Introduction to the theory of currents

Tien-Cuong Dinh and Nessim Sibony

September 21, 2005

This course is an introduction to the theory of currents. We give here the main notions with examples, exercises and we state the basic results. The proofs of theorems are not written. We hope that this will be done in the next version. All observations and remarks are welcome.

`dinh@math.jussieu.fr` and `nessim.sibony@math.u − psud.fr`

Contents

Notation

X, X' open sets in euclidian spaces $\mathbb{R}^N$, $\mathbb{R}^{N'}$ or in other manifolds
$\mathrm{B}_N(a,r) := \{x \in \mathbb{R}^N :\ \|x-a\| < r\}$ open ball of radius r centered at a
$\mathrm{B}_N(r)$ open ball $\mathrm{B}_N(0,r)$
B_N unit open ball $\mathrm{B}_N(0,1)$
$\mathscr{B}(X)$ Borel σ-algebra of X
$x = (x_1,\ldots,x_N)$ and $y = (y_1,\ldots,y_N)$ coordinates of $\mathbb{R}^N$
$x' = (x'_1,\ldots,x'_{N'})$ coordinates of $\mathbb{R}^{N'}$
$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$
$\mathbb{R}_+ := \{x \in \mathbb{R},\ x \geq 0\}$ and $\mathbb{R}_- := \{x \in \mathbb{R},\ x \leq 0\}$
$\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\}$ and $\overline{\mathbb{R}}_- := \mathbb{R}_- \cup \{-\infty\}$
δ_a Dirac mass at a
$\mathscr{L}^N$ Lebesgue measure on $\mathbb{R}^N$
$\mathscr{H}^\alpha$ Hausdorff measure of dimension α
$\mathscr{C}(X)$ space of continuous functions on X
$\mathscr{C}_c(X)$ space of continuous functions with compact support in X
$\mathscr{L}^1(\mu)$ space of functions integrable with respect to μ
$\mathscr{L}^1_{loc}(\mu)$ space of functions locally integrable with respect to μ
$\mathscr{L}^1(X)$ space of functions integrable with respect to the Lebesgue measure on X
$\mathscr{L}^1_{loc}(X)$ space of functions locally integrable with respect to the Lebesgue measure on X
$\mathscr{E}_{[k]}(X)$ space of functions of classes $\mathscr{C}^k$ on X
$\mathscr{E}(X)$ space of functions of classes $\mathscr{C}^\infty$ on X
$\mathscr{E}^p_{[k]}(X)$ space of p-forms of classes $\mathscr{C}^k$ on X
$\mathscr{E}^p(X)$ space of p-forms of classes $\mathscr{C}^\infty$ on X
$\mathscr{E}^{p,q}_{[k]}(X)$ space of (p,q)-forms of classes $\mathscr{C}^k$ on X
$\mathscr{E}^{p,q}(X)$ space of (p,q)-forms of classes $\mathscr{C}^\infty$ on X
$\mathscr{D}_{[k]}(X)$ space of functions of class $\mathscr{C}^k$ with compact support in X
$\mathscr{D}(X)$ space of functions of class $\mathscr{C}^\infty$ with compact support in X
$\mathscr{D}^p_{[k]}(X)$ space of p-forms of class $\mathscr{C}^k$ with compact support in X
$\mathscr{D}^p(X)$ space of p-forms of class $\mathscr{C}^\infty$ with compact support in X
$\mathscr{D}^{p,q}_{[k]}(X)$ space of (p,q)-forms of class $\mathscr{C}^k$ with compact support in X
$\mathscr{D}^{p,q}(X)$ space of (p,q)-forms of class $\mathscr{C}^\infty$ with compact support in X
$\mathscr{M}^+(X)$ cone of positive locally finite measures on X

$\mathscr{D}'(X)$ space of distributions on X
$\mathscr{D}'_p(X)$ space of currents of degree p on X
$\mathscr{E}'_p(X)$ space of currents of degree p with compact support in X
$\mathscr{D}'_{p,q}(X)$ space of currents of bidegree (p,q) on X
$\mathscr{E}'_{p,q}(X)$ space of currents of bidegree (p,q) with compact support in X
$\log^+(\cdot) := \max\big(\log(\cdot), 0\big)$
$\mathscr{N}_K(\cdot)$ normal semi-norm on K
$\mathscr{N}^p_K(X)$ space of normal p-currents on X with support in K
$\mathscr{N}^p(X)$ space of normal p-currents on X
$\mathscr{N}^p_{loc}(X)$ space of locally normal p-currents on X
$\mathscr{F}_K(\cdot)$ flat semi-norm on K
$\mathscr{F}^p_K(X)$ space of p-currents on X flat on K
$\mathscr{F}^p(X)$ space of flat p-currents on X
$\mathscr{F}^p_{loc}(X)$ space of locally flat p-currents on X

Chapter 1

Measures

We will recall the fundamental notions in measure theory on $\mathbb{R}^N$. We will define the Lebesgue and Hausdorff measures and the operations on measures: multiplication by a function, push-forward by a continuous map and product of measures. We will also give some basic properties of locally finite measures: compactness and approximation by Dirac masses.

1.1 Borel σ-algebra and measurable maps

Recall that a subset X of $\mathbb{R}^N$ is *open* if for every point a in X there is a ball $\mathrm{B}_N(a, r)$ which is contained in X. In other words, open sets are unions of open balls. Complements of open sets are said to be *closed*. Roughly speaking, an open set in $\mathbb{R}^N$ does not contain any point of its boundary and in constrast a closed set contains all points of its boundary. Only $\varnothing$ and $\mathbb{R}^N$ are open and closed. Unions and finite intersections of open sets are open. Intersections and finite unions of closed sets are closed.

From now on, X is a non-empty open set in $\mathbb{R}^N$. An open subset of X is the intersection of X with an open set of $\mathbb{R}^N$, a closed subset of X is the intersection of X with a closed subset of $\mathbb{R}^N$. So, open subsets of X are open in $\mathbb{R}^N$, but in general closed subsets of X, for example X itself, are not closed in $\mathbb{R}^N$. Compact subsets of X are closed in X and in $\mathbb{R}^N$. If E is a subset of X *the closure* $\overline{E}$ of E in X is the intersection of the closed subsets containing E. We say that E is *relatively compact in* X and we write $E \Subset X$ if $\overline{E}$ is a compact subset of X, i.e. for every sequence $(a_n) \subset \overline{E}$ one can extract a subsequence (a_{n_i}) converging to a point of $\overline{E}$.

Definition 1.1.1. A family $\mathscr{A}$ of subsets of X is called *an algebra* if

1. $\varnothing$ is an element of $\mathscr{A}$.

2. If A is an element of $\mathscr{A}$ then $X \setminus A \in \mathscr{A}$.

3. $\mathscr{A}$ is stable under finite union: if A_n, $n = 1, \dots, m$, are elements of $\mathscr{A}$ then $\bigcup_{n=1}^{m} A_n$ is in $\mathscr{A}$.

An algebra $\mathscr{A}$ is said to be *σ-algebra* if

3'. $\mathscr{A}$ is stable under countable union: if A_n, $n \in \mathbb{N}$, are elements of $\mathscr{A}$ then $\bigcup_{n\in\mathbb{N}} A_n$ is in $\mathscr{A}$.

If $\mathscr{A}$ is an algebra, then X is an element of $\mathscr{A}$ and $\mathscr{A}$ is stable under finite intersection. If $\mathscr{A}$ is a σ-algebra, then it is stable under countable intersection. The smallest σ-algebra is $\{\varnothing, X\}$ and the largest σ-algebra is the family of all the subsets of X. The intersection of a family of algebras (resp. σ-algebras) is an algebra (resp. σ-algebra).

Definition 1.1.2. *The Borel σ-algebra* of X is the smallest σ-algebra $\mathscr{B}(X)$ of subsets of X which contains all the open subsets of X. An element of $\mathscr{B}(X)$ is called a *Borel subset of* X or simply a *Borel set*.

