.

We also define the notion of the *universal enveloping algebra* $\mathcal{U}(\mathfrak{a})$ associated to an arbitrary Lie antialgebra. We describe explicitly the algebra $\mathcal{U}(\mathrm{asl}_2(\mathbb{K}))$ and relate it to a quotient of $\mathcal{U}(\mathrm{osp}(1|2))$. A more general treatment of enveloping algebras of Lie antialgebras will be given in a subsequent article.

This article is organized as follows. In Section 2, we recall the general definitions of Lie antialgebras and their representations. In Section 3, we obtain preliminary results on representations of $\mathrm{asl}_2(\mathbb{K})$ and define the universal enveloping algebra $\mathcal{U}(\mathrm{asl}_2(\mathbb{K}))$; a link with the Lie algebra $\mathrm{osp}(1|2)$ and its (ghost) Casimir elements is established. We complete the proof of Theorem 3 in Section 3.4. In Section 4, we introduce the notion of weighted representations and construct the family of irreducible weighted representations $V(\ell)$, $\ell \in \mathbb{K}$; we study the representations $V(\ell)$ and complete the proofs of Theorem 1 and Corollary 1.2. At the end of the article, we discuss some general aspects of the representation theory of $\mathrm{asl}_2(\mathbb{K})$, such as the tensor product of two representations.

2 Lie Antialgebras and Their Representations

Let us give a definition of a Lie antialgebra equivalent to the original definition given in [5]. Throughout this article the ground vector field is $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. A Lie antialgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$, equipped with a bilinear product satisfying the following conditions:

(1) it is even: $\mathfrak{a}_i \cdot \mathfrak{a}_j \subset \mathfrak{a}_{i+j}$;

(2) it is supercommutative, i.e. for all homogeneous elements $x, y \in \mathfrak{a}$,

$$x \cdot y = (-1)^{p(x)p(y)}\, y \cdot x,$$

where p is the parity function defined by $p(x) = i$ for $x \in \mathfrak{a}_i$;

(3) the subspace $\mathfrak{a}_0$ is a commutative associative algebra;

(4) for all $x_1, x_2 \in \mathfrak{a}_0$ and $y \in \mathfrak{a}_1$, one has

$$x_1 \cdot (x_2 \cdot y) = \tfrac{1}{2}(x_1 \cdot x_2) \cdot y,$$

in other words, the subspace $\mathfrak{a}_1$ is a module over $\mathfrak{a}_0$, the homomorphism $\varrho : \mathfrak{a}_0 \to \mathrm{End}(\mathfrak{a}_1)$ being given by $\varrho_x\, y = 2\, x \cdot y$ for all $x \in \mathfrak{a}_0$ and $y \in \mathfrak{a}_1$;

(5) for all $x \in \mathfrak{a}_0$ and $y_1, y_2 \in \mathfrak{a}_1$, the Leibniz identity

$$x \cdot (y_1 \cdot y_2) = (x \cdot y_1) \cdot y_2 + y_1 \cdot (x \cdot y_2)$$

is satisfied;

(6) for all $y_1, y_2, y_3 \in \mathfrak{a}_1$, the Jacobi-type identity

$$y_1 \cdot (y_2 \cdot y_3) + y_2 \cdot (y_3 \cdot y_1) + y_3 \cdot (y_1 \cdot y_2) = 0$$

is satisfied. □

Example 2.2. It is easy to see that the above axioms are satisfied for $\mathrm{asl}_2(\mathbb{K})$. In this case, the element ε spans the even part, $\mathrm{asl}_2(\mathbb{K})_0$, while the elements a, b span the odd part, $\mathrm{asl}_2(\mathbb{K})_1$. □

Given a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$, the space $\mathrm{End}(V)$ of linear endomorphisms of V is a $\mathbb{Z}_2$-graded associative algebra:

$$\mathrm{End}(V)_0 = \mathrm{End}(V_0) \oplus \mathrm{End}(V_1), \quad \mathrm{End}(V)_1 = \mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0).$$

Following [5], we define the following "anticommutator" on $\mathrm{End}(V)$:

$$]X, Y[:= X\,Y + (-1)^{p(X)\,p(Y)}\,Y\,X, \tag{2.5}$$

where p is the parity function on $\mathrm{End}(V)$ and $X, Y \in \mathrm{End}(V)$ are homogeneous (purely even or purely odd) elements. Note that the sign rule in (2.5) is opposite to that of the usual commutator:

$$[X, Y] = X\,Y - (-1)^{p(X)\,p(Y)}\,Y\,X. \tag{2.6}$$

Remark 2.3. Let us stress that the operation (2.5) does *not* define a Lie antialgebra structure on the full space $\mathrm{End}(V)$; it is not known for which subspaces of $\mathrm{End}(V)$ this is the case. This operation provides, however, a definition of the notion of representation of a Lie antialgebra. □

Definition 2.4. (a) A *representation* of a Lie antialgebra $\mathfrak{a}$ is a pair (V, χ) where $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space and $\chi : \mathfrak{a} \to \mathrm{End}(V)$ is an even linear map such that

$$]\chi_x, \chi_y[= \chi_{x \cdot y} \tag{2.7}$$

for all $x, y \in \mathfrak{a}$.

(b) A *subrepresentation* is a $\mathbb{Z}_2$-graded subspace $V' \subset V$ stable under χ_x for all $x \in \mathfrak{a}$.

(c) A representation is called *irreducible* if it does not have proper subrepresentations.

