UNBOUNDED PSEUDODIFFERENTIAL CALCULUS ON LIE GROUPOIDS

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Abstract

We develop an abstract theory of unbounded longitudinal pseudodifferential calculus on smooth groupoids (also called Lie groupoids) with compact basis. We analyze these operators as unbounded operators acting on Hilbert modules over $C^*(G)$, and we show in particular that elliptic operators are regular. We construct a scale of Sobolev modules which are the abstract analogues of the ordinary Sobolev spaces, and analyze their properties. Furthermore, we show that complex powers of positive elliptic pseudodifferential operators are still pseudodifferential operators in a generalized sense.

1 Introduction

The use of groupoids to analyze the properties of noncommutative objects goes back to the foundational work of Connes [8, 9] on foliations, where the longitudinal pseudodifferential calculus was linked with the holonomy groupoid of the foliation. Since then, groupoids have appeared as very rich structures which encode the singularities of the considered objects. For pseudodifferential calculus in particular, a general framework was introduced by Monthubert, Pierrot and Nistor, Weinstein, Xu in [25, 26], which allows the definition of a pseudodifferential calculus attached to any smooth groupoid. Monthubert [24] also used this framework to show that the *b*-calculus developed by Melrose for manifolds with boundary or with corners can be described fully in terms of groupoids. This is equally true for *cusp*-calculus. A general aim would be to know how singular problems can be translated in the language of groupoids, which would unify the approach to this kind of problems.

However, both articles [25, 26] deal mainly with the case of bounded operators, i.e. of pseudodifferential operators of order less than or equal to 0. To complete the picture, one needs to be able to deal with unbounded calculus which is necessary, for example to treat differential operators and their functional calculus (in particular complex powers). Understanding the complex powers of a (pseudo)differential operator as pseudodifferential operators is a classical problem solved by Seeley [34] for compact manifolds and extended since to various situations. This question is also very important from the noncommutative point of view. Indeed, for the definition of the noncommutative residue, a key tool in noncommutative geometry, one needs to

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construct zeta functions of operators, i.e. to construct the complex powers of a pseudodifferential operator. In [38] we will give a construction of the noncommutative residue for foliations

The aim of this article is to give a general and abstract framework to develop unbounded pseudodifferential calculus on Lie groupoids and complex powers for such operators. Recall [25, 26] that pseudodifferential operators of negative order are bounded operators on the C^{*}-algebra of the groupoid $C^*(G)$, i.e. morphisms on the Hilbert $C^*(G)$ -module $E = C^*(G)$. Thus positive order operators should be treated as unbounded operators in the powerful framework of Hilbert modules. The key result is the fact that elliptic operators are *regular* operators in the sense of Baaj (see [4] and [5]). Hence an elliptic operator which is normal (as a regular operator) admits functional calculus, and we can define the complex powers of a pseudodifferential operator.

To interpret these complex powers as pseudodifferential operators, one needs to enlarge the class of compactly supported pseudodifferential operators. Indeed, even the resolvent of a compactly supported pseudodifferential operator is not compactly supported. To do so we define a new class of (non compactly supported) smoothing operators and show that this definition is natural in the following sense. We give the definition of a natural scale of Sobolev Hilbert modules, which are the abstract analogues of ordinary Sobolev spaces. As in the classical case, pseudodifferential operators act naturally on these Sobolev modules as morphims and a smoothing operator is one that acts between any pair of such Sobolev modules. Next we show that this class of smoothing operators is stable under holomorphic functional calculus in the C*-algebra of the groupoid.

We then develop the theory of complex powers for a positive definite pseudodifferential elliptic operator and show that for every $s \in \mathbb{C}$ the regular operator A^s can be written as the sum of a compactly supported pseudodifferential operator and of a non compactly supported smoothing operator.

We would like to stress the fact that though this theory might seem quite abstract, one gets a new approach to several singular problems. Indeed, the theory of regular operators on Hilbert modules is well behaved with respect to taking representations of the C*-algebra. One obtains by this procedure concrete Sobolev spaces, since the image of a regular operator by a representation is a closed operator. This yields applications to a wide class of problems, since the class of Lie groupoids contains for example compact manifolds, Lie groups, foliations, and deformation objects like the tangent groupoid of a compact manifold...

We briefly sketch at the end of this paper some applications to the foliated case. We show for example, using regularity, that an elliptic operator on a foliation is closable and that its closure is maximal [8]. We also recover the result of Kordyukov [18] on the complex powers for operators on a foliated manifold M, understood as acting on $L^2(M)$. We use the full force of our approach in another paper [38] where we analyze the noncommutative residue for foliated manifolds.

Let us briefly review the content of each section.

In section 2, we recall some basic definitions and facts (without proofs) on Lie groupoids and regular operators.

In section 3 we deal with pseudodifferential calculus on Hausdorff Lie groupoids with compact basis $G^{(0)}$. We take a more general definition than [25, 26] to include holomorphic families in the framework. Further, we introduce our class of smoothing operators, which allows to define a "generalized" pseudodifferential calculus.

In section 4 we construct a scale $(H_s)_{s\in\mathbb{R}}$ of Sobolev modules associated to each elliptic

pseudodifferential operator of positive order, and we show that these Sobolev modules are independent of the chosen operator. We then study their properties and get the following analogues of classical ones.

- 1. The modules H^s et H^{-s} are dual C^{*}-modules.
- 2. We have $H^s \subset H^{s'}$ whenever $s \ge s'$. The inclusion map is a compact morphism in the sense of C^{*}-modules if s > s'.
- 3. A pseudodifferential operator with order $m \in \mathbb{C}$ defines for any $s \in \mathbb{R}$ a morphism $H^s \to H^{s-m_0}$, where $m_0 = \Re e m$.
- 4. An operator R is smoothing if and only if it is in $\bigcap_{s,t} \mathcal{L}(H^s, H^t)$.

In section 5 we construct the complex powers of a positive elliptic pseudodifferential operator of integral order, following the strategy of Guillemin [14] for the proof in the case of a compact manifold. We show in particular that the complex powers of such an operator are pseudodifferential operators in our generalized sense. Note however that the class of smoothing operator we defined is, in some sense, the biggest natural class. Indeed, smoothing operators are just continuous in the transverse direction, and have *a priori* no better decay at infinity than any element in the C^{*}- algebra of the groupoid. One can hope to find in particular cases a smaller class of smoothing operator with a good topology and stable under holomorphic functional calculus. We explain at the end of the paper the conditions needed for such a sub-algebra.

Finally, we briefly sketch in section 6 how to recover from our work some results of Connes [8] and Kordyukov [18] in the case of foliated manifolds.

In an appendix, we give the proof of a technical result used to construct our class of pseudodifferential operators.

2 Preliminaries

2.1 Lie groupoids

Recall that a *groupoid* is a small category G (this means that the morphism class of G is a set) in which all morphisms are invertible. For the sake of simplicity, all our groupoids will be assumed to be Hausdorff. Here is a more explicit definition.

Definition 2.1.1 — A groupoid is given by two sets $G^{(1)} = G$ and $G^{(0)} = M$ and the following maps :

- $u: M \to G^{(1)}$, the diagonal imbedding,
- an involution $\kappa: G^{(1)} \to G^{(1)}$ called *inversion* denoted by $\kappa(\gamma) = \gamma^{-1}$,
- source (s) and range (r) maps from $G^{(1)}$ into M ,
- a multiplication m taking values in $G^{(1)}$ and defined on the set $G^{(2)} \subset G^2$ of pairs (γ, γ') for which $r(\gamma') = s(\gamma)$, denoted by $m(\gamma, \gamma') = \gamma \gamma'$,

satisfying the following conditions :

1. r(u(x)) = s(u(x)) = x, and $\gamma u(s(\gamma)) = u(r(\gamma))\gamma = \gamma$.

2.
$$r(\gamma^{-1}) = s(\gamma)$$
 and $\gamma\gamma^{-1} = u(r(\gamma))$

3.
$$s(\gamma\gamma') = s(\gamma')$$
 and $r(\gamma\gamma') = r(\gamma)$.

4. $\gamma_1(\gamma_2\gamma_3) = (\gamma_1\gamma_2)\gamma_3$ if $s(\gamma_1) = r(\gamma_2)$ and $s(\gamma_2) = r(\gamma_3)$.

The set $G^{(1)}$ is the set of *arrows*, and we will often refer to it as G, by a common abuse of notation. A *topological groupoid* is then a groupoid for which G and M are locally compact topological spaces and r, s, m, u are continuous maps, κ is a homeomorphism, and r and s are open maps.

A Lie groupoid is a groupoid where G and M are smooth manifolds, and where m, u are smooth maps, κ is a smooth diffeomorphism, and r and s are submersions. Recall that to any Lie groupoid G of basis M can be associated a Lie algebroid A(G) over the basis M as follows : A(G) is the bundle over M of longitudinal tangent spaces T_xG_x to G_x for $x \in M$. The bundle of longitudinal cotangent spaces $T_x^*G_x$ is denoted by $A^*(G)$, and $S^*(G)$ denotes the quotient by the action of \mathbb{R}^*_+ of $A^*(G) - \{0\}$.

A continuous (respectively smooth) left invariant Haar system, is a family $\{\lambda^x, x \in M\}$ of positive measures on G with support in G^x such that :

- 1. For all $f \in C_c(G), \delta \in G$ we have $\int f(\delta \gamma) d\lambda^{s(\delta)}(\gamma) = \int f(\gamma) d\lambda^{r(\delta)}(\gamma)$;
- 2. For all $f \in C_c(G)$, the map $x \mapsto \int_{\gamma \in G^x} f(\gamma) d\lambda^x(\gamma)$ is continuous (respectively smooth) on $G^{(0)}$.

A smooth section of the bundle of 1-densities on A(G) gives rise to a smooth Haar system λ . Moreover the measure λ_x is in the Lebesgue class for all x in this case.

2.2 C*-algebras of a groupoid

Let G be a topological groupoid which is Hausdorff and locally compact, and equipped with a continuous Haar system. The set $C_c(G)$ of compactly supported continuous functions on G is endowed with a structure of *-algebra : multiplication is defined by $(f*g)(\gamma) = \int f(\gamma')g(\gamma'^{-1}\gamma)d\lambda^{r(\gamma)}(\gamma')$ and involution by $f^*(\gamma) = \overline{f(\gamma^{-1})}$. The algebra obtained from $C_c(G)$ by completion for the norm

$$\|f\|_1 = \sup_{x \in G^{(0)}} \{ \int |f(\gamma)| \, d\lambda^x(\gamma), \ \int \left| f(\gamma^{-1}) \right| d\lambda^x(\gamma) \}$$

is a Banach *-algebra denoted by $L^1(G, \lambda)$.

For $f, g \in C_c(G)$ and $x \in G^{(0)}$, the left regular representation π_x is the $||||_1$ -bounded *representation of $C_c(G)$ on $L^2(G^x, \lambda^x)$ given by $(\pi_x(f)g)(\gamma) = (f*g)(\gamma)$. The reduced C*-algebra of G is the completion of $C_c(G)$ with respect to the norm $||f||_r = \sup_{x \in G^{(0)}} ||\pi_x(f)||$.

The maximal (or full) C^* -algebra of G is the completion w.r.t the norm defined taking the supremum of the norms over all $||||_1$ -bounded *-representations of $C_c(G)$ on Hilbert spaces. There is a natural epimorphism $C^*(G) \to C_r^*(G)$. Note that the definitions of $C^*(G)$ and $C_r^*(G)$ are independent of the choice of the Haar system in the sense that the obtained C*algebras are isomorphic [32]. Furthermore, in the smooth case it is also possible to construct all objects without having to choose a Haar system, working with half densities (see e.g. [8, 25]).

2.3 Regular operators on a Hilbert module [36, 21, 39]

Recall that a Hilbert module on a C^{*}-algebra A is a right A-module E together with a sesquilinear positive map $\langle , \rangle : E \times E \to A$, such that $||\langle x, x \rangle||$ turns E into a Banach space. Unlike the case of Hilbert spaces (when $A = \mathbb{C}$), a closed submodule F of a Hilbert module E does not have an orthogonal complement F^{\perp} such that $F \oplus F^{\perp} = E$. If this is nevertheless the case, we say that F is orthocomplemented in E.

A morphism between two Hilbert modules E, E' on A is an A-linear operator admitting an adjoint (for the involved A-valued scalar products). Morphims from E to E' are bounded and we denote the space of morphisms by $\mathcal{L}(E, E')$. A bounded A-linear map $T : E \to E'$ is a morphism if and only if the graph of T is orthocomplemented in $E \oplus E'$. We will use the following easy fact from [36].

Proposition 2.3.1 — Let $T \in \mathcal{L}(E, E')$.

- (a) If T is surjective then TT* is invertible in L(E') and E = KerT ⊕ ImT*.
 (b) If T is bijective, then so is T*. We have T⁻¹ ∈ L(E', E) and (T⁻¹)* = (T*)⁻¹.
- $2. \ \ {\ \ The following conditions are equivalent:}$

a) ImT is closed in E'; b) ImT^* is closed in E; c) 0 is isolated in the spectrum of T^*T . If these conditions are satisfied, then ImT and ImT^* are orthocomplemented submodules of E' and E with $ImT \oplus KerT^* = E'$ and $ImT^* \oplus KerT = E$.

Regular operators are unbounded operators between Hilbert modules resembling as much as possible to morphisms. This class of operators defined by Baaj [4] in his thesis is very rich and useful, as the properties proved later by Woronowicz show. The reader is referred to [36, 21, 39] for details and proofs.

An unbounded operator $T: E \to E'$ is defined by its graph $G(T) = \{(x, Tx), x \in \text{Dom}T\}$ which is a sub-A-module of $E \oplus E'$. If $(\text{Dom}T)^{\perp} = 0$, then there is a natural definition for T^* by its graph. A densely defined operator T, with densely defined adjoint is said to be *regular* if its graph G(T) is orthocomplemented. This is the case if and only if $1 + T^*T$ is surjective and then $(1 + T^*T)^{-1}$ is a morphism. Taking the square root we have that $(1 + T^*T)^{-\frac{1}{2}}$ is of image DomT and such that $Q(T) = T(1 + T^*T)^{-\frac{1}{2}}$ is a morphism.

The map $T \mapsto Q(T)$ is a 1-1 correspondence between regular operators and morphisms Q with norm less than one and such that $\operatorname{Im}(1 - Q^*Q)$ is dense in E, called the Woronowicz transform. A regular operator T is self-adjoint if $T^* = T$, respectively normal if $T^*T = TT^*$. This is the case if and only if Q(T) is self-adjoint, respectively normal.

The resolvent $R_{\lambda}(T) = (T - \lambda)^{-1}$ of a regular operator T is an analytic map from $\mathbb{C} - \operatorname{Sp} T$ to $\mathcal{L}(E)$ and a regular operator T (with spectrum $\neq \mathbb{C}$) is normal if and only if $R_{\lambda}(T)$ is. The natural correspondence between regular operators and morphisms allows to define a continuous functional calculus for normal regular operators [19].

Theorem 2.3.2 — Let T be a regular normal operator on a Hilbert module E and X be a closed set in \mathbb{C} with $\operatorname{Sp} T \subset X$. Then any map $f \in C(X)$ defines a normal regular operator f(T), with $(f(T))^* = \overline{f}(T^*)$, such that

- 1. For any pair (f,g) of continuous functions, (f+g)(T) is the closure of f(T) + g(T) and (fg)(T) the closure of f(T)g(T).
- 2. If f is continuous and bounded, we have $f(T) \in \mathcal{L}(E)$ et $||f(T)|| = \sup \{|f(\lambda)|, \lambda \in SpT\}$.
- 3. $\operatorname{Sp} f(T)$ is the closure in \mathbb{C} of $f(\operatorname{Sp} T)$.
- 4. For $f, g \in C(\mathbb{C})$, we have $(f \circ g)(T) = f(g(T))$.
- 5. $id_X(T) = T$ and q(T) = Q(T) if q is the map $q(z) = \frac{z}{1 + |z|^2}$.
- 6. If $T \in \mathcal{L}(E)$, the map $f \mapsto f(T)$ coincides with the continuous functional calculus in the C^* -algebra $\mathcal{L}(E)$.

Now let E be a Hilbert module on a C^{*}-algebra A and $\pi : A \to \mathcal{L}(H)$ a representation of A on a Hilbert module H on a C^{*}-algebra B. A morphism $T \in \mathcal{L}(E)$ can be pushed forward to a morphism $T \otimes_{\pi} 1 \in \mathcal{L}(E \otimes_{\pi} H)$. This extends to regular operators.

