Foliations and noncommutative longitudinal residue

Polyhomogeneous Pseudodifferential operators

$$\mathcal{S}^{m_0}(U) = \{a(x,\xi) \in \mathcal{C}^\infty(U \times \mathbb{R}^n), \forall K, \alpha, \beta \ \exists C_{K,\alpha,\beta}$$
$$\sup_{x \in K} \left| \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right| \leq C_{K,\alpha,\beta}(1+|\xi|)^{m_0 - |\beta|}\}$$
$$\mathcal{S}^m_{hom}(U) = \{a(x,\xi) \in \mathcal{S}^{m_0}, \exists a_{m-j} \text{ hom. of } d^\circ \ m-j$$
$$a(x,\xi) \sim \sum_j a_{m-j}(x,\xi)\}$$

$P =$ Op$(a) : \mathcal{C}^\infty_c(U) \to \mathcal{C}^\infty(U)$ defined by

$$(Pu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi)\hat{u}(\xi) e^{i<x,\xi>} d\xi.$$

$\mathcal{P}^m(U)$ stable under diffeomorphisms
$\Rightarrow$ can be defined on manifolds.
a_m is the *principal symbol* of P. For a manifold, $a_m \in \mathcal{C}^\infty(S^*M)$.

Wodzicki Residue of a PDO P on a compact manifold M of dim. n.

1. Local Definition : $\mathrm{Res}\,(P) = \int_M \mathrm{res}\,_x(P)$

$$\mathrm{res}\,_x(P) = (2\pi)^{-n} \int_{|\xi|=1} a_{-n}(x,\xi)d\xi$$

2. Given by the log. coef. in the asymptotic expansion of Schwartz kernel $k_P(x,y)$ around $x = y$.

3. Global Def : $\mathrm{Res}\,(P) = m\,\mathrm{res}\,_{z=0} Tr(PA^{-z})$

4. Given by the log. coef. in asymptotic expansion of $m\mathrm{Tr}(Pe^{-tA})$ when $t \to 0$.

5. Gives a residual trace on $\mathcal{P}^{\mathbb{Z}}(M)$, unique if M connected and $n > 1$.

Goal :
Develop an analogue of Wodzicki residue for longitudinal pseudodifferential operators along the leaves of a compact measured foliation

1. Foliations
2. Longitudinal PDOs and C^*-algebras
3. Complex powers and Sobolev modules
4. Noncommutative longitudinal residue

