DISTRIBUTIONS FOR WHICH $\text{div} \ v = F$ HAS A CONTINUOUS SOLUTION

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Abstract. The equation $\text{div} \ v = F$ has a continuous weak solution in an open set $U \subset \mathbb{R}^m$ if and only if the distribution $F$ satisfies the following condition: $F(\varphi_i)$ converge to zero for each sequence $\{\varphi_i\}$ of test functions such that the supports of $\varphi_i$ are contained in a fixed compact subset of $U$, and in the $L^1$ norm, $\{\varphi_i\}$ converges to zero and $\{\nabla \varphi_i\}$ is bounded.

If $F$ is a distribution in $\mathbb{R}^m$, then a vector field $v \in L^1(\mathbb{R}^m; \mathbb{R}^m)$ is a solution of the equation $\text{div} \ v = F$ whenever

$$F(\varphi) = -\int_{\mathbb{R}^m} v(x) \cdot \nabla \varphi(x) \, dx$$

for each test function $\varphi \in \mathcal{D}(\mathbb{R}^m)$. If such a $v$ is continuous and $\varepsilon > 0$, we can find a $w \in C^1(\mathbb{R}^m; \mathbb{R}^m)$ so that $|v(x) - w(x)| < \varepsilon$ for each $x$ in the ball $B(1/\varepsilon)$ of radius $1/\varepsilon$ about the origin. Selecting $\varphi$ supported in $B(1/\varepsilon)$ and integrating by parts, we obtain

$$|F(\varphi)| \leq \left| \int_{B(1/\varepsilon)} \varphi \text{div} \ w \right| + \left| \int_{B(1/\varepsilon)} (w - v) \cdot \nabla \varphi \right|$$

$$\leq |\varphi|_1 \sup_{x \in B(1/\varepsilon)} |\text{div} \ w(x)| + \varepsilon |\nabla \varphi|_1,$$

which implies a stronger continuity of $F$. In other words, the following continuity property of $F$ is necessary for the equation $\text{div} \ v = F$ to have a continuous solution.

**Continuity.** Given $\varepsilon > 0$ there is a $\theta > 0$ such that

$$|F(\varphi)| \leq \theta |\varphi|_1 + \varepsilon |\nabla \varphi|_1 \quad (\ast)$$

for each $\varphi \in \mathcal{D}(\mathbb{R}^m)$ with $\text{supp} \ \varphi \subset B(1/\varepsilon)$.

Our main result is Theorem 3.7 below, which asserts that this necessary continuity property is also sufficient. For historical reasons (see below), a distribution $F$ satisfying the above continuity property is called a strong charge.

An example of a strong charge is the distribution associated with a function $f \in L^m_{\text{loc}}(\mathbb{R}^m)$ (Proposition 2.9 below). J. Bourgain and H. Brezis [1, Proposition 1] proved that a continuous solution of $\text{div} \ v = f$ exists for a $\mathbb{Z}^m$ periodic function $f \in L^m_{\text{loc}}(\mathbb{R}^m)$. The continuity of $v$ is the main point — establishing the existence of a solution $v \in L^\infty(\mathbb{R}^m; \mathbb{R}^m)$ is appreciably easier (Proposition 2.11 below). In

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general, neither a continuous nor essentially bounded solution is obtainable by solving the Poisson equation $\Delta u = f$ and letting $v := \nabla u$; a pertinent example is due to L. Nirenberg [1, Remark 7]. The absence of such a solution is related to the role of $p = m$ as the critical exponent for representing elements of $W^{1,p}$ by continuous functions [7, Chapter 5, Theorem 5].

We outline the proof of Theorem 3.7, which is inspired by the above mentioned proof of Bourgain and Brezis. The linear spaces $S$ of all strong charges, and $C$ of all continuous vector fields $v : \mathbb{R}^m \to \mathbb{R}^m$, are equipped with the Fréchet topologies of locally uniform convergence. For a $v \in C$, we define a strong charge $F_v$ by

$$F_v(\varphi) := -\int_{\mathbb{R}^m} v(x) \cdot \nabla \varphi(x) \, dx$$

for each $\varphi \in \mathcal{D}(\mathbb{R}^m)$, and observe that the linear map $\Gamma : v \mapsto F_v$ from $C$ to $S$ is continuous. Showing that

1. $\Gamma(C)$ is a dense subspace of $S$ (Lemma 3.1 below),
2. if $\Gamma^* : S^* \to C^*$ is the adjoint map of $\Gamma$, then $\Gamma^*(S^*)$ is closed in the strong topology of $C^*$ (Proposition 3.6 below),

completes the argument: (ii) and the Closed Range Theorem imply that $\Gamma(C)$ is closed in $S$, and hence $\Gamma(C) = S$ by (i).

Because the space $S$ is topologized so that its dual $S^*$ is isomorphic to the linear space $BV_c$ of all compactly supported BV functions in $\mathbb{R}^m$ (Proposition 3.2 below), the adjoint map $\Gamma^*$ of $\Gamma$ has an intuitive geometric meaning. Indeed, interpreting the continuous vector fields as $(m-1)$-forms and strong charges as $m$-forms, we can think of $\Gamma$ as the exterior derivative; note that by definition, $\Gamma$ is a weak divergence operator. Thus $\Gamma^*$ is a boundary operator which maps $g \in BV_c$ to a compactly supported Radon measure $Dg$ in $\mathbb{R}^m$. Clearly, $Dg$ belongs to the dual space $C^*$ of $C$; see diagram (3.2) below.

In the obvious way, the balls $B(i)$, $i = 1, 2, \ldots$, determine seminorms $s_i$ and $c_i$ which define the topologies of locally uniform convergence in $S$ and $C$, respectively. Theorem 3.8 below shows that given a strong charge $F$ and an integer $i \geq 1$, we can find a solution $v \in C$ of $\text{div} \, v = F$ so that $c_i(v)$ is as close to $s_i(F)$ as we wish.

If $F$ is a strong charge, then the set $\Gamma^{-1}(F)$ of all continuous solutions of the equation $\text{div} \, v = F$ has many elements. In Section 4 we consider continuous vector fields and strong charges that are invariant with respect to the special orthogonal group $SO(m)$, and produce constructively an isometry $\Upsilon : S_{\text{inv}} \to C_{\text{inv}}$ that is a right inverse of $\Gamma$ (Proposition 4.3 below). The construction depends on showing that a rotation invariant strong charge on the sphere $S^{m-1}$ is a multiple of a strong charge induced on $S^{m-1}$ by the Hausdorff measure $H^{m-1}$ in $\mathbb{R}^m$ (Proposition 4.2 below).

A strong charge is a special case of a charge, i.e., of a distribution $F$ with the above continuity property where inequality $(\ast)$ is replaced by the inequality

$$|F(\varphi)| \leq \theta |\varphi|_1 + \varepsilon (|\nabla \varphi|_1 + |\varphi|_{\infty}).$$

Every distribution associated with an $f \in L^1_{\text{loc}}(\mathbb{R}^m)$ is a charge, called an absolutely continuous charge. Elaborating on the proof of Theorem 3.7, we show that
each charge is the sum of a strong charge and an absolutely continuous charge (Theorem 5.2 below).

The concepts of charges and strong charges originate from our previous work on generalized Riemann integrals and the Gauss-Green theorem [12, 3, 2, 13, 5, 4]. In Section 6, we indicate how a substantial generalization of the classical Gauss-Green theorem (Theorem 6.5 below) can be obtained by means of charges and their derivatives. This version of the Gauss-Green theorem admits further generalizations that can be applied to removale sets of PDEs in divergence form [5, 4, Sections 4].

1. Preliminaries

The set of all real numbers is denoted by \( \mathbb{R} \). In the Cartesian product \( \mathbb{R}^n \) where \( n \geq 1 \) is an integer, we denote by \( x \cdot y \) the usual inner product, which induces the norm \( |x| \). The zero vector in \( \mathbb{R}^n \) is denoted by \( 0 \). All functions we consider are real valued. For a map \( f : A \to B \) and an \( x \in A \), we use the symbols \( f(x) \) and \( \langle f, x \rangle \) interchangeably; both denote the value of \( f \) at \( x \).

The ambient space of this paper is \( \mathbb{R}^m \) where \( m \geq 2 \) is a fixed integer. Restricting to dimensions larger than one merely eliminates trivialities. The closure, interior, and diameter of a set \( E \subset \mathbb{R}^m \) are denoted by \( \text{cl} \ E \), int \( E \), and \( d(E) \), respectively. The open and closed balls of radius \( r > 0 \) centered at \( x \in \mathbb{R}^m \) are denoted by \( B(x,r) \) and \( B[x,r] \), respectively. We write \( B(r) \) instead of \( B(0,r) \), and \( B[r] \) instead of \( B[0,r] \).

In \( \mathbb{R}^m \) we use Lebesgue measure \( \mathcal{L} := \mathcal{L}^m \) and the Hausdorff measure \( \mathcal{H} := \mathcal{H}^{m-1} \). For \( E \subset \mathbb{R}^m \), we write \( |E| \) instead of \( \mathcal{L}(E) \), and define the restricted measures \( \mathcal{L}|_E \) and \( \mathcal{H}|_E \) in the usual way [8, Section 1.1.1]. Unless specified otherwise, the words “measure,” “measurable,” and “negligible,” as well as the expressions “almost everywhere” and “almost all” refer to Lebesgue measure \( \mathcal{L} \). Symbols \( \int f \) and \( \int f(x) \, dx \) denote the Lebesgue integral \( \int d\mathcal{L} \).

Throughout, \( U \subset \mathbb{R}^m \) is a fixed nonempty open set. For \( 1 \leq p \leq \infty \) and an integer \( n \geq 1 \), we give \( \mathcal{L}^p_{\text{loc}}(U;\mathbb{R}^n) \) a topology induced by the seminorms

\[
|f|_{p,K} := \left| f \upharpoonright K \right|_p
\]

where \( f \in \mathcal{L}^p_{\text{loc}}(U;\mathbb{R}^n) \) and \( K \subset U \) is a compact set. As there is an increasing sequence of compact subsets of \( U \) whose interiors cover \( U \), the space \( \mathcal{L}^p_{\text{loc}}(U;\mathbb{R}^n) \) is a Fréchet space. Clearly, \( C(U;\mathbb{R}^m) \) topologized as a subspace of \( \mathcal{L}^p_{\text{loc}}(U;\mathbb{R}^m) \) is a Fréchet space as well. We write \( \mathcal{L}^p_{\text{loc}}(U) \) instead of \( \mathcal{L}^p_{\text{loc}}(U;\mathbb{R}) \), and denote by \( \mathcal{L}^p(U) \) the linear space of all functions in \( \mathcal{L}^p_{\text{loc}}(U) \) whose support is a compact subset of \( U \).

We denote by \( BV(U) \) the linear space of all BV functions in \( U \), and let

\[
BV_c(U) := BV(U) \cap \mathcal{L}^1(U) \quad \text{and} \quad BV^\infty_c(U) := BV(U) \cap L^\infty_c(U).
\]

If \( g \in BV(U) \), then \( \|g\| \) is the total variation of the distributional gradient \( Dg \) of \( g \).

The essential boundary, perimeter and exterior normal of a BV set \( E \) in \( U \) are denoted by \( \partial_* E \), \( |E| \) and \( n_E \), respectively. Note that \( \|E\| = \mathcal{H}(\partial_* E) = \|\chi_E\| \) where \( \chi_E \) is the indicator of \( E \) in \( U \).
Definitions and basic properties

**Definition 2.1.** A distribution \( F \in \mathcal{D}'(U) \) is called fluxing, or simply a flux, if the equation \( \text{div} \, v = F \) has a continuous solution, i.e., if there is a vector field \( v \in C(U; \mathbb{R}^m) \) such that for each \( \varphi \in \mathcal{D}(U) \),

\[
F(\varphi) = - \int_U v(x) \cdot \nabla \varphi(x) \, dx \tag{2.1}
\]

The linear space of all fluxing distributions in \( U \) is denoted by \( \mathcal{F}(U) \). A distribution \( F \) defined by equality (2.1) is called the flux of \( v \), denoted by \( F_v \).

