UNBOUNDED PSEUDODIFFERENTIAL CALCULUS ON LIE GROUPOIDS

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Abstract

We develop an abstract theory of unbounded longitudinally pseudodifferential calculus on smooth groupoids (also called Lie groupoids) with compact basis. We analyse these operators as unbounded operators acting on Hilbert modules over $C^*(G)$, and we show in particular that elliptic operators are regular. We construct a scale of Sobolev modules which are the abstract analogues of the ordinary Sobolev spaces, and analyse their properties. Furthermore, we show that complex powers of positive elliptic pseudodifferential operators are still pseudodifferential operators.

1 Introduction

The use of groupoids to analyse the properties of noncommutative objects goes back to the founding work of Connes[Con79, Con94] on foliations, when the longitudinal pseudodifferential calculus was linked with the holonomy groupoid of the foliation. Since then, the groupoids have appeared as very rich structures which encode the singularities of the considered objects. For pseudodifferential calculus in particular, a general framework was introduced in [MP97, NWX99], which allows the definition of a pseudodifferential calculus attached to any smooth groupoid. Monthubert [Mon98] also used this framework to show that the $b$-calculus developed by Melrose for manifolds with boundary or with corners could be developed fully in terms of groupoids. This is equally true for cusp-calculus, and significative progresses have also been made by Claire Debord [Deb00] and Mariusz Crainic and Rui L. Fernandes [CF03] for singular foliations and integration of Lie algebroids. A general aim would be to know how singular problems can be translated in the language of groupoids, which would unify the approach to this kind of problems.

However, both articles [MP97, NWX99] deal mainly with the case of bounded operators, i.e. of pseudodifferential operators of order less than or equal to 0. To complete the picture, one needs to be able to deal with unbounded calculus which is necessary, for example to treat differential operators and their functional calculus (in particular complex powers). The aim of this article is to give a general and abstract framework to develop this unbounded pseudodifferential calculus on Lie groupoids. As shown in [MP97, NWX99], pseudodifferential operators of negative orders are bounded operators on the $C^*$-algebra of the groupoid $C^*(G)$, i.e. morphisms

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on the Hilbert $C^*(G)$-module $E = C^*(G)$. This leads us to treat positive order operators as unbounded operators in the very powerful framework of Hilbert modules. The key result is the fact that elliptic operators are regular operators in the sense of Baaj (see [Baa80] and [BJ83]). Regular operators on Hilbert modules are particularly nice unbounded operators, as they are deeply linked with bounded morphisms by the Woronowicz transform. In particular, the normal ones admit functional calculus, and we will develop the complex powers using this calculus. One of the main results of this paper is the definition of a natural scale of Sobolev Hilbert modules, which are abstract analogues of ordinary Sobolev spaces. As in the classical case, pseudodifferential operators act naturally on these Sobolev modules.

In order to include the resolvent and the complex powers of a differential operator in this calculus, we have to enlarge the class of compactly supported pseudodifferential operators. We then define a new class of (non compactly supported) smoothing operators, which is the intersection of the morphisms between all Sobolev modules. We show that this class is stable under holomorphic functionnal calculus in the $C^*$-algebra of the groupoid. However, this class of smoothing operator is quite wide, as the smoothing operators are a priori continuous solely in the transverse direction, and have a priori no better decay at infinity than any element in the $C^*$- algebra of the goupoid. One could hope to find in particular cases a smaller class of smoothing operator.

We then develop the theory of complex powers for a definite positive pseudodifferential elliptic operator in this generalized calculus, and show that for every $s \in \mathbb{C}$ the regular operator $A^s$ can be written as the sum of a compactly supported pseudodifferential operator and of a non compactly supported smoothing operator.

We would like to stress the fact that though this theory might seem quite abstract, one gets a new approach to several singular problems by applying these objects in a particular context. Indeed, the theory of regular operators on Hilbert modules is well behaved with respect to taking representations of the $C^*$-algebra. One obtains by this procedure concrete Sobolev spaces, as the image of a regular operator by a representation is still a regular operator. We will briefly sketch at the end of this paper the application to the foliated case, that we develop fully in another paper [Vas04] where we analyse the noncommutative residue for foliated manifold.

We should also add that here we only use the smoothness in the longitudinal sense, that is on each $G_x$, and not on all of $G$. We intend to treat in a forthcoming paper the case of more general groupoids, namely groupoids which are smooth in the longitudinal direction but solely continuous in the transverse one, as the continuous family groupoids presented by Paterson [Pat00, Pat01, LMN00].

We shall now briefly review the content of each section.

In section 2, we briefly recall some definitions and facts (without proofs) on the theory of Hilbert modules and regular operators, as this will be central in our approach to pseudodifferential operators on groupoids.

In section 3, we review definitions and basic properties of Lie groupoids and ordinary pseudodifferential calculus on a smooth manifold.

In section 4 we deal with pseudodifferential calculus on Hausdorff Lie groupoids with compact basis $G^{(0)}$. We begin by recalling the definition and properties developed in [MP97, NWX99]. We will allow the families of pseudodifferential operators to have a $C^k$-dependance on the parameter rather than a smooth one. Further, we introduce a new class of smoothing operators, which allows to define a “generalized” pseudodifferential calculus.

In section 5 we construct a scale $(H_s)_{s \in \mathbb{R}}$ of Sobolev modules associated to each elliptic
pseudodifferential operator of positive order, and we then show that these Sobolev modules are independent of the chosen operator. We then study their properties and get the following analogues of classical ones.

1. The modules $H^s$ et $H^{-s}$ are dual $C^*$-modules.

2. We have $H^s \subset H^{s'}$ whenever $s \geq s'$. The inclusion map is a compact morphism in the sense of $C^*$-modules if $s > s'$.

3. A pseudodifferential operator with order $m \in \mathbb{C}$ defines for any $s \in \mathbb{R}$ a morphism $H^s \rightarrow H^{s-m_0}$, where $m_0 = \Re m$.

4. An operator $R$ is smoothing if and only if it is in $\cap_{s,t} \mathcal{L}(H^s, H^t)$.

In section 6 we construct the complex powers of a positive elliptic pseudodifferential operator of integral order, using an analogue of the proof of Guillemin [Gui85] in the case of a compact manifold. We show in particular that the complex powers of such an operator are pseudodifferential operators in our generalized sense.

Finally, we briefly sketch in section 7 how to recover from our work some results of Connes [Con79] and Kordyukov [Kor95] in the case of foliated manifolds.

2 Preliminaries on Hilbert modules

We recall here some “classical” results on the theory of operators on a Hilbert $C^*$-module. For more details and proofs, the reader is referred to [Ska96, Lan95, WN92].

2.1 Morphisms of Hilbert Modules

Suppose we are given a $C^*$-algebra $A$, and $E$ a right $A$-module. A sesquilinear positive map is a map

$$\langle \ , \ \rangle : E \times E \rightarrow A$$

which is antilinear in the first variable and linear in the second and such that for any $x \in E$, $\langle x, x \rangle \geq 0$. A scalar product on $E$ is a sesquilinear positive map such that the map $y \rightarrow \langle x, y \rangle$ is $A$-linear.

**Property** — Let $\langle \ , \ \rangle$ be a sesquilinear positive map.

1. For all $x, y \in E$, we have $\langle y, x \rangle = \langle x, y \rangle^*$.

2. The map $p : x \mapsto \|\langle x, x \rangle\|^{\frac{1}{2}}$ defines a semi-norm on $E$.

3. The sesquilinear map $\langle \ , \ \rangle$ admits an extension to the Hausdorff completion $E_p$ of $E$ for the semi-norm $p$.

We then call *pre-Hilbert $A$-module* an $A$-module endowed with a scalar product. A *Hilbert $A$-module* is then a pre-Hilbert $A$-module which is Hausdorff and complete.

**Examples** —
1. If $A = \mathbb{C}$, a Hilbert module is simply a Hilbert space.

2. The $C^*$-algebra $A$ endowed with the scalar product $\langle a, b \rangle = a^*b$ is a Hilbert module over itself.

3. If $E$ and $E'$ are Hilbert $A$-modules, their Hilbert sum $E \oplus E'$ is the Hilbert module defined over the $A$-module $E \oplus E'$ by the scalar product
   \[ \langle x \oplus y, x' \oplus y' \rangle = \langle x, x' \rangle + \langle y, y' \rangle. \]

4. The Hilbert sum can be extended to an infinite sum. In particular, if $E$ is a Hilbert $A$-module we denote by $H^E$ the Hilbert module $\ell_2(N, E)$ formed of the sequences $(x_n)_{n \in \mathbb{N}}$ in $E$ such that $\sum_{n \in \mathbb{N}} \langle x_n, x_n \rangle$ is convergent in $A$:
   \[ H^E = \ell_2(N, E) = \bigoplus_{n \in \mathbb{N}} E \]
   is a Hilbert module for the scalar product
   \[ \langle (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \rangle = \sum_{n \in \mathbb{N}} \langle x_n, y_n \rangle. \]

5. A closed submodule of a Hilbert $A$-module is itself a Hilbert $A$-module for the inherited scalar product.

Let $E$ be a Hilbert $A$-module. The orthogonal $S^\perp$ of a part $S$ in $E$ is defined as the closed submodule of $E$ such that
   \[ S^\perp = \{ y \in E, \forall x \in S, \langle x, y \rangle = 0 \}. \]

Note that even if $F$ is a closed submodule, we do not have in general $F \oplus F^\perp = E$. If this is the case, we say that $F$ is orthocomplemented.

**Definition 2.1.1 —** Let $A$ be a $C^*$-algebra, $E$ and $E'$ be Hilbert $A$-modules ; a map $T : E \to E'$ is called a morphism if there exists a map $S : E' \to E$ such that for all $(x, y) \in E \times E'$, we have
   \[ \langle Tx, y \rangle_{E'} = \langle x, Sy \rangle_E. \]

We denote by $\mathcal{L}(E, E')$ and $\mathcal{L}(E) = \mathcal{L}(E, E)$ the sets of these morphisms.

This definition is all we need to have morphisms, but it implies many other properties. For example, the operator $S$ is uniquely defined ; it is called the adjoint of $T$ and noted $T^*$. Both $T$ and $T^*$ are $A$-linear continuous operators. Their operator norm satisfies $\|T\|^2 = ||T^*||^2 = ||T^*T||$. The composition of two morphisms is still a morphism. Note that $\mathcal{L}(E, E')$ is a closed subspace of the space of bounded linear operators $L(E, E')$ and that $\mathcal{L}(E)$, is a $C^*$-algebra for the operator norm. We denote by $\mathcal{K}(E, E')$ the space of compact morphisms : it is the closure in $\mathcal{L}(E, E')$ of the linear span of rank-one operators (operators of the form $\Theta_{x,z}y \mapsto \langle x, y \rangle z$) ; as previously, $\mathcal{K}(E)$ is a a $C^*$-algebra for the operator norm. In particular, in the case where
A = E, then \( L(A) = M(A) \) coincides with the multiplier algebra of \( A \), whereas \( K(E) = A \).

The \( C^* \)-algebra structure of \( L(E) \) is expressed by its Hilbert \( A \)-module structure. In particular \( T \) is selfadjoint (respectively positive) if and only if for all \( x \in E \), \( \langle Tx, x \rangle = \langle x, Tx \rangle^* \) (respectively \( \langle Tx, x \rangle \in A^+ \)).

Note that the morphisms can be characterized by properties of their graphs.

**Proposition 2.1.2** — A \( A \)-linear operator \( T \) from the \( A \)-module \( E \) into the \( A \)-module \( E' \) is a morphism if and only if the graph \( G(T) \) of \( T \) is an orthocomplemented submodule of \( E \oplus E' \).

Furthermore, we have the following properties

**Proposition 2.1.3** — Let \( T \in L(E, E') \).

1. (a) If \( T \) is surjective then \( TT^* \) is invertible in \( L(E') \) and \( E = \text{Ker}T \oplus \text{Im}T^* \).
   
   (b) If \( T \) is bijective, then so is \( T^* \). We have \( T^{-1} \in L(E', E) \) and \( (T^{-1})^* = (T^*)^{-1} \).

2. The following conditions are equivalent:
   
   (a) \( \text{Im}T \) is closed in \( E' \).
   
   (b) \( \text{Im}T^* \) is closed in \( E \).
   
   (c) \( 0 \) is isolated in the spectrum of \( T^*T \).
   
   (d) \( 0 \) is isolated in the spectrum of \( TT^* \).

If these conditions are satisfied, then \( \text{Im}T \) and \( \text{Ker}T^* \) are orthocomplemented submodules of \( E' \) and \( \text{Im}T^* \) and \( \text{Ker}T \) are orthocomplemented submodules of \( E \).

### 2.2 Composition of Hilbert modules

**Proposition 2.2.1** — Let \( A \) and \( B \) be two \( C^* \)-algebras, \( E \) a Hilbert \( A \)-module and \( H \) a Hilbert \( B \)-module, and \( \varphi : A \to L(H) \) a completely positive map. For \( x, y \in E, \xi, \eta \in H \) and \( b \in B \), we set \( (x \otimes \xi)b = x \otimes \xi b \) and \( \langle x \otimes \xi, y \otimes \eta \rangle = \langle \xi, \varphi(\langle x, y \rangle)\eta \rangle \). The algebraic tensor product \( E \otimes B \) endowed with the structure of right \( B \)-module and of the above scalar product is then a pre-Hilbert \( B \)-module.

We will denote \( E \otimes_B H \) (or \( E \otimes_A H \) if there is no ambiguity on \( \varphi \)) the corresponding Hilbert \( B \)-module.

**Proposition 2.2.2** — Let \( A \) and \( B \) be two \( C^* \)-algebras, \( E \) a Hilbert \( A \)-module, \( H \) a Hilbert \( B \)-module, and \( \pi : A \to L(H) \) a representation.

- For any \( T \in L(E) \) the map \( x \otimes \xi \mapsto Tx \otimes \xi \) defines a morphism \( T \otimes \pi 1 \in L(E \otimes_B H) \).

  The map \( T \mapsto T \otimes \pi 1 \) is a \( \ast \)-homomorphism, and is injective when \( \pi \) is.
• If $S \in \mathcal{L}(H)$ commutes with $\pi(A)$, the map $x \otimes \xi \mapsto x \otimes S\xi$ defines a morphism
  \[
  1 \otimes_\pi S \in \mathcal{L}(E \otimes_\pi H)
  \]
  which commutes with $T \otimes_\pi 1$ for any $T \in \mathcal{L}(E)$.

Note that in the particular case where $E = A$, the Hilbert module $E \otimes \pi H$ is equal to $\pi(A)H$.
Then, if $\pi$ is non-degenerate, i.e. $\pi(A)H = H$, any $T \in \mathcal{L}(A) \simeq \mathcal{M}(A)$ defines a bounded morphism on $H$.

2.3 Regular operators

This class of operators defined by Baaj [Baa80] in his thesis is very rich and handy, as the properties proved later by Woronowicz show. The reader is again referred to [Ska96, Lan95, WN92] for details and proof.

2.3.1 Definition and elementary properties

Recall that an unbounded operator $T$ from a Banach space $E$ to another $E'$ is a linear map from a linear subspace of $E$, called the domain of $T$ and denoted by $\text{Dom}T$, into $E'$. Equivalently, $T$ is characterized by its graph $G(T) = \{(x, Tx), x \in \text{Dom}T\}$ which is a linear subspace of $E \times E'$. Let $A$ be a $C^*$-algebra, $E$ and $E'$ two Hilbert $A$-modules. An unbounded operator from $E$ to $E'$ is said to be $A$-linear when its graph is a sub-$A$-module of $E \oplus E'$. Let $T$ be an unbounded operator such that $(\text{Dom}T)^\perp = \{0\}$ in $E$; the adjoint $T^*$ of $T$ is then the $A$-linear operator which has graph

\[G(T^*) = \{(y, x) \in E' \times E, \forall z \in \text{Dom}T, \langle x, z \rangle = \langle y, Tz \rangle\} .\]

Let $U_0 \in \mathcal{L}(E' \oplus E, E \oplus E')$ be the morphism sending $(y, x) \in E' \oplus E$ to $(x, -y)$. Then $G(T^*)$ is the orthogonal of $U_0^*(G(T))$ in $E' \oplus E$.

Note that if $(\text{Dom}T)^\perp = \{0\}$ and $(\text{Dom}T^*)^\perp = \{0\}$, then $T \subset (T^*)^*$ and so $T$ is closeable. Recall that if $S$ and $T$ are unbounded operators, then $S + T$ is the operator with domain $\text{Dom}S \cap \text{Dom}T$ such that $(S + T)x = Sx + Tx$. Similarly $ST$ is the operator of domain \{\(x \in \text{Dom}T, Tx \in \text{Dom}S\)\} such that $(ST)x = S(Tx)$.

If $T : E \to E'$ is injective, then we can define an operator $T^{-1} : E' \to E$, which has domain $\text{Dom}(T^{-1}) = \text{Im}T$ and graph $G(T^{-1}) = \{(y, x) \in E' \times E, (x, y) \in G(T)\}$.

**Proposition 2.3.1** — Let $E, E', E''$ be Hilbert $A$-modules.

1. Let $S$ and $T$ be operators from $E$ to $E'$ such that $(\text{Dom}S \cap \text{Dom}T)^\perp = \{0\}$. Then $S^* + T^* \subset (S + T)^*$. If $S \in \mathcal{L}(E, E')$ then $S^* + T^* = (S + T)^*$.

2. Let $T$ be an injective operator from $E$ to $E'$ such that $(\text{Dom}T)^\perp = \{0\}$ and $(\text{Im}T)^\perp = \{0\}$. Then $T^*$ is injective and $(T^*)^{-1} = (T^{-1})^*$.

3. Let $T$ be an operator from $E$ to $E'$ and $S$ an operator from $E'$ to $E''$ such that $(\text{Dom}ST)^\perp = \{0\}$ and $(\text{Dom}S)^\perp = \{0\}$.

Then we have $T^*S^* \subset (ST)^*$. If $S$ is a morphism or if $T$ is the inverse of an injective morphism then $T^*S^* = (ST)^*$.
An operator $T$ from $E$ to itself is said to be \textit{self-adjoint} if $T = T^*$. Such an operator is \textit{positive} if for any $x \in \text{Dom}T$, we have $(x, Tx) \in A^+$. 

\textbf{Proposition 2.3.2} — Let $T$ be a closed and densely defined operator from $E$ to $E'$, such that $T^*$ is also densely defined. The image of $(1 + T^*T)$ is then equal to $\{x \in E, (x,0) \in G(T) + U_0 G(T^*)\}$, with $U_0$ as above. Furthermore, the following properties are equivalent.

1. The image of $(1 + T^*T)$ is dense in $E$.
2. $1 + T^*T$ is surjective.
3. The graph of $T$ is an orthocomplemented submodule of $E \oplus E'$.

Note that if these conditions are verified, so are they for $T^*$, and then $(T^*)^* = T$. Such an operator $T$ is called \textit{regular}.

\textbf{Operations on regular operators}

\textbf{Sum} Let $S \in \mathcal{L}(E, E')$ and $T$ be a regular operator from $E$ to $E'$. The operator $S + T$ is regular.

\textbf{Product} Let $T$ be a regular operator from $E$ to $E'$ and $S \in \mathcal{L}(E', E')$, be an invertible morphism. The operator $ST$ is regular. The same is true if $S$ is a regular operator from $E'$ to $E''$ and $T$ an invertible morphism from $E$ to $E'$.

\textbf{Inverse} Let $T$ be a regular operator from $E$ to $E'$.

1. If $\text{Dom}T = E$, then $T \in \mathcal{L}(E, E')$.
2. If the images of $T$ and $T^*$ are dense in $E'$ and $E$ respectively, then $T$ is injective and $T^{-1}$ is a regular operator from $E'$ to $E$.
3. If $T$ is bijective from $\text{Dom}T$ to $E'$, $T^{-1} \in \mathcal{L}(E', E)$.
4. If $E = E'$, the map $\lambda \mapsto R_\lambda(T) = (T - \lambda)^{-1}$ is analytic from $\mathbb{C} - \text{Sp}T$ into $\mathcal{L}(E)$, where $\text{Sp}T = \{\lambda \in \mathbb{C}, (T - \lambda) \text{ is non bijective}\}$.

\subsection*{2.3.2 Woronowicz transform}

\textbf{Proposition 2.3.3} — Let $T$ be a regular operator from $E$ to $E'$. Denote by $p : G(T) \to E$ the first projection.

