UNIFORM NON AMENABILITY AND ℓ^2 BETTI NUMBERS

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Abstract. It is shown that $\beta_1(\Gamma) \leq h(\Gamma)$ for any countable group Γ, where $\beta_1(\Gamma)$ is the first ℓ^2 Betti number and $h(\Gamma)$ the uniform Cheeger constant. In particular a countable group with non vanishing first ℓ^2 Betti number is uniformly non amenable.

Cheeger constants are then defined in the framework of measured equivalence relations. For an ergodic measured equivalence R of type II_1 the uniform Cheeger constant $h(R)$ of R is invariant under orbit equivalence and satisfies $\beta_1(R) \leq h(R)$, where $\beta_1(R)$ is the first L^2 Betti number of R in the sense of Gaboriau (in particular $h(R)$ is a non trivial invariant). In contrast with the group case, uniformly non amenable measured equivalence relations of type II_1 always contain non amenable subtreeings.

Let Γ be a countable group and α be an essentially free ergodic and measure preserving actions of Γ on a probability space (X, μ). Denote by R_α the associated equivalence relation on X (the partition into Γ-orbits). A well-known theorem of Gaboriau implies that $\beta_1(R_\alpha) = \beta_1(\Gamma)$, while the inequality $h(R_\alpha) \leq h(\Gamma)$ is immediate from the definitions. It may happen that $h(R_\alpha) < h(\Gamma)$.

An ergodic version $h_e(\Gamma)$ of the uniform Cheeger constant $h(\Gamma)$ is defined as the infimum over all essentially free ergodic and measure preserving actions α of Γ of the uniform Cheeger constant $h(R_\alpha)$ of the equivalence relation R_α associated to α. By establishing a connection with the cost of measure preserving equivalence relations one proves that $h_e(\Gamma) = 0$ for any lattice Γ in a semi-simple Lie group of real rank at least 2 (while $h(\Gamma)$ doesn't vanish in general).

1. Introduction

The Cheeger constant of a graph offers a simple way to capture the isoperimetric behavior of finite sets in this graph. For a finitely generated countable group it reflects isoperimetry in Cayley graphs associated to finite generating sets of this group and is related to amenability. In the present paper we introduce an analog of this constant for measured equivalence relations of type II_1. As we shall see, it behaves in a very different manner from a measured dynamic point of view.

Let R be a measured equivalence relation of type II_1 on a standard probability space (X, μ). Our main interest lies in two geometric invariants that have recently been attached to R: the cost and the L^2 Betti numbers. The cost of R is a real number with values in $[1, \infty]$ (assuming R to be ergodic) denoted by $C(R)$. From its definition one can readily infer that it is invariant under isomorphism — i.e., orbit equivalence — of R and the main problem is to compute it. In [8] Damien Gaboriau established an explicit formula relating the cost of an amalgamated free product (over amenable equivalence subrelations) to the costs of their components; this allowed him to solve the long standing problem of distinguishing the free groups up to orbit equivalence. In [9] he went further and introduced the so-called L^2 Betti numbers of R. They are non negative numbers $\beta_0(R), \beta_1(R), \beta_2(R), \ldots$ in $[0, \infty]$ defined by geometric means using an approximation process (as for their analogs for countable groups, see Cheeger and Gromov [5]) and one of the main problem here

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(solved in [9]) was to show that the resulting numbers only depend on the isomorphism class of R. The first L^2 Betti number also allows to distinguish the free groups up to orbit equivalence. The relation between L^2 Betti numbers and the cost is still unclear, but the inequality $C(R) \geq \beta_1(R) + 1$ is known to hold true for any ergodic equivalence relation of type II_1 [9]. Recall that L^2 Betti numbers were first introduced by Atiyah [3] in 1976 in his work on the index of equivariant elliptic operators on coverings spaces of Riemannian manifolds. In the present paper we consider a new isomorphism invariant for R, the uniform Cheeger constant, $h(R)$. It takes values in $[0, \infty]$ and is defined as an infimal value of isoperimetric ratios for 'finite sets' in the Cayley graphs of R (see Section 4). For compact Riemannian manifolds (and their coverings) the Cheeger constant was considered by Cheeger when he proved his well-known 'Cheeger inequality' relating it to the bottom spectrum of the Laplacian.

Our first main theorem asserts that for any measured equivalence relation of type II_1, and in fact for any r-discrete measured groupoid G of type II_1, one has the inequality

$$\beta_1(G) \leq h(G).$$

Here we assume G to be finitely generated, in which case the uniform Cheeger constant $h(G)$ has a finite value (for infinitely generated groupoids G we simply set $h(G) = \infty$ so that, for instance, a measured equivalence relation R given by a measure preserving and essentially free action of the free group F_∞ on infinity many generators will satisfy $\beta_1(R) = h(R) = \beta_1(F_\infty) = h(F_\infty) = +\infty$ [9]). For the sake of clarity and for the convenience of the reader only interested in the group setting we present in Section 3 a complete proof of this result in the particular case of a finitely generated countable group. This will also facilitate the comparison with the proof of Cheeger-Gromov's celebrated vanishing theorem in [5], to which our arguments owe much. The latter asserts that for an amenable countable group Γ the sequence

$$\beta_0(\Gamma), \beta_1(\Gamma), \beta_2(\Gamma), \ldots$$

of all ℓ^2 Betti numbers vanishes identically. Note that for an amenable Γ one does have $h(\Gamma) = 0$ but there exist non amenable groups with $h(\Gamma) = 0$ (see [16, 2]). We then show how to handle the case of measured equivalence relations of type II_1 in Section 6 and that of measured groupoids in Section 7.

We call an ergodic measured groupoid (of type II_1) uniformly non amenable if its Cheeger constant $h(G)$ is non zero. The class of uniformly non amenable groups is quite large and has been studied recently by Osin (see e.g. [16, 17]) and Arzhantseva, Burillo, Lustig, Reeves, Short, Ventura ([2]). Breuillard and Gelander [4] have shown that for an arbitrary field K, any non amenable and finitely generated subgroup of $\mathrm{GL}_n(K)$ is uniformly non amenable.

The class of uniformly non amenable measured equivalence relation turns out to be "much smaller" than its corresponding group-theoretic analog. For instance if an equivalence relation R is the partition into the orbits of an essentially free measure preserving action of a (non-uniform) lattice in a higher rank Lie group we have $h(R) = 0$ (Corollary 17) and thus R is not uniformly non amenable (note that R has the property T of Kazhdan in that case). As well equivalence relations which are decomposable as a direct product of two infinite equivalence subrelation have trivial uniform Cheeger constant (Corollary 18). These results are derived by establishing a relation between the cost and the uniform Cheeger constant and by appealing to some of Gaboriau's results in [8]. Explicitly we show (in Section 8) that an ergodic equivalence relation with cost 1 has a vanishing Cheeger constant. The proof is reminiscent of the (non) concentration of measure property for measured equivalence relations ([20]) which is implemented here via the Rokhlin Lemma. We also show that uniformly non amenable measured equivalence relations have trivial fundamental groups (see Section 10, this will alternatively follow from [8]) and always contain a non amenable subtreeing (see Section 9). The latter is related to the measure-theoretic analog of the Day–von Neumann problem. Recall that in the group case this problem (i.e., is it true that every non amenable group contains a non amenable free group?) is well-know to have a negative answer, as was proved Ol'shanskii. Osin [16] showed that the answer was negative as well for uniformly non amenable groups. To prove that it is positive in our situation, we combine our results with the corresponding known fact for equivalence relations with cost greater than 1 (which was proved independently by the first author and by Kechris-Miller, see the discussion in Section 9).