Proposition 2.3.3 — Let T be a regular operator on E and $\pi : A \to \mathcal{L}H$) a representation of A on a B-Hilbert module H. Then $T \otimes_{\pi} 1$ is a regular operator on the B-Hilbert module $E \otimes_{\pi} H$ and we have

- 1. $Q(T \otimes_{\pi} 1) = Q(T) \otimes_{\pi} 1$, $(T \otimes_{\pi} 1)^* = T^* \otimes_{\pi} 1$, and $(T \otimes_{\pi} 1)^* (T \otimes_{\pi} 1) = T^*T \otimes_{\pi} 1$;
- 2. if T is normal or self-adjoint, then so is $T \otimes_{\pi} 1$;
- 3. If $D \subset \text{Dom}T$ is a core for T, and \mathcal{H} a dense subspace in H, then $D \otimes_{\text{alg}} \mathcal{H}$ is a core for $T \otimes_{\pi} 1$.
- 4. We have $\operatorname{Sp}(T \otimes_{\pi} 1) \subset \operatorname{Sp}T$, with equality when π is injective ;
- 5. if T is normal and $f \in C(X)$ with X closed and $\operatorname{Sp} T \subset X$ then $f(T \otimes_{\pi} 1) = f(T) \otimes_{\pi} 1$;

3 Pseudodifferential calculus on Lie groupoids

In this section, we define pseudodifferential operators on Lie groupoids following Monthubert Pierrot [25] and Nistor, Weinstein and Xu [26]. Note that the calculus developed here is a bit more general as we define families of operators with non-constant order, in order to include in this framework holomorphic families of pseudodifferential operators.

3.1 Classical symbols of complex orders

Recall the definition [35], of symbols of type (1,0). For U an open set in \mathbb{R}^n and $m_0 \in \mathbb{R}$, denote by $\mathcal{S}^{m_0}(U, \mathbb{R}^p)$ (the space of *symbols*) the set of smooth complex valued functions on $U \times \mathbb{R}^p$ such that :

$$\left|\partial_{y}^{\alpha}\partial_{\xi}^{\beta}a(y,\xi)\right| \leq C_{K,\alpha,\beta}(1+|\xi|)^{m_{0}-|\beta|} \tag{1}$$

for any compact $K \subset U$, α , β multi-indices, $(y,\xi) \in K \times \mathbb{R}^p$. Taking the smallest possible constant $C_{K,\alpha,\beta}$ in the above inequality (for fixed K, α and β), we get a family of semi-norms that defines a natural Fréchet topology on $S^{m_0}(U, \mathbb{R}^p)$; the space $S^{m_0}(U, \mathbb{R}^n)$ will be simply denoted by $S^{m_0}(U)$. We will in fact use a particular class of symbols : the polyhomogeneous ones. A function $f \in C^{\infty}(\mathbb{R}^p - \{0\})$ is *positively homogeneous* of degree $l \in \mathbb{C}$ whenever $f(t\xi) = t^l f(\xi)$ for all $\xi \neq 0$ and all t > 0. Let m be a complex number of real part m_0 . We say that a symbol $a \in S^{m_0}(U, \mathbb{R}^p)$ admits a polyhomogeneous expansion if there exists, for every $j \in \mathbb{N}$, a function $a_{m-j} \in C^{\infty}(U \times (\mathbb{R}^p - \{0\}))$ that is positively homogeneous of degree m - jin its second variable such that if a C^{∞} cut-off function χ on \mathbb{R}^p , with $\chi(\xi) = 0$ if $|\xi| < 1/2$ and $\chi(\xi) = 1$ if $|\xi| \ge 1$ then for all $N \in \mathbb{N}$

$$a(y,\xi) - \chi(\xi) \sum_{k=0}^{N-1} a_{m-k}(y,\xi) \in \mathcal{S}^{m_0-N}(U).$$

We call $\mathcal{S}_{hom}^m(U, \mathbb{R}^p)$ the set of such polyhomogeneous symbols of order m. Note that this property does not depend on the cut-off function χ . Note also that for each j, we have $\chi(\xi)a_{m-j} \in \mathcal{S}^{m_0-j}(U)$, and that the functions a_{m-j} are uniquely determined for $|\xi| \geq 1$. We can then associate to each j a smooth function, still denoted a_{m-j} on the sphere bundle $U \times S^{p-1}$. The natural topology on $C^{\infty}(U \times S^{p-1})$ is the topology of uniform convergence on compact subsets of the function and all its derivatives. This allows us to define the correct topology on the spaces $\mathcal{S}_{hom}^m(U, \mathbb{R}^p)$. Indeed, these sets $\mathcal{S}_{hom}^m(U, \mathbb{R}^p)$ are not closed in $\mathcal{S}^m(U, \mathbb{R}^p)$ for the Fréchet topology of semi-norms defined previously. Let χ be a cut-off function as above. The topology on $\mathcal{S}_{hom}^m(U, \mathbb{R}^p)$ is the weakest topology making the following maps continuous

3.2 Families of classical operators

We now come to the notion of families of ordinary pseudodifferential operators. To begin we recall the definition of a map of class $C^{\infty,k}$, as given by Atiyah and Singer [3]. By convention if $k \in \mathbb{N} \cup \{\infty\}$ a C^k -space is a manifold of class C^k except in the case when k = 0, where we allow M to be any Hausdorff locally compact space.

Definition 3.2.1 — Let M be a C^k -space (for any $k \ge 0$), and U an open set in \mathbb{R}^p . A map ψ from $U \times M$ to \mathbb{R}^n is said to be of class $C^{\infty,0}$ if the map $x \mapsto \psi(., x)$ is continuous from M to $C^{\infty}(U, \mathbb{R}^n)$, endowed with the topology of uniform convergence on compact subsets of the function and of its derivatives. If M is a C^k -space, a map ψ from $U \times M$ to \mathbb{R}^n is said to be of class $C^{\infty,k}$, with $k \in \mathbb{N} \cup \{\infty\}$ if the map $v \mapsto \psi(., v)$ is of class C^k from M to $C^{\infty}(U, \mathbb{R}^n)$.

We can now define the notion of a C^k family of polyhomogeneous symbols

Definition 3.2.2 — Let M be a C^k -space and m be a map of class C^k from M to \mathbb{C} . A C^k -family

of polyhomogeneous symbols of order m is a map $M \to \mathcal{S}_{hom}^{m(.)}(U)$, $x \mapsto a(u, \xi, x)$ such that for any cut-off function χ null in a neighborhood of 0 and equal to 1 in a neighborhood of ∞ , the map $M \to \mathcal{S}_{hom}^0(U), x \mapsto \chi(\xi) \|\xi\|^{-m(x)} a(u, \xi, x)$ is of class C^k , where the space $\mathcal{S}_{hom}^0(U)$ is endowed with its natural topology described above.

These symbols define, as in the classical case, C^k -families of pseudodifferential operators

Definition 3.2.3 — Let U be an open set in \mathbb{R}^p , M a locally compact Hausdorff space, and m a map of class C^k from M to \mathbb{C} . A C^k -family of classical pseudodifferential operators of order m with compact support in $U \times M$ is a family $P_x \in \mathcal{P}_c^{m(x)}(U)$ such that for all $f \in C_c^{\infty}(U)$, the operator $P_x f$ is given, for $x \in M$, by

$$(P_x f)(u) = \frac{1}{(2\pi)^p} \int_U \int_{\mathbb{R}^p} a(u',\xi,x) f(u') e^{i\langle u'-u,\xi\rangle} d\xi du'$$

with the condition that the map $M \to \mathcal{S}_{hom}^{m(.)}(U)$, $x \mapsto a(u', x, \xi)$ is of class C^k .

Proposition 3.2.4 — We have the following analogues of classical properties

- 1. The adjoint of a C^k -family of classical pseudodifferential operators of order m(.) with compact support in $U \times M$ is still a C^k -family of classical pseudodifferential operators of order m(.) with compact support in $U \times M$.
- 2. Let m and n be maps of class C^k from M to \mathbb{C} then, if $A \in \mathcal{S}_{hom,k}^{m(.)}(U \times M)$ and $B \in \mathcal{S}_{hom,k}^{n(.)}(U \times M)$ then $AB \in \mathcal{S}_{hom,k}^{m(.)+n(.)}(U \times M)$.
- 3. Let κ be a C^{∞} -diffeomorphism from U onto itself. Take $a \in \mathcal{S}_{hom,k}^{m(.)}(U)$, and denote by A the corresponding family of pseudodifferential operator and by a_{κ} the map defined by

$$a_{\kappa}(\kappa(x),\eta) = e^{-i\langle\kappa(x),\eta\rangle} A e^{i\langle\kappa(x),\eta\rangle}.$$

Then we have $a_{\kappa} \in \mathcal{S}^{m(.)}_{hom,k}(U).$

For a proof, the reader is referred to the appendix.

3.3 Pseudodifferential *G*-operators

The definition of a $C^{\infty,k}$ -function can be extended to the situation of a groupoid in a natural way. Let $p: X \to M$ be a submersion between smooth manifolds. We say that a function f on X is $C^{\infty,k}$ with respect to the submersion when for any trivializing open set for the submersion of the form $\Omega \simeq U \times V$ with V an open set of M, the restriction of f to $U \times V$ is $C^{\infty,k}$. If G is a Lie groupoid, we say that a function on G is $C^{\infty,k}$, when it is $C^{\infty,k}$ with respect to the submersion $s: G \to G^{(0)}$.

In the special case of Lie groupoids, we have a notion of invariant families of operators (non necessarily pseudodifferential). An element $\gamma \in G$ acts by right translation in the following way

$$U_{\gamma}: C^{\infty}(G_{r(\gamma)}) \to C^{\infty}(G_{s(\gamma)}); \ (U_{\gamma}f)(\gamma') = f(\gamma'\gamma),$$

A *G*-operator of class C^k is then an operator P acting on $C_c^{\infty,k}(G)$, the space of compactly supported $C^{\infty,k}$ -functions on G such that there exists a family $(P_x)_{x\in M}$ of operators acting respectively on $C_c^{\infty}(G_x)$, with

$$(Pf)(\gamma) = (P_{s(\gamma)}f_{s(\gamma)})(\gamma)$$
(2)

$$P_{s(\gamma)}U_{\gamma} = U_{\gamma}P_{r(\gamma)}, \qquad (3)$$

with $\gamma \in G$, $f \in C_c^{\infty,k}(G)$, and f_x the restriction of f to G_x . Such a *G*-operator of class C^k is characterized by a distributional kernel k_P on G, which is a C^k -family of distributions k_x on G_x :

$$(Pf)(\gamma) = \int_{G_{s(\gamma)}} k_P(\gamma \gamma'^{-1}) f(\gamma') d\lambda_{s(\gamma)}(\gamma') = \int_{G^{r(\gamma)}} k_P(\gamma') f(\gamma'^{-1} \gamma) d\lambda^{r(\gamma)}(\gamma').$$

It is said to be compactly (or uniformly) supported when k_P is compactly supported in G, and smoothing with compact support when $k_P \in C_c^{\infty,k}(G)$.

Before giving the definition of a G-pseudodifferential operator in general, we begin by studying the special case when the groupoid G is the groupoid of a submersion.

Let $p: X \to M$ be a submersion between smooth manifolds. To any such submersion is naturally associated a Lie groupoid G(X, p, M) which is the closed subspace of the groupoid of couples $X \times X$ made out of couples (y, y') such that p(y) = p(y'). The source map is s((y,y')) = y' and the range map is r((y,y')) = y. The composition is the one of couples $(y,z) \circ (z,t) = (y,t).$

If G = G(X, p, M) is the groupoid of a submersion, then a G-operator is a family of operators indexed by X invariant under the action of G. Hence it is in fact a family indexed by M as the invariance condition imposes exactly that $P_x = P_y$ if p(x) = p(y).

Definition 3.3.1 — Let $p: X \to M$ be a submersion and m be a $C^{\infty,k}$ map. A (properly supported) C^k -family of classical pseudodifferential operators of order m is a compactly supported G(X, p, M)-operator such that the family $P_x \in \mathcal{P}^m(p^{-1}(x))$ satisfies :

- For each trivializing open set $\Omega \simeq U \times V$ and all $\phi, \psi \in C_c^{\infty,k}(X)$ with support in Ω , the operator $\phi P_x \psi$ viewed as an operator on $U \times V$ is a compactly supported C^k -family of classical pseudodifferential operators of order m in the sense defined above.
- For all maps $\phi,\psi\in C^{\infty,k}_c(X)$ with disjoint supports, the operator $\phi P_x\psi$ is a compactly supported smoothing G-operator.

We denote by $\mathcal{P}_k^m(X, p, M)$ the set of these operators.

Then we can compose two such operators.

 $\textbf{Proposition 3.3.2-lf } P \in \mathcal{P}_k^m(X,p,M) \text{ and } Q \in \mathcal{P}_k^n(X,p,M) \text{ are pseudodifferential operators}$ then $PQ \in \mathcal{P}_k^{m+n}(X, p, M)$.

Proof— Consider the operator $\phi PQ\psi$ with ϕ and ψ in $C_c^{\infty,k}(X)$. Since P and Q are compactly

supported in G(X, p, M), the operators ϕP and $Q\psi$ are compactly supported so that there are functions ϕ' and $\psi' \in C_c^{\infty,k}(X)$ such that $\phi P = \phi P \psi'$ and $Q\psi = \phi' Q \psi$. Hence the result is clear, using a partition of unity and the following easy facts :

- The product PQ of two compactly supported C^k product-families of pseudodifferential operators in a given trivializing open set is still a compactly supported C^k product-family of pseudodifferential operators.
- The product of a compactly supported smoothing operator by a compactly supported C^k product-family of pseudodifferential operators is a compactly supported smoothing operator.

Definition 3.3.3 — We will say that a compactly supported *G*-operator *P* is a *pseudodifferential* operator of class C^k if the family $(P_x)_{x \in M}$ is a C^k -family of pseudodifferential operators of order *m* for the submersion $s: G \to G^{(0)}$. We denote by $\Psi^m_{c,k}(G)$ this space of operators.

Note that this definition implies that m is a C^k map on $G^{(0)}$, with m(x) = m(y) whenever G_x^y is non empty. For us, the case where m is constant is the more interesting. Nevertheless, this generalization, with m varying is straightforward, and allows us to give a simpler definition for holomorphic families of pseudodifferential operators.

This definition also implies that the pseudodifferential operators P_x on G_x vary in a C^k -manner : for any open chart $\Omega \subset G$ diffeomorphic to $U \times s(\Omega)$, and for any $\phi \in C_c^{\infty,k}(\Omega)$, there exists a C^k -family $(a_x)_{x \in s(\Omega)}$ of polyhomogeneous symbols of order m on U such that the compactly supported operator $\phi P_x \phi$ corresponds to $Op(a_x)$ under the diffeomorphism $\Omega \cap G_x \simeq U$.

3.4 Principal and total symbol of a pseudodifferential G-operator

Note that in general a C^k -family of symbols $(a_x)_{x \in s(\Omega)}$ of polyhomogeneous symbols of order m on U in any chart Ω is not enough to define a G-operator, since this family has to satisfy an invariance property. We can associate to any G-operator a total symbol, which is given as a family of symbols $a_x \in S^m_{hom}(T^*_xG_x)$, but not in a unique way. However, we can associate canonically a principal symbol to any pseudodifferential operator.

The principal symbol ([25, 26]) of a compactly supported pseudodifferential operator is defined by $\sigma_m(P)(\xi) = \sigma_m(P_x)(\xi)$ for $\xi \in A_x^*(G) = T_x^*G_x$. Note that using homogeneity, it can be defined as an element

$$\sigma_m(P) \in C^{\infty,k}(S^*(G)),$$

with $S^*(G)$ the "co-sphere bundle" of G, i.e. the quotient of $A^*G - \{0\}$ by the action of \mathbb{R}^*_+ .

From [25, Theorem 1], we know the following analogs of classical results

Theorem 3.4.1 — [25]

- 1. $\Psi_{c,k}^{m(.)}(G) \circ \Psi_{c,k}^{m'(.)}(G) \subset \Psi_{c,k}^{m(.)+m'(.)}(G).$
- 2. $\sigma_{m+m'}(PQ) = \sigma_m(P)\sigma_{m'}(Q)$,

3. σ_m gives rise to the following short exact sequence.

$$0 \to \Psi_{c,k}^{m(.)-1}(G) \to \Psi_{c,k}^{m(.)}(G) \xrightarrow{\sigma_m} C^{\infty,k}(S^*(G)) \to 0.$$

Note that $\sigma_m(P)$ is a good model for a "global" homogeneous symbol of order m on $A^*(G)$. We can generalize this by defining "global" total symbols on $A^*(G)$.

Definition 3.4.2 — Let m be a C^k map on $G^{(0)}$ such that m(x) = m(y) whenever G^y_x is non empty. We denote by $\mathcal{S}^{m(.)}_{hom,k}(A^*(G))$ the subspace of $C^{\infty,k}(A^*(G))$ such that $\forall x \in M, a_x \in \mathcal{S}^{m(x)}_{hom}(\{x\}, T^*_x G_x)$ and such that for any trivializing open set $\Omega \subset A^*(G)$, with $\Omega \simeq U \times s(\Omega)$, the map $s(\Omega) \to \mathcal{S}^{m(.)}_{hom}(U), x \mapsto a_x$ is a C^k map, in the sense defined previously.