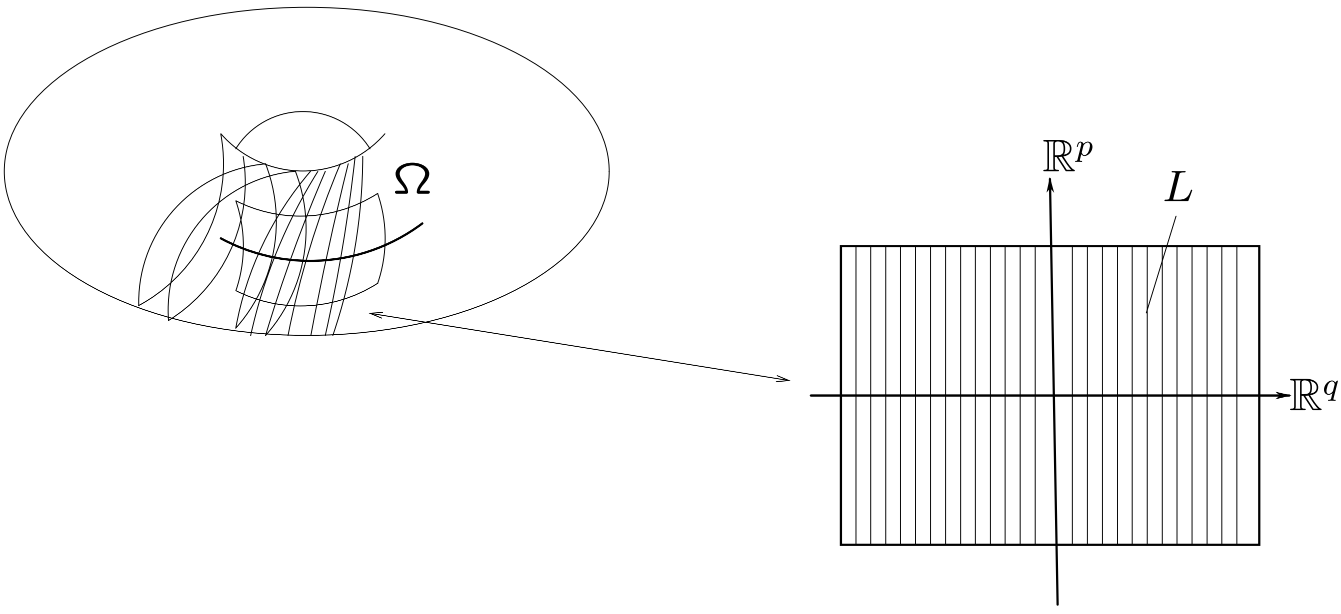
Foliations

Definition

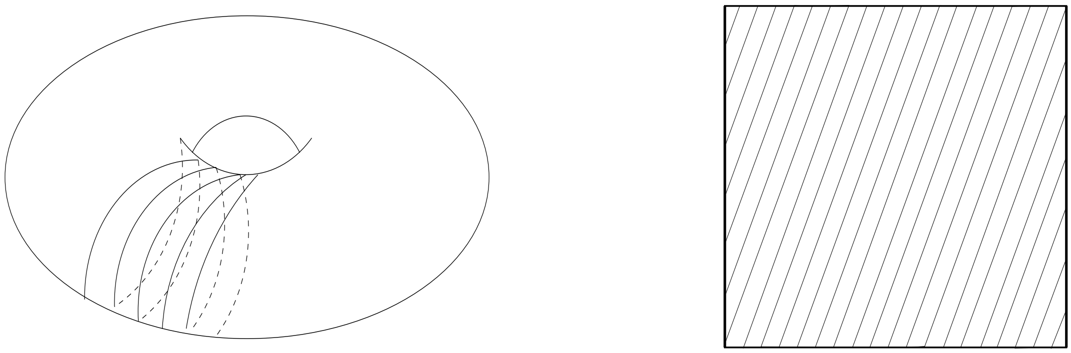
Foliation $= (M, \mathcal{F})$ with M smooth manifold of dim $p+q$ and atlas of *trivializing* open sets $\Omega \simeq L \times T$, $L \subset \mathbb{R}^p$ and $T \subset \mathbb{R}^q$ with coordinate changes

$$l' = \psi(l, t) \quad t' = \Gamma(t)$$

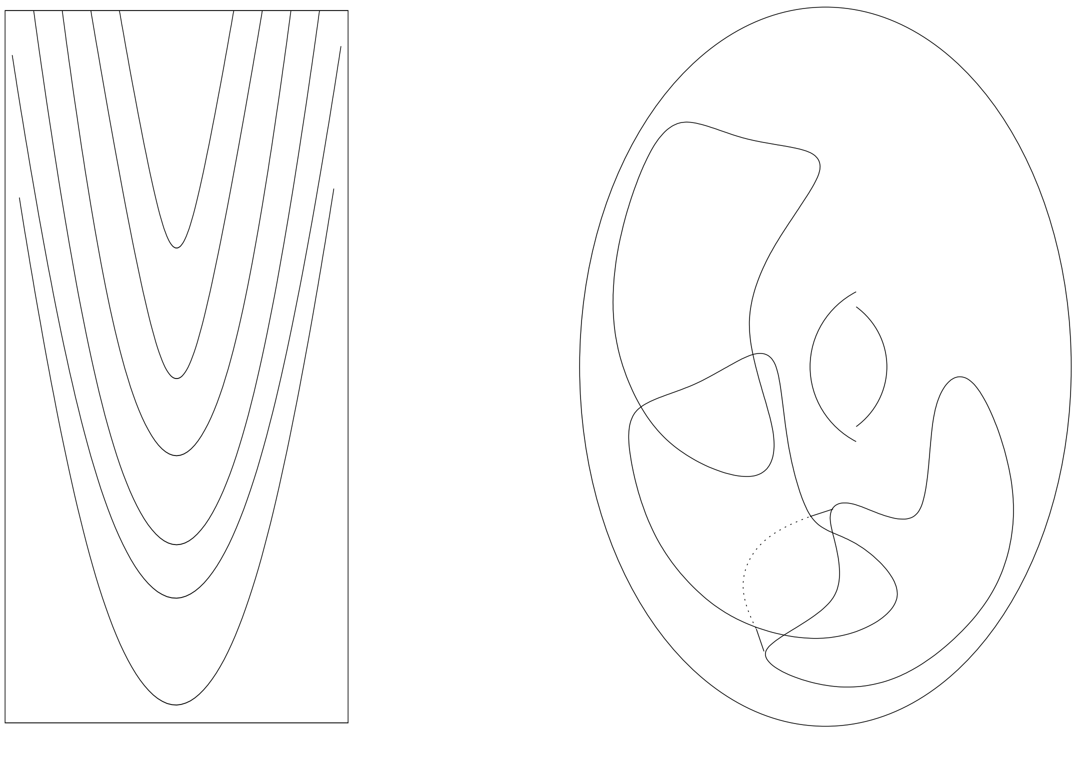
ψ $\mathcal{C}^\infty$ and Γ local diffeo.