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2. Cheeger constants and countable groups

Let Y be a locally finite graph and A be a subset of vertices of Y. Define the *boundary of A in Y* to be the set $\partial_Y A$ of edges of Y with one extremity in A and the other one in $Y \backslash A$. The *Cheeger isoperimetric constant* of Y is the non negative number

$$h(Y) = \inf_{A \subset Y} \frac{\# \partial_Y A}{\# A},$$

where the infimum runs over all finite subsets A of vertices of Y.
Let Γ be a finitely generated group and S be a finite generating set of Γ. Recall that the *Cayley graph* of Γ with respect to S is the graph Y whose vertices are the elements of Γ and whose edges are given by right multiplication by elements of S. The Cheeger constant of this graph is called the *Cheeger constant of Γ with respect to S* and is denoted it by $h_S(\Gamma)$. Følner's theorem asserts that a finitely generated group Γ is amenable if and only if $h_S(\Gamma) = 0$ for some (hence every) finite genereting set S [12].
By definition the *uniform Cheeger constant* of Γ is the infimum

$$h(\Gamma) = \inf_S h_S(\Gamma)$$

over all finite generating sets S of Γ.

Example 1. Osin [16] has given examples of non amenable groups with $h(\Gamma) = 0$. For instance he proved that the Baumslag-Solitar groups, with presentation

$$\mathrm{BS}_{p,q} = \langle a, t \mid t^{-1} a^p t = a^q \rangle$$

where $p, q > 1$ are relatively prime, have vanishing uniform Cheeger constant (note that the uniform Cheeger constant considered in [16] is defined in terms of the regular representation of the given group: compare Section 13 in the paper of Arzhantseva et al. [2]).

Finitely generated groups with $h(\Gamma) > 0$ are called *uniformly non-amenable* (see [2]). In fact our definition slightly differs from the one given in [2] due to a different choice for the boundary of a finite subset of vertices in a graph (basically the present paper deals with the "external boundary" while the definition in [2] involves the "internal boundary", see Appendix A for more details). The Baumslag-Solitar groups have vanishing uniform Cheeger constant for any reasonable definition of the boundary.

3. The uniform Cheeger constant and the first ℓ^2 Betti number

The ℓ^2 Betti numbers of a countable group Γ are non negative real numbers $\beta_0(\Gamma)$, $\beta_1(\Gamma), \ldots$ coming from ℓ^2 (co-)homology as Γ-dimension (also known as Murray-von Neumann dimension). We refer to [11, 5, 18, 14] for their precise definition. By a well-known theorem due to Cheeger and Gromov [5] the ℓ^2 Betti numbers of a countable amenable group vanish identically. By elaborating on ideas of [5] (see in particular §3 in [5]) in the case of non amenable groups we obtain the following explicit relation between the first ℓ^2-Betti number and the uniform Cheeger constant.

Theorem 2. Let Γ be a finitely generated group. Then $\beta_1(\Gamma) \leq h(\Gamma)$.

Proof. Let S be a finite generating set of Γ and Y be the Cayley graph of Γ with respect to S. Write $C_i^{(2)}(Y)$, $i = 0, 1$, for the space of square integrable functions on the i-cells (vertices and edges) of Y. Associated to the simplicial boundary ∂_Y on Y we have a bounded operator

$$\partial_1^{(2)} : C_1^{(2)}(Y) \to C_0^{(2)}(Y).$$

Denote $Z_1(Y)$ for the space of finite 1-cycles and $Z_1^{(2)}(Y)$ square integrable 1-cycles on Y. Thus $Z_1^{(2)}(Y)$ is the kernel of $\partial_1^{(2)}$ while $Z_1(Y)$ is space of functions with finite support in this kernel. The first ℓ^2 Betti number $\beta_1(\Gamma)$ of Γ coincides with the Murray-von Neumann dimension

$$\beta_1(\Gamma) = \dim_\Gamma \bar{H}_1^{(2)}(Y)$$

of the orthogonal complement $\bar{H}_1^{(2)}(Y) \subset C_1^{(2)}(Y)$ of the closed subspace $\overline{Z_1(Y)}$ in $Z_1^{(2)}(Y)$, where the closure of $Z_1(Y)$ is taken with respect to the Hilbert norm. A proof of this fact, together with the basic definitions used here, can be found in [19, section 3] (in particular this step takes care of the approximation process involved in Cheeger-Gromov's definition of ℓ^2-Betti numbers; compare to [5]).

Let $\Omega \subset Y^{(1)}$ be a finite subset of edges of Y and consider the space

$$\bar{H}_1^{(2)}(Y)_{|\Omega} = \{\sigma_{|\Omega} : \sigma \in \bar{H}_1^{(2)}(Y)\}$$

of restrictions of harmonic chains to Ω. This is a linear subspace of the space $C_1(\Omega)$ of complex functions on Ω. Let

$$P : C_1^{(2)}(Y) \to C_1^{(2)}(Y)$$

be the (equivariant) orthogonal projection on $\bar{H}_1^{(2)}(Y)$ and

$$\chi_\Omega : C_1^{(2)}(Y) \to C_1^{(2)}(Y)$$

be the orthogonal projection on $C_1(\Omega)$.

Given a finite set A of Γ we denote by A_S the set of edges with a vertex in A. One has $\partial_S A \subset A_S$, where $\partial_S A$ is the boundary of A in Y defined in Section 2. Let us now prove that, for every non empty finite set A of Γ,

$$\beta_1(\Gamma) \leq \frac{1}{\#A} \dim_{\mathbf{C}} \bar{H}_1^{(2)}(Y)_{|A_S}.$$

Write M_{A_S} for the composition $\chi_{A_S}P$, considered as an operator from $C_1(A_S)$ to itself with range $\bar{H}_1^{(2)}(Y)_{|A_S}$. We have

$$\begin{aligned}\dim_\Gamma \bar{H}_1^{(2)}(Y) &= \sum_{s\in S}\langle P\delta_{(e,s)} \mid \delta_{(e,s)}\rangle = \frac{1}{\#A}\sum_{a\in A, s\in S}\langle P\delta_{(a,as)} \mid \delta_{(a,as)}\rangle \\ &\leq \frac{1}{\#A}\sum_{u\in A_S}\langle P\delta_u \mid \delta_u\rangle = \frac{1}{\#A}\operatorname{Tr} M_{A_S} \\ &\leq \frac{1}{\#A}\dim_{\mathbf{C}} \bar{H}_1^{(2)}(Y)_{|A_S}\end{aligned}$$

where the last inequality follows from the fact that $\|M_{A_S}\| \leq 1$. This gives the desired inequality.
We now observe that every harmonic 1-chain (i.e. element of $\bar{H}_1^{(2)}(Y)$) which "enters" a subset A_S has to intersect its boundary $\partial_S A$:

Lemma 3. Let A be a finite subset of Γ. The canonical (restriction) map

$$r : \bar{H}_1^{(2)}(Y)_{|A_S} \to \bar{H}_1^{(2)}(Y)_{|\partial_S A}$$

is injective.