1. $1 + T^*T$ is invertible and $(1 + T^*T)^{-1} = pp^* \in \mathcal{L}(E)$. We then define $(1 + T^*T)^{-\frac{1}{2}}$ by $(1 + T^*T)^{-\frac{1}{2}} = \left[(1 + T^*T)^{-1}\right]^{\frac{1}{2}} = |p^*|$. 
2. The domain of $T^*T$ is a core for $T$.
3. The operator $T^*T$ is self-adjoint, regular, and $\text{Sp}(T^*T) \subset \mathbb{R}_+$.
4. We have $\text{Dom}T = \text{Im}(1 + T^*T)^{-\frac{1}{2}}$ and $T(1 + T^*T)^{-\frac{1}{2}} \in \mathcal{L}(E, E')$. 

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This gives us the following theorem, which is essential in dealing with regular operators.

**Theorem 2.3.4** — Let $T$ be a regular operator from $E$ to $E'$. Set $Q(T) = T(1 + T^*T)^{-\frac{1}{2}}$.

1. We have $Q(T) \in \mathcal{L}(E)$, $1 - Q(T)^*Q(T) = (1 + T^*T)^{-1}$ and $Q(T)^* = Q(T^*)$.

2. The map $T \mapsto Q(T)$ is a 1–1 correspondence between the set of regular operators from $E$ to $E'$ and the set of morphisms $Q \in \mathcal{L}(E, E')$ with $\|Q\| \leq 1$ and $\text{Im}(1 - Q^*Q)$ dense in $E$. The inverse map is given by

   $$Q \mapsto Q(1 - Q^*Q)^{-\frac{1}{2}}$$

From this theorem, we can deduce the following properties.

1) **Image** We have $\text{Im}T = \text{Im}Q(T)$ and $\text{Ker}T = \text{Ker}Q(T)$. In particular, $T$ is bijective if and only if $Q(T)$ is.

2) **Self-adjointness** If $E = E'$, $T$ is self-adjoint if and only if $Q(T)$ is.

2) **Normal operators** If $E = E'$, $T$ is normal if and only if $Q(T)$ is.

4) **Polar decomposition** There exists a unique regular operator $|T|$ such that $Q(|T|) = |Q(T)|$. This operator is self-adjoint, positive, of domain $\text{Dom}T$, and $|T|^2 = T^*T$. If $Q(T)$ has polar decomposition $Q(T) = u|Q(T)|$, then $T$ has polar decomposition $T = u|T|$.

### 2.3.3 Image of a regular operator by a representation

Let $A$ and $B$ be two $C^*$-algebras, $E$ and $E'$ Hilbert $A$-modules, $T$ regular operator from $E$ to $E'$, $H$ a Hilbert $B$-module, and $\pi : A \to \mathcal{L}(H)$ a representation of $A$.

**Proposition 2.3.5** — There exists an operator $T_0$ from $E \otimes_\pi H$ to $E' \otimes_\pi H$, whose domain is given by the image in $E \otimes_\pi H$ of the algebraic tensor product of $\text{Dom}T$ by $H$ such that for any $x \in \text{Dom}T$ and any $\xi \in H$, we have

$$T_0(x \otimes \xi) = T(x) \otimes \xi.$$ 

This operator is closeable and its closure is a regular operator from $E \otimes_\pi H$ to $E' \otimes_\pi H$ denoted by $T \otimes_\pi 1$. Furthermore, we have $Q(T \otimes_\pi 1) = Q(T) \otimes_\pi 1$ and $(T \otimes_\pi 1)^* = T^* \otimes_\pi 1$.

**Proposition 2.3.6** —

1. We have $(T \otimes_\pi 1)^* = T^* \otimes_\pi 1$ and $(T \otimes_\pi 1)^*(T \otimes_\pi 1) = T^*T \otimes_\pi 1$.

2. If $E = E'$ and $T$ is normal or self-adjoint, then so is $T \otimes_\pi 1$.

3. If $D \subset \text{Dom}T$ is a core for $T$, and $\mathcal{H}$ a dense subspace in $H$, then the image in $E \otimes_\pi H$ of the algebraic tensor product of $D$ by $\mathcal{H}$ is a core for $T \otimes_\pi 1$.

4. If $E = E'$, we have $\text{Sp}(T \otimes_\pi 1) \subset \text{Sp}T$, with equality when $\pi$ is injective.
2.3.4 Functionnal calculus for normal regular operators

Let $A$ and $B$ be two $C^*$-algebras, $H$ a Hilbert $B$-module, and $\pi : A \to \mathcal{L}(H)$ a non-degenerate representation. Denote $\pi : \mathcal{M}(A) \to \mathcal{L}(H)$ the extension of this representation to the multiplier algebra. The $B$-module $A \otimes_B H$ can be identified to $H$ using the unitary operator $a \otimes x \mapsto \pi(a)x$.

For a given morphism $T \in \mathcal{L}(A) = \mathcal{M}(A)$, we get $\tilde{\pi}(T) = T \otimes_B 1$ through this identification. If $T$ is now a regular operator on the the Hilbert $A$-module $A$, we denote by $\hat{\pi}(T) = T \otimes_B 1$ the corresponding regular operator on the Hilbert $B$-module $H$. The domain of this regular operator is the image of $\hat{\pi}\left[(1 + T^*T)^{-1}\right]$ i.e. the set $\{\pi(a)x, a \in \text{Dom}T, x \in H\}$. We then have $\tilde{\pi}(T)\pi(a)x = \pi(Ta)x$.

\textbf{Theorem 2.3.7} — Let $A$ be a $C^*$-algebra, $E$ a Hilbert $A$-module, $X$ a locally compact Hausdorff space and $\pi : C_0(X) \to \mathcal{L}(E)$ a non degenerate representation. We denote $\tilde{\pi} : C_b(X) \to \mathcal{L}(E)$ the extension of this representation to the multiplier algebra and $\hat{\pi}$ the map from $C(X)$ into the set of regular operators on $E$ defined as previously.

1. For any function $f \in C(X)$, the operator $\tilde{\pi}(f)$ is normal. We have
   \[\tilde{\pi}(f)^* = \tilde{\pi}(\bar{f}), \quad \text{and} \quad Q(\tilde{\pi}(f)) = \pi[f(1 + |f|^2)^{-\frac{1}{2}}].\]
   Moreover, we have $\text{Sp}(\hat{\pi}(f)) \subset \overline{f(X)}$ with equality if $\pi$ is injective.
2. For any function $g \in C_c(X)$, we have $\tilde{\pi}(f)\pi(g) = \pi(fg)$ and the set $\{\pi(g)x, g \in C_c(X), x \in E\}$ is a core for $\hat{\pi}(f)$.
3. If $f \in C_b(X)$, then $\tilde{\pi}(f) = \hat{\pi}(f)$.
4. If $T$ is a regular operator on $E$ such that $T\pi(g) = \pi(fg)$ for any function $g \in C_c(G)$, then $T = \hat{\pi}(f)$.
5. For $f, g \in C(X)$, we have $\tilde{\pi}(f + g) = \tilde{\pi}(f) + \tilde{\pi}(g)$ and $\tilde{\pi}(fg) = \tilde{\pi}(f)\tilde{\pi}(g)$.
   If $g(|f + g| + 1)^{-1} \in C_b(X)$, then $\tilde{\pi}(f + g) = \tilde{\pi}(f) + \tilde{\pi}(g)$; if $g(|fg| + 1)^{-1} \in C_b(X)$, then $\tilde{\pi}(fg) = \tilde{\pi}(f)\tilde{\pi}(g)$.

\textbf{Proposition 2.3.8} — Let $T$ be a regular normal operator on a Hilbert $A$-module $E$ and $X$ a closed part of $C$ containing $\text{Sp}T$. Denote by $z \in C(X)$ the identity map $\lambda \mapsto \lambda$. There exists a unique non-degenerate representation $\pi : C_0(X) \to \mathcal{L}(E)$ such that $\tilde{\pi}(z) = T$. The kernel of $\pi$ is formed by the functions in $C_0(X)$ which are null on $\text{Sp}T$.

Now let $T$ be a normal regular operator on a Hilbert $A$-module $E$ and $X$ a closed part of $C$ containing $\text{Sp}T$. Denote by $z \in C(X)$ the identity map $\lambda \mapsto \lambda$ and by $\pi : C_0(X) \to \mathcal{L}(E)$ the representation such that $\tilde{\pi}(z) = T$. For $f \in C(X)$, we denote $f(T)$ the operator $\tilde{\pi}(f)$. If $f$ is a function defined on a part of $C$ containing $\text{Sp}T$ and which has a continuous restriction $g$ to $\text{Sp}T$, we set $f(T) = g(T)$.

Finally, we get the following theorem.

\textbf{Theorem 2.3.9} — The map $f \mapsto f(T)$ so defined has the following properties:
1. For any function \( f \) continuous on \( X \) containing \( \text{Sp}T \), the operator \( f(T) \) is a normal regular operator on \( E \) and we have \( f(T)^* = f(T) \).

2. \( \text{Sp}(f(T)) \) is the closure in \( \mathbb{C} \) of \( f(\text{Sp}T) \).

3. If \( f \) is continuous and bounded, we have \( f(T) \in \mathcal{L}(E) \) and \( \|f(T)\| = \sup \{|f(\lambda)|, \lambda \in \text{Sp}T\} \).

4. For any pair \((f, g)\) of continuous functions, \((f + g)(T)\) is the closure of \( f(T) + g(T) \) and \((fg)(T)\) the closure of \( f(T)g(T) \). Moreover, if \( g(|f + g| + 1)^{-1} \) is bounded, we have \((f + g)(T) = f(T) + g(T)\). Equally, if \( g(|fg| + 1)^{-1} \) is bounded, we have \((fg)(T) = f(T)g(T)\).

5. For \( f, g \in C(\mathbb{C}) \), we have \( f(T) = g(T) \) if and only if \( f \) and \( g \) coincide on \( \text{Sp}T \).

6. For \( f, g \in C(\mathbb{C}) \), we have \( f \circ g(T) = f(g(T)) \).

7. If \( T \in \mathcal{L}(E) \), the map so defined coincides with the continuous functionnal calculus on the \( C^*\)-algebra \( \mathcal{L}(E) \).

3  Recalls on Lie groupoids and ordinary pseudodifferential calculus

We begin this section by recalling the definition of a smooth groupoid, and we then review the bases of the ordinary pseudodifferential calculus on a smooth manifold.

3.1 Lie groupoids and their \( C^*\)-algebras

3.1.1 Topological groupoids

Recall that a groupoid is a small category \( G \) (this means that the morphism class of \( G \) is a set) in which all morphisms are invertible. Here is a more explicit definition.

**Definition 3.1.1** — A groupoid is given by two sets \( G^{(1)} = G \) and \( G^{(0)} = M \) and the following maps :

- \( u : M \to G^{(1)} \), the diagonal imbedding,
- an involution \( \kappa : G^{(1)} \to G^{(1)} \) called inversion denoted by \( \kappa(\gamma) = \gamma^{-1} \),
- source \((s)\) and range \((r)\) maps from \( G^{(1)} \) into \( M \),
- a multiplication \( m \) valued in \( G^{(1)} \), and defined on the set \( G^{(2)} \subset G^2 \) of pairs \((\gamma, \gamma')\) for which \( r(\gamma') = s(\gamma) \), denoted by \( m(\gamma, \gamma') = \gamma \gamma' \),

satisfying the following conditions :

1. \( r(u(x)) = s(u(x)) = x \), and \( \gamma u(s(\gamma)) = u(r(\gamma)) \gamma = \gamma \).
2. \( r(\gamma^{-1}) = s(\gamma) \) and \( \gamma \gamma^{-1} = u(r(\gamma)) \).
3. \( s(\gamma \gamma') = s(\gamma') \) and \( r(\gamma \gamma') = r(\gamma) \).
4. \( \gamma_1(\gamma_2\gamma_3) = (\gamma_1\gamma_2)\gamma_3 \) if \( s(\gamma_1) = r(\gamma_2) \) and \( s(\gamma_2) = r(\gamma_3) \).

The set \( G^{(1)} \) is the set of arrows, and we will often refer to it as \( G \), by a common abuse of notation. A topological groupoid is then a groupoid for which \( G \) and \( M \) are locally compact topological spaces and \( r, s, m, u \) are all continuous maps, \( \kappa \) a homeomorphism, and \( r \) and \( s \) are open maps.

### 3.1.2 Lie groupoids

A Lie groupoid is a groupoid where \( G \) and \( M \) are smooth manifolds, and where \( m, u \) are smooth maps, \( \kappa \) a smooth diffeomorphism, and \( r \) and \( s \) are submersions. It is clear that a Lie groupoid is a particular example of a topological groupoid.

A Lie algebroid \( A \) over a manifold \( M \) is a vector bundle over \( M \) together with a Lie algebra structure on the space \( \Gamma(A) \) of smooth sections and a bundle map ("the anchor map") \( \rho : A \to TM \) which extends to a map between smooth sections of this bundle and such that for \( X, Y \in \Gamma(A) \), \( f \in C^\infty(M) \):

1. \( \rho([X,Y]) = [\rho(X),\rho(Y)] \);
2. \( [X,fY] = f[X,Y] + (\rho(X)f)Y \).

Recall that to any Lie groupoid \( G \) of basis \( M \) can be associated a Lie algebroid \( A(G) \) over the basis \( M \) as follows. \( A(G) \) is the bundle over \( M \) of the longitudinal tangent spaces \( T_xG_x \) to \( G_x \) for \( x \in M \). The bundle of longitudinal cotangent spaces \( T^*_xG_x \) is denoted by \( A^*(G) \), and \( S^*(G) \) designs the quotient by the action of \( \mathbb{R}^*_+ \) of \( A^*(G) \) minus \{0\}.

Note that in general the converse is not true, see [CF03] for a characterization of Lie algebroids which integrates into Lie groupoids.

### 3.1.3 Haar systems

A continuous left invariant Haar system, is a family \( \{\lambda^x, x \in M\} \) of measures on \( G \) such that :

1. \( \lambda^x \) is a positive measure supported on \( G^x \);
2. \( \forall x, y \in G^{(0)}, f \in C_c(G), \delta \in G^y_x \)
   \[ \int_{\gamma \in G^x} f(\delta\gamma)d\lambda^x(\gamma) = \int_{\gamma \in G^y} f(\gamma)d\lambda^y(\gamma); \]

3. \( \forall f \in C_c(G), \) the map \( x \mapsto \int_{\gamma \in G^x} f(\gamma)d\lambda^x(\gamma) \) is continuous on \( G^{(0)} \).

A smooth section of the bundle of 1-densities on \( A(G) \) gives rise to such a Haar system \( \lambda \). Moreover, for such a system \( \lambda \), we have that for all \( x \), the measure \( \lambda_x \) is in the Lebesgue class and that \( \lambda \) is a smooth Haar system, in the following sense

\( \forall f \in C^\infty_c(G), \) the map \( x \mapsto \int_{\gamma \in G^x} f(\gamma)d\lambda^x(\gamma) \) is smooth on \( G^{(0)} \). The construction above shows that any Lie groupoid is endowed with a smooth Haar system.
3.1.4 C*-algebras of a groupoid

We now give the construction of the C*-algebras associated to a groupoid. We do it in general for a topological groupoid which is Hausdorff and locally compact, and equipped with such a continuous Haar system, as this framework is the most natural one, though we will only be using it here for Lie groupoids. Note that these constructions are independent of the choice of the Haar system in the sense that the obtained C*-algebras are isomorphic [Ren80]. Furthermore, it is also possible to construct all objects without having to choose a Haar system, working with half densities (see e.g. [Con79, MP97]).

The set $C_c(G)$ of compactly supported continuous functions on $G$ is endowed with a structure of *-algebra:

- the multiplication is defined by:
  $$ (f * g)(\gamma) = \int f(\gamma') g(\gamma'^{-1} \gamma) d\lambda(\gamma')(\gamma') $$

- the involution is given by
  $$ f^*(\gamma) = \overline{f(\gamma^{-1})} $$

The algebra obtained from $C_c(G)$ by completion for the norm
$$ \|f\|_1 = \sup_{x \in G(0)} \left\{ \int |f(\gamma)| d\lambda(\gamma), \int |f(\gamma^{-1})| d\lambda(\gamma) \right\} $$

is a Banach *-algebra. A bounded *-representation of $C_c(G)$ in a Hilbert space is a *-homomorphism $\pi : C_c(G) \to B(H)$ such that $\|\pi(f)\| \leq \|f\|_1$. It is non-degenerate when the linear span $\{\pi(f)\xi, f \in C_c(G), \xi \in H\}$ is dense in $H$ for all $f$.

An important family of such representations is given by the left regular representations. It is a set $\{\pi_x\}_{x \in G(0)}$ of bounded *-representations on (resp.) $L^2(G^x, \lambda^x)$ defined by:
$$ (\pi_x(f)g)(\gamma) = \int f(\gamma') g(\gamma'^{-1} \gamma) d\lambda^x(\gamma') = f * g(\gamma) $$

This leads us to the definition of two natural C*-completions of $C_c(G)$. The first one, known as the reduced C*-algebra of $G$ is the completion w.r.t. the norm $\|f\|_r = \sup_{x \in G(0)} \|\pi_x(f)\|$. The second one, known as the maximal (or full) C*-algebra of $G$ is the completion w.r.t the norm defined taking the supremum of the norms over all bounded *-representations. This implies in particular a natural epimorphism $C^*(G) \to C^*_r(G)$.

3.2 Ordinary pseudodifferential calculus

We recall the classical definition and properties of pseudodifferential calculus [Trè80, AG91, Shu87, Hör90, Hör83, Hör94a, Hör94b].
3.2.1 Symbols of complex orders

We begin by defining, as in [Shu87], the sets of symbols of type \((1, 0)\). For \(U\) an open set in \(\mathbb{R}^n\) and \(m_0 \in \mathbb{R}\), denote by \(S^{m_0}(U, \mathbb{R}^p)\) the space of symbols to be the set of smooths complex valued functions on \(U \times \mathbb{R}^p\) such that:

\[
|\partial_y^a \partial_\xi^\beta a(y, \xi)| \leq C_{K,\alpha,\beta}(1 + |\xi|)^{m_0-|\beta|}
\]  

(1)

for any compact \(K \subset U\), \(\alpha, \beta\) multi-indices, \((y, \xi) \in K \times \mathbb{R}^p\). Taking the smallest possible constant \(C_{K,\alpha,\beta}\) in the above inequality (for fixed \(K, \alpha\) and \(\beta\)), we get a family of semi-norms which form a natural Frechet topology on \(S^{m_0}(U, \mathbb{R}^p)\); the space \(S^{m_0}(U, \mathbb{R}^n)\) will be simply denoted by \(S^{m_0}(U)\). We will in fact use a particular class of symbols: the polyhomogeneous ones. A function \(f \in C^\infty(\mathbb{R}^p - \{0\})\) is positively homogeneous of degree \(l \in \mathbb{C}\) whenever \(f(t\xi) = t^l f(\xi)\) for all \(\xi \neq 0\) and all \(t > 0\). Let \(m\) be a complex number of real part \(m_0\). We say that a symbol \(a \in S^{m_0}(U, \mathbb{R}^p)\) admits a polyhomogeneous expansion if there exists, \(\forall j \in \mathbb{N}\), functions \(a_{m-j} \in C^\infty(U \times (\mathbb{R}^p - \{0\}))\) positively homogeneous of degree \(m - j\) in the second variable, and a \(C^\infty\) cut-off function \(\chi\) on \(\mathbb{R}^p\), with \(\chi(\xi) = 0\) if \(|\xi| < 1/2\) and \(\chi(\xi) = 1\) if \(|\xi| \geq 1\), such that, for all \(N \in \mathbb{N}\)

\[
a(y, \xi) - \chi(\xi) \sum_{k=0}^{N-1} a_{m-k}(y, \xi) \in S^{m_0-N}(U).
\]

We call \(S^m_{hom}(U, \mathbb{R}^p)\) the set of such polyhomogeneous symbols of order \(m\). Note that this property does not depend on the cut-off function \(\chi\). Note also that for each \(j\), we have \(\chi(\xi)a_{m-j} \in S^{m_0-j}(U)\), and that the functions \(a_{m-j}\) are uniquely determined for \(|\xi| \geq 1\).