(d) Two representations (V, χ) and (V', χ') are called *equivalent* if there exists a linear isomorphism $\Phi : V \to V'$ such that $\Phi \circ \chi_x = \chi'_x \circ \Phi$, for every $x \in \mathfrak{a}$. □

Remark 2.5. There is a notion of a module over a Lie antialgebra, which is different from that of a representation. For instance, the "adjoint action" defined as usual by $\mathrm{ad}_x\, y = x \cdot y$ is *not* a representation, but it does define an $\mathfrak{a}$-module structure on $\mathfrak{a}$. □

Given a Lie antialgebra $\mathfrak{a}$, it was shown in [5] that there exists a Lie superalgebra, $\mathfrak{g}_{\mathfrak{a}}$, canonically associated to $\mathfrak{a}$. Every representation of $\mathfrak{a}$ extends to a representation of $\mathfrak{g}_{\mathfrak{a}}$. In the case of $\mathrm{asl}_2(\mathbb{K})$, the corresponding Lie superalgebra is the classical simple Lie antialgebra $\mathrm{osp}(1|2)$. We will give the explicit construction of this Lie superalgebra in the next section and use it as the main tool for our study.

3 Representations of $\mathrm{asl}_2(\mathbb{K})$ and the (Ghost) Casimir of osp(1|2)

In this section we provide general information about the representations of $\mathrm{asl}_2(\mathbb{K})$, and prove Proposition 1.1. We also introduce the action of $\mathrm{osp}(1|2)$ and prove Theorem 3.

3.1 Generators of $\mathrm{asl}_2(\mathbb{K})$ and the $\mathbb{Z}_2$-grading

Consider an $\mathrm{asl}_2(\mathbb{K})$-representation $V = V_0 \oplus V_1$ with $\chi : \mathrm{asl}_2(\mathbb{K}) \to \mathrm{End}(V)$. The homomorphism condition (2.7) can be written explicitly in terms of the basis elements:

$$\begin{cases} \chi_a \chi_b - \chi_b \chi_a = \frac{1}{2}\, \chi_\varepsilon \\ \chi_a \chi_\varepsilon + \chi_\varepsilon \chi_a = \frac{1}{2}\, \chi_a \\ \chi_b \chi_\varepsilon + \chi_\varepsilon \chi_b = \frac{1}{2}\, \chi_b \\ \chi_\varepsilon \chi_\varepsilon = \frac{1}{2} \chi_\varepsilon. \end{cases}$$

Let us simplify the notation by fixing the following elements of $\mathrm{End}(V)$:

$$A = 2\, \chi_a, \quad B = 2\, \chi_b, \quad \mathcal{E} = 2\, \chi_\varepsilon.$$

The above relations are then equivalent to (1.2).

The element $\mathcal{E}$ is a projector in V. This leads to a decomposition of V into eigenspaces $V = V^{(0)} \oplus V^{(1)}$ defined by

$$V^{(\lambda)} = \{ v \in V \mid \mathcal{E} v = \lambda\, v \}, \quad \lambda = 0, 1.$$

This decomposition is not necessarily the same as the initial one, $V = V_0 \oplus V_1$. Since V_i, where $i = 0, 1$, is stable under the action of $\mathcal{E}$, we obtain a refinement

$$V = V_0^{(0)} \oplus V_0^{(1)} \oplus V_1^{(0)} \oplus V_1^{(1)},$$

where

$$V_i^{(\lambda)} = \{v \in V_i \mid \mathcal{E}v = \lambda\, v\}, \quad \lambda = 0, 1, \quad i = 0, 1.$$

We are now ready to prove Proposition 1.1. Using the relations (1.2) it is easy to see that A and B send the spaces $V_i^{(\lambda)}$ into $V_{1-i}^{(1-\lambda)}$, where $\lambda = 0, 1,\ i = 0, 1$. Thus, changing the $\mathbb{Z}_2$-grading of V to $V' = V_0' \oplus V_1'$, where

$$\begin{aligned} V_0' &= V_0^{(0)} \oplus V_1^{(0)}, \\ V_1' &= V_0^{(1)} \oplus V_1^{(1)} \end{aligned}$$

does not change the parity of the operators A, B, and $\mathcal{E}$ viewed as elements of the Z_2-graded space $\mathrm{End}(V')$. In other words, the map $\chi' : \mathfrak{a} \to \mathrm{End}(V')$ defined by $\chi'_x = \chi_x$ for all $x \in \mathfrak{a}$, is still an even map satisfying the condition 2.7. Consequently, (V', χ') is also a representation. It is then clear that (V', χ') is equivalent (in the sense of Definition 2.4 (d)) to (V, χ).

Proposition 1.1 is proved.

3.2 The action of osp(1|2)

The construction of this section is a special case of the general construction of [5], where, however, some proofs are missing (cf. Proposition 4.7). For the sake of completeness, we give here the complete proofs.

Given an $\mathrm{asl}_2(\mathbb{K})$-representation with generators A, B, we define operators E, F, and H by

$$E := A^2, \quad F := -B^2, \quad H := -(AB + BA). \tag{3.8}$$

Lemma 3.1. Given an $\mathrm{asl}_2(\mathbb{K})$-representation, the operators A, B and E, F, H span an action of the Lie superalgebra $\mathrm{osp}(1|2)$. □

Proof. Recall that the Lie superalgebra osp(1|2) contains three even generators E, F, H and two odd generators A, B satisfying the relations:

$$\begin{array}{lll} [H, E] = 2E, & [H, F] = -2F, & [E, F] = H, \\ [H, A] = A, & [E, A] = 0, & [F, A] = B, \\ [H, B] = -B, & [E, B] = A, & [F, B] = 0, \\ [A, B] = -H, & [A, A] = 2E, & [B, B] = -2F. \end{array} \tag{3.9}$$