Proof. Recall that

$$Z_1^{(2)}(Y) = \bar{H}_1^{(2)}(Y) \oplus_\perp \overline{Z_1(Y)}.$$

Let $\sigma \in \bar{H}_1^{(2)}(Y)$. If σ vanishes on $\partial_S A$, then $\sigma_{|A_S}$ is a finite 1-cycle as the boundary operator commutes with the restriction to A_S in that case. Thus $\sigma_{|A_S}$ vanishes identically. □

Back to the proof of Theorem 2. The above Lemma 3 gives

$$\dim_{\mathbf{C}} \bar{H}_1^{(2)}(Y)_{|A_S} = \dim_{\mathbf{C}} \bar{H}_1^{(2)}(Y)_{|\partial_S A} \leq \#\partial_S A$$

which immediately yields the theorem:

$$\dim_\Gamma \bar{H}_1^{(2)}(Y) \leq \frac{\#\partial_S A}{\#A}$$

and thus $\beta_1(\Gamma) \leq h(\Gamma)$. □

Remark 4. Lück's generalization of Cheeger-Gromov theorem to arbitrary module coefficients (Theorem 5.1 in [15]) does not hold for groups with vanishing uniform Cheeger constant (as these groups may contain non abelian free groups, compare Remark 5.14 in [15]).

Remark 5. The inequality $\beta_1(\Gamma) \leq h(\Gamma)$ is not optimal in general. Consider for example the case of the free group F_k on k generators. As is well-known one has $\beta_1(F_k) = k-1$ and the uniform Cheeger constant $h(F_k) = 2k-2$ can be computed by considering large balls with respect to a fixed generating set, *e.g.* the usual system S_k of k free generators (see Appendix C). Let us concentrate on the inequality

$$\dim_\Gamma \bar{H}_1^{(2)}(Y) \leq \frac{1}{\#A}\sum_{u\in A_S}\langle P\delta_u \mid \delta_u\rangle$$

(derived in the proof above) which is quite generous in the case of F_k (for $k \geq 2$). Given a ball B with respect to S_k we have

$$\#\partial_{S_k} B \geq (2k-2)\#B.$$

Let $\partial_{S_k^-} B$ be the set of edges of $\partial_{S_k} B$ of the form (a, as) with $as \in B$ and $a \notin B$. Then

$$\#\partial_{S_k^-} B = \frac{1}{2}\#\partial_{S_k} B \geq (k-1)\#B.$$

On the other hand the difference between $\displaystyle\frac{1}{\#B}\sum_{u\in B_{S_k}} \langle P\delta_u \mid \delta_u\rangle$ and $\beta_1(F_k)$ is

$$\frac{1}{\#B}\sum_{\partial_{S_k^-} B} \langle P\delta_u \mid \delta_u\rangle = \frac{\beta_1(F_k)}{k}\frac{\#\partial_{S_k^-} B}{\#B},$$

where the equality comes from equidistribution on the generating edges for $\beta_1(F_k)$: for any edge in the graph, one has

$$\langle P\delta_u \mid \delta_u\rangle = \frac{k-1}{k}.$$

So we get

$$\frac{1}{\#B}\sum_{\partial_{S_k^-} B} \langle P\delta_u \mid \delta_u\rangle \geq \frac{k-1}{k}\beta_1(F_k),$$

which is already of the order of $\beta_1(F_k)$. Thus one can easily propose a better inequality than the one in Theorem 2 for free groups, but this is too particular to improve the general case.

4. Cheeger constants and measured equivalence relations

The aim of this section is to define the uniform Cheeger constant for measured equivalence relations of type II_1.

Let (X, μ) be a standard probability space. An equivalence relation R with countable classes on X is called Borel if its graph $R \subset X \times X$ is a Borel subset of $X \times X$. It is called a measured equivalence relation if the saturation of a negligible subset of X is again negligible. For instance if Γ is a countable group and α is a non singular action of Γ on (X, μ), then the associated equivalence relation R_α, defined as the partition of X into Γ-orbits, is a measured equivalence relation on X. See [7] for more details.

Let R be a measured equivalence relation on a standard probability space (X, μ). We endow R with the horizontal counting measure $\mathfrak{h}$ given by

$$\mathfrak{h}(K) = \int_X \#K^x d\mu(x)$$

where K is a measurable subset of R and K^x is the subset of $X \times X$ defined as $K^x = \{(x, y) \in K\}$. A partial automorphism of R is a partial automorphism of (X, μ) whose graph is included in R. One says that R is of type II_1 if the measure μ is invariant under every partial automorphism of R.

Recall that a *graphing* of a measured equivalence relation can be described in either one the following two ways (cf. [8]):

(1) a family $\Phi = \{\varphi_i\}_{i\geq 1}$ of partial automorphisms of R such that for almost every $(x, y) \in R$, there exists a finite sequence $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ of elements of $\Phi \cup \Phi^{-1}$ such that $y = \varphi_n \ldots \varphi_1(x)$,
(2) a measurable subset K of R such that R coincide with $\cup_1^\infty K^n$ up to a negligible set, where K^n is the n-th convolution product of K.

Let K be a graphing of R. Let us first define the Cheeger constant $h_K(R)$ of R with respect to K. Consider the measurable field of graphs $\Sigma = \amalg_{x\in X}\Sigma^x$ over X defined as follows (see [9]): the vertices of Σ^x are elements of R^x and the set of edges of Σ^x is the family of triple $(x, y, z) \in R * R$ such that $(y, z) \in K$, where $R * R$ is the fibered product of R with itself over X. There is an obvious action of R on Σ by permutation of fibers. In concordance with group theory we will call Σ the *Cayley graph* of R associated to K. This is an example of a "quasi-periodic metric space" associated to R ([21]). The canonical projection $r : \Sigma \to X$ is called the realization map. One writes $\Sigma^{(0)}$ for the set of vertices and $\Sigma^{(1)}$ for the set of edges of Σ (thus $\Sigma^{(0)} = R$). We define points in Σ as follows (this strengthens the corresponding definition in [21] so as to fit our present purposes).

Definition 6. Let Σ be the Cayley graph of R with respect to a graphing K. By a *point of* Σ we mean the graph in $\Sigma^{(0)}$ of an automorphism of the equivalence relation R. We will say that two points of Σ are *distinct* if the corresponding graphs in $\Sigma^{(0)}$ have null intersection (with respect to the measure $\mathfrak{h}$).

Given a finite set A of points of Σ one denotes by $\partial_K A \subset \Sigma^{(1)}$ the set of edges of Σ with one vertex in A and the other one outside of A. We endow $\Sigma^{(1)}$ with the measure $\nu^{(1)}$ defined by,

$$\nu^{(1)}(E) = \int_X \#(E \cap \Sigma^x) d\mu(x)$$

for a measurable subset E of $\Sigma^{(1)}$.

A graphing K of R is said to be *finite* if it can be partitioned into a finite number of partial automorphisms of R. A type II_1 equivalence relation R on (X, μ) is said to be *finitely generated* if it admits a finite graphing (this is equivalent to saying that R has finite cost, see [8]; the definition of the cost of an equivalence relation is recalled in section 8).

Definition 7. Let R be a finitely generated equivalence relation of type II_1 on (X, μ) and K be a finite graphing of R. The *Cheeger Constant* of R with respect to K is the non negative number

$$h_K(R) = \inf_{A\subset\Sigma} \frac{\nu^{(1)}(\partial_K A)}{\#A},$$

where the infimum is taken over all finite sets of pairwise distinct points of Σ.