Closed subsets of X are Borel sets. To get the picture, we can construct Borel sets as follows. We start with open and closed sets, then we take their countable unions or intersections. We can produce new Borel sets by taking countable unions or intersections of the sets from the fisrt step and iterate the process. Borel sets could be very complicated and there exist sets which are not Borel sets. In practice, except for some very exceptional cases, all sets that one constructs are Borel sets.

Let $x = (x_1, \dots, x_N)$ denote the coordinates of $\mathbb{R}^N$. Let I_i be an interval in $\mathbb{R}$ not necessarily open, closed or bounded. The set

$$W := \prod_{i=1}^{N} I_i = \big\{x \in \mathbb{R}^N,\ x_i \in I_i,\ i = 1, \dots, N\big\}$$

is called *N-cell*.

Proposition 1.1.3. *The family $\mathscr{A}_0(X)$ of all finite unions of mutually disjoint N-cells in X is an algebra which generates $\mathscr{B}(X)$. More precisely, $\mathscr{B}(X)$ is the smallest σ-algebra containing $\mathscr{A}_0(X)$.*

Definition 1.1.4. A map $f : X \to X'$ is *measurable* if $f^{-1}(B)$ is a Borel set for every Borel set B in X'.

To verify that f is measurable, it is sufficient to verify that f^{-1}(B) is a Borel set for every open ball B in X'. It is easy to show that continuous maps and compositions of measurable maps are measurable. In practice, most of maps that one constructs are measurable.

Definition 1.1.5. A function $f : X \to \overline{\mathbb{R}}$ is *measurable* if $f^{-1}(B)$ and $f^{-1}(\pm\infty)$ are measurable for every Borel set B in $\mathbb{R}$. A measurable function $f : X \to \mathbb{R}$ is called *simple* if it takes only a finite number of values.

Proposition 1.1.6. *Let $f : X \to \overline{\mathbb{R}}_+$ be a positive measurable function. Then there exists an increasing sequence (f_n) of positive simple functions converging pointwise to f.*

Exercise 1.1.7. *Let $f_1 : X_1 \to \mathbb{R}^{m_1}$ and $f_2 : X_2 \to \mathbb{R}^{m_2}$ be measurable maps. Show that the map $f : X_1 \times X_2 \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ given by $f(x_1, x_2) = \big(f(x_1), f(x_2)\big)$ is measurable.*

Exercise 1.1.8. *Let $f_i : X \to \mathbb{R}^m$ be measurable maps. Show that the maps $f_1 \pm f_2$ are measurable.*

Exercise 1.1.9. *Let f_1 and f_2 be real-valued measurable functions. Show that $f_1 f_2$ is measurable.*

Exercise 1.1.10. *Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Show that the following functions are measurables:*

$$\sup_{n\in\mathbb{N}} f_n, \quad \inf_{n\in\mathbb{N}} f_n, \quad \limsup_{n\to\infty} f_n \quad \text{and} \quad \liminf_{n\to\infty} f_n.$$

In particular, if f is measurable, then $f^{\pm} := \max(0, \pm f)$ and $|f|$ are measurable.

Exercise 1.1.11. *A function $f : X \to \mathbb{R} \cup \{-\infty\}$ is said to be upper semi-continuous (u.s.c. for short) if the set $f^{-1}([-\infty, \alpha[)$ is an open set for every α.*

1. *If f is u.s.c. show that $f(a) \geq \limsup_{x\to a} f(x)$ for every $a \in X$ and f reaches its maximum value on any compact set.*

2. *Let (f_n) be a decreasing sequence of u.s.c. functions. Show that $\inf_n f_n$ is u.s.c.*

3. *Assume that f is u.s.c. and bounded from above. Define for $n \in \mathbb{N}$*

$$f_n(x) := \sup_y \big\{u(y) - n\mathrm{dist}(y, x)\big\}.$$

   Prove that the functions f_n are Lipschitz and decrease to f.

4. *If f is a function locally bounded from above define $f^*(x) := \limsup_{y\to x} f(y)$. Show that f^* is the smallest u.s.c. function which is larger than f.*

1.2 Positive measures and integrals

All the measures that we consider in this notes are Borel measures, i.e. measures defined on Borel sets.

Definition 1.2.1. A function $\mu : \mathscr{B}(X) \to \overline{\mathbb{R}}_+$ is called *additive* if $\mu(\varnothing) = 0$ and if for every multually disjoint Borel sets B_n in X, $1 \leq n \leq m$, we have

$$\mu\Big(\bigcup_{n=1}^{m} B_n\Big) = \sum_{n=1}^{m} \mu(B_n).$$

A positive measure is a **countably** additive function μ, that is for all multually disjoint Borel sets B_n in X, $n \in \mathbb{N}$, we have

$$\mu\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) = \sum_{n\in\mathbb{N}} \mu(B_n).$$

When $\mu(K)$ is finite for every compact subset K of X, we say that μ is *locally finite*. When $\mu(X)$ is finite, we say that μ is *finite* or *of finite mass*. The constant $\mu(X)$ is *the mass* of μ. If $\mu(X) = 1$, then μ is a *probability measure*.

Definition 1.2.2. *The support of* μ is the smallest closed subset $\mathrm{supp}(\mu)$ of X such that μ vanishes in a neighbourhood of every point a in $X \setminus \mathrm{supp}(\mu)$. More precisely, there exists a ball $\mathrm{B}_N(a, r)$ of zero μ measure.

Let μ_1 and μ_2 be two positive measures and $c \geq 0$ be a constant. Define $\mu_1 + \mu_2$ and $c\mu$ by $(\mu_1 + \mu_2)(B) := \mu_1(B) + \mu_2(B)$ and $(c\mu)(B) := c\mu(B)$ for every Borel set B. It is clear that $\mathrm{supp}(\mu_1 + \mu_2) = \mathrm{supp}(\mu_1) \cup \mathrm{supp}(\mu_2)$ and $\mathrm{supp}(c\mu) = \mathrm{supp}(\mu)$ for $c > 0$.

Example 1.2.3. Let a be a point in X. The *Dirac mass at* a is the probability measure δ_a defined as follows. If a Borel set B contains a then $\delta_a(B) = 1$, otherwise $\delta_a(B) = 0$. The support of δ_a is reduced to $\{a\}$.

Example 1.2.4. Let $f : X \to X'$ be a measurable map and let μ be a positive measure on X. *The push-forward* $f_*(\mu)$ *of* μ is a positive measure on X' defined by

$$f_*(\mu)(B) := \mu(f^{-1}(B)) \quad \text{for } B \in \mathscr{B}(X').$$

If μ is finite then $f_*(\mu)$ is finite. If μ is locally finite and f is proper on $\mathrm{supp}(\mu)$, i.e. $f^{-1}(K) \cap \mathrm{supp}(\mu)$ is compact for any compact K, then $f_*(\mu)$ is locally finite.

We know define the integral of functions with respect to an arbitrary measure.