We say a sequence \( \{f_i\} \) of functions defined on \( U \) is compactly supported if there is a compact set \( K \subset U \) such that \( \{f_i \neq 0\} \subset K \) for \( i = 1, 2, \ldots \). If the compact set \( K \) is specified a priori, we say that \( \{f_i\} \) is supported in \( K \). A sequence \( \{A_i\} \) of subsets of \( U \) is called compactly supported, or supported in a compact set \( K \subset U \), whenever the sequence \( \{\chi_{A_i}\} \) has the respective property.

**Observation 2.2.** If \( F \in \mathcal{D}'(U) \) is a flux, then \( \lim F(\varphi_i) = 0 \) for every compactly supported sequence \( \{\varphi_i\} \) in \( \mathcal{D}(U) \) for which

\[
\lim |\varphi_i|_1 = 0 \quad \text{and} \quad \sup \|\varphi_i\| < \infty. \tag{2.2}
\]

**Proof.** Let \( F = F_v \) for a \( v \in C(U; \mathbb{R}^m) \), and let \( \{\varphi_i\} \) be a sequence in \( \mathcal{D}(U) \) supported in a compact set \( K \subset U \) satisfying conditions (2.2). Find a sequence \( \{w_j\} \) in \( C^1_c(\mathbb{R}^m; \mathbb{R}^m) \) converging to \( v \) uniformly in \( K \), and observe

\[
|F(\varphi_i)| \leq \int_K |v(x) - w_j(x)| \cdot |\nabla \varphi(x)| \, dx + \left| \int_K \varphi_i(x) \text{div} \, w_j(x) \, dx \right|
\]

\[
\leq (\sup_n \|\varphi_n\|) \sup_{x \in K} |v(x) - w_j(x)| + |\varphi_i|_1 \sup_{x \in K} |\text{div} \, w_j(x)|
\]

for \( i, j = 1, 2, \ldots \). Choosing a sufficiently large \( j \) and then a sufficiently large \( i \), we can make \( F(\varphi_i) \) arbitrarily small. \qed

Observation 2.2 motivates in part the following definition.

**Definition 2.3.** A linear functional \( F : \mathcal{D}(U) \to \mathbb{R} \) is called

(i) a charge if \( \lim F(\varphi_i) = 0 \) for every compactly supported sequence \( \{\varphi_i\} \) in \( \mathcal{D}(U) \) for which

\[
\lim |\varphi_i|_1 = 0 \quad \text{and} \quad \sup \|\varphi_i\| + |\varphi_i|_\infty < \infty;
\]

(ii) a strong charge (abbreviated as s-charge) if \( \lim F(\varphi_i) = 0 \) for every compactly supported sequence \( \{\varphi_i\} \) in \( \mathcal{D}(U) \) for which

\[
\lim |\varphi_i|_1 = 0 \quad \text{and} \quad \sup \|\varphi_i\| < \infty.
\]
For each compact set $K \subset U$ and $n = 1, 2, \ldots$, the convex sets

$$BV(K, n) := \{g \in BV_c^\infty(U) : \{g \neq 0\} \subset K \text{ and } \|g\| + |g|_\infty \leq n\},$$

$$BV_s(K, n) := \{g \in BV_c(U) : \{g \neq 0\} \subset K \text{ and } \|g\| \leq n\}$$

are compact subsets of $L^1(U)$ [8, Section 5.2, Theorem 4]. Give $BV_c^\infty(U)$ and $BV_c(U)$, respectively, the largest topology $\mathcal{T}$ and $\mathcal{T}_s$ for which all inclusion maps

$$BV(K, n) \hookrightarrow BV_c^\infty(U) \quad \text{and} \quad BV_s(K, n) \hookrightarrow BV_c(U)$$

are continuous. Since $U$ is the union of an increasing sequence of compact sets, it follows from [13, Proposition 1.2.2] that the topologies $\mathcal{T}$ and $\mathcal{T}_s$ are locally convex, sequential, and sequentially complete. Moreover $\mathcal{T}_s \subset \mathcal{T}$, and $\mathcal{D}(U)$ is a dense subset of both $(BV_c^\infty(U), \mathcal{T})$ and $(BV_c(U), \mathcal{T}_s)$ [8, Section 5.2, Theorem 2].

**Observation 2.4.** A linear functional $F : \mathcal{D}(U) \to \mathbb{R}$ is, respectively, a charge or an $s$-charge if and only if it is $\mathcal{T}$ or $\mathcal{T}_s$ continuous. In particular, each charge has a unique $\mathcal{T}$ continuous extension to $BV_c^\infty(U)$, and each $s$-charge has a unique $\mathcal{T}_s$ continuous extension to $BV_c(U)$. These extensions are linear.

**Remark 2.5.** Observe that the flux $F_v$ of a locally bounded Borel vector field $v : U \to \mathbb{R}^m$, which need not be a charge, still extends to

$$F_v : g \mapsto -\int_U v \cdot d(Dg) : BV_c(U) \to \mathbb{R}.$$  

In view of Observation 2.4, we always think of charges as defined on $BV_c^\infty(U)$, and of $s$-charges as defined on $BV_c(U)$. If $F$ is a charge and $E$ is a bounded BV set whose closure is contained in $U$, we let $F(E) := F(\chi_E)$. Note that

$$F_v(E) = -\int_U v \cdot d(D\chi_E) = \int_{\partial_s E} v \cdot \nu_E \, d\mathcal{H}.$$  

**Proposition 2.6.** If $F : BV_c(U) \to \mathbb{R}$ is a linear functional, then the following properties are equivalent.

(i) The functional $F$ is an $s$-charge.

(ii) Given $\varepsilon > 0$ and compact set $K \subset U$, there is a $\theta > 0$ such that

$$|F(g)| \leq \theta |g|_1 + \varepsilon \|g\|$$

for each $g \in BV_c(U)$ with $\{g \neq 0\} \subset K$.

(iii) For each compactly supported sequence $\{B_i\}$ of BV sets in $U$,

$$\lim_{\|B_i\|} \frac{F(B_i)}{\|B_i\|} = 0 \quad \text{whenever} \quad \lim |B_i| = 0.$$  

**Proof.** (i) $\Rightarrow$ (ii). Suppose $F$ is an $s$-charge, and choose an $\varepsilon > 0$ and a compact set $K \subset U$. There is an $\eta > 0$ such that $|F(g)| < \varepsilon/2$ for each $g \in BV_c(U)$ with $|g|_1 < \eta$, $\|g\| < 1$, and $\{g \neq 0\} \subset K$. Let $\theta := \varepsilon/(2\eta)$ and select a $g \in \mathcal{D}(U)$ with $\{g \neq 0\} \subset K$. With no loss of generality, we may assume $g \geq 0$; see [13, Theorem 1.8.12].
Let \( p \) and \( q \) be the smallest positive integers with \(|g|_1/p < \eta\) and \( \|g\|/q < 1\). Note \( p \leq |g|_1/\eta + 1 \) and \( q \leq \|g\| + 1 \). Since

\[
s \mapsto \int_0^s |\{ g > t \}| \, dt \quad \text{and} \quad s \mapsto \int_0^s \|\{ g > t \}\| \, dt
\]

are continuous increasing functions which map \([0, \infty]\) onto \([0, |g|_1]\) and \([0, \|g\|]\), respectively, there are points \( 0 = a_0 < \cdots < a_p = \infty \) and \( 0 = b_0 < \cdots < b_q = \infty \) such that

\[
\int_{a_i-1}^{a_i} |\{ g > t \}| \, dt = \frac{1}{p} |g|_1 < \eta \quad \text{and} \quad \int_{b_j-1}^{b_j} \|\{ g > t \}\| \, dt = \frac{1}{q} \|g\| < 1
\]

for \( i = 1, \ldots, p \) and \( j = 1, \ldots, q \). Order the set \( \{a_0, \ldots, a_p, b_0, \ldots, b_q\} \) into a sequence \( 0 = c_0 < \cdots < c_r = \infty \). Then \( r \leq p + q - 1 \), and

\[
g_k := \max\{\min\{g, c_k\}, c_{k-1}\} - c_{k-1}, \quad k = 1, \ldots, r,
\]

are BV functions vanishing outside \( K \). As each \([c_{k-1}, c_k]\) is contained in some \([a_{i-1}, a_i) \cap [b_{j-1}, b_j)\], the previous inequalities imply \(|g_k|_1 < \eta\) and \( \|g_k\| < 1 \). Since \( g = \sum_{k=1}^r g_k \), we obtain

\[
|F(g)| \leq \sum_{k=1}^r |F(g_k)| < r\frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} (p + q - 1)
\]

\[
\leq \frac{\varepsilon}{2}\left(1 + \frac{1}{\eta} |g|_1 \right) = \theta \|g\|_1 + \frac{\varepsilon}{2} \|g\| + \frac{\varepsilon}{2},
\]

and inequality (2.3) follows whenever \( \|g\| \geq 1 \). If \( 0 < \|g\| < 1 \), we apply the previous result to \( h := g/\|g\|\):

\[
|F(g)| = \|g\| \cdot |F(h)| \leq \|g\| (\theta \|h\|_1 + \varepsilon \|h\|) = \theta |g|_1 + \varepsilon \|g\|.
\]

As the case \( \|g\| = 0 \) is trivial, the desired inequality is established.

(ii) \(\Rightarrow\) (iii). By [13, Proposition 2.1.7], given \( \varepsilon > 0 \) and a compact set \( K \subset U \), there is a \( \theta > 0 \) such that

\[
|F(B)| \leq \theta |B| + \varepsilon \|B\|
\]

for each BV set \( B \subset K \). Now it follows from [13, Proposition 2.2.6 and Section 4.1] that \( F \) satisfies (ii).

The implications (ii) \(\Rightarrow\) (i) and (ii) \(\Rightarrow\) (iii) are obvious. \(\square\)

**Remark 2.7.** Charges in \( U \) are characterized by an inequality similar to (2.3). Indeed, it follows from [13, Proposition 2.2.6 and Section 4.1] that a linear functional \( F : BV_{c}^\infty(U) \to \mathbb{R} \) is a charge if and only if given \( \varepsilon > 0 \) and a compact set \( K \subset U \), there is a \( \theta > 0 \) such that

\[
|F(g)| \leq \theta |g|_1 + \varepsilon (\|g\| + |g|_\infty)
\]

for each \( g \in BV_{c}^\infty(U) \) with \( \{g \neq 0\} \subset K \). A direct proof of this fact is analogous to that of Proposition 2.6; see also [5, Proposition 2.4].

**Remark 2.8.** It follows from [13, Section 4.1] that charges are uniquely determined by their values on the indicators of bounded BV sets. As bounded BV sets can be approximated by finite unions of nondegenerate compact intervals [13, Proposition 1.10.3], charges, and a fortiori s-charges, are uniquely determined by their values on the indicators of nondegenerate compact intervals.
The linear spaces of all charges in $U$ and all $s$-charges in $U$ are denoted by $CH(U)$ and $CH_s(U)$, respectively. By Observation 2.2, 

$$F(U) \subset CH_s(U) \subset CH(U) \subset D'(U).$$

If $f \in L^1_{\text{loc}}(U)$, then the distribution $\Lambda(f)$ in $D'(U)$ defined by

$$\langle \Lambda(f), \varphi \rangle := \int_U f(x)\varphi(x) \, dx$$

for each $\varphi \in D(U)$ is a charge, called an absolutely continuous charge (abbreviated as $ac$-charge). Denoting by $CH_{ac}(U)$ the linear space of all ac-charges in $U$, we have a linear isomorphism

$$\Lambda : f \mapsto \Lambda(f) : L^1_{\text{loc}}(U) \rightarrow CH_{ac}(U)$$

In particular, each $F \in CH_{ac}$ has an obvious linear extension to $L^\infty_{\text{loc}}(U)$.