Kronecker Foliation of $\mathbb{T}^2$



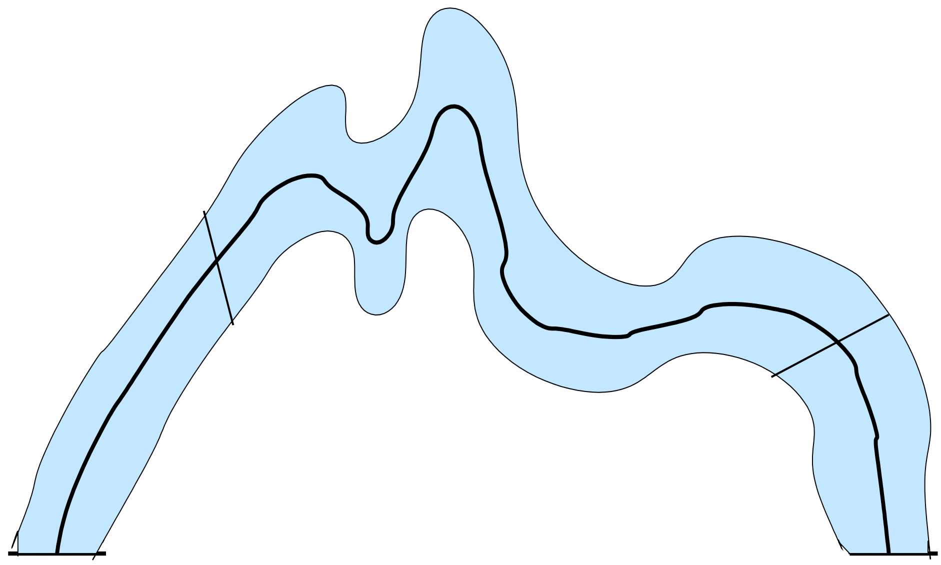
Reeb Foliation of $D^2 \times S^1$



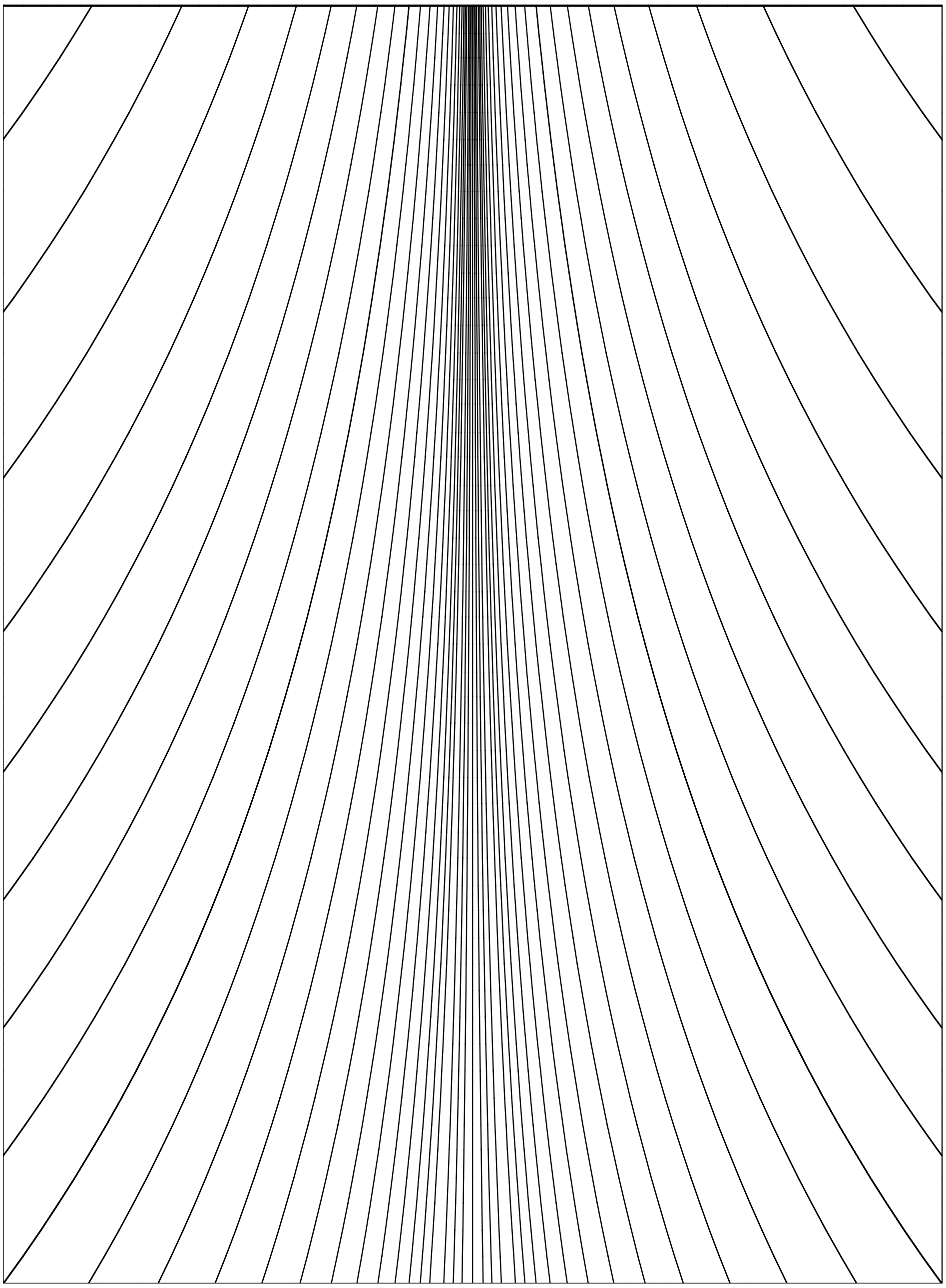
Holonomy Groupoid

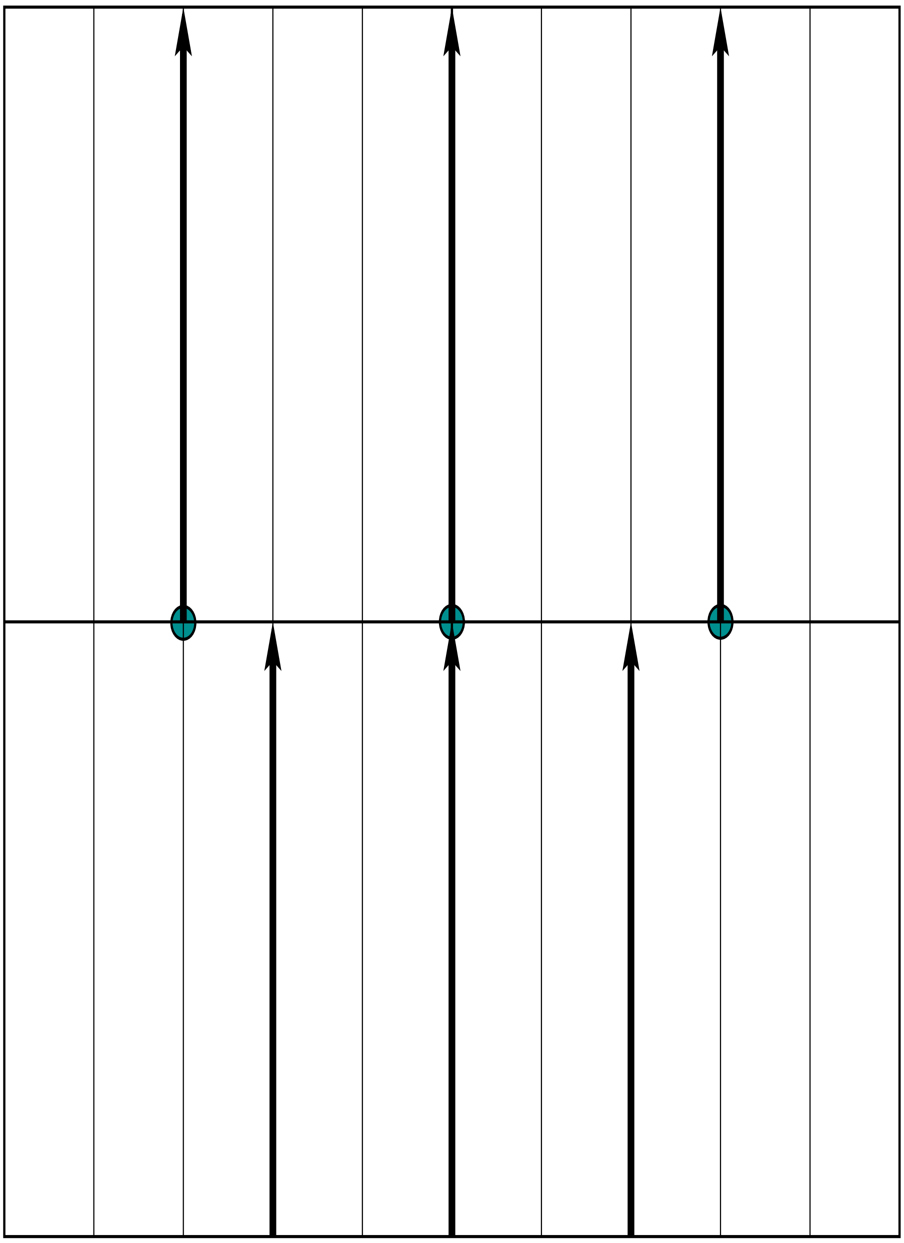
Holonomy groupoid = desingularisation of the groupoid of the equivalence relation

Holonomy of a path =
" Parallel transport of a transverse neighbourhood"



$G = \mathcal{G}^{fond}/\{h(\gamma_1) = h(\gamma_2)\}$.
G Lie groupoid of dim $2p + q$, non Hausdorff in general.
G^x holonomy covering over the leaf of x.
1-density on $\mathcal{F}$
$\rightsquigarrow \lambda$ fixed $\mathcal{C}^\infty$-Haar system on G.