Definition 1.2.5. Let $f : X \to \overline{\mathbb{R}}_+$ be a positive measurable function. *The integral of f with respect to μ* is denoted by $\int f d\mu$, $\langle \mu, \varphi \rangle$ or $\mu(f)$. If f is a simple positive function with values $a_1, \ldots, a_m$ then

$$\int f d\mu := \sum_{n=1}^{m} a_n \mu(f^{-1}(a_n)).$$

For an arbitrary positive measurable function f we have

$$\int f d\mu := \sup \left\{ \int g d\mu, \quad g \text{ simple such that } 0 \leq g \leq f \right\}.$$

One can check that the integral $\int f d\mu$ depends linearly on f and on μ, i.e.

$$\int (f_1 + f_2) d\mu = \int f_1 d\mu + \int f_2 d\mu, \quad \int c f d\mu = c \int f d\mu = \int f d(c\mu)$$

and

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2.$$

Definition 1.2.6. A measurable function $f : X \to \overline{\mathbb{R}}$ or $f : X \to \mathbb{C}$ is *integrable* if $\int |f| d\mu$ is finite. For such a function define

$$\int f d\mu := \int f^+ d\mu - \int f^- d\mu$$

where $f^\pm := \max(\pm f, 0)$, or if f is complex-valued

$$\int f d\mu := \int \Re(f) d\mu + i \int \Im(f) d\mu.$$

Let $\mathscr{L}^1(\mu)$ denote the space of integrable functions where we identifie functions which are equal outside a Borel set of zero μ measure. Such functions have the same integral. The integral defines a linear form on this space. For the Dirac mass δ_a, we have $\int f d\delta_a = f(a)$. The function f is integrable if and only if $f(a)$ is finite. We also have $\mathscr{L}^1(\delta_a) \simeq \mathbb{C}$.

A measurable function f on X is *locally integrable* with respect to μ if f restricted to each compact subset of X is integrable. The space of such functions is denoted by $\mathscr{L}^1_{loc}(\mu)$ where we identify also functions which are equal out of a set of zero μ measure.

The following theorem due to Beppo-Levi is useful in order to prove the convergence of integrals of positive functions.

Theorem 1.2.7. *Let (f_n) be an increasing sequence of positive measurable functions converging to a function f. Then*

$$\int f_n d\mu \to \int f d\mu.$$

A consequence of this theorem is *Fatou's lemma.*

Corollary 1.2.8. *Let (f_n) be a sequence of positive measurable functions. Then*

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

The following result of Lebesgue is often used in the case of signed and complex-valued functions.

Theorem 1.2.9. *Let (f_n) be a sequence of measurable functions converging pointwise to a function f outside a set of μ measure zero. Assume that there exists a positive integrable function g such that $|f_n| \leq g$ for every n. Then*

$$\int f_n d\mu \to \int f d\mu.$$

Exercise 1.2.10. *Construct a subset of $\mathbb{R}$ which is not a Borel set.*
Hint: consider a set E such that each class of $\mathbb{R}/\mathbb{Q}$ contains exactly one point of E.

Exercise 1.2.11. *Let (B_n) be a sequence of Borel sets in X. Show that*

$$\mu(\bigcup_{n\in\mathbb{N}} B_n) \leq \sum_{n\in\mathbb{N}} \mu(B_n).$$

In particular, countable unions of Borel sets of zero measure are of zero measure.

Exercise 1.2.12. *Let (B_n) be an increasing sequence of Borel sets in X, i.e. $B_n \subset B_{n+1}$. Show that*

$$\mu(\bigcup_{n\in\mathbb{N}} B_n) = \lim_{n\to\infty} \mu(B_n) = \sup_{n\in\mathbb{N}} \mu(B_n).$$

Exercise 1.2.13. *Let (B_n) be a sequence of Borel sets in X. Show that*

$$\mu(\bigcap_{n\in\mathbb{N}} B_n) \leq \liminf_{n\to\infty} \mu(B_n).$$

Assume that (B_n) is decreasing, i.e. $B_n \supset B_{n+1}$. Assume also that $\mu(B_n)$ is finite for some n. Prove that

$$\mu(\bigcap_{n\in\mathbb{N}} B_n) = \lim_{n\to\infty} \mu(B_n) = \inf_{n\to\infty} \mu(B_n).$$

What happens when $\mu(B_n)$ is infinite for every n?

Exercise 1.2.14. *Find all positive measures supported on a finite set.*

Exercise 1.2.15. *A positive measure μ is called extremal if the following holds. If μ_1, μ_2 are positive measures such that $\mu_1 + \mu_2 = \mu$ then μ_1 and μ_2 are multiples of μ, i.e. $\mu_1 = c_1\mu$ and $\mu_2 = c_2\mu$ for some constants c_1 and c_2. Show that μ is extremal if and only if it is a Dirac mass.*

Exercise 1.2.16. *Let (f_n) be a sequence of positive measurable functions on X. Show that*

$$\int \Big(\sum_{n\in\mathbb{N}} f_n\Big) d\mu = \sum_{n\in\mathbb{N}} \Big(\int f_n d\mu\Big).$$

Exercise 1.2.17. *Let μ be a positive measure on X and $p \geq 1$ an real number. Let $\mathscr{L}^p(\mu)$ denote the set of measurable functions f such that $\int |f|^p d\mu < \infty$ where we identify two functions if they are equal outside a set of zero μ measure.*

1. *Show that $\mathscr{L}^p(\mu)$ is a vector space and $\|f\|_{\mathscr{L}^p(\mu)} := (\int |f|^p d\mu)^{1/p}$ is a norm.*

2. *Suppose f_n and f are functions in $\mathscr{L}^p(\mu)$ and $f_n \to f$ μ-almost everywhere. Suppose also that $\|f_n\|_{\mathscr{L}^p(\mu)} \to \|f\|_{\mathscr{L}^p(\mu)}$. Show that $\lim \|f_n - f\|_{\mathscr{L}^p(\mu)} = 0$.*

Hint: apply Fatou's lemma to $g_n := |f_n|^p + |f|^p - 2|f_n - f|^p$.

1.3 Locally finite measures

From now on, all the measure that we consider, except the Hausdorff measures that we will define latter, are supposed to be locally finite. The terminology "positive measure" will mean locally finite positive measure. We give some properties of the cone $\mathscr{M}^+(X)$ of all the positive measures and introduce signed and complex measures. We first construct basic examples using the following Carathéodory criterion.

Theorem 1.3.1. *Let $\mathscr{A}$ be an algebra which generates $\mathscr{B}(X)$. Let $\mu : \mathscr{A} \to \overline{\mathbb{R}}_+$ be an additive function. Assume that $\mu(A)$ is finite for every $A \in \mathscr{A}$ relatively compact in X. Then, there exists at most one positive measure $\widetilde{\mu}$ which extends μ, i.e. $\widetilde{\mu} = \mu$ on $\mathscr{A}$. This measure exists if and only if $\lim_{n\to\infty} \mu(A_n) = 0$ for every decreasing sequence $(A_n) \subset \mathscr{A}$ such that $\mu(A_0)$ is finite and $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$.*

Example 1.3.2. *The Lebesgue measure $\mathscr{L}^N$* of a finite union of mutually disjoint N-cells is simply the sum of volumes of these N-cells. Proposition 1.1.3 and the Carathéodory criterion allow to extends $\mathscr{L}^N$ to all the Borel sets in X. The spaces of integrable and locally integrable functions with respect to $\mathscr{L}^N$ are denoted by $\mathscr{L}^1(X)$ and $\mathscr{L}^1_{loc}(X)$ respectively.

Example 1.3.3. Let μ be a positive measure on X and let μ' be a positive measure on X'. Define *the tensor product $\mu \otimes \mu'$* as follows. If W be an n-cell in X and W' be an N'-cell in X' then

$$(\mu \otimes \mu')(W \times W') := \mu(W)\mu(W').$$

The measure of a finite union of mutually disjoint $(N+N')$-cells is equal to the sum of the measures of these cells. Using Proposition 1.1.3 and the Carathéodory criterion, one can extend μ to a measure on $X \times X'$.

The following theorem shows that positive measures are uniquely determined by their values on compact or open sets.

Theorem 1.3.4. *Let μ be a positive measure on X. Then μ is regular in the following sense: for every Borel set B in X and every $\epsilon > 0$ there exist a compact subset K and an open subset U of X such that*

$$K \subset B \subset U \quad \textit{and} \quad \mu(U \setminus K) < \epsilon.$$

We define the *weak topology* on $\mathscr{M}^+(X)$.