Now given such a symbol, we can define a compactly supported G-pseudodifferential operator associated to it. Unlike the case of manifolds, there is not a unique way to do so, since in general there is no canonical Fourier transform on G. We can determine a formula as follows : suppose we are given a diffeomorphism ϕ from a neighborhood W of $G^{(0)}$ in G to a neighborhood of the 0 section in A(G), with $d\phi = Id$, and a cut-off map χ , with support in W. Then set, for $\xi \in A_x^*(G)$ and $\gamma \in G_x$, $e_{\xi}(\gamma) = \chi(\gamma)e^{i\langle\phi(\gamma),\xi\rangle}$. We then have the following.

Proposition 3.4.3 — Let $a \in S_{hom,k}^{m(.)}(A^*(G))$. Denote by Op(a) the *G*-operator defined by its kernel

$$k(\gamma) = \frac{1}{(2\pi)^n} \int_{A^*_{r(\gamma)}(G)} e_{-\xi}(\gamma^{-1}) a(r(\gamma), \xi) d\xi$$

Then Op(a) is in $\Psi_{c,k}^{m(.)}(G)$.

Moreover, if we denote by a_m the homogeneous principal symbol, then we have $\sigma_m(Op(a)) = a_m$.

Proof— By definition Op(a) is a *G*-operator, so it remains to show that Op(a) is locally a C^k -family of pseudodifferential operators. To check this, we fix an open chart $\Omega \subset G$, with $\Omega \simeq U \times s(\Omega)$. Denote by κ the diffeomorphism from $U \times s(\Omega)$ to Ω and by κ_x its restriction from $U \times \{x\}$ to $\Omega \cap G_x$. Consider now a map $\varphi \in C_c^{\infty,k}(\Omega)$ and denote by P_x the operator $\varphi Op(a)$ considered as an operator on $U \times \{x\}$. Then, if $f \in C_c^{\infty}(U)$, one has :

$$(P_{x}f)(u) = \varphi(\kappa_{x}(u) \int_{G_{x}} k_{a}(\kappa_{x}(u)\gamma'^{-1}) f(\kappa_{x}^{-1}(\gamma')) d\lambda_{x}(\gamma')$$

$$= (2\pi)^{-n} \int_{G_{x}} \int_{A_{r(\kappa_{x}(u))}^{*}} \varphi(\kappa_{x}(u))\chi(\gamma'[\kappa_{x}(u)]^{-1}) a(r(\kappa_{x}(u)),\xi) e^{i\langle\phi(\gamma'[\kappa_{x}(u)]^{-1}),\xi\rangle} f(\kappa_{x}^{-1}(\gamma')) d\xi d\lambda_{x}(\gamma')$$

$$= (2\pi)^{-n} \int_{U} \int_{A_{r(\kappa_{x}(u))}^{*}} \varphi(\kappa_{x}(u))\chi(\kappa_{x}(u')[\kappa_{x}(u)]^{-1}) a(r(\kappa_{x}(u)),\xi) e^{-i\langle\phi(\kappa_{x}(u')[\kappa_{x}(u)]^{-1}),\xi\rangle} f(u') |J_{x}(u')| d\xi du'$$

This shows that a priori the operator P_x is a Fourier Integral Operator. It is in fact a pseudodifferential operator, thanks to a theorem of Hörmander and Kuranishi. (theorem 2.1.2 p107 in [16]). Indeed, we know that there exists a smooth map ψ from $U \times U$ to $GL(\mathbb{R}^n, A^*_{r(\kappa_x(u))}(G))$ such that $\langle \phi(\kappa_x(u')[\kappa_x(u)]^{-1}), \psi_x(u, u')\xi \rangle = \langle u' - u, \xi \rangle$. Hence one can write :

$$(P_x f)(u) = (2\pi)^{-n} \int_U \int_{\mathbb{R}^n} \tilde{a}(u, u', x, \xi) f(u') e^{i\langle u - u' \rangle, \xi \rangle} d\xi du'$$

with

$$\tilde{a}(u,u',x,\xi) = \varphi(\kappa_x(u))\chi(\kappa_x(u')[\kappa_x(u)]^{-1})a(r(\kappa_x(u)),\psi_x(u,u')\xi) \left| J_x(u') \right| \left| J_{\psi_x(u,u')} \right|.$$

The map $\tilde{a}(u, u', x, \xi)$ is a C^k family of classical amplitudes and so gives rise to a pseudodifferential operator, as in classical theory (see [16]). To find the principal symbol of this operator, we need to take the first term in the homogeneous expansion in ξ on the diagonal u' = u. By *G*-invariance of the symbol, we can reduce this to the case where u = 0. As $\psi_x(0,0)$ is simply given by transposition of $d\kappa_x(0)$, and by hypothesis $d\phi = Id$, one gets that $\sigma_m(P_x)(0,\xi) = a_m(x, t(d\kappa_x)(0)\xi)$, and hence the principal symbol of Op(a) is a_m .

Remarks

- 1. It follows from the definition that Op(a) has compact support in W
- 2. As the principal symbol of Op(a) is a_m , different choices for ϕ in the above formula give rise to pseudodifferential operators of same order m which coincide at first order, with difference an operator of order m - 1.
- 3. Examples of maps e_{ξ} are given in [26]. First, fix an invariant connection ∇ on $A(G) \to M$, so that one can define an exponential map exp from a neighborhood V_0 of the zero section in A(G) to a neighborhood V of M in G, which maps the zero section to M and which is a local diffeomorphism. Then define a cut-off map $\chi \in C_c^{\infty}(V)$ such that $\chi = 1$ in a smaller neighborhood of G. Denote by ϕ a local inverse of exp in the support of χ , and by $e_{\xi}(\gamma) = \chi(\gamma) \exp(i\langle \phi(\gamma), \xi \rangle)$, for $\xi \in A_{s(\gamma)}^*(G) = T_{s(\gamma)}^*G_{s(\gamma)}$. Then e_{ξ} satisfies the required conditions.
- 4. The original proof of Hörmander shows that more general type of maps e_{ξ} are allowed to provide a formula that associates a pseudodifferential operator to a symbol.

Observe that if $a \in \mathcal{S}_{hom,k}^{-\infty}(A^*(G))$ then Op(a) is smoothing. Hence Op defines a map from $\mathcal{S}_{hom,k}^{m(.)}(A^*(G))/\mathcal{S}_{hom,k}^{-\infty}(A^*(G))$ to $\Psi_{c,k}^{m(.)}(G)/\Psi_{c,k}^{-\infty}(G)$, which is injective. Indeed if $a = (a_{m(.)-j})_{j \in \mathbb{N}}$ and $b = (b_{m(.)-j})_{j \in \mathbb{N}}$ are two sequences of homogeneous symbols in $\mathcal{S}_{hom,k}^{m(.)}(A^*(G))/\mathcal{S}_{hom,k}^{-\infty}(A^*(G))$, then Op(a) = Op(b) implies that the principal symbol of Op(a-b) is 0 hence $a_{m(.)-j} = b_{m(.)-j}$ for all $j \in \mathbb{N}$.

Moreover, this map is surjective and admits an inverse σ_{tot} , defined as follows. Let $P \in \Psi_{c,k}^{m(.)}(G)$, then we can define $\sigma_{tot}(P) = (\sigma_{m-j}(P_j))_{j \in \mathbb{N}}$ with $P_j \in \Psi_{c,k}^{m(.)-j}(G)/\Psi_{c,k}^{-\infty}(G)$ defined recursively by $P_0 = P$ and

$$P_j = P_{j-1} - Op(\sigma_{m-j+1}(P_{j-1})).$$

This defines a map from $\Psi_{c,k}^{m(.)}(G)/\Psi_{c,k}^{-\infty}(G)$ to $\mathcal{S}_{hom,k}^{m(.)}(A^*(G))/\mathcal{S}_{hom,k}^{-\infty}(A^*(G))$ such that for all $a \in \mathcal{S}_{hom,k}^{m(.)}(A^*(G))$, one has $\sigma_{tot}(Op(a)) \equiv a$. Indeed, one gets in this situation that $P_N = Op(a - \sum_{j=0}^{N-1} a_{m-j})$. Hence we have constructed an inverse for Op.

Proposition 3.4.4 — Assume we have defined a map $Op : S_{hom,k}^{m(.)}(A^*(G))/S_{hom,k}^{-\infty}(A^*(G)) \rightarrow \Psi_{c,k}^{m(.)}(G)/\Psi_{c,k}^{-\infty}(G)$ as in Proposition 3.4.3. Then this map is a 1-1 correspondence and it admits an inverse denoted by σ_{tot} .

As there is no canonical definition for an Op map, there is none for σ_{tot} either. Two different formulas accord only in general on the first term, which is the principal symbol. Hence, when we will speak later on of the total symbol of an operator, this will suppose that we have fixed a formula for Op, what we assume from now on.

From the proposition 3.4.4 we can deduce the following lemma which will be useful for us.

LEMMA 3.4.5 — Let $(P_j)_{j \in \mathbb{N}}$ be a family of *G*-pseudodifferential operators of order m - j with compact support in a fixed compact *W*. Then there exists a pseudodifferential *G*-operator $P \in \Psi_{c,k}^{m(.)}(G)$ with compact support in *W* such that $P \sim \sum P_j$, which means that $\forall N \in \mathbb{N}$, $P - \sum_{j=0}^{N} P_j \in \Psi_{c,k}^{m(.)-N-1}(G).$

Proof of the lemma— In view of the 1-1 correspondence between symbols and G-operators, it suffices to show that there exists a symbol $a_P \in \mathcal{S}_{hom,k}^{m(.)}(A^*(G))$ such that $a_P \sim \sum a_{P_j}$. This is a classical result showed using an analogy of the Borel lemma [1][Prop. 2.3].

Following the original idea of Connes in [8], we can restate the theorem proved by Monthubert and Pierrot in [25] for the classical pseudodifferential operators (those with integer order), which can immediately be extended to polyhomogeneous operators of complex order. Let Edenote the Hilbert $C^*(G)$ -module $C^*(G)$.

Theorem 3.4.6 — Let $P \in \Psi_{c,k}^{m(.)}(G)$ be a compactly supported C^{k} - pseudodifferential operator on G, and $m_0 = \max \Re e m$.

- 1. If $m_0 < 0$, then P extends to an operator $\overline{P} \in \mathcal{K}(E) = C^*(G)$.
- 2. If $m_0 = 0$, then P extends to a bounded morphism $\overline{P} \in \mathcal{L}(E)$.

3.5 Ellipticity

From now on, we assume that $M = G^{(0)}$ is a compact set, since we want to study compactly supported elliptic operators. Recall that an operator is *elliptic* when its principal symbol is invertible. As in the classical setting, we want ellipticity to imply that there exists a parametrix, i.e. a pseudodifferential quasi-inverse for an elliptic operator.

Proposition 3.5.1 — Let m be a complex map on $G^{(0)}$ constant on the orbits of G and $P \in$

 $\Psi_{c,k}^{m(.)}(G)$ be an elliptic operator. Then there exists an operator $Q \in \Psi_{c,k}^{-m(.)}(G)$ which is a parametrix for P:

$$PQ - I = R$$
 and $QP - I = R'$,

with R and R' compactly supported smoothing operators.

Proof— By definition of ellipticity, we know that the principal symbol $\sigma_m(P) \in C^{\infty,k}(S^*(G))$ is invertible. Hence we have $(\sigma_m(P))^{-1} \in C^{\infty,k}(S^*(G))$. By theorem 3.4.1, this means that there exists a *G*-pseudodifferential operator Q_0 of order -m with principal symbol $(\sigma_m(P))^{-1}$. Moreover, we may assume that Q_0 is supported in a compact neighborhood *W* of *M* in *G*, containing the support of *P*. We now construct a sequence $(P_j)_{j\in\mathbb{N}}$ of operators supported in *W* and of orders -m - j by setting $Q_j = Q_0(I - PQ_0)^j$. Using the lemma 3.4.5, we then know that there exists an operator $Q \in \Psi_{c,k}^{-m(.)}(G)$ with support in *W* and such that

$$Q \sim \sum Q_j = Q_0 \sum_{j=0}^{\infty} (I - PQ_0)^j.$$

For any $N \in \mathbb{N}$, we then have that $PQ - I \in \Psi_{c,k}^{-N}(G)$. Indeed, we have $Q - \sum_{j=0}^{N-1} Q_j \in \Psi_{c,k}^{-m(.)-N}(G)$, and $P\left(\sum_{j=0}^{N-1} Q_j\right) - I = -(I - PQ_0)^{-N} \in \Psi_{c,k}^{-N}(G)$, from which we deduce that $PQ - I \in \Psi_{c,k}^{-\infty}(G)$. We can do the same for the left parametrix, and by the classical argument, show that left and right parametrix coincide modulo a smoothing operator with compact support in W.

3.6 Unbounded operators

We now wish to consider compactly supported G-pseudodifferential operators as unbounded operators on the Hilbert C^* -module $E = C^*(G)$. We show that in the case where the operator is elliptic it is regular, as an unbounded operator, in the sense of Baaj [4, 5]. The material in this subsection is taken from a graduate course by Georges Skandalis [36] and has also been written by François Pierrot in [27].

Consider now a compactly supported pseudodifferential operator P of C^k -type on G, with order m of real part $m_0 > 0$. This operator with domain $C^{\infty,k}(G)$ can be viewed as an unbounded, densely defined operator on the Banach space $E = C^*(G)$. Recall also that such an operator admits a formal adjoint P^{\natural} which is again a compactly supported pseudodifferential operator, with order \overline{m} . This operator is characterized by the equality, $\langle Pu, v \rangle = \langle u, P^{\natural}v \rangle$, which holds for all $u, v \in C_c^{\infty,k}(G)$. As both P and P^{\natural} are densely defined operators, P and P^{\natural} are closable. We denote by \overline{P} the closure of P. Recall that it is the smallest extension of Pwith its graph being a closed sub- $C^*(G)$ -module of E, and that its graph is given by

$$G(\overline{P}) = \overline{G(P)} = \left\{ (x, y) \in (C^*(G)^2, \ \exists (u_n) \in C_c^{\infty, k}(G), \|u_n - x\| \to 0 \text{ and } \|Pu_n - y\| \to 0 \right\}.$$

Note in particular that \overline{P} is a densely defined operator with a densely defined adjoint P^* such that $\overline{P^{\natural}} \subset P^*$. We begin by a very useful lemma :

LEMMA 3.6.1 — Let $A, B \in \Psi_{c,k}(G)$, such that max $\Re e(\operatorname{ord} A + \operatorname{ord} B) \leq 0$ and max $\Re e \operatorname{ord} B \leq 0$. Then we have $\overline{AB} = \overline{AB}$ and this operator is in $\mathcal{L}(E)$.

Proof of the lemma— It is enough to show that \overline{AB} is a closed operator. Indeed we know that $AB \subset \overline{AB}$ and that AB is of order with real part less or equal to 0 and so extends to a continuous morphism $\overline{AB} \in \mathcal{L}(E)$, of domain E, by proposition 3.4.6.

We know that $G(AB) = \{(x, z) \in E \times E, (Bx, z) \in G(A)\}$. As G(A) is a closed subspace of $E \times E$ and as the map \overline{B} is continuous from E to E by proposition 3.4.6, the set $G(\overline{AB})$ is closed.

We now come to the main proposition of this section.

Proposition 3.6.2 — Let P be an elliptic, compactly supported pseudodifferential operator of C^k -type on G. Then the operator \overline{P} is a regular operator on E.

Proof— It is enough to consider the case when $m_0 = \max \Re e m > 0$, as we have seen that $\overline{P} \in \mathcal{L}(E)$ otherwise. Note that both \overline{P} and P^* are densely defined so that we only have to prove that G(P) is orthocomplemented.

Now let Q be a parametrix of order -m for P, and R and S be the compactly supported smoothing operators such that

$$QP = 1 - S$$
 and $PQ = 1 - R$.

Applying proposition 3.4.6 to Q, R and S, we know that these operators extend to compact morphisms in $\mathcal{L}(E)$. We then have :

Lemma 3.6.3 -

- a) $\overline{P} \overline{Q} = \overline{PQ}$ and $\overline{P} \overline{S} = \overline{PS}$. Moreover these operators have domain E.
- b) $\operatorname{Dom}\overline{P} = \operatorname{Im}\overline{Q} + \operatorname{Im}\overline{S}$.
- c) $\overline{P^{\natural}} = P^*$.