While easy examples show that neither of the spaces $CH_s(U)$ and $CH_{ac}(U)$ contains the other, they have a sizable intersection.

**Proposition 2.9.** $\Lambda[L^m_{\text{loc}}(U)] \subset CH_s(U)$.

**Proof.** Choose an $f \in L^m_{\text{loc}}(U)$, and let $F := \Lambda(f)$. For $g \in BV_c(U)$ and a measurable set $B \subset U$, the Hölder and Poincaré inequalities imply

$$\int_B |fg| \leq \left( \int_B |f|^m \right)^{\frac{1}{m}} \left( \int_B |g|^{\frac{m}{m-1}} \right)^{\frac{m-1}{m}} \leq \kappa \|g\| \left( \int_B |f|^m \right)^{\frac{1}{m}} \quad (2.4)$$

where $\kappa$ is a positive constant depending only on the dimension $m$ [8, Section 5.6, Theorem 1.(i)]. In particular

$$|F(g)| \leq \kappa \|g\| \left( \int_{\{g \neq 0\}} |f|^m \right)^{\frac{1}{m}} < \infty, \quad (2.5)$$

and it follows that $F$ is a linear functional on $BV_c(U)$. To show that $F$ is an $s$-charge, select a sequence $\{g_i\}$ in $BV_c(U)$ supported in a compact set $K \subset U$, and assume that $\lim |g_i| = 0$ and $\sup \|g_i\| < \infty$. Applying inequality (2.4) to the set $B_\theta := \{x \in K : |f(x)| > \theta\}$ with $\theta \geq 0$, we obtain

$$|F(g_i)| \leq \int_{K-B_\theta} |fg_i| + \int_{B_\theta} |fg_i| \leq \theta |g_i|_1 + \kappa \|g_i\| \left( \int_{B_\theta} |f|^m \right)^{\frac{1}{m}}$$

$$\leq \theta |g_i|_1 + \kappa \left( \sup \|g_n\| \right) \left( \int_{B_\theta} |f|^m \right)^{\frac{1}{m}}.$$

As $\lim_{\theta \to \infty} \left( \int_{B_\theta} |f|^m \right)^{1/m} = 0$, choosing a sufficiently large $\theta$ and then a sufficiently large $i$, we can make $F(g_i)$ arbitrarily small. \qed

**Note.** We proved Proposition 2.9 directly from the definition of $s$-charges. Using Proposition 2.6, the second part of the proof can be simplified by choosing a compactly supported sequence $\{B_i\}$ of $BV$ sets in $U$, and applying inequality (2.5) to $g := \chi_{B_i}$. Indeed, we obtain

$$|F(B_i)| \leq \kappa \|B_i\| \left( \int_{B_i} |f|^m \right)^{1/m}.$$
Remark 2.12. It follows from Proposition 2.11 that for each $i = 1, 2, \ldots$, and hence $\lim [F(B_i)/\|B_i\|] = 0$ whenever $\lim |B_i| = 0$.

The next example shows that the inclusion $\Lambda[L^m_{\text{loc}}(U)] \subset CH_u(U) \cap CH_{\text{ac}}(U)$ established in Proposition 2.9 is generally proper.

Example 2.10. Assume $m = 2$, and let $f(\xi, \eta) := \xi^{-\eta} + \eta^{-\xi}$ for each $(\xi, \eta)$ in $U := (0,1)^2$. If $p \geq 1$ then
\[\xi^{-\rho \eta} + \eta^{-\rho \xi} \leq [f(\xi, \eta)]^p \leq 2^p (\xi^{-\rho \eta} + \eta^{-\rho \xi})\]
for each $(\xi, \eta) \in U$. Since for every $0 < a \leq 1/p$
\[\int_{[0,a]^2} (\xi^{-\rho \eta} + \eta^{-\rho \xi}) \, d\xi \, d\eta = \frac{2}{p} \int_1^{1-a} t^{-1} a^t \, dt,\]
we see that $f \in L^p_{\text{loc}}(U)$ if and only if $p = 1$. On the other hand, the formula
\[v(\xi, \eta) := \left( \frac{\xi^{1-\eta}}{1-\eta}, \frac{\eta^{1-\xi}}{1-\xi} \right)\]
for $(\xi, \eta) \in U$ defines a $v \in C^\infty(U; \mathbb{R}^2)$ with $\text{div} \, v = f$. Integration by parts shows that $\Lambda(f)$ is the flux of $v$, and hence an $s$-charge according to Observation 2.2.

Proposition 2.11. Given $f \in L^m(U)$, there is a $v \in L^\infty(U; \mathbb{R}^m)$ such that $\Lambda(f)$ is the flux $F_v$ of $v$, and $|v|_\infty \leq \kappa |f|_m$ where $\kappa$ is a constant depending only on the dimension $m$.

Proof. Since it suffices to prove the proposition in each connected component of $U$, we may assume $U$ is connected. Let $X := \{ \nabla \varphi : \varphi \in \mathcal{D}(U) \}$, and for $w \in X$, let
\[G(w) := \int_U f(x) \varphi(x) \, dx\]
where $\varphi$ is the unique element of $\mathcal{D}(U)$ with $\nabla \varphi = w$. By the Hölder and Poincaré inequalities, there is a constant $\kappa$ depending only on the dimension $m$ such that
\[|G(w)| \leq \|f\|_m \|\varphi\|_{m-1} \leq \kappa |f|_m |w|_1,\]
for each $w \in X$. Applying Hahn-Banach theorem, extend $G$ to a linear functional $H : L^1(U; \mathbb{R}^m) \to \mathbb{R}$ so that $|H(w)| \leq \kappa |f|_m |w|_1$ for each $w \in L^1(U; \mathbb{R}^m)$. Using the duality of $L^p$ spaces, find a $v \in L^\infty(U; \mathbb{R}^m)$ so that $|v|_\infty \leq \kappa |f|_m$, and
\[H(w) = \int_U v(x) \cdot w(x) \, dx\]
for each $w \in L^1(U; \mathbb{R}^m)$. In particular, for each $\varphi \in \mathcal{D}(U)$,
\[\langle \Lambda(f), \varphi \rangle = \int_U f(x) \varphi(x) \, dx = G(\nabla \varphi) = H(\nabla \varphi) = \int_U v(x) \cdot \nabla \varphi(x) \, dx = \langle F_v, \varphi \rangle.\]

Remark 2.12. It follows from Proposition 2.11 that for each $f \in L^m(U)$, the equation $\text{div} \, v = f$ has a solution in $L^\infty(U; \mathbb{R}^m)$. We included this result because it has a simple proof. Using a more elaborate argument, Brezis and Bourgain established the existence of a bounded continuous solution [1, Proposition 1]. The same, and more, follows from Section 3 below.
3. $S$-charges

A Lipschitz domain is an open set $\Omega \subset \mathbb{R}^m$ with Lipschitz boundary [8, Section 4.2.1]. Note that each Lipschitz domain is a locally BV set. If $\Omega \subset U$ is a Lipschitz domain and $g \in BV_c(U)$ with support in $B(r)$, then it follows from [16, Remark 5.10.2 and Lemma 5.10.4] that $g\chi_{\Omega} \in BV_c(U)$, and that
\[
\|g\chi_{\Omega}\| \leq \kappa(\|g\|_1 + \|g\|)
\]
where $\kappa > 0$ depends only on $\Omega \cap B(r)$.

Let $F$ be an $s$-charge in $U$, and let $\Omega \subset U$ be a Lipschitz domain. In view of the previous paragraph and Proposition 2.6, the linear functional
\[
F \downarrow \Omega : g \mapsto F(g\chi_{\Omega}) : BV_c(U) \to \mathbb{R}
\]
is an $s$-charge in $U$. If $c\Omega \subset U$, we view $F \downarrow \Omega$ as an $s$-charge in $\mathbb{R}^m$; since $F(g\chi_{\Omega})$ is defined for each $g \in BV_c(\mathbb{R}^m)$.

If $f : U \to \mathbb{R}$ is locally Lipschitz and $g \in BV_c(U)$, then $fg \in BV_c(U)$ and
\[
\|fg\| \leq L\|g\|_1 + c\|g\|
\]
where $L := \text{Lip}(f \mid \text{supp } g)$ and $c := |f \mid \text{supp } g|_{\infty}$. Thus by Proposition 2.6,
\[
F \downarrow f : g \mapsto F(fg) : BV_c(U) \to \mathbb{R}
\]
is an $s$-charge in $U$ whenever $F$ is an $s$-charge in $U$.

We give $CH_s(U)$ a Fréchet topology induced by the seminorms
\[
\|F\|_{s,K} := \sup \{ F(g) : g \in BV(U), \ {g \neq 0} \subset K, \ \text{and} \ \|g\| \leq 1 \}
\]
where $F \in CH_s(U)$ and $K \subset U$ is a compact set. In view of Observation 2.2, there is a linear map
\[
\Gamma : v \mapsto F_v : C(U; \mathbb{R}^m) \to CH_s(U),
\]
which is continuous. Indeed, given a compact set $K \subset U$, we have
\[
|F_v(g)| \leq \|g\| \cdot |v|_{\infty,K}
\]
for every $g \in BV(U)$ with $\{g \neq 0\} \subset K$; thus $\|F_v\|_{s,K} \leq |v|_{\infty,K}$ for each compact set $K \subset U$. Note that $\mathcal{F}(U)$ is the image of $\Gamma$.

By Proposition 2.9, the restriction $\Lambda_s := \Lambda \upharpoonright L_{loc}^m(U)$ maps $L_{loc}^m(U)$ to $CH_s(U)$. It follows from inequality (2.5) that there is a constant $\kappa$, depending only on the dimension $m$, such that for each $f \in L_{loc}^m(U)$ and each compact set $K \subset U$,
\[
\|\Lambda(f)\|_{s,K} \leq \kappa|f|_{m,K}.
\]
(3.1)
In particular, the map $\Lambda_s : L_{loc}^m(U) \to CH_s(U)$ is continuous.

**Lemma 3.1.** Given $F \in CH_s(U)$, there is a sequence $\{v_i\}$ in $C^\infty(U; \mathbb{R}^m)$ such that the support of each $\text{div } v_i$ is a compact subset of $U$, and
\[
\lim \|F - \Gamma(v_i)\|_{s,K} = 0
\]
for every compact set $K \subset U$. In particular, the spaces $\mathcal{F}(U)$ and $\Lambda[\mathcal{D}(U)]$ are dense in $CH_s(U)$. 

Proof. There are bounded Lipschitz domains $\Omega_i$ such that $\text{cl} \Omega_i \subset \Omega_{i+1}$ and $U = \bigcup_{i=1}^\infty \Omega_i$. Since every compact set $K \subset U$ is contained in some $\Omega_i$, we have
\[
\lim \|F - F\res\Omega_i\|_{s,K} = 0
\]
for each compact set $K \subset U$. Thus it suffices to prove the lemma for an $s$-charge $F$ such that $F = F\res\Omega$ for a bounded Lipschitz domain $\Omega$ with $\text{cl} \Omega \subset U$. Select a bounded Lipschitz domain $\Omega_0$ with $\text{cl} \Omega \subset \Omega_0$ and $\text{cl} \Omega_0 \subset U$. There is a convergent (in the distributional sense) sequence $\{\eta_i\}$ of standard mollifiers such that the convolutions $\varphi_i := F \ast \eta_i$ have support in $\Omega_0$. By [15, Theorems 6.30 and 6.32], each $\varphi_i$ belongs to $C^\infty(\mathbb{R}^m)$, the $s$-charges $F_i := \Lambda(\varphi_i)$ converge to $F$ in the distributional sense, and $F_i(g) = F(\eta_i \ast g)$ for every $g \in BV_c(\mathbb{R}^m)$.