Definition 1.3.5. Let μ_n and μ be positive measures. We say that μ_n *converge weakly to* μ and we write $\mu_n \to \mu$ if for any function φ in $\mathscr{C}_c(X)$ we have

$$\int \varphi d\mu_n \to \int \varphi d\mu.$$

The following theorem shows that combinations of Dirac masses are dense in $\mathscr{M}^+(X)$ for the weak topology.

Theorem 1.3.6. *Let μ be a positive measure on X. Then there exists a sequence of measures μ_n, which are finite combinations with positive coefficients of Dirac masses, such that $\mu_n \to \mu$.*

The cone $\mathscr{M}^+(X)$ satisfies also the following compactness property.

Theorem 1.3.7. *Let (μ_n) be a sequence of positive measures on X. Assume that $\mu_n(K)$ is uniformly bounded for every compact subset K of X. Then there exists an increasing sequence of integers (n_i) and a positive measure μ such that $\mu_{n_i} \to \mu$.*

We now define real and complex measures.

Definition 1.3.8. Let $\mu^\pm$ be positive measures on X. Define $\mu = \mu^+ - \mu^-$ by $\mu(B) := \mu^+(B) - \mu^-(B)$ for every Borel set B in X such that $\mu^\pm(B)$ are finite. In particular, $\mu(B)$ is defined if $B \Subset X$. We say that μ is a *real measure*. If μ_1 and μ_2 are two real measures, define $\mu' = \mu_1 + i\mu_2$ by $\mu'(B) := \mu_1(B) + i\mu_2(B)$. We say that μ' is a *complex measure*.

Theorem 1.3.9. *Let μ be a real measure on X. There there exist a unique couple μ^+ and μ^- of positive measures on X such that $\mu = \mu^+ - \mu^-$, and $\|\mu^+\| + \|\mu^-\|$ is minimal.*

An analog result for complex measures can be deduced easily since the real and imaginary parts of a complex measure are uniquely determined.

Definition 1.3.10. Let μ, $\mu^{\pm}$ be as in Theorem 1.3.9. Define *the total variation* of μ by $|\mu| := \mu^{+} + \mu^{-}$. This is a positive measure on X. The total variation of a complex measure $\mu_1 + i\mu_2$ is equal to $|\mu_1| + |\mu_2|$. We say that a sequence of measures (μ_n) *converge weakly* to a measure μ if $|\mu_n|(K)$ is uniformly bounded for every compact set $K \subset X$ and if

$$\int \varphi d\mu_n \to \int \varphi d\mu \quad \text{for } \varphi \in \mathscr{C}_c(X).$$

The sup-norm $\|\cdot\|_\infty$ on $\mathscr{C}_c(X)$ is defined as:

$$\|\varphi\|_\infty := \sup_X |\varphi(x)| \quad \text{for } \varphi \in \mathscr{C}_c(X).$$

We say that a sequence (φ_n) *converges* to φ in $\mathscr{C}_c(X)$ if there is a compact subset K of X such that $\mathrm{supp}(\varphi_n) \subset K$ for every n and $\|\varphi_n - \varphi\|_\infty \to 0$. Recall that *the support* $\mathrm{supp}(\varphi)$ of φ is the closure of the set $\{a \in X,\ \varphi(a) \neq 0\}$.

The following theorem due to Riesz, gives a new point of view on measures. In particular, it shows that the integrals of functions in $\mathscr{C}_c(X)$ determine the measure.

Theorem 1.3.11. *Let μ be a complex measure on X. Then the integration with respect to μ defines a continuous linear form on $\mathscr{C}_c(X)$. Conversely, every continuous linear form on $\mathscr{C}_c(X)$ is defined by integration with respect to a complex measure. Moreover, μ is real (resp. positive) if and only if the associated linear form is real-valued, i.e. it takes real values on real-valued functions (resp. positive, i.e. it takes positive values on positive functions).*

Exercise 1.3.12. *Let (μ_n) be a sequence of positive measures converging to a positive measure μ. If u is an u.s.c function with compact support show that*

$$\limsup_{n\to\infty} \int u d\mu_n \leq \int u d\mu.$$

Show that for every compact subset K and open subset U of X

$$\liminf_{n\to\infty} \mu_n(U) \geq \mu(U) \quad \textit{and} \quad \limsup_{n\to\infty} \mu_n(K) \leq \mu(K).$$

Hint: use Exercise 1.1.11.

Exercise 1.3.13. *Let $f : X \to X'$ be a measurable map and let μ be a probability measure on X. Show that $\nu := f_*(\mu)$ is a probability measure and if φ is a ν-integrable function on X' then $\int \varphi d\nu = \int (\varphi \circ f) d\mu$.*

Exercise 1.3.14. *Let $\mathbb{S}^1$ be the unit circle in $\mathbb{C}$ and let $R_\theta(z) := e^{i\theta}z$ be the rotation of angle θ.*

1. *Detemine the probability measure λ with support in $\mathbb{S}^1$ which is invariant with respect to all the R_θ, i.e. $(R_\theta)_*(\lambda) = \lambda$.*
2. *Let $f(z) := z^n$ with $n \geq 1$. Determine $f_*(\lambda)$.*
3. *Let ν be a real measure on $\mathbb{S}^1$ such that $\langle \nu, z^n \rangle = 0$ for any $n \in \mathbb{N}^*$. Show that ν is proportional to λ. Does this hold for complex measures?*

Hint: trigonometrical polynomials are dense in $\mathscr{C}(\mathbb{S}^1)$.

Exercise 1.3.15. *Let $f : X \to X$ be a continuous map and let K be a compact set such that $f(K) \subset K$. Let μ be a probability measure on K. Define*

$$\sigma_n := \frac{1}{n}\sum_{i=1}^{n}(f^i)_*(\mu)$$

where $f^n := f \circ \cdots \circ f$ (n times). Show that any limit measure ν of (σ_n) is an invariant measure, i.e. $f_*(\nu) = \nu$.

Exercise 1.3.16. *Let (μ_n) be a sequence of measures. Assume that $|\mu_n|(K)$ is uniformly bounded for every compact subset K of X. Show that one can extract a subsequence (μ_{n_i}) converging to a measure on X.*

Exercise 1.3.17. *Let ν_1 and ν_2 be two positive measures. We say that ν_1 is singular with respect to ν_2 if there exists a Borel set B such that $\nu_1(B) = 0$ and $\nu_2(X \setminus B) = 0$. Observe that ν_1 is singular with respect to ν_2 if and only if ν_2 is singular with respect to ν_1. Show that in Theorem 1.3.9, μ^+ is singular with respect to μ^-. Show that (μ^+, μ^-) is the unique couple of positive measures such that $\mu = \mu^+ - \mu^-$ and μ^+ is singular with respect to μ^-.*

Exercise 1.3.18. *Let μ be a real (resp. complex) measure. Show that μ can be approximated by finite combinations of Dirac masses with real (resp. complex) coefficients.*

Exercise 1.3.19. *Let μ be a complex measure on X and let $K \subset X$ be a compact set. Then there exists a positive constant c such that*

$$|\langle \mu, \varphi \rangle| \leq c\|\varphi\|_\infty \quad \text{for every } \varphi \in \mathscr{C}_c(X) \text{ with } \operatorname{supp}(\varphi) \subset K.$$

1.4 Outer measures and Hausdorff measures

Definition 1.4.1. *An outer measure* ν on X is a map defined on all the subsets of X with values in $\overline{\mathbb{R}}_+$ satisfying the following properties:

1. $\nu(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \nu(A_n)$.
2. $\nu(\varnothing) = 0$.

A set A is *ν-measurable* if

$$\nu(C) = \nu(C \cap A) + \nu(C \setminus A) \quad \text{for every } C.$$

It follows that a set A for which $\nu(A) = 0$ is always measurable. The following is Carathéodory criterion for measurability.

Theorem 1.4.2. *If ν is an outer measure then the class $\mathscr{A}_\nu$ of ν-measurable sets is a σ-algebra. This σ-algebra contains $\mathscr{B}(X)$ if and only if*

$$\nu(A \cup B) \geq \nu(A) + \nu(B) \quad \textit{when } \operatorname{dist}(A, B) > 0.$$

In this case ν defines a positive measure on X.