Proof of the lemma—

- a) This is a direct application of lemma 3.6.1 above.
- b) Let $x \in \text{Dom}\overline{P}$, $\overline{P}x \in E$; there exists a sequence u_n in $C_c^{\infty,k}(G)$ converging to x in norm and such that Pu_n converges in norm to $\overline{P}x$. As we have $QPu_n = u_n - Su_n$, with Q and S continuous, we get $\overline{Q}(\overline{P}x) = x - \overline{S}x$, from which we deduce

$$x = \overline{Q}(\overline{P}x) + \overline{S}x \in \operatorname{Im}\overline{Q} + \operatorname{Im}\overline{S}.$$

On the other hand, we know from a) that: $\operatorname{Im}\overline{Q} \subset \operatorname{Dom}\overline{P}$ and $\operatorname{Im}\overline{S} \subset \operatorname{Dom}\overline{P}$.

c) We have already noticed that $\overline{P^{\natural}} \subset P^*$. It remains to show that $\text{Dom}P^* \subset \text{Dom}\overline{P^{\natural}}$. But we know that PQ = I + R, and so $(PQ)^* = I + R^*$. As we have $Q^*P^* \subset (PQ)^*$, we get, for any $x \in \text{Dom}P^*$, that $x = Q^*(P^*x) - R^*x$, and so that $\text{Dom}P^* \subset \text{Im}Q^* + \text{Im}R^*$. As Q and R are negative order operators, we have $\overline{Q^{\natural}} = Q^*$ et $\overline{R^{\natural}} = R^*$. Applying b) to $\overline{P^{\natural}}$, we get :

$$\mathrm{Dom}\overline{P^{\natural}} = \mathrm{Im}\overline{Q^{\natural}} + \mathrm{Im}\overline{R^{\natural}},$$

which suffices to conclude.

 \square

We have proven that $G(\overline{P}) = \{(\overline{Q}x + \overline{S}y, \overline{PQ}x + \overline{PS}y), (x, y) \in E \times E\}$. Consider now the operator on $E \oplus E$ defined by

$$U = \left(\begin{array}{cc} \overline{Q} & \overline{S} \\ \overline{PQ} & \overline{PS} \end{array}\right).$$

It is a morphism in $\mathcal{L}(E \oplus E)$ as Q, S, PQ, PS and their adjoints are compactly supported pseudodifferential operators with real part of the order less or equal to zero, and so their closures are elements of $\mathcal{L}(E)$. The range of U is then exactly equal to the graph of \overline{P} , and we get the result using proposition 2.3.1.

We get an immediate corollary of proposition 3.4.6

Corollary 3.6.4 — Let P_1 and P_2 be respectively in $\Psi_{c,k}^{m_1(.)}(G)$ and $\Psi_{c,k}^{m_2(.)}(G)$, with $\max \Re(m_2 - m_1) \ge 0$ and P_2 elliptic. Then there exists c > 0 such that, using the norm of $C^*(G)$, for any $u \in C_c^{\infty,k}(G)$ we have

$$||P_1u|| \leq c(||P_2u|| + ||u||)$$
 and $\mathsf{Dom}\overline{P_2} \subset \mathsf{Dom}\overline{P_1}$

Proof of the corollary— Let $Q_2 \in \Psi_{c,k}^{-m_2(.)}(G)$ be a parametrix for P_2 . As $P_1Q_2 \in \Psi_{c,k}^{m_1(.)-m_2(.)}(G)$, it is bounded and there exists $c_1 > 0$ s.t. $\|P_1Q_2(P_2u)\| \leq c_1 \|P_2u\|$. Moreover, $P_1(Q_2P_2-I)$ is a compactly supported smoothing operator and so there exists c_2 s.t. $\|P_1Q_2P_2u - P_1u\| \leq c_2 \|u\|$. Finally, we get

$$||P_1u|| \le ||P_1Q_2P_2u - P_1u|| + ||P_1Q_2(P_2u)|| \le c(||P_2u|| + ||u||).$$

 \square

3.7 The algebra $\Psi_k(G)$ of pseudodifferential operators of C^k -type

As we intend to develop functional calculus with our operators, we need a class of smoothing operators that are not any more compactly supported. We show later that this definition is not artificial, and that it fits well with the framework of our Sobolev modules, although this class is quite big.

If P is a compactly supported pseudodifferential G-operator and $T \in \mathcal{L}(E)$, we write $\overline{TP} \in \mathcal{L}(E)$ when $\operatorname{Im} T^* \subset \operatorname{Dom} P^*$ which implies that \overline{TP} is a morphism, as we can extend by continuity the equality $\langle TPx, y \rangle = \langle x, P^*T^*y \rangle$ which is true for any $(x, y) \in C_c^{\infty,k}(G) \times E$. With the same notations, we write $\overline{PT} \in \mathcal{L}(E)$ whenever $\overline{T^*P^{\natural}} \in \mathcal{L}(E)$, with P^{\natural} the formal adjoint of P. Finally, we write $\overline{P_1TP_2} \in \mathcal{L}(E)$ when $\overline{P_1T} \in \mathcal{L}(E)$ and $\overline{(\overline{P_1T})P_2} \in \mathcal{L}(E)$ and when $\overline{TP_2} \in \mathcal{L}(E)$ and $\overline{P_1(\overline{TP_2})} \in \mathcal{L}(E)$.

Definition 3.7.1 — A *smoothing operator* is an operator $R \in \mathcal{L}(E)$ such that for any compactly supported pseudodifferential *G*-operators P_1, P_2 of C^k -type, we have $\overline{P_1RP_2} \in \mathcal{L}(E)$. We denote by $\Psi^{-\infty}(G)$ the algebra formed by these operators.

Remarks

1. As the property $P_1RP_2 \in \mathcal{L}(E)$ should be true for all pseudodifferential operators, we can easily deduce a handier characterization of smoothing operators.

Proposition 3.7.2 — An operator R is smoothing if and only if it fulfills the two following conditions

- (a) $\forall P \in \Psi_{c,k}(G), \ \mathsf{Im}R \subset \mathsf{Dom}\overline{P} \text{ and } \ \mathsf{Im}R^* \subset \mathsf{Dom}\overline{P}.$
- (b) The operator $\overline{P_1}RP_2$ defined on $C_c^{\infty,k}(G)$ is bounded on E.
- 2. Note that the letter k denoting the transversal class of regularity has disappeared, as we will show this set is independent of k. Indeed, our class of smoothing operators appears to be only continuous in the direction transverse to G^x in G. In general, we do not know a better result, for transverse regularity, though in particular cases we can ask better transverse regularity, provided there are enough transverse vector fields.

To give a more precise idea on this set $\Psi^{-\infty}(G)$, we can state the following.

Proposition 3.7.3 — The set $\Psi^{-\infty}(G)$ is a sub-algebra of $\mathcal{L}(E)$ and has the following properties :

- 1. $\Psi^{-\infty}(G) \subset \mathcal{K}(E);$
- 2. $\forall P_1, P_2 \in \Psi_{c,k}(G), \forall R \in \Psi^{-\infty}(G), P_1RP_2 \in \Psi^{-\infty}(G)$;
- 3. $\forall R_1, R_2 \in \Psi^{-\infty}(G), \forall T \in \mathcal{L}(E), R_1TR_2 \in \Psi^{-\infty}(G)$;
- 4. $\Psi_{c,k}^{-\infty}(G) = C_c^{\infty,k}(G) \subset \Psi^{-\infty}(G).$

Proof— The properties 2 and 3 are direct consequences of the definition of $\Psi^{-\infty}(G)$, while property 4 comes from the definition of compactly supported smoothing operators on G. We show the first one. We first of all remark that the definition of $\Psi^{-\infty}(G)$ implies that for any P_1, P_2 in $\Psi_{c,k}(G)$, we have $\overline{P_1RP_2} \in \mathcal{K}(E)$. Indeed, let P be an elliptic operator with order m, with $\Re e m > 0$, and denote Q a parametrix for P and S = QP - I. S is a compactly supported smoothing operator and by theorem 3.4.6, we know that $\overline{Q} \in \mathcal{K}(E)$ and $\overline{S} \in \mathcal{K}(E)$. By hypothesis, we know that $\overline{PP_1RP_2} \in \mathcal{L}(E)$ so that using the closure of the equality $P_1RP_2 = QPP_1RP_2 - SP_1RP_2$, we get $\overline{P_1RP_2} \in \mathcal{K}(E)$. In the case where $P_1 = P_2 = 1$, this shows that $R \in \mathcal{K}(E)$.

One can prove, using parametrices, that the statements

$$\{PR \in \mathcal{K}(E), \text{ for all } P \in \Psi_{c,k}(G)\}$$

and

$$\{PR \in \mathcal{K}(E), \text{ for all } P \in \Psi_{c,k}(G), P \text{ elliptic } \}$$

are equivalent. Hence, for the definition of smoothing operators, we may consider only elliptic pseudodifferential operators.

Proposition 3.7.4 — Let $R \in \mathcal{L}(E)$, and P_1 and P_2 two elliptic operators with constant order of strictly positive real part. Then the following are equivalent.

- 1. $R \in \Psi^{-\infty}(G)$.
- 2. $\forall n \in \mathbb{N}, \overline{P_1^n R P_2^n} \in \mathcal{L}(E).$

Proof-

• It is clear that $1 \Rightarrow 2$). It remains to show that $2 \Rightarrow 1$). We use the characterization in proposition 3.7.2 above to show this property. Note that we need only to show that if the two conditions in 3.7.2 are true for an operator R and for P_1^n, P_2^n for any $n \in \mathbb{N}$, then they are true for all pseudodifferential operators A_1 and A_2 . For i = 1, 2, fix a parametrix $Q_i(n)$ for P_i^n , and denote by $R_i(n)$ and $S_i(n)$ the compactly supported smoothing operators defined by

$$S_i(n) = I - Q_i(n)P_i^n$$
 and $R_i(n) = I - P_i^n Q_i(n)$.

Note by the way that by elementary calculus we can show that Q_i^n is a parametrix for P_i^n whenever Q_i is a parametrix for P_i .

- The assumption (a) in proposition 3.7.2 is then a consequence of the lemma 3.6.3 which states that $\text{Dom}P_i^n = \text{Im}\overline{Q_i(n)} + \text{Im}\overline{S_i(n)}$. Choose $A \in \Psi_{c,k}^{m(.)}$ and $n \in \mathbb{N}$ such that $\Re e \text{ ord } P_i^n > \max \Re e \text{ ord } A$. Then we have $\text{Im}\overline{Q_i(n)} \subset \text{Dom}\overline{A}$ and $\text{Im}\overline{S_i(n)} \subset \text{Dom}\overline{A}$. Indeed, this comes from the lemma 3.6.1 : if Q is a compactly supported pseudodifferential operator with order -m, with $\Re e m > \Re e \text{ ord } A$, then one has $\overline{AQ} = \overline{AQ}$ and this morphism has domain E. This implies in particular that $\text{Im}\overline{Q} \subset \text{Dom}\overline{A}$.
- For assumption (b) in proposition 3.7.2, suppose we are given $A_1, A_2 \in \Psi_{c,k}^m$, and choose $n \in \mathbb{N}$ such that max $\Re e$ ord A_i is strictly less than the minimum of $n \Re e$ ord P_i for i = 1, 2. With the previous notations, we may then write

$$A_1RA_2 = (A_1Q_1(n))(P_1^nRP_2^n)(Q_2(n)A_2) + (A_1S_1(n))(RP_2^n)(Q_2(n)A_2) + (A_1Q_1(n))(P_1(n)R)(R_2(n)A_2) + (A_1S_1(n))R(R_2(n)A_2).$$

As the operators $A_1Q_1(n)$, $A_1S_1(n)$, $Q_2(n)A_2$ and $R_2(n)A_2$ are compactly supported pseudodifferential operators with order of negative real part, their closures are elements in $\mathcal{K}(E)$ by proposition 3.4.6. Hence the assumptions for P_1^n and P_2^n imply the assumptions for all compactly supported operators A_1 and A_2 .

This allows us to enlarge the class of pseudodifferential G-operators of C^k -type. Set $\Psi_k(G)$ to be the linear span generated by operators of constant order in $\Psi_{c,k}(G)$ and by operators in $\Psi^{-\infty}(G)$. Then, we have

Proposition 3.7.5 —

- 1. For any $m, n \in \mathbb{C} \cup \{-\infty\}$ we have : $\Psi_k^m(G) \circ \Psi_k^n(G) \subset \Psi_k^{m+n}(G)$.
- 2. The space $\Psi_k(G)$ endowed with composition is a *-algebra, filtered by \mathbb{R} and graded by \mathbb{C}/\mathbb{Z} . Note that the previous property turns $\Psi^{-\infty}(G)$ into a two-sided ideal of this algebra.
- 3. The set $\Psi_k^{\mathbb{Z}}(G)$ of operators with integer order is a *-sub-algebra of $\Psi_k(G)$.

REMARK — For the sake of simplicity, we have dealt in this section only with scalar operators. The case of operators acting on sections of vector bundles is a straightforward generalization. For the trivial bundles \mathbb{C}^j and \mathbb{C}^l on $G^{(0)}$, a *G*-operator from \mathbb{C}^j to \mathbb{C}^l is simply a matrix $l \times j$ of *G*-operators

Suppose now that we are given a smooth finite dimensional vector bundle \mathcal{E} over $G^{(0)} = M$. We construct from \mathcal{E} a vector bundle $r^*(\mathcal{E})$ over G_x for any x, simply by pull-back of the range map r. The fiber of $\gamma \in G_x$ is given by $r^*(\mathcal{E})_{\gamma} = \mathcal{E}_{r(\gamma)}$.

The manifold $M = G^{(0)}$ being compact, a classical result states that any complex finite dimensional vector bundle \mathcal{E} admits a supplementary vector bundle \mathcal{E}^{\sharp} in a trivial fibre bundle

$$\mathcal{E} \oplus \mathcal{E}^{\sharp} = \mathbb{C}^{j}.$$

Then there exists a section

$$e_0 \in C^{\infty}(M, M_j(\mathbb{C})) \simeq M_j(C^{\infty}(M))$$

which is an orthogonal projection, with image \mathcal{E} and with kernel \mathcal{E}^{\sharp} . A map $f \in C^{k}(M)$ acts naturally by multiplication on $C_{c}^{\infty,k}(G)$, and can so be considered, by composition with the range map r, as an element of the multiplier algebra $\mathcal{M}(C^{*}(G))$. We then denote by $e = e_{0} \circ r$ the corresponding projection of $M_{j}(\mathcal{M}(C^{*}(G)))$, so that we have constructed a module over $C^{*}(G)$ by $\mathcal{E} \otimes_{C(M)} C^{*}(G) = e(C^{*}(G))^{j}$.

Suppose now we are given two smooth finite dimensional vector bundles \mathcal{E} and \mathcal{E}' , over M. We can suppose that these are subbundles of the same trivial bundle \mathbb{C}^j . We denote by e and e' the corresponding projections of $M_j(\mathcal{M}(C^*(G)))$.

Definition 3.7.6 — A *G*-operator *P* from \mathcal{E} to \mathcal{E}' is defined by

$$P = e'\dot{P}e$$

with \check{P} a *G*-operator acting on the sections of the trivial vector bundle \mathbb{C}^{j} .

In particular, a compactly supported G-operator of order m and class C^k from \mathcal{E} to \mathcal{E}' is an operator of the form $P = e'\check{P}e$, with \check{P} a matrix in $M_j(\Psi^m_{c,k}(G))$. We will denote by $\Psi^m_{c,k}(G, \mathcal{E}, \mathcal{E}')$ the space of these operators.

To recover the results of this paper in the case of an operator acting on the sections of such a vector bundle \mathcal{E} , we need just to replace in the statements the Hilbert $C^*(G)$ -module $E = C^*(G)$ by the projective $C^*(G)$ -module $E = e((C^*(G))^j)$, where e is the projection corresponding to \mathcal{E} as above.

4 Sobolev modules

4.1 Definition

Let P be a C^k -elliptic operator of constant order m, with $\Re e m = s \ge 0$. When there is no risk for confusion we denote by P as well the closure \overline{P} of P. Since the operator P is regular, we have $\text{Dom}P = (1 + P^*P)^{-1/2}E$, so that it is clear that DomP is a sub- $C^*(G)$ -module of $C^*(G)$. Moreover it can be equipped with a Hilbert module structure using the scalar product $\langle x, y \rangle_s = \langle Px, Py \rangle + \langle x, y \rangle$.

Recall that if A is a C^* -algebra and \mathcal{E} an A-module, two scalar products \langle, \rangle_1 and \langle, \rangle_2 , such that $(\mathcal{E}, \langle, \rangle_1)$ and $(\mathcal{E}, \langle, \rangle_2)$ are Hilbert A-modules, are said to be *compatible* whenever there exists an operator T, A-linear and invertible, such that for any $\xi, \zeta \in \mathcal{E}, \langle \xi, \zeta \rangle_1 = \langle \xi, T\zeta \rangle_2$. Rather than looking at the above Sobolev module as a Hilbert module on its own, we are interested in its equivalence class of compatible scalar products (it is a *Hilbertizable module* in the sense of [15] [p75].) The advantage of this notion is that if \mathcal{E} and \mathcal{F} are two given Hilbert modules, the spaces $\mathcal{K}(\mathcal{E}, \mathcal{F})$ or $\mathcal{L}(\mathcal{E}, \mathcal{F})$ do not vary if one takes another scalar product on \mathcal{E} and \mathcal{F} provided they are compatible with the original ones. (cf. [15, 17]). Of course the adjoint of an operator in these spaces depends on the scalar products, but then, changing the scalar products is equivalent to composing the original *-operation with bijective A-linear operators.