These identities can be obtained by straightforward computations. For instance,

$$\begin{aligned} [H, E] &= -(AB + BA)A^2 + A^2(AB + BA) \\ &= -ABA^2 - BA^3 + A^3B + A^2BA \\ &= 2A(AB - BA)A + (AB - BA)A^2 + A^2(AB - BA) \\ &= 2A\mathcal{E}A + \mathcal{E}A^2 + A^2\mathcal{E} \\ &= (A\mathcal{E} + \mathcal{E}A)A + A(\mathcal{E}A + A\mathcal{E}) \\ &= A^2 + A^2 \\ &= 2E. \end{aligned}$$

In the same way, one obtains $[E, F] = H$, etc. ∎

With similar computations, one can establish the following additional relations:

$$[H, \mathcal{E}] = 0, \quad [E, \mathcal{E}] = 0, \quad [F, \mathcal{E}] = 0, \tag{3.10}$$

which will also be useful.

3.3 The universal enveloping algebra

Given a Lie antialgebra $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$, we define an associative $\mathbb{Z}_2$-graded algebra $\mathcal{U}(\mathfrak{a})$, which plays the role of *universal enveloping algebra* of $\mathfrak{a}$. Consider the tensor algebra

$$T(\mathfrak{a}) := \bigoplus_{n \geq 0} \mathfrak{a}^{\otimes n}$$

together with the natural $\mathbb{Z}_2$-grading. Denote by $\mathcal{R}$ the two-sided ideal of $T(\mathfrak{a})$ generated by

$$\{x \otimes y + (-1)^{p(x)p(y)} y \otimes x - xy\,, \quad x, y \in \mathfrak{a}_0 \cup \mathfrak{a}_1\}$$
$$\bigcup \{x \otimes y - y \otimes x\,, \quad x, y \in \mathfrak{a}_0\}.$$

We define

$$\mathcal{U}(\mathfrak{a}) := T(\mathfrak{a})/\mathcal{R}.$$

We believe this definition is natural since the properties of the algebra $\mathcal{U}(\mathfrak{a})$ are similar to the properties of the universal enveloping algebras associated to Lie algebras or superalgebras. The properties of $\mathcal{U}(\mathfrak{a})$ will be studied elsewhere.

In the case $\mathfrak{a} = \mathrm{asl}_2(\mathbb{K})$, one readily obtains:

Proposition 3.2. The universal enveloping algebra $\mathcal{U}(\mathrm{asl}_2(\mathbb{K}))$ is the associative algebra generated by three elements: A, B, and $\mathcal{E}$ satisfying the relations (1.2). □

One can show that $\mathcal{U}(\mathfrak{a})$ is, indeed, a quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathfrak{a})$ of the corresponding superalgebra $\mathfrak{g}_\mathfrak{a}$. In particular, the algebra $\mathcal{U}(\mathrm{asl}_2)$ can be viewed as a quotient of $\mathcal{U}(\mathrm{osp}(1|2))$. Recall that the algebra $\mathcal{U}(\mathrm{osp}(1|2))$ is the associative graded algebra generated by the even elements E, F, H and the odd elements A, B satisfying the relations 3.9.

Proposition 3.3. One has

$$\mathcal{U}(\mathrm{asl}_2) \simeq \mathcal{U}(\mathrm{osp}(1|2)) \big/ (\mathcal{E}^2 = \mathcal{E}),$$

where $\mathcal{E} = AB - BA$. □

Proof. Using computations similar to those in the proof of Lemma 3.1, one checks that the algebra $\mathcal{U}(\mathrm{osp}(1|2))$ admits the following alternative presentation: two odd generators A, B and one even generator $\mathcal{E}$ satisfying the three first relations of (1.2). ■

3.4 The twisted adjoint action and ghost Casimir element

The notion of the *twisted adjoint action* of Lie superalgebras was introduced in [2]. For X an element of $\mathfrak{g}$ and Y an element of $\mathcal{U}(\mathfrak{g})$,

$$\widetilde{\mathrm{ad}}_X Y := XY - (-1)^{p(X)(p(Y)+1)}\, YX. \tag{3.11}$$

Remarkably, $\widetilde{\mathrm{ad}}$ defines an $\mathrm{osp}(1|2)$-action on $\mathcal{U}(\mathrm{osp}(1|2))$.

The relation of the twisted adjoint action and representations of $\mathrm{asl}_2(\mathbb{K})$ is elucidated by the following elementary but important observation.

Lemma 3.4. If X is an odd element of $\mathrm{osp}(1|2)$, then the action (3.11) coincides with the anticommutator (2.5). □

The *ghost Casimir* elements are the invariants of the twisted adjoint action; see [2], and also [4]. In the case of $\mathrm{osp}(1|2)$, the ghost Casimir element is particularly simple:

$$\Gamma = AB - BA - \tfrac{1}{2}\,\mathrm{Id}. \tag{3.12}$$

It satisfies $\widetilde{\mathrm{ad}}_X\Gamma = 0$ for all $X \in \mathrm{osp}(1|2)$.

We are ready to prove Formula (1.4) of Theorem 3. The operator $\mathcal{E}$ and the ghost Casimir Γ are obviously related by

$$\Gamma = \mathcal{E} - \tfrac{1}{2}\,\mathrm{Id}.$$

It follows that the second and third relations in (1.2) are equivalent to $\widetilde{\mathrm{ad}}_A\Gamma = 0$ and $\widetilde{\mathrm{ad}}_B\Gamma = 0$, respectively, while the relation $\mathcal{E}^2 = \mathcal{E}$ is equivalent to $\Gamma^2 = \frac{1}{4}\,\mathrm{Id}$.