The following construction gives geometrically important outer measures. Suppose ξ is a function defined on a family $\mathscr{F}$ of subsets of X with values in $\overline{\mathbb{R}}_+$. We want to construct an outer measure ν and hence a measure so that the measure of A is related to the covering properties of A by elements of $\mathscr{F}$. The basic example to have in mind is the function

$$\xi(F) := c(\operatorname{diam}(F))^s \quad \text{with } s \geq 0 \text{ fixed and } c > 0 \text{ a constant}$$

which is defined on all subsets of X.

We define first for any $A \subset X$ and $\epsilon > 0$

$$\nu_\epsilon(A) := \inf \Big\{ \sum_n \xi(A_n),\ A \subset \bigcup_n A_n,\ \xi(A_n) \leq \epsilon \Big\}.$$

It is clear that $\nu_\epsilon(A)$ is decreasing with respect to ϵ so we can define an outer measure ν as

$$\nu(A) := \lim_{\epsilon \to 0} \nu_\epsilon(A) = \sup_{\epsilon > 0} \nu_\epsilon(A).$$

Assume $\xi(A) = c_s(\operatorname{diam}(A))^s$ where $\operatorname{diam}(\cdot)$ is with respect to the euclidian distance in $\mathbb{R}^N$ and $c_s := 2^{-s}\Gamma(1/2)^s\Gamma(s/2+1)^{-1}$. Then we denote the corresponding outer measure as $\mathscr{H}^s$ and we call it *the Hausdorff measure of dimension s*. These measures can be defined as well by considering only covering by open sets or by closed sets. It is clear that Carathéodory's criterion is satisfied. Hence $\mathscr{H}^s$ is also a positive measure.

For every set A there is an $s_0 \geq 0$ such that $\mathscr{H}^s(A) = +\infty$ if $s < s_0$ and $\mathscr{H}^s(A) = 0$ is $s > s_0$. The number s_0 is called *the Hausdorff dimension* of A. If $s = N$ then $\mathscr{H}^N = \mathscr{L}^N$ on $\mathscr{B}(X)$. When $s < N$ then $\mathscr{H}^s$ is not locally finite. The Hausdorff measure is very useful to prove geometric inequalities. If $f : X \to X'$ is a map and A is a subset of X we denote the number of preimages of x' in A by

$$N(f_{|A}, x') := \#\big\{x \in A,\ f(x) = x'\big\}.$$

We have the following theorem.

Theorem 1.4.3. *Let $f : X \to X'$ be a Lipschitz map between two open subsets in $\mathbb{R}^N$ and $\mathbb{R}^{N'}$. Let A be a Borel subset of X. Then*

$$\int N(f_{|A}, x') d\mathscr{H}^s(x') \leq (\mathrm{Lip}(f))^s \mathscr{H}^s(A).$$

Here

$$\mathrm{Lip}(f) := \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$

Exercise 1.4.4. *Let μ be a positive measure on X. Define for any subset A of X*

$$\mu^*(A) := \inf \big\{ \mu(A),\ U \supset A,\ U \text{ open} \big\}.$$

Show that μ^* is an outer measure, the associated algebra $\mathscr{A}$ contains $\mathscr{B}(X)$ and sets of outer measure zero.

Exercise 1.4.5. *Recall that the Cantor set M is the union of real numbers which can be written as $\sum_{n \geq 1} a_n 3^{-n}$ with $a_n = 0$ or 2. Compute the Hausdorff dimension of F.*

Chapter 2

Distributions

The notion of distributions was introduced by Laurent Schwartz. It generalizes the notion of functions. In this chapter, we introduce the distributions in $\mathbb{R}^N$ and we describe the fundamental operations on distributions. We also show that every distribution can be approximated by smooth functions. Finally we define subharmonic functions and describe some properties of this class of functions.

2.1 Definitions

Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be a collection of positive integers. Define

$$|\alpha| := \alpha_1 + \cdots + \alpha_N \quad \text{and} \quad \partial^\alpha := \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_N} x_N}.$$

We endow the space $\mathscr{E}_{[k]}(X)$ of functions of class $\mathscr{C}^k$ on X a family of semi-norms. If K is a compact subset of X, define for every function φ in $\mathscr{E}_{[k]}(X)$

$$\|\varphi\|_{k,K} := \max_\alpha \left\{ \max_K |\partial^\alpha \varphi|,\ 0 \leq |\alpha| \leq k \right\}.$$

Define also a topology on $\mathscr{D}_{[k]}(X)$ associated to the previous semi-norms. We say that (φ_n) *converges to* φ *in* $\mathscr{D}_{[k]}(X)$ if there is a compact subset $K \subset X$ such that φ_n and φ are supported in K and $\|\varphi_n - \varphi\|_{k,K}$ converge to 0.

Proposition 2.1.1. *The space $\mathscr{D}(X)$ of smooth functions with compact support in X is dense in $\mathscr{D}_{[k]}(X)$. More precisely, for every function φ in $\mathscr{D}_{[k]}(X)$ there exists a sequence $(\varphi_n) \subset \mathscr{D}(X)$ such that $\varphi_n \to \varphi$ in $\mathscr{D}_{[k]}(X)$.*

The elements of $\mathscr{D}(X)$ are called *test functions*. We say that *a sequence* (φ_n) *converges to* φ *in* $\mathscr{D}(X)$ if there is a compact subset $K \subset X$ such that φ_n and φ are supported in K and $\|\varphi_n - \varphi\|_{k,K}$ tend to 0 for every $k \geq 0$.

Definition 2.1.2. *A distribution on* X is a linear form $T : \mathscr{D}(X) \to \mathbb{C}$, continuous with respect to the topology on $\mathscr{D}(X)$. The value of T on φ is denoted by $T(\varphi)$ or $\langle T, \varphi\rangle$. Denote by $\mathscr{D}'(X)$ the set of distributions on X. We say that *a sequence* $(T_n) \subset \mathscr{D}'(X)$ *converges to* $T \in \mathscr{D}'(X)$ *in the sense of distributions* if $\langle T_n, \varphi\rangle \to \langle T, \varphi\rangle$ for every function $\varphi \in \mathscr{D}(X)$.

Proposition 2.1.3. *Let* T *be a distribution on* X *and let* $K \subset X$ *be a compact set. Then there exist a positive integer* k *and a positive constant* c *such that*

$$|\langle T, \varphi\rangle| \leq c\|\varphi\|_{k,K} \quad \textit{for every } \varphi \in \mathscr{D}(X) \textit{ with } \operatorname{supp}(\varphi) \subset K.$$

Definition 2.1.4. If the integer k in Proposition 2.1.3 can be chosen independently on K we say that T *is of finite order* and the smallest integer k satisfying this property is called *order of* T.

Example 2.1.5. Let a be a point in X. Consider the distribution T_α given by

$$\langle T_\alpha, \varphi\rangle := \partial^\alpha \varphi(a).$$

It is easy to check that T_α is of order $|\alpha|$.

Example 2.1.6. Consider a sequence $(a_k) \subset \mathbb{R}$ converging to infinity and define

$$\langle T, \varphi\rangle := \sum_{k\in\mathbb{N}} \varphi^{(k)}(a_k).$$

Then T is a distribution in $\mathbb{R}$ of infinite order.

Example 2.1.7. Let f be a function in $\mathscr{L}^1_{loc}(X)$. Then $f\mathscr{L}^N$ is a measure on X. In particular, it is a distribution of order zero. We have

$$\langle f\mathscr{L}^N, \varphi\rangle := \int_X f\varphi d\mathscr{L}^N.$$

From now on, we often identify the function f to the distribution $f\mathscr{L}^N$.

The following result is a consequence of Proposition 2.1.1.

