arXiv:1808.05026v1 [math.RA] 15 Aug 2018

GRÖBNER-SHIRSHOV BASES FOR TEMPERLEY-LIEB ALGEBRAS OF TYPES B AND D

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Abstract. For Temperley-Lieb algebras of types B and D, we construct their Gröbner-Shirshov bases and the corresponding standard monomials.

1. Introduction

Originally, the Temperley-Lieb algebra appears in the context of statistical mechanics [15], and later its structure has been studied in connection with knot theory, where it is known to be a quotient of the Hecke algebra of type A [7].

Our method for understanding the structure of Temperley-Lieb algebras is from the non-commutative Gröbner basis theory, called the *Gröbner-Shirshov basis theory*, which provides a powerful tool for understanding the structure of (non)associative algebras and their representations, especially in computational aspects. With the ever-growing power of computers, it is now viewed as a universal engine behind algebraic or symbolic computation.

The main interest of the notion of Gröbner-Shirshov bases stems from Shirshov's Composition Lemma and his algorithm [11, 12] for Lie algebras and independently from Buchberger's algorithm [4] of computing Gröbner bases for commutative algebras. In [2], Bokut applied Shirshov's method to associative algebras, and Bergman mentioned the diamond lemma for ring theory [1].

The Gröbner-Shirshov bases for Coxeter groups of classical and exceptional types were completely determined in [3, 9, 10, 14]. The cases for Hecke algebras and Temperley-Lieb algebras of type A were calculated in [8].

In this paper, we deal with Temperley-Lieb algebras of types B and D, extending the result in [8, §6]. By completing the relations coming from a presentation of the Temperley-Lieb algebra, we compute its Gröbner-Shirshov basis to obtain the corresponding set of standard monomials.

Our approach gives the following interesting properties :

1. The number of standard monomials is the dimension of the Temperley-Lieb algebra, computed in the work of Fan and Stembridge [5, 13].

2. The standard monomials reside themselves inside the Temperley-Lieb algebra. In this sense, the standard monomials could be more interesting than the fully commutative elements.

3. The product of two standard monomials becomes a standard monomial up to a scalar multiple. We give some examples for each type in the following sections.

2010 *Mathematics Subject Classification.* Primary 20F55, Secondary 05E15, 16Z05.

Key words and phrases. Temperley-Lieb algebra, Gröbner-Shirshov basis.

[1] She is grateful to KIAS for its hospitality during this work.

[2] This research was supported by NRF Grant # 2014R1A1A2054811 and a research grant from Seoul Women's University(2017).

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2. Preliminaries

In this section, we recall a basic theory of *Gröbner-Shirshov bases* for associative algebras so as to make the paper self-contained. There will be some properties listed without proofs which are well-known and necessary for this paper.

Let X be a set and let $\langle X\rangle$ be the free monoid of associative words on X. We denote the empty word by 1 and the *length* (or *degree*) of a word u by $l(u)$. We define a total-order $<$ on $\langle X\rangle$, called a *monomial order* as follows ;

$$\textit{if } x<y \textit{ implies } axb<ayb \textit{ for all } a,b\in\langle X\rangle.$$

Fix a monomial order $<$ on $\langle X\rangle$ and let $\mathbb{F}\langle X\rangle$ be the free associative algebra generated by X over a field $\mathbb{F}$. Given a nonzero element $p\in\mathbb{F}\langle X\rangle$, we denote by $\overline{p}$ the monomial (called the *leading monomial*) appearing in p, which is maximal under the ordering $<$. Thus $p=\alpha\overline{p}+\sum\beta_i w_i$ with $\alpha,\beta_i\in\mathbb{F}$, $w_i\in\langle X\rangle$, $\alpha\neq 0$ and $w_i<\overline{p}$ for all i. If $\alpha=1$, p is said to be *monic*.

Let S be a subset of monic elements in $\mathbb{F}\langle X\rangle$, and let I be the two-sided ideal of $\mathbb{F}\langle X\rangle$ generated by S. Then we say that the algebra $A=\mathbb{F}\langle X\rangle/I$ is *defined by* S.