3.5 The classical Casimir elements

The operator $\mathcal{E}$ is also related to the usual Casimir elements C of $\mathcal{U}(\mathrm{osp}(1|2))$ and C_0 of $\mathcal{U}(\mathrm{sl}_2)$. Recall

$$C = EF + FE + \tfrac{1}{2}(H^2 + AB - BA),$$

$$C_0 = EF + FE + \tfrac{1}{2}H^2.$$

We easily see

$$\mathcal{E} = 2(C - C_0).$$

This implies that if V is an irreducible representation of $\mathrm{asl}_2(\mathbb{K})$, then $\mathcal{E}|_{V_0}$ and $\mathcal{E}|_{V_1}$ are proportional to Id.

Moreover, straightforward computation in $\mathcal{U}(\mathrm{osp}(1|2))$ gives the following relation (first obtained in [6])

$$4(C - C_0)^2 = 4C - 2C_0.$$

It follows that in $\mathcal{U}(\mathrm{osp}(1|2))$ the relation $\mathcal{E}^2 = \mathcal{E}$ is equivalent to $C = 0$.

Theorem 3 is proved.

4 Weighted Representations of $\mathbf{asl}_2(\mathbb{K})$

In this section, we introduce the notion of weighted representation of the Lie antialgebra $\mathrm{asl}_2(\mathbb{K})$. This class of representation is characterized by the property that the action of the Cartan element H of $\mathrm{osp}(1|2)$ has at least one eigenvector. We do not require *a priori* the eigenspaces to be finite-dimensional.

4.1 The definition

Let V be a representation of $\mathrm{asl}_2(\mathbb{K})$. We introduce the subspaces

$$V_\ell = \{v \in V \mid Hv = \ell\, v\}, \quad \ell \in \mathbb{K}.$$

When $V_\ell \neq \{0\}$, we call it the *weight space* of V with weight ℓ. We denote by $\Pi_H(V)$ the set of weights of the representation V. The following statement is straightforward.

Lemma 4.1. With the above notation,

(i) The element A (resp. B) maps V_ℓ into $V_{\ell+1}$ (resp. $V_{\ell-1}$).
(ii) The sum $\sum_{\ell\in\Pi_H(V)} V_\ell$ is direct in V.

(iii) The space

$$Wt(V) := \bigoplus_{\ell \in \Pi_H(V)} V_\ell$$

is a subrepresentation of V. □

Corollary 4.2. If V is an irreducible representation, then either

$$Wt(V) = \{0\} \quad \text{or} \quad Wt(V) = V. \qquad \square$$

Definition 4.3. Any representation V of $\mathrm{asl}_2(\mathbb{K})$ such that $Wt(V) \neq \{0\}$, is called a *weighted representation*. □

4.2 The family of weighted representations $V(\ell)$

For every $\ell \in \mathbb{K}$, we construct an irreducible weighted representation of $\mathrm{asl}_2(\mathbb{K})$, which we denote by $V(\ell)$. This representation contains an odd vector e_1 such that $He_1 = \ell e_1$ and, by irreducibility, every element of $V(\ell)$ is a result of the (iterated) $\mathrm{asl}_2(\mathbb{K})$-action on e_1.

(a) **The case where ℓ is not an odd integer.** Consider a family of linearly independent vectors $\{e_k\}_{k\in\mathbb{Z}}$. We set $V(\ell) = \bigoplus_{k\in\mathbb{Z}} \mathbb{K}e_k$ and define the operators A and B on $V(\ell)$ by

$$\begin{aligned} A\, e_k &= e_{k+1}, \\ B\, e_k &= ((1-\ell)/2 - [k/2])e_{k-1}, \end{aligned}$$

where $k \in \mathbb{Z}$ and $[k/2]$ denotes the integral part of $k/2$. The operator $\mathcal{E}$ is defined by $\mathcal{E} = AB - BA$. Introduce the following $\mathbb{Z}_2$-grading on $V(\ell)$:

$$V(\ell)_0 = \bigoplus_{k \text{even}} \mathbb{K}e_k, \quad V(\ell)_1 = \bigoplus_{k \text{odd}} \mathbb{K}e_k. \tag{4.13}$$

It is easy to see that the operators A and B are odd operators with respect to this grading, whereas $\mathcal{E}$ is even.

Proposition 4.4. The space $V(\ell)$ is an $\mathrm{asl}_2(\mathbb{K})$-representation. □

Proof. By simple straightforward computations, we obtain:

$$A\mathcal{E} + \mathcal{E}A = A, \quad B\mathcal{E} + \mathcal{E}B = B.$$

Moreover, on the basis elements e_k of $V(\ell)$, one has $\mathcal{E}e_k = e_k$ if k is odd, and $\mathcal{E}e_k = 0$ if k is even, so $\mathcal{E}^2 = \mathcal{E}$. ■

It is easy to see that the basis elements e_k are weight vectors. Indeed, one checks

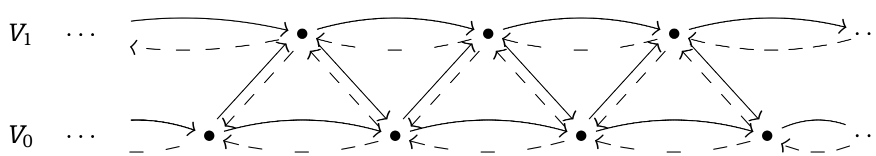
$$He_k = (\ell + k - 1)\, e_k, \quad k \in \mathbb{Z}. \tag{4.14}$$

In particular, the element e_1 is a weight vector of weight ℓ and generates the representation $V(\ell)$.