Proposition 2.1.8. *Let* T *be a distribution of order* k *on* X. *Then we can extend* T *to a continuous linear form on* $\mathscr{D}_{[k]}(X)$. *Moreover, the extension is unique.*

Proposition 1.3.11 implies that distributions of order zero are measures (see also Exercise 1.3.19).

Proposition 2.1.9. *Let* T *be a distribution on* X. *Assume that* T *is positive, i.e.* $\langle T, \varphi\rangle$ *is positive for every positive function* $\varphi \in \mathscr{D}(X)$. *Then* T *is a positive measure on* X.

Definition 2.1.10. *The support* $\mathrm{supp}(T)$ *of a distribution* T is the smallest closed subset of X such that T vanishes on $X \setminus \mathrm{supp}(T)$. That is, $\langle T, \varphi\rangle = 0$ for every test function $\varphi \in \mathscr{D}(X)$ with compact support in $X \setminus \mathrm{supp}(T)$.

We consider the following topology on $\mathscr{E}(X)$. We say that $(\varphi_n) \subset \mathscr{E}(X)$ *converges to* $\varphi \in \mathscr{E}(X)$ if $\|\varphi_n - \varphi\|_{k,K} \to 0$ for every compact set $K \subset X$ and every positive integer k.

Theorem 2.1.11. *Let T be a distribution with compact support in X. Then T can be extended to a continuous linear form on $\mathscr{E}(X)$. Moreover, the extension is unique.*

Exercise 2.1.12. *Let T_α be the distributions in Example 2.1.5. Let $(\alpha^{(n)})$ such that $(T_{\alpha^{(n)}})$ converges. Show that $(\alpha^{(n)})$ is stationary.*

Exercise 2.1.13. *Let T be the distribution in Example 2.1.6. Determine the support of T. Let a be a point of X. Determine all the distributions with support $\{a\}$.*

Exercise 2.1.14. *Let T be a distribution of order k on X. Show that T can be extended uniquely to a linear continuous form on the space of functions $\varphi \in \mathscr{E}_{[k]}(X)$ such that* $\mathrm{supp}(T) \cap \mathrm{supp}(\varphi)$ *is compact. In particular, if T has compact support it defines a linear continuous form on $\mathscr{E}_{[k]}(X)$.*

2.2 Operations on distributions

The set $\mathscr{D}'(X)$ of distributions on X is a vector space. Let a and b be complex numbers and let T and R be distributions on X. Define

$$\langle aT + bR, \varphi\rangle := a\langle T, \varphi\rangle + b\langle R, \varphi\rangle \quad \text{for } \varphi \in \mathscr{D}(X).$$

If g is a function in $\mathscr{E}(X)$ define the product fT by

$$\langle gT, \varphi\rangle := \langle T, g\varphi\rangle.$$

One checks easily that $aT + bR$ and gT are distributions on X.

Recall that a map $f : X \to X'$ is proper on a closed subset Y of X if $f^{-1}(K) \cap Y$ is compact for every compact set $K \subset X'$.

Definition 2.2.1. Let T be a distribution on X. Let $f : X \to X'$ be a smooth map proper on the support of T. *The push-forward* $f_*(T)$ is a distribution on X' given by

$$\langle f_*(T), \varphi\rangle := \langle T, \varphi \circ f\rangle \quad \text{for } \varphi \in \mathscr{D}(X').$$

If $f : X \to X'$ is proper then $T \mapsto f_*(T)$ is continuous.

The interest with distributions is that several operations like derivation, make sense for more general functions than the usual differentiable ones. So we extend some operations on smooth functions to distributions. In order to use the new calculus we need these operations to be continuous for the topology on distributions.

Definition 2.2.2. Define *the partial derivative* $\frac{\partial T}{\partial x_i}$ by

$$\left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle := -\left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle.$$

Then $\partial T/\partial x_i$ defines a distribution and $T \mapsto \partial T/\partial x_i$ is continuous on $\mathscr{D}'(X)$.

Proposition 2.2.3. *Let T be a distribution on X. Then*

$$\frac{\partial^2 T}{\partial x_i \partial x_j} = \frac{\partial^2 T}{\partial x_j \partial x_i}.$$

Proposition 2.2.4. *Let T be a distribution on X and $f : X \to X'$ be smooth bijective map. Then, the partial derivatives of the push-forward satisfy the following formula*

$$\left(\frac{\partial f_*(T)}{\partial x_{i'}}\right)_{1\le i'\le N'} = \mathrm{Jac}(f^{-1}) \left(f_* \left(\frac{\partial T}{\partial x_i}\right)\right)_{1\le i\le N}$$

where $\mathrm{Jac}(f^{-1})$ denotes the jacobian matrix of f^{-1}.

Proposition 2.2.5. *The space of functions generated by the functions of the form*

$$\varphi(x)\varphi'(x') \quad \text{with } \varphi \in \mathscr{D}(X) \text{ and } \varphi' \in \mathscr{D}(X')$$

is dense in $\mathscr{D}(X \times X')$.

We then deduce the following theorem which gives the definition of *tensor product of distributions.*

Theorem 2.2.6. *Let T be a distribution on X and let T' be a distribution on X'. Then there exists a unique distribution $T \otimes T'$ on $X \times X'$, called tensor product of T and T', such that*

$$\langle T \otimes T', \varphi(x)\varphi'(x') \rangle = \langle T, \varphi \rangle \langle T', \varphi' \rangle \quad \text{for } \varphi \in \mathscr{D}(X) \text{ and } \varphi' \in \mathscr{D}(X').$$

Proposition 2.2.7. *The tensor product $T \otimes T'$ depends continuously on T and T'. More precisely, if $T_n \to T$ and $T'_n \to T'$ in the sense of distributions then $T_n \otimes T'_n \to T \otimes T'$ in the sense of distributions.*

The following proposition allows us to compute the partial derivatives of a tensor product.

Proposition 2.2.8. *Let T and T' be as in Theorem 2.2.6. Then*

$$\frac{\partial(T \otimes T')}{\partial x_i} = \frac{\partial T}{\partial x_i} \otimes T' \quad \textit{and} \quad \frac{\partial(T \otimes T')}{\partial x'_j} = T \otimes \frac{\partial T'}{\partial x'_j}.$$

Exercise 2.2.9. *Show that* $\operatorname{supp}(\frac{\partial T}{\partial x_i}) \subset \operatorname{supp}(T)$*,* $\operatorname{supp}(gT) = \operatorname{supp}(g) \cap (\operatorname{supp}(T)$*,* $\operatorname{supp}(f_*(T)) \subset f(\operatorname{supp}(T))$ *and* $\operatorname{supp}(T \otimes T') = \operatorname{supp}(T) \times \operatorname{supp}(T')$*.*

Exercise 2.2.10. *Let φ be a function in $\mathscr{D}(\mathbb{R})$ such that $\varphi(0) = 0$. Show that $\frac{\varphi(x)}{x}$ is a function in $\mathscr{D}(\mathbb{R})$. Show that for every distribution T on $\mathbb{R}$ there is a distribution S such that $xS = T$.*
Hint: compute $\frac{d}{dt}(\varphi(tx))$.