For $i = 1, 2, \ldots$ and $x = (\xi_1, \ldots, \xi_m) \in U$, let
\[
f_i(t, \xi_1, \ldots, \xi_m) := \int_{-\infty}^t \varphi_i(s, \xi_1, \ldots, \xi_m) \, ds.
\]
Since $v_i := (f_i, 0, \ldots, 0)$ belongs to $C^\infty(U; \mathbb{R}^m)$ and $\text{div} v_i = \varphi_i$, integration by parts shows that $F_i = \Gamma(v_i)$. As $F = F\res\Omega_0$ and $F_i = F_i\res\Omega_0$, it remains to prove $\lim \|F - F_i\|_{s,K} = 0$ for $K := \text{cl} \Omega_0$. To this end choose an $\varepsilon > 0$, and use Proposition 2.6 to find a $\theta > 0$ so that
\[
|F(g)| \leq \theta |g|_1 + \varepsilon \|g\|
\]
for each $g \in BV(\mathbb{R}^m)$ with $\{g \neq 0\} \subset K$. Select such a $g$, and let $g_x(z) := g(x - z)$ for all $x, z \in \mathbb{R}^m$. By an argument identical to the proof of [13, Lemma 4.2.1],
\[
|g_x - g_y|_1 \leq |x - y| \cdot \|g\|
\]
for all $x, y \in \mathbb{R}^m$. This and Fubini’s theorem yield
\[
|g - g \ast \eta_i|_1 = \int_{\mathbb{R}^m} \left| g(x) \int_{\mathbb{R}^m} \eta_i(y) \, dy - \int_{\mathbb{R}^m} g_x(y) \eta_i(y) \, dy \right| \, dx
\]
\[
\leq \int_{\mathbb{R}^m} \eta_i(y) \left( \int_{\mathbb{R}^m} |g_0(x) - g_0(-x)| \, dx \right) \, dy
\]
\[
\leq \int_{\mathbb{R}^m} \eta_i(y) \|g_0 - g_y\|_1 \, dy \leq \int_{B(1/4)} \eta_i(y) |y| \cdot \|g\| \, dy \leq \frac{1}{i} \|g\|.
\]
Combining the above inequalities, we obtain
\[
|F(g) - F_i(g)| = |F(g - g \ast \eta_i)| \leq \theta |g - g \ast \eta_i|_1 + \varepsilon \|g - g \ast \eta_i\|
\]
\[
\leq \frac{\theta}{i} \|g\| + \varepsilon (\|g\| + \|g \ast \eta_i\|) = \|g\| \left( \frac{\theta}{i} + 2\varepsilon \right)
\]
for $i = 1, 2, \ldots$, and the lemma follows from the arbitrariness of $\varepsilon$. \qed

The dual space of a topological vector space $X$ is denoted by $X^*$. Aside from the $w^*$-topology on $X^*$, we will also use the strong topology defined by the uniform convergence on the family of all bounded subsets of $X$ [6, Section 1.8.7 and 8.4].

**Proposition 3.2.** There is a linear bijection $\Phi : BV_c(U) \rightarrow CH_s(U)^*$ defined by
\[
\langle \Phi(g), F \rangle := \langle F, g \rangle
\]
for each $g \in BV_c(U)$ and each $F \in CH_s(U)$.

\[\text{For traditional reasons we still write } \mathcal{D}'(U) \text{ rather than } \mathcal{D}(U)^*.\]
Proof. Clearly \( \Phi \) is a linear map. Since
\[
\left| \langle \Phi(g), F \rangle \right| = \left| \langle F, g \rangle \right| \leq \|g\| \cdot \|F\|_{s,K}
\]
for each \( F \in CH_s(U) \), each compact set \( K \subset U \), and each \( g \in BV_c(U) \) with \( \{g \neq 0\} \subset K \), we see that \( \Phi \) maps \( BV_c(U) \) to \( CH_s(U)^* \). If \( g \in BV_c(U) \) and \( \Phi(g) = 0 \), then
\[
\int_B g(x) \, dx = \int_U \chi_B(x) g(x) \, dx = \langle \Lambda(\chi_B), g \rangle = \langle \Phi(g), \Lambda(\chi_B) \rangle = 0
\]
for each bounded measurable set \( B \subset U \). Consequently \( \Phi \) is injective.

Let \( T \in CH_s(U)^* \). As \( \Lambda_s : L^m_{loc}(U) \rightarrow CH_s(U) \) is continuous, \( T \circ \Lambda_s \in L^m_{loc}(U)^* \). Using the duality of \( L^p \) spaces [14, Theorem 6.16], find a \( g \in L^{m/(m-1)}_c(U) \) so that
\[
\langle T, \Lambda(f) \rangle = \langle T \circ \Lambda_s, f \rangle = \int_U f(x)g(x) \, dx
\]
for every \( f \in L^m_{loc}(U) \). Now choose a \( v \in C^1(U; \mathbb{R}^m) \) with \( |v|_{\infty} \leq 1 \). Since
\[
\langle \Lambda(\text{div } v), h \rangle = \int_U h(x) \text{div } v(x) \, dx \leq \|h\|
\]
for each \( h \in BV_c(U) \), we infer \( \|\Lambda(\text{div } v)\|_{s,C} \leq 1 \) for every compact set \( C \subset U \). By the continuity of \( T \), there are a \( c > 0 \) and a compact set \( K \subset U \) such that
\[
\int_U g(x) \text{div } v(x) \, dx = \langle T, \Lambda(\text{div } v) \rangle \leq c \|\Lambda(\text{div } v)\|_{s,K} \leq c.
\]
Thus \( g \in BV_c(U) \) by the arbitrariness of \( v \), and for each \( f \in L^m_{loc}(U) \),
\[
\langle \Phi(g), \Lambda(f) \rangle = \langle \Lambda(f), g \rangle = \int_U f(x)g(x) \, dx = \langle T, \Lambda(f) \rangle.
\]
As \( \Lambda[L^m_{loc}(U)] \) is dense in \( CH_s(U) \) by Lemma 3.1, we conclude \( T = \Phi(g) \).
\( \square \)

A set \( C \subset U \) is called amiable if it is compact, and for each connected component \( V \) of \( U - C \), either \( d(V) = \infty \) or \( \partial V \cap \partial U \neq \emptyset \).

**Lemma 3.3.** Each compact set \( K \subset U \) is contained in an amiable set \( C \subset U \).

**Proof.** Denote by \( W \) the collection of all bounded connected components \( W \) of \( U - K \) with \( \partial W \subset K \), and by \( V \) the collection of all other connected components of \( U - K \). Given \( W \in \mathcal{W} \), observe that
\[
\text{dist}(W, \partial U) = \text{dist}(\partial W, \partial U) \geq \text{dist}(K, \partial U),
\]
and as \( W \) is bounded, also \( d(W) = d(\partial W) \leq d(K) \). Consequently
\[
d(\bigcup \mathcal{W}) \leq 3d(K) \quad \text{and} \quad \text{dist}(\bigcup \mathcal{W}, \partial U) \geq \text{dist}(K, \partial U).
\]
Since \( \bigcup \mathcal{W} \) is a relatively closed subset of \( U - K \), the previous inequalities imply that \( C := K \cup \bigcup \mathcal{W} \) is a compact subset of \( U \). If \( V \in \mathcal{V} \) is bounded, then \( \partial V \) is a subset of \( \partial(U - K) = \partial U \cap \partial K \), but not a subset of \( K \). Thus \( V \) is either unbounded, or its boundary meets the boundary of \( U \). But \( \mathcal{V} \) is the collection of all connected components of \( U - C \), and hence \( C \) is amiable. \( \square \)

**Observation 3.4.** Let \( g \in BV_c(U) \), and let the support of \( Dg \) be contained in an amiable set \( C \subset U \). Then the support of \( g \) is contained in \( C \).
Proposition 3.6. Observe that $g$ is constant in each connected component of $U - C$. If the support of $g$ meets a connected component $V$ of $U - C$, then $V \cap (\text{supp } g)$ is a proper subset of $V$; since $V$ is either unbounded or $\partial V \cap \partial U \neq \emptyset$. As $V$ is open, $V \cap \{g \neq 0\}$ is also a proper subset of $V$, a contradiction. \hfill \Box

Lemma 3.5. Let $\{g_i\}$ be a sequence in $BV_c(U)$ such that

$$\sup \left\{ \int_{U} v \cdot d(Dg_i) : v \in B \text{ and } i = 1, 2, \ldots \right\} < \infty$$

for each bounded set $B \subset C(U; \mathbb{R}^m)$. Then $\{g_i\}$ is compactly supported.

Proof. There are open sets $U_i$ such that $K_i := \text{cl } U_i$ is contained in $U_{i+1}$, and $U = \bigcup_{i=1}^{\infty} U_i$. If the sequence $\{\text{supp } Dg_i\}$ is not compactly supported, then we can construct inductively subsequences of $\{U_i\}$ and $\{g_i\}$, still denoted by $\{U_i\}$ and $\{g_i\}$, so that $\text{supp } Dg_i$ meets the open set $U_{i+1} - K_i$. Consequently, there are $v_i \in C(U; \mathbb{R}^m)$ supported in $U_{i+1} - K_i$ such that $|v|_{\infty} \leq 1$ and $a_i := \int_{U_i} v_i \cdot d(Dg_i)$ is different from zero. Let $b_i = \max\{|a_1|^{-1}, \ldots, |a_i|^{-1}\}$. The bounded set

$$B := \left\{ v \in C(U; \mathbb{R}^m) : |v|_{\infty, K_{i+1}} \leq ib_i \text{ for } i = 1, 2, \ldots \right\}$$

contains $w_i := (ib_i)v_i$, $i = 1, 2, \ldots$. As $|\int_{U_i} w_i \cdot Dg_i| \geq i$, we have a contradiction. Thus there is a compact set $K \subset U$ containing the support of each $Dg_i$. An application of Lemma 3.3 and Observation 3.4 completes the argument. \hfill \Box

Proposition 3.6. If $\Gamma^*$ is the adjoint map of

$$\Gamma : C(U; \mathbb{R}^m) \rightarrow CH_s(U),$$

then $\Gamma^*[CH_s(U)^*]$ is sequentially closed in the strong topology of $C(U; \mathbb{R}^m)^*$. 

Proof. Denote by $M_c(U; \mathbb{R}^m)$ the linear space of all compactly supported Radon measures in $U$ with values in $\mathbb{R}^m$, and define maps

$$D : g \mapsto Dg : BV_c(U) \rightarrow M_c(U; \mathbb{R}^m),$$

$$\varphi : \mu \mapsto T_\mu : M_c(U; \mathbb{R}^m) \rightarrow C(U; \mathbb{R}^m)^*,$$

where $T_\mu(v) := -\int_{U} v \cdot d\mu$ for each $v \in C(U; \mathbb{R}^m)$. Observe that the diagram

$$\begin{array}{ccc}
BV_c(U) & \xrightarrow{D} & M_c(U; \mathbb{R}^m) \\
\Phi \downarrow & \quad & \downarrow e \\
CH_s(U)^* & \xrightarrow{T^*} & C(U; \mathbb{R}^m)^*
\end{array}$$

(3.2)

commutes. Hence for $S \in CH_s(U)^*$, $v \in C(U; \mathbb{R}^m)$, and $g := \Phi^{-1}(S)$,

$$\langle \Gamma^*(S), v \rangle = -\int_{U} v \cdot d(Dg).$$

(3.3)