The actions on the basis elements e_k's can be pictured as follows:

$$\begin{tikzcd}
\mathbb{K}e_k \arrow[rr, bend left, "E"] \arrow[dr, "A"] & & \mathbb{K}e_{k+2} \arrow[ll, dashed, bend left, "F"] \arrow[dl, dashed, "B"] \\
& \mathbb{K}e_{k+1} \arrow[ul, dashed, "B"] \arrow[ur, "A"] &
\end{tikzcd}$$

The entire space $V(\ell)$ can be pictured as an infinite chain of the above diagrams.



(b) **Construction of $V(\ell)$ for ℓ a positive odd integer.** Consider a family of linearly independent vectors $\{e_k\}_{k\in\mathbb{Z}, k\geq 2-\ell}$. We set $V(\ell) = \bigoplus_{k\geq 2-\ell} \mathbb{K}e_k$, and we define the operators A and B on $V(\ell)$ by similar formulas:

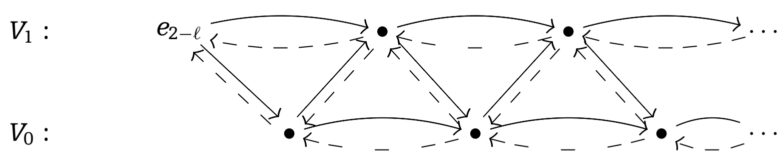
$$\begin{aligned}
A\,e_k &= e_{k+1}, \quad k \geq 2-\ell, \\
B\,e_k &= ((1-\ell)/2 - [k/2])e_{k-1}, \quad k > 2-\ell, \\
B\,e_{2-\ell} &= 0.
\end{aligned}$$

The operator $\mathcal{E}$ is again determined by $\mathcal{E} = AB - BA$. The $\mathbb{Z}_2$-grading on $V(\ell)$ is defined by the same formula (4.13). The result of Proposition 4.4 holds true.

The element e_1 is an odd weight vector of weight ℓ, which generates the representation $V(\ell)$. However, the vector $e_{2-\ell}$ is more interesting.

Definition 4.5. A lowest (resp. highest) weight representation is one that contains a weight vector v, such that $Bv = 0$ (resp. $Av = 0$) and the vectors $A^n v$ (resp. $B^n v$) span V; the vector v is called a lowest (resp. highest) weight vector. □

Clearly, the vector $e_{2-\ell}$ is a lowest weight vector of the representation $V(\ell)$ if ℓ a positive odd integer. One obtains the following diagram.

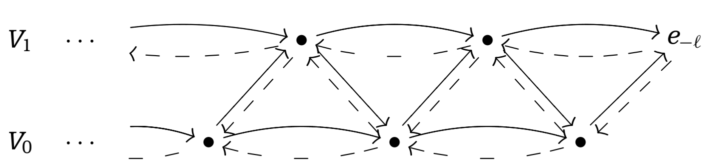


Viewed as a representation of osp(1|2), $V(\ell)$ is a Verma module.

(c) **Construction of $V(\ell)$ for ℓ a negative odd integer.** Consider a family of linearly independent vectors $\{e_k\}_{k\in\mathbb{Z},k\leq-\ell}$. We set $V(\ell) = \bigoplus_{k\leq-\ell}\mathbb{K}e_k$, and we define the operators A and B on $V(\ell)$ by

$$\begin{aligned}
Ae_k &= ((1-\ell)/2 + [k/2])e_{k+1}, \quad \forall k < -\ell,\\
Ae_{-\ell} &= 0,\\
Be_k &= e_{k-1}, \quad \forall k \leq -\ell.
\end{aligned}$$

As before, these operators define an $\mathrm{asl}_2(\mathbb{K})$-representation. The vector $e_{-\ell}$ is a highest weight vector of $V(\ell)$.



4.3 Geometric realization

It was shown in [5] that $\mathrm{asl}_2(\mathbb{K})$ has a representation in terms of vector fields on $1|1$-dimensional space. More precisely, consider $\mathscr{F} = C^{\infty}_{\mathbb{K}}(\mathbb{R})$, the set of $\mathbb{K}$-valued C^{∞}-functions of one real variable x. Introduce $\mathscr{A} = \mathscr{F}[\xi]/(\xi^2)$ with the $\mathbb{Z}_2$-grading $\mathscr{A}_0 = \mathscr{F}$, $\mathscr{A}_1 = \mathscr{F}\xi$. Define the vector field

$$\mathcal{D} = \frac{\partial}{\partial \xi} + \xi \, \frac{\partial}{\partial x}.$$

It is easy to check that the vector fields

$$A = \mathcal{D}, \quad B = x\mathcal{D}, \quad \mathcal{E} = \xi\, \mathcal{D} \tag{4.15}$$

satisfy the relations 1.2. Therefore they define an $\mathrm{asl}_2(\mathbb{K})$-action on $\mathscr{A}$.

Consider the function

$$e_1 = x^{\lambda}\, \xi,$$

where $\lambda \in \mathbb{K}$. It turns out that this function generates a weighted representation of $\mathrm{asl}_2(\mathbb{K})$, isomorphic to $V(-2\lambda - 1)$. Note that the case λ is an integer gives the highest and lowest weight irreducible representations.

4.4 Classification of weighted representations

The following statement shows that the representations $V(\ell)$ are, indeed, irreducible; it also classifies all the isomorphisms between these representations.

Proposition 4.6. (i) The representation $V(\ell)$ is irreducible for every $\ell \in \mathbb{K}$.

(ii) If ℓ and ℓ' are not odd integers, then $V(\ell) \simeq V(\ell')$ if and only if $\ell' - \ell = 2\,m$ for some $m \in \mathbb{Z}$.