Exercise 2.2.11. *Let T be a distribution of order k on X. Let f be a function in $\mathscr{E}_{[k]}(X)$. Show that fT defines a distribution of order $\leq k$ on X.*

Exercise 2.2.12. *Let f be a function in $\mathscr{E}(X)$ and T be the associated distribution. Compute $\partial^\alpha T$.*

Exercise 2.2.13. *Let f be a function on $\mathbb{R}$. Assume that f vanishes outside an interval $[a, b] \subset \mathbb{R}$ and that $f_{|[a,b]}$ is a function of class $\mathscr{C}^1$. Let T be the distribution on $\mathbb{R}$ associated to f. Compute $\frac{\partial T}{\partial x}$.*

Exercise 2.2.14. *Let T be a distribution on X. Assume that $\frac{\partial T}{\partial x_i} = 0$ for every i. Show that T is defined by a constant function. Find all the distributions T satisfying $\partial^\alpha T = 0$ for every α with $|\alpha| = k$.*

Exercise 2.2.15. *Let T be the distribution on $\mathbb{R}$ defined by the characteristic function $\mathbf{1}_{\mathbb{R}_+}$ of $\mathbb{R}_+$. Show that $\frac{\partial T}{\partial x} = \delta_0$. Find all the distributions T on $\mathbb{R}$ such that $\frac{\partial T}{\partial x} = \delta_0$.*

Exercise 2.2.16. *Let f be a real-valued increasing function on $\mathbb{R}$. Show that $f' := \frac{\partial f}{\partial x}$ in the sense of distributions is a positive measure.*

2.3 Convolution and regularization

We first consider a distribution T defined on $\mathbb{R}^N$. Let φ be a function in $\mathscr{E}(\mathbb{R}^N)$. Assume that either $\operatorname{supp}(T)$ or $\operatorname{supp}(\varphi)$ is compact (in fact it is enough to assume that the set of $(x, y) \in \operatorname{supp}(T) \times \operatorname{supp}(\varphi)$ with $|x + y| \leq R$ is compact for every R). Define for $x \in \mathbb{R}^N$ an affine invertible map $\tau_x : \mathbb{R}^N \to \mathbb{R}^N$ by $\tau_x(y) := x - y$. *The convolution of T and φ* is a function on $\mathbb{R}^N$ given by

$$(T * \varphi)(x) := \langle T, \varphi \circ \tau_x \rangle = \langle (\tau_x)_* T, \varphi \rangle.$$

Note that in the last formula x is fixed.

Proposition 2.3.1. *Let T and φ be as above. Then $T*\varphi$ is a function in $\mathscr{E}(\mathbb{R}^N)$. Moreover, we have*

$$\partial^\alpha(T*\varphi) = T*\partial^\alpha\varphi = (\partial^\alpha T)*\varphi.$$

For every function $f \in \mathscr{L}^1_{loc}(\mathbb{R}^N)$ and φ in $\mathscr{E}(\mathbb{R}^N)$ such that either $\operatorname{supp}(f)$ or $\operatorname{supp}(\varphi)$ is compact, define *the convolution of f and φ* by

$$f*\varphi := (f\mathscr{L}^N)*\varphi$$

or more precisely

$$(f*\varphi)(x) := \int_{y\in\mathbb{R}^N} f(y)\varphi(x-y)d\mathscr{L}^N(y).$$

We have the following proposition.

Proposition 2.3.2. *Let T and φ be as above. Then*

$$\operatorname{supp}(T*\varphi) \subset \operatorname{supp}(T) + \operatorname{supp}(\varphi)$$

and if ψ is a function in $\mathscr{D}(\mathbb{R}^N)$ we have

$$(T*\varphi)*\psi = T*(\varphi*\psi).$$

Let χ be a positive function in $\mathscr{D}(\mathbb{R}^N)$ such that $\int \chi d\mathscr{L}^N = 1$. For $\epsilon > 0$, define

$$\chi_\epsilon(x) := \epsilon^{-N}\chi\left(\frac{x}{\epsilon}\right).$$

Theorem 2.3.3. *Let T, χ and χ_ϵ be as above. Then $\chi_\epsilon \to \delta_0$ for the weak topology on measures and $T*\chi_\epsilon \to T$ in the sense of distributions when $\epsilon \to 0$. Moreover, $\operatorname{supp}(\chi_\epsilon) \to \{0\}$ and $\operatorname{supp}(T*\chi_\epsilon) \to \operatorname{supp}(T)$ in the Hausdorff sense.*

Recall that a sequence of closed sets (F_n) in X converges in the Hausdorff sense to a closed set F if for $x \in F$ there are $x_n \in F_n$ such that $x_n \to x$ and for $x \notin F$ there is a neighbourhood of x which intersects only a finite number of F_n. The previous theorem shows that every distribution in $\mathbb{R}^N$ can be approximated by smooth functions. This property holds for distributions on every open set X in $\mathbb{R}^N$.

Theorem 2.3.4. *Let T be a distribution in $\mathscr{D}'(X)$. Then there exist distributions T_n defined by functions in $\mathscr{E}(X)$ such that $T_n \to T$ in the sense of distributions and $\operatorname{supp}(T_n) \to \operatorname{supp}(T)$ in the Hausdorff sense.*

Exercise 2.3.5. *Let f be a function in $\mathscr{L}^1(\mathbb{R}^N)$. Compute in $\mathscr{D}'(\mathbb{R}^N)$ the limit*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^N} f\left(\frac{x}{\epsilon}\right).$$

Exercise 2.3.6. *Let φ and ψ be two functions in $\mathscr{D}(\mathbb{R}^N)$. Show that $\varphi * \psi = \psi * \varphi$.*

Exercise 2.3.7. *Let μ be a probability measure with compact support in $\mathbb{R}^N$. Show that $\mu * \chi_\epsilon$ is a probability measure.*

Exercise 2.3.8. *Find all the distribution T with compact support in $\mathbb{R}$ such that $T*\varphi = 0$ where $\varphi(x) = x$. Same question when φ is a polynomial on x.*

Exercise 2.3.9. *Let P denote a differential operator with constant coefficients in $\mathbb{R}^N$. Assume that there is a distribution E smooth out of the origin such that $P(E) = \delta_0$. Let v be a distribution on an open set $X \subset \mathbb{R}^N$. Show that if $P(v)$ is smooth on X then v is smooth on X.*
Hint: Choose an open set $X' \Subset X$ and a smooth function χ with $0 \leq \chi \leq 1$ and $\chi = 1$ on X'. Write $\chi v = P(E) * \chi v$.

2.4 Laplacian and subharmonic functions

The Laplace operator on $\mathbb{R}^N$ is defined as $\Delta := \sum \frac{\partial^2}{\partial x_n^2}$. It acts on smooth functions and also on distributions.

Definition 2.4.1. An upper semi-continuous (u.s.c) function $u : X \to \mathbb{R}\cup\{-\infty\}$ is *subharmonic* if it belongs to $\mathscr{L}^1_{loc}(X)$ and if Δu is a positive measure. The function u is *harmonic* if u and $-u$ are subharmonic.

Let $\mathrm{SH}(X)$ denote the convex cone of subharmonic functions on X. In order to study subharmonic functions it is useful to describe explicitely a fundamental solution of the Laplace equation.

Theorem 2.4.2. *Let s_{N-1} denote the area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Define*

$$E(x) := \frac{-1}{(N-2)s_{N-1}\|x\|^{N-2}} \quad \text{if} \quad N \geq 2 \quad \text{and} \quad \frac{1}{2\pi}\log\|x\| \quad \text{if} \quad N = 2.$$

Then $\Delta E = \delta_0$ where δ_0 is the Dirac mass at 0.

Theorem 2.4.3. *Let ν be a positive measure with compact support on X then $u := \nu * E$ is a subharmonic function such that $\Delta u = \nu$.*

Corollary 2.4.4. *Harmonic functions are smooth.*

Theorem 2.4.5. *Let $u : X \to \mathbb{R} \cup \{-\infty\}$ be an u.s.c. function not identically $-\infty$. Then u is subharmonic if and only if it satisfies the following submean inequality: for every ball $\overline{\mathrm{B}}(x_0, r)$ contained in X we have*

$$u(x_0) \leq \frac{1}{s_{N-1}} \int_{\|\xi\|=1} u(x_0 + r\xi) d\sigma(\xi)$$

where σ is the invariant probability measure on the unit sphere of $\mathbb{R}^N$. This function is harmonic if and only if for every ball $\overline{\mathrm{B}}(x_0, r)$ contained in X we have

$$u(x_0) = \frac{1}{s_{N-1}} \int_{\|\xi\|=1} u(x_0 + r\xi) d\sigma(\xi).$$

The previous integral is the mean value of u on the sphere $b\mathrm{B}(x_0, r)$. The theorem is also valid if we replace the mean of u on the sphere by the mean on the ball $\mathrm{B}(x_0, r)$.