We begin with the definition of the Sobolev module $H^s(P)$ associated with an operator, and we then show that all operators provide compatible scalar products so that the notion of H^s as a Hilbertizable module is well-defined.

Definition 4.1.1 — Let s be a positive real number and let P be a C^k -elliptic operators of order m, with $\Re e m = s$. The *Sobolev module* of rank s associated with P is the Hilbert $C^*(G)$ -module $H^s(P) = \text{Dom}\overline{P}$ endowed with the scalar product

$$\langle x, y \rangle_s = \langle Px, Py \rangle + \langle x, y \rangle.$$

We now prove that these Sobolev modules are in fact independent of the operator P chosen to define them.

Proposition 4.1.2 — Let P and P' be two compactly supported elliptic operator of order m and m', with $\Re e m = \Re e m' = s$. Then the Sobolev modules $H^s(P)$ and $H^s(P')$ are compatible.

Proof— We know from corollary 3.6.4 that DomP = DomP'. Set $T = (1+P'^*P')^{-1}(1+P^*P)$. It remains to show that T is an invertible element in $\mathcal{L}(E)$. Using a parametrix Q for the elliptic operator P'^*P' (note that a priori $1+P^*P$ is not a polyhomogeneous pseudodifferential operator) such that $P'^*P'Q + R = I$ with R a compactly supported smoothing operator, we see that one can write $(1+P'^*P')^{-1} = Q + (1+P'^*P')^{-1}R - (1+P'^*P')^{-1}Q$, and so

$$T = Q(1 + P^*P) + (1 + P'^*P')^{-1}R(1 + P^*P) - (1 + P'^*P')^{-1}Q(1 + P^*P)$$

The first part on the right hand side is a pseudodifferential of order m - m', so that the real part of its order is zero, and it is a bounded operator. The second part of it is the product of a bounded operator $(1 + P'^*P')^{-1}$ by a smoothing operator $R(1 + P^*P)$, which is also bounded, and the third pard is also the product of two bounded operators, so that T is bounded. The inverse of T is simply $(1 + P^*P)^{-1}(1 + P'^*P')$ which is bounded by the same proof.

We can then proceed to define the negative rank Sobolev modules by duality. First observe that E is naturally included in $\mathcal{K}(H^s, C^*(G))$ by the following map $\xi \mapsto \langle \xi, . \rangle$ where the scalar product is taken in E.

Definition 4.1.3 — Let s > 0, the Sobolev (Hilbertizable) module H^{-s} is the completion of E with respect to the norm of $\mathcal{K}(H^s, C^*(G))$.

Then to any C^k -elliptic operator P of order m, with $\Re e m = s \ge 0$, one associates the Hilbert module $H^{-s}(P)$: it is exactly the completion of E with respect to the norm induced by the scalar product : $\langle \xi, \zeta \rangle_{-s} = \langle (1 + P^*P)^{-\frac{1}{2}} \xi, (1 + P^*P)^{-\frac{1}{2}} \zeta \rangle_E$. Indeed, using the fact that $\text{Dom}P = \text{Im}(1 + P^*P)^{-\frac{1}{2}}$: if $\xi \in E$ then, we have

$$\begin{split} \|\xi\|_{-s} &= \sup \left\{ \|\langle \xi, x \rangle \|, \|x\|_{s} \leq 1 \right\} \\ &= \sup \left\{ \|\langle \xi, x \rangle \|, \left\| (1+P^{*}P)^{\frac{1}{2}}x \right\| \leq 1 \right\} \\ &= \sup \left\{ \left\| \langle \xi, (1+P^{*}P)^{-\frac{1}{2}}y \rangle \right\|, \|y\| \leq 1 \right\} \\ &= \sup \left\{ \left\| \langle (1+P^{*}P)^{-\frac{1}{2}}\xi, y \rangle \right\|, \|y\| \leq 1 \right\} \\ &= \left\| (1+P^{*}P)^{-\frac{1}{2}}\xi \right\|. \end{split}$$

4.2 Properties of the Sobolev modules

We will from now on denote :

$$H^{\infty} = \bigcap_{s \in \mathbb{R}} H^s$$
 and $H^{-\infty} = \bigcup_{s \in \mathbb{R}} H^s$.

First note that we have, by definition, duality between H^s and H^{-s} in this framework.

Proposition 4.2.1 — Let s > 0. The $C^*(G)$ -sesquilinear continuous map defined by $H^s \times E \to C^*(G)$

$$(u,v) \mapsto \langle u, v \rangle_E$$

extends into a $C^*(G)$ -sesquilinar continuous map $H^s \times H^{-s} \to C^*(G)$.

As in the classical setting, we have imbeddings between Sobolev modules.

Proposition 4.2.2 — Let s > s'. The identity on $C_c(G)$ extends to an imbedding $i_{s,s'} : H^s \hookrightarrow H^{s'}$, which is a compact morphism between these C^* Hilbert modules.

Proof-

1. If $s > s' \ge 0$, Let P and P' be respectively elliptic pseudodifferential operators of order with real part s and s' (in case s' = 0 we assume that P' = 0). Then by corollary 3.6.4, we have an imbedding $\text{Dom}P \subset \text{Dom}P'$ so that $H^s \subset H^{s'}$ and the map $i_{s,s'}$ is well defined. By the definition of a morphism between Hilbert modules, it suffices to show that there exists a map $i^*_{s,s'} : H^{s'} \to H^s$ such that for any $x \in H^s$, and any $y \in H^{s'}$, one has $\langle i_{s,s'}(x), y \rangle_{s'} = \langle x, i^*_{s,s'}(y) \rangle_s$. Rewriting this equality, we get :

$$\langle P'x, P'y \rangle + \langle x, y \rangle = \langle Px, Pi^*_{s,s'}(y) \rangle \text{ i.e. } \langle x, (1+P'^*P')y \rangle = \langle x, (1+P^*P)i^*_{s,s'}(y) \rangle.$$

Putting $i_{s,s'}^*(y) = (1+P^*P)^{-1}(1+P'^*P')y$ solves the equality. Moreover the operator $(1+P^*P)^{-1}(1+P'^*P')$ is compact. Indeed, let Q be a parametrix for P^*P , such that $P^*PQ = I+R$ with $R \in \Psi_{c,k}^{-\infty}$. We know that $Q, R \in \mathcal{K}(E)$, and from the equality $(1+P^*P)^{-1} = Q - (1+P^*P)^{-1}R - (1+P^*P)^{-1}Q$, we get that $(1+P^*P)^{-1} \in \mathcal{K}(E)$. Finally we have that $(1+P^*P)^{-1}(1+P'^*P') = Q(1+P'^*P') - (1+P^*P)^{-1}R(1+P'^*P') - (1+P^*P)^{-1}Q(1+P'^*P')$. The first term on the right hand side is a pseudodifferential operator of negative order, so it is in $\mathcal{K}(E)$. The second term is in $\mathcal{K}(E)$ by the properties of the smoothing operators and the third is a product of an element in $\mathcal{K}(E)$ by a pseudodifferential operator of order with real part 0 (hence bounded).

2. If $0 \ge s > s'$, the result is immediate using the definition of negative rank Sobolev modules, i.e. the duality between H^s and H^{-s} . We have that an operator $T \in \mathcal{K}(H^s, H^{s'})$ if and only if the operator $\tilde{T} \in \mathcal{K}(H^{-s'}, H^{-s})$ where \tilde{T} is defined by the following equality for any $\eta \in H^{-s'}$ and $\xi \in H^s$:

$$\langle T\eta, \xi \rangle = \langle \eta, T\xi \rangle.$$

We will need the following lemma.

LEMMA 4.2.3 — Let P be a pseudodifferential operator of order m with real part m_0 . Then P is in $\mathcal{L}(H^{m_0}, E)$ and $\mathcal{L}(E, H^{-m_0})$.

Proof of the lemma— The case where $m_0 = 0$ is clear.

1. Assume that $m_0 > 0$. The first assertion simply comes from the definition of H^{m_0} and corollary 3.6.4. Let P' be an elliptic pseudodifferential operator with order of real part m_0 . Then we know by the corollary 3.6.4 that $\text{Dom}P' \subset \text{Dom}P$ and that there exists c > 0 for any $u \in H^s$, we have $||Pu|| \leq c(||u|| + ||P'u||)$, so that there exists C > 0 such that $||Pu|| \leq C ||u||_s$. The second one comes from the duality between H^{m_0} and H^{-m_0} . Let $\xi \in E$. Then Px is defined as an element in H^{m_0} by duality by the following equality for any $\zeta \in H^{m_0}$

$$\langle Px, \zeta \rangle = \langle x, P^*\zeta \rangle$$

and $P \in \mathcal{L}(E, H^{-m_0})$.

2. If $m_0 < 0$ then the second assertion comes from the fact that if P' is an elliptic pseudodifferential operator with order of real part $-m_0$, then P'P is a pseudodifferential operator of order 0 so an element in $\mathcal{L}(E)$. As $P' \in \mathcal{L}(H^{-m_0}, E)$, and H^{-m_0} is exactly the domain of P', this implies that $P \in \mathcal{L}(E, H^{-m_0})$. Then we can once again deduce the first inclusion by duality.

We can now, using this lemma, show that our definition of smoothing operators is coherent with the natural one arising from this Sobolev scale.

Proposition 4.2.4 — An operator R is smoothing if and only if it is in the intersection of all $\mathcal{L}(H^s, H^t)$ for $s, t \in \mathbb{R}$. Moreover, the algebra $\Psi^{-\infty}$ is stable under holomorphic functional calculus and contains $\Psi_{c,k}^{-\infty}$ as a dense sub-algebra.

Proof-

- Suppose that $R \in \bigcap_{s,t} \mathcal{L}(H^s, H^t)$. Then, we want to show that for any two elliptic operators P_1 and P_2 of strictly positive order s and t, the operator P_1RP_2 is in $\mathcal{L}(E)$. We know, from the previous proposition that $P_1 \in \mathcal{L}(H^s, E)$ and $P_2 \in \mathcal{L}(E, H^{-t})$. As $R \in \mathcal{L}(H^{-t}, H^s)$, we can conclude that P_1RP_2 is in $\mathcal{L}(E)$.
- Suppose now that R is smoothing. Then for any $t \in \mathbb{R}$, if P is an elliptic pseudodifferential operator of order with real part t, then PR is a smoothing operator. In particular, for any real $s, PR \in \Psi_{c,k}^s(G)$ so that $PR \in \mathcal{L}(H^s, E)$. As above, this implies that $R \in \mathcal{L}(H^s, H^t)$ for all $s, t \in \mathbb{R}$.
- To show that this algebra $\Psi^{-\infty}$ is stable under holomorphic functional calculus, it suffices to show that it is hereditary in $\mathcal{K}(E)$. This means that $AXB \in \Psi^{-\infty}$ whenever $A, B \in$ $\Psi^{-\infty}$ and $X \in \mathcal{K}(E)$, which is clear in our case. Applying the classical formula f(z) =az + zg(z)z (for any holomorphic function f such that f(0) = 0) to the operator R, we then get stability under holomorphic functional calculus.

Finally, one can see that, as in the classical setting, the Sobolev scale is a natural framework for the action of pseudodifferential G operators.

 \square

Proposition 4.2.5 — Let P be a compactly supported pseudodifferential G-operator of order m of real part m_0 . Then P defines for any real s a morphism from H^s into H^{s-m_0} .

Proof— Let P_1 be an elliptic pseudodifferential operator of order of real part s and Q_1 a parametrix of P_1 so that $Q_1P_1 + R_1 = I$ with R_1 a smoothing operator. Then one can write $P = PQ_1P_1 + PR_1$. But as, P_1R is smoothing it is in $\mathcal{L}(H^s, H^{s-m_0})$. On the other hand we know by the preceding lemma that $P_1 \in \mathcal{L}(H^s, E)$ as it is of order with real part s, and that $PQ_1 \in \mathcal{L}(E, H^{s-m_0})$, as PQ_1 is of order with real part $m_0 - s$. This gives us that $PQ_1P_1 \in \mathcal{L}(H^s, H^{s-m_0})$ and P_1R too, so that it is true for P.

Remarks

- 1. We have supposed here that the order of P is constant, but it is not necessary and the above proposition can be adapted for operators with non-constant order, if we set $m_0 = \max \Re e(m)$.
- 2. Remark that we have used here the class of classical pseudodifferential operators, as they are the ones we are interested in, for example in the case of the complex powers, but our results remain valid if one replaces classical symbols by (1,0)-symbols or even (ρ, δ) -symbols.
- 3. We have considered all over the section, that $E = C^*(G)$, but we can with no change consider that E is a Hilbert module on $C^*(G)$ coming from a vector bundle over G as explained at the end of the previous section, with pseudodifferential operators acting on this vector bundle. In this case, we denote by $H^s(E)$ the corresponding Sobolev modules.

5 Complex powers of a positive elliptic pseudodifferential operator

We consider the classical problem of understanding the complex powers of a positive elliptic pseudodifferential operator A. The problem was first solved, in the case of compact manifolds, by Seeley [34] in a rather technical way. Shubin later on gave in his book [35] a beautiful framework to analyze the resolvent as a pseudodifferential operator, but his approach was limited to the powers of differential operators. Then Guillemin [14] proposed a very elegant way to bypass the detailed analysis of the resolvent of the operator, that we will adapt to our situation.

Various generalizations of these techniques have been used to take the complex powers in quite different geometric situations. Let us cite, without being exhaustive, Rempel and Schulze [29, 30, 31] for manifolds with boundary (see also Grubb [13]); Kordyukov for foliations [18]; Ponge [28] for Heisenberg manifolds and contact structures (the problem here is the lack of elliptic operators, so he studies the complex powers for hypoelliptic operators); Loya [23]for manifolds with conical singularities; Schrohe [33] and Amman, Lauter, Nistor and Vasy [2] for some classes of noncompact manifolds and manifolds with singularities.

Here we take a *G*-pseudodifferential operator *A* of strictly positive (fixed) order *m*, elliptic, invertible and positive, with (positive definite) principal symbol $\sigma = \sigma_m(A)$. We then know that the spectrum of *A* is included in $[\epsilon, +\infty[$, for some $\epsilon > 0$. Recall that the principal symbol $\sigma = \sigma_m(A)$ of *A* is a map in $C^{\infty,k}(S^*(G))$ which is positive definite in $C(S^*(G))$ and so we can define the *s*-th power σ^s of σ for $s \in \mathbb{C}$. Moreover the map $s \mapsto \sigma^s$ is holomorphic from \mathbb{C} to $C^{\infty,k}(S^*(G))$.

Using holomorphic functional calculus for normal regular operators, we know that one can write the complex powers of the regular unbounded operator A by putting

$$A^{s} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{s} (A - \lambda)^{-1} d\lambda$$

with Γ a contour of the form Γ_{ρ} for $\rho < \varepsilon$ $\Gamma_{\rho}^{+} = \{iv, v \in \mathbb{R}, \rho < v < +\infty\}$ $\Gamma_{\rho}^{0} = \{z = \rho e^{i\theta}, \ \pi/2 \le \theta \le 3\pi/2\}, \ \Gamma_{\rho}^{-} = -\Gamma_{\rho}^{+}.$

This operator is defined a priori as an unbounded regular operator on E. We prove the following theorem.

Theorem 5.0.1 — Let A be a positive definite elliptic operator of positive order m in $\Psi_k^m(G)$ as defined above. Then, for any $t \in \mathbb{R}$, the operator A^s is in $\mathcal{L}(H^{t+m \Re e s}, H^t)$ and there exists a holomorphic family A_s of pseudodifferential operators of order ms such that the operator $A^s - A_s$ is a holomorphic family in $\Psi^{-\infty}(G)$.

To achieve this, we will follow the general strategy developped by Guillemin [14].

- 1. We first show that there exists a family of symbols $\sigma(s)$, holomorphic in $s \in \mathbb{C}$, such that the corresponding operators A(s) fulfill the one-parameter group relation A(s)A(t) = A(s+t) modulo smoothing operators, with A(0) = 1 and such that A(1) A is smoothing. Moreover we will show that this family is unique modulo smoothing symbols.
- 2. Then we show that there exists a holomorphic family of pseudodifferential operators A_s which fulfills exactly the one-parameter group relation $A_sA_t = A_{s+t}$ with $A_0 = 1$ and such that $A_1 A$ is smoothing.
- 3. Finally we show that the difference between the operator A^s obtained via functional calculus and the operator A_s is a smoothing operator depending holomorphically on s

Note that, as in the rest of the paper, we have omitted explicit reference to vector bundles, as they do not introduce any change in the theory, except at the point where we construct the family A(s). Indeed, we treat there explicitly the case of vector bundles as they introduce non commutativity of the product of principal symbols. Elsewhere, the generalization to the case of operators acting on sections of a vector bundle is a straightforward.