Definition 2.1. Given a subset S of monic elements in $\mathbb{F}\langle X\rangle$, a monomial $u\in\langle X\rangle$ is said to be *S-standard* (or *S-reduced*) if $u\neq a\overline{s}b$ for any $s\in S$ and $a,b\in\langle X\rangle$. Otherwise, the monomial u is said to be *S-reducible*.

Lemma 2.2 ([1, 2])**.** *Every $p\in\mathbb{F}\langle X\rangle$ can be expressed as*

$$p=\sum\alpha_i a_i s_i b_i+\sum\beta_j u_j, \tag{2.1}$$

where $\alpha_i,\beta_j\in\mathbb{F}$, $a_i,b_i,u_j\in\langle X\rangle$, $s_i\in S$, $a_i\overline{s_i}b_i\leq\overline{p}$, $u_j\leq\overline{p}$ and u_j are S-standard.

Remark. The term $\sum\beta_j u_j$ in the expression (2.1) is called a *normal form* (or a *remainder*) of p with respect to the subset S (and with respect to the monomial order $<$). In general, a normal form is not unique.

As an immediate corollary of Lemma 2.2, we obtain:

Proposition 2.3. *The set of S-standard monomials spans the algebra $A=\mathbb{F}\langle X\rangle/I$ defined by the subset S, as a vector space over $\mathbb{F}$.*

Let p and q be monic elements in $\mathbb{F}\langle X\rangle$ with leading monomials $\overline{p}$ and $\overline{q}$. We define the *composition* of p and q as follows.

Definition 2.4. (a) If there exist a and b in $\langle X\rangle$ such that $\overline{p}a=b\overline{q}=w$ with $l(\overline{p})>l(b)$, then the *composition of intersection* is defined to be $(p,q)_w=pa-bq$.

(b) If there exist a and b in $\langle X\rangle$ such that $a\neq 1$, $a\overline{p}b=\overline{q}=w$, then the *composition of inclusion* is defined to be $(p,q)_{a,b}=apb-q$.

Let $p,q\in\mathbb{F}\langle X\rangle$ and $w\in\langle X\rangle$. We define the *congruence relation* on $\mathbb{F}\langle X\rangle$ as follows: $p\equiv q \mod (S;w)$ if and only if $p-q=\sum\alpha_i a_i s_i b_i$, where $\alpha_i\in\mathbb{F}$, $a_i,b_i\in\langle X\rangle$, $s_i\in S$, $a_i\overline{s_i}b_i<w$.

Definition 2.5. A subset S of monic elements in $\mathbb{F}\langle X\rangle$ is said to be *closed under composition* if $(p,q)_w\equiv 0 \mod (S;w)$ and $(p,q)_{a,b}\equiv 0 \mod (S;w)$ for all $p,q\in S$, $a,b\in\langle X\rangle$ whenever the compositions $(p,q)_w$ and $(p,q)_{a,b}$ are defined.

The following theorem is a main tool for our results in the subsequent sections.

Theorem 2.6 ([1, 2])**.** *Let S be a subset of monic elements in $\mathbb{F}\langle X\rangle$. Then the following conditions are equivalent:*

(a) *S is closed under composition.*
(b) *For each $p \in \mathbb{F}\langle X\rangle$, a normal form of p with respect to S is unique.*
(c) *The set of S-standard monomials forms a linear basis of the algebra $A = \mathbb{F}\langle X\rangle/I$ defined by S.*

Definition 2.7. A subset S of monic elements in $\mathbb{F}\langle X\rangle$ is a *Gröbner-Shirshov basis* if S satisfies one of the equivalent conditions in Theorem 2.6. In this case, we say that S is a *Gröbner-Shirshov basis* for the algebra A defined by S.