Select a sequence $\{S_i\}$ in $CH_s(U)^*$ so that $\{\Gamma^*(S_i)\}$ converges strongly to a $T$ in $C(U; \mathbb{R}^m)^*$, and note that $\{\Gamma^*(S_i)\}$ is uniformly bounded on each bounded subset of $C(U; \mathbb{R}^m)$. Applying (3.3) to $g_i := \Phi^{-1}(S_i)$, Lemma 3.5 implies that the sequence $\{g_i\}$ in $BV_c(U)$ is compactly supported. As

$$B := \left\{ v \in C(U; \mathbb{R}^m) : |v|_{\infty} \leq 1 \right\}$$

Theorem 3.7. is a bounded subset of $C(U; \mathbb{R}^m)$, we have

$$\|R\| := \sup\{\langle R, v \rangle : v \in B\} < \infty$$

for each $R \in C(U; \mathbb{R}^m)^*$. Since $\lim\|\Gamma^*(S_i) - T\| = 0$, there is a $c > 0$ such that

$$\|g_i\| = \sup\left\{ \int_U v \cdot d(Dg_i) : v \in C^1_c(U; \mathbb{R}^m) \text{ and } |v|_\infty \leq 1 \right\}$$

$$\leq \sup\{ \langle \Gamma^*(S_i), v \rangle : v \in B\} = \|\Gamma^*(S_i)\| \leq c$$

for $i = 1, 2, \ldots$. By Poincaré inequality, there is a constant $\kappa > 0$, depending only on the dimension $m$, such that $|g_i|_{\frac{m}{m-1}} \leq \kappa \|g_i\|$; in particular $g_i \in L^{\frac{m}{m-1}}(U)$. Since $L^{\frac{m}{m-1}}(U)$ is the dual space of $L^m(U)$, and since

$$V := \left\{ h \in L^m(U) : |h|_m \leq \frac{1}{\kappa c} \right\}$$

is a neighborhood of zero in $L^m(U)$, the Banach-Alaoglu theorem [15, Section 3.15] shows that

$$\mathcal{K} := \left\{ f \in L^{\frac{m}{m-1}}(U) : \left| \int_U f(x) h(x) \, dx \right| \leq 1 \text{ for each } h \in V \right\}$$

is w*-compact subset of $L^{\frac{m}{m-1}}(U)$. By the Hölder and Poincaré inequalities,

$$\left| \int_U g_i(x) h(x) \, dx \right| \leq |g_i|_{\frac{m}{m-1}} |h|_m \leq \kappa \|g_i\| \cdot |h|_m \leq \kappa c |h|_m \leq 1$$

for each $g_i$ and each $h \in V$. Thus the sequence $\{g_i\}$ has a w*-cluster point $g \in \mathcal{K}$. As $\{g_i\}$ is compactly supported, supp $g$ is a compact subset of $U$; in particular $g \in L^1(U)$. Equality (3.3) implies

$$\lim \langle \Gamma^*(S_i), v \rangle = \lim \int_U g_i(x) \div v(x) \, dx = \int_U g(x) \div v(x) \, dx$$

for each $v \in C^1_c(U; \mathbb{R}^m)$; the last equality holds, since $\int_U g \div v$ is the cluster point of a convergent sequence $\{\int_U g_i \div v\}$. Hence for $v \in C^1_c(U; \mathbb{R}^m)$ with $|v|_\infty \leq 1,$

$$\int_U g(x) \div v(x) \, dx \leq \sup \|g_i\| \leq c.$$

We infer $g \in BV_c(U)$, and let $S := \Phi(g)$. By equalities (3.4) and (3.3),

$$\langle T, v \rangle = \lim \langle \Gamma^*(S_i), v \rangle = -\lim \int_U v \cdot d(Dg_i)$$

$$= -\int_U v \cdot d(Dg) = \langle \Gamma^*(S), v \rangle$$

for each $v \in C^1_c(U; \mathbb{R}^m)$. As $C^1_c(U; \mathbb{R}^m)$ is a dense subspace of $C(U; \mathbb{R}^m)$, we see that $T = \Gamma^*(S)$ belongs to $\Gamma^* [CH_s(U)^*]$.

\[ \square \]

**Theorem 3.7.** $\mathcal{F}(U) = CH_s(U)$.

**Proof.** According to the Closed Range Theorem [6, Theorem 8.6.13], the following claims are equivalent:

(a) $\Gamma^* [CH_s(U)^*]$ is strongly closed in $C(U; \mathbb{R}^m)^*$;

(b) $\Gamma^* [CH_s(U)^*]$ is w*-closed in $C(U; \mathbb{R}^m)^*$;

Proof. According to the Closed Range Theorem [6, Theorem 8.6.13], the following claims are equivalent:
Theorem 3.8. Let $F \in CH_s(U)$. For each $\varepsilon > 0$ and each amiable set $K \subset U$, there is a $v \in C(U; \mathbb{R}^m)$ such that $\Gamma(v) = F$ and

$$\|F\|_{s,K} \leq |v|_{\infty,K} \leq (1 + \varepsilon)\|F\|_{s,K}.$$ 

Proof. The first inequality, which holds for any compact set $K \subset U$, is obvious. Choose an $\varepsilon > 0$ and an amiable set $K \subset U$. We simplify the notation by letting $|v| := |v|_{\infty,K}$ for each $v \in C(U; \mathbb{R}^m)$, and $\|F\| := \|F\|_{s,K}$. To avoid a triviality, assume that $\|F\| > 0$. It suffices to show that the nonempty convex sets

$$A := \{v \in C(U; \mathbb{R}^m) : |v| < (1 + \varepsilon)\|F\|\},$$

$$B := \{v \in C(U; \mathbb{R}^m) : \Gamma(v) = F\}$$

have a nonempty intersection. Proceeding toward a contradiction suppose that $A \cap B = \emptyset$. As $A$ is open, it follows from the Hahn-Banach theorem that there are $T \in [C(U; \mathbb{R}^m)]^*$ and $\gamma \in \mathbb{R}$ such that

$$\langle T, v \rangle < \gamma \leq \langle T, w \rangle \quad (3.5)$$

for each $v \in A$ and each $w \in B$ [15, Theorem 3.4, (a)]. Note $\gamma > 0$, because $v = 0$ belongs to $A$. For the reminder of the proof, select a $w \in B$. If $u \in \Gamma^{-1}(0)$, then $w + tu$ belongs to $B$ for each $t \in \mathbb{R}$. Hence $t\langle T, u \rangle \geq \gamma - \langle T, w \rangle$ for each $t \in \mathbb{R}$, and consequently $\langle T, u \rangle = 0$. Therefore $\Gamma^{-1}(0) \subset T^{-1}(0)$. Since $\Gamma$ is surjective, and hence open by the Open Mapping Theorem [15, Corollary 2.12, (a)], there is an $S \in [CH_s(U)]^*$ with $T = S \circ \Gamma$. The function $g := \Phi^{-1}(S)$ belongs to $BV_c(U)$, and

$$\int_U v \cdot d(Dg) = \langle \Gamma(v), g \rangle = \langle S, \Gamma(v) \rangle = \langle T, v \rangle \quad (3.6)$$

for each $v \in C(U; \mathbb{R}^m)$. If $v \in C(U; \mathbb{R}^m)$ and $\{v \neq 0\} \cap K = \emptyset$, then $|v| = 0$. Thus both $tv$ and $-tv$ belong to $A$ for each $t \in \mathbb{R}$, and inequality (3.5) implies $Tv = 0$. By equality (3.6), the support of $Dg$ is contained in $K$, and by Observation 3.4, so is the support of $g$. Choose a positive $\eta < \varepsilon$ and a $u \in C^1_c(\mathbb{R}^m; \mathbb{R}^m)$ with $|u|_{\infty} \leq 1$. Clearly $v := -(1 + \eta)\|F\|u$ belongs to $A$, and by (3.6) and (3.5),

$$\int_U g(x) \text{ div } u(x) \, dx = -\int_U u \cdot d(Dg) = \frac{1}{(1 + \eta)\|F\|} \int_U v \cdot d(Dg)$$

$$= \frac{1}{(1 + \eta)\|F\|} \langle T, v \rangle < \frac{\gamma}{(1 + \eta)\|F\|}.$$ 

We infer $\|g\| \leq \gamma/[(1 + \eta)\|F\|]$. As the support of $g$ is contained in $K$, a contradiction follows from (3.5) and (3.6):

$$\gamma \leq \langle T, w \rangle = \langle \Gamma(w), g \rangle = \langle F, g \rangle \leq \|g\| \cdot \|F\| \leq \frac{\gamma}{1 + \eta} < \gamma.$$ 

\[\square\]
Let \( K \subset \mathbb{R}^m \) be a compact set, and let \( BV(K) \) be the linear space of all functions \( g \in BV(\mathbb{R}^m) \) with \( \{ g \neq 0 \} \subset K \). A linear functional \( F : BV(K) \to \mathbb{R} \) is called an \( s \)-charge in \( K \) if given \( \varepsilon > 0 \), there is a \( \theta > 0 \) such that
\[
|F(g)| \leq \theta |g_1| + \varepsilon \|g\|
\]
for each \( g \in BV(K) \). The linear space of all \( s \)-charges in \( K \), denoted by \( CH_s(K) \), is equipped with the Banach norm
\[
\|F\|_s := \sup \{ F(g) : g \in BV(K) \text{ and } \|g\| \leq 1 \}
\]
for \( F \in CH_s(K) \). Given \( K \subset U \), the restriction map \( \rho_s : F \mapsto F|_{BV(K)} : CH_s(U) \to CH_s(K) \) is linear and continuous. If \( \Omega \) is a bounded Lipschitz domain, then \( CH_s(\text{cl} \Omega) \) is linearly homeomorphic to \( \{ F \in CH_s(\mathbb{R}^m) : F = F|_{\Omega} \} \) topologized as a subspace of \( CH_s(\mathbb{R}^m) \).

As the definitions of \( s \)-charges in an open set \( U \) and a in compact set \( K \) are similar, most of the properties established for \( s \)-charges in \( U \) hold also for \( s \)-charges in \( K \), and the corresponding proofs are analogous. Since \( CH_s(K) \) is a Banach space, proving properties of \( s \)-charges in \( K \) is often less technical.

Let \( K \subset \mathbb{R}^m \) be a compact set. If \( v \in C(K; \mathbb{R}^m) \), then the functional
\[
F_v : g \mapsto \int_K v \cdot d(Dg) : BV(K) \to \mathbb{R}
\]
is an \( s \)-charge in \( K \), still called the flux of \( v \). Topologizing \( C(K; \mathbb{R}^m) \) by the Banach norm \( |v|_\infty \), we have a continuous linear surjection
\[
\Gamma_K : v \mapsto F_v : C(K; \mathbb{R}^m) \to CH_s(K)
\]
(cf. Theorem 3.7), and the following diagram commutes
\[
\begin{array}{ccc}
C(\mathbb{R}^m; \mathbb{R}^m) & \xrightarrow{\rho} & C(K; \mathbb{R}^m) \\
\Gamma \downarrow & & \downarrow \Gamma_K \\
CH_s(\mathbb{R}^m) & \xrightarrow{\rho_s} & CH_s(K)
\end{array}
\]
As the restriction map \( \rho : v \mapsto v|_K \) is surjective, so is \( \rho_s \); in particular
\[
CH_s(K) = \{ F : BV(K) : F \in CH_s(\mathbb{R}^m) \}.
\]
However, note that for an \( F \in CH_s(\mathbb{R}^m) \), the inclusion
\[
\{ v \mid K : v \in \Gamma^{-1}(F) \} \subset \Gamma_K^{-1}[F \mid BV(K)]
\]
may be proper. The next proposition, whose proof is analogous to that of Theorem 3.8, holds for any compact set \( K \subset \mathbb{R}^m \).

**Proposition 3.9.** Let \( F \) be an \( s \)-charge in a compact set \( K \), and let \( \varepsilon > 0 \). There is a \( v \in C(K; \mathbb{R}^m) \) such that \( F = \Gamma_K(v) \) and
\[
\|F\|_s \leq |v|_\infty \leq (1 + \varepsilon)\|F\|_s.
\]
4. Rotation invariant charges

In this section we consider $\Gamma$ restricted to a map from the space of all rotation invariant vector fields to the space of all rotation invariant $s$-charges, and construct a continuous right inverse of $\Gamma$.

