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Standard Monomials for Temperley-Lieb Algebras

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We deal with Temperley-Lieb algebras of type B, extending the result in [3, §6]. By completing the relations coming from a presentation of the Temperley-Lieb algebra, we find its Gröbner-Shirshov basis to obtain the corresponding set of standard monomials. The explicit multiplication table between the monomials follows naturally.

First, we review the results on Temperley-Lieb algebras $\mathcal{T}(A_{n-1})$ $(n \geq 2)$. Define $\mathcal{T}(A_{n-1})$ to be the associative algebra over the complex field $\mathbb{C}$, generated by $X = \{E_1, E_2, \ldots, E_{n-1}\}$ with defining relations:

$$R_{\mathcal{T}(A_{n-1})}: \quad \begin{array}{lll} E_i^2 = \delta E_i & \text{for } 1 \leq i \leq n-1, & \text{(idempotent relations)} \\ E_iE_j = E_jE_i & \text{for } i > j+1, & \text{(commutative relations)} \\ E_iE_jE_i = E_i & \text{for } j = i \pm 1, & \text{(untwisting relations)} \end{array}$$

where $\delta \in \mathbb{C}$ is a parameter. Our monomial order $<$ is taken to be the degree-lexicographic order with

$$E_1 < E_2 < \cdots < E_{n-1}.$$

We write $E_{i,j} = E_iE_{i-1}\cdots E_j$ for $i \geq j$ (hence $E_{i,i} = E_i$). By convention $E_{i,i+1} = 1$ for $i \geq 1$.

Proposition 1 ([3, Proposition 6.2]) *The Temperley-Lieb algebra $\mathcal{T}(A_{n-1})$ has a Gröbner-Shirshov basis as follows:*

$$\widehat{R}_{\mathcal{T}(A_{n-1})}: \quad \begin{array}{ll} E_i^2 - \delta E_i & \text{for } 1 \leq i \leq n-1, \\ E_iE_j - E_jE_i & \text{for } i > j+1, \\ E_{i,j}E_i - E_{i-2,j}E_i & \text{for } i > j, \\ E_jE_{i,j} - E_jE_{i,j+2} & \text{for } i > j. \end{array} \tag{1}$$

The corresponding $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard monomials are of the form

$$E_{i_1,j_1}E_{i_2,j_2}\cdots E_{i_p,j_p} \quad (0 \leq p \leq n-1) \tag{2}$$

where

$$1 \leq i_1 < i_2 < \cdots < i_p \leq n-1, \quad 1 \leq j_1 < j_2 < \cdots < j_p \leq n-1,$$
$$i_1 \geq j_1,\ i_2 \geq j_2,\ \ldots,\ i_p \geq j_p$$

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(the case of $p = 0$ is the monomial 1). We denote the set of $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard monomials by $M_{\mathcal{T}(A_{n-1})}$. Note that the number $|M_{\mathcal{T}(A_{n-1})}|$ of $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard monomials is

$$\frac{1}{n+1}\binom{2n}{n} := C_n,$$

the n^{th} Catalan number.

Let $\mathcal{T}(B_n)$ $(n \geq 2)$ be the Temperley-Lieb algebra of type B_n, that is, the associative algebra over the complex field $\mathbb{C}$, generated by $X = \{E_0, E_1, \ldots, E_{n-1}\}$ with defining relations:

$$R_{\mathcal{T}(B_n)}: \quad \begin{array}{rl} E_i^2 = \delta E_i & \text{for } 0 \leq i \leq n-1, \\ E_iE_j = E_jE_i & \text{for } i > j+1, \\ E_iE_jE_i = E_i & \text{for } j = i \pm 1,\ i,j > 0, \\ E_iE_jE_iE_j = 2E_iE_j & \text{for } \{i,j\} = \{0,1\}, \end{array} \tag{3}$$

where $\delta \in \mathbb{C}$ is a parameter. Fix our monomial order $<$ to be the degree-lexicographic order with

$$E_0 < E_1 < \cdots < E_{n-1}.$$

We write $E_{i,j} = E_iE_{i-1}\cdots E_j$ for $i \geq j \geq 0$, and $E^{i,j} = E_iE_{i+1}\cdots E_j$ for $i \leq j$. By convention, $E_{i,i+1} = 1$ and $E^{i+1,i} = 1$ for $i \geq 0$.