(iii) If ℓ and ℓ' are odd integers, then $V(\ell) \simeq V(\ell')$ if and only if $\ell\ell' > 0$. □

Proof. (i) Suppose $V' \subset V(\ell)$ is a subrepresentation. For any nonzero $v \in V'$, write

$$v = \sum_{1 \le i \le N} \alpha_i e_{k_i},$$

with $\alpha_i \neq 0$, for all $1 \leq i \leq N$. Using (4.14), we obtain

$$H^p v = \sum_{1 \leq i \leq N} \alpha_i (\ell + k_i - 1)^p e_{k_i},$$

where $0 \leq p \leq N-1$. We may assume the coefficients $(\ell + k_i - 1)$ are nonzero, i.e. the basis elements occurring in the decomposition of v are not of weight zero.

The above equations form a linear system of Vandermonde type with distinct nonzero coefficients, and so they are solvable. We can express the elements e_{k_i} as linear combinations of $H^p v$. It follows that the vectors e_{k_i} are in V'. Applying A and B to e_{k_i}, we produce all the vectors $e_k, k \in \mathbb{Z}$. Hence, the basis vectors e_k are in V' for all $k \in \mathbb{Z}$. This implies $V' = V(\ell)$.

(ii) Denote by $\{e_k\}_k$ the standard basis of $V(\ell)$, and by $\{e'_k\}_k$ the standard basis of $V(\ell')$. Suppose there exists an isomorphism $\Phi : V(\ell') \to V(\ell)$. The vector $\Phi(e'_1)$ is a weighted vector (of weight ℓ') in $V(\ell)$. Since the weights in $V(\ell)$ are of the form $\ell + k$ for some $k \in \mathbb{Z}$ and the corresponding weight space is $\mathbb{K}e_{k+1}$, we have $\Phi(e'_1) = \alpha\, e_{k+1}$ for some $\alpha \in \mathbb{K}^*, k \in \mathbb{Z}$, and $\ell' = \ell + k$. Moreover, $\mathcal{E}\,\Phi(e'_1) = \Phi(e'_1)$, so e_{k+1} is an odd vector, i.e. k is even. This proves that the condition $\ell' = \ell + 2m$ for some $m \in \mathbb{Z}$ is necessary for $V(\ell) \simeq V(\ell')$.

Conversely, suppose that $\ell' = \ell + 2m$, for some $m \in \mathbb{Z}$. It is easy to check that the linear map $\Phi : V(\ell') \mapsto V(\ell)$ defined by

$$\Phi(e'_k) = e_{k+2m} \tag{4.16}$$

for all $k \in \mathbb{Z}$ is an isomorphism of representation.

(iii) If ℓ and ℓ' have opposite sign then $V(\ell)$ and $V(\ell')$ cannot be isomorphic, since on one of the spaces A acts injectively and on the other it has a nontrivial kernel. If ℓ and ℓ' have same sign then $\ell' - \ell = 2m$ and (4.16) is again an isomorphism. ■

4.5 The structure of weighted representations

In this section we establish two combinatorial lemmas. Consider a nontrivial irreducible representation V of $\mathrm{asl}_2(\mathbb{K})$. Recall that the $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$ corresponds to the decomposition with respect to the eigenvalues of $\mathcal{E}$ (see Lemma 1.1).

Lemma 4.7. If there exists a nonzero element $v \in V$ such that $Hv = \ell v, \ell \in \mathbb{K}$, then there exist nonzero elements $v_i \in V_i$ and $\ell_i \in \mathbb{K}$, $i = 0, 1$, such that $Hv_i = \ell_i v_i$. □

Proof. In the case that v is homogeneous, i.e. $v \in V_i$, we can choose for v_{1-i} the vector Av or Bv. Indeed, one has

$$\begin{aligned} HAv &= [H, A]v + AHv = Av + \ell Av = (\ell + 1)Av, \\ HBv &= [H, B]v + BHv = -Bv + \ell Av = (\ell - 1)Bv. \end{aligned} \tag{4.17}$$

Note that the two vectors Av and Bv are not simultaneously zero, otherwise $V = \mathbb{K}v$ is the trivial representation.

If v is not a homogeneous element, we can write $v = v_0 + v_1$ with $v_0 \neq 0 \in V_0$ and $v_1 \neq 0 \in V_1$. One then has

$$Hv = Hv_0 + Hv_1 \quad \text{and} \quad Hv = \ell v = \ell v_0 + \ell v_1.$$

Furthermore, Hv_0 is an element of V_0 and Hv_1 is an element of V_1, since the operator $H = -(AB + BA)$ is even. Therefore, by uniqueness of the writing in $V_0 \oplus V_1$, one has $Hv_0 = \ell v_0$ and $Hv_1 = \ell v_1$. ■

Lemma 4.8. If $v \in V_i$ is such that $Hv = \ell v$, then

(i) for all $k \geq 1$,

$$AB^k v = \lambda_k B^{k-1} v,$$

with $\lambda_k = [\frac{k-i}{2}] + \frac{i-\ell}{2}$, where $[\frac{k-i}{2}]$ is the integral part of $(k-i)/2$;

(ii) for all $k \geq 1$,

$$BA^k v = \mu_k A^{k-1} v,$$

with $\mu_k = -[\frac{k-i}{2}] - \frac{i+\ell}{2}$. □

Proof. We first establish the formulas for $k = 1$. On the one hand,

$$ABv = -Hv - BAv = -\ell v - BAv$$

and on the other hand,

$$ABv = \mathcal{E}v + BAv = iv + BAv.$$

By adding or subtracting these two identities, we deduce

$$2ABv = (i - \ell)v \quad \text{and} \quad 2BAv = (-i - \ell)v.$$

Thus, $\lambda_1 = (i-\ell)/2$ and $\mu_1 = (-i-\ell)/2$; so, the $k=1$ case is established.

