The Knaster problem and the geometry of high-dimensional cubes

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Abstract

We study questions of the following type: *Given positive semi-definite matrix $\mathcal{G}$, does there exist a sequence of vectors in $\mathbb{R}^n$ whose Grammian equals to $\mathcal{G}$ and which has some specified additional properties (typically related to the sup norm)?* In particular, we show that the answer to the 1947 Knaster problem about real functions on spheres is negative for sufficiently large dimensions.

We consider the Euclidean space $\mathbb{R}^n$ endowed with the scalar product $\langle\cdot,\cdot\rangle$ and the induced norm $|\cdot|$. As usual, $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere and $B^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ is the unit ball. For $x = (x_j)_{j=1}^n \in \mathbb{R}^n$ we set $\|x\|_\infty := \max_{1\leq j\leq n} |x_j|$ and $\|x\|_1 := \sum_{j=1}^n |x_j|$. The same notation $\|\cdot\|_p$ shall be used for the norm on any $L_p(\mu)$-space. We shall write L_p for $L_p(0,1)$ and ℓ_p^n for $(\mathbb{R}^n, \|\cdot\|_p)$. For a sequence $\mathcal{Z} = (z_1,\ldots,z_p)$ in $\mathbb{R}^n$, $\mathcal{G}_\mathcal{Z} := [\langle z_i, z_k\rangle]_{i,k=1}^p$ is its Gram matrix.

In this note we consider problems of the following type: what are the conditions on $\mathcal{Z}$ so that there exists a sequence of elements $\mathcal{F} = (f_1,\ldots,f_p)$ in an $L_2(\mu)$-space for which $\mathcal{G}_\mathcal{F} = \mathcal{G}_\mathcal{Z}$ and which has some additional prescribed properties? [Note that, by an elementary argument, the condition $\mathcal{G}_\mathcal{F} = \mathcal{G}_\mathcal{Z}$ is equivalent to existence of a linear isometry σ from the linear span of $\mathcal{Z}$ into $L_2(\mu)$ such that $\sigma(z_j) = f_j$ for $j = 1,\ldots,p$.] Such questions arise naturally in various areas of analysis, geometry and applied mathematics, and serve as a motivation for two results presented here. The first one is related to a fundamental question about orthogonal systems and solves a problem that arose independently in control theory. The second result answers, in the negative and for sufficiently large dimensions, a problem about functions on spheres posed by Knaster in 1947.

The starting point of the first result is the following question: given sequence $\Phi = (\varphi_j)_{j=1}^p$ of functions in $L_2(-1,0)$ (where $p \in \mathbb{N}\cup\{\infty\}$), when can we extend it to an orthonormal sequence on $L_2(-1,1)$? An elementary argument leads to the Schur criterion: $I - \mathcal{G}_\Phi \geq 0$. If it is satisfied, one can choose as the extensions of φ_j's any sequence $\Psi = (\psi_j)_{j=1}^p$ in $L_2(0,1)$ for which $\mathcal{G}_\Psi = I - \mathcal{G}_\Phi$. Olevskiĭ ([12], p. 58) asked whether it is possible to additionally require that the functions ψ_j be uniformly bounded. This question makes sense also for finite systems and can be rephrased as

Does there exist $C > 0$ such that, for any $p \in \mathbb{N}$ and for any positive semi-definite $p\times p$ matrix $\mathcal{G}$ verifying $0 \leq \mathcal{G} \leq I$, there is a sequence $\mathcal{F} = (f_1,\ldots,f_p)$ in L_∞ such that $\mathcal{G}_\mathcal{F} = \mathcal{G}$ and $\max_{1\leq j\leq p} \|f_j\|_\infty \leq C$?