$G \rightsquigarrow \ C_r^*(G),\ C^*(G)$.





Ruelle-Sullivan currents and transverse distributions

Current of dim. d on M = Linear Form $\Omega^d_{\mathbb{C}}(M)$.
Foliated Current= dim p and st :

$$\langle C, \omega\rangle = 0 \text{ if } \omega_{|F} = 0.$$

Theorem[R-S] (1) $\Leftrightarrow$ (2)

1. Closed foliated current
2. Holonomy inv. transverse distribution

Positivity $\Longleftrightarrow$ (Distribution = Measure)

Transverse measure μ + Haar system λ
$\rightsquigarrow$ Measures ν on M and $\nu \circ \lambda$ on G
$\rightsquigarrow$ Lower semi-cont. trace on $C^*_r(G)$:

$$\mathrm{Tr}_\mu(f^* * f) = \int_{x\in M} (f^* * f) d\nu(x) = \|f\|^2_{\mathcal{L}^2(G,\nu\circ\lambda)}$$

Longitudinal pseudodifferential operators [Connes]

Global definition

P operator on $\mathcal{C}_c^\infty(G)$ such that

- $\exists (P_x)_{x\in M}$, $(Pf)(\gamma) = (P_{s(\gamma)}f)(\gamma)$,
- $P_{s(\gamma)}U_\gamma = U_\gamma P_{r(\gamma)}$,
- $(P_x)_{x\in M}$ $\mathcal{C}^\infty$ PDO family with uniform support.

$\exists k$ compactly supported s.t.

$$Pf(\gamma) = \int_{G_x} k(\gamma\gamma'^{-1})f(\gamma')d\lambda_x(\gamma') = k * f(\gamma).$$

Local definition

$\Omega \simeq L \times T$, $P = (P_t)_{t\in T}$ $\mathcal{C}^\infty$ PDO family on $L \times \{t\}$ with compact support ($\subset L \times L \times T$).