Lemma 1 *The following relation holds in $\mathcal{T}(B_n)$:*

$$E_{i,0}E^{1,j}E_i = E_{i-2,0}E^{1,j}E_i$$

for $i > j+1 \geq 1$.

Let $\widehat{R}_{\mathcal{T}(B_n)}$ be the set of defining relations (3) combined with (1) and the relation in Lemma 1. From this, we define $M_{\mathcal{T}(B_n)}$ by the set of $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials. Among the monomials in $M_{\mathcal{T}(B_n)}$, we consider the monomials which are not $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard. That is, we take only $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials which are not of the form (2). This set is denoted by $M^0_{\mathcal{T}(B_n)}$. Note that each monomial in $M^0_{\mathcal{T}(B_n)}$ contains E_0. We decompose the set $M^0_{\mathcal{T}(B_n)}$ into two parts as follows :

$$M^0_{\mathcal{T}(B_n)} = M^{0+}_{\mathcal{T}(B_n)} \amalg M^{0-}_{\mathcal{T}(B_n)}$$

where the monomials in $M^{0+}_{\mathcal{T}(B_n)}$ are of the form

$$E_0E_{i_1,j_1}E_{i_2,j_2}\cdots E_{i_p,j_p} \quad (0 \leq p \leq n-1)$$

with

$$\begin{array}{l} 1 \leq i_1 < i_2 < \cdots < i_p \leq n-1, \quad 0 \leq j_1 \leq j_2 \leq \cdots \leq j_p \leq n-1, \\ i_1 \geq j_1,\ i_2 \geq j_2,\ \ldots,\ i_p \geq j_p, \text{ and} \\ j_k > 0\ (1 \leq k < p) \text{ implies } j_k < j_{k+1} \end{array}$$

(the case of $p = 0$ is the monomial E_0), and the monomials in $M^{0-}_{\mathcal{T}(B_n)}$ are of the form

$$E'_{i_1,j_1}E_{i_2,j_2}\cdots E_{i_p,j_p} \quad (1 \leq p \leq n-1)$$

with

$$E'_{i,j} = E_{i,0}E^{1,j}$$

and the same restriction on i's and j's as above. It can be easily checked that $M^0_{\mathcal{T}(B_n)}$ is the set of $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials which are not $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard.

Counting the number of elements in $M^0_{\mathcal{T}(B_n)}$, we obtain the following theorem.

Theorem 1 *The algebra $\mathcal{T}(B_n)$ has a Gröbner-Shirshov basis $\widehat{R}_{\mathcal{T}(B_n)}$ with respect to our monomial order $<$:*

$$\widehat{R}_{\mathcal{T}(B_n)}: \quad \begin{array}{cl} E_i^2 - \delta E_i & \text{for } 0 \le i \le n-1, \\ E_iE_j - E_jE_i & \text{for } i > j+1, \\ E_{i,j}E_i - E_{i-2,j}E_i & \text{for } i > j > 0, \\ E_jE_{i,j} - E_jE_{i,j+2} & \text{for } i > j > 0. \\ E_iE_jE_iE_j - 2E_iE_j & \text{for } \{i,j\} = \{0,1\}, \\ E_{i,0}E^{1,j}E_i - E_{i-2,0}E^{1,j}E_i & \text{for } i > j+1 \ge 1. \end{array}$$

The corresponding $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials are exactly the ones in $M_{\mathcal{T}(B_n)}$.

Remark. (1) We note that $\dim \mathcal{T}(B_n) = (n+2)C_n - 1$ as in [4, §5].

(2) From the works in [1, 2], we notice that

$$\begin{aligned} &2^{n-2} + \tfrac{n}{n+1}\tbinom{n+1}{n}2^{n-3} + \tfrac{n-1}{n+1}\tbinom{n+2}{n}2^{n-4} + \tfrac{n-2}{n+1}\tbinom{n+3}{n}2^{n-5} + \cdots + \tfrac{3}{n+1}\tbinom{2n-2}{n}2^0 \\ &= \tfrac{1}{2}\left(\textstyle\sum_{k=0}^{n-2} C(n,k)|\mathcal{P}_B(n,k)| + 1\right) = \textstyle\sum_{k=0}^{n-2} C(n,k)|\mathcal{P}_D(n,k)| + 1 \; = \; \tfrac{n-1}{2}C_n \end{aligned}$$

where $C(n,k)$ is the (n,k)-entry of the Catalan triangle, and $|\mathcal{P}(n,k)|$ is the number of elements in the (n,k)-packet.

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