This problem has been investigated already in [10] and solved, in the affirmative, under some additional (rather strong) assumptions. We point out that statements similar to the one above are closely related to several deep facts from functional analysis. For example, it follows from the Grothendieck theorem and the Pietsch factorization theorem that, given sequence $\mathcal{V} = (v_1, \ldots, v_p)$ in B^n there is an $\mathcal{F} = (f_1, \ldots, f_p)$ in L_∞ with $\|f_j\|_\infty \leq \sqrt{\pi/2}$ for $j = 1, \ldots, p$ and such that $\mathcal{G}_\mathcal{V} \leq \mathcal{G}_\mathcal{F}$ (see, e.g., [13], Theorem 5.10). Consequently, $\mathcal{G}_\mathcal{V} = \mathcal{G}_\mathcal{F} - \Delta$, where Δ is positive semi-definite. However, as can be easily seen (e.g., from Lemma 3 below), one can not, in general, have $\mathcal{G}_\mathcal{V} = \mathcal{G}_\mathcal{F}$ with $\|f_j\|_\infty$ bounded by a universal constant. A natural "next best try" is to aim for $\mathcal{G}_\mathcal{V} = \mathcal{G}_\mathcal{F} - \Delta$ with Δ diagonal; this question was also posed by Megretski (see [9]). However, this is not possible, either; we have

Theorem 1 *Given $p \in \mathbb{N}$ there exist vectors $v_1, \ldots, v_p$ in a Euclidean space with $|v_j| \leq 1$ for $j = 1, \ldots, p$ such that whenever $f_1, \ldots, f_p \in L_\infty$ verify $\langle f_i, f_k \rangle = \langle v_i, v_k \rangle$ for all $1 \leq i < k \leq p$, then*

$$\max_{1 \leq j \leq p} \|f_j\|_\infty \geq c (\log p)^{1/4},$$

where $c > 0$ is a universal constant.

Sketch of the construction. Let $d \in \mathbb{N}$ and let $u_1, \ldots, u_m$ be 1-net of S^{d-1} (i.e., if every $u \in S^{d-1}$ is within 1 of one of the u_j's); by a standard volumetric argument, it is possible to achieve that with $m \leq 2(4/\sqrt{3})^d$. The sequence (v_j) is constructed by repeating each u_j s times, where s is exponential in d. One shows then that the conditions $\langle f_i, f_k \rangle = \langle v_i, v_k \rangle$ for $i \neq k$ and $\|f_j\|_\infty \leq K$ for $1 \leq j \leq p := ms$ are inconsistent if $K/(\log p)^{1/4}$ is small enough. The details of the argument will be presented in [6]. □

Let us point out that if we replace quadratic expressions by bilinear ones in the considerations above, the situation is quite different. For example, the following statement is true, and is easily seen to be equivalent to the famous Grothendieck inequality. [Cf. [3] and see Appendix for details and a derivation of the statement from the existence of very large nearly Euclidean subspaces of ℓ_1^n.]

There exists $C > 0$ such that, for any $p, n \in \mathbb{N}$ and for any sequences $(u_1, \ldots, u_p)$, $(v_1, \ldots, v_p)$ in B^n there exist sequences $(f_1, \ldots, f_p)$, $(g_1, \ldots, g_p)$ in $L_\infty(0,1)$ such that $\langle f_i, g_k \rangle = \langle u_i, v_k \rangle$, $\|f_i\|_\infty \leq C$ and $\|g_k\|_\infty \leq C$ for all $1 \leq i, k \leq p$.

The remainder of the note will be devoted to the Knaster problem, for which we shall provide a complete, even if not fully optimized argument. The problem, as stated in the *New Scottish Book* in 1946 and in the 1947 note [7], asks

Given a continuous function $F : S^{n-1} \to \mathbb{R}^m$ and a configuration of $p = n - m + 1$ points $q_1, \ldots, q_p$ in S^{n-1}, is there a rotation $U \in SO(n)$ such that $F(Uq_1) = \ldots = F(Uq_p)$?