Working with the standard orthonormal base in $\mathbb{R}^m$, we view the special orthogonal group $SO := SO(m)$ as the multiplicative group of orthogonal matrices with positive determinants, and employ the usual matrix multiplication. Vectors and one-forms are viewed as one-column and one-row matrices, respectively. In particular, $x \in \mathbb{R}^m$ is a one-column matrix, and the gradient $\nabla \varphi$ of a $\varphi \in C^1(\mathbb{R}^m)$ is a one-row matrix; in this interpretation, $x \cdot \nabla \varphi(x) = [\nabla \varphi(x)]^T x$. The Haar probability on $SO$ is denoted by $\theta$.

Throughout this section, select a positive $R \leq \infty$, and let

$$U := \{ x \in \mathbb{R}^m : |x| < R \} \quad \text{and} \quad U_0 := \{ x \in \mathbb{R}^m : 0 < |x| < R \}.$$ 

The group $SO$ acts linearly and continuously on the spaces $BV_c(U)$, $CH_s(U)$, and $C(U; \mathbb{R}^m)$ by the following rules:

$$(A \bullet g, x) := \langle g, Ax \rangle, \quad (A \bullet F, g) := \langle F, A \bullet g \rangle, \quad (A \bullet v, x) := A^{-1}(v, Ax)$$

for every $A \in SO$, $g \in BV_c(U)$, $F \in CH_s(U)$, $v \in C(U; \mathbb{R}^m)$, and $x \in U$. Let

$$CH^\text{inv}_s(U) := \{ F \in CH_s(U) : A \bullet F = F \text{ for each } A \in SO \},$$

$$C^\text{inv}(U; \mathbb{R}^m) := \{ v \in C(U; \mathbb{R}^m) : A \bullet v = v \text{ for each } A \in SO \},$$

and give these spaces the subspace topology. If $v \in C^\text{inv}(U; \mathbb{R}^m)$ then $v(0) = 0$, since $A^{-1}v(0) = v(0)$ for each $A \in SO$. Observe

$$\langle \Gamma(A \bullet v), \varphi \rangle = - \int_U \nabla \varphi(x) \big[A(v, A^{-1}x)\big] \, dx = - \int_U \big[\nabla \varphi(Ay)A\big] v(y) \, dy$$

$$= - \int_U \nabla(A \bullet \varphi)(y)v(y) \, dy = \langle \Gamma(v), A \bullet \varphi \rangle = \langle A \bullet \Gamma(v), \varphi \rangle$$

(4.1)

for every $A \in SO$, $v \in C(U; \mathbb{R}^m)$, and $\varphi \in \mathcal{D}(U)$. Thus $\Gamma(A \bullet v) = A \bullet \Gamma(v)$, and it follows that $\Gamma$ maps $C^\text{inv}(U; \mathbb{R}^m)$ into $CH^\text{inv}_s(U)$.

**Observation 4.1.** The map $\Gamma : C^\text{inv}(U; \mathbb{R}^m) \to CH^\text{inv}_s(U)$ is surjective.

**Proof.** If $F \in CH^\text{inv}_s(U)$ then by Theorem 3.7, there is a $v \in C(U; \mathbb{R}^m)$ such that $\Gamma(v) = F$. Defining a $w \in C^\text{inv}(U; \mathbb{R}^m)$ by the formula

$$w := \int_{SO} A \bullet v \, d\theta(A),$$

we have $\Gamma(w) = F$. Indeed for each $\varphi \in \mathcal{D}(U)$, Fubini’s theorem and (4.1) yield

$$\langle \Gamma(w), \varphi \rangle = \int_U w(x) \cdot \nabla \varphi(x) \, dx = \int_{SO} \int_U (A \bullet v)(x) \cdot \nabla \varphi(x) \, dx \, d\theta(A)$$

$$= \int_{SO} \langle \Gamma(A \bullet v), \varphi \rangle \, d\theta(A) = \int_{SO} \langle A \bullet \Gamma(v), \varphi \rangle \, d\theta(A)$$

$$= \int_{SO} \langle A \bullet F, \varphi \rangle \, d\theta(A) = \int_{SO} \langle F, \varphi \rangle \, d\theta(A) = \langle F, \varphi \rangle.$$

$\square$
Throughout the reminder of this section, we let \( B_r := B[r] \) for \( r > 0 \). View the sphere \( S := \partial B_1 \) as a Riemannian submanifold of \( \mathbb{R}^m \), and denote by \( T_x S \) its tangent space at \( x \in S \). The measure \( \mathcal{H}/\|B_1\| \) defines an \( SO \) invariant probability in \( S \), denoted by \( \sigma \). For \( x \in S \) and \( \varphi \in \mathcal{D}(S) \), let
\[
|\nabla \varphi|(x) := \sup \{|X\varphi| : X \in T_x S \text{ and } |X| = 1\},
\]
\[
\|\varphi\| := \int_S |\nabla \varphi|(x) \, d\sigma(x).
\]
With this notation at hand, we can introduce charges and s-charges in \( S \) and s-charges in the obvious modification of Definition 2.3. Observation 2.4 readily translates to charges if Proposition 4.2. (cf. Remark 2.8). A charge \( G \) in \( S \) is called invariant if
\[
\langle G, A \cdot \varphi \rangle = \langle G, \varphi \rangle
\]
for each \( A \in SO \) and each \( \varphi \in \mathcal{D}(S) \).

**Proposition 4.2.** If \( G \) is an invariant charge in \( S \), then
\[
G(g) = G(S) \int_S g(x) \, d\sigma(x)
\]
for each \( g \in BV(S) \).

**Proof.** In view of Observation 2.4, it suffices to prove the proposition when \( g \) is a test function. Choose a \( \varphi \in \mathcal{D}(S) \), and for each \( x \in S \), let
\[
f(x) := \int_{SO} \varphi(Ax) \, d\theta(A).
\]
Since \( f(Bx) = f(x) \) for each \( B \in SO \), and since \( SO \) acts transitively on \( S \), the function \( f \) equals a constant \( c \). By Fubini’s theorem
\[
c = \int_S f(x) \, d\sigma(x) = \int_{SO} \left[ \int_S \varphi(Ax) \, d\sigma(x) \right] \, d\theta(A)
\]
\[
= \int_{SO} \left[ \int_S \varphi(x) \, d\sigma(x) \right] \, d\theta(A) = \int_S \varphi(x) \, d\sigma(x),
\]
and hence
\[
G(f) = G(c\chi_S) = cG(S) = G(S) \int_S \varphi(x) \, d\sigma(x).
\]
We complete the proof by showing that \( G(\varphi) = G(f) \). To this end, consider collections \( P := \{(E_1, A_1), \ldots, (E_p, A_p)\} \) such that \( E_1, \ldots, E_p \) are disjoint Borel subsets of \( SO \) whose union is \( SO \), and \( A_i \in E_i \) for \( i = 1, \ldots, p \). Given such a collection \( P \), define a test function \( f_P := \sum_{i=1}^p (A_i \cdot \varphi) \theta(E_i) \), and observe
\[
|f_P|_{\infty} \leq \sum_{i=1}^p |A_i \cdot \varphi|_{\infty} \theta(E_i) \leq |\varphi|_{\infty} \sum_{i=1}^p \theta(E_i) = |\varphi|_{\infty}, \tag{4.2}
\]
\[
\|f_P\| \leq \sum_{i=1}^p \|A_i \cdot \varphi\| \theta(E_i) \leq \|\varphi\| \sum_{i=1}^p \theta(E_i) = \|\varphi\|.
\]
The first inequality is obvious, and since \( |\nabla (A_i \cdot \varphi)|(x) = |\nabla \varphi|(A_i x) \) for each \( x \in S \) and \( i = 1, \ldots, p \), the second one follows. The function \( (A, x) \mapsto \varphi(Ax) \) is uniformly continuous on \( SO \times S \). Thus making the diameter of each \( E_i \) sufficiently small,
Proposition 4.3. The map $f$ approximates $f$ uniformly with an arbitrary precision; in particular, $f_P$ can be arbitrarily close to $f$ in the $L^1$ norm of $L^1(S,\sigma)$. In view of Remark 2.7, this and inequalities (4.2) imply that $G(f_P)$ can be arbitrarily close to $G(f)$. Since

$$G(f_P) = \sum_{i=1}^P \theta(E_i)G(A_i\cdot\varphi) = G(\varphi)\sum_{i=1}^P \theta(E_i) = G(\varphi).$$

for each $P$, we obtain $G(f) = G(\varphi)$. □

Proposition 4.3. The map $\Gamma : C_{\text{inv}}^\text{inv}(U;\mathbb{R}^m) \rightarrow C_{\text{inv}}^\text{inv}(U)$ has a linear right inverse $\Upsilon : C_{\text{inv}}^\text{inv}(U) \rightarrow C_{\text{inv}}^\text{inv}(U;\mathbb{R}^m)$ defined for each $F \in C_{\text{inv}}^\text{inv}(U)$ by the formulas

$$\langle \Upsilon(F), x \rangle := \frac{F(B_r)}{\|B_r\|} \cdot \frac{x}{r} \quad (4.3)$$

if $r := |x| > 0$, and $\langle \Upsilon(F), 0 \rangle := 0$. The equality $|\Upsilon(F)|_{\infty, B_r} = \|F\|_{s,B_r}$ holds for each positive $r < R$; in particular $\Upsilon$ is continuous.

Proof. Clearly $\Upsilon$ is a linear map. Select an $F \in C_{\text{inv}}^\text{inv}(U)$, and note that $v := \Upsilon(F)$ belongs to $C_{\text{inv}}^\text{inv}(U;\mathbb{R}^m)$ by Proposition 2.6. We show that $F = F_v$. If $E$ is a BV set in $S$, then it is easy to verify that

$$C_E := \{sx : x \in E \text{ and } 0 \leq s \leq 1\}$$

is a bounded BV subset of $\mathbb{R}^m$, called a BV cone in $B_1$. Assuming that $R > 1$ and using Remark 2.8, we can define an invariant charge $G$ in $S$ by letting $G(E) := F(C_E)$ for each BV set $E$ in $S$. By Proposition 4.2,

$$F(C_E) = G(E) = G(S)\frac{\Upsilon(E)}{\|B_1\|} = \frac{F(B_1)}{\|B_1\|} \Upsilon(E)$$

$$= \int_{\partial_s C_E} v \cdot \nu_{C_E} \, d\mathcal{H} = F_v(C_E)$$

for every BV set $E$ in $S$. Thus $R > 1$ implies that $F$ and $F_v$ coincide on all BV cones in $B_1$. By the obvious extrapolation, $F$ and $F_v$ coincide on all BV cones in $B_r$ with $0 < r < R$. Cover $U_0$ by charts $(J_1, \phi_1), \ldots, (J_n, \phi_n)$ where $J_i$ are open subintervals of $\mathbb{R}^m$ and $\phi_i : J_i \rightarrow U_0$ are defined by means of the spherical coordinates. If $K$ is a compact subinterval of $J_i$, we call $\phi_i(K)$ a “rectangle” in $U_0$. Up to a negligible set, each “rectangle” in $U_0$ has the form $C_s - (C_s \cap B_r)$ where $0 < r < s < R$ and $C_s$ is a BV cone in $B_s$. By additivity, $F$ and $F_v$ coincide on all “rectangles” in $U_0$, and in view of Remark 2.8, they coincide on all bounded BV sets $E$ with $\text{cl} \, E \subset U_0$. If $E$ is a bounded BV subset of $U$, then

$$F(E) = \lim_{r \rightarrow 0^+} F(E - B_r) = \lim_{r \rightarrow 0^+} F_v(E - B_r) = F_v(E)$$

and Remark 2.8 implies $F = F_v$. Since $|v(x)| \leq \|F\|_{s,B_r}$ for each $x \in B_r$, we have $|\Upsilon(F)|_{\infty,B_r} = |v|_{\infty,B_r} \leq \|F\|_{s,B_r}$. The reverse inequality has been established prior to Lemma 3.1:

$$\|F\|_{s,B_r} = \|\langle F, \Upsilon(F) \rangle\|_{\infty,B_r} \leq |\Upsilon(F)|_{\infty,B_r}.$$

Noting that each compact subset of $U$ is contained in $B_r$ for some $r < R$ completes the argument. □
Remark 4.4. We present a different proof of Proposition 4.3, which is available in dimension $m = 2$, but may not generalize to higher dimensions.