By induction on k,

$$\begin{aligned} ABB^{k-1}v &= -HB^{k-1}v - BAB^{k-1}v \\ &= -(\ell-(k-1))B^kv - \lambda_{k-1}B^{k-1}v \\ &= (-\ell+k-1-\lambda_{k-1})B^{k-1}v. \end{aligned}$$

We deduce the relations

$$\begin{aligned} \lambda_k &= -\ell+k-1-\lambda_{k-1} \\ &= -\ell+k-1-(-\ell+(k-2)-\lambda_{k-2}) \\ &= 1+\lambda_{k-2}. \end{aligned}$$

Knowing $\lambda_1 = \frac{i-\ell}{2}$, we can now obtain the explicit expression of λ_k:

$$\lambda_k = \left[\frac{k-i}{2}\right] + \frac{i-\ell}{2}.$$

Hence part (i).

Part (ii) can be proved in a similar way. ■

4.6 Proof of Theorem 1

Let V be an irreducible finite-dimensional representation of $\mathrm{asl}_2(\mathbb{K})$. Under the actions of the elements $E = A^2$, $F = -B^2$ and H, the space V is a sl_2-module. Therefore, there exists a weight vector v such that $Hv = \ell\, v$ for some $\ell \in \mathbb{Z}$.

By lemma 4.7, we can assume that v is a homogeneous element: $v \in V_i, i = 0, 1$. Let us consider the family of vectors

$$\mathcal{F} = \{\dots, B^kv, \dots, Bv, v, Av, \dots, A^kv, \dots\}.$$

From formula (4.17) we know that all the nonzero vectors of $\mathcal{F}$ are eigenvectors of H with distinct eigenvalues, $\ell \pm k, k \in \mathbb{N}$. Therefore they are linearly independent.

Hence there exists $N \geq 1$ such that $B^{N-1}v \neq 0$ and $B^kv = 0$ for all $k \geq N$, and $M \geq 1$ such that $A^{M-1}v \neq 0$ and $A^kv = 0$ for all $k \geq M$. Using Lemma 4.8, we deduce $\lambda_N = 0$

and $\mu_M = 0$. This leads to

$$\left[\frac{N-i}{2}\right] + \frac{i-\ell}{2} = 0, \quad -\left[\frac{M-i}{2}\right] - \frac{i+\ell}{2} = 0.$$

Subtracting these equations, we obtain:

$$\left[\frac{N-i}{2}\right] + \left[\frac{M-i}{2}\right] + i = 0.$$

Since $N, M \geq 1$ and $i = 0, 1$, we conclude $N = M = 1$ and $i = 0$, so V is the trivial representation.

Finally, if V is an arbitrary finite-dimensional representation, then V is completely reducible, by the classical theorem in the $\mathrm{osp}(1|2)$-case.

Theorem 1 is proved.

4.7 Proof of Corollary 1.2

Lemma 4.9. $V(-1)$ is the only nontrivial irreducible highest weight representation of $\mathrm{asl}_2(\mathbb{K})$. □

Proof. Consider an irreducible representation V containing a weight vector v of weight ℓ such that $Av = 0$. We write $v = v_0 + v_1$ with $v_i \in V_i, i = 0, 1$. We also have $Av_i = 0$ and $Hv_i = \ell v_i$ for $i = 0, 1$.

We first show that $v_0 = 0$. Since $Av_0 = 0$, we get $Hv_0 = ABv_0 = -\mathcal{E}v_0 = 0$. The vector v_0 is a highest weight vector of weight 0 for the action of $\mathrm{sl}_2(\mathbb{K})$. In consequence, v_0 is also a lowest weight vector, i.e. $B^2 v_0 = 0$. Thus, the space $\mathrm{Span}\,(v_0, Bv_0)$ is stable, hence $v_0 = 0$.

Thus v belongs to V_1. Let us use Lemma 4.8, part (ii). From the relation $BAv = 0$, we deduce $\mu_1 = 0$ and thus $\ell = -1$. This implies that all the constants λ_k from Lemma 4.8, part (i) are nonzero. By induction we deduce, using Lemma 4.8, part (i), that all the vectors $B^k v, k \in \mathbb{N}$ are nonzero. Moreover, these vectors are linearly independent since they are eigenvectors for H associated to distinct eigenvalues. By setting $e_k = B^{1-k}v,\ k \in \mathbb{Z},\ k \leq 1$, we obtain $V(-1)$ as a subrepresentation of V. The irreducibility assumption then gives: $V \simeq V(-1)$. ■

In the same way, we show that $V(1)$ is the only nontrivial irreducible lowest weight representation of $\mathrm{asl}_2(\mathbb{K})$.

4.8 Complete classification

The following proposition, together with Proposition 4.6, provides a complete classification of weighted representations of $\mathrm{asl}_2(\mathbb{K})$.

We now prove Theorem 1.2.

Fix a weight vector $v \in V_1$ (such a vector exists by Lemma 4.7) of some weight $\ell \in \mathbb{K}$. Consider the family $\mathcal{F} = \{A^k v,\ B^k v,\ k \in \mathbb{N}\}$.

(a) Suppose that ℓ is not an odd integer. It is easy to see that the constants λ_k and $\mu_k, k \in \mathbb{N}$, from Lemma 4.8 never vanish. Indeed,

$$\lambda_k = 0 \Rightarrow \ell = 2\left[\frac{k-1}{2}\right] + 1, \quad \mu_k = 0 \Rightarrow \ell = -2\left[\frac{k-1}{2}\right] - 1. \tag{4.18}$$

By induction, we deduce that all the elements of $\mathcal{F}$ are nonzero. Moreover, they are eigenvectors of the operator H with distinct eigenvalues $\ell \pm k$, where $k \in \mathbb{N}$. Hence they are linearly independent. Setting

$$e_k = \begin{cases} A^{k-1}v, & k \geq 1 \\ B^{1-k}v, & k \leq 0, \end{cases}$$

we see that $V(\ell)$ is a subrepresentation of V. By irreducibility, $V \simeq V(\ell)$.

(b) Suppose that ℓ is a positive odd integer. From the first statement of 4.18, we deduce the existence of an integer $N \geq 1$ such that $\lambda_N = 0$ and $\lambda_k \neq 0$ for all $k < N$. Hence

$$B^k v \neq 0, \quad \forall\, k < N, \quad AB^N v = 0.$$

If $B^N v \neq 0$, then it is a highest weight vector. By Lemma 4.9, we obtain $V \simeq V(-1)$. But, in the highest weight representation $V(-1)$, the set of weights is the set of negative integers. We obtain a contradiction as v has positive weight. It follows that $B^N v = 0$, and so $B^{N-1}v$ is a lowest weight vector. We conclude $V \simeq V(1)$.

(c) Suppose finally that ℓ is a negative odd integer. Then similar arguments show: $V \simeq V(-1)$.

Theorem 1.2 is proved.

Remark 4.10. We have proven that any irreducible weighted representation is an irreducible Harish-Chandra representation (i.e. its weight spaces are all finite-dimensional). A classification of the irreducible Harish-Chandra representations of osp(1|2) over the field of complex numbers is given in [3]. The correspondence between the representations $V(\ell)$ and the representations given in Theorem 5.13 of [3] is the following:

(a) If $\ell \in \mathcal{P}^+$ is not an odd integer, then $V(\ell) \simeq \mathscr{D}(0, \ell/2)$;

(b) The lowest weight representation is $V(1) \simeq [-1/2] \downarrow$;

(c) The highest weight representation is $V(-1) \simeq [1/2] \uparrow$. □

Appendix: The Tensor Product of Two Representations

It is clear that the representations of $\mathrm{asl}_2(\mathbb{K})$ are not a tensor product category. Indeed, given two representations V and W of osp(1|2) with trivial action of the Casimir element,$V \otimes W$ does not have trivial Casimir element, and so by Theorem 3 it cannot be an$\mathrm{asl}_2(\mathbb{K})$-representation.

An attempt to define an action of $\mathrm{asl}_2(\mathbb{K})$ on $V \otimes W$ leads to a deformation of the asl_2-relations by the Casimir element of $\mathcal{U}(\mathrm{osp}(1|2))$. The algebraic meaning of this deformation is not yet clear.

The operators A and B have canonical lifts to $V \otimes W$ according to the Leibniz rule:

$$\widetilde{A} = A \otimes \mathrm{Id} + \mathrm{Id} \otimes A, \quad \widetilde{B} = B \otimes \mathrm{Id} + \mathrm{Id} \otimes B,$$

since they belong to the osp(1|2)-action. It is then natural to define the lift of operator $\mathcal{E}$ by $\widetilde{\mathcal{E}} := \widetilde{A}\widetilde{B} - \widetilde{B}\widetilde{A}$. One immediately obtains the explicit formula

$$\widetilde{\mathcal{E}} = \mathcal{E} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathcal{E} + 2(A \otimes B - B \otimes A).$$

The following statement is straightforward.

Proposition 4.11. The operators $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{\mathcal{E}}$ satisfy the following relations:

$$\begin{aligned} \widetilde{A}\widetilde{B} - \widetilde{B}\widetilde{A} &= \widetilde{\mathcal{E}}, \\ \widetilde{A}\widetilde{\mathcal{E}} + \widetilde{\mathcal{E}}\widetilde{A} &= \widetilde{A}, \\ \widetilde{B}\widetilde{\mathcal{E}} + \widetilde{\mathcal{E}}\widetilde{B} &= \widetilde{B}, \\ \widetilde{\mathcal{E}}^2 &= \widetilde{\mathcal{E}} + 4\bar{C}, \end{aligned} \tag{4.19}$$

where

$$\bar{C} = E \otimes F + F \otimes E + \tfrac{1}{2}(H \otimes H + A \otimes B - B \otimes A). \qquad \square$$

This means that two of the relations (1.2) are satisfied, but not the last asl_2-relation $\mathcal{E}^2 = \mathcal{E}$.

Let us recall that the element $C \in \mathcal{U}(\mathrm{osp}(1|2))$ given by

$$C = EF + FE + \tfrac{1}{2}(H^2 + AB - BA)$$

is nothing but the classical Casimir element. The operator $\bar{C}$ is the diagonal part of the standard lift of C to $V \otimes W$. In particular, the operator $\bar{C}$ commutes with the action of $\mathrm{asl}_2(\mathbb{K})$ and $\mathrm{osp}(1|2)$:

$$[\bar{C}, \widetilde{A}] = [\bar{C}, \widetilde{B}] = [\bar{C}, \widetilde{\mathcal{E}}] = 0.$$

This is how the Casimir operator of $\mathrm{osp}(1|2)$ appears in the context of representations of $\mathrm{asl}_2(\mathbb{K})$.

The relations (4.19) look like a “deformation” of the asl_2-relations (1.2) with one parameter that commutes with all the generators. It would be interesting to find a precise algebraic sense of this deformation.

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