3. Review of results for the Temperley-Lieb algebra of type A_{n-1}

First, we review the results on Temperley-Lieb algebras $\mathcal{T}(A_{n-1})$ $(n \geq 2)$. Define $\mathcal{T}(A_{n-1})$ to be the associative algebra over the complex field $\mathbb{C}$, generated by $X = \{E_1, E_2, \ldots, E_{n-1}\}$ with defining relations:

$$R_{\mathcal{T}(A_{n-1})}: \quad \begin{aligned} E_i^2 &= \delta E_i && \text{for } 1 \leq i \leq n-1, \\ E_iE_j &= E_jE_i && \text{for } i > j+1 \quad \text{(commutative relations)}, \\ E_iE_jE_i &= E_i && \text{for } j = i \pm 1, \end{aligned}$$

where $\delta \in \mathbb{C}$ is a parameter. Our monomial order $<$ is taken to be the degree-lexicographic order with

$$E_1 < E_2 < \cdots < E_{n-1}.$$

We write $E_{i,j} = E_iE_{i-1}\cdots E_j$ for $i \geq j$ (hence $E_{i,i} = E_i$). By convention $E_{i,i+1} = 1$ for $i \geq 1$.

Proposition 3.1. ([8, Proposition 6.2]) *The Temperley-Lieb algebra $\mathcal{T}(A_{n-1})$ has a Gröbner-Shirshov basis as follows:*

$$\widehat{R}_{\mathcal{T}(A_{n-1})}: \quad \begin{aligned} &E_i^2 - \delta E_i && \text{for } 1 \leq i \leq n-1, \\ &E_iE_j - E_jE_i && \text{for } i > j+1, \\ &E_{i,j}E_i - E_{i-2,j}E_i && \text{for } i > j, \\ &E_jE_{i,j} - E_jE_{i,j+2} && \text{for } i > j. \end{aligned} \tag{3.1}$$

The corresponding $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard monomials are of the form

$$E_{i_1,j_1}E_{i_2,j_2}\cdots E_{i_p,j_p} \quad (0 \leq p \leq n-1) \tag{3.2}$$

where

$$\begin{aligned} &1 \leq i_1 < i_2 < \cdots < i_p \leq n-1, \quad 1 \leq j_1 < j_2 < \cdots < j_p \leq n-1, \\ &i_1 \geq j_1, \ i_2 \geq j_2, \ \ldots, \ i_p \geq j_p \end{aligned}$$

(the case of $p = 0$ is the monomial 1). We denote the set of $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard monomials by $M_{\mathcal{T}(A_{n-1})}$ and the number $|M_{\mathcal{T}(A_{n-1})}|$ of $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard monomials is the n^{th} Catalan number,

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

Example 3.2. Note that $|M_{\mathcal{T}(A_3)}| = C_4 = 14$. Explicitly, the $\widehat{R}_{\mathcal{T}(A_3)}$-standard monomials are as follows:

$$1, E_1, E_{2,1}, E_2, E_1E_2, E_{3,1}, E_{3,2}, E_3,$$
$$E_1E_{3,2}, E_1E_3, E_{2,1}E_{3,2}, E_{2,1}E_3, E_2E_3, E_1E_2E_3.$$

Remark. (1) One interesting point of considering standard monomials is that the product of two standard monomials becomes a standard monomial up to a scalar multiple. As an example, if we multiply E_1E_2 by $E_{2,1}E_{3,2}$ in the previous example then we obtain

$$(E_1E_2)(E_{2,1}E_{3,2}) = \delta E_1E_2E_1E_{3,2} = \delta E_1E_{3,2},$$

a multiple of another standard monomial $E_1E_{3,2}$. For another one, the multiplication of $E_{2,1}$ by $E_{3,1}$ leads us to have

$$E_{2,1}E_{3,1} = E_2(E_1E_{3,1}) = E_2(E_1E_3) = E_{2,1}E_3$$

by the Gröbner-Shirshov basis (3.1).

(2) One can also notice that the number of standard monomials equals the number of fully commutative elements, which is the dimension of the Temperley-Lieb algebra of type A.

In the following sections, keeping the same strategy and notations, we give analogous results for the Temperley-Lieb algebra of types B and D.