Principal symbol

$a_m(P) \in C^\infty(T^*\mathcal{F} - \{0\})$ hom. of d°m.

Ellipticity

If a_m is invertible, P is called elliptic and $\exists Q$ G-PDO of order $-m$ s.t.

$$PQ - I \text{ and } QP - I \in \Psi_c^{-\infty}(G).$$

C^*-algebraic interpretation

$E = C^*(G)$ Hilbert mod. for $\langle a, b\rangle = a^* b$.

Proposition[Connes,Monthubert-Pierrot]
$P \in \Psi_c^m(G)$

1. If $\Re e\, m < 0$, P extends into $\overline{P} \in \mathcal{K}(E)$.
2. If $\Re e\, m = 0$, P extends $\overline{P} \in \mathcal{L}(E)$.

Proposition
P elliptic $\Rightarrow$ P regular operator

$\Rightarrow$ $(1 + P^*P)^{-\frac{1}{2}} \in C^*(G)$

$\Rightarrow$ Functionnal calculus

$\Rightarrow$ Image of P/repr. $\rightsquigarrow$ P_π closed op. s.t. $(P_\pi)^* = P_\pi^*$, spectral equalities,...

Non compactly supported smoothing operators
$R \in C^*(G)$, $PRP \in C^*(G)$

Complex Powers

G-operators depending on a parameter = generalization of Shubin's work

Symbols depend on 2 parameters : $\sigma(x, \xi, \lambda, t)$

Proposition

A diff. ell. c.s. op , Sp$(A) \subset [\varepsilon, +\infty[$.

$(A - \lambda)$ ell. G-op with par. $\Rightarrow \exists B_\lambda \in \Psi_c^{-m}(G, \Lambda)$

$(A - \lambda)B_\lambda = I + R_\lambda$ and $B_\lambda(A - \lambda) = I + S_\lambda$

R_λ and S_λ smoothing with compact support indep. of λ

$\|PR_\lambda P\| = \bigcirc(|\lambda|^{-N})$ and $\|PS_\lambda P\| = \bigcirc(|\lambda|^{-N})$.

Theorem

$$A^s = B(s) + R(s)$$

$B(s)$ G-PDO with compact support,
$R(s)$ smoothing (generalized).

Sobolev modules

$H^s(P) =$ Dom$(1 + P^*P)^{s/2m}$ scal prod :

$$\langle x,\, y\rangle_s = \langle (1+P^*P)^{\frac{s}{2m}} x,\, (1+P^*P)^{\frac{s}{2m}} y\rangle.$$

Theorem

$H^s(P)$ does not depend on P.

Properties of Sobolev modules

- Smoothing : R smooth. $\Leftrightarrow R \in \cap_{s,t}\mathcal{L}(H^s, H^t)$.
- Inclusions : $s > s',\ H^s \hookrightarrow H^{s'}$ compact.
- Duality : H^s dual of H^{-s}.
- Action of a PDO : If $P \in \Psi^m(G)$,
  $P \in \mathcal{L}(H^s, H^{s-m_0})$.

Algebra of longitudinal PDOs

$\Psi^m(G)$ gen. by $\Psi^{-\infty}(G)$ and $\Psi^m_c(G)$.
$\Psi^m(G) \circ \Psi^n(G) \subset \Psi^{m+n}(G)$.
$\rightsquigarrow$ Nat. action on $H^{-\infty}$
$\Psi^{-\infty}(G)$ ideal of the inv. alg. $\Psi(G)$.

Noncommutative longitudinal residue

Local Def. $\text{Res}_{\mu}(P) = \int_{t\in T} \text{Res}\,(P_t) d\mu(t)$.

Logarithmic Coef.

The coef $c_P(x,t) \rightsquigarrow$ 1-density on $\mathcal{F}$.

$$\begin{aligned} k(x,y,t) &= \sum_{j=0}^{r} a_{j-m-p}(x, x-y, t) \\ &\quad - c_P(x,t) \log|x-y| + R(x,y,t) \end{aligned}$$

Theorem[cf Kontsevich-Vishik]

1. $P \mapsto \text{Tr}_{\mu}(P)$ holom. on $\Psi^{tr}(G)$ and $\exists!$ anal. prol. TR_{μ} on $\Psi^{\mathbb{C}\setminus\mathbb{Z}}(G)$.
2. $z \mapsto P(z)$ hol. s.t. ord $P(z) = z$. $z \mapsto \text{TR}_{\mu}(P(z))$ merom. on $\mathbb{C}$ with simple poles $\in \mathbb{Z}$.