We can then associate to each \(j\) a smooth function, still denoted \(a_{m-j}\) on the sphere bundle \(U \times S^{p-1}\). The natural topology on \(C^\infty(U \times S^{p-1})\) is the topology of uniform convergence and all its derivatives on every compact set. This allows us to define the correct topology on the spaces \(S^m_{hom}(U, \mathbb{R}^p)\). Indeed, these sets \(S^m_{hom}(U, \mathbb{R}^p)\) are not closed in \(S^m(U)\) for the Frechet topology of semi-norms. Let \(\chi\) be a cut-off function as above. The topology on \(S^m_{hom}(U, \mathbb{R}^p)\) is the weakest topology making the following maps continuous, for all \(j, N \in \mathbb{N}\)

- \(a \mapsto a_{m-j} \in C^\infty(U \times S^{p-1})\) for all \(j \in \mathbb{N}\) (with its natural topology);

- \(a \mapsto a - \chi \sum_{j=0}^N a_{m-j} \in S^{m_0-N-1}(U, \mathbb{R}^p)\) (for the above Frechet topology), for all \(N \in \mathbb{N}\).

3.2.2 Pseudodifferential operators

Given any symbol \(a \in S^m_{hom}(U)\) we can construct an operator \(P = Op(a) : C^\infty_c(U) \rightarrow C^\infty(U)\), by

\[
P u(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iy \xi} a(y, \xi) \hat{u}(\xi) d\xi,
\]

(2)

with \(\hat{u}\) the Fourier transform of \(u\).

This leads us to define pseudodifferential operators to be operators from \(C^\infty_c(U)\) to \(C^\infty(U)\) which are given by \(P = Op(a)\). We denote by \(P^m(U)\) the space of such operators with
We say that the operator is properly supported (resp. compactly supported) when its Schwartz kernel is. This means that the projections from the support on both copies of $U$ are proper maps (respectively that the support is itself compact in $U \times U$). When $P$ is proper, it maps $C^\infty_c(U)$ to itself and can be extended as an operator from $C^\infty(U)$ to itself.

In the formula (2) above, the map $a$ is called the total symbol of $P$, whereas the first term $a_m$ in the asymptotic expansion called the principal symbol of $P$. The composition $PQ$ of two properly supported pseudodifferential operators $P$ and $Q$ is still a properly supported pseudodifferential operator and its total symbol is obtained by a composition formula from the total symbols of $P$ and $Q$:

$$\sigma(PQ)(y, \xi) \sim \sum_{a=(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n} \frac{[\partial_\xi^a \sigma(P)(y, \xi)][\partial_\eta^a \sigma(Q)(y, \eta)]}{\alpha!}.$$ 

Recall that a smoothing operator $R : C^\infty_c(U) \to C^\infty(U)$ is an operator with a smooth Schwartz kernel, and that any pseudodifferential operator can be written as the sum of a properly supported pseudodifferential operator and of a smoothing operator.

The formal adjoint of a pseudodifferential operator $P$ is defined to be the operator $P^\dagger$ such that for any $u, v \in C^\infty_c(U)$, $\int (Pu)v = \int u(P^\dagger v)$.

### 3.2.3 Properties of pseudodifferential calculus

An operator is elliptic when its principal symbol is invertible on $S^*U$. An important property of an elliptic operator $P$ is that it admits a parametrix i.e. an inverse in the category of pseudodifferential operators modulo smoothing operators: there exists a pseudodifferential operator $Q$ such that $PQ - I$ and $QP - I$ are smoothing operators.

Note that the class of pseudodifferential operators is invariant under taking diffeomorphisms. Indeed, if $\Phi : U \to V$ is a diffeomorphism, and $P \in \mathcal{P}_m(U)$ a pseudodifferential operator, the operator $(\Phi^\dagger P) : C^\infty_c(V) \to C^\infty(V)$ defined by

$$(\Phi^\dagger P)(u) = [P(u \circ \Phi)] \circ \Phi^{-1}$$

is a pseudodifferential operator in $\mathcal{P}_m(V)$. From this, it is clear that the class of pseudodifferential operators can be extended to smooth manifolds: a pseudodifferential operator $P$ of order $m$ on a smooth manifold $M$ will be a linear operator from $C^\infty_c(M)$ to $C^\infty(M)$ such that in each coordinate chart for the manifold, the operator defined by $\Phi^\dagger P$ is a pseudodifferential operator of order $m$. We will refer to these operators as ordinary pseudodifferential operators, to distinguish them from pseudodifferential operators on groupoids; we will denote by $\mathcal{P}_m(M)$ the space of all properly supported operators of order $m$, by $\mathcal{P}^{-\infty}(M)$ the intersection of all the spaces $\mathcal{P}_m(M)$, which are the smoothing operators, and by $\mathcal{P}(M)$ the space generated by all linear combinations of elements of $\mathcal{P}_m(M)$ for $m \in \mathbb{C}$. This space $\mathcal{P}(M)$ is endowed with an involutive algebra structure.

**Proposition 3.2.1** —
1. For any $m, m' \in \mathbb{C} \cup \{-\infty\}$ we have: $P_m(M) \circ P_{m'}(M) \subset P_{m+m'}(M)$ and $(P_m(M))^* = P_{-m}(M)$

2. The space $P(M)$ endowed with composition is an involutive algebra, filtered by $\mathbb{R}$ and graded by $\mathbb{C}/\mathbb{Z}$. The previous property turns $P_{-\infty}(M)$ into a two-sided ideal of this involutive algebra.

3. The space $P^Z(M)$ of operators with integer order is an involutive subalgebra of $P(M)$.

4 Pseudodifferential calculus on Lie groupoids

In this section, we define pseudodifferential operators on Lie groupoids following the articles of Monthubert and Pierrot [MP97] and Nistor, Weinstein and Xu [NWX99]. For sake of simplicity, we have chosen to deal only with scalar operators at first. We will explain at the end of this section how to deal with operators acting on sections of vector bundles.

4.1 Families of classical operators

We now come to the notion of families of ordinary pseudodifferential operators. To begin with, we recall the definition of a map of class $C^\infty,k$, as given by Atiyah and Singer [AS71];

**Definition 4.1.2** — Let $M$ be a locally compact Hausdorff space, and $U$ an open set in $\mathbb{R}^p$. A map $\psi$ from $U \times M$ to $\mathbb{R}^n$ is said to be of class $C^\infty,0$ if the map $x \mapsto \psi(.,x)$ is continuous from $M$ to $C^\infty(U, \mathbb{R}^n)$, endowed with the topology of uniform convergence of the function and of its derivatives on any compact set. If $M = V$ is an open set in $\mathbb{R}^q$, a map $\psi$ from $U \times V$ to $\mathbb{R}^n$ is said to be of class $C^\infty,k$, with $k \in \mathbb{N} \cup \{\infty\}$ if the map $v \mapsto \psi(.,v)$ is of class $C^k$ from $M$ to $C^\infty(U, \mathbb{R}^n)$.

We can now define the notion of a $C^k$ family of polyhomogeneous symbols

**Definition 4.1.3** — A $C^k$-family of polyhomogeneous symbols of order $m$ is a map $M \to S^{m(\cdot)}_{\text{hom}}(U)$, $x \mapsto a(u, \xi, x)$ such that for any cut-off function $\chi$ null in a neighborhood of 0 and equal to 1 in a neighborhood of $\infty$, the map $M \to S^0_{\text{hom}}(U), x \mapsto \chi(\xi) \|\xi\|^{-m(x)} a(u, \xi, x)$ is of class $C^k$, where the space $S^0_{\text{hom}}(U)$ is endowed with its natural topology described above.

We can now define $C^k$-families of pseudodifferential operators

**Definition 4.1.4** — Let $U$ be an open set in $\mathbb{R}^p$, $M$ a locally compact Hausdorff space, and $m$ a map of class $C^k$ from $M$ to $\mathbb{C}$. A $C^k$-family of classical pseudodifferential operators of order $m$ with compact support in $U \times M$ is a family $P_x \in \mathcal{P}_c^{m(x)}(U)$ such that for all $f \in C^\infty_c(U)$, the operator $P_x f$ is given, for $x \in M$, by

$$(P_x f)(u) = \frac{1}{(2\pi)^p} \int_{\mathbb{R}^p} \int_U a(u', \xi, x) f(u') e^{i(u'-u, \xi)} d\xi du'$$

with the condition that the map $M \to S^{m(\cdot)}_{\text{hom}}(U), x \mapsto a(u', x, \xi)$ is of class $C^k$. 

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Proposition 4.1.5 — We have the following analogues of classical properties

1. The adjoint of a $C^k$-family of classical pseudodifferential operators of order $m(.)$ with compact support in $U \times M$ is still $C^k$-family of classical pseudodifferential operators of order $m(.)$ with compact support in $U \times M$.

2. Let $m$ and $n$ be maps of class $C^k$ from $M$ to $\mathbb{C}$ then, if $A \in S^{m(.)\text{hom.,}k}(U \times M)$ and $B \in S^{n(.)\text{hom.,}k}(U \times M)$ then $AB \in S^{m(.)+n(.)\text{hom.,}k}(U \times M)$.

3. Let $\kappa$ be a $C^\infty$-diffeomorphism from $U$ onto itself. Take $a \in S^{m(.)\text{hom.,}k}(U)$, and denote by $A$ the corresponding family of pseudodifferential operator and by $a_\kappa$ the map defined by

$$a_\kappa(\kappa(x), \eta) = e^{-i\kappa(x), \eta} A e^{i\kappa(x), \eta}.$$

Then we have $a_\kappa \in S^{m(.)\text{hom.,}k}(U)$.

Proof — From classical theory (see [Trè80]), it is easily seen that all these properties are true if the order is constant. So we only have to prove that they respect the topology when $m$ is not constant. Moreover as differentiating with respect to $x \in M$ does not change the estimates in $S^0_{\text{hom.,}k}(U)$, it is enough that we look at the continuity of these operations. Recall that two types of continuity are to consider: the one of homogeneous parts of the symbol, seen as maps on $C^\infty(U \times M)$, and the other for the topology of semi-norms for $(1,0)$-symbols in $S^{m_0}$, with $m_0$ the real part of $m$.

- From classical theory we know that the homogeneous parts of the symbol are given in all three cases by formulae which involve only a finite number of homogeneous symbols and there derivatives, so that if these homogeneous symbols are $C^\infty$ maps, then so is the resulting symbol. To give an explicit example, we know that if we compose two symbols $a$ and $b$, then we have, for $c = a \ast b$ that

$$c_{m(x)+n(x)−p}(u, \xi, x) = \sum_{j+l=|\alpha|=p} \partial^{\alpha}_{\xi} a_{m(x)−j}(u, \xi, x) \partial^{\alpha}_{u} b_{n(x)−l}(u, \xi, x)/\alpha!$$

is in $C^\infty(U \times M)$ for all $p \in \mathbb{N}$ as $a_{m(x)−j}$ and $b_{n(x)−l}$ are, for all $j$ and $l$.

- We are left to show the continuity for $(1,0)$-symbols topology, sowe can suppose that $m(.)$ is real. To see this continuity we use the classical theory of amplitudes associated with pseudodifferential operators (see [Trè80]). In the same way we defined $S^{m(.)\text{hom.,}k}(U \times M)$, we can define the space $S^{m(.)}_{k}(U \times U \times M)$ of amplitudes depending in a $C^k$-way of a parameter $x \in M$, and compactly supported in $U \times U$.

From the formula linking compactly supported amplitudes to compactly supported symbols

$$\sigma(u, \xi, x) = \int \int a(u, u', \eta, x)e^{-i(u-u', \eta-\xi)} du' d\eta$$

we can define the space $S^{m(.)\text{hom.,}k}(U \times U \times M)$ of amplitudes depending in a $C^k$-way of a parameter $x \in M$, and compactly supported in $U \times U$.
we deduce the fact that if $\chi(\xi) \|\xi\|^{-m(x)} a(u, u', \xi, x)$ is a $C^k$-map from $M$ to $S^0_k(U \times U \times M)$, then $\chi(\xi) \|\xi\|^{-m(x)} a(u, \xi, x)$ is a $C^k$-map from $M$ to $S^0_k(U \times M)$. Indeed, the classical formula
\[
\sigma(u, \xi, x) \sim \sum_{\alpha} (-i)^{|\alpha|} \partial^\alpha_x \partial^\alpha_u a(u, u, \xi, x)
\]
shows that homogeneous components of $\sigma$ are $C^k$-maps if homogenous components of $a$ are. To show that it is true for the $(1,0)$-topology, we can as well consider operators of order as negative as needed, as we may subtract the homogenous components. Then the estimates on $a$ and its derivatives with respect to $\eta, u, x$ implies the estimates for $\chi(\xi) \|\xi\|^{-m(x)} \sigma(u, \xi, x)$ as the formula above is an absolutely convergent integral for sufficiently negative order. This proves that the operators associated in the usual way to the $C^k$-amplitudes are exactly the $C^k$-families of $(1,0)$-pseudodifferential operators. This ends the proof for the first property as an amplitude of $A^*$ is given by $a(u', u, -\xi, x)$ when $a(u, u', \xi, x)$ is an amplitude for $A$.

To prove the property for composition of operators, we need only to use a classical trick: if $a(u, \xi, x)$ is an amplitude for $A$, not depending on $u'$ and $b(u', \xi, x)$ an amplitude for $B$, not depending on $u$, then $c(u, u', \xi, x) = a(u, \xi, x) b(u', \xi, x)$ is an amplitude for $AB$, and hence satisfy the required estimates as $a$ and $b$ do.

Finally, for the change of coordinates one needs to prove directly the estimates, the way through amplitudes being more complicated than the direct one. We know from classical computations that
\[
a_\kappa(u, \eta, x) = a(\kappa^{-1}(u), J^{-1}(u) \eta, x)
\]
with $J$ is the Jacobian matrix associated with $\kappa^{-1}$. Set $\xi = J^{-1}(u) \eta$. Then we have
\[
\|\eta\|^{-m(x)} a_\kappa(u, \eta, x) = \left( \frac{\|J(u)\|}{\|\xi\|} \right)^{m(x)} \|\xi\|^{-m(x)} a(\kappa^{-1}(u), J^{-1}(u) \eta, x).
\]
Hence, from estimates on $a$ we get estimates on $a_\kappa$, using the fact that if $u$ and $x$ vary in compact sets, then the expression $\left( \frac{\|J(u)\|}{\|\xi\|} \right)^{m(x)}$ and its derivatives are bounded. Indeed derivation with respect to $\eta$ or $x$ does not affect the estimates, while derivation with respect to $u$ introduces multiplication by $\eta$ which is counterbalanced by the facts that derivatives of $a$ with respect to the second variable lowers the order of the estimates in $\xi = J^{-1}(u) \eta$. Anyway these estimates remain uniform with respect to $x$ (what is the new thing here).

\[\blacksquare\]

4.2 Pseudodifferential $G$-operators

The definition of a $C^{\infty, k}$ function can be extended to the situation of a groupoid in a natural way. Let $p : X \to M$ be a submersion between smooth manifolds. We will say that a function $f$ on $X$ is $C^{\infty, k}$ with respect to the submersion when for any trivializing open set for the submersion of the form $\Omega \supset U \times V$ with $V$ an open set of $M$, the restriction of $f$ to $U \times V$ is $C^{\infty, k}$. If $G$ is now a Lie groupoid, we will say that a function is $C^{\infty, k}$, when it is $C^{\infty, k}$ with respect to
the submersion $s : G \to G^{(0)}$.

In the special case of Lie groupoids, we have a notion of invariant families of operators (non necessarily pseudodifferential). An element $\gamma \in G$ acts by right translation in the following way

$$U_{\gamma} : C^\infty(G_{r(\gamma)}) \to C^\infty(G_{s(\gamma)}) : (U_{\gamma}f)(\gamma') = f(\gamma'\gamma).$$

A $G$-operator of class $C^k$ is then an operator $P$ acting on $C^\infty_c(G)$, the space of compactly supported $C^\infty,k$-functions on $G$ such that there exists a family $(P_x)_{x \in M}$ of operators acting respectively on $C^\infty_c(G_x)$, with

$$\begin{align*}
(Pf)(\gamma) &= (P_{s(\gamma)}f_{s(\gamma)})(\gamma) \\
P_{s(\gamma)}U_{\gamma} &= U_{\gamma}P_{r(\gamma)},
\end{align*}$$

with $\gamma \in G$, $f \in C^\infty_c(G)$, and $f_x$ the restriction of $f$ to $G_x$.

Such a $G$-operator of class $C^k$ is characterized by a distributionnal kernel $k_P$ on $G$, which is a $C^k$-family of distributions $k_x$ on $G_x$:

$$(Pf)(\gamma) = \int_{G_{r(\gamma)}} k_P(\gamma'\gamma^{-1})f(\gamma')d\lambda_{s(\gamma)}(\gamma') = \int_{G_{r(\gamma)}} k_P(\gamma')f(\gamma'^{-1}\gamma)d\lambda(\gamma')(\gamma').$$

It is said to be compactly (or uniformly) supported when $k_P$ is compactly supported in $G$, and smoothing with compact support when $k_P \in C^\infty_c(G)$.

Before giving the definition of a $G$-pseudodifferential operator in general, we begin by studying the special case when the groupoid $G$ is the groupoid of a submersion.

Let $p : X \to M$ be a submersion between smooth manifolds. To any such submersion is naturally associated a Lie groupoid $G(X, p, M)$ which is the closed subspace of the groupoid of couples $X \times X$ made out of couples $(y, y')$ such that $p(y) = p(y')$. The source map is $s((y, y')) = y$ and the range map is $r((y, y')) = y$. The composition is the one of couples $(y, z) \circ (z, t) = (y, t)$.

If $G = G(X, p, M)$ is the groupoid of a submersion, then a $G$-operator is a family of operators indexed by $X$ invariant under the action of $G$. Hence it is in fact a family indexed by $M$ as the invariance condition imposes exactly that $P_x = P_y$ if $p(x) = p(y)$.

**Definition** 4.2.6 — Let $p : X \to M$ be a submersion and $m$ be a $C^{\infty,k}$ map. A (properly supported) $C^k$-family of classical pseudodifferential operators of order $m$ is a compactly supported $G(X, p, M)$-operator such that the family $P_x \in \mathcal{P}^m(p^{-1}(x))$ satisfies:

- In each trivialising open set $U \simeq U \times V$, for all $\phi, \psi \in C^\infty_c(X)$ with support in $U$, the operator $\phi P_x \psi$ viewed as an operator on $U \times V$ is a compactly supported $C^k$-family of classical pseudodifferential operators of order $m$ in the sense defined previously.
- For any maps $\phi, \psi \in C^\infty_c(X)$ with disjoint supports, the operator $\phi P_x \psi$ is a compactly supported smoothing $G$-operator.

We denote by $\mathcal{P}^m_k(X, p, M)$ the space of these operators.
Then we can compose such operators on a submersion.

**Proposition 4.2.7** — If $P \in \mathcal{P}_m(X, p, M)$ and $Q \in \mathcal{P}_n(X, p, M)$ are pseudodifferential operators then $PQ \in \mathcal{P}_{m+n}(X, p, M)$.

**Proof** — Consider the operator $\phi P Q \psi$ with $\phi$ and $\psi$ in $C^\infty_c(X)$. First of all, as $P$ and $Q$ are compactly supported in $G(X, p, M)$, the operators $\phi P$ and $Q \psi$ are compactly supported so that there are functions $\phi'$ and $\psi'$ in $C^\infty_c(X)$ such that $\phi P = \phi P \psi'$ and $Q \psi = \phi' Q \psi$. Hence, using partition of unity, we are left to show that the product of two compactly supported $C^k$ product-families of pseudodifferential operators in a given trivialising open set is still a compactly supported $C^k$ product-family of pseudodifferential operators and that the product of a compactly supported smoothing operator by a compactly supported $C^k$ product-family of pseudodifferential operators is a compactly supported smoothing operator, which is clear. ■

**Definition 4.2.8** — We will say that a compactly supported $G$-operator $P$ is a pseudodifferential operator of class $C^k$ if the family $(P_x)_{x \in M}$ is a $C^k$-family of pseudodifferential operators of order $m$ for the submersion $s : G \to G^{(0)}$. We denote by $\Psi^m_{c,k}(G)$ this space of operators.

Note that this definition implies that $m$ is a $C^k$ map on $G^{(0)}$, with $m(x) = m(y)$ whenever $G_\phi^y$ is non empty. What interests us at most is the case when $m$ is constant. Nevertheless, this generalization, with $m$ varying is straightforward, and allows us to give a simpler definition for holomorphic families of pseudodifferential operators. This definition also implies that the pseudodifferential operators $P_x$ on $G_x$ vary in a $C^k$ way : for any open chart $\Omega \subset G$ differentomorphic to $U \times s(\Omega)$, and for any $\phi \in C^\infty_{c, k}(\Omega)$, there exists a $C^k$-family $(a_x)_{x \in s(\Omega)}$ of polyhomogeneous symbols of order $m$ on $U$ such that the compactly supported operator $\phi P_x \phi$ corresponds to $Op(a_x)$ under the diffeomorphism $\Omega \cap G_x \simeq U$.