4. Gröbner-Shirshov bases for the Temperley-Lieb algebras of type B_n

Let $\mathcal{T}(B_n)$ $(n \geq 2)$ be the Temperley-Lieb algebra of type B_n, that is, the associative algebra over the complex field $\mathbb{C}$, generated by $X = \{E_0, E_1, \ldots, E_{n-1}\}$ with defining relations:

$$(4.1) \qquad R_{\mathcal{T}(B_n)}: \quad \begin{aligned} E_i^2 &= \delta E_i && \text{for } 0 \leq i \leq n-1, \\ E_iE_j &= E_jE_i && \text{for } i > j+1, \\ E_iE_jE_i &= E_i && \text{for } j = i \pm 1,\ i, j > 0, \\ E_iE_jE_iE_j &= 2E_iE_j && \text{for } \{i, j\} = \{0, 1\}, \end{aligned}$$

where $\delta \in \mathbb{C}$ is a parameter. .

Fix our monomial order $<$ to be the degree-lexicographic order with

$$E_0 < E_1 < \cdots < E_{n-1}.$$

We write $E_{i,j} = E_iE_{i-1}\cdots E_j$ for $i \geq j \geq 0$, and $E^{i,j} = E_iE_{i+1}\cdots E_j$ for $i \leq j$. By convention, $E_{i,i+1} = 1$ and $E^{i+1,i} = 1$ for $i \geq 0$.

Lemma 4.1. *The following relation holds in* $\mathcal{T}(B_n)$*:*

$$E_{i,0}E^{1,j}E_i = E_{i-2,0}E^{1,j}E_i$$

for $i > j+1 \geq 1$*.*

Proof. Since $2 \leq i \leq n-1$ and $0 \leq j \leq i-2$, we calculate that

$$E_{i,0}E^{1,j}E_i = (E_iE_{i-1}E_i)E_{i-2,0}E^{1,j} = E_iE_{i-2,0}E^{1,j} = E_{i-2,0}E^{1,j}E_i$$

by the commutative relations and $E_iE_{i-1}E_i = E_i$. □

Let $\widehat{R}_{\mathcal{T}(B_n)}$ be the set of defining relations (4.1) combined with (3.1) and the relation in Lemma 4.1. From this, we define $M_{\mathcal{T}(B_n)}$ by the set of $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials. Among the monomials in $M_{\mathcal{T}(B_n)}$, we consider the monomials which are not $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard. That is, we take only $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials which are not of the form (3.2). This set is denoted by $M^0_{\mathcal{T}(B_n)}$. Note that each monomial in $M^0_{\mathcal{T}(B_n)}$ contains E_0. We decompose the set $M^0_{\mathcal{T}(B_n)}$ into two parts as follows :

$$M^0_{\mathcal{T}(B_n)} = M^{0+}_{\mathcal{T}(B_n)} \amalg M^{0-}_{\mathcal{T}(B_n)}$$

where the monomials in $M^{0+}_{\mathcal{T}(B_n)}$ are of the form

$$E_0 E_{i_1,j_1} E_{i_2,j_2} \cdots E_{i_p,j_p} \quad (0 \le p \le n-1) \tag{4.2}$$

with

$$\begin{aligned} &1 \le i_1 < i_2 < \cdots < i_p \le n-1, \quad 0 \le j_1 \le j_2 \le \cdots \le j_p \le n-1, \\ &i_1 \ge j_1,\ i_2 \ge j_2,\ \ldots,\ i_p \ge j_p, \ \text{ and} \\ &j_k > 0\ (1 \le k < p) \ \text{ implies } j_k < j_{k+1} \end{aligned}$$

(the case of $p = 0$ is the monomial E_0), and the monomials in $M^{0-}_{\mathcal{T}(B_n)}$ are of the form

$$E'_{i_1,j_1} E_{i_2,j_2} \cdots E_{i_p,j_p} \quad (1 \le p \le n-1)$$

with

$$E'_{i,j} = E_{i,0} E^{1,j}$$

and the same restriction on i's and j's as above. It can be easily checked that $M^0_{\mathcal{T}(B_n)}$ is the set of $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials which are not $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard.