From now on, we consider that we are given fixed Op and symbol maps, as in section 3. Recall that in this case, we have a bijective map between totals symbols modulo smoothing ones and *G*-pseudodifferential operators modulo smoothing ones (theorem 3.4.4), and that the operators constructed using this formula have compact support in a fixed compact set $W \subset G$.

5.1 Holomorphic families of pseudodifferential operators

First of all, we need to define correctly the notion of an holomorphic family of (generalized) pseudodifferential operators. Thanks to our approach in section 3 where we allowed the order of a family to be non constant, we have a simple description for such holomorphic families.

Let $K \subset \mathbb{C}$ be a compact set and, consider the groupoid $G_K = G \times K$ with units $G_K^{(0)} = G^{(0)} \times K$, with r and s being the identity on K.

Definition 5.1.1 — Let $m: K \to \mathbb{C}, z \mapsto m(z)$ be a holomorphic map. We consider m as a map on $G_K^{(0)}$, constant on $G^{(0)}$.

1. We say that a map $a : K \to S_{hom,k}^{m(.)}(A^*(G))$ is a holomorphic family of polyhomogeneous symbols when the symbol a is a C^k -family of polyhomogeneous symbols on the groupoid G_K , and satisfies Cauchy identity for some contour $\Gamma \subset K$ around s for any s in the interior of K.

$$a(s) = \frac{1}{2\pi i} \int_{\Gamma} \frac{a(z)}{z-s} dz.$$

2. Let $A: K \to \Psi_{c,k}^{m(.)}(G)$ be a family of pseudodifferential operators with support in a fixed compact set W'. We say that $s \mapsto A(s)$ is a holomorphic family of pseudodifferential operators with compact support in W' if A is a C^k -family of pseudodifferential operators on the groupoid G_K and if for any $f \in C_c^{\infty,k}(G)$ the map A(s)f satisfies Cauchy equality $A(s)f = \frac{1}{2\pi i} \int_{\Gamma} \frac{A(z)f}{z-s} dz$ for any s in the interior of K.

By extension we say that these families are holomorphic on \mathbb{C} if they are holomorphic on any compact subset $K \subset \mathbb{C}$ (with the support not depending on K, for the operators). Then we have the following properties.

Proposition 5.1.2 — Let $m : \mathbb{C} \to \mathbb{C}, z \mapsto m(z)$ be a holomorphic map.

- 1. If $a : \mathbb{C} \to \mathcal{S}_{hom,k}^{m(.)}(A^*(G))$ is a holomorphic family of polyhomogeneous symbols, then for any $j \in \mathbb{N}$, the map $s \mapsto a_{m(s)-j}(s)$ is holomorphic from \mathbb{C} to $C^{\infty,k}(S^*(G))$.
- 2. Conversely, if we are given any family $(a_{m(s)-j})_{j\in\mathbb{N}}$ such that the maps $s \mapsto a_{m(s)-j}(s)$ are holomorphic from \mathbb{C} to $C^{\infty,k}(S^*(G))$, then there exists a holomorphic family of polyhomogeneous symbols $a : \mathbb{C} \to \mathcal{S}_{hom,k}^{m(.)}(A^*(G))$ such that $a \sim \sum_j a_{m(s)-j}$ modulo smoothing symbols.

Proof—

1. First, it suffices to prove that $s \mapsto a_{m(s)}(s)$ is holomorphic, as we can repeat the argument to deduce it inductively for any $j \in \mathbb{N}$, considering $a(x, s, \xi) - \chi(\xi) \sum_{l=0}^{j-1} a_{m(s)-l}(x, s, \xi)$. As a(s) is a C^k -family of symbols on G, we know that $a_{m(s)} \in C^{\infty,k}(S^*(G))$. We can write

$$a_{m(s)}(x,s,\xi) = \lim_{t \to +\infty} \frac{a(x,s,t\xi)}{t^{m(s)}}$$

Hence, $a_{m(s)}(x, s, \xi)$ satisfies Cauchy identity if we can prove that $\tilde{a}(s,t) = \frac{a(x,s,t\xi)}{t^{m(s)}}$ is continuous in s, uniformly with respect to t, i.e. that $\sup_{t>1} |\tilde{a}(s,t) - \tilde{a}(z,t)| \to 0$ when $z \to s$. This comes simply from the fact that the map $s \mapsto \chi(\xi) \|\xi\|^{-m(s)} a(x,s,\xi)$ is of class C^k hence continuous in $\mathcal{S}^0_{hom,k}(A^*(G))$.

2. This is a holomorphic version of the classical proof (Borel lemma). Using the usual formula,

$$a(x,s,\xi) = \sum_{j=0} \chi(\frac{\xi}{t_j}) \|\xi\|^{m(s)-j} a_{m(s)-j}(x,s,\frac{\xi}{\|\xi\|}),$$

with χ a cut-off map and t_j going quickly to ∞ , we get a C^k -family of polyhomogeneous symbols. It remains to check that $a(x, s, \xi)$ satisfies Cauchy equality, which is clear as for fixed ξ the sum defining a is finite and all the terms in the sum satisfy the Cauchy equality.

Proposition 5.1.3 — Let $m : \mathbb{C} \to \mathbb{C}, z \mapsto m(z)$ be a holomorphic map.

- 1. If $a : \mathbb{C} \to \mathcal{S}_{hom,k}^{m(.)}(A^*(G))$ is a holomorphic family of polyhomogeneous symbols, then the family $s \mapsto Op(a(s))$ is a holomorphic family of pseudodifferential operators.
- 2. If $s \mapsto A(s) \in \Psi_{c,k}^{m(s)}(G)$ is a holomorphic family of pseudodifferential operators, then there exists a holomorphic family of polyhomogeneous symbols $\tilde{\sigma}_{tot}(A(s)) : \mathbb{C} \to \mathcal{S}_{hom,k}^{m(.)}(A^*(G))$ such that the class of $\tilde{\sigma}_{tot}(A(s))$ modulo smoothing symbols is $\sigma_{tot}(A(s))$.

Proof-

1. Set A(s) = Op(a(s)). Then we have, for any $f \in C_c^{\infty,k}(G)$,

$$(A(s)f)(\gamma) = \frac{1}{(2\pi)^n} \int_{G_{s(\gamma)}} \int_{A^*_{r(\gamma)}(G)} e_{-\xi}(\gamma'\gamma^{-1})a(s)(r(\gamma),\xi)f(\gamma')d\xi d\lambda_{s(\gamma)}(\gamma').$$

Using the Fubini theorem, we know that $(A(s)f)(\gamma) = \int_{A_{r(\gamma)}^*(G)} a(s)(r(\gamma),\xi)g(\gamma,\xi)d\xi$ where $g(\gamma,\xi) = \frac{1}{(2\pi)^n} \int_{G_{s(\gamma)}} e_{-\xi}(\gamma'\gamma^{-1})f(\gamma')d\lambda_{s(\gamma)}(\gamma')$ belongs to the Schwartz class as a function in ξ . The map $s \mapsto A(s)$ is of class C^k since $s \mapsto a(s)$ is. Using once again the Fubini theorem, it is clear that A(s)f satisfies the Cauchy equality $A(s)f = \frac{1}{2\pi i} \int_{\Gamma} \frac{A(z)f}{z-s} dz$ if a(s) does, which proves the holomorphicity of the family A(s).

2. In view of the previous proposition, it is enough to show that $s \mapsto \sigma_{tot}(A(s))$ is holomorphic in $\mathcal{S}_{hom,k}^{m(.)}(A^*(G))/\mathcal{S}_{hom,k}^{-\infty}(A^*(G))$, i.e. that the homogeneous parts of the symbols are holomorphic. As they are defined inductively see 3.4.4 using the principal symbol and Opwhich respect holomorphicity, this is clear by induction. **Proposition** 5.1.4 — If for any $j \in \mathbb{N}$, the family $s \mapsto A_j(s) \in \Psi_{c,k}^{m(s)-j}(G)$ is a holomorphic family of elliptic pseudodifferential operators, there exists a holomorphic family $s \mapsto A(s) \in \Psi_{c,k}^{m(s)}(G)$ such that for any $N \in \mathbb{N}$, we have $A(s) - \sum_{j=0}^{N-1} A_j(s) \in \Psi_{c,k}^{m(s)-N}(G)$.

Proof— In view of the 1 - 1 correspondence between symbols and operators, it is enough to prove the proposition for symbols. Using again the usual formula,

$$a(x,s,\xi) = \sum_{j=0} \chi(\frac{\xi}{t_j}) a_j(x,s,\xi),$$

with χ a cut-off map and t_j going quickly to ∞ , we get a C^k -family of polyhomogeneous symbols. It remains to check that $a(x, s, \xi)$ satisfies the Cauchy equality, which is clear as for fixed ξ the sum defining a is finite and all the terms in the sum satisfy the Cauchy equality.

This implies the following proposition.

Proposition 5.1.5 — If $s \mapsto A(s) \in \Psi_{c,k}^{m(s)}(G)$ is a holomorphic family of elliptic pseudodifferential operators, there exists a holomorphic family $s \mapsto B(s) \in \Psi_{c,k}^{-m(s)}(G)$ such that B(s) is a parametrix for A(s).

5.2 First step : construction of A(s)

We take a *G*-pseudodifferential operator *A* acting on the sections of a vector bundle $\tilde{\mathcal{E}} = r^*(\mathcal{E})$ over *G*, coming from a hermitian vector bundle \mathcal{E} over $G^{(0)} = M$ and we denote by *E* the corresponding Hilbert module. Moreover, we suppose that *A* is of strictly positive order *m*, elliptic, invertible and positive, with (positive definite) principal symbol $\sigma = \sigma_m(A)$. So, the spectrum of *A* is included in $[\epsilon, +\infty[$, for some $\epsilon > 0$.

Recall that the principal symbol $\sigma = \sigma_m(A)$ of A is a $C^{\infty,k}$ section of the fibre bundle $\mathcal{L}(\mathcal{E})$ pulled-back over $S^*(G)$, i.e. $\sigma \in C^{\infty,k}(S^*(G), \mathcal{L}(\mathcal{E}))$. In our case, it follows that σ takes values in positive definite operators in $\mathcal{L}(\mathcal{E})$. Then, by holomorphic functional calculus we know that we can define the s-th power σ^s of σ for $s \in \mathbb{C}$. Moreover, the map $s \mapsto \sigma^s$ is holomorphic from \mathbb{C} to $C^{\infty,k}(S^*(G), \mathcal{L}(\mathcal{E}))$.

We wish to construct a holomorphic family of pseudodifferential operators A(s) for $s \in \mathbb{C}$ with principal symbol $\sigma(A(s)) = \sigma^s$ such that A(0) = Id and $A(s)A(t) \equiv A(s+t)$ modulo smoothing operators and the difference $A_1 - A$ being a smoothing operator. To construct the family $(A(s))_{s\in\mathbb{C}}$ we need to consider the cohomology of the group $(\mathbb{C}, +)$ with coefficients in the representation of $(\mathbb{C}, +)$ on the space of sections $C^{\infty,k}(S^*(G), \mathcal{L}(\mathcal{E}))$. This construction generalizes the cohomology considered by Guillemin in [14] for the trivial representation of $(\mathbb{C}, +)$ on the space of smooth functions on $S^*(M)$ and the extension of Bucicovschi [6] to fibre bundles on a smooth compact manifold. To do so we consider an even more general situation : consider a C*algebra A and a sub-algebra B, which is a projective limit of Banach algebras and stable under holomorphic functional calculus (in our case we will consider $A = C(S^*(G), \mathcal{L}(\mathcal{E}))$ and $B = C^{\infty,k}(S^*(G), \mathcal{L}(\mathcal{E}))$).

Let σ be an element of B which is invertible and positive in A. The representation of $(\mathbb{C}, +)$ on A that we consider is the following : any $s \in \mathbb{C}$ acts on A by $s \cdot g = \sigma^{-s} g \sigma^s$. (Note that B is stable under this action).

Let $C^r = C^r(\mathbb{C}; B)$ be the space of functions

$$f: \underbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}_{r \text{ times}} \to B$$

that are holomorphic and such that $f(s_1, \ldots, s_r) = 0$ if at least one s_i is equal to zero.

Let $\delta^r : C^r \to C^{r+1}$ defined as:

$$(\delta^r f)(s_0, s_1, \dots, s_r) = s_0 \cdot f(s_1, \dots, s_r) + \sum_{i=1}^r (-1)^i f(s_0, \dots, s_{i-1} + s_i, \dots, s_r) + (-1)^{r+1} f(s_0, \dots, s_{r-1}).$$

Let $\mathcal{H}^r(\mathbb{C}; B) = \operatorname{Ker} \delta^r / \operatorname{Im} \delta^{r-1}$.

Proposition 5.2.1 —

We have $\mathcal{H}^2(\mathbb{C}; B) = 0$.

Moreover, for each 2-cocycle f and each $b \in B$, there exists a unique 1-cochain h such that $\delta h = f$ and h(1) = b.

Proof— Let $f : \mathbb{C} \times \mathbb{C} \to B$ so that for all $a, b, c \in \mathbb{C}$

$$\begin{cases} f(0,b) = f(a,0) = 0, \\ (\delta^2 f)(a,b,c) = a \cdot f(b,c) - f(a+b,c) + f(a,b+c) - f(a,b) = 0 \end{cases}$$

We try to find $h : \mathbb{C} \to B$ such that

$$(\delta^{1}h)(a,b) = \sigma^{-a}h(b)\sigma^{a} - h(a+b) + h(a) = f(a,b).$$

The existence of an h as above implies:

$$h'(a) = \sigma^{-a} h'(0) \sigma^a - \frac{\partial f}{\partial b}(a, 0).$$
(4)

Consider h to be the unique solution of the previous equation with h(0) = 0 and with a fixed prescribed value at 1, h(1). Then h can be found in the following way:

Let $\Phi(t)$ be the operator in $\mathcal{L}(A)$ given by $u \to \sigma^{-t} u \sigma^t$ for $u \in A$. Then

$$h(a) = -\int_0^a \frac{\partial f}{\partial b}(t,0) dt + \int_0^a \Phi(t)(h'(0)) dt.$$

If $T(a)u = \int_0^a \Phi(t)u \, dt$, then, in order to get any prescribed value for h(1), we need to show that T(1)u can be any element of B. Indeed, for any t we have that $\Phi(t)$ is a strictly positive

operator in $\mathcal{L}(A)$, and so this is also true for T(a), so that T(1) is invertible on A. Now it is clear that B is stable by T(1) as it is stable by $\Phi(t)$, and this shows that $T(1)_{|B}$ is bijective.

Thus we obtain a holomorphic map $h : \mathbb{C} \to B$ such that $h \in C^1$. We will show that $\delta h = f$ so f is a coboundary. To see this, let

$$g(a,b) = f(a,b) - (\sigma^{-a}h(b)\sigma^{a} - h(a+b) + h(a))$$

Clearly $\delta h = f$ if and only if $g \equiv 0$. Denote by $\frac{\partial}{\partial b}$ the partial derivative with respect to the second variable. Then:

$$\frac{\partial g}{\partial b}(a,b) = \frac{\partial f}{\partial b}(a,b) - \sigma^{-a}h'(b)\sigma^{a} + h'(a+b).$$
(5)

From (4) we get:

$$h'(b) = \sigma^{-b}h'(0)\sigma^{b} - \frac{\partial f}{\partial b}(b,0) \text{ and}$$
$$h'(a+b) = \sigma^{-(a+b)}h'(0)\sigma^{(a+b)} - \frac{\partial f}{\partial b}(a+b,0).$$

These two equalities and (5) imply

$$\begin{split} \frac{\partial g}{\partial b}(a,b) &= \frac{\partial f}{\partial b}(a,b) - \sigma^{-a} \left(\sigma^{-b} h'(0) \sigma^{b} - \frac{\partial f}{\partial b}(b,0) \right) \sigma^{a} + \sigma^{-(a+b)} h'(0) \sigma^{(a+b)} \\ &- \frac{\partial f}{\partial b}(a+b,0) \\ &= \sigma^{-a} \frac{\partial f}{\partial b}(b,0) \sigma^{a} - \frac{\partial f}{\partial b}(a+b,0) + \frac{\partial f}{\partial b}(a,b) \\ &= \frac{\partial}{\partial c} \left[(\delta^{2} f)(a,b,c) \right]_{|_{c=0}}. \end{split}$$

So $\frac{\partial g}{\partial b} = 0$; hence g(a, b) is constant in b. When b = 0 we have

$$g(a,0) = f(a,0) - \left(\sigma^{-a}h(0)\sigma^{a} - h(a) + h(a)\right) = 0.$$

So $g \equiv 0$. Because f was chosen arbitrarily we conclude $\mathcal{H}^2(\mathbb{C}; B) = 0$.