### 4.3 Principal and total symbol of a pseudodifferential $G$-operator

Note that in general the collection of such $C^k$-families of symbols $(a_x)_{x \in s(\Omega)}$ of polyhomogeneous symbols of order $m$ on $U$ in any chart $\Omega$ is not enough to define a $G$-operator, as these families have to satisfy an invariance property. We can nevertheless associate, to any $G$-operator a total symbol, which is given as a family of symbols $a_x \in S^m_{\text{hom}}(T^*_x G_x)$, but not in a unique way. However, it can be associated canonically a principal symbol to any pseudodifferential operator.

The principal symbol ([MP97, NWX99]) of a compactly supported pseudodifferential operator is defined by $\sigma_m(P)(\xi) = \sigma_m(P_x)(\xi)$ for $\xi \in A^*_x(G) = T^*_x G_x$. Note that using homogeneity, it can be defined as an element

$$
\sigma_m(P) \in C^\infty_k(S^*(G)),
$$

with $S^*(G)$ the ”cosphere bundle” of $G$, ie the quotient of $A^* G - 0$ by the action of $\mathbb{R}^*_+.$

From [MP97]Theorem 1, we know the following analogs of classical results.

**Theorem 4.3.9** — [MP97]
1. $\Psi_{c,k}^{m(\cdot)}(G) \circ \Psi_{c,k}^{m' (\cdot)}(G) \subset \Psi_{c,k}^{m(\cdot)+m'(\cdot)}(G)$.

2. $\sigma_{m+m'}(PQ) = \sigma_m(P)\sigma_{m'}(Q)$.

3. $\sigma_m$ gives rise to the following short exact sequence of locally compact spaces.

$$0 \to \Psi_{e,k}^{m(\cdot)-1}(G) \to \Psi_{e,k}^{m(\cdot)}(G) \xrightarrow{\sigma_m} C^{\infty, k}(S^* (G)) \to 0.$$ 

Note that $\sigma_m(P)$ is a good model for a "global" homogeneous symbol of order $m$ on $A^*(G)$. We can generalize this by defining "global" total symbols on $A^*(G)$.

**Definition 4.3.10** — Let $m$ be a $C^k$ map on $G^{(0)}$ such that $m(x) = m(y)$ whenever $G^y_x$ is nonempty. We denote by $S_{hom,k}^m(A^*(G))$ the subspace of $C^{\infty, k}(A^*(G))$ such that $\forall x \in M, a_x \in S_{hom}^m(\{x\}, T_x^* G_x)$ and such that for any trivializing open set $\Omega \subset A^*(G)$, with $\Omega \simeq U \times s(\Omega)$, the map $s(\Omega) \to S_{hom,k}^m(U), x \mapsto a_x$ is a $C^k$ map, in the sense defined previously.

Now given such a symbol, we can define a compactly supported $G$-pseudodifferential operator associated to it. There is not a unique way to do so, and different formulas give $a$ priori rise to pseudodifferential operators of same order $m$ with difference an operator of order $m - 1$. We can determine a formula as follows: suppose we are given a diffeomorphism $\phi$ from a neighborhood $W$ of $G^{(0)}$ in $G$ to a neighborhood of the 0 section in $A(G)$, with $d\phi = Id$, and a cut-off map $\chi$, with support in $W$. Then set, for $\xi \in A^*_x(G)$ and $\gamma \in G_x$, $e_{\xi}(\gamma) = \chi(\gamma) e^{i(\phi(\gamma), \xi)}$. We then have the following.

**Proposition 4.3.11** — Let $a \in S_{hom,k}^m(A^*(G))$. Denote by $Op(a)$ the $G$-operator defined by its kernel

$$k(\gamma) = \frac{1}{(2\pi)^n} \int_{A^*_x(G)} e^{-\xi}(\gamma^{-1})a(r(\gamma), \xi) d\xi.$$ 

Then $Op(a)$ is in $\Psi_{c,k}^{m(\cdot)}(G)$. Moreover, if we denote by $a_m$ the homogeneous principal symbol, then we have $\sigma_m(Op(a)) = a_m$.

**Proof** — By definition $Op(a)$ is a $G$-operator, so we are left to show that $Op(a)$ is locally a $C^k$-family of pseudodifferential operators in a local chart. To check this, we fix an open chart $\Omega \subset G$, with $\Omega \simeq U \times s(\Omega)$. Denote by $\kappa$ the diffeomorphism from $U \times s(\Omega)$ to $\Omega$ and by $\kappa_x$ its restriction from $U \times \{x\}$ to $\Omega \cap G_x$. Consider now a map $\phi \in C^{\infty, k}_c(\Omega)$ and denote by $P_x$ the operator $\phi Op(a)$ considered as an operator on $U \times \{x\}$. Then, if $f \in C^{\infty}_c(U)$, one has:

$$(P_x f)(u) = \phi(\kappa_x(u)) \int_{G_x} k_{\kappa}(\kappa_x(u) \gamma^{-1}) f(\kappa_{x}^{-1}(\gamma')) d\lambda_x(\gamma') = (2\pi)^{-n} \int_{G_x} \int_{A^*_x(\kappa_x(u))} \phi(\kappa_x(u)) \chi(\gamma'[\kappa_x(u)]^{-1}) a(r(\kappa_x(u)), \xi) e^{i(\phi(\gamma'[\kappa_x(u)]^{-1}), \xi)} f(\kappa_{x}^{-1}(\gamma')) d\xi d\lambda_x(\gamma').$$
Moreover, this map is surjective and admits an inverse $\sigma_{op}$. Indeed, we know that there exists a smooth map $\psi$ from $U \times U$ to $GL(\mathbb{R}^n, A_{\psi}(\kappa_x(u)))$ such that $(\phi(\kappa_x(u'))|\kappa_x(u))^{-1}, \psi_x(u, u') \xi = (u' - u, \xi)$. Hence one can write:

$$(P_x f)(u) = (2\pi)^{-n} \int_{U} \int_{\mathbb{R}^n} \tilde{a}(u, u', x, \xi) f(u') e^{i(u-u') \xi} d\xi du'$$

with

$$\tilde{a}(u, u', x, \xi) = \phi(\kappa_x(u)) \chi(\kappa_x(u')|\kappa_x(u))^{-1} a(r(\kappa_x(u)), \xi) e^{-i(\phi(\kappa_x(u'))|\kappa_x(u))^{-1}, \psi_x(u, u') \xi} |J_x(u')| |J_{\psi_x(u, u')}|.$$ 

The map $\tilde{a}(u, u', x, \xi)$ is a $C^k$ family of classical amplitudes and so gives rise to a pseudodifferential operator, as in classical theory (see [Hör71]). To find the principal symbol of this operator, we need to take the first term in the homogeneous expansion in $\xi$ on the diagonal $u' = u$. By $G$-invariance of the symbol, we can reduce this to the case where $u = 0$. As $\psi_x(0, 0)$ is simply given by transposition of $d\kappa_x(0)$, and by hypothesis $d\phi = Id$, one gets that $\sigma_m(P_x)(0, \xi) = a_m(x, \xi (d\kappa_x(0)\xi), \xi)$, and hence the principal symbol of $Op(a)$ is $a_m$.

**Remarks**

1. It follows from the definition that $Op(a)$ as compact support in $W$

2. As the principal symbol of $Op(a)$ is $a_m$, it implies that all formulas of this type coincide at first order.

3. Examples of maps $e_{\xi}$ are given in [NWX99]. First, fix an invariant connection $\nabla$ on $A(G) \to M$, so that one can define an exponential map from a neighborhood $V_0$ of the zero section in $A(G)$ to a neighborhood $V$ of $M$ in $G$, which sends the zero section on $M$ and which is a local diffeomorphism. Then define a cut-off map $\chi \in C_c^\infty(V)$ such that $\chi = 1$ in a smaller neighborhood of $G$. Denote by $\phi$ a local inverse of $\nabla$ in the support of $\chi$, and by $e_{\xi}(\gamma) = \chi(\gamma) \exp(i(\phi(\gamma), \xi))$, for $\xi \in A^*_{\chi}(G) = T^*_{\chi}(G) G_{\chi}$. Then $e_{\xi}$ satisfies the required conditions.

4. The original proof of Hörmander shows that more general type of maps $e_{\xi}$ are allowed to provide a formula that associates a pseudodifferential operator to a symbol.

Observe that if $a \in S_{hom,k}^\infty(A^*(G))$ then $Op(a)$ is smoothing. Hence $Op$ defines a map from $S_{hom,k}^{m,c,k}(A^*(G))/S_{hom,k}^{\infty}(A^*(G))$ to $\Psi_{c,k}^{m,c,k}(G)/\Psi_{c,k}^{\infty}(G)$, which is injective.

Indeed if $a = (a_{m(-j)})_{j \in \mathbb{N}}$ and $b = (b_{m(-j)})_{j \in \mathbb{N}}$ are two sequences of homogenous symbols in $S_{hom,k}^{m,c,k}(A^*(G))/S_{hom,k}^{\infty}(A^*(G))$, then $Op(a) = Op(b)$ implies that the principal symbol of $Op(a - b)$ is 0 hence $a_{m(-j)} = b_{m(-j)}$ for all $j \in \mathbb{N}$.

Moreover, this map is surjective and admits an inverse $\sigma_{tot}$, defined as follows. Let $P \in \Psi_{c,k}^{m,c,k}(G)$,
then we can define \( \sigma_{\text{tot}}(P) = (\sigma_{m-j}(P_j))_{j \in \mathbb{N}} \) with \( P_j \in \Psi^{m(-j)}_{c,k}(G) / \Psi^{-\infty}_{c,k}(G) \) defined recursively by \( P_0 = P \) and

\[
P_j = P_{j-1} - Op(\sigma_{m-j+1}(P_{j-1})).
\]

This defines a map from \( \Psi^{m(-j)}_{c,k}(G) / \Psi^{-\infty}_{c,k}(G) \) to \( \mathcal{S}^{m(-j)}_{\text{hom},k}(A^*(G)) / \mathcal{S}^{-\infty}_{\text{hom},k}(A^*(G)) \) such that for all \( a \in \mathcal{S}^{m(-j)}_{\text{hom},k}(A^*(G)) \), one has \( \sigma_{\text{tot}}(Op(a)) \equiv a \). Indeed, one gets in this situation that \( P_N = Op(a - \sum_{j=0}^{N-1} a_{m-j}) \). Hence we have constructed an inverse for \( Op \).

**Proposition 4.3.12** — Assume we have defined a map \( Op : \mathcal{S}^{m(-j)}_{\text{hom},k}(A^*(G)) / \mathcal{S}^{-\infty}_{\text{hom},k}(A^*(G)) \to \Psi^{m(-j)}_{c,k}(G) / \Psi^{-\infty}_{c,k}(G) \) as in Proposition 4.3.11. Then this map is a 1-1 correspondence and it admits an inverse denoted by \( \sigma_{\text{tot}} \).

As there were no canonical definition for \( Op \) map nor there is one for \( \sigma_{\text{tot}} \). Two different formulae accord only in general on the first term, which is the principal symbol. Hence, when we will speak later on of the total symbol of an operator, this will suppose that we have fixed a formula for \( Op \), what we assume from now on.

From the proposition 4.3.12 we can deduce the following lemma which will be useful for us.

**Lemma 4.3.13** — Let \( (P_j)_{j \in \mathbb{N}} \) be a family of \( G \)-pseudodifferential operators of order \( m-j \) with compact support in a fixed compact \( W \). Then there exists a pseudodifferential \( G \)-operator \( P \in \Psi^{m(-j)}_{c,k}(G) \) with compact support in \( W \) such that \( P \sim \sum P_j \), which means that for all \( N \in \mathbb{N} \), \( P - \sum_{j=0}^{N} P_j \in \Psi^{m(-j-N-1)}_{c,k}(G) \).

**Proof of the lemma** — In view of the 1-1 correspondence between symbols and \( G \)-operators, it is enough to show that there exists a symbol \( a_P \in \mathcal{S}^{m(-j)}_{\text{hom},k}(A^*(G)) \) such that \( a_P \sim \sum a_{P_j} \). This is a classical result showed using an analogous of Borel lemma [AG91][Prop. 2.3].

Following the original idea of Connes in [Con79], we can restate the theorem proved by Monthubert and Pierrot in [MP97] for the classical pseudodifferential operators (those with integer order), which can immediately be extended to polyhomogeneous operators of complex order. Let \( E \) denote the Hilbert \( C^*(G) \)-module \( C^*(G) \).

**Theorem 4.3.14** — Let \( P \in \Psi^{m(-j)}_{c,k}(G) \) be a compactly supported \( C^k \)-pseudodifferential operator on \( G \), and \( m_0 = \max \{ m \} \).

1. If \( m_0 < 0 \), then \( P \) extends to an operator \( \overline{P} \in \mathcal{K}(E) = C^*(G) \).
2. If \( m_0 = 0 \), then \( P \) extends to a bounded morphism \( \overline{P} \in \mathcal{L}(E) \).

### 4.4 Ellipticity

We now on assume that \( M = G^{(0)} \) is a compact set, as we are willing to look at compactly supported elliptic operators. Recall that an operator is **elliptic** when its principal symbol is
invertible. As in the classical setting, we want ellipticity to imply that there exists a parametrix, i.e. a pseudodifferential quasi-inverse for an elliptic operator.

**Proposition 4.4.1** — Let \( m \) be a complex map on \( G^{(0)} \) constant under the orbits of \( G \) and \( P \in \Psi_{c,k}^{m(\cdot)}(G) \) be an elliptic operator. Then there exists an operator \( Q \in \Psi_{c,k}^{-m(\cdot)}(G) \) which is a parametrix for \( P \):

\[
PQ - I = R \quad \text{and} \quad QP - I = R',
\]

with \( R \) and \( R' \) compactly supported smoothing operators.

**Proof** — By definition of ellipticity, we know that the principal symbol \( \sigma_m(P) \in C^\infty_{c,k}(S^*(G)) \) is invertible. Hence we have \( (\sigma_m(P))^{-1} \in C^\infty_{c,k}(S^*(G)) \). By theorem 4.3.9, this means that there exists a \( G \)-pseudodifferential operator \( Q_0 \) of order \(-m\) with principal symbol \( (\sigma_m(P))^{-1} \). Moreover, we can ask that \( Q_0 \) would be supported in a compact neighborhood \( W \) of \( M \) in \( G \), containing the support of \( P \). We now construct a sequence \( (P_j)_{j \in \mathbb{N}} \) of operators supported in \( W \) and of orders \(-m - j\) by setting \( Q_j = Q_0(I - PQ_0)^j \). Using the lemma 4.3.13, we then know that there exists an operator \( Q \in \Psi_{c,k}^{-m(\cdot)}(G) \) with support in \( W \) and such that

\[
Q \sim \sum Q_j = Q_0 \sum_{j=0}^{\infty} (I - PQ_0)^j.
\]

For any \( N \in \mathbb{N} \), we then have that \( PQ - I \in \Psi_{c,k}^{-N}(G) \). Indeed, we have \( Q - \sum_{j=0}^{N-1} Q_j \in \Psi_{c,k}^{-m(\cdot)-N}(G) \), and \( P \left( \sum_{j=0}^{N-1} Q_j \right) - I = -(I - PQ_0)^{-N} \in \Psi_{c,k}^{-N}(G) \), from which we deduce that \( PQ - I \in \Psi_{c,k}^{-\infty}(G) \). We can do the same for the left parametrix, and by the classical argument, show that left and right parametrix coincide modulo a smoothing operator with compact support in \( W \).

\[\blacksquare\]

### 4.5 Unbounded operators

We now wish to consider compactly supported \( G \)-pseudodifferential operators as unbounded operators on the Hilbert \( C^* \)-module \( E = C^*(G) \). We are going to show that in the case where the operator is elliptic, then it is regular, as an unbounded operator, in the sense of Baaj [Baa80, BJ83] (cf definition 2.3.2). The material in this subsection is coming from a graduate course of Georges Skandalis [Ska96] which has also been exposed by François Pierrot in [Pie96].

Consider now a compactly supported pseudodifferential operator \( P \) of \( C^k \)-type on \( G \), with order \( m \) of real part \( m_0 > 0 \). This operator, of domain \( C^\infty_{c,k}(G) \) can be viewed as an unbounded, densely defined operator on the Banach space \( E = C^*(G) \). Recall also that such an operator admits a formal adjoint \( P^\natural \) which is again a compactly supported pseudodifferential operator, with order \( \overline{m} \). This operator is characterized by the equality, \( \langle Pu, v \rangle = \langle u, P^\natural v \rangle \), which holds for all \( u, v \in C^\infty_{c,k}(G) \). As both \( P \) and \( P^\natural \) are densely defined operators, \( P \) and \( P^\natural \) are closeable. We denote by \( \overline{P} \) the closure of \( P \). Recall that it is the smallest extension of \( P \) with its graph being a closed sub-\( C^*(G) \)-module of \( E \), and that its graph is given by

\[
G(\overline{P}) = G(P) = \{(x, y) \in (C^*(G))^2, \exists (u_n) \in C^\infty_{c,k}(G), \|u_n - x\| \to 0 \text{ and } \|Pu_n - y\| \to 0\}.
\]
Note in particular that \( \mathcal{P} \) is a densely defined operator with a densely defined adjoint \( P^* \) such that \( \overline{\mathcal{P}} \subset P^* \). We begin by a very useful lemma:

**Lemma 4.5.1** — Let \( A, B \in \Psi_{c,k}(G) \), such that \( \max \Re(\text{ord} A + \text{ord} B) \leq 0 \) and \( \max \Re \text{ord} B \leq 0 \). Then we have \( A\overline{B} = \overline{A}B \) and this operator has domain \( E \).

Proof of the lemma — It is enough to show that \( A\overline{B} \) is a closed operator. Indeed we know that \( AB \subset A\overline{B} \) and that \( AB \) is of order with real part less or equal to 0 and so extends to a continuous morphism \( \overline{AB} \in \mathcal{L}(E) \), of domain \( E \), by proposition 4.3.14.

We know that \( G(\overline{AB}) = \{ (x, z) \in E \times E, (\overline{B}x, z) \in G(\overline{A}) \} \). As \( G(\overline{A}) \) is a closed subspace of \( E \times E \) and as the map \( \overline{B} \) is continuous from \( E \) to \( E \) by proposition 4.3.14, the set \( G(\overline{AB}) \) is closed.

We now come to the main proposition of this section.

**Proposition 4.5.2** — Let \( P \) be an elliptic, compactly supported pseudodifferential operator of \( C^k \)-type on \( G \). Then the operator \( \mathcal{P} \) is a regular operator on \( E \).

Proof — It is enough to consider the case when \( m_0 = \max \Re m > 0 \), as we have seen that \( \overline{P} \in \mathcal{L}(E) \) otherwise. Note that both \( \overline{P} \) and \( P^* \) are densely defined so that we only have to prove that \( G(\overline{P}) \) is orthocomplemented.

Now let \( Q \) be a parametrix of order \( -m \) for \( P \), and \( R \) and \( S \) be the compactly supported smoothing operators such that

\[
QP = 1 - S \quad \text{and} \quad PQ = 1 - R.
\]

Applying the proposition (4.3.14) to \( Q, R \) and \( S \), we know that these operators extend to compact morphisms in \( \mathcal{L}(E) \). We then have:

**Lemma 4.5.3** —

a) \( \mathcal{P}Q = Q\overline{P} \) and \( \mathcal{P}S = S\overline{P} \). Moreover these operators have domain \( E \).

b) \( \text{Dom}\overline{P} = \text{Im}Q + \text{Im}S \).

c) \( \overline{P}^* = P^* \).