To each monomial $E_0 E_{i_1,0} E_{i_2,0} \cdots E_{i_k,0} E_{i_{k+1},j_{k+1}} \cdots E_{i_p,j_p}$ in $M^{0+}_{\mathcal{T}(B_n)}$ with $j_{k+1} > 0$, we can associate a unique path

$$(0,0) \to (i_1, 0) \to (i_2, 0) \to \cdots \to (i_k, 0) \to (i_{k+1}, j_{k+1}) \to \cdots \to (i_p, j_p) \to (n, n).$$

Here, a path consists of moves to the east or to the north, not above the diagonal in the lattice plane. The move from (i,j) to (i',j') ($i < i'$ and $j < j'$) is a concatenation of eastern moves followed by northern moves. As an example, the monomial $E_0 E_{1,0} E_{2,1} \in M^{0+}_{\mathcal{T}(B_3)}$ corresponds to

$$(0,0) \to (1,0) \to (2,1) \to (3,3).$$

Counting the number of elements in $M^0_{\mathcal{T}(B_n)}$, we obtain the following theorem.

Theorem 4.2. *The algebra $\mathcal{T}(B_n)$ has a Gröbner-Shirshov basis $\widehat{R}_{\mathcal{T}(B_n)}$ with respect to our monomial order $<$:*

$$\widehat{R}_{\mathcal{T}(B_n)}: \quad \begin{array}{ll} E_i^2 - \delta E_i & \text{for } 0 \le i \le n-1, \\ E_i E_j - E_j E_i & \text{for } i > j+1, \\ E_{i,j} E_i - E_{i-2,j} E_i & \text{for } i > j > 0, \\ E_j E_{i,j} - E_j E_{i,j+2} & \text{for } i > j > 0. \\ E_i E_j E_i E_j - 2 E_i E_j & \text{for } \{i,j\} = \{0,1\}, \\ E_{i,0} E^{1,j} E_i - E_{i-2,0} E^{1,j} E_i & \text{for } i > j+1 \ge 1. \end{array}$$

The cardinality of the set $M_{\mathcal{T}(B_n)}$, i.e. the set of $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials, is

$$\dim \mathcal{T}(B_n) = (n+2) C_n - 1.$$

Proof. First, we consider a mapping

$$\phi : M^{0+}_{\mathcal{T}(B_n)} \setminus \{E_0\} \to M^{0-}_{\mathcal{T}(B_n)}$$

defined by $\phi(E_0 E_{i_1,j_1} E_{i_2,j_2} \cdots E_{i_p,j_p}) = E'_{i_1,j_1} E_{i_2,j_2} \cdots E_{i_p,j_p}$. Then this map is a bijection. In order to compute $|M^0_{\mathcal{T}(B_n)}|$, it is enough to count the the number of elements in $M^{0+}_{\mathcal{T}(B_n)}$. For this, we consider the following procedure.

In the lattice plane, we plot the sequence of points $(i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p)$ corresponding to the monomial $E_0E_{i_1,j_1}E_{i_2,j_2}\cdots E_{i_p,j_p}$ in (4.2). Set $\ell > 0$ to be the largest i such that $(i, 0)$ belongs to the sequence of plotted points. Then the number of sequences of plotted points between $(\ell, 0)$ and (n, n) is the number of paths from $(\ell + 1, 0)$ and (n, n) .

Counting the number of these paths, we have

$$\binom{2n-\ell-1}{n} - \binom{2n-\ell-1}{n+1} = \frac{\ell+2}{n+1}\binom{2n-\ell-1}{n}.$$

Thus the number of monomials of the form $E_0E_{i_1,0}\cdots E_{i_p,j_p}$ (4.2) is

$$\sum_{\ell=1}^{n-1} \frac{\ell+2}{n+1}\binom{2n-\ell-1}{n}2^{\ell-1},$$

which is the same quantity as $\frac{1}{2}\left(\sum_{k=0}^{n-2} C(n,k)|\mathcal{P}_B(n,k)| + 1\right) = \frac{n-1}{2}C_n$, as in [6, Corollary 2.14].