For $x = (\xi_1, \xi_2)$ in $\mathbb{R}^2$, let $\bar{x} = (-\xi_2, \xi_1)$. Given $v \in C^{\text{inv}}(U; \mathbb{R}^m)$, there are continuous functions $a_1, a_2$ defined on $[0, R)$ such that $a_1(0) = a_2(0) = 0$, and

$$v(x) = a_1(|x|)x + a_2(|x|)\bar{x}$$

for each $x \in U$. Define vector fields $\pi_i v \in C^{\text{inv}}(U; \mathbb{R}^2)$, $i = 1, 2$, by

$$\pi_1 v(x) := a_1(|x|)x \quad \text{and} \quad \pi_2 v(x) := a_2(|x|)\bar{x}$$

for every $x \in U$. Interpreting derivatives in the distributional sense, observe that $\text{div} \pi_2 v = 0$, and that $\text{div} \pi_1 v = 0$ implies $t a_1'(t) + 2a_1(t) = 0$ for $0 < t < R$. In $(0, R)$, the continuous distributional solutions of the last equation are the same as the classical solutions $a_1(t) = ct^{-2}$ where $c \in \mathbb{R}$. As $a_1$ is bounded in the neighborhood of zero, $\text{div} \pi_1 v = 0$ implies $\pi_1 v = 0$. Now

$$\langle \Gamma(v), \varphi \rangle = -\int_U \pi_1 v(x) \cdot \nabla \varphi(x) \, dx - \int_U \pi_2 v(x) \cdot \nabla \varphi(x) \, dx$$

$$= \int_U \varphi(x) \text{div} \pi_1 v(x) \, dx + \int_U \varphi(x) \text{div} \pi_2 v(x) \, dx$$

$$= \int_U \varphi(x) \text{div} \pi_1 v(x) \, dx$$

for each $\varphi \in \mathcal{D}(U)$, and we conclude that $\Gamma(v) = 0$ implies $\pi_1 v = 0$.

Choose an $F \in CH_a^{\text{inv}}(U)$, and use Observation 4.1 to find a $v \in C^{\text{inv}}(U; \mathbb{R}^2)$ with $\Gamma(v) = F$. By the previous paragraph, $\pi_1 v \in C^{\text{inv}}(U; \mathbb{R}^2)$ does not depend on the choice of $v$. Thus letting $\Upsilon(F) = \pi_1 v$ for any $v \in C^{\text{inv}}(U; \mathbb{R}^2)$ with $\Gamma(v) = F$, we have defined a right inverse $\Upsilon$ of $\Gamma : C^{\text{inv}}(U; \mathbb{R}^2) \rightarrow CH^a_{s}(U)$. Since

$$F(B_r) = \int_{\partial B_r} \pi_1 v \cdot \nu_{B_r} \, dH = ra_1(r)\|B_r\|$$

for $0 < r < R$, the vector field $\Upsilon(F)$ is defined by formula (4.3).

5. Charges

Under the name “continuous additive functions”, charges were introduced in [12] as a common generalization of ac-charges and fluxing distributions. They facilitate a definition of a multidimensional Riemann type integral that provides a Gauss-Green theorem for any differentiable vector field (cf. Section 6 below). In this section, we show that the common generalization given by charges is minimal: the space $CH(U)$ of all charges in $U$ is the smallest linear space containing both $CH_{ac}(U)$ and $CH_s(U)$. The idea of the proof is similar to that of Theorem 3.7.

We give $CH(U)$ a Fréchet topology defined by the seminorms

$$\|F\|_K := \sup \{ F(g) : g \in BV(U), \, \{g \neq 0\} \subset K, \text{ and } |g|_\infty + \|g\| \leq 1 \}$$

where $F \in CH(U)$, and $K \subset U$ is a compact set. Since $\|F\|_K \leq \|F\|_{s,K}$ for each $s$-charge $F$, the inclusion map $CH_a(U) \hookrightarrow CH(U)$ is continuous. However, $CH_s(U)$ is not topologized as a subspace of $CH(U)$.

The product topology in $L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)$ is defined by the seminorms

$$\|(f, v)\|_K := \max \{|f|_{1,K}, |v|_{\infty,K}\}.$$
A linear map $\Theta : L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m) \to CH(U)$, defined by the formula
\[
\langle \Theta(f, v), g \rangle := \langle \Lambda(f) + \Gamma(v), g \rangle = \int_U fg d\mathcal{L}^m - \int_U v \cdot d(Dg)
\]
for $(f, v) \in L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)$ and $g \in BV^\infty_c(U)$, is continuous. Indeed,
\[
\|\Theta(f, v)\| \leq |g|_{\infty} \cdot |f|_{1, K} + \|g\| \cdot |v|_{\infty, K} \leq (|g|_{\infty} + \|g\|) \|(f, v)\|_K
\]
whenever $K \subset U$ is compact and $\{g \neq 0\} \subset K$, and hence $\|\Theta(f, v)\|_K \leq \|(f, v)\|_K$.

**Proposition 5.1.** If $\Theta^*$ is the adjoint map of $\Theta$, then $\Theta^*[CH(U)^*]$ is sequentially closed in the strong topology of $[L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)]^*$.

**Proof.** While the proof is similar to that of Proposition 3.6, it is more technical. Denote by $M_c(U; \mathbb{R}^m)$ the linear space of all compactly supported Radon measures with values in $\mathbb{R}^m$, and define maps
\[
i \times D : g \mapsto (g, Dg) : BV^\infty_c(U) \to L^\infty_c(U) \times M_c(U; \mathbb{R}^m),
\]
\[
t : (g, \mu) \mapsto T_{g, \mu} : L^\infty_c(U) \times M_c(U; \mathbb{R}^m) \to [L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)]^*,
\]
where $T_{g, \mu}(f, v) := \int_U fg d\mathcal{L} - \int_U v \cdot d\mu$ for each $(f, v)$ in $L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)$. By [13, Theorem 4.3.5], there is a linear bijection $\Psi : BV^\infty_c(U) \to CH(U)^*$ such that
\[
\langle \Psi(g), F \rangle = \langle F, g \rangle.
\]
for each $g \in BV^\infty_c(U)$ and each $F \in CH$. The diagram
\[
\begin{array}{ccc}
BV^\infty_c(U) & \xrightarrow{\iota \times D} & L^\infty_c(U) \times M_c(U; \mathbb{R}^m) \\
\downarrow \psi & & \downarrow \tau \\
CH(U)^* & \xrightarrow{\Theta^*} & [L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)]^*
\end{array}
\]
commutes. Hence for $S \in CH(U)^*$, $(f, v) \in L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)$, and $g := \Psi^{-1}(S)$,
\[
\langle \Theta^*(S), (f, v) \rangle = \int_U f(x)g(x) dx - \int_U v \cdot d(Dg). \tag{5.1}
\]
Select a sequence $\{S_i\}$ in $CH(U)^*$ so that $\{\Theta^*(S_i)\}$ converges strongly to a $T$ in $[L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)]^*$, and note that $\{\Theta^*(S_i)\}$ is uniformly bounded on each bounded subset of $L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)$. Applying (5.1) to $g_i = \Psi^{-1}(S_i)$ and $(0, v)$, Lemma 3.5 implies that the sequence $\{g_i\}$ in $BV^\infty_c(U)$ is compactly supported. As
\[
B := \{(f, v) \in L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m) : \|f\|_1 \leq 1 \text{ and } |v|_{\infty} \leq 1\}
\]
is a bounded subset of $L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)$, we have
\[
\|R\| := \sup \left\{ \langle R, (f, v) \rangle : (f, v) \in B \right\} < \infty
\]
for each $R \in [L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)]^*$. Since $\lim \|\Theta^*(S_i) - T\| = 0$, there is a $c > 0$ such that $\|\Theta^*(S_i)\| \leq c$ for $i = 1, 2, \ldots$. From
\[
|g_i|_{\infty} = \sup \left\{ \int_U f(x)g_i(x) dx : f \in L^1(U) \text{ and } |f|_1 \leq 1 \right\},
\]
\[
\|g_i\| = \sup \left\{ \int_U v \cdot d(Dg_i) : v \in C^1_c(U; \mathbb{R}^m) \text{ and } |v|_{\infty} \leq 1 \right\},
\]
and equality (5.1), we obtain
\[ |g_i|_\infty + \|g_i\| \leq \sup \left\{ \langle \Theta^*(S_i), (f, v) \rangle : (f, v) \in B \right\} = \|\Theta^*(S_i)\| \leq c. \]

Now \( L^\infty(U) \) is the dual of \( L^1(U) \), and \( V := \{ h \in L^1(U) : |h|_1 \leq 1/c \} \) is a neighborhood of zero in \( L^1(U) \). According to the Banach-Alaoglu theorem,
\[ K := \left\{ f \in L^\infty(U) : \left| \int_U f(x)h(x) \, dx \right| \leq 1 \text{ for each } h \in V \right\} \]
is w*-compact subset of \( L^\infty(U) \). Every \( g_i \) belongs to \( BV_c \subset L^\infty(U) \), and
\[ \left| \int_{\mathbb{R}^m} g_i(x)h(x) \, dx \right| \leq |g_i|_\infty \cdot |h|_1 \leq 1 \]
for each \( h \in V \). Thus the sequence \( \{g_i\} \) has a w*-cluster point \( g \in K \). As \( \{g_i\} \) is compactly supported, \( supp g \) is a compact subset of \( U \). The sequence \( \{\Theta^*(S_i)\} \) converges strongly to \( T \), and a fortiori, it w*-converges to \( T \). Equality (5.1) implies
\[ \lim \langle \Theta^*(S_i), (f, 0) \rangle = \lim \int_U f(x)g_i(x) \, dx = \int_U f(x)g(x) \, dx, \]
\[ \lim \langle \Theta^*(S_i), (0, v) \rangle = \lim \int_U g_i(x) \text{div} v(x) \, dx = \int_U g(x) \text{div} v(x) \, dx \]
for each \( (f, v) \in L^1(U) \times C^1_c(U; \mathbb{R}^m) \); the last equalities hold, since the right hand sides are cluster points of convergent sequences \( \{ \int_U f h \} \) and \( \{ \int_U g_i \text{div} v \} \). For each \( v \in C^1_c(U; \mathbb{R}^m) \) with \( |v|_\infty \leq 1 \), the second equality in (5.2) implies
\[ \int_U g(x) \text{div} v(x) \, dx = \int_U g_i(x) \text{div} v(x) \, dx \leq \sup \|g_i\| \leq c. \]

We infer \( g \in BV_c(U) \), and let \( S := \Psi(g) \). By equalities (5.2) and (5.1),
\[ \langle T, (f, v) \rangle = \lim \langle \Theta^*(S_i), (f, v) \rangle \]
\[ = \lim \langle \Theta^*(S_i), (f, 0) \rangle + \lim \langle \Theta^*(S_i), (0, v) \rangle \]
\[ = \lim \int_U f(x)g_i(x) \, dx - \lim \int_U v \cdot d(Dg_i) \]
\[ = \int_U f(x)g(x) \, dx - \int_U v \cdot d(Dg) = \langle \Theta^*(S), (f, v) \rangle \]
for each \( (f, v) \) in \( L^1(U) \times C^1_c(U; \mathbb{R}^m) \). As \( L^1(U) \times C^1_c(U; \mathbb{R}^m) \) is a dense subspace of \( L^1_{loc}(U) \times C(U; \mathbb{R}^m) \), we see that \( T = \Theta^*(S) \) belongs to \( \Theta^*\left[ CH(U)^* \right] \).