Proof of the lemma —

a) This is a direct application of the lemma 4.5.1 above.

b) Let \( x \in \text{Dom}\overline{P} \), \( \overline{P}x \in E \); there exists a sequence \( u_n \) in \( C^\infty_c(G) \) converging to \( x \) in norm and such that \( Pu_n \) converges in norm to \( \overline{P}x \). As we have \( QPu_n = u_n - Su_n \), with \( Q \) and \( S \) continuous, we get \( Q(\overline{P}x) = x - \overline{S}x \), from which we deduce

\[
x = Q(\overline{P}x) + \overline{S}x \in \text{Im}Q + \text{Im}S.
\]
On the other hand, we know from a) that: $\text{Im}Q \subset \text{Dom}P$ and $\text{Im}S \subset \text{Dom}P$.

c) We have already noticed that $\overline{P^*} \subset P^*$. We are then left to show that $\text{Dom}P^* \subset \text{Dom}\overline{P^*}$. But we know that $PQ = I + R$, and so $(PQ)^* = I + R^*$. As we have $Q^*P^* \subset (PQ)^*$, we get, for any $x \in \text{Dom}P^*$, that $x = Q^*(P^*x) - R^*x$, and so that $\text{Dom}P^* \subset \text{Im}Q^* + \text{Im}R^*$. As $Q$ and $R$ are negative order operators, we have $\overline{Q} = Q^*$ et $\overline{R} = R^*$. Applying b) to $\overline{P^*}$, we get:

$$\text{Dom}\overline{P^*} = \text{Im}\overline{Q^*} + \text{Im}\overline{R^*},$$

which is enough to conclude.

We have proven that $G(\overline{P}) = \{(\overline{Q}x + \overline{S}y, \overline{PQ}x + \overline{PS}y), (x, y) \in E \times E\}$. Consider now the operator on $E \oplus E$ defined by

$$U = \left( \begin{array}{cc} \overline{Q} & \overline{S} \\ \overline{PQ} & \overline{PS} \end{array} \right).$$

It is a morphism in $L(E \oplus E)$ as $Q$, $S$, $PQ$, $PS$ and their adjoints are compactly supported pseudodifferential operators with real part of the order less or equal to zero, and so their closures are elements of $L(E)$. The range of $U$ is then exactly equal to the graph of $\overline{P}$, and we get the result using proposition (2.1.3).

We then get an immediate corollary of the proposition 4.3.14

**Corollary 4.5.4** — Let $P_1$ and $P_2$ be respectively in $\Psi_{c,k}^{m_1}(G)$ and $\Psi_{c,k}^{m_2}(G)$, with $\max \Re(m_2 - m_1) \geq 0$ and $P_2$ elliptic. Then there exists $c > 0$ such that for any $u \in C_0^\infty(G)$, we have, for the norm of $C^*(G)$,

$$\|P_1u\| \leq c(\|P_2u\| + \|u\|)$$

and $\text{Dom}P_2 \subset \text{Dom}P_1$.

Proof of the corollary— Let $Q_2 \in \Psi_{c,k}^{-m_2}(G)$ be a parametrix for $P_2$. As $P_1Q_2 \in \Psi_{c,k}^{m_1-m_2}(G)$, it is bounded and there exists $c_1 > 0$ s.t. $\|P_1Q_2(P_2u)\| \leq c_1 \|P_2u\|$. Moreover, $P_1(Q_2P_2 - I)$ is a compactly supported smoothing operator and so there exists $c_2$ s.t. $\|P_1Q_2P_2u - P_1u\| \leq c_2 \|u\|$. Finally, we get

$$\|P_1u\| \leq \|P_1Q_2P_2u - P_1u\| + \|P_1Q_2(P_2u)\| \leq c(\|P_2u\| + \|u\|).$$

4.6 **The algebra $\Psi_k(G)$ of pseudodifferential operators of $C^k$-type**

As we intend to make functional calculus with our operators, we will need a class of smoothing operators which are not any more compactly supported. We will later show that this definition is not artificial, and that it fits well with the framework of our Sobolev modules, thought this class is quite big.

If $P$ is a compactly supported pseudodifferential $G$-operator and $T \in L(E)$, then we write
$TP \in \mathcal{L}(E)$ when $\text{Im} T^* \subset \text{Dom} P^*$ which implies that $TP$ is a morphism, as we can extend by continuity the equality $(TPx, y) = (x, P^*T^*y)$ which is true for any $(x, y) \in C_c^\infty(G) \times E$.

With the same notations, we write $\overline{TP} \in \mathcal{L}(E)$ whenever $\overline{T^*P^*} \in \mathcal{L}(E)$, with $P^*$ the formal adjoint of $P$. Finally, we write $\overline{P_1TP_2} \in \mathcal{L}(E)$ when $\overline{P_1T} \in \mathcal{L}(E)$ and $(\overline{P_1T})P_2 \in \mathcal{L}(E)$ and when $\overline{TP_2} \in \mathcal{L}(E)$ and $P_1(\overline{TP_2}) \in \mathcal{L}(E)$.

**Definition 4.6.1** — A smoothing operator is an operator $R \in \mathcal{L}(E)$ such that for any compactly supported pseudodifferential $G$-operators $P_1, P_2$ of $C^k$-type, we have $P_1RP_2 \in \mathcal{L}(E)$. We denote by $\Psi^{-\infty}(G)$ the algebra formed by these operators.

**Remarks**

1. As the property $P_1RP_2 \in \mathcal{L}(E)$ should be true for all pseudodifferential operators, we can easily deduce a handier characterization of smoothing operators.

**Proposition 4.6.2** — An operator $R$ is smoothing if and only if it fulfills the two following conditions

(a) $\forall P \in \Psi_{c,k}(G)$, $\text{Im} R \subset \text{Dom} \overline{P}$ and $\text{Im} R^* \subset \text{Dom} \overline{P}$.

(b) The operator $\overline{P_1TP_2}$ defined on $C_c^\infty(G)$ is bounded on $E$.

2. Note that the letter $k$ denoting the transversal class of regularity has disappeared, as we will show this set is independent of $k$. Indeed, our class of smoothing operators appears to be only continuous in the direction transverse to $G^x$ in $G$. In general, we do not know a better result, for tranverse regularity, though in particular cases we can ask better transverse regularity, provided there are enough transverse vector fields.

To give a more precise idea on this set $\Psi^{-\infty}(G)$, we can state the following.

**Proposition 4.6.3** — The set $\Psi^{-\infty}(G)$ is a sub-algebra of $\mathcal{L}(E)$ and has the following properties:

1. $\Psi^{-\infty}(G) \subset \mathcal{K}(E)$;

2. $\forall P_1, P_2 \in \Psi_{c,k}(G), \forall R \in \Psi^{-\infty}(G), P_1RP_2 \in \Psi^{-\infty}(G)$;

3. $\forall R_1, R_2 \in \Psi^{-\infty}(G), \forall T \in \mathcal{L}(E), R_1TR_2 \in \Psi^{-\infty}(G)$;

4. $\Psi^{-\infty}_{c,k}(G) = C_c^{\infty,k}(G) \subset \Psi^{-\infty}(G)$.

**Proof** — The properties 2 and 3 are direct consequences of the definition of $\Psi^{-\infty}(G)$, while property 4 comes from the definition of compactly supported smoothing operators on $G$. We
are only left to show the first one. We can first of all remark that the definition of \( \Psi^{-\infty}(G) \) implies that for any \( P_1, P_2 \) in \( \Psi_{c,k}(G) \), we have \( P_1 R P_2 \in \mathcal{K}(E) \). Indeed, let \( P \) be an elliptic operator with order \( m \), with \( \Re m > 0 \), and denote \( Q \) a parametrix for \( P \) and \( S = Q P - I \). \( S \) is a compactly supported smoothing operator and by theorem 4.3.14, we know that \( \overline{Q} \in \mathcal{K}(E) \) and \( \overline{S} \in \mathcal{K}(E) \). By hypothesis, we know that \( P P_1 R P_2 \in \mathcal{L}(E) \) so that using the closure of the equality \( P_1 R P_2 = Q P P_1 R P_2 - S P_1 R P_2 \), we get \( P_1 R P_2 \in \mathcal{K}(E) \). In the case where \( P_1 = P_2 = 1 \), this shows that \( R \in \mathcal{K}(E) \). 

One can prove, using parametrices, that the statements

\[
\{ PR \in \mathcal{K}(E), \text{ for all } P \in \Psi_{c,k}(G) \}
\]

and

\[
\{ PR \in \mathcal{K}(E), \text{ for all } P \in \Psi_{c,k}(G), P \text{ elliptic } \}
\]

are equivalent. Hence, for the definition of smoothing operators, we may consider only elliptic pseudodifferential operators.

**Proposition 4.6.4** — Let \( R \in \mathcal{L}(E) \), and \( P_1 \) and \( P_2 \) two elliptic operators with constant order of strictly positive real part. Then the following are equivalent.

1. \( R \in \Psi^{-\infty}(G) \).
2. \( \forall n \in \mathbb{N}, \hat{P} P_1 R P_2 P_i \in \mathcal{L}(E) \).

**Proof** —

- It is clear that 1) \( \Rightarrow \) 2). We are left to show that 2) \( \Rightarrow \) 1). We are using the characterization in the proposition 4.6.2 above to show this property. Note that we need only to show that if the two conditions in 4.6.2 are true for an operator \( R \) and for \( P_1^n, P_2^n \) for any \( n \in \mathbb{N} \), then they are true for all pseudodifferential operators \( A_1 \) and \( A_2 \). For \( i = 1, 2 \), fix a parametrix \( Q_i(n) \) for \( P_i^n \), and denote by \( R_i(n) \) and \( S_i(n) \) the compactly supported smoothing operators defined by

\[
S_i(n) = I - Q_i(n) P_i^n \quad \text{and} \quad R_i(n) = I - P_i^n Q_i(n).
\]

Note by the way that by elementary calculus we can show that \( Q_i^n \) is a parametrix for \( P_i^n \) whenever \( Q_i \) is a parametrix for \( P_i \).

- The assumption (a) is then a consequence of the lemma 4.5.3 which states that \( \text{Dom} P_i^n = \text{Im} Q_i(n) + \text{Im} S_i(n) \). Choose \( A \in \Psi^{m(n)} \) and \( n \in \mathbb{N} \) such that \( \Re \text{ ord } P_i^n > \max \Re \text{ ord } A \). Then we have \( \text{Im} Q_i(n) \subset \text{Dom} A \) and \( \text{Im} S_i(n) \subset \text{Dom} A \). Indeed, this comes from the lemma 4.5.1 : if \( Q \) is a compactly supported pseudodifferential operator with order \(-m\), with \( \Re m > \Re \text{ ord } A \), then one has \( \overline{A Q} = \overline{A Q} \) and this morphism has domain \( E \). This implies in particular that \( \text{Im} Q \subset \text{Dom} A \).
We construct from Proposition 4.6.5 — \( \Psi \) to be the linear span generated by operators of constant order in \( \Psi_{c,k} \). Let us begin by the case when we have trivial vector bundle over \( M \). Then, we have

\[
A_1 R A_2 = (A_1 Q_1(n))(P_1^m R P_2^n)(Q_2(n) A_2) + (A_1 S_1(n))(R P_1^n)(Q_2(n) A_2) + (A_1 Q_1(n))(P_1(n) R)(R Q_2 n A_2) + (A_1 S_1(n)) R(R Q_2 n A_2).
\]

As the operators \( A_1 Q_1(n), A_1 S_1(n), Q_2(n) A_2 \) and \( R Q_2(n) A_2 \) are compactly supported pseudodifferential operators with order of negative real part, their closures are elements in \( \mathcal{K}(E) \) by proposition 4.3.14. Hence the assumptions for \( P_1^m \) and \( P_2^n \) imply the assumptions for all compactly supported operators \( A_1 \) and \( A_2 \).

This allows us to enlarge the class of pseudodifferential \( G \)-operators of \( C^k \)-type. Set \( \Psi_k(G) \) to be the linear span generated by operators of constant order in \( \Psi_{c,k}(G) \) and by operators in \( \Psi^{-\infty}(G) \). Then, we have

**Proposition 4.6.5** —

1. For any \( m, n \in \mathbb{C} \cup \{-\infty\} \) we have : \( \Psi_k^m(G) \circ \Psi_k^n(G) \subset \Psi_k^{m+n}(G) \).
2. The space \( \Psi_k(G) \) endowed with composition is a \( * \)-algebra, filtered by \( \mathbb{R} \) and graded by \( \mathbb{C}/\mathbb{Z} \).
   Note that the previous property turns \( \Psi^{-\infty}(G) \) into a two-sided ideal of this algebra.
3. The set \( \Psi_k^\infty(G) \) of operators with integer order is a \( * \)-subalgebra of \( \Psi_k(G) \).

### 4.7 Vector Bundles

We can define, as in the classical case, pseudodifferential \( G \)-operators acting on the sections of a vector bundle. We give here some hints to treat the case of operators acting over the sections of vector bundles in the above framework.

Let us begin by the case when we have trivial vector bundle over \( G^{(0)} = M \). In this case we can easily deduce the definition of a \( G \)-operator from \( C^j \) to \( C^l \).

**Definition 4.7.1** — A \( G \)-operator from \( C^j \) to \( C^l \) is a matrix \( l \times j \) of \( G \)-operators.

In particular, it is clear that a compactly supported pseudodifferential \( G \)-operator of order \( m \) from the \( C^{\infty,k} \)-sections of \( C^j \) to the \( C^{\infty,k} \)-sections of \( C^l \) is defined by a matrix \( l \times j \) of compactly supported pseudodifferential \( G \)-operators of order \( m \), \( P \in \mathcal{M}_{j \times i}(\Psi_{c,k}^m(G)) \).

Suppose now we are given a smooth finite dimensional vector bundle \( \mathcal{E} \) over \( G^{(0)} = M \). We construct from \( \mathcal{E} \) a vector bundle \( r^*(\mathcal{E}) \) over \( G_x \) for any \( x \), simply by pull-back of the range map \( r \). The fibre of \( \gamma \in G_x \) is given by \( r^*(\mathcal{E}) \gamma = \mathcal{E}_{r(\gamma)} \).

The manifold \( M = G^{(0)} \) being compact, a classical result states that any complex finite dimensional vector bundle \( \mathcal{E} \) admits a supplementary vector bundle \( \mathcal{E}^\sharp \) in a trivial fibre bundle

\[
\mathcal{E} \oplus \mathcal{E}^\sharp = C^j.
\]
Then there exists a section
\[ e_0 \in C^\infty(M, M_j(\mathbb{C})) \simeq M_j(C^\infty(M)) \]
which is an orthogonal projection, with image \( E \). A map \( f \in C^k(M) \) acts naturally by multiplication on \( C^\infty_c(G) \), and can so be considered, by composition with the range map \( r \), as an element of the multiplier algebra \( M(C^*(G)) \). We then denote by \( e = e_0 \circ r \) the corresponding projection of \( M_j(M(C^*(G))) \), so that we have constructed a module over \( C^*(G) \) by \( E \otimes_{C(M)} C^*(G) = e(C^*(G))^\sharp \).

Suppose now we are given two smooth finite dimensional vector bundles \( E \) and \( E' \), over \( M \). We can suppose that these are subbundles of the same trivial bundle \( \mathbb{C}^j \). We denote by \( e \) and \( e' \) the corresponding projections of \( M_j(M(C^*(G))) \).

**Definition 4.7.2** — A \( G \)-operator \( P \) from \( E \) to \( E' \) is defined by
\[ P = e' \hat{P} e \]
with \( \hat{P} \) a \( G \)-operator acting on the sections of the trivial vector bundle \( \mathbb{C}^j \).

In particular, a compactly supported \( G \)-operator of order \( m \) and class \( C^k \) from \( E \) to \( E' \) is an operator of the form \( P = e' \hat{P} e \), with \( \hat{P} \) a matrix in \( M_j(\Psi_{c,k}^m(G)) \). We will denote by \( \Psi_{c,k}^m(G, E, E') \) the space of these operators.

As said above, we can also associate a Hilbert \( C^* \)-module over \( C^*(G) \) to any such fibre bundle \( E \). Indeed, consider the module over \( C_c(G) \) formed by the sections \( C_c(G, r^*(E)) \), where \( C_c(G) \) acts by convolution. We define a scalar product on \( C_c(G, r^*(E)) \) with values in \( C_c(G) \) by making the pointwise scalar product of the sections
\[ \langle \eta, \eta' \rangle(\gamma) = \langle \eta(\gamma), \eta'(\gamma) \rangle_{r^*(E)}. \]
This gives us a norm on \( C_c(G, r^*(E)) \) by letting
\[ \|\eta\| = \|\langle \eta, \eta \rangle\|_{C^*(G)}. \]
We then define a Hilbert \( C^* \)-module \( E = C^*(G, E) \) over \( C^*(G) \) by completing \( C_c(G, r^*(E)) \) for this norm. We can give another description of this Hilbert module, by using the projection \( e \in M_j(M(C^*(G))) \) associated to \( E \). Indeed, in this framework, the module \( E = C^*(G, E) \) is simply the sub-\( C^*(G) \)-module \( E = e(C^*(G))^\sharp \) of the module \( (C^*(G))^\sharp \), which is a Hilbert module over \( C^*(G) \) for the scalar product \( \langle X, Y \rangle = \sum_{i=1}^j X_i^* Y_i \).

To be complete, we need to give the definition of the formal adjoint of such an operator. If \( P \) is a \( G \)-operator from \( E \) to \( E' \), it can be written in the form \( e' \hat{P} e \); its kernel \( K(\gamma, \gamma') \) is then for the form \( e'(\gamma) \hat{K}(\gamma, \gamma') e(\gamma') \). Then, the kernel \( K^\sharp(\gamma, \gamma') \) defined by
\[ \bar{e}'(\gamma'^{-1}) \overline{\hat{K}(\gamma', \gamma)} \bar{e}(\gamma^{-1}) \]
defines an operator \( \hat{P}^\sharp = \bar{e}' \hat{P} e' \), the formal adjoint of \( P \), such that for any \( x \in M \) and any sections \( v \in C^\infty_c(G_x, r^*(E)) \) and \( u \in C^\infty_c(G_x, r^*(E')) \) we have
\[ \langle \hat{P}^\sharp u, v \rangle_{r^*(E)} = \langle u, P v \rangle_{r^*(E')} \tag{5} \]
For pseudodifferential operators, this formal adjoint coincides on $C_c^\infty$-sections with the adjoint $P^*$ of $P$, in the following sense:

**Lemma 4.7.3** — For all sections $v \in C_c^\infty(G, \pi(G))$ and $u \in C_c^\infty(G, r_*(\mathcal{E}))$, we have the following equality in $C^*(G)$:

$$\langle P^* u, v \rangle = \langle u, P v \rangle.$$  

**Proof of the lemma** — We know that $u$ and $v$ can be written $u = e^U$ et $v = e^V$, with $U, V$ elements in $(C_c^\infty(G))^j$. Now we have

$$\langle P^* u, v \rangle = \sum_{i,k} (P^*_{i,k} U_i) V_k = \sum_{i,k} U_i^* P_{k,i} V_k = \langle u, P v \rangle,$$

by definition of $P^*$. □

A compactly supported $G$-operator of class $C^k$ from $\mathcal{E}$ to $\mathcal{E}'$ then appears, as before, as an unbounded (in general) operator on a Hilbert module, which is densely defined, and has a densely defined adjoint. By adding $\mathcal{E}$ and $\mathcal{E}'$, we can always reduce to the case where $\mathcal{E} = \mathcal{E}'$. To recover the results of this paper in the case of an operator acting on the sections of such a vector bundle $\mathcal{E}$, we need just to replace in the statements the Hilbert $C^*(G)$-module $E = C^*(G)$ by the projective $C^*(G)$-module $E = e((C^*(G))^j)$, where $e$ is the projection corresponding to $\mathcal{E}$ as above.

## 5 Sobolev modules

### 5.1 Definition

Consider now a $C^k$-elliptic operator $P$ of constant order $m$, with $\Re m = s \geq 0$. We will from now also denote by $P$ the closure $\overline{P}$ of $P$ when there is no ambiguity. As the operator $P$ is regular, we have $\text{Dom} P = (1 + P^* P)^{-1/2} E$, so that it is clear that $\text{Dom} P$ is a sub-$C^*(G)$-module of $C^*(G)$. Moreover it can be equipped with a Hilbert module structure using the scalar product $\langle x, y \rangle_s = \langle P x, P y \rangle + \langle x, y \rangle$.

Actually the right way to look at this Sobolev module is to consider it not as a precise Hilbert module, associated with a well-defined scalar product, but rather as a class of such (a Hilbertizable module), provided that the vector spaces are equal and that the scalar products are compatible in the sense of [HS92] p75. Recall that if $A$ is a $C^*$-algebra and $\mathcal{E}$ an $A$-module, two scalar products $\langle , \rangle_1$ and $\langle , \rangle_2$, such that $(\mathcal{E}, \langle , \rangle_1)$ and $(\mathcal{E}, \langle , \rangle_2)$ are Hilbert $A$-modules, are said to be compatible whenever there exists an operator $T$, $A$-linear and invertible, such that for any $\xi, \zeta \in \mathcal{E}$, $\langle \xi, \zeta \rangle_1 = \langle \xi, T \zeta \rangle_2$. The advantage of this notion is that if $\mathcal{E}$ and $\mathcal{F}$ are two given Hilbert modules, the spaces $\mathcal{K}(\mathcal{E}, \mathcal{F})$ or $\mathcal{L}(\mathcal{E}, \mathcal{F})$ do not vary if one takes another scalar product on $\mathcal{E}$ and $\mathcal{F}$ provided they are compatible with the original ones. (cf [HS92, KM85]). Of course the adjoint of an operator in these spaces depends on the scalar products, but then, changing the scalar products is equivalent to composing the original $*$-operation with bijective $A$-linear operators.