Therefore we have

$$|M^{0+}_{\mathcal{T}(B_n)}| = C_n + \tfrac{n-1}{2}C_n = \tfrac{n+1}{2}C_n.$$

Then, the number of $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials becomes

$$|M_{\mathcal{T}(A_{n-1})}| + 1 + 2|M^{0+}_{\mathcal{T}(B_n)} \setminus \{E_0\}| = C_n + 1 + 2\left(\tfrac{n+1}{2}C_n - 1\right),$$

which gives exactly the number equal to

$$\dim \mathcal{T}(B_n) = (n+2)C_n - 1$$

as mentioned in [13, §5] and [5, §7]. Theorem 2.6 yields that $\widehat{R}_{\mathcal{T}(B_n)}$ is a Gröbner-Shirshov basis for $\mathcal{T}(B_n)$. □

Example 4.3. (1) We enumerate the $\widehat{R}_{\mathcal{T}(B_3)}$-standard monomials containing E_0:

$$E_0,\ E_0E_{1,0}, E_{1,0},\ E_0E_1, E_1',\ E_0E_{2,0}, E_{2,0},\ E_0E_{2,1}, E_{2,1}',\ E_0E_2, E_2',$$
$$E_0E_{1,0}E_{2,0}, E_{1,0}E_{2,0}, E_0E_{1,0}E_{2,1}, E_{1,0}E_{2,1}, E_0E_{1,0}E_2, E_{1,0}E_2, E_0E_1E_2, E_1'E_2.$$

(2) The product of two $\widehat{R}_{\mathcal{T}(B_3)}$-standard monomials is a scalar multiple of a standard monomial. For instance, we multiply $E_0E_{1,0}E_{2,0}$ by E_2 from the left:

$$E_2(E_0E_{1,0}E_{2,0}) = E_0E_{2,0}E_{2,0} = E_0E_0E_2E_{1,0} = \delta E_0E_{2,0}.$$

Note that the second equality comes from the Lemma 4.1.

5. Gröbner-Shirshov bases for the Temperley-Lieb algebras of type D_n

Now we consider $\mathcal{T}(D_n)$ $(n \geq 4)$, the Temperley-Lieb algebra of type D_n, which is the associative algebra over the complex field $\mathbb{C}$, generated by $X = \{E_0, E_1, E_2, \ldots, E_{n-1}\}$ with defining relations:

$$R_{\mathcal{T}(D_n)}: \begin{array}{rl} E_i^2 = \delta E_i & \text{for } 0 \leq i \leq n-1, \\ E_iE_j = E_jE_i & \text{for } 1 < j+1 < i \leq n-1, \\ E_iE_0 = E_0E_i & \text{for } i \neq 2 \\ E_iE_jE_i = E_i & \text{for } j = i \pm 1,\ i,j > 0, \\ E_iE_jE_i = E_i & \text{for } \{i,j\} = \{0,2\}, \end{array} \tag{5.1}$$

where $\delta \in \mathbb{C}$ is a parameter.

Take the degree-lexicographic monomial order $<$ with

$$E_0 < E_1 < E_2 < \cdots < E_{n-1}.$$

We write $E_{i,j} = E_i E_{i-1} \cdots E_{j+1} E_j$ for $i \geq j > 0$, and let $E_{i,0} = E_i E_{i-1} \cdots E_3 E_2 E_0$ for $i \geq 2$, and $E^{i,j} = E_i E_{i+1} \cdots E_j$ for $1 \leq i \leq j$. By convention, $E_{1,0} = E_0$ and $E_{i,i+1} = 1$ for $i \geq 0$.