The *uniform Cheeger constant* of a finitely generated equivalence relation R is the non negative number

$$h(R) = \inf_{K\subset R} h_K(R)$$

where the infimum is taken over the finite graphings K of R. Note that by definition the uniform Cheeger constant of an equivalence relation is invariant under isomorphism.

Remark 8. Other definitions of the Cheeger constant would a priori be possible in the measure-theoretic context, for instance

$$h'(R) = \inf_{K\subset R} \inf_{A\subset\Sigma^{(0)}} \frac{\nu^{(1)}(\partial_K A)}{\mathfrak{h}(A)}$$

where the second infimum is now taken over all subsets $A \subset \Sigma^{(0)}$ which, say, have finite and non zero $\mathfrak{h}$-measure. One has to be careful here to indeed define a non trivial invariant in that way. For example the above definition gives $h'(R) = 0$ for any ergodic equivalence relation R of type II_1 (see Section 7 in [20]).

In fact it is not clear either that our definition of $h(R)$ can achieve non trivial numbers. It involves an infimum over all possible (finite) graphings of R, in the spirit of the cost of equivalence relations [8, 13], so that a proof that $h(R)$ is a non trivial invariant requires an homotopy invariance type argument. The one we will give below relies on Gaboriau's homotopy invariance theorem [9] for L^2 Betti numbers (see section 5).

Definition 9. An ergodic equivalence relation of type II_1 is called *uniformly non amenable* if $h(R) > 0$.

Proposition 10. A uniformly non amenable ergodic equivalence relation of type II_1 is not amenable.

Proof. This is an easy consequence of Connes-Feldman-Weiss theorem [6]. □

5. The first L^2 Betti number of a type II_1 equivalence relation

We sketch the definition of the first L^2 Betti numbers for measure preserving equivalence relations. For more details and proofs see the paper of Gaboriau [9].
Let (X, μ) be a standard probability space and Σ be a measurable field of oriented 2-dimensional cellular complexes. For $i = 0, 1, 2$ we write $\mathbf{C}[\Sigma^{(i)}]$ for the algebras of functions on the i-cells of Σ which have uniformly finite support: a function $f : \Sigma^{(i)} \to \mathbf{C}$ is in $\mathbf{C}[\Sigma^{(i)}]$ if and only if there exists a constant C_f such that for almost every $x \in X$, the number of i-cell σ of Σ^x such $f(\sigma) \neq 0$ is bounded by C_f. The completion of $\mathbf{C}[\Sigma^{(i)}]$ for the norm

$$\|f\|_2^2 = \int_X \sum_{\sigma \ i-\text{cells in } \Sigma^x} |f(\sigma)|^2 d\mu(x)$$

is a Hilbert space which we denote by $C_i^{(2)}(\Sigma)$. If Σ is uniformly locally finite, i.e. if the number of cells attached a vertex $\sigma \in \Sigma^{(0)}$ is almost surely bounded by a constant C, then the natural (measurable fields of) boundary operators $\partial_i : \mathbf{C}[\Sigma^i] \to \mathbf{C}[\Sigma^{i-1}]$ coming from the 'attaching cells maps' extend to bounded operators $\partial_i^{(2)} : C_i^{(2)}(\Sigma) \to C_i^{(2)}(\Sigma)$. The *first reduced L^2 homology space* of Σ is the quotient space

$$\overline{H}_1^{(2)}(\Sigma) = \ker \partial_1^{(2)} / \overline{\operatorname{Im} \partial_2^{(2)}}$$

of the kernel of $\partial_1^{(2)}$ by the (Hilbert) closure of the image of $\partial_2^{(2)}$. It is naturally isometric to the orthogonal complement of $\overline{\operatorname{Im} \partial_2^{(2)}}$ in $\ker \partial_1^{(2)}$.
Let now R be a measured equivalence relation of type II_1 on (X, μ). Consider a measurable field of oriented 2-dimensional cellular complex Σ endowed with a measurable action of R with fundamental domain (a fundamental domain D in Σ is a measurable set of cells of Σ intersecting almost each R-orbit at a single cell of D). Assume that Σ is uniformly locally finite and denote by N the von Neumann algebra of R . The first L^2 homology space $\overline{H}_1^{(2)}(\Sigma)$ is then a Hilbert module and it has a Murray-von Neumann dimension over N. This dimension is called the first L^2 Betti number of Σ and is denoted by $\beta_1(\Sigma, R)$.
Gaboriau [9] has extended this definition to the non necessarily uniformly locally finite case by using an approximation technique in the spirit of Cheeger-Gromov [5]. He then proved that *the associated L^2 Betti number $\beta_1(\Sigma, R)$ is independent of the choice of Σ provided that Σ is simply connected* (i.e. almost each fiber is simply connected). We refer to this result as Gaboriau's homotopy invariance theorem (it holds for all L^2 Betti numbers, see [9, Théorème 3.13]). The number $\beta_1(\Sigma, R)$ where Σ is simply connected (for instance one can take for Σ the Cayley graph of R with respect to a given graphing K as

defined in the previous section), is called the first L^2 Betti number of R and is denoted by $\beta_1(R)$.

6. The uniform Cheeger constant and the first L^2 Betti number

Theorem 11. Let R be a finitely generated measured equivalence relation of type II_1 on a probability space (X,μ). Then $\beta_1(R) \leq h(R)$.

Proof. Let $K \subset R$ be a finite graphing of R and Σ be the Cayley graph of R associated to K. Denote by $C_0^{(2)}(\Sigma)$ and $C_1^{(2)}(\Sigma)$ the spaces of square integrable functions on the vertices and edges of Y (as in section 5) and consider the bounded fibred boundary operator $\partial_1^{(2)} : C_1^{(2)}(\Sigma) \to C_0^{(2)}(\Sigma)$ coming from the graph structure of Σ. Let $Z_1^{(2)}(\Sigma)$ be the kernel of $\partial_1^{(2)}$ (the space of square integrable 1-cycles) and $Z_1(\Sigma)$ the vector space of functions with uniformly finite support in $Z_1^{(2)}(\Sigma)$. Then $\beta_1(R)$ is the Murray-von Neumann dimension of the orthogonal complement $\bar{H}_1^{(2)}(\Sigma) \subset C_1^{(2)}(\Sigma)$ of $\overline{Z_1(\Sigma)}$ in $Z_1^{(2)}(\Sigma)$ where the closure is understood with respect to the Hilbert norm. Let us explain how to adapt the proof of theorem 2 to this case.
Given a measurable subset $\Omega \subset \Sigma^{(1)}$ for which the function $x \mapsto \#\Omega^x$ is essentially bounded we define

$$\bar{H}_1^{(2)}(\Sigma)_{|\Omega} = \{\sigma_{|\Omega} : \sigma \in \bar{H}_1^{(2)}(\Sigma)\}$$

to be the space of restrictions of harmonic chains to Ω. This is a fibred linear subspace of the space $C_1(\Omega)$ of complex functions with uniformly finite support on Ω. Let $P : C_1^{(2)}(\Sigma) \to C_1^{(2)}(\Sigma)$ be the (equivariant) orthogonal projection onto $\bar{H}_1^{(2)}(\Sigma)$ and $\chi_\Omega : C_1^{(2)}(\Sigma) \to C_1^{(2)}(\Sigma)$ be the orthogonal projection on $C_1(\Omega)$.
Let A be a finite subset of points of Σ (Def. 6). We write A_K for the set of edges of Σ with a vertex in A, and $\partial_K A$ the set of edges of Σ with exactly one extremity in A. Then for every finite subset of pairwise disjoints points of Σ one has

$$\beta_1(R) \leq \frac{1}{\#A} \int_X \dim_{\mathbf{C}} \bar{H}_1^{(2)}(\Sigma)^x_{|A_K} d\mu(x).$$