$$\text{res}_{z=k_0} \text{TR}_{\mu}(P(z)) = -\,\text{Res}_{\mu}(P(k_0)).$$

In particular

$$\text{res}_{s=0} \text{TR}_{\mu}(PA^{-s}) = \frac{1}{m} \text{Res}_{\mu}(P).$$

ζ function and asymptotics of heat kernels

$s \mapsto \zeta_A(s) = \mathrm{Tr}_\mu(A^{-s})$ is meromorphic with at most simple poles in $s_j = \frac{j-p}{m}$ and

$$\mathrm{res}_{s=s_j}\zeta_A(s) = \frac{1}{m}\mathrm{Res}_\mu(A^{\frac{p-j}{m}}).$$

We have $\mathcal{M}(Tr_\mu(e^{-tA}))(s) = \Gamma(s)\zeta_{A,\mu}(s) \Rightarrow$

$$\mathrm{Tr}_\mu(Pe^{-tA}) \overset{t\to 0}{\sim} \alpha_n(P,A) + \sum_{n\neq j=0}^{\infty} \alpha_j(A)t^{\frac{j-n-r}{m}}$$

$$+ \sum_{k=0}^{\infty} \beta_k(A,P)t^k \ln t.$$

In particular $\beta_0(A,P) = \frac{1}{m}\mathrm{Res}_\mu(P)$.

Residues and traces on $\Psi^{\mathbb{Z}}(G)$

Proposition

NC Residue gives a trace on $\Psi^{\mathbb{Z}}(G)$.

Theorem [cf Benameur-Nistor]

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3)

1. Trace on $\Psi^{\mathbb{Z}}_{\infty}(G)$ zero on $\Psi^{-\infty}_{\infty}(G)$
2. Holonomy inv. transverse distribution
3. Closed foliated current

Residue and Dixmier trace

Theorem[Benameur-Fack,V]

P PDo of order $-p \Rightarrow P \in L^{(1,\infty)}(W^*_\mu(M,\mathcal{F}))$

$$\mathrm{Tr}_{\omega,\mu}(P) = \frac{1}{p}\,\mathrm{Res}\,_\mu(P).$$

Spectral vN-triple

$(\mathcal{A}, \mathcal{M}, D)$, of dim d and simple dim. spectrum Sd with :

- $\mathcal{M}$ vN + faithful, normal, sf trace τ,
- $\mathcal{A}$ dense sub-alg. of $\mathcal{M}$,
- D selfadj. unbounded aff. $\mathcal{M}$,

1. $a\,$Dom$D \subset$ Dom$D$ et $[D, a] \in \mathcal{M}$;
2. a and $[D, a] \in \bigcap_{n\in\mathbb{N}}$ Dom(δ^n)
   avec $\delta(b) = [|D|\,, b]$;
3. $a(D + i)^{-d} \in \mathcal{L}^1(\mathcal{M})$;
4. $\mathcal{B} =$ alg. generated by $\delta^n(a)$ and $[D, a]$, Sd discrete, $\zeta(b) = \tau(b\,|D|^{-z})$ holom. on $\mathbb{C} - Sd$ with at most simple poles $\in Sd$.

The local Index formula [Connes-Moscovici]

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with simple dim. spectrum.

1. We define an odd cocycle in the (b, B)-bi-complex by letting,

$$\varphi_n(a^0, \dots, a^n) = \sum_{k_i \geq 0} \sqrt{2i}\, c_{n,k}\, \mathrm{Res}\,(a^0 (da^1)^{(k_1)} \dots (da^n)^{(k_n)} \, |D|^{-(n+2|k|)})$$

   with :

   - $\mathrm{Res}\,(b)$ the residue in $z = 0$ of ζ_b ;
   - $|k| = \sum k_j$ and

$$c_{n,k} = \frac{(-1)^{|k|}\Gamma(|k| + \frac{n}{2})}{k!(k_1+1)(k_1+k_2+2)\dots(k_1+\dots+k_n+n)}$$

2. The cyclic cohomology class of φ_n coincides with the Chern character in cyclic cohomol. $ch_*(\mathcal{A}, \mathcal{H}, D)$.