Lemma 5.1. *The following relations hold in $\mathcal{T}(D_n)$.*

(a) *For $i > 2$, we have*

$$E_{i,0} E_i = E_{i-2,0} E_i.$$

(b) *For $i \geq 2$, we have*

$$E_0 E_{i,0} = E_0 E_{i,3}.$$

(c) *For $i \geq 2$, we have*

$$E_0 E_1 E_{i,0} = E_0 E_1 E_{i,3}.$$

(d) *For $i > j + 1 > 1$, we have*

$$E_{i,0} E^{1,j} E_i = E_{i-2,0} E^{1,j} E_i.$$

Proof. (a) By the commutative relations and $E_i E_{i-1} E_i = E_i$, we calculate that

$$E_{i,0} E_i = (E_i E_{i-1} E_i) E_{i-2,0} = E_{i-2,0} E_i.$$

(b) It follows that

$$E_0 E_{i,0} = E_{i,3}(E_0 E_2 E_0) = E_0 E_{i,3}$$

from the commutative relations and $E_0 E_2 E_0 = E_0$.

(c) In the same way as the previous relation,

$$E_0 E_1 E_{i,0} = E_1 E_{i,3}(E_0 E_2 E_0) = E_0 E_1 E_{i,3}.$$

(d) We get that

$$E_{i,0} E^{1,j} E_i = (E_i E_{i-1} E_i) E_{i-2,0} E^{1,j} = E_{i-2,0} E^{1,j} E_i,$$

as desired. □

We let $\widehat{R}_{\mathcal{T}(D_n)}$ be the set of defining relations (5.1) combined with (3.1) and the relations in Lemma 5.1. The set $M_{\mathcal{T}(D_n)}$ is defined to be the collection of $\widehat{R}_{\mathcal{T}(D_n)}$-standard monomials. Among the monomials in $M_{\mathcal{T}(D_n)}$, we consider the set of monomials which are not $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard and denote this set by $M^0_{\mathcal{T}(D_n)}$. That is, we take only $\widehat{R}_{\mathcal{T}(D_n)}$-standard monomials which are not of the form (3.2). Thus we have

$$M_{\mathcal{T}(D_n)} = M_{\mathcal{T}(A_{n-1})} \amalg M^0_{\mathcal{T}(D_n)}.$$

The monomials in $M^0_{\mathcal{T}(D_n)}$ are of the form

$$\varphi(E_{i_1,j_1}) E_{i_2,j_2} \cdots E_{i_p,j_p} \quad (1 \leq p \leq n-1) \tag{5.2}$$

where

$$\begin{aligned}
&1 \leq i_1 < i_2 < \cdots < i_p \leq n-1, \\
&i_1 \geq j_1,\ i_2 \geq j_2,\ \ldots,\ i_p \geq j_p, \quad \text{and} \\
&j_k = 0 \ (1 \leq k < p) \ \text{ implies } j_{k+1} \geq 1, \quad \text{and} \\
&j_k = 1 \ (1 \leq k < p) \ \text{ implies } j_{k+1} = 0 \text{ or } j_{k+1} \geq 2, \quad \text{and} \\
&j_k > 1 \ (1 \leq k < p) \ \text{ implies } j_k < j_{k+1}
\end{aligned}$$

where

$$\varphi(E_{i_1,j_1}) := \begin{cases} E_{i_1,j_1} & \text{if } j_1 j_2 = 0, \\ E'_{i_1,j_1} = E_{i_1,0}E^{1,j_1} & \text{otherwise.} \end{cases}$$

We can easily check that $M^0_{\mathcal{T}(D_n)}$ is the set of $\widehat{R}_{\mathcal{T}(D_n)}$-standard monomials which are not $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard.

We count $|M^0_{\mathcal{T}(D_n)}|$ to obtain the following theorem.