Indeed denote by $M_{A_K} = \chi_{A_K} P$ (considered as a fibered operator from $C_1(A_K)$ to itself with range $\bar{H}_1^{(2)}(\Sigma)_{|A_K}$). We have

$$\begin{aligned}
\dim_N \bar{H}_1^{(2)}(\Sigma) &= \int_K \langle P^{r(\gamma)} \delta_\gamma \mid \delta_\gamma \rangle d\mathfrak{h}(\gamma) \\
&\leq \frac{1}{\#A} \int_{A_K} \langle P^{r(u)} \delta_u \mid \delta_u \rangle d\nu^{(1)}(u) \\
&= \frac{1}{\#A} \int_X \mathrm{Tr}\, M^x_{A_K} d\mu(x) \\
&\leq \frac{1}{\#A} \int_X \dim_{\mathbf{C}} \bar{H}_1^{(2)}(\Sigma)^x_{|A_K} d\mu(x)
\end{aligned}$$

where the last inequality follows from the a.e. inequality $\|M^x_{A_K}\| \leq 1$. Now as the boundary operators are fibered one can apply Lemma 3 to conclude that

$$\dim_{\mathbf{C}} \bar{H}_1^{(2)}(\Sigma)^x_{|A_K} = \dim_{\mathbf{C}} \bar{H}_1^{(2)}(\Sigma)^x_{|\partial A_K} \leq \#\partial_K A^x$$

for almost every $x \in X$. Thus $\beta_1(R) \leq h_K(R)$ and the Theorem follows. $\square$

Remark 12 (See also Section 9)**.** Keeping the notations of the proof, the number

$$\tau(R) = \inf_{K \subset R} \dim_N \overline{Z_1(\Sigma_K)},$$

where the infimum is taken over all finite graphings K of R (and Σ_K is the Cayley graph of R with respect ot K) was called the *rate of cycles of* R in [21]. It was introduced there because of its relations to the following (still open) question of Gaboriau [9, section 3.6]: is it true that $C(R) = \beta_1(R) + 1$ for every finitely generated ergodic equivalence relation of type II_1 ? (here $C(R)$ is the cost of R, see section 8). One has indeed

$$C(R) = \tau(R) + \beta_1(R) + 1$$

for any finitely generated ergodic equivalence relation of type II_1 (see [21], this is not hard to show and can be seen exactly as in the group case [19]). Thus there is no known examples of equivalence relations with non trivial rate of cycles. "Erasing cycles" of a given graphing leads to the existence of non amenable subtreeings in equivalence relation with non trivial cost ([21]).

Proposition 13. Let Γ be a countable group and α be an ergodic essentially free measure preserving action of Γ on a probability space (X, μ). Let R_α be the orbit partition of X into the orbits of α. Then $\beta_1(\Gamma) \leq h(R_\alpha) \leq h(\Gamma)$

Proof. The first equality follows from [9] and Theorem 11 while the second follows from the definitions (simply note that for any Cayley graph Y of Γ and any distinct points $\gamma_1, \gamma_2 \in \Gamma = Y^{(0)}$ the graphs of $\alpha(\gamma_1^{-1})$ and $\alpha(\gamma_2^{-1})$ are distinct points in the corresponding Cayley graph of R_α because α is essentially free). □

7. The case of discrete groupoids

Let G be an r-discrete measured groupoid of type II_1 with base space (X, μ) and counting Haar system $\mathfrak{h} = \#$ (see [1]). We write r and s for the range and the source map, respectively. Examples of r-discrete measured groupoids include countable groups and measured equivalence relations with countable classes.

By a graphing of G we mean a measurable subset K of G such that $\cup_1^\infty K^n = G$ up to a negligible set, where K^n is the n-th convolution product of K. To a graphing is associated a *Cayley graph* of G, which is constructed as for measured equivalence relations and coincide with the classical definition when G is a countable group. Let Σ be a Cayley graph of G, associated to a graphing K. In order to define $h_K(G)$ we only need to precise the notion of a point in Σ (compare Def. 6), as we can then copy the equivalence relations definition (see Section 4). Recall that the vertex set $\Sigma^{(0)}$ of Σ coincides with G. By a *point of* Σ we mean *a subset a of $\Sigma^{(0)} = G$ such that the restrictions $s_{|a}$ and $r_{|a}$ of both the range and the source maps are measurable isomorphisms.* Note that these definitions coincide with the ones given for groups and equivalence relations. We say that G is finitely generated if it admits a finite graphing, i.e. a graphing K for which the functions $x \mapsto \#K^x$ and $x \mapsto \#K_x$ are in $L^\infty(X, \mu)$, where $K^x = K \cap r^{-1}(x)$ and $K_x = K \cap s^{-1}(x)$. For a finitely generated measured groupoid we set

$$h(G) = \inf_{K \subset G} h_K(G)$$

where the infimum ranges over all finite graphings K of G. As far as quasi-periodicity is concerned the difference between measured equivalence relations and measured groupoids when it comes to the definition of a 'notion of quasi-periodicity' (see [21]) is that the latter forces some parts of the quasi-periodic metric space under consideration to actually be periodic.

As noted by Gaboriau in [9] the notion of L^2 Betti numbers for measured equivalence relations also extends to r-discrete measured groupoids of type II_1 and gives an isomorphism invariant.
With these definitions at hand it is now an easy exercise to adapt the proof of Theorem 11 to the present situation. As in the case of equivalence relations the above definition of $h(G)$ is conceived to imply the following estimate to hold,

$$\beta_1(G) \leq \frac{1}{\#A} \int_X \dim_{\mathbf{C}} \bar{H}_1^{(2)}(\Sigma)^x_{|A_K} d\mu(x)$$

(with obvious notations), from which the desired result follows:

Theorem 14. Let G be a finitely generated r-discrete measured groupoid of type II_1. Then $\beta_1(G) \leq h(G)$.

8. Ergodic Cheeger constant of countable groups

Let Γ be a finitely generated group. We define the *ergodic Cheeger constant* of Γ by the expression

$$h_e(\Gamma) = \inf_\alpha h(R_\alpha)$$

where the infimum is taken over all ergodic essentially free measure preserving actions α of Γ on a probability space, R_α is the partition of X into the orbits of α, and $h(R_\alpha)$ is the uniform Cheeger constant of R_α. Thus we have $\beta_1(\Gamma) \leq h_e(\Gamma) \leq h(\Gamma)$ by Proposition 13. Examples of groups for which $h_e(\Gamma) < h(\Gamma)$ are given below. Let us first recall the notion of cost of an equivalence relation [8].