**Theorem 5.2.** Each charge is the sum of an ac-charge and an s-charge.

**Proof.** As in the proof of Theorem 3.6, we deduce from Proposition 5.1 and the Closed Range Theorem that \( \Theta[L^1_{loc}(U) \times C(U; \mathbb{R}^m)] \) is a closed subspace of \( CH(U) \). By [13, Proposition 4.2.2], the space \( CH_{ac}(U) = \Theta[L^1_{loc}(U) \times \{0\}] \) is dense in \( CH(U) \). Consequently
\[ CH_{ac}(U) + \mathcal{F}(U) = \Theta[L^1_{loc}(U) \times C(U; \mathbb{R}^m)] = CH(U), \]
and the theorem follows from Theorem 3.7.

**Remark 5.3.** From [13, Proposition 4.2.2] and Lemma 3.1, we see that both spaces \( CH_{ac}(U) \) and \( CH_s(U) \) are dense in \( CH(U) \).
6. The Gauss-Green theorem

According to Definition 2.1, the distributional divergence of \( v \in C(U; \mathbb{R}^m) \) is defined as the flux \( F_v \) of \( v \). In this framework the Gauss-Green theorem is a mere tautology, which gains its usual meaning when the distribution \( F_v \) is given by a function \( f \in L^1_{\text{loc}}(U) \) [11, Proposition 4.1]. This is a well-known case: the flux \( F_v \) is an ac-charge whose density \( f \) is obtained by derivating \( F_v \) with respect to a suitable derivation basis. However, one may wish to look at a more general situation when \( F_v \) is not an ac-charge, but still has a density \( f \) obtained by derivation. Then \( f \) is not in \( L^1_{\text{loc}}(U) \), and two questions arise.

(i) When is \( F_v \) determined uniquely by its density \( f \)?
(ii) If \( F_v \) is determined uniquely by its density \( f \), then how can \( F_v \) be recovered from \( f \)?

Answers to these questions lead to extensions of the classical Gauss-Green theorem — a topic to which we devote the remainder of our paper.

For a bounded BV set \( A \) contained in \( U \), let

\[
r(A) := \begin{cases} \frac{|A|}{d(A)||A||} & \text{if } |A| > 0, \\ 0 & \text{otherwise}. \end{cases}
\]

We say that a sequence \( \{A_i\} \) of bounded BV sets contained in \( U \) tends to \( x \in U \) if \( x \) belongs to each \( A_i \), \( \lim d(A_i) = 0 \), and \( \inf r(A_i) > 0 \). A charge \( F \) in \( U \) is derivable at \( x \in U \) whenever a finite limit

\[
DF(x) := \lim F(A_i) / |A_i|
\]

exists for each sequence \( \{A_i\} \) of bounded BV sets contained in \( U \) that tends to \( x \). The number \( DF(x) \), called the derivative of \( F \) at \( x \), does not depend on a particular sequence \( \{A_i\} \). If \( v \in C(U; \mathbb{R}^m) \) is differentiable at \( x \in U \), the it is easy to verify that the flux \( F_v \) of \( v \) is derivable at \( x \) and \( DF_v(x) = \text{div} v(x) \).

Denote by \( CH_D(U) \) the linear space of all charges in \( U \) that are derivable at almost all \( x \in U \), and by \( L^0(U) \) the space of all measurable functions defined on \( U \). According to Luzin’s theorem for charges [10], the map

\[
D_U : F \mapsto DF : CH_D(U) \rightarrow L^0(U),
\]

is surjective, and we call it the derivation in \( U \). By our choice of derivation basis, the derivation \( D_U \) is a natural transformation of the functors \( CH_D : U \mapsto CH_D(U) \) and \( L^0 : U \mapsto L^0(U) \) defined on the category \( \text{Lip}_{\text{loc}} \) of open subsets of \( \mathbb{R}^m \) and proper local lipomorphisms [13, Section 4.6]. The map \( D_U \) has a nontrivial kernel \( CH_{\text{sing}}(U) := D_U^{-1}(0) \), whose elements are called singular charges in \( U \). Consequently, \( D_U \) has no natural right inverse. Notwithstanding, we may find a functorial subspace \( X(U) \) of \( CH_D(U) \) so that \( X(U) \cap CH_{\text{sing}}(U) = \{0\} \), in which case the restriction \( D_U \mid X(U) \) is a bijection from \( X(U) \) onto \( J_X(U) := D_U(X) \). We denote the inverse map

\[
(D_U \mid X)^{-1} : J_X(U) \rightarrow X(U)
\]
by $I_{X,U}$, and call it the integration in $U$ induced by $X$. Clearly, the integration $I_{X,U}$ is a natural transformation of the functors $X : U \to X(U)$ and $\mathcal{I}_X : U \to \mathcal{I}_X(U)$ defined on $\text{Lip}_{\text{loc}}$.

The following are classical examples of the procedure we described.

(1) Letting $X(U) := CH_{ac}(U)$, we obtain $\mathcal{I}_X(U) = L^1_{\text{loc}}(U)$ and $I_{X,U}$ is the Lebesgue integration in $U$.

(2) Let $X(U) := CH_{DD}(U)$ be the linear space of all charges in $U$ that are derivable at each $x \in U$. Then $X(U) \cap CH_{\text{sing}}(U) = \{0\}$ by [13, Section 2.6], and the resulting integration $I_{X,U}$ generalizes the Newton integral of elementary calculus.

Since neither of the spaces $CH_{ac}(U)$ and $CH_{DD}(U)$ contains the other, it is inviting to look for a functorial space $X(U) \subseteq CH_D(U)$ such that

$$CH_{ac}(U) + CH_{DD}(U) \subseteq X(U) \quad \text{and} \quad X(U) \cap CH_{\text{sing}}(U) = \{0\}.$$ 

While such a space $X(U)$ is by no means unique, practical considerations limit the choices. We seek an $X(U)$ that is large and well behaved — a delicate balancing act still open for investigation. Below we describe a particular definition of $X(U)$ that proved useful in applications.

A gage on a set $E \subseteq \mathbb{R}^m$ is a nonnegative function $\delta$ defined on $E$ such that the measure $\mathcal{H} \{\delta = 0\}$ is $\sigma$-finite (see Remark 6.6 below for the motivation). Given $F \in CH(U)$ and $E \subseteq U$, let

$$V_*F(E) := \sup_{\eta > 0} \inf_{\delta} \sup \sum_{i=1}^p |F(A_i)|$$

where $\delta$ is a gage on $E$ and the supremum is taken over all collections

$$\{(A_1, x_1), \ldots, (A_p, x_p)\}$$

such that $A_1, \ldots, A_p$ are disjoint BV sets in $U$, and $x_i \in A_i$, $d(A_i) < \delta(x_i)$, and $r(A_i) > \eta$ for $i = 1, \ldots, p$.

It is not difficult to prove that $V_*F : E \mapsto V_*F(E)$ is a Borel regular measure in $U$ [13, Proposition 3.5.1]. It follows from [13, Proposition 3.5.3] that $V_*F$ restricted to BV subsets of a compact interval $J \subseteq U$ is the least additive function larger than or equal to $|F \downarrow J|$. In particular $|F(J)| \leq V_*F(J)$ for each compact interval $J \subseteq U$. An easy argument reveals that $F$ is an ac-charge if and only if $V_*F$ is absolutely continuous and locally finite [13, Proposition 3.6.1]. This fact suggests the following definition.

**Definition 6.1.** An $F \in CH(U)$ is called an ac-charge if the measure $V_*F$ is absolutely continuous.

Denoting by $CH_*^a(U)$ the linear space of all ac-charges, it is immediate that $CH_{ac}^a(U) \subseteq CH_*^a(U)$; in fact, it follows from Theorems 5.2 and 3.7 that

$$CH_*^a(U) = CH_{ac}^a(U) + \mathcal{F}(U) \cap CH_*^a(U).$$

A direct verification of the inclusion $CH_{DD}(U) \subseteq CH_*^a(U)$ is straightforward [13, Theorem 3.6.7]. Establishing the functoriality of $CH_*^a : U \mapsto CH_*^a(U)$ on the category $\text{Lip}_{\text{loc}}$ is not difficult, but requires some work [13, Section 4.6]. On the
other hand, proving the next fundamental theorem is hard. We refer the interested reader to [13, Sections 3.5 and 3.6].

**Theorem 6.2.** $CH_*(U) \subset CH_D(U)$ and

$$V_* F(E) = \int_E |DF(x)| \, dx$$

for each $F \in CH_*(U)$ and each measurable set $E \subset U$.

If $F \in CH_*(U) \cap CH_{sing}(U)$, then Theorem 6.2 yields $|F(J)| \leq V_* F(J) = 0$ for each compact interval $J \subset U$. From this and Remark 2.8, we obtain the following essential corollary.

**Corollary 6.3.** $CH_*(U) \cap CH_{sing}(U) = \{0\}$.

The next theorem, proved in [13, Section 4.5], is important for applications [13, Sections 5.2 and 5.3]. It indicates a good behavior of the space $CH_*(U)$.

**Theorem 6.4.** Let $F \in CH_*(U)$ and $g \in BV_{loc}^\infty(U)$. Then $F \llcorner g \in CH_*(U)$ and $D(F \llcorner g)(x) = DF(x)g(x)$ for almost all $x \in U$.

A vector field $v : U \to \mathbb{R}^m$ is called *pointwise Lipschitz* in a set $E \subset U$ if

$$\limsup_{y \to x} \frac{|v(y) - v(x)|}{|y - x|} < \infty$$

for each $x \in E$. By Stepanoff’s theorem [9, Theorem 3.1.9], a vector field $v$ that is pointwise Lipschitz in $E \subset U$ is differentiable at almost all $x \in E$; in particular, the classical div $v$ is defined almost everywhere in $E$. Now we can generalize the classical Gauss-Green theorem.

**Theorem 6.5.** Let $E \subset U$ be such that the measure $\mathcal{H} \llcorner E$ is $\sigma$-finite, and let $v \in C(U; \mathbb{R}^m)$ be pointwise Lipschitz in $U - E$. Then within $CH_*(U)$, the flux $F_v$ of $v$ is uniquely determined by the classical div $v$. If div $v$ belongs to $L_{loc}^1(U)$, then

$$F_v(A) = \int_A \text{div } v(x) \, dx$$

for each bounded BV set $A$ with $\text{cl } A \subset U$.

**Proof.** Since $v$ is pointwise Lipschitz almost everywhere in $U$, Stepanoff’s theorem implies $DF_v(x) = \text{div } v(x)$ for almost all $x \in U$. However, more is true. Utilizing that $v$ is pointwise Lipschitz in $U - E$ and that the measure $\mathcal{H} \llcorner E$ is $\sigma$-finite, it is easy to find gages on negligible sets which demonstrate the absolute continuity of the measure $V_* F_v$. Consequently $F_v \in CH_*(U)$, and the first claim follows from Corollary 6.3. If div $v$ belongs to $L_{loc}^1(U)$, then the charge $G : A \mapsto \int_A \text{div } v(x) \, dx$ belongs to $CH_{ac}(U)$, and hence to $CH_*(U)$. By the classical derivability result,

$$DG(x) = \text{div } v(x) = DF_v(x)$$

for almost all $x \in U$, and another application of Corollary 6.3 completes the argument.
Remark 6.6. The simplicity of the previous proof is due to an application of Corollary 6.3. Of course, if $\text{div } v$ belongs to $L^1_{\text{loc}}(U)$, then the conclusion of Theorem 6.5 tells us that $F_v$ is an ac-charge. However, proving this directly from the assumptions of Theorem 6.5 is appreciably harder than proving that $F_v$ is an $\text{ac}_*$-charge. The latter proof is facilitated by our definition of gages.

If under the assumptions of Theorem 6.5, $\text{div } v$ does not belong to $L^1_{\text{loc}}(U)$, we must address question (ii) concerning the recovery of $F_v$ from $\text{div } v$. The answer is affirmative: each $F \in CH^*(U)$ can be recovered from $DF$ by means of an averaging process akin to the generalized Riemann integral of Henstock and Kurzweil [13, Section 5.5].

References


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