Theorem 5.2. *The algebra $\mathcal{T}(D_n)$ has a Gröbner-Shirshov basis $\widehat{R}_{\mathcal{T}(D_n)}$ with respect to our monomial order $<$:*

$$\widehat{R}_{\mathcal{T}(D_n)}: \quad \begin{array}{ll} E_i^2 - \delta E_i & \text{for } 0 \le i \le n-1, \\ E_iE_j - E_jE_i & \text{for } 1 < j+1 < i \le n-1, \\ E_iE_0 - E_0E_i & \text{for } i \ne 2, \\ E_iE_jE_i - E_i & \text{for } \{i,j\} = \{0,2\}, \\ E_{i,j}E_i - E_{i-2,j}E_i & \text{for } i > j > 0, \\ E_{i,0}E_i - E_{i-2,0}E_i & \text{for } i > 2, \\ E_jE_{i,j} - E_jE_{i,j+2} & \text{for } i > j > 0, \\ E_0E_{i,0} - E_0E_{i,3} & \text{for } i \ge 2, \\ E_0E_1E_{i,0} - E_0E_1E_{i,3} & \text{for } i \ge 2, \\ E_{i,0}E^{1,j}E_i - E_{i-2,0}E^{1,j}E_i & \text{for } i > j+1 > 1. \end{array}$$

The cardinality of the set $M_{\mathcal{T}(D_n)}$, i.e. the set of $\widehat{R}_{\mathcal{T}(D_n)}$-standard monomials, is

$$\dim \mathcal{T}(D_n) = \tfrac{n+3}{2} C_n - 1.$$

Proof. Let $r > 1$ be the biggest such that $j_r = 0$. In order to count the number of monomials in $M^0_{\mathcal{T}(D_n)}$, we change the indices in (5.2) into $j_s = 0$ for $s \le r$. As we did for type B, we count the number of sequences of points $(i_1, j_1), \dots, (i_p, j_p)$, which is exactly the number of monomials of the form (4.2) with $p \ge 1$. Thus,

$$|M^0_{\mathcal{T}(D_n)}| = \tfrac{n+1}{2} C_n - 1 = \tfrac{1}{2}\binom{2n}{n} - 1.$$

Counting the $\widehat{R}_{\mathcal{T}(D_n)}$-standard monomials by

$$|M_{\mathcal{T}(A_{n-1})}| + |M^0_{\mathcal{T}(D_n)}| = C_n + \tfrac{n+1}{2} C_n - 1$$

leads us exactly the number equal to

$$\dim \mathcal{T}(D_n) = \tfrac{n+3}{2} C_n - 1$$

as proved in [5, §6] and [13, §10]. By Theorem 2.6, we conclude that $\widehat{R}_{\mathcal{T}(D_n)}$ is a Gröbner-Shirshov basis for $\mathcal{T}(D_n)$. □

Example 5.3. (1) The $\widehat{R}_{\mathcal{T}(D_4)}$-standard monomials containing E_0 are as follows:

$$E_{1,0} = E_0, E'_1 = E_0E_1, E_{2,0} = E_2E_0, E'_{2,1} = E_2E_0E_1, E'_2 = E_2E_0E_1E_2,$$
$$E_1E_{2,0}, E_0E_{2,1}, E_0E_2, E'_1E_2, E_{3,0}, E'_{3,1}, E'_{3,2}, E'_3, E_1E_{3,0}, E_0E_{3,1}, E_0E_{3,2}, E_0E_3,$$
$$E'_1E_{3,2}, E'_1E_3, E_{2,1}E_{3,0}, E_{2,0}E_{3,1}, E_{2,0}E_{3,2}, E_{2,0}E_3, E'_{2,1}E_{3,2}, E'_{2,1}E_3, E'_2E_3,$$
$$E_0E_{2,1}E_{3,0}, E_1E_{2,0}E_{3,1}, E_1E_{2,0}E_{3,2}, E_1E_{2,0}E_3, E_0E_{2,1}E_{3,2}, E_0E_{2,1}E_3, E_0E_2E_3, E'_1E_2E_3.$$

(2) We multiply $E_0E_{2,1}E_{3,0}$ by E_3 from the left:

$$E_3(E_0E_{2,1}E_{3,0}) = E_0(E_{3,1}E_3)E_{2,0} = E_0E_1E_{3,0} = E_0E_1E_3 = E_1'E_3.$$

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