Let R be an ergodic equivalence relation of type II_1 on a probability space (X, μ). The cost of a partial automorphism $\varphi : A \to B$ of R is the measure of its domain, $C(\varphi) = \mu(A)$. The cost of a graphing $\Phi = \{\varphi_i\}_{i\geq 1}$ of R (see section 4) is defined to be

$$C(\Phi) = \sum_{i\geq 1} C(\varphi_i)$$

while the *cost of* R is the infimum

$$C(R) = \inf_\Phi C(\Phi)$$

where Φ runs among all graphings Φ of R. The *cost of a countable group* Γ is the infimum

$$C(\Gamma) = \inf_\alpha C(R_\alpha)$$

where α runs over all ergodic essentially free measure preserving actions α of Γ on a probability space. This definition has been introduced by Levitt. See [13] for a survey.

Proposition 15. Let R be a ergodic equivalence relation with cost 1. Then $h(R) = 0$.

Proof. Let R be an ergodic equivalence relation with cost 1. Let φ be an ergodic automorphism of R and for each real number $\varepsilon > 0$ let ψ_ε be a partial automorphism of R of cost ε such that

$$\Phi_\varepsilon = (\varphi, \psi_\varepsilon)$$

is a graphing of R. The existence of such graphings of R is proved in [8]. Denote by R_φ be the equivalence relation generated by φ and fix $n_0 \in \mathbf{N}$, $\varepsilon_0 > 0$. By Rokhlin Lemma there exists a family $B_1, \ldots, B_{n_0}$ of disjoints subset of X such that

$$\varphi(B_i) = B_{i+1} \text{ for } i = 1 \ldots n_0 - 1 \text{ and } \mu(X \setminus \cup_1^{n_0} B_i) \leq \varepsilon_0.$$

Suppose that $\varepsilon < \mu(B_1)$ and consider two partial automorphisms $\theta_{\varepsilon,1}$ and $\theta_{\varepsilon,2}$ of the equivalence relation R_φ such that $\theta_{\varepsilon,1}(\operatorname{dom}\psi_\varepsilon) \subset B_1$ and $\theta_{\varepsilon,2}(\operatorname{Im}\psi_\varepsilon) \subset B_1$. Set

$$\psi'_\varepsilon = \theta_{\varepsilon,2}\psi_\varepsilon\theta_{\varepsilon,1}^{-1}.$$

It is not hard to check that $\Phi'_\varepsilon = (\varphi, \psi'_\varepsilon)$ is a graphing of R. By letting $n_0 \to \infty$ we get the following property: for any integer $n \in \mathbf{N}$ and any $\varepsilon > 0$ sufficiently small there exists a graphing $\Phi_{\varepsilon,n}$ of R of the form $\Phi_{\varepsilon,n} = (\varphi, \psi_{\varepsilon,n})$ whose cost is less than $1+\varepsilon$ and such that for almost every $x \in X$, the intersection of either the domain or the image of $\psi_{\varepsilon,n}$ with the finite set

$$(x, \varphi(x), \ldots, \varphi^n(x))$$

is at most one point (this should be compared to the fact that the "concentration of measure" property fails for automorphisms of standard probability spaces by Rokhlin Lemma, see [20]). Let us now consider the Cayley graph $\Sigma_{\varepsilon,n}$ associated to $\Phi_{\varepsilon,n}$ as in section 4. Let A_n be the finite set of pairwise disjoints points of $\Sigma_{\varepsilon,n}$ given by $A_n = \{\varphi^i\}_{i=0}^n$. Then the boundary of A_n^x in $\Sigma_{\varepsilon,n}^x$ consists of at most 4 points for almost every $x \in X$. It follows that $h(R) = 0$. □

Corollary 16. If a group has cost 1, then its ergodic Cheeger constant is zero.

Proof. This follows from the fact that the infimum over the actions of the group occurring in the definition of the cost is attained [8]. Note that the infimum in [8] is taken over all (not necessarily ergodic) measure preserving essentially free actions but this infimum is attained (and thus is a minimum) for an ergodic action, as one easily sees by replacing the infinite product measure by an ergodic joining [10] in the proof of Proposition VI.21 in [8]. □

Thus the class of uniformly non amenable groups appears to be much larger from the geometric point of view than from the ergodic point of view. Breuillard and Gelander proved in [4] that for an arbitrary field K, any non amenable and finitely generated subgroup of $\mathrm{GL}_n(K)$ is uniformly non amenable. This is the case for instance for $\mathrm{SL}_3(\mathbf{Z})$ while this group has cost 1 (see [8]) and thus $h_e(\mathrm{SL}_3(\mathbf{Z})) = 0$. More generally if Γ is a lattice in a semi-simple Lie group of real rank at least 2, then Γ has cost 1 by Corollaire VI.30 in [8]. Note that non uniform lattices have fixed price [8] and in that case any measure preserving action gives an equivalence relation with trivial Cheeger constant.

Corollary 17 (Compare [4])**.** Lattices in a semi-simple Lie group of real rank at least 2 have trivial ergodic Cheeger constant.

For the case of direct product of groups one gets the following result.

Corollary 18. Finitely generated groups which are decomposable as a direct product of two infinite groups have trivial ergodic Cheeger constant. Finitely generated equivalence relation which are decomposable as a direct product of two equivalence relations with infinite classes have trivial uniform Cheeger constant.

Proof. This follows from the fact that the cost of a direct product is 1 [8]. See also [13]. □

9. On the Day-von Neumann problem for uniformly non amenable equivalence relations

Using Adian's theorem, Osin [17] proved that free Burnside groups with large (odd) exponent are uniformly non amenable (and even that the regular representation has non vanishing uniform Kazhdan's constant). As this groups do not contain free groups he

deduced the existence of finitely generated groups which are uniformly non amenable and do not contain any free group on two generators [17, Theorem 1.3]. The question of the existence of non abelian free groups in non amenable groups is often referred to as the (Day-)von Neumann problem. It has been solved negatively by Ol'shanskii in 1980 ; Adian (1982) has shown that the above mentioned Burnside groups are non amenable.
The Day-von Neumann problem is an open question for measured equivalence relations. It can be formulated in the following way. Let (X, μ) be standard probability space and R be a non amenable ergodic equivalence relation of type II_1 on (X, μ). Is it true that R contains a non amenable subtreeing ? Recall that a treeing is a graphing without non negligible cycle (see [8]).
It has been proved independently (in 2001) by the first author [21] and by Kechris-Miller [13] that every ergodic equivalence relation of type II_1 with cost greater than 1 (and thus non amenable) contains a non amenable subtreeing. Combining with Theorem 15 we get the following result.

Corollary 19 (See [17] for the group case)**.** Let R be a uniformly non amenable ergodic equivalence relation of type II_1. Then R contains a non amenable subtreeing.

10. Fundamental groups

Let (X, μ) be a probability space and R be an ergodic equivalence relation of type II_1 on (X, μ). The so-called *fundamental group* of R is the multiplicative subgroup of $\mathbf{R}_+^*$ generated by the measure of all Borel subsets Y of X such that the restricted equivalence relation $R_{|Y}$ is isomorphic to R. The next proposition is a corollary of Theorem 15 and the fact that equivalence relations with non trivial cost have trivial fundamental group, which is proved in [8]. We now give a direct proof of this result.

Proposition 20. An ergodic equivalence relation of type II_1 which is uniformly non amenable has a trivial fundamental group.