We begin with the definition of the Sobolev module $H^s(P)$ associated with an operator, and
we then show that all operators provide compatible scalar products so that the notion of \( H^s \) as a Hilbertizable module is well-defined.

**Definition 5.1.1** — Let \( s \) be a positive real number and let \( P \) be a \( C^k \)-elliptic operator of order \( m \), with \( \Re m = s \). The Sobolev module of rank \( s \) associated with \( P \) is the Hilbert \( C^s(G) \)-module \( H^s(P) = \operatorname{Dom} \overline{P} \) endowed with the scalar product 

\[
\langle x, y \rangle_s = \langle Px, Py \rangle + \langle x, y \rangle.
\]

We are now going to prove that these Sobolev modules are in fact independant of the operator \( P \) chosen to define them.

**Proposition 5.1.2** — Let \( P \) and \( P' \) be two compactly supported elliptic operator of order \( m \) and \( m' \), with \( \Re m = \Re m' = s \). Then the Sobolev modules \( H^s(P) \) and \( H^s(P') \) are compatible.

**Proof** — We know from corollary 4.5.4 that we have \( \operatorname{Dom} P \subset \operatorname{Dom} P' \) and that \( \operatorname{Dom} P' \subset \operatorname{Dom} P \), so that the modules are equal.

Set \( T = (1 + P^* P')^{-1}(1 + P^* P) \). It is left to show that \( T \) is an invertible element in \( \mathcal{L}(E) \).

Using a parametrix \( Q \) for the elliptic operator \( P^* P' \) (note that \textit{a priori} \( 1 + P^* P \) is not a polyhomogeneous pseudodifferential operator in general) such that \( P^* P'Q + R = I \) with \( R \) a compactly supported smoothing operator, we see that one can write \( (1 + P^* P')^{-1} = Q + (1 + P^* P')^{-1}R - (1 + P^* P')^{-1}Q \), and so

\[
T = Q(1 + P^* P) + (1 + P^* P')^{-1}R(1 + P^* P) - (1 + P^* P')^{-1}Q(1 + P^* P).
\]

The first part on the right hand side is a pseudodifferential of order \( m - m' \), so that the real part of its order is zero, and it is a bounded operator. The second part of it is the product of a bounded operator \( (1 + P^* P')^{-1} \) by a smoothing operator \( R(1 + P^* P) \), which is also bounded, and the third part is also the product of two bounded operators, so that \( T \) is bounded. The inverse of \( T \) is simply \( (1 + P^* P)^{-1}(1 + P^* P') \) which is bounded by the same proof. □

We can then proceed to define the negative rank Sobolev modules by duality. First observe that \( E \) is naturally included in \( \mathcal{K}(H^s, C^*(G)) \) by the following map \( \xi \mapsto \langle \xi, \cdot \rangle \) where the scalar product is taken in \( E \).

**Definition 5.1.3** — Let \( s > 0 \), the Sobolev (Hilbertizable) module \( H^{-s} \) is the completion of \( E \) with respect to the norm of \( \mathcal{K}(H^s, C^*(G)) \).

Then to any \( C^k \)-elliptic operator \( P \) of order \( m \), with \( \Re m = s \geq 0 \), one can describe the Hilbert module \( H^{-s}(P) \) : it is exactly the completion of \( E \) with respect to the norm induced by the scalar product : \( \langle \xi, \zeta \rangle_{-s} = \langle (1 + P^* P)^{-\frac{1}{2}} \xi, (1 + P^* P)^{-\frac{1}{2}} \zeta \rangle_E \). Indeed, using the fact that \( \operatorname{Dom} P = \operatorname{Im} (1 + P^* P)^{-\frac{1}{2}} \) : if \( \xi \in E \) then, we have
\[ \|\xi\|_{-s} = \sup \{ \| (\xi, x)\|, \| x\|_s \leq 1 \} \]
\[ = \sup \left\{ \| (\xi, x)\|, \left\| (1 + P^* P)^{\frac{1}{2}} x \right\| \leq 1 \right\} \]
\[ = \sup \left\{ \| (1 + P^* P)^{-\frac{1}{2}} \xi\|, \| y\| \leq 1 \right\} \]
\[ = \left\| (1 + P^* P)^{-\frac{1}{2}} \xi \right\|. \]

### 5.2 Properties of the Sobolev modules

We will from now on denote:

\[ H^\infty = \cap_{s \in \mathbb{R}} H^s \text{ and } H^{-\infty} = \cup_{s \in \mathbb{R}} H^s. \]

First note that we have, by definition, duality between \( H^s \) and \( H^{-s} \) in this framework.

**Proposition 5.2.1** — Let \( s > 0 \). The \( C^*(G) \)-sesquilinear continuous map defined by \( H^s \times E \to C^*(G) \)

\[ (u, v) \mapsto \langle u, v \rangle_E \]

extends into a \( C^*(G) \)-sesquilinear continuous map \( H^s \times H^{-s} \to C^*(G) \).

As in the classical setting, we have embeddings between Sobolev modules.

**Proposition 5.2.2** — Let \( s > s' \). The identity on \( C_c(G) \) extends itself in an imbedding \( i_{s,s'} : H^s \hookrightarrow H^{s'} \), which is a compact morphism between these \( C^* \) Hilbert modules.

**Proof**

1. If \( s > s' \geq 0 \), Let \( P \) and \( P' \) be respectively elliptic pseudodifferential operators of order with real part \( s \) and \( s' \) (in case \( s' = 0 \) we assume that \( P' = 0 \)). Then by corollary 4.5.4, we have an embedding \( \text{Dom}P \subset \text{Dom}P' \) so that \( H^s \subset H^{s'} \) and the map \( i_{s,s'} \) is well defined. By definition of a morphism between Hilbert modules, it is enough to show that there exists a map \( i_{s,s'}^* : H^{s'} \to H^s \) such that for any \( x \in H^s \), and any \( y \in H^{s'} \), one has \( \langle i_{s,s'}(x), y \rangle_{s'} = \langle x, i_{s,s'}^*(y) \rangle_s \). Rewriting this equality, we get:

\[ \langle P' x, P' y \rangle + \langle x, y \rangle = \langle P x, P^s_{s',s}(y) \rangle \text{ ie } \langle x, (1 + P^* P)^{-1} (1 + P^* P)y \rangle = \langle x, (1 + P^* P) i_{s,s'}^*(y) \rangle. \]

We see that setting \( i_{s,s'}^*(y) = (1 + P^* P)^{-1} (1 + P^* P)y \) solves the equality. Moreover the operator \((1 + P^* P)^{-1} (1 + P^* P)\) is compact. Indeed, let \( Q \) be a parametrix for \( P^* P \), such that \( P^* P Q = I + R \) with \( R \in \Psi_{c,k}^\infty \). We know that \( Q, R \in \mathcal{K}(E) \), and by
the following equality \((1 + P^*P)^{-1} = Q - (1 + P^*P)^{-1}R - (1 + P^*P)^{-1}Q\), we get that \((1 + P^*P)^{-1} \in K(E)\). Finally we have that \((1 + P^*P)^{-1}(1 + P^*P^r) = Q(1 + P^*P) - (1 + P^*P)^{-1}R(1 + P^*P)^{-1}Q(1 + P^*P^r)\). The first term on the right hand side is a pseudodifferential operator of negative order, so it is in \(K(E)\). The second term is in \(K(E)\) by the properties of the smoothing operators and the third is a product of an element in \(K(E)\) by a pseudodifferential operator of order with real part 0 (hence bounded).

2. If \(0 \geq s > s'\), then it is immediate using the definition of negative rank Sobolev modules, ie the duality between \(H^s\) and \(H^{-s}\). We have that an operator \(T \in K(H^s, H^{s'})\) if and only if the operator \(\tilde{T} \in K(H^{-s'}, H^{-s})\) where \(\tilde{T}\) is defined by the following equality for any \(\eta \in H^{-s'}\) and \(\xi \in H^s\):

\[
\langle \tilde{T} \eta, \xi \rangle = \langle \eta, T \xi \rangle.
\]

\(\blacksquare\)

We can now, using this lemma, show that our definition of smoothing operators is coherent with the natural one arising from this Sobolev scale.

**Proposition 5.2.4** — An operator \(R\) is smoothing if and only if it is in the intersection of all \(L(H^s, H^t)\) for \(s, t \in \mathbb{R}\). Moreover, the algebra \(\Psi^{-\infty}\) is stable under holomorphic functionnal calculus and contains \(\Psi_{c,k}^{-\infty}\) as a dense subalgebra.
Proof—

- Suppose that $R \in \cap_{s,t} \mathcal{L}(H^s, H^t)$. Then, we want to show that for any two elliptic operators $P_1$ and $P_2$ of strictly positive order $s$ and $t$, the operator $P_1RP_2$ is in $\mathcal{L}(E)$. We know, from the previous proposition that $P_1 \in \mathcal{L}(H^s, E)$ and $P_2 \in \mathcal{L}(E, H^{-t})$. As $R \in \mathcal{L}(H^{-t}, H^s)$, we can conclude that $P_1RP_2$ is in $\mathcal{L}(E)$.

- Suppose now that $R$ is smoothing. Then for any $t \in \mathbb{R}$, if $P$ is an elliptic pseudodifferential operator of order with real part $t$, then $PR$ is a smoothing operator. In particular, for any real $s$, $PR \in \Psi^s_{c,k}(G)$ so that $PR \in \mathcal{L}(H^s, E)$. This implies, as previously, that $R \in \mathcal{L}(H^s, H^t)$, and this is true for any $s, t$ real numbers.

- To show that this algebra $\Psi^{-\infty}$ is stable under holomorphic under functional calculus, it is enough to say it is hereditary in $\mathcal{K}(E)$. This means that $AXB \in \Psi^{-\infty}$ whenever $A, B \in \Psi^{-\infty}$ and $X \in \mathcal{K}(E)$, which is clear in our case. Applying then the classical formula $f(z) = az + zg(z)z$ (for any holomorphic function $f$ such that $f(0) = 0$) to the operator $R$, we then get stability under holomorphic functional calculus.

Finally, one can see that, as in the classical setting, the Sobolev scale is a natural framework for the action of pseudodifferential $G$ operators.

Proposition 5.2.5 — Let $P$ be a compactly supported pseudodifferential $G$-operator of order $m$ of real part $m_0$. Then $P$ defines for any real $s$ a morphism from $H^s$ into $H^{s-m_0}$.

Proof — Let $P_1$ be an elliptic pseudodifferential operator of order of real part $s$ and $Q_1$ a parametrix of $P_1$ so that $Q_1P_1 + R_1 = I$ with $R_1$ a smoothing operator. Then one can write $P = PQ_1P_1 + PR_1$. But as, $P_1R$ is smoothing it is in $\mathcal{L}(H^s, H^{s-m_0})$. On the other hand we know by the preceding lemma that $P_1 \in \mathcal{L}(H^s, E)$ as it is of order with real part $s$, and that $PQ_1 \in \mathcal{L}(E, H^{s-m_0})$, as $PQ_1$ is of order with real part $m_0 - s$. This gives us that $PQ_1P_1 \in \mathcal{L}(H^s, H^{s-m_0})$ and $P_1R$ too, so that it is true for $P$.

Remarks

1. We have supposed here that the order of $P$ is constant, but it is not necessary and the above proposition can be adapted for operators with nonconstant order, if we set $m_0 = \max \text{Re}(m)$.

2. Remark that we have used here the class of classical pseudodifferential operators, as they are the ones we are interested in, for example in the case of the complex powers, but our results remain valid if one replaces classical symbols by $(1,0)$-symbols or even $(\rho, \delta)$-symbols.

3. We have considered all over the section, that $E = C^*(G)$, but we can with no change consider that $E$ is a Hilbert module on $C^*(G)$ coming from a vector bundle over $G$ as in

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section 4.7, with pseudodifferential operators acting on this vector bundle. In this case, we denote by $H^s(E)$ the corresponding Sobolev modules.

6 Complex powers of a positive elliptic pseudodifferential operator

We now consider the classical problem of understanding the complex powers of a positive elliptic pseudodifferential operator $A$. The problem was first solved, in the case of compact manifolds by Seeley [See67] in a rather technical way. Shubin later on gave in his book [Shu87] a beautiful framework to analyse the resolvent as a pseudodifferential operator, but its approach was limited to the powers of differential operators. Then Guillemin [Gui85] proposed a very elegant way to bypass the detailed analysis of the resolvent of the operator, that we will adapt to our situation.

Various generalizations of these techniques have been used to perform the complex powers in quite different geometric situations. Let us cite, without being exhaustive, Rempel and Schulze ([RS82, RS83, RS84] for manifolds with boundary (see also Grubb [Gru87]) ; Kordyukov for foliations [Kor95] ; Ponge [Pon00] for Heisenberg manifolds and contact structures (the problem here is the lack of elliptic operators, so he studies the complex powers for hypoelliptic operators) ; Loya [Loy03] for manifolds with conical singularities ; Schrohe [Sch88] and Amman, Lauter, Nistor and Vasy [ALNV04] for some classes of noncompact manifolds and manifolds with singularities.

Here we take a $G$-pseudodifferential operator $A$ of strictly positive (fixed) order $m$, elliptic, invertible and positive, with (positive definite) principal symbol $\sigma = \sigma_m(A)$. We then know that the spectrum of $A$ is included in $[\epsilon, +\infty[$, for some $\epsilon > 0$. Recall that the principal symbol $\sigma = \sigma_m(A)$ of $A$ is a map in $C^\infty_{\text{cont}}(S^*(G))$ which is positive definite in $C(S^*(G))$ and so we can define the $s$-th power $\sigma^s$ of $\sigma$ for $s \in \mathbb{C}$. Moreover, this depends holomorphically on $s$ : the map $s \mapsto \sigma^s$ is holomorphic from $\mathbb{C}$ to $C^{\infty, k}(S^*(G))$.

Using holomorphic functional calculus for normal regular operators, we know that one can define the complex powers of the regular unbounded operator $A$ by setting

$$A^s = \frac{1}{2\pi i} \int_\Gamma \lambda^s(A - \lambda)^{-1} d\lambda$$

with $\Gamma$ a contour of the form $\Gamma_\rho$ for $\rho < \varepsilon$

$\Gamma^+_\rho = \{iv, v \in \mathbb{R}, \rho < v < +\infty\}$

$\Gamma^0_\rho = \{z = \rho e^{i\theta}, \pi/2 \leq \theta \leq 3\pi/2\}$, $\Gamma^-_\rho = -\Gamma^+_\rho$.

This operator is defined a priori as an unbounded regular operator on $E$.

The theorem we want to show is the following

**Theorem 6.0.1** — Let $A$ be a positive definite elliptic operator of positive order $m$ in $\Psi^m_k(G)$ as defined above. Then, for any $t \in \mathbb{R}$, the operator $A^t$ is in $\mathcal{L}(H^{t+m} \cap H^s, H^s)$ and there exists an holomorphic family $A_s$ of pseudodifferential operators of order $ms$ such that the operator $A^s - A_s$ is an holomorphic family in $\Psi^{-\infty}(G)$.

To achieve this, we will follow the general path drawn by Guillemin [Gui85].
1. We will first show that there exists a family of symbols $\sigma(s)$ holomorphic in $s \in \mathbb{C}$ such that the corresponding operators $A(s)$ fulfill the one-parameter group relation $A(s)A(t) = A(s + t)$ modulo smoothing operators, with $A(0) = 1$ and such that $A(1) - A$ is smoothing. Moreover we will show that this family is unique modulo smoothing symbols.

2. Then we will show that there exists a holomorphic family of pseudodifferential operators $A_s$ which fulfills exactly the one-parameter group relation $A_sA_t = A_{s+t}$ with $A_0 = 1$ and such that $A_1 - A$ is smoothing.

3. Finally we will show that the difference between the operator $A_s$ obtained via functional calculus and the operator $A_s$ is a smoothing operator depending holomorphically on $s$.

The theorems and proofs of Guillemin [Gui85] and Bucicovschi [Buc99] can be applied in our case quasi-verbatim. Nevertheless, as the proofs are very elegant and quite short, we chose to reproduce them for reader’s convenience.

Note that, as in the rest of the paper, we have chosen to simplify notations in omitting explicit reference to vector bundles, as they do not introduce any change in the theory, except at the point when we construct the family $A(s)$. Indeed, we treat there explicitly the case of vector bundles as they introduce non commutativity of the product of principal symbols. Elsewhere, the generalization to the case of operators acting on sections of a vector bundle is a straightforward.

From now on, we consider that we are given fixed $Op$ and symbol maps, as in section 4. Recall that in this case, we have a bijective map between totals symbols modulo smoothing ones and $G$-pseudodifferential operators modulo smoothing ones (theorem 4.3.12), and that the operators constructed using this formula have compact support in a fixed compact $W \subset G$.

### 6.1 Holomorphic families of pseudodifferential operators

Let $K \subset \mathbb{C}$ be a compact set and, consider the groupoid $G_K = G \times K$ with units $G_K^{(0)} = G^{(0)} \times K$, with $r$ and $s$ being the identity on $K$.

**Definition 6.1.2** — Let $m : K \rightarrow \mathbb{C}, z \mapsto m(z)$ be an holomorphic map. We consider $m$ as a map on $G_K^{(0)}$, constant on $G^{(0)}$.

1. We say that a map $a : K \rightarrow S_{\text{pol},k}^n(A^*(G))$ is a holomorphic family of polyhomogeneous symbols when the symbol $a$ is a $C^k$-family of polyhomogeneous symbols on the groupoid $G_K$, and satisfies Cauchy identity for some contour $\Gamma \subset K$ around $s$ for any $s$ in the interior of $K$.

   $$a(s) = \frac{1}{2\pi i} \int_{\Gamma} \frac{a(z)}{z - s} \, dz.$$

2. Let $A : K \rightarrow \Psi_{c,k}^m(G)$ be a family of pseudodifferential operators with support in a fixed compact set $W'$. We will say that $s \mapsto A(s)$ is a holomorphic family of pseudodifferential operators with compact support in $W'$ if $A$ is a $C^k$-family of pseudodifferential operators on the groupoid $G_K$ and if for any $f \in C^\infty_c(G)$ the map $A(s)f$ satisfies Cauchy equality $A(s)f = \frac{1}{2\pi i} \int_{\Gamma} \frac{A(z)f}{z - s} \, dz$ for any $s$ in the interior of $K$. 

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By extension we will say that these families are holomorphic on \( \mathbb{C} \) if they are holomorphic on any compact subset \( K \subset \mathbb{C} \) (with the support not depending on \( K \), for the operators). Then we have the following propositions

**Proposition 6.1.3** — Let \( m : \mathbb{C} \to \mathbb{C}, z \mapsto m(z) \) be an holomorphic map.

1. If \( a : \mathbb{C} \to S_{\text{hom},k}^m(A^*(G)) \) is a holomorphic family of polyhomogeneous symbols, then for any \( j \in \mathbb{N} \), the map \( s \mapsto a_{m(s)-j}(s) \) is holomorphic from \( \mathbb{C} \) to \( C^{\infty,k}(S^*(G)) \).

2. Conversely, if we are given any family \( (a_{m(s)-j})_{j \in \mathbb{N}} \) such that the maps \( s \mapsto a_{m(s)-j}(s) \) are holomorphic from \( \mathbb{C} \) to \( C^{\infty,k}(S^*(G)) \), then there exists a holomorphic family of polyhomogeneous symbols \( a : \mathbb{C} \to S_{\text{hom},k}^m(A^*(G)) \) such that \( a \sim \sum_j a_{m(s)-j} \) modulo smoothing symbols.

**Proof** —

1. First, it is enough to do it for the principal symbol, as we can repeat the argument to deduce it inductively for any \( j \in \mathbb{N} \), considering \( a(x, s, \xi) - \chi(\xi) \sum_{l=0}^{j-1} a_{m(s)-l}(x, s, \xi) \).