We now prove the existence of the family A(s).

Proposition 5.2.2 — There exists a holomorphic family of pseudodifferential operators $A(s), s \in \mathbb{C}$ with compact support in W and principal symbol $\sigma(A(s)) = \sigma^s$ such that A(0) = Id, $A(s)A(t) \equiv A(s+t)$ modulo smoothing operators and the difference $A_1 - A$ being a smoothing operator.

Proof-

As we know that an elliptic family of pseudodifferential operators admits a holomorphic family of operators as parametrix, and as we will be working in this proof modulo smoothing operators, we will denote by $A(s)^{-1}$ a holomorphic parametrix for an elliptic, holomorphic

A(s). The statement of the proposition is then equivalent to finding a holomorphic family of pseudodifferential operators A(s) with compact support in W for $s \in \mathbb{C}$ with principal symbol $\sigma(A(s)) = \sigma^s$ such that:

$$\begin{cases}
A(s)A(t)A(s+t)^{-1} \equiv Id \quad (\text{mod } \Psi^{-\infty}), \\
A^{-1}A(1) \equiv Id \quad (\text{mod } \Psi^{-\infty}), \\
A(0) \equiv Id \quad (\text{mod } \Psi^{-\infty}).
\end{cases}$$
(6)

(we denoted the space of smoothing operators by $\Psi^{-\infty}$).

To prove Proposition 5.2.2, we will construct A(s) inductively in $k \in \mathbb{N}$, such that $\forall s, t \in \mathbb{C}$ in a neighborhood of 0,

$$\begin{cases} A_k(s)A_k(t)A_k(s+t)^{-1} \equiv Id \quad \pmod{\Psi^{-k}}, \\ A^{-1}A_k(1) \equiv Id \quad \pmod{\Psi^{-k}}, \\ A_k(0) \equiv Id \quad \pmod{\Psi^{-k}}. \end{cases}$$
(7)

Let χ be a cut-off map on $A^*(G)$, $\chi(x,\xi) = \omega(||\xi||)$, with $\omega \in C^{\infty}(\mathbb{R})$ a positive map, null if t < 1/2 and $\omega(t) = 1$ if $t \ge 1$. Given any element $a \in C^{\infty,k}(S^*(G), \mathcal{L}(\mathcal{E}))$ and any complex number z, construct an element \tilde{a} in $\mathcal{S}^z_{hom,k}(A^*(G), \mathcal{L}(\mathcal{E}))$ by putting

$$\tilde{a}(x,\xi) = \chi(x,\xi) \|\xi\|^z a(x,\frac{\xi}{\|\xi\|}).$$

Recall that we fixed a map $Op: S^z_{hom,k}(A^*(G), \mathcal{L}(\mathcal{E})) \to \Psi^z_{c,k}(G, \mathcal{E})$ which associates an operator to any given total symbol. Composing these two maps, we get a map θ_z from $C^{\infty,k}(S^*(G), \mathcal{L}(\mathcal{E}))$ to $\Psi^z_{c,k}(G, \mathcal{E})$ which maps any a to an operator of degree z with principal symbol equal to a. Moreover, it is clear, that if we take a holomorphic map $s \mapsto a(s)$ in $C^{\infty,k}(S^*(G), \mathcal{L}(\mathcal{E}))$, and a holomorphic map $f: \mathbb{C} \to \mathbb{C}$, the operators $\theta_{f(s)}(a(s))$ form a holomorphic family of operators from $H^{t+r}(E)$ to $H^r(E)$ for any real $t > \Re e(f(s))$ and any r. Indeed the Cauchy equality

$$A(s)u = \int_{\Gamma} \frac{A(z)u}{z-s} dz \tag{8}$$

holds for any $u \in C_c^{\infty,k}(G)$, because it holds for a(s). Now, for a fixed s, both operators A(s)and $\int_{\Gamma} \frac{A(z)u}{z-s} dz$ define bounded operators from $H^{t+r}(E)$ to $H^r(E)$ for any real $t > \Re e(f(s))$ and any r, provided the contour Γ is well chosen. So equation (8) extends by continuity.

For k = 1 we want $(A_1(s))_{s \in \mathbb{C}}$ to be a holomorphic family of pseudodifferential operators of order ms with compact support in W, with the principal symbol equal to σ^s where σ is the principal symbol of A. We can construct such a family in the following way . Let P(s) be the family of operators with compact support in W defined by $P(s) = \theta_{ms}(\sigma^s)$. We know that P(s) is a holomorphic family of elliptic operators. We may assume that P(0) is invertible, since we can add to it a smoothing operator without changing the holomorphicity of the family. Denote by Q(s) a parametrix for P(s) and set

$$A_1(s) = P(s) \left(sQ(1)A + (1-s)Q(0) \right).$$

It is clear that $A_1(s)$ satisfies all required properties (7) modulo Ψ^{-1} , so we are done with step one.

Now suppose that the relations (7) hold for a certain $k \in \mathbb{N}$. We will construct a new family $(A_{k+1}(s))_{s\in\mathbb{C}}$ that satisfies (7) for k+1. We set :

$$A_{k+1}(s) = A_k(s)(Id - H(s)), \qquad H(s) \in \Psi^{-k}.$$
(9)

In this way $A_{k+1}(s) - A_k(s) \in \Psi^{ms-k}$. We have the following equalities (mod Ψ^{-k-1}):

$$\begin{aligned} A_{k+1}(s)A_{k+1}(t) & A_{k+1}(s+t)^{-1} \equiv A_k(s)(Id-H(s))A_k(t)(Id-H(t))(Id+H(s+t))A_k(s+t)^{-1} \\ \equiv A_k(s)A_k(t)A_k(s+t)^{-1} - A_k(s)H(s)A_k(t)A_k(s+t)^{-1} - A_k(s)A_k(t)H(t)A_k(s+t)^{-1} \\ & + A_k(s)A_k(t)H(s+t)A_k(s+t)^{-1} \\ \equiv Id + F(s,t) - A_k(s)H(s)A_k(t)A_k(s+t)^{-1} - A_k(s)A_k(t)H(t)A_k(s+t)^{-1} \\ & + A_k(s)A_k(t)H(s+t)A_k(s+t)^{-1} \end{aligned}$$

where $F(s,t) = A_k(s)A_k(t)A_k(s+t)^{-1} - Id$, $F(s,t) \in \Psi^{-k}$ by the induction step. To proceed with the induction we have to find a family $(H(s))_{s\in\mathbb{C}}$ that makes the right hand side of the previous equivalence equal to the identity modulo Ψ^{-k-1} . If $\sigma_{pr}(F(s,t))$ and $h(s) = \sigma_{pr}(H(s))$ are the principal symbols, then the condition on H(s) is equivalent to:

$$\sigma_{pr}(F(s,t)) = \sigma^s h(s)\sigma^{-s} + \sigma^{s+t}h(t)\sigma^{-(s+t)} - \sigma^{s+t}h(s+t)\sigma^{-(s+t)}$$

or

:

$$\sigma^{-(s+t)}\sigma_{pr}(F(s,t))\sigma^{s+t} = \sigma^{-t}h(s)\sigma^t - h(s+t) + h(t).$$
(10)

Set $f(s,t) = \sigma^{-(s+t)}\sigma_{pr}(F(s,t))\sigma^{s+t}$. We show that $f \in C^2(\mathbb{C}; C^{\infty,k}(S^*(G), \mathcal{L}(\mathcal{E})))$ and that $\delta^2 f = 0$. Then h defined in (10) is a 1-cochain with $\delta h = f$.

We also want the second condition of (7) to be satisfied, i.e. $A^{-1}A_{k+1}(1) \equiv Id \pmod{\Psi^{-k-1}}$. We know from the induction step that $(A^{-1}A_k(1) - Id) \in \Psi^{-k}$, and we have

$$A^{-1}A_{k+1}(1) - Id = A^{-1}A_k(1)(Id - H(1)) - Id = (A^{-1}A_k(1) - Id) - A^{-1}A_k(1)H(1).$$

In the last part of the equation above, both terms are operators in Ψ^{-k} , so that $(A^{-1}A_k(1) - Id) - A^{-1}A_k(1)H(1) \in \Psi^{-k-1}$ if and only if its principal symbol is null. This holds if and only if

$$h(1) = \sigma_{pr}(A^{-1}A_k(1) - Id)$$
(11)

We still have to show that f is a cocycle in C^2 . Obviously, f(0,t) = f(s,0) = 0 and we have

$$(\delta^2 f)(s,t,r) = \sigma^{-s} f(t,r) \sigma^s - f(s+t,r) + f(s,t+r) - f(s,t)$$

Recall that $f(s,t) = \sigma^{-(s+t)}\sigma_{pr}(F(s,t))\sigma^{s+t}$, so that $(\delta^2 f)(s,t,r) = 0$ is equivalent to

$$\sigma_{pr}(F(r,t)) - \sigma_{pr}(F(r,s+t)) + \sigma_{pr}(F(t+r,s)) - \sigma^{r}\sigma_{pr}(F(t,s))\sigma^{-r} = 0.$$
(12)

As by definition $F(s,t) \in \Psi^{-k}$, equation (12) is equivalent to the following equation for operators

$$F(r,t) - F(r,s+t) + F(t+r,s) - A_k(r)F(t,s)A_k(r)^{-1} \equiv 0. \quad (\text{mod } \Psi)^{-k}$$

To see that this one holds, consider the following equivalences modulo Ψ^{-k} :

$$(Id + F(r,t))(Id + F(t+r,s))(Id - F(r,s+t))A_k(r)(Id - F(t,s))A_k(r)^{-1} \equiv A_k(r)A_k(t)A_k(t+r)^{-1}A_k(t+r)A_k(s)A_k(s+t+r)^{-1}A_k(s+t+r) \times A_k(s+t)^{-1}A_k(r)^{-1}A_k(r)A_k(s+t)A_k(s)^{-1}A_k(t)^{-1}A_k(r)^{-1} \equiv Id$$

and the first term is also equivalent to

$$Id + F(r,t) - F(r,s+t) + F(t+r,s) - A_k(r)F(t,s)A_k(r)^{-1}$$

which proves (12). So $f(s,t) = \sigma^{-(s+t)}\sigma_{pr}(F(s,t))\sigma^{s+t}$ is a cocycle.

Proposition 5.2.1 provides us with a family h(s) such that $\delta h = f$. We can choose this family so that (11) holds as well. This determines h in a unique way. If $(H(s))_{s \in \mathbb{C}}$ is a holomorphic family of pseudodifferential operators of fixed order -k with principal symbol h(s) constructed as before, $H(s) = \theta_{-k}(h(s))$, then $A_{k+1}(s) = A_k(s)(Id - H(s))$ satisfies the equivalences (7) modulo Ψ^{-k-1} .

In this way we obtain a sequence of families of operators $(A_k(s))_{s\in\mathbb{C}}$ that satisfy the relations (7) for each $k \in \mathbb{N}$. Moreover, $A_{k+1}(s) - A_k(s) \in \Psi^{ms-k}$. Then, by proposition 5.1.4, we know that there exists a holomorphic family family $(A(s))_{s\in\mathbb{C}}$ such that $A(s) \sim$ $A_1(s) + \sum_{k>1} (A_{k+1}(s) - A_k s)$. The family $(A(s))_{s\in\mathbb{C}}$ then satisfies the following properties.

- 1. $\sigma(A(s)) = \sigma^s$.
- 2. $A(s)A(t) \equiv A(s+t)$ modulo smoothing operators.
- 3. $A_1 A$ is a smoothing operator.
- 4. $A(0) \equiv Id$ modulo smoothing operators.

Adding Id - A(0) to A(s) we can impose that A(0) = Id. Moreover, $(A(s))_{s \in \mathbb{C}}$ is unique up to smoothing operators because it must satisfy the relations (7) for all $k \in \mathbb{N}$ and so it must be equal to $(A_k(s))_{s \in \mathbb{C}}$ modulo Ψ^{-k} .

Suppose now that we are given a holomorphic family A(s) as constructed above. We can ask that it satisfies the relation $A^*(\overline{s}) = A(s)$.

Indeed one observes that the family $(A^*(\bar{s}))_{s\in\mathbb{C}}$ fulfills exactly the relations (7) for all $k \in \mathbb{N}$ and so we have that $A^*(\bar{s}) \equiv A(s)$ modulo smoothing operators. Hence the family $\tilde{A}(s) = \frac{1}{2}(A^*(\bar{s}) + A(s))$ satisfies the additional condition and is equal to A(s) modulo smoothing operators. We assume from now on that $A^*(\bar{s}) = A(s)$.

5.3 Second Step : Construction of A_s

We know from the previous section that there exists a holomorphic family A(s) of compactly supported operators such that A(s+t) - A(s)A(t) is a compactly supported smoothing operator, and that A(0) = Id. This implies that there exists an open neighborhood Ω of 0 in \mathbb{C} such that A(s) is invertible if $s \in \Omega$. Indeed we know that A(-s)A(s) = A(0) + R(s) = Id + R(s)where R(s) is a compactly supported smoothing operator going to 0 when s goes to 0. Then we know that (Id + R(s)) is invertible for s in some neighborhood Ω of 0. From the stability under holomorphic functional calculus of $\Psi^{-\infty}(G)$, we can write $(Id+R(s))^{-1} = Id+S(s)$ with $S(s) \in \Psi^{-\infty}(G)$. Hence A(s) is invertible in the algebra of pseudodifferential operators around s = 0, of inverse $A(s)^{-1} = A(-s) + S(s)A(-s)$.

Put $F(s,t) = A(t) - A(s)^{-1}A(s+t)$. By construction, this is a bi-holomorphic smoothing operator (not compactly supported). We are now searching for a holomorphic family of pseudodifferential operators $A_s = A(s)C(s)$, with C(s) - Id smoothing, such that $A_sA_t = A_{s+t}$ for s and t in some neighborhood of 0. If such a $C \in \mathcal{L}(E)$ exists, then, it should fulfill :

$$A(s)C(s)A(t)C(t) = A(s+t)C(s+t)$$
, i.e. $C(s)A(t)C(t) = A(t)C(s+t) - F(s,t)C(s+t)$.

Taking the derivative in s = 0, we obtain $C'(0)A(t)C(t) = A(t)C'(t) - \partial_1 F(0,t)C(t)$. We can suppose that C'(0) = 0, and then we get that for $t \in \Omega$,

$$C'(t) = R(t)C(t)$$
 with $R(t) = A(t)^{-1}\partial_1 F(0, t)$.

Note that R(t) is smoothing, so that this differential equation holds in $\mathcal{L}(E)$. By standard theory of differential equations in Banach spaces, we know it has a unique solution in a neighborhood of 0 with C(0) = Id.

Proposition 5.3.1 — Put C(s) be the solution around 0 of the differential equation C'(t) = R(t)C(t) in $\mathcal{L}(E)$ with $R(t) = A(t)^{-1}\partial_1 F(0,t)$ such that C(0) = Id. Then if $A_s = A(s)C(s)$, we have that $A_sA_t = A_{s+t}$ and $A_s - A(s)$ is smoothing.

Proof— Set $\tilde{C}(t) = C(t) - Id$. Then \tilde{C} is solution of the differential equation X'(t) = R(t)X(t) + R(t). This equation holds in $\mathcal{L}(H^{-\infty}, H^{+\infty})$ and has a unique solution satisfying X(0) = 0, which shows that $\tilde{C}(t)$ is smoothing. Fix $s \in \Omega$ and put $B(t) = A(t)^{-1}A_{s+t}A_s^{-1}$. We have B(t) holomorphic in $\mathcal{L}(E)$, B(t) - Id is smoothing(as C - Id is) and B(0) = Id. If we show that B'(t) = R(t)B(t), we get B(t) = C(t), hence $A_tA_s = A(t)B(t)A_s = A_{s+t}$. We can write $B(t) = [A(s) - F(t, s)]C(s + t)A_s^{-1}$. Differentiating in t = 0, we get

$$B'(t) = -\partial_1 F(t,s)C(s+t)A_s^{-1} + [A(s) - F(t,s)]C'(s+t)A_s^{-1} = [-\partial_1 F(t,s) + A(t)^{-1}\partial_1 F(0,s+t)]C(s+t)A(s)^{-1} + [A(s) - F(t,s)]C'(s+t)A_s^{-1} = [-\partial_1 F(t,s) + A(t)^{-1}\partial_1 F(0,s+t)]C(s+t)A(s)^{-1} + [A(s) - F(t,s)]C'(s+t)A_s^{-1} = [-\partial_1 F(t,s) + A(t)^{-1}\partial_1 F(0,s+t)]C(s+t)A(s)^{-1} + [A(s) - F(t,s)]C'(s+t)A_s^{-1} = [-\partial_1 F(t,s) + A(t)^{-1}\partial_1 F(0,s+t)]C(s+t)A(s)^{-1} + [A(s) - F(t,s)]C'(s+t)A_s^{-1} = [-\partial_1 F(t,s) + A(t)^{-1}\partial_1 F(0,s+t)]C(s+t)A(s)^{-1} + [A(s) - F(t,s)]C'(s+t)A(s)^{-1} + [A(s$$

On the other hand, we have

$$R(t)B(t) = A(t)^{-1}[\partial_1 F(0,t)(A(s) - F(t,s))]C(s+t)A_s^{-1}.$$

Hence B'(t) = R(t)B(t) if and only if

$$-A(t)\partial_1 F(t,s) + \partial_1 F(0,s+t) - \partial_1 F(0,t)A(s) + \partial_1 F(0,t)F(t,s) = 0$$
(13)

To prove this, observe that

On the other hand

$$A(u+t+s) = A(u)A(t+s) - A(u)F(u,t+s) = A(u)A(t)A(s) - A(u)A(t)F(t,s) - A(u)F(u,t+s).$$

Subtracting these equalities, and using the fact that $A(u)^{-1}A(u+t) = A(t) - F(u,t)$, we get

$$-F(u,t)A(s) - A(t)F(u+t,s) + F(u,t)F(u+t,s) + A(t)F(t,s) + F(u,t+s) = 0$$
(14)

Differentiating with respect to u in 0, we get exactly (13), and this ends the proof.