Proof. This is straightforward from the following compression formula. Let R be an ergodic equivalence relation on (X, μ) and $Y \subset X$ be a non negligible measurable subset of X. Let S be the restriction of R to Y. Then

$$h(R) \leq \mu(Y) h(S).$$

Note that (by definition) $h(S)$ should be computed with respect to the normalized measure $\mu_1 = \frac{\mu}{\mu(Y)}$. Let us prove this formula.
Fix $\varepsilon \in (0, \mu(Y))$. Let K be a graphing of S such that $h_K(S) \leq h(S) + \varepsilon/4$ and let A be a finite set of pairwise distinct points of Σ_K such that

$$\frac{\nu_1^{(1)}(\partial_K A)}{\# A} < h_K(S) + \varepsilon/4$$

and $\# A > 12/\varepsilon$, where $\nu_1^{(1)}$ is the counting measure on $\Sigma_K^{(1)}$ associated to μ_1. Write $A = \{\psi_1, \ldots, \psi_k\}$ where $\psi_j \in [[S]]$, $j = 1 \ldots k$.
Let n be an integer greater than k and $\{Y_i\}_{i \in \mathbf{Z}/n\mathbf{Z}}$ be a partition of $X \backslash Y$ into n measurable subsets of equal measure $\delta < \varepsilon/4$. Choose $Y_{-1} \subset Y$ such that $\mu(Y_{-1}) = \delta$ and consider partial isomorphisms

$$\varphi_{-1} : Y_{-1} \to Y_0$$

and

$$\varphi[i] : Y_i \to Y_{i+1}, \ i \in \mathbf{Z}/n\mathbf{Z},$$

whose graphs are included in R and such that the automorphism $\varphi = \amalg_{i\in \mathbf{Z}/n\mathbf{Z}}\varphi_i$ induces an action of $\mathbf{Z}/n\mathbf{Z}$ on $X\backslash Y$. Then

$$K' = K \cup \{\varphi_{-1}\} \cup \{\varphi\}$$

is a graphing of R. Denote by Σ_K the Cayley graph of S associated to K and $\Sigma_{K'}$ the Cayley graph of R associated to K'. For $j = 1 \ldots k$ consider the automorphism of X defined by

$$\psi'_j = \varphi^j \amalg \psi_j$$

and observe that the graphs of ψ_j, $j = 1 \ldots k$, are pairwise disjoint. Set $A' = \{\psi'_j\}_{j=1\ldots k}$. Then for $y \in Y$ one has

$$\#(\partial_{K'}A')^y \leq \#(\partial_K A)^y + \#\{\psi \in A, \psi(y) \in Y_{-1}\}$$

and for $y \in X\backslash Y$ one has $\#(\partial_{K'}A')^y \leq 3$. Thus

$$\nu^{(1)}(\partial_{K'}A') \leq \mu(Y)\nu_1^{(1)}(\partial_K A) + k\delta + 3\mu(Y\backslash X)$$

where $\nu^{(1)}$ is the counting measure on $\Sigma^{(1)}$ associated to μ. It follows that

$$\begin{aligned} h(R) &\leq \frac{\nu^{(1)}(\partial_{K'}A')}{\#A'} \\ &\leq \mu(Y)(h(S) + \varepsilon/2) + \varepsilon/2 \\ &\leq \mu(Y)h(S) + \varepsilon. \end{aligned}$$

So $h(R) \leq \mu(Y)h(S)$ as required. □

Appendix A. Comparison with alternative definitions of uniform amenability

In this section we analyze the differences and analogies between the different notions of uniform non amenability. In [2], Arzhantseva, Burillo, Lustig, Reeves, Short and Ventura give a definition of *Følner constants*, which is very close to our definition of Cheeger constants. The only difference in the definition is the fact that they consider the inner boundary while we consider the geometric boundary. For a finitely generated group Γ, with generating system S, and A a finite part of the Cayley graph of Γ, one has

$$\text{Føl}_X(\Gamma, A) = \frac{\#\partial_S^{int}A}{\#A} \text{ with } \partial_S^{int}A = \{a \in A, \exists x \in S \cup S^{-1}, ax \notin A\},$$

while our boundary is $\partial_S A = \{(a, ax), x \in S \cup S^{-1}, a \in A, ax \notin A\}$ with (a, ax) the edge between the vertices a and ax.

Considering sets A without isolated points, one immediately gets that

$$\text{Føl}_S(\Gamma, A) \leq h_S(\Gamma, A) \leq (2\#S - 1)\text{Føl}_S(\Gamma, A) \tag{1}$$

hence one has the following proposition.

Proposition 21. Let $\text{Føl}_S(\Gamma)$ be the Følner constant defined in [2], then one has

$$\text{Føl}_S(\Gamma) \leq h_S(\Gamma) \leq (2\#S - 1)\text{Føl}_S(\Gamma).$$

In particular one gets that $\text{Føl}(\Gamma) \leq h(\Gamma)$.

This means that a uniformly non amenable group in the sense of [2] is uniformly non amenable in our sense. It is unclear whether the converse is true : there might exist non amenable groups that are not uniformly non amenable in the sense of [2] but uniformly non amenable in our sense. However, for all known examples of such groups, there exists a maximal bound for the size of the generating systems used to reach the infimum, so these groups are also non uniformly non amenable in our sense. This can be summarized as follows.

Claim 22. Suppose there exists an integer n such that $\inf_S \mathrm{Føl}_S(\Gamma) = \inf_{S, \#S \leq n} \mathrm{Føl}_S(\Gamma)$, then both notions of uniform non amenability coincide.

Another notion of uniform non amenability was introduced by Osin in [16], linked with the Kazhdan estimates for the regular representation λ. A group is said to be uniformly non amenable if $\alpha(\Gamma) = \inf_S \alpha(\Gamma, S) > 0$, where S runs over all finite generating systems of Γ and

$$\alpha(\Gamma, S) = \inf_{u \in \ell^2(\Gamma), u \neq 0} \max_{x \in S} \frac{\|\lambda(x)u - u\|}{\|u\|}.$$

(Note that this constant is also presented as a Kazhdan constant for the regular representation in [2] and denoted by $K(\Lambda_\Gamma, \Gamma)$.)
Consider the Laplacian associated with the Cayley graph Y generated by a given symetric system of generators S, *id est* the operator Δ_S acting on $\ell^2(\Gamma)$ by

$$\Delta_S(u)(g) = \frac{1}{\#S} \sum_{x \in S} [u(g) - u(xg)].$$

Let A be a finite subset in Y and consider the normalized characteristic function $u = \frac{\chi_{A^{-1}}}{\sqrt{\#A}}$. Then one has

$$\|\Delta_S(u)\|^2 = \frac{1}{\#S\#A} \sum_{x \ \in S, g \in \Gamma} [\chi_{A^{-1}}(g) - \chi_{A^{-1}}(xg)]^2 = \frac{\#\partial_S A}{\#S\#A}.$$

Hence, one has that

$$\frac{\alpha(\Gamma, S)^2}{\#S} \leq \inf_{u \in \ell^2(\Gamma), \|u\|=1} \|\Delta_S(u)\|^2 \leq \frac{h_S(\Gamma)}{\#S},$$

so we have proven the following proposition.

Proposition 23. Let $\alpha(\Gamma)$ be the constant introduced by Osin [16]. Then one has

$$\alpha(\Gamma) \leq \sqrt{h(\Gamma)}.$$

This means that a non uniformly amenable group in the sense of Osin is still non amenable in our case, while the converse seems to be an open question [2].

Appendix B. Some properties of Cheeger constants

In this section we restate properties showed in [2] on Følner constants for the Cheeger constants we introduce. We have ommited the proofs as they are *quasi verbatim* the ones in [2]. First of all, Cheeger constants are linked to exponential growth. Recall that the exponential growth rate of a group Γ finitely generated by S is defined as the limit $\omega_S(\Gamma) = \lim_{n \to \infty} \sqrt[n]{\#B_S(n)}$ where $B_S(n)$ is the ball of elements in Γ with geodesic distance as S words at most n.

Proposition 24 (Cheeger constant and exponential growth)**.**
Let Γ be a finitely generated group and S a finite generating system, then

$$h(\Gamma) \leq h_S(\Gamma) \leq (2\#S - 1)[1 - \frac{1}{\omega_S(\Gamma)}].$$

Note that we don't know in general in our case if uniform non amenability implies uniform exponential growth. This is the case when we know that the infimum can be attained for systems of generators with uniform bound on the cardinals of these systems.
About the Cheeger constants for subgroups and quotients the theorems of [2] are exactly the same.

Theorem 25 (Subgroups)**.**
Let Γ be a finitely generated group and S a finite system of generators.

(1) Consider the subgroup Γ' generated by a subsystem $S' \subset S$. Then one has $h_S(\Gamma) \geq h_{S'}(\Gamma')$.
(2) Let H be a subgroup of γ generated by a system T with m elements and such that the length of each element of T as a word in S is at most L. Then $h_S(\Gamma) \geq \dfrac{h_T(H)}{1+mL}$.

Theorem 26 (Quotients)**.** Let Γ be a finitely generated group and S a finite system of generators. Denote by N a normal subgroup and by $\pi : \Gamma \to \Gamma/N$ the natural projection. Then $h_S(\Gamma) \geq h_{\pi(S)}(\Gamma/N)$ hence $h(\Gamma) \geq h(\Gamma/N)$

Appendix C. Some bounds on Cheeger constants

In this section we give an explicit calculation for the free group, as in [2], and derive some bounds for groups by quotient and subgroup theorems.

Proposition 27. For the free group on k generators, one has $h(F_k) = 2k - 2$.

First, we prove that this a lower bound.

Lemma 28. Let A be a finite part of Y the Cayley graph of F_k generated by S a system of k free generators. Then one has

$$\frac{\#\partial_S A}{\#A} \geq 2k-2.$$

Proof. Denote by V_j the number of vertices in A which have j edges sitting completely in A. Then A has Euler characteristic null being a tree, so that

$$1 - \sum_{j=1}^{2k} V_j + 1/2 \sum_{j=1}^{2k} V_j = 0.$$

Then one has $\#\partial_S A = \displaystyle\sum_{j=1}^{2k}(2k-j)V_j$ hence

$$\frac{\#\partial_S A}{\#A} = 2k - \frac{\sum_{j=1}^{2k} jV_j}{\sum_{j=1}^{2k} V_j} = 2k-2+\frac{2}{\#A} \geq 2k-2.$$

∎

Now using comparison with the exponential rate, we know that

$$h(F_k) \leq (2k-1)(1-\frac{1}{\omega_S(F_k)})$$

and that $\omega_S(F_k) = 2k-1$. Hence we showed that

Lemma 29.
Let F_k be the free group on k generators, then $h(F_k) \leq 2k-2$.

To conclude we only need to know that if S' is any generating system for F_k then it contains k elements which generates freely a subgroup H of F_k which is isomorphic to F_k. This is shown in proposition 5.4 of [2], and we get in the same way Proposition 27.
From this result and the quotients and subgroup theorem, we can derive a certain number of results for groups, that are analogous to the ones in [2].

Proposition 30.

(1) Let Γ be a finitely generated group which admits a system S of k generators. Then $h(\Gamma) \leq 2k - 2$ with equality if only if Γ is a free group F_k.
(2) Let Γ be a finitely generated group such that $h(\Gamma) \leq 2k - 2$ for a given $k \geq 2$. Then the rank of G is at least k.

N.B. The present paper substitutes and extends a short note by the first author circulating during his Ph. D. under the same title, where the group case only (Theorem 2) was considered.

References

[1] C. Anantharaman-Delaroche and J. Renault. "Amenable Groupoids", L'Enseignement Mathématique, Geneva, *2000*.

[2] Arzhantseva G., Burillo J., Lustig M., Reeves L., Short H., Ventura E., "Uniform non amenability", to appear in Adv. in Math.

[3] Atiyah M. F., "Elliptic operators, discrete groups and von Neumann algebras", Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974), pp. 43–72. Asterisque, No. 32-33, Soc. Math. France, Paris, *1976*.

[4] Breuillard E., Gelander T., "Cheeger constant and algebraic entropy of linear groups", Int. Math. Res. Not., no. 56, 3511–3523, *2005*.

[5] Cheeger J., Gromov M., "L_2-cohomology and group cohomology", Topology 25, no. 2, 189–215, *1986*.

[6] Connes A., Feldman J., Weiss B., "An amenable equivalence relation is generated by a single transformation", Ergod. Th. & Dynam. Sys., 1, 431-450, *1981*.

[7] Feldman J., Moore C., "Ergodic equivalence relations, cohomology, and von Neumann algebras. I", Trans. Amer. Math. Soc., 234(2), 289-324, *1977*.

[8] Gaboriau D., "Coût des relations d'équivalence et des groupes", Invent. Math. 139, no. 1, 41–98, *2000*.

[9] Gaboriau D., "Invariants l^2 de relations d'équivalence et de groupes", Publ. Math. Inst. Hautes Études Sci. No. 95, 93–150, *2002*.

[10] Glasner E., "Ergodic Theory via Joinings", American Mathematical Society, Providence, Rhode Island, *2003*.

[11] Gromov M., "Asymptotic invariants of infinite groups", Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge, *1993*.

[12] de la Harpe P., "Topics in geometric group theory", Chicago Lectures in Math. Series, *2000*.

[13] Kechris A., Miller B., "Topics in orbit equivalence", Lecture Notes in Mathematics 1852. Berlin: Springer.

[14] Lück W., "L^2-invariants: theory and applications to geometry and K-theory", Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 44. Springer-Verlag, Berlin, *2002*.

[15] Lück W. "Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers, I Foundations", J. Reine Angew. Math. 495, 135–162, *1998*.

[16] Osin D., "Weakly amenable groups", Computational and statistical group theory (Las Vegas, NV/Hoboken, NJ, 2001), 105–113, Contemp. Math., 298, Amer. Math. Soc., Providence, RI, *2002*.

[17] Osin D., "Uniform non amenability of free Burnside groups", preprint.

[18] Pansu P., "Introduction to L^2 Betti numbers", Riemannian geometry (Waterloo, ON, 1993), 53–86, Fields Inst. Monogr., 4, Amer. Math. Soc., Providence, RI, *1996*.

[19] Pichot M., "Semi-continuity of the first ℓ^2 Betti number on the space of finitely generated groups", Comm. Math. Helv., 81, No. 3, 643–652, *2006*.

[20] Pichot M., "Espaces mesurés singuliers fortement ergodiques – Étude métrique-mesurée", to appear in Ann. Inst. Fourier.
[21] Pichot M., "Quasi-périodicité et théorie de la mesure", Ph. D. Thesis.

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