We obtain the following *maximum principle.*

Corollary 2.4.6. *Let u be a subharmonic function on X and let a be a point of X. Assume that $u(a) = \max_X u$. Then u is constant.*

The cone $\mathrm{SH}(X)$ satisfies the following compacteness properties.

Theorem 2.4.7.

1. *Let (u_n) be a sequence of subharmonic functions on X decreasing to a function u. Then either u is subharmonic or is identically $-\infty$.*
2. *If (u_n) is a sequence of subharmonic functions on X locally bounded from above, then $(\sup u_n)^*$ and $(\limsup u_n)^*$ are subharmonic or identically $-\infty$.*
3. *If u_n are subharmonic and if $\chi : (\mathbb{R} \cup \{-\infty\})^m \to \mathbb{R} \cup \{-\infty\}$ is a convex function which is increasing in each variable then $\chi(u_1, \ldots, u_p)$ is subharmonic or identically $-\infty$. In particular $\max(u_1, \ldots, u_m)$ is subharmonic.*

Theorem 2.4.8. *The cone $\mathrm{SH}(X)$ is closed for the $\mathscr{L}^1_{loc}$ topology. Every bounded set in the $\mathscr{L}^1_{loc}$ topology is relatively compact. Let (u_n) be a sequence in $\mathrm{SH}(X)$ locally bounded from above. Then either there is a subsequence of (u_n) which converges in $\mathscr{L}^1_{loc}(X)$ to a subharmonic function or (u_n) converges locally uniformly to $-\infty$.*

The following result is called *Hartogs lemma.*

Theorem 2.4.9. *Let (u_n) be a sequence of subharmonic functions on X. Assume that u is a subharmonic function on X and $u_n \to u$ in $\mathscr{L}^1_{loc}(X)$. Then $\limsup u_n(x) \leq u(x)$ for $x \in X$ with equality almost everywhere. For any compact set K in X and every continuous function f on K we have*

$$\limsup_{n\to\infty} \sup_K (u_n - f) \leq \sup_K (u - f).$$

In particular if $f \geq u$ and $\epsilon > 0$ then $u_n \leq f + \epsilon$ on K for n large enough.

Definition 2.4.10. A set $A \subset X$ is said to be *polar* if there is a subharmonic function on X such that $A \subset \{u = -\infty\}$. Since u is u.s.c. $\{u = -\infty\}$ is always a G_δ set, i.e. a countable intersection of open sets.

Theorem 2.4.11. *Let A be a polar closed set in X and let u be a subharmonic function on $X \setminus A$. Assume that u is locally bounded form above near A. Then there is a subharmonic function $\widetilde{u}$ on X such that $\widetilde{u} = u$ on $X \setminus A$.*

Exercise 2.4.12. *Find a real-valued function E on $\mathbb{R}$ such that $\frac{\partial^2}{\partial x^2} E = \delta_0$.*

Exercise 2.4.13. *Let u be a subharmonic function on X. Show that u is in $\mathscr{L}^p_{loc}(X)$ for $1 \leq p < \frac{N}{N-2}$ and ∇u is in $\mathscr{L}^p_{loc}(X)$ for $1 \leq p < \frac{N}{N-1}$.*

Exercise 2.4.14. *Let $K \Subset L$ be two compact subsets of X. Show that there is a constant $c_{KL} > 0$ such that*

$$\|\Delta u\|_K \leq c_{KL} \|u\|_{\mathscr{L}^1(L \setminus K)}$$

for every subharmonic function u on X.

Exercise 2.4.15. *Let (u_n) be a sequence of subharmonic functions uniformly bounded on X and let K be a compact subset of X. Assume that $\sup_X u_{n+1} \leq \sup_K u_n$ for every n. Show that u_n converge in $\mathscr{L}^1_{loc}$ to a constant. Does u_n converge pointwise to this constant?*

Exercise 2.4.16. *Let u_n and u be subharmonic functions on X such that $u_n \to u$ in $\mathscr{L}^1_{loc}(X)$. Let ν is a probability measure with compact support in X. Assume that $\int u_n d\nu \to \int u d\nu$. Show that*

1. $\limsup_{n\to\infty} u_n = u$*, ν-almost everywhere.*
2. *If u_n are uniformly bounded then $u_n \to u$ in $\mathscr{L}^p(\nu)$ for $1 \leq p < \infty$.*
3. *There is a subsequence of (u_n) converging to u in $\mathscr{L}^1(\nu)$.*

Hint:

1. *Use Fatou's lemma and Hartog's lemma.*
2. *Fix $\delta > 0$ and consider $A^\delta_n := \{u_n < u - \delta\}$.*
3. *If $\nu(A^\delta_n) \to 0$ one can extract a subsequence such that $u_{n_j} \to u$ ν-almost everywhere. Then use Fatou's lemma again.*

Exercise 2.4.17. *Let ν be a positive measure with compact support in X. We say that ν is SPB if every subharmonic on X is ν-integrable.*

1. *Give examples of SPB probability measure.*

2. *Let ν be a probability measure with compact support in $\mathbb{R}^2$. Prove that if $u := E*\nu$ is locally bounded then ν is SPB. Give an analogous statement in $\mathbb{R}^N$, $N > 2$.*

3. *Let ν be an SPB measure with compact support in X. Show that there exists a constant $c > 0$ such that for every subharmonic function u on X*

$$\int |u| d\nu \leq c \|u\|_{\mathscr{L}^1(X)}.$$

Hint: if this does not hold there are u_n subharmonic such that $\|u_n\|_{\mathscr{L}^1(X)} = 1$ and $\int |u_n| d\nu \geq n^3$. Show that u_n are locally unifomly bounded from above.

Exercise 2.4.18. *Let $z = x + iy$ be the complex coordinate on $\mathbb{C} \simeq \mathbb{R}^2$. Define*

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The second operator is called the Cauchy-Riemann operator on $\mathbb{C}$. Show that

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}} \quad \text{and} \quad \frac{\partial}{\partial \overline{z}}\left(\frac{1}{\pi z}\right) = \delta_0.$$

If f is a distribution on an open set $X \subset \mathbb{C}$ such that $\frac{\partial f}{\partial \overline{z}} = 0$, show that f is a holomorphic function on X. If φ is a function in $\mathscr{L}^1(\mathbb{C})$ with compact support in $\mathbb{C}$, show that

$$\frac{\partial}{\partial \overline{z}}\left(\frac{1}{\pi z} * \varphi\right) = \varphi.$$

Chapter 3

Currents

The notion of currents was introduced by Georges de Rham. It generalizes the notion of distribution. In fact, distributions are currents of maximal degree. We will define differential forms and currents on an open set X in $\mathbb{R}^N$ and state basic properties of currents. We will show that currents can be approximated by smooth forms and show that every closed current is locally exact (Poincaré's lemma).

3.1 Differential forms and currents

Consider a function f of class $\mathscr{C}^1$ on X and fix a point $a \in X$. To a vector $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ we associate the derivation $\xi_1 \frac{\partial}{\partial x_1} + \cdots + \xi_N \frac{\partial}{\partial x_N}$ at the point a. The space T_aX of such derivations is the *tangent space* of X at a. *The differential* df_a of f at a is the linear form on T_aX defined by

$$df_a(\xi) := \left(\xi_1 \frac{\partial f}{\partial x_1} + \cdots + \xi_N \frac{\partial f}{\partial x_N} \right)(a).$$

Consider the coordinate x_i as a function on X. Its differential at a is given by

$$(dx_i)_a(\xi) = \xi_i.$$

Hence we can write

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \xi_N \frac{\partial f}{\partial x_N} dx_N