To extend, A_s to $s \in \mathbb{C}$, we simply set $A_s = (A_{\frac{s}{n}})^n$ for some $n \in \mathbb{N}$ big enough. Recall that, in previous section, we have imposed that $A(s) = A(\overline{s})^*$. This implies that $A_s = A_{\overline{s}}^*$. Indeed, set $B(s) = A(s)^{-1}A_{\overline{s}}^*$. We have that B(s) is in $\mathcal{L}(E)$ and holomorphic. Moreover B(0) = Id and

$$A(s)B(s)A(t)B(t) = A_{\overline{s}}^*A_{\overline{t}}^* = A_{\overline{t+s}}^* = A(s+t)B(s+t),$$

hence B(s) satisfies the same differential equation as C(s) and we get B(s) = C(s), and finally

$$A_s = A(s)C(s) = A(s)B(s) = A_{\overline{s}}^*.$$

Last step : The operator $A_s - A^s$ is smoothing 5.4

First of all, recall that $A_1 = A_{1/2}A_{1/2}^*$ is a positive definite elliptic pseudodifferential operator, so we can define its complex powers. We then have

Proposition 5.4.2 — For any $s \in \mathbb{C}$, we have

 $A_1^s = A_s$

as operators in $\mathcal{L}(H^{t+m \Re e s}, H^t)$ for any $t \in \mathbb{R}$

Proof— Using the facts that $A_{p/q} = (A_{1/q})^p$ and that $A_{1/q} = A_1^{\frac{1}{q}}$, we obtain that for any $r \in \mathbb{Q}$, we have $A_1^r = A_r$. This implies in particular that the domains of these operators are equal. But for any $u \in C_c^{\infty,k}(G)$, the map $s \mapsto (A_s - A_1^s)u(\gamma)$ is holomorphic on \mathbb{C} and null on \mathbb{Q} , hence null everywhere. The restriction to $C_c^{\infty,k}(G)$ of A_1^s and A_s are equal, and as they are both regular operators, the domain of $A_1^{s+\overline{s}}$ is a core for A_1^s , while the domain of $A_{s+\overline{s}}$ is a core for A_s . From the equality of the domains of A_1^r and A_r we deduce the fact that the domain of each of these two operators contains a core for the other one. Finally they have same domain $H^{m \Re e s}$ (because A_s is an elliptic operator of order ms), and they are equal as operators in $\mathcal{L}(H^{t+m\Re e_s}, H^t)$ if t = 0. To extend this to any $t \in \mathbb{R}$, we just write $A_s = A_{-t}A_{s+t}$ and $A_1^s = A_1^{-t} A_1^{s+t}$ and use the previous result and duality for Sobolev modules.

It remains to show that $A^s - A_1^s$ is a smoothing operator. But we know that $A - A_1 = R$ is a a smoothing operator, and we can write

$$A^{s} - A_{1}^{s} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{s} [(A - \lambda)^{-1} - (A_{1} - \lambda)^{-1}] d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{s} (A - \lambda)^{-1} (A_{1} - A) (A_{1} - \lambda)^{-1} d\lambda,$$

as $(A - \lambda)^{-1} - (A_1 - \lambda)^{-1} = (A - \lambda)^{-1}(A_1 - A)(A_1 - \lambda)^{-1}$. The operator $(A - \lambda)^{-1}(A_1 - A)(A_1 - \lambda)^{-1}$ is smoothing and its Sobolev norm $\|(A - \lambda)^{-1}(A_1 - A)(A_1 - \lambda)^{-1}\|_{t,r}$ is bounded independently from λ for any real numbers r, t so if $\Re e s < -1$, the integral converges to a smoothing operator. Then if $\Re e s < -1$, $A^s - A_1^s$ is a smoothing operator. But we have, for any integer n that $A^n - A_1^n$ is a smoothing operator, so $A^{s+n} - A_1^{s+n} = A^n(A^s - A_1^s) + (A^n - A_1^n)A_1^s$ is a smoothing operator. This ends the proof of theorem 5.0.1.

REMARK — We have constructed the complex powers of a positive pseudodifferential operator in our framework of generalized pseudodifferential calculus which is quite a wide algebra, as we already mentioned. Now suppose we are given an algebra $\mathcal{P}^{-\infty}(G)$ with the following properties.

- The algebra $\mathcal{P}^{-\infty}(G)$ is a sub- $\Psi^{-\infty}(G)$ -algebra, containing $\Psi_{c,k}^{-\infty}(G)$.
- The algebra $\mathcal{P}^{-\infty}(G)$ is stable under holomorphic functional calculus.
- For all P, Q compactly supported pseudodifferential operators and $R \in \mathcal{P}^{-\infty}(G)$, then $PQR \in \mathcal{P}^{-\infty}(G)$.
- The algebra $\mathcal{P}^{-\infty}(G)$ has a Frechet topology, stronger than the topology of $\Psi^{-\infty}(G)$.

Then we can define a "smaller" generalized pseudodifferential calculus on G, by saying that a pseudodifferential operator is the sum of a compactly supported pseudodifferential operator and of an element of $\mathcal{P}^{-\infty}(G)$. The conditions listed above then show that the complex powers of an operator lie in this smaller algebra of pseudodifferential. This can be useful in applications as one can be interested in having better regularity conditions on the smoothing operators involved. Such smaller algebras of smoothing operators, as the Schwartz algebra, appear for example in the work of Lauter, Monthubert and Nistor [22] or in the work of Lafforgue [20].

6 Application to the foliated case

We briefly recall how to recover previous results of Connes [8, 7] and Kordyukov [18] on unbounded pseudodifferential calculus on smooth compact foliations. In his work [8, 7], Connes considered G-pseudodifferential operators on the reduced C*-algebra of the foliation, $C_r^*(G)$, and analyzed the operators as acting on Sobolev spaces that are defined in the ordinary way from the Hilbert space $L^2(G) = \bigoplus_x L^2(G_x, \lambda_x)$. In the work of Kordyukov, operators are acting in the global space $L^2(M)$ and they can be defined as G-operators acting on the full C^{*}-algebra of the foliation $C^*(G)$. Both cases can be seen as particular cases of the results above, by composition of our Hilbert module formulation with a representation of the considered C^{*}-algebra of the foliation. To recover results of Connes on complex powers or Sobolev spaces, it suffices to use the left-regular representation of $C_r^*(G)$ in $L^2(G) = \bigoplus_x L^2(G_x, \lambda_x)$, whereas we recover the complex powers and Sobolev spaces of Kordyukov using the left-regular representation of $C^*(G)$ on $L^2(M)$. To illustrate this machinery, let us give an example of a new proof of the fact that a longitudinal elliptic operator which is formally self-adjoint defines a self adjoint operator in $L^2(G_x)$. Suppose we are given such an elliptic longitudinal operator $P = (P_x)$ on the foliation. The restriction $P_{x|D_x}$ of the operator P_x to $D_x = C_c^{\infty}(G_x)$ can be considered as an unbounded linear operator on $L^2(G_x)$.

Proposition 6.0.1 — [8] The operator $P_{x|D_x}$ is closable and the domain of its closure is maximal.

Proof— Let π_x be the left regular representation of the C*-algebra $C_r^*(M, \mathcal{F})$ in $L^2(G_x, \lambda_x)$. This representation π_x is non-degenerate and the operator \overline{P} is regular on $E_r = C_r^*(G)$. We can then apply proposition 2.3.3 to the operator \overline{P} : There exists an operator $P_{0,x}$ from $E_r \otimes_{\pi_x} L^2(G_x, \lambda_x) = L^2(G_x, \lambda_x)$ whose domain is the image of the algebraic tensor product $\operatorname{Dom}\overline{P} \otimes D_x$ which is closable, and whose closure $\overline{P} \otimes_{\pi_x} 1$ is a regular operator. Moreover, we know that for any $f \in \operatorname{Dom}\overline{P}, \xi \in D_x$, we have

$$\overline{P} \otimes_{\pi_x} 1(f \otimes \xi) = P_{0,x}(f \otimes \xi) = \overline{P}(f) \otimes \xi.$$

The operator $\overline{P} \otimes_{\pi_x} 1$ then coincides with the operator P_x on $D \otimes D_x$, with $D = C_c^{\infty,k}(G)$, and so defines a closed extension of P_x .

It remains to show that $\overline{P_x} = \overline{P} \otimes_{\pi_x} 1$. It suffices to prove that the algebraic tensor product $D \otimes D_x$ is a core for $\overline{P} \otimes_{\pi_x} 1$. But D is a core for \overline{P} , and from proposition 2.3.3, we know that the image of the algebraic tensor product $D \otimes D_x$ in $E_r \otimes_{\pi_x} L^2(G_x, \lambda_x)$ is a core for P_x .

These techniques can also be used to show spectral properties for operators, in more precise geometric situations, using the results of Fack and Skandalis [10, 12, 11]. Suppose now that the foliation is minimal (i.e. all leaves are dense) and that its holonomy groupoid is Hausdorff, then by [10, 12, 11] we know that $C_r^*(G)$ is simple.

- If G is amenable, the maximal and reduced C^{*}-algebras coincide and the spectrum of a regular operator is the same in any representation associated to the $C^*(G)$. In particular the spectrum as an operator in $L^2(G_x)$ and the spectrum as an operator in $L^2(M)$ coincide.
- If the C*-algebra has no projections, a positive definite elliptic operator has no gap in its spectrum. Indeed, by functional calculus, a gap in the spectrum gives rise to a projection in the C*-algebra. What is interesting is that this result remains valid in any faithful representation of the C*-algebra. For example, the foliation induced by the horocyclic flow on the quotient $V = SL(2, \mathbb{R})/\Gamma$, with Γ a discrete cocompact subgroup of $SL(2, \mathbb{R})$, and defined by the left action of the subgroup of lower triangular matrices of the form $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ with $t \in \mathbb{R}$ is a minimal foliation and it can be shown that its C*-algebra has no non trivial projections(cf. [9], p135). Hence this gives a connectivity result for the spectrum of positive elliptic operators on the foliation viewed as operators on each leaf. Let *i* denote the injection of \mathbb{R} in *V* as a generic leaf. The preceding shows in particular a connectivity result for the spectrum of Schrödinger operators of the form $-\frac{d^2}{dx^2} + V$ on $L^2(\mathbb{R})$ for potentials *V* of the form $V = f \circ i$, where *f* is a continuous positive (or even real valued) continuous function *f* on *V*.

A A proof of proposition 3.2.4

From classical theory (see [37]), it is easily seen that all these properties are true if the order is constant. So we only have to prove that they respect the topology when m is not constant.

Moreover since differentiating with respect to $x \in M$ does not change the estimates in $\mathcal{S}^0_{hom,k}(U)$, it suffices to show continuity of these operations. Recall that two types of continuity should be considered : the one of the homogeneous parts of the symbol, seen as maps on $C^{\infty,k}(S^*U \times M)$ and the other for the topology of semi-norms for (1,0)-symbols in \mathcal{S}^{m_0} , with m_0 the real part of m.

• From classical theory we know that the homogeneous parts of the symbol are given in all three cases by formulas which involve only a finite number of homogeneous symbols and their derivatives, so that if these homogeneous symbols are $C^{\infty,k}$ maps, then so is the resulting symbol. To give an explicit example, composing two symbols a and b and writing $c = a \star b$, we have that

$$c_{m(x)+n(x)-p}(u,\xi,x) = \sum_{j+l+|\alpha|=p} \frac{\partial_{\xi}^{\alpha} a_{m(x)-j}(u,\xi,x) \partial_{u}^{\alpha} b_{n(x)-l}(u,\xi,x)}{\alpha!}$$

So $c_{m(x)+n(x)-p}(u,\xi,x)$ is in $C^{\infty,k}(S^*U \times M)$ for all $p \in \mathbb{N}$ since for all j and l, $a_{m(x)-j}$ and $b_{n(x)-l}$ belong to $C^{\infty,k}(S^*U \times M)$.

• It remains to show the continuity for the (1, 0)-symbols topology, so we can suppose that m(.) is real. We use the classical theory of amplitudes associated with pseudodifferential operators (see [37]). In the same way as we defined $S_{hom,k}^{m(.)}(U \times M)$, we define the space $S_k^{m(.)}(U \times U \times M)$ of amplitudes compactly supported in $U \times U$ and depending in a C^k -manner of a parameter $x \in M$.

From the formula linking compactly supported amplitudes to compactly supported symbols

$$\sigma(u,\xi,x) = \int \int a(u,u',\eta,x) e^{-i\langle u-u',\eta-\xi\rangle} du' d\eta$$
(15)

we deduce the fact that if $\chi(\xi) \|\xi\|^{-m(x)} a(u, u', \xi, x)$ is a C^k -map from M to $\mathcal{S}_k^0(U \times U \times M)$, then $\chi(\xi) \|\xi\|^{-m(x)} a(u, \xi, x)$ is a C^k -map from M to $\mathcal{S}_k^0(U \times M)$. Indeed, the classical formula

$$\sigma(u,\xi,x) \sim \sum_{\alpha} (-i)^{|\alpha|} \partial_{\xi}^{\alpha} \partial_{u'}^{\alpha} a(u,u,\xi,x)$$

shows that the homogeneous components of σ are C^k -maps if homogeneous components of a are. To show the continuity for the (1, 0)-topology, we can as well consider operators of arbitrary negative order since we may subtract the homogeneous components. Then the estimates on a and its derivatives with respect to η, u, x imply the estimates for $\chi(\xi) \|\xi\|^{-m(x)} \sigma(u, \xi, x)$ since the formula 15 is an absolutely convergent integral for sufficiently negative order.

This proves that the operators associated in the usual way to the C^k -amplitudes are exactly the C^k -families of (1,0)-pseudodifferential operators. This ends the proof for the first property since an amplitude for A^* is given by $\overline{a(u', u, -\xi, x)}$ when $a(u, u', \xi, x)$ is an amplitude for A.

To prove the property for composition of operators, we just use a classical trick : if $a(u,\xi,x)$ is an amplitude for A, not depending on u' and $b(u',\xi,x)$ an amplitude for B, not depending on u, then $c(u, u', \xi, x) = a(u, \xi, x)b(u', \xi, x)$ is an amplitude for AB, and hence satisfies the required estimates since a and b do.

Finally, for the change of coordinates one needs to prove directly the estimates, the way through amplitudes being more complicated than the direct one. We know from classical computations that

$$a_{\kappa}(u,\eta,x) = a(\kappa^{-1}(u), {}^{t}J^{-1}(u)\eta,x)$$

where J is the Jacobian matrix associated with κ^{-1} . Set $\xi = {}^{t}J^{-1}(u)\eta$. Then we have

$$\|\eta\|^{-m(x)} a_{\kappa}(u,\eta,x) = \left(\frac{\|\mathcal{U}(u)\xi\|}{\|\xi\|}\right)^{m(x)} \|\xi\|^{-m(x)} a(\kappa^{-1}(u),\mathcal{U}^{-1}(u)\eta,x).$$

Hence, from estimates on a we get estimates on a_{κ} , using the fact that if u and x vary in compact sets, then the expression $||^{t}J(u)||^{-m(x)}$ and its derivatives are bounded. Indeed derivation with respect to η or x does not affect the estimates, while derivation with respect to u introduces multiplication by η which is counterbalanced by the fact that derivatives of a with respect to the second variable lower the order of the estimates in $\xi = {}^{t}J^{-1}(u)\eta$. Anyway these estimates remain uniform with respect to x (which is the new thing here).

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