As \( a(s) \) is a \( C^k \)-family of symbols on \( G \), we know that \( a_{m(s)}(x, s, \xi) \in C^{\infty,k}(S^*(G)) \). We can write

\[
a_{m(s)}(x, s, \xi) = \lim_{t \to +\infty} \frac{a(x, s, t\xi)}{t^{m(s)}}.
\]

Hence, \( a_{m(s)}(x, s, \xi) \) satisfies Cauchy identity if we can prove that \( \tilde{a}(s, t) = \frac{a(x, s, t\xi)}{t^{m(s)}} \) is continuous in \( s \), uniformly with respect to \( t \), i.e., that \( \sup_{t > 1} |\tilde{a}(s, t) - \tilde{a}(z, t)| \to 0 \) when \( z \to s \).

This comes simply from the fact that the map \( s \mapsto \chi(\xi) \|\xi\|^{-m(s)} a(x, s, \xi) \) is of class \( C^k \) hence continuous in \( S_{\text{hom},k}^0(A^*(G)) \).

2. It is a holomorphic version of classical proof (Borel lemma). Using the usual formula,

\[
a(x, s, \xi) = \sum_{j=0} \chi_{t_j}(\xi) \|\xi\|^{-m(s)-j} a_{m(s)-j}(x, s, \xi),
\]

with \( \chi \) a cut-off map as usual and \( t_j \) going quickly to \( \infty \), we get a \( C^k \)-family of polyhomogeneous symbols. We are then left to check that \( a(x, s, \xi) \) satisfies Cauchy equality, which is clear as for fixed \( \xi \) the sum defining \( a \) is finite and all the terms in the sum satisfy Cauchy equality.

**Proposition 6.1.4** — Let \( m : \mathbb{C} \to \mathbb{C}, z \mapsto m(z) \) be an holomorphic map.

1. If \( a : \mathbb{C} \to S_{\text{hom},k}^m(A^*(G)) \) is a holomorphic family of polyhomogeneous symbols, then the family \( s \mapsto Op(a(s)) \) is a holomorphic family of pseudodifferential operators.
2. If \( s \mapsto A(s) \in \Psi^{m(s)}_{c,k}(G) \) is a holomorphic family of pseudodifferential operators, then there exists a holomorphic family of polyhomogeneous symbols \( \tilde{\sigma}_{\text{tot}}(A(s)) : \mathbb{C} \to \mathcal{S}^{m(\cdot)}_{\text{hom},k}(A^+(G)) \) such that the class of \( \tilde{\sigma}_{\text{tot}}(A(s)) \) modulo smoothing symbols is \( \sigma_{\text{tot}}(A(s)) \).

Proof—

1. Set \( A(s) = Op(a(s)) \). Then we have, for any \( f \in C^\infty_c(G) \),
   \[
   (A(s)f)(\gamma) = \frac{1}{(2\pi)^n} \int_{G_{s,\gamma}} \int_{A^*_r(G)} e^{-\xi(\gamma'\gamma^{-1})} a(s)(r(\gamma), \xi) f(\gamma') d\lambda_{s(\gamma)}(\gamma').
   \]
   Using Fubini theorem, we know that \( (A(s)f)(\gamma) = \int_{A^*_r(G)} a(s)(r(\gamma), \xi) g(\gamma, \xi) d\xi \) with \( g(\gamma, \xi) = \frac{1}{(2\pi)^n} \int_{G_{s,\gamma}} e^{-\xi(\gamma'\gamma^{-1})} f(\gamma') d\lambda_{s(\gamma)}(\gamma') \) a map in the Schwartz class in \( \xi \). The map \( s \mapsto A(s) \) is of class \( C^k \), as \( s \mapsto a(s) \) is. Using once again the Fubini theorem, it is clear that \( A(s)f \) satisfies Cauchy equality \( A(s)f = \frac{1}{2\pi i} \int_{\Gamma} \frac{A(z)f}{z-s} dz \) if \( a(s) \) does, which proves the holomorphicity of the family \( A(s) \).

2. In view of previous proposition, it is enough to show that \( s \mapsto \sigma_{\text{tot}}(A(s)) \) is holomorphic in \( \mathcal{S}^{m(\cdot)}_{\text{hom},k}(A^+(G))/\mathcal{S}^{-\infty}_{\text{hom},k}(A^+(G)) \), i.e. that the homogeneous parts of the symbols are holomorphic. As they are defined inductively see 4.3.12 using the principal symbol and \( Op \) which respect holomorphicity, this is clear by induction.

\[ \square \]

Proposition 6.1.5 — If for any \( j \in \mathbb{N} \), the family \( s \mapsto A_j(s) \in \Psi^{m(s)-j}_{c,k}(G) \) is a holomorphic family of elliptic pseudodifferential operators there exists a holomorphic family \( s \mapsto A(s) \in \Psi^{m(s)}_{c,k}(G) \) such that for any \( N \in \mathbb{N} \), we have \( A(s) - \sum_{j=0}^{N-1} A_j(s) \in \Psi^{m(s)-N}_{c,k}(G) \).

Proof— In view of the \( 1 \rightarrow 1 \) correspondance between symbols and operators, it is enough to see it for the symbols. Using again the usual formula,
   \[
   a(x, s, \xi) = \sum_{j=0}^{\chi} \chi\left(\frac{\xi}{t_j}\right) a_j(x, s, \xi),
   \]
   with \( \chi \) a cut-off map as usual and \( t_j \) going quickly to \( \infty \), we get a \( C^k \)-family of polyhomogeneous symbols. We are then left to check that \( a(x, s, \xi) \) satisfies Cauchy equality, which is clear as for fixed \( \xi \) the sum defining \( a \) is finite and all the terms in the sum satisfy Cauchy equality. \[ \square \]

This implies in particular the following

Proposition 6.1.6 — If \( s \mapsto A(s) \in \Psi^{m(s)}_{c,k}(G) \) is a holomorphic family of elliptic pseudodifferential operators there exists an holomorphic family \( s \mapsto B(s) \in \Psi^{-m(s)}_{c,k}(G) \) such that \( B(s) \) is a parametrix for \( A(s) \).
6.2 First step: construction of $A(s)$

Now we take a $G$-pseudodifferential operator $A$ acting on the sections of a vector bundle $\tilde{\mathcal{E}} = r^*(\mathcal{E})$ over $G$, coming from a hermitian vector bundle $\mathcal{E}$ over $G^{(0)} = M$. We denote by $E$ the corresponding Hilbert module (see section 4.7). Moreover, we consider that $A$ is of strictly positive order $m$, elliptic, invertible and positive, with (positive definite) principal symbol $\sigma = \sigma_m(A)$. We then know that the spectrum of $A$ is included in $[\epsilon, +\infty[$, for some $\epsilon > 0$.

Recall that the principal symbol $\sigma = \sigma_m(A)$ of $A$ is a $C^{\infty, k}$ section of the fibre bundle $\mathcal{L}(\mathcal{E})$ pull-backed over $S^*(G)$, i.e $\sigma \in C^{\infty, k}(S^*(G), \mathcal{L}(\mathcal{E}))$. In our case, it follows that $\sigma$ takes values in positive definite operators in $\mathcal{L}(\mathcal{E})$. Then, by holomorphic functionnal calculus we know that we can define the $s$-th power $\sigma^s$ of $\sigma$ for $s \in \mathbb{C}$. Moreover, this depends holomorphically on $s$ : the map $s \mapsto \sigma^s$ is holomorphic from $\mathbb{C}$ to $C^{\infty, k}(S^*(G), \mathcal{L}(\mathcal{E}))$.

We wish to construct an holomorphic family of pseudodifferential operators $A(s)$ for $s \in \mathbb{C}$ with principal symbol $\sigma(A(s)) = \sigma^s$ such that $A(0) = Id$ and $A(s)A(t) \equiv A(s + t)$ modulo smoothing operators and the difference $A_1 - A$ is a smoothing operator. To construct the family $(A(s))_{s \in \mathbb{C}}$ we need to consider the cohomology of the group $(\mathbb{C}, +)$ with coefficients in the representation of $(\mathbb{C}, +)$ on the space of sections $C^{\infty, k}(S^*(G), \mathcal{L}(\mathcal{E}))$. This construction generalizes the cohomology considered by Guillemin in [Gui85] for the trivial representation of $(\mathbb{C}, +)$ on the space of smooth functions on $S^*(M)$ and the extension of Bucicovschi [Buc99] to fibre bundles on a smooth compact manifold. To do so we will consider an even more general situation: consider a $C^*$algebra $A$ and a sub-algebra $B$, which is a projective limit of Banach algebras and stable under holomorphic functionnal calculus (in our case we will consider $A = C((S^*(G), \mathcal{L}(\mathcal{E}))$ and $B = C^{\infty, k}(S^*(G), \mathcal{L}(\mathcal{E})))$.

Let $\sigma$ be an element of $B$ which is invertible and positive in $A$. The representation of $(\mathbb{C}, +)$ on $A$ we consider is the following one: any $s \in \mathbb{C}$ acts on $A$ by $s \cdot g = \sigma^{-s}g\sigma^s$. (Note that $B$ is stable under this action).

Let $C^r = C^r(\mathbb{C}; B)$ be the space of functions

$$f : \underbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}_{r \text{ times}} \to B$$

that are holomorphic and such that $f(s_1, \ldots, s_r) = 0$ if at least one $s_i$ is equal to zero.

Let $\delta^r : C^r \to C^{r+1}$ defined as:

$$(\delta^r f)(s_0, s_1, \ldots, s_r) = s_0 \cdot f(s_1, \ldots, s_r) + \sum_{i=1}^{r} (-1)^i f(s_0, \ldots, s_{i-1} + s_i, \ldots, s_r)$$

$$+ (-1)^{r+1} f(s_0, \ldots, s_{r-1}).$$

Let $\mathcal{H}^r(\mathbb{C}; B) = \text{Ker}\delta^r / \text{Im}\delta^{r-1}$.

**Proposition 6.2.7**

We have $\mathcal{H}^2(\mathbb{C}; B) = 0$.

Moreover, for each 2-cocycle $f$ there exists a unique 1-cocahin $h$ such that $\delta h = f$ and $h$ has a prescribed value at 1, $h(1)$.
Proof— Let \( f : \mathbb{C} \times \mathbb{C} \to B \) so that for all \( a, b, c \in \mathbb{C} \)
\[
\begin{align*}
\begin{cases}
f(0, b) = f(a, 0) = 0, \\
(\delta^2 f)(a, b, c) = a \cdot f(b, c) - f(a + b, c) + f(a, b + c) - f(a, b) = 0.
\end{cases}
\end{align*}
\]
We will try to find \( h : \mathbb{C} \to B \) such that
\[
(\delta^1 h)(a, b) = \sigma^{-a} h(b) \sigma^a - h(a + b) + h(a) = f(a, b).
\]
The existence of an \( h \) as above implies:
\[
h'(a) = \sigma^{-a} h'(0) \sigma^a - \frac{\partial f}{\partial b}(a, 0).
\]
Consider \( h \) to be the unique solution of the previous equation with \( h(0) = 0 \) and with a fixed prescribed value at 1, \( h(1) \). \( h \) can be found in the following way:

Let \( \Phi(t) \) be the operator in \( \mathcal{L}(A) \) given by \( u \to \sigma^{-t} u \sigma^t \) for \( u \in A \). Then
\[
h(a) = -\int_0^a \frac{\partial f}{\partial b} (t, 0) \, dt + \int_0^a \Phi(t)(h'(0)) \, dt.
\]
If \( T(a)u = \int_0^a \Phi(t)u \, dt \), then, in order to get any prescribed value for \( h(1) \), we need to show that \( T(1)u \) can be any element of \( B \). Indeed, for any \( t \) we have that \( \Phi(t) \) is a strictly positive operator in \( \mathcal{L}(A) \), and so this is also true for \( T(a) \), so that \( T(1) \) is invertible on \( A \). Now it is clear that \( B \) is stable by \( T(1) \) as it is stable by \( \Phi(t) \), and this shows that \( T(1)_B \) is bijective.

Thus we obtain a holomorphic map \( h : \mathbb{C} \to B \) such that \( h \in C^1 \). We will show that \( \delta h = f \) so \( f \) is a coboundary. To see this, let
\[
g(a, b) = f(a, b) - (\sigma^{-a} h(b) \sigma^a - h(a + b) + h(a)).
\]
Clearly \( \delta h = f \) if and only if \( g \equiv 0 \). Denote by \( \frac{\partial}{\partial b} \) the partial derivative with respect to the second variable. Then:
\[
\frac{\partial g}{\partial b}(a, b) = \frac{\partial f}{\partial b}(a, b) - \sigma^{-a} h'(b) \sigma^a + h'(a + b).
\]
From (6) we get:
\[
\begin{align*}
h'(b) &= \sigma^{-b} h'(0) \sigma^b - \frac{\partial f}{\partial b}(b, 0) \quad \text{and} \\
h'(a + b) &= \sigma^{-(a+b)} h'(0) \sigma^{a+b} - \frac{\partial f}{\partial b}(a + b, 0).
\end{align*}
\]
These two equalities and (7) imply
\[
\frac{\partial g}{\partial b}(a, b) = \frac{\partial f}{\partial b}(a, b) - \sigma^{-a} \left( \sigma^{-b} h'(0) \sigma^b - \frac{\partial f}{\partial b}(b, 0) \right) \sigma^a + \sigma^{-(a+b)} h'(0) \sigma^{a+b}
\]
\[
- \frac{\partial f}{\partial b}(a + b, 0) \\
= \sigma^{-a} \frac{\partial f}{\partial b}(b, 0) \sigma^a - \frac{\partial f}{\partial b}(a + b, 0) + \frac{\partial f}{\partial b}(a, b)
\]
\[
= \frac{\partial}{\partial c} \left[ (\delta^2 f)(a, b, c) \right] \bigg|_{c=0}.
\]
So \( \frac{\partial g}{\partial b} = 0 \); hence \( g(a, b) \) is constant in \( b \). When \( b = 0 \) we have
\[
g(a, 0) = f(a, 0) - (\sigma^{-\alpha} h(0) \sigma^\alpha - h(a) + h(a)) = 0.
\]
So \( g \equiv 0 \). Because \( f \) was chosen arbitrarily we conclude \( H^2(\mathbb{C}; B) = 0 \). \( \blacksquare \)

We can now prove the existence of the family \( A(s) \).

**Proposition 6.2.8** — There exists a holomorphic family of pseudodifferential operators \( A(s) \) with compact support in \( W \) for \( s \in \mathbb{C} \) with principal symbol \( \sigma(A(s)) = \sigma^s \) such that \( A(0) = Id \), \( A(s)A(t) \equiv A(s + t) \) modulo smoothing operators and the difference \( A_1 - A \) is a smoothing operator.

**Proof** —

As we know that an elliptic family of pseudodifferential operators admits a holomorphic family of operators as parametrix, and as we will be working in this proof modulo smoothing operators, we will denote by \( A(s)^{-1} \) a holomorphic parametrix for an elliptic, holomorphic \( A(s) \). The statement of the proposition is then equivalent to finding a holomorphic family of pseudodifferential operators \( A(s) \) with compact support in \( W \) for \( s \in \mathbb{C} \) with principal symbol \( \sigma(A(s)) = \sigma^s \) such that:
\[
\begin{align*}
A(s)A(t)A(s + t)^{-1} & \equiv Id \quad \text{mod } \Psi^{-\infty}, \\
A^{-1}A(1) & \equiv Id \quad \text{mod } \Psi^{-\infty}, \\
A(0) & \equiv Id \quad \text{mod } \Psi^{-\infty}.
\end{align*}
\]
(proof denoted the space of smoothing operators by \( \Psi^{-\infty} \)).

To prove Proposition 6.2.8, we will construct \( A(s) \) inductively in \( k \in \mathbb{N} \), such that \( \forall s, t \in \mathbb{C} \) in a neighborhood of \( 0 \),
\[
\begin{align*}
A_k(s)A_k(t)A_k(s + t)^{-1} & \equiv Id \quad \text{mod } \Psi^{-k}, \\
A^{-1}A_k(1) & \equiv Id \quad \text{mod } \Psi^{-k}, \\
A_k(0) & \equiv Id \quad \text{mod } \Psi^{-k}.
\end{align*}
\]

Let \( \chi \) be a cut-off map on \( A^s(G) \), \( \chi(x, \xi) = \omega(\|\xi\|) \), with \( \omega \in C^\infty(\mathbb{R}) \) a positive map, null if \( t < 1/2 \) and \( \omega(t) = 1 \) if \( t \geq 1 \). Then we know that, given any element \( a \in C^\infty,k(S^*(G), \mathcal{L}(\mathcal{E})) \) we can, for any complex number \( z \) construct from it an element \( \tilde{a} \) in \( S^\infty_{hom,k}(A^s(G), \mathcal{L}(\mathcal{E})) \) setting
\[
\tilde{a}(x, \xi) = \chi(x, \xi) \|\xi\|^2 a(x, \xi/\|\xi\|).
\]
Recall that we have fixed a map \( Op : S^\infty_{hom,k}(A^s(G), \mathcal{L}(\mathcal{E})) \rightarrow \Psi^\infty_{c,k}(G, \mathcal{E}) \) which associates an operator to any given total symbol. Composing these two maps, we get a map \( \theta_z \) from \( C^\infty,k(S^*(G), \mathcal{L}(\mathcal{E})) \) to \( \Psi^\infty_{c,k}(G, \mathcal{E}) \) which maps any \( a \) to an operator of degree \( z \) with principal symbol equal to \( a \). Moreover, it is clear, that if we take a holomorphic map \( s \mapsto a(s) \) in \( C^\infty,k(S^*(G), \mathcal{L}(\mathcal{E})) \), and a holomorphic map \( f : \mathbb{C} \rightarrow \mathbb{C} \), the operators \( \theta_{f(s)}(a(s)) \) is a holomorphic family of operators from \( H^{1+r}(E) \) to \( H^r(E) \) for any real \( t > \Re(f(s)) \) and any \( r \). Indeed the Cauchy equality
\[
A(s)u = \int_{\Gamma} \frac{A(z)u}{z - s} dz
\]
holds for any \( u \in C_c^\infty(G) \), because it is true for \( a(s) \). Now, for a fixed \( s \) both operators \( A(s) \) and \( f_\Gamma \frac{A(s)\bar{z}}{2}dz \) define bounded operators from \( H^{t+r}(E) \) to \( H^r(E) \) for any real \( t > \Re(f(s)) \) and any \( r \), provided the contour \( \Gamma \) is well chosen. So the equation (10) extends by continuity.

For \( k = 1 \) we want \( (A_1(s))_{s \in \mathbb{C}} \) to be a holomorphic family of pseudodifferential operators of order \( ms \) with compact support in \( W \), with the principal symbol equal to \( \sigma^s \) where \( \sigma \) is the principal symbol of \( A \). We can construct such a family in the following way . Let \( P(s) \) be the family of operators with compact support in \( W \) defined by \( P(s) = \theta_{ms}(\sigma^s) \). We know that \( P(s) \) is a holomorphic family , in the previous sense, of elliptic operators. We can with no harm assume that \( P(0) \) is invertible, as we can add to it a smoothing operator without trouble for the holomorphicity of the family.Denote by \( Q(s) \) a parametrix for \( P(s) \) and set

\[
A_1(s) = P(s) (sQ(1)A + (1 - s)Q(0)).
\]

It is clear that \( A_1(s) \) satisfies all required properties (9) modulo \( \Psi^{-1} \), so we are done with step one.