Special cases of the problem go back to earlier inquiries by Steinhaus and Rademacher, cf. [4]. It has been since determined that the answer to the Knaster problem is negative for $m > 2$ and for some values of n for $m = 2$; see [8] for the first such result and [1] for an update and references. On the other hand, in the central case of real-valued functions (i.e., $m = 1$ and $p = n$) positive partial results were obtained: $n = 2$ is elementary, $n = 3$ was shown in [2]; the answer is also positive if the points $q_1, \ldots, q_p$ form an orthonormal sequence ([4], [15]) or, more generally, an orbit of an abelian group of rotations. Let us also recall that the case $m = 1$ of the problem, if true, would provide an alternate proof of Dvoretzky theorem on almost spherical sections of convex bodies with very good dependence on parameters (cf. [11]). However, we show here that, for sufficiently large n, the answer to the Knaster problem is negative also for $m = 1$.

We start the proof by introducing some more notation. If $d, n \in \mathbb{N}$ with $d \leq n$, $\mathcal{V} \subset \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^n$, we shall say that σ is a *Knaster embedding* of $\mathcal{V}$ into ℓ_∞^n if σ is an isometry of ℓ_2^d into ℓ_2^n and $\|\sigma(\cdot)\|_\infty$ is constant on $\mathcal{V}$. [One could also require that σ is an isometry just on the linear span of $\mathcal{V}$.] We then have

Theorem 2 *Given $p \in \mathbb{N}$, there exists $\mathcal{V} = \{v_1, \ldots, v_p\} \subset S^{p-1}$ such that there is no Knaster embedding σ of $\mathcal{V}$ into ℓ_∞^n with $n \leq p\lfloor\log(p/2)\rfloor/32$. In particular, the sequence $(\|\sigma v_i\|_\infty)_{i=1}^p$ can not be constant if $p = n$ and and n is sufficiently large.*

The answer to the Knaster problem is thus negative (for large n) even for convex functions. However, it is conceivable that one may still obtain a proof of Dvoretzky theorem via a Knaster problem-like argument. Our example leaves open the possibility that the following is true

Given a continuous real-valued function F on S^{n-1} and a configuration of points $v_1, \ldots, v_p$ in S^{d-1} there is an isometry $\sigma : \mathbb{R}^d \to \mathbb{R}^n$ such that $F(\sigma v_1) = \ldots = F(\sigma v_p)$ provided $n \geq pd$ (resp., if pd/n is small enough, or if $p = d$ and p^2/n is small enough).

In fact, this statement *does hold* if $F = \|\cdot\|_\infty$, as can be checked following our analysis. We separated the roles of p and d in the above since in fact in our example the set $\mathcal{V}$ is concentrated on a small subspace of $\mathbb{R}^p$, specifically of dimension $d = O(\log p)$. Of course, a small perturbation of the set $\mathcal{V}$ yields a *basis* of $\mathbb{R}^p$ with no Knaster embedding into ℓ_∞^n. However, the condition number of that basis is very large and, in view of [15], such basis *can not* be orthonormal.

For the proof of Theorem 2 we need two lemmas. The first of them is well known.

Lemma 3 *Let $d \in \mathbb{N}$ and let E be a d-dimensional subspace of L_2 and $S_E := \{f \in E, \|f\|_2 = 1\}$ its sphere. Then*

$$\max\{\|f\|_\infty \ : \ f \in S_E\} \ \geq \ \sqrt{d},$$

Consequently, if $\delta > 0$ and if S is any δ-net of S_E, then $\max\{\|f\|_\infty : f \in S\} \geq (1 - \delta^2/2)\sqrt{d}$. If E is a d-dimensional subspace of ℓ_2^n, the corresponding lower bounds are respectively $\sqrt{d/n}$ and $(1 - \delta^2/2)\sqrt{d/n}$.

Sketch of the proof. Let $(\varphi_j)_{j=1}^d$ be an orthonormal basis of E. Then $\mathcal{S}_\varphi := (\sum_{j=1}^d |\varphi_j|^2)^{1/2}$ is, modulo easily remedied measurability issues, the pointwise supremum of $|f|$ over $f \in S$. On the other hand, it is directly verified that $\sqrt{d} = \|\mathcal{S}_\varphi\|_2 \leq \|\mathcal{S}_\varphi\|_\infty$. If S is a δ-net of S_E, then the convex hull of S contains $(1 - \delta^2/2)S_E$, which yields the second estimate. The above argument works for L_2 and L_∞ on any probability space; the variant for ℓ_2^n and ℓ_∞^n follows by rescaling. □

Next, let $w_1, \dots, w_N, \dots, w_{2N} \in S^1$ be consecutive vertices of of a regular $2N$-gon (in particular $w_{N+j} = -w_j$ for $j = 1, \dots, N$) and set $\mathcal{P}_N := \{w_1, \dots, w_N\}$. We then have

Lemma 4 *Let $n, N \in \mathbb{N}$ and let $\sigma : \mathbb{R}^2 \to \mathbb{R}^n$ be a Knaster embedding of $\mathcal{P}_N$ into ℓ_∞^n. Let A be the common value of $\|\sigma(w_j)\|_\infty$, $j = 1, \dots, N$. Then $A \le 2/\sqrt{N}$.*

Versions of the above statement can be proved for dimensions higher than 2. They show that for a class of "tight near-embeddings" of S^{d-1} into ℓ_∞^n the lower bound from Lemma 3 gives in fact the correct order. This implies that, for small d, such embedding can not be "random" since a "typical" value of $\|\cdot\|_\infty$ on S^{n-1} is of order $\sqrt{(1 + \log n)/n}$.

We shall postpone the proof of Lemma 4 for a moment and show how the two lemmas imply Theorem 2. Set $d := \lfloor \log (p/2) \rfloor$ and let $\mathcal{S}$ be a set of cardinality $m < (4/\sqrt{3})^d < p/2$ such that $\mathcal{S} \cup (-\mathcal{S})$ is a 1-net of S^{d-1} (as indicated earlier, the existence of $\mathcal{S}$ follows by a standard volumetric argument; the bound 3^d that usually appears in more general statements would work here also). Next, let $N := p - m$ (note that $N > p/2$) and consider the set P_N of Lemma 4. Identifying $\mathbb{R}^d$ and $\mathbb{R}^2$ with appropriate d- and 2-dimensional subspaces of $\mathbb{R}^p$ allows to think of all these sets as subsets of S^{p-1} and to define $\mathcal{V} = \{v_1, \dots, v_p\} := \mathcal{S} \cup \mathcal{P}_N$. Now let $n \in \mathbb{N}$ and let σ be any isometry of ℓ_2^p (or the linear span of $\mathcal{V}$) into ℓ_2^n. Since $\sigma(\mathcal{S}) \cup (-\sigma(\mathcal{S}))$ is a 1-net in the sphere of the d-dimensional space $E = \sigma(\mathbb{R}^d)$, it follows from Lemma 3 that

$$\max_{1 \le i \le p} \|\sigma v_i\|_\infty \ge \frac{1}{2}\sqrt{\frac{d}{n}}.$$

On the other hand, Lemma 4 implies that

$$\min_{1 \le i \le p} \|\sigma v_i\|_\infty \le \frac{2}{\sqrt{N}} < 2\sqrt{\frac{2}{p}}.$$

Thus the sequence $(\|\sigma v_i\|_\infty)$ can not be constant if $n \le pd/32 = p\lfloor \log (p/2) \rfloor/32$, which proves Theorem 2. An examination of the above argument shows that the smallest value $p = n$ for which it yields a counterexample to the Knaster problem is of order 10^{12}. A more careful (but still using only volumetric methods for estimating sizes of nets) calculation along the same lines allows to reduce this to approximately 3.2×10^4. We expect that working with nets obtained by more efficient constructions and otherwise fine-tuning the argument may give a counterexample for n of order 10^2. It would be of interest to narrow the gap between these values and those corresponding to positive results (to date, $n \le 3$). $\square$

Proof of Lemma 4. Set $\sigma_1 := \sigma/A$. Then σ_1 is necessarily of the form $\sigma_1(x) = (\langle x, y_s \rangle)_{s=1}^n$ for some $y_1, \dots, y_n \in \mathbb{R}^2$ verifying

(a) $|\langle w_1, y_s \rangle| \le 1$ for $i = 1, \dots, N$ and $s = 1, \dots, n$

(b) for any $i = 1, \dots, N$ there exists $s_i \in \{1, \dots, n\}$ such that $|\langle w_i, y_{s_i} \rangle| = 1$.

Let Q be the convex hull of $\mathcal{P}_N \cup (-\mathcal{P}_N)$ and Q° its polar. Then Q° is a regular $2N$-gon circumscribed around the unit circle. The condition (a) above is equivalent to "$y_s \in Q^\circ$

for $s = 1, \ldots, n$." The condition (b) says that for every side of Q° there is an s such that either y_s or $-y_s$ belong to that side, namely s_i for the side tangent to the unit circle at w_i. Since a point may belong to at most two sides of Q° (it does when it is a vertex), there must be at least $k := \lceil N/2 \rceil$ distinct points among the y_{s_i}'s. Note that, by (b), $|y_{s_i}| \geq 1$ for $i = 1, \ldots, N$.

Let us now calculate the Hilbert-Schmidt norm $\|\sigma_1\|_{HS} := (\mathrm{tr}\,(\sigma_1^* \sigma_1))^{1/2}$ in two ways. First, $\|\sigma_1\|_{HS} = (\sum_{s=1}^n |y_s|^2)^{1/2}$, which, by the observations above, is $\geq (N/2)^{1/2}$. Next, σ_1 being a multiple of an isometry of $\mathbb{R}^2$, $\|\sigma_1\|_{HS} = \sqrt{2}/A$. Comparing the two quantities we obtain $A \leq 2/\sqrt{N}$, as required. □

Appendix. As is well known ([5], see also [14]), for any $\theta \in (0,1)$ there is a constant $c(\theta) > 0$ such that, for any $N \in \mathbb{N}$, nearly all (in the sense of the invariant measure on the corresponding Grassmanian) $\lceil \theta N \rceil$-dimensional subspaces $E \subset \mathbb{R}^N$ verify the property

$$\forall x \in E \quad c(\theta)|x| \leq \|x\|_1/\sqrt{N} \leq |x|.$$

By duality, the above condition is equivalent to

$$\forall x \in E \; \exists y \in E^\perp \;\; \|x + y\|_\infty \leq c(\theta)^{-1}|x|/\sqrt{N}.$$

Applying this with $\theta = 2/3$ yields, for each $n \in \mathbb{N}$, an orthogonal decomposition $F_0 \oplus F_1 \oplus F_2$ of $\mathbb{R}^{3n}$ with $\dim F_i = n$, $i = 0, 1, 2$, such that the above properties hold for $N = 3n$ and for all subspaces of the form $E = F_i + F_k$, $i, k \in \{0, 1, 2\}$ (cf. [13], Corollary 7.4).

Let now $(u_1, \ldots, u_p)$, $(v_1, \ldots, v_p)$ be sequences in $B^n \subset \mathbb{R}^n$, which we shall identify (as an inner product space) with F_0. Applying the preceding with $E = F_0 + F_2$ we obtain a sequence $(y_1, \ldots, y_p) \in F_1 = (F_0 + F_2)^\perp$ such that $f_i := u_i + y_i$ verify $\|f_i\|_\infty \leq c(2/3)^{-1}/\sqrt{3n}$ for $i = 1, \ldots, p$. Similarly, there are z_k's in F_2 such that $g_k := v_k + z_k$, $k = 1, \ldots, p$ satisfy the same estimate. Since clearly $\langle f_i, g_k \rangle = \langle u_i, v_k \rangle$ for all $1 \leq i, k \leq p$, it is now enough to identify $\mathbb{R}^{3n}$ with, say, the appropriate space of step functions on $(0,1)$ to obtain the required assertion with $C = c(2/3)^{-1}$. Note that the argument above yields $f_i, g_k \in \ell_\infty^N$ with $N = O(n)$, as opposed to $N = O(p^2)$) which follows from the Caratheodory theorem.

The fact that the Grothendieck inequality is related to existence of near-Euclidean subspaces of L_1-spaces has been well known to experts in the area (see, e.g., [3] and its references). The argument above lays out a specific very direct and conceptually simple link.

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