Now suppose that the relations (9) hold for a certain \( k \in \mathbb{N} \). We will construct a new family \( (A_{k+1}(s))_{s \in \mathbb{C}} \) that satisfies (9) for \( k + 1 \). We set :

\[
A_{k+1}(s) = A_k(s)(Id - H(s)), \quad H(s) \in \Psi^{-k}. \tag{11}
\]

In this way \( A_{k+1}(s) - A_k(s) \in \Psi^{ms-k} \). We have the following equalities \( (\bmod \Psi^{-k-1}) \):

\[
A_{k+1}(s)A_{k+1}(t)A_k(s+t)^{-1} = A_k(s)(Id - H(s))A_k(t)(Id - H(t))(Id + H(s+t))A_k(s+t)^{-1}
\]

\[
\equiv A_k(s)A_k(t)A_k(s+t)^{-1} - A_k(s)H(s)A_k(t)A_k(s+t)^{-1} - A_k(s)A_k(t)H(t)A_k(s+t)^{-1}
\]

\[
+ A_k(s)A_k(t)H(s+t)A_k(s+t)^{-1}
\]

\[
\equiv Id + F(s,t) - A_k(s)H(s)A_k(t)A_k(s+t)^{-1} - A_k(s)A_k(t)H(t)A_k(s+t)^{-1}
\]

\[
+ A_k(s)A_k(t)H(s+t)A_k(s+t)^{-1}
\]

where \( F(s,t) = A_k(s)A_k(t)A_k(s+t)^{-1} - Id \), \( F(s,t) \in \Psi^{-k} \) by the induction step. To proceed with the induction we have to find a family \( (H(s))_{s \in \mathbb{C}} \) that makes the right hand side of the previous equivalence equal to the identity modulo \( \Psi^{-k-1} \). If \( \sigma_{pr}(F(s,t)) \) and \( h(s) = \sigma_{pr}(H(s)) \) are the principal symbols, then the condition on \( H(s) \) is equivalent to:

\[
\sigma_{pr}(F(s,t)) = \sigma^s h(s) \sigma^{-s} + \sigma^{s+t} h(t) \sigma^{-(s+t)} - \sigma^{s+t} h(s+t) \sigma^{-(s+t)}
\]

or

\[
\sigma^{-(s+t)} \sigma_{pr}(F(s,t)) \sigma^{s+t} = \sigma^{-t} h(s) \sigma^t - h(s+t) + h(t). \tag{12}
\]

Set \( f(s,t) = \sigma^{-(s+t)} \sigma_{pr}(F(s,t)) \sigma^{s+t} \). We will show that \( f \in \mathcal{C}^2(\mathbb{C};C^\infty(k(S^*(G), \mathcal{L}(\mathcal{E}))) \) and \( \delta^2 f = 0 \). Then \( h \) as in (12) will be a 1-cochain with \( \delta h = f \).

We would also want the second condition of (9) to be satisfied, namely that

\[
A^{-1}A_{k+1}(1) \equiv Id \pmod{\Psi^{-k-1}}.
\]

We know from the induction step that \( (A^{-1}A_k(1) - Id) \in \Psi^{-k} \), and we have

\[
A^{-1}A_{k+1}(1) - Id = A^{-1}A_k(1)(Id - H(1)) - Id = (A^{-1}A(1) - Id) - A^{-1}A(1)H(1).
\]

In the last part of the equation above, both terms are operators in \( \Psi^{-k} \), so that this is in \( \Psi^{-k-1} \) if and only if the principal symbol of this is null. This holds if

\[
h(1) = \sigma_{pr}(A^{-1}A_k(1) - Id) \tag{13}
\]
We now have to show that $f$ is a cocycle in $C^2$. Obviously, $f(0, t) = f(s, t) = 0$ and we have

$$(\delta^2 f)(s, t, r) = \sigma^{-s} f(t, r) \sigma^s - f(s + t, r) + f(s, t + r) - f(s, t).$$

Recall that $f(s, t) = \sigma^{-(s+t)} \sigma_{pr}(F(s, t)) \sigma^{s+t}$, so that $(\delta^2 f)(s, t, r) = 0$ is equivalent to

$$\sigma_{pr}(F(r, t)) - \sigma_{pr}(F(r, s + t)) + \sigma_{pr}(F(t + r, s)) - \sigma^r \sigma_{pr}(F(t, s)) \sigma^{-r} = 0. \quad (14)$$

As by definition $F(s, t) \in \Psi^{-k}$ the equation (14) is equivalent to the following for the operators :

$$F(r, t) - F(r, s + t) + F(t + r, s) - A_k(r) F(t, s) A_k(r)^{-1} \equiv 0. \quad (\mod \Psi)^{-k}$$

To see this holds, consider the following equivalences modulo $\Psi^{-k}$:

$$(Id + F(r, t))(Id + F(t + r, s))(Id - F(r, s + t)) A_k(r)(Id - F(t, s)) A_k(r)^{-1}$$

$$\equiv A_k(r) A_k(t) A_k(t + r)^{-1} A_k(t + r) A_k(s) A_k(s + t + r)^{-1} A_k(s + t + r)$$

$$\times A_k(s + t)^{-1} A_k(r)^{-1} A_k(r) A_k(s + t) A_k(s)^{-1} A_k(t)^{-1} A_k(r)^{-1}$$

$$\equiv Id$$

and the first term is also equivalent to

$$Id + F(r, t) - F(r, s + t) + F(t + r, s) - A_k(r) F(t, s) A_k(r)^{-1}$$

which proves (14). So $f(s, t) = \sigma^{-(s+t)} \sigma_{pr}(F(s, t)) \sigma^{s+t}$ is a cocycle.

Proposition 6.2.7 provides us with a family $h(s)$ such that $\delta h = f$. We can choose this family so that (13) holds as well. This determines $h$ in a unique way. If $(H(s))_{s \in \mathbb{C}}$ is a holomorphic family of pseudodifferential operators of fixed order $-k$ with principal symbol $h(s)$ constructed as before, $H(s) = \theta_{-k}(h(s))$, then $A_{k+1}(s) = A_k(s)(Id - H(s))$ satisfies the equivalences (9) modulo $\Psi^{-k-1}$.

In this way we obtain a sequence of families of operators $(A_k(s))_{s \in \mathbb{C}}$ that satisfy the relations (9) for each $k \in \mathbb{N}$. Moreover, $A_{k+1}(s) - A_k(s) \in \Psi^{ms-k}$. Then, by proposition 6.1.5, we know that there exists a holomorphic family $(A(s))_{s \in \mathbb{C}}$ such that $A(s) \sim A_1(s) + \sum_{k \geq 1} (A_{k+1}(s) - A_k(s))$. The family $(A(s))_{s \in \mathbb{C}}$ then satisfies the following

1. $\sigma(A(s)) = \sigma^s$ ,

2. $A(s) A(t) \equiv A(s + t)$ modulo smoothing operators,

3. $A_1 - A$ is a smoothing operator,

4. $A(0) \equiv Id$ modulo smoothing operators.

Adding $Id - A(0)$ to $A(s)$ we can impose that $A(0) = Id$. Moreover, $(A(s))_{s \in \mathbb{C}}$ is unique up to smoothing operators because it must satisfy the relations (9) for all $k \in \mathbb{N}$ and so it must be equal to $(A_k(s))_{s \in \mathbb{C}}$ modulo $\Psi^{-k}$. ■

Suppose now that we are given such an holomorphic family. We can ask that it satisfies the relation $A^*(\pi) = A(s)$. Indeed one can observe that the family $(A^*(\pi))_{s \in \mathbb{C}}$ fulfills exactly the relations (9) for all $k \in \mathbb{N}$ and so we have that $A^*(\pi) \equiv A(s)$ modulo smoothing operators. Hence the family $\hat{A}(s) = \frac{1}{2}(A^*(\pi) + A(s))$ satisfies the additional condition and is equal to $A(s)$ modulo smoothing operators. We from now on assume that $A^*(\pi) = A(s)$. 

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6.3 Second Step: Construction of $A_s$

We know from previous section that there exists a holomorphic family $A(s)$ of compactly supported operators such that $A(s + t) - A(s)A(t)$ is a compactly smoothing operator, and that $A(0) = Id$. This implies that there exists an open neighborhood $\Omega$ of $0$ in $\mathbb{C}$ such that $A(s)$ is invertible if $s \in \Omega$, and such that $F(s, t) = A(t) - A(s)^{-1}A(s + t)$ is a bi-holomorphic smoothing operator (not compactly supported). We are now searching for a holomorphic family of pseudodifferential operators $A_s = A(s)C(s)$, with $C(s) - Id$ smoothing, such that $A_s A_t = A_{s+t}$ for $s$ and $t$ in some neighborhood of 0. If such a $C \in \mathcal{L}(E)$ exists, then, it should fulfill:

$$A(s)A(t)C(t) = A(s + t)C(s + t), \text{ ie } C(s)A(t)C(t) = A(t)C(s + t) - F(s, t)C(s + t).$$

Taking the derivative in $s = 0$, we obtain $C'(0)A(t)C(t) = A(t)C'(t) - \partial_1 F(0, t)C(t)$. We can suppose that $C'(0) = 0$, and then we get that for $t \in \Omega$,

$$C'(t) = R(t)C(t) \text{ with } R(t) = A(t)^{-1}\partial_1 F(0, t).$$

Note that $R(t)$ is smoothing, so that this differential equation holds in $\mathcal{L}(E)$. By standard theory on differential equation in Banach spaces, we know it has a unique solution in a neighborhood of 0 with $C(0) = Id$.

**Proposition 6.3.9** — Let $C(s)$ be in a neighborhood of 0 the solution of the differential equation $C'(t) = R(t)C(t)$ in $\mathcal{L}(E)$ with $R(t) = A(t)^{-1}\partial_1 F(0, t)$ such that $C(0) = Id$. Then if $A_s = A(s)C(s)$, we have that $A_s A_t = A_{s+t}$ and $A_s - A(s)$ is smoothing.

**Proof** — Set $\tilde{C}(t) = C(t) - Id$. Then $\tilde{C}$ is solution of the differential equation $X'(t) = R(t) X(t) + R(t)$. This equation holds in $\mathcal{L}(H^{-\infty}, H^{+\infty})$ and has there a unique solution such that $X(0) = 0$, so it shows that $\tilde{C}(t)$ is smoothing. Fix $s \in \Omega$, set $B(t) = A(t)^{-1} A_{s+t} A_s^{-1}$. We have $B(t)$ holomorphic in $\mathcal{L}(E)$, $B(t) - Id$ is smoothing (as $C - Id$ is) and $B(0) = Id$. If we show that $B'(t) = R(t)B(t)$, we get $B(t) = C(t)$, hence $A_t A_s = A(t)B(t)A_s = A_{s+t}$.

We can write $B(t) = [A(s) - F(t, s)]C(s + t)A_s^{-1}$. Differentiating in $t = 0$, we get

$$B'(t) = -\partial_1 F(t, s)C(s + t)A_s^{-1} + [A(s) - F(t, s)]C'(s + t)A_s^{-1} = [-\partial_1 F(t, s)A(s) + A(t)^{-1}\partial_1 F(0, s + t)]C(s + t)A(s)^{-1}.$$

On the other hand, we have

$$R(t) B(t) = A(t)^{-1} [\partial_1 F(0, t)(A(s) - F(t, s))] C(s + t) A_s^{-1}.$$

Hence $B'(t) = R(t) B(t) \iff -A(t) \partial_1 F(t, s) + \partial_1 F(0, s + t) - \partial_1 F(0, t) A(s) + \partial_1 F(0, t) F(t, s) = 0 \quad (15)$

To find this, observe that

$$A(u + t + s) = A(u + t) A(s) - A(u + t) F(u + t, s) = A(u) A(t) A(s) - A(u) F(u, t) A(s) - A(u + t) F(u + t, s).$$

On the other hand

$$A(u + t + s) = A(u) A(t + s) - A(u) F(u, t + s) = A(u) A(t) A(s) - A(u) A(t) F(t, s) - A(u) F(u, t + s).$$

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Subtracting these equalities, and using the fact that $A(u)^{-1}A(u + t) = A(t) - F(u, t)$, we get

$$-F(u, t)A(s) - A(t)F(u + t, s) + F(u, t)F(u + t, s) + A(t)F(t, s) + F(u, t + s) = 0 \quad (16)$$

Differentiating with respect to $u$ in 0, we get exactly (15), and this ends the proof. ■

To extend, $A_s$ to $s \in \mathbb{C}$, we simply set $A_s = (A_s^n)^n$ for some $n \in \mathbb{N}$ big enough. Recall that, in previous section, we have imposed that $A(s) = A(\bar{s})^*$. This implies that $A_s = A_{s^*}$. Indeed, set $B(s) = A(s)^{-1}A_s^*$. We have that $B(s)$ is in $\mathcal{L}(E)$ and holomorphic. Moreover $B(0) = Id$ and

$$A(s)B(s)A(t)B(t) = A_s^*A_s^* = A_{s^*} = A(s + t)B(s + t),$$

hence $B(s)$ satisfies the same differential equation as $C(s)$ and we get $B(s) = C(s)$, and finally

$$A_s = A(s)C(s) = A(s)B(s) = A_s^*.$$

### 6.4 Last step : The operator $A_s - A^s$ is smoothing

First of all, recall that $A_1 = A_{1/2}A_{1/2}$ is a positive definite elliptic pseudodifferential operator, so we can define its complex powers. We then have

**Proposition 6.4.10** — For any $s \in \mathbb{C}$, we have

$$A_s^* = A_s$$

as operators in $\mathcal{L}(H^{t+m \Re s}, H^l)$ for any $t \in \mathbb{R}$

**Proof** — Using the facts that $A_{p/q} = (A_{1/q})^p$ and that $A_{1/q} = A_1^{\frac{1}{q}}$, we obtain that for any $r \in \mathbb{Q}$, we have $A_s^r = A_r$. This implies in particular that the domains of these operators are equal. But for any $u \in C^{\infty,k}(G)$, the map $s \mapsto (A_s - A_1)u(\gamma)$ is holomorphic on $\mathbb{C}$ and null on $\mathbb{Q}$, hence null everywhere. The restriction to $C^{\infty,k}(G)$ of $A_s^r$ and $A_s$ are equal, and as they are both regular operators, the domain of $A_s^{1+\pi}$ is a core for $A_s^1$, while the domain of $A_s^{1+\pi}$ is a core for $A_s$. From the equality of the domains of $A_s^1$ and $A_r$ we deduce the fact that the domain of each of these two operators contains a core for the other one. Finally they have same domain $H^{m \Re s}$ (because $A_s$ is an elliptic operator of order $m \Re s$), and they are equal as operators in $\mathcal{L}(H^{t+m \Re s}, H^l)$ if $t = 0$. To extend this to any $t \in \mathbb{R}$, we just write $A_s = A_{-t}A_{s+t}$ and $A_s^* = A_{1-t}A_s^{1+t}$ and use the previous result and duality for Sobolev modules. ■

We are now left to show that $A^s - A_1^s$ is a smoothing operator. But we know that $A - A_1 = R$ is a a smoothing operator, and we can write

$$A^s - A_1^s = \frac{1}{2\pi i} \int_{\Gamma} \lambda^s((A - \lambda)^{-1} - (A_1 - \lambda)^{-1})d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \lambda^s(A - \lambda)^{-1}(A_1 - A)(A_1 - \lambda)^{-1}d\lambda,$$

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as \((A - \lambda)^{-1} - (A_1 - \lambda)^{-1} = (A - \lambda)^{-1}(A_1 - A)(A_1 - \lambda)^{-1}\). The operator \((A - \lambda)^{-1}(A_1 - A)(A_1 - \lambda)^{-1}\) is smoothing and its Sobolev norm \(\|(A - \lambda)^{-1}(A_1 - A)(A_1 - \lambda)^{-1}\|_{t,r}\) is bounded independently from \(\lambda\) for any real numbers \(t, r\) so if \(\Re s < -1\), the integral converges to a smoothing operator. Then if \(\Re s < -1\), \(A^n - A_1^n\) is a smoothing operator. But we have, for any integer \(n\) that \(A^n - A_1^n\) is a smoothing operator, so \(A^{s+n} - A_1^{s+n} = A^n(A^s - A_1^s) + (A^n - A_1^n)A^s\) is a smoothing operator. This ends the proof of theorem 6.0.1.

7 Application to the foliated case

We now briefly recall how to recover previous results of Connes [Con79, Con82] and Kordyukov [Kor95] on unbounded pseudodifferential calculus on smooth compact foliations. In his work [Con79, Con82], Connes considered \(G\)-pseudodifferential operators on the reduced \(C^*\)-algebra of the foliation, \(C^*_r(G)\), and analysed the operators as acting on Sobolev spaces that are defined in the ordinary way from the Hilbert space \(L^2(G) = \oplus_x L^2(G_x, \lambda_x)\). In the work of Kordyukov, operators are acting in the global space \(L^2(M)\) and they can be defined as \(G\)-operators acting on the full \(C^*\)-algebra of the foliation \(C^*(G)\). Both cases can be seen as particular cases of the results above, by composition of our Hilbert module formulation with a representation of the considered \(C^*\)-algebra of the foliation, as explained in subsection 2.2. To recover results of Connes on complex powers or Sobolev spaces, it suffices to use the left-regular representation of \(C^*_r(G)\) in \(L^2(G) = \oplus_x L^2(G_x, \lambda_x)\), whereas we recover the complex powers and Sobolev spaces of Kordyukov using the left-regular representation of \(C^*(G)\) on \(L^2(M)\). To illustrate this machinery, let us give an example of a new proof of the fact that a longitudinal elliptic operator which is formally self-adjoint defines a self-adjoint operator in \(L^2(G_x)\). Suppose we are given such an elliptic longitudinal operator \(P = (P_x)\) on the foliation. The restriction \(P_x|_{D_x}\) of the operator \(P_x\) to \(D_x = C^\infty_c(G_x)\) can be considered as unbounded linear operator on \(L^2(G_x)\).

**Proposition 7.0.1** — [Con79] The operator \(P_x|_{D_x}\) is closeable and the domain of its closure is maximal.

**Proof** — Let \(\pi_x\) be the left regular representation of the \(C^*\)-algebra \(C^*_r(M, \mathcal{F})\) in \(L^2(G_x, \lambda_x)\). This representation \(\pi_x\) is non degenerated and the operator \(\overline{P}\) is regular on \(E_r = C^*_r(G)\). We can then apply proposition (2.3.5) to the operator \(\overline{P}\): There exists an operator \(P_0|_{D_x}\) from \(E_r \otimes \pi_x L^2(G_x, \lambda_x) = L^2(G_x, \lambda_x)\) whose domain is the image of the algebraic tensor product \(\text{Dom}\overline{P} \otimes D_x\) which is closeable, and whose closure \(\overline{P} \otimes \pi_x 1\) is a regular operator. Moreover, we know that for any \(f \in \text{Dom}\overline{P}, \xi \in D_x\), we have

\[
\overline{P} \otimes \pi_x 1(f \otimes \xi) = P_0|_{D_x}(f \otimes \xi) = \overline{P}(f) \otimes \xi.
\]

The operator \(\overline{P} \otimes \pi_x 1\) then coincides with the operator \(P_x\) on \(D \otimes D_x\), with \(D = C^\infty_c(G)\), and so defines a closed extension of \(P_x\).

We are left to show that \(\overline{P}_x = \overline{P} \otimes \pi_x 1\). It suffices to prove that the algebraic tensor product \(D \otimes D_x\) is a core for \(\overline{P} \otimes \pi_x 1\). But \(D\) is a core for \(\overline{P}\), and from proposition 2.3.6, we know that the image of the algebraic tensor product \(D \otimes D_x\) in \(E_r \otimes \pi_x L^2(G_x, \lambda_x)\) is a core for \(P_x\). This ends the proof as it is dense in \(D_x\).

\[\Box\]
These techniques can also be used to show spectral properties for operators, in more precise geometric situations, using the results of Fack and Skandalis [FS82a, FS82b, FS80]. Suppose now the foliation is minimal (ie all leaves are dense) and that the groupoid is Hausdorff, then by [FS82a, FS82b, FS80] we know that $C^*_r(G)$ is simple.

- If $G$ is amenable, the maximal and reduced C*-algebras coincide then the spectrum of a regular operator is the same in any representation associated to the $C^*(G)$. In particular the spectrum as an operator in $L^2(G_x)$ and the spectrum as an operator in $L^2(M)$ coincide.

- If the C*-algebra has no projections, it implies that a positive definite elliptic operator has no gap in its spectrum. Indeed, by functionnal calculus, a gap in the spectrum gives rise to a projection of the $C^*$-algebra. What is interesting is that this result remains valid in any faithful representation of the $C^*$-algebra. For example, the foliation induced by the horocyclic flow on the quotient $V = SL(2,\mathbb{R})/\Gamma$, with $\Gamma$ a discrete cocompact subgroup of $SL(2,\mathbb{R})$, and defined by the left action of the subgroup of lower triangular matrices of the form $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ with $t \in \mathbb{R}$ is a minimal foliation and it can be shown that its C*-algebra has no non trivial projections (cf [Con94], p135). Hence this gives a connexity result for the spectrum of positive elliptic operators on the foliations viewed as operators on on each leaf. Let $i$ denote the injection of $\mathbb{R}$ in $V$ as a generic leaf. The preceding shows in particular a connexity result for the spectrum of Schrödinger operators of the form $-\frac{d^2}{dx^2} + V$ on $L^2(\mathbb{R})$ for potentials $V$ of the form $V = f \circ i$, where $f$ is a continuous positive (or even real valued) continuous function $f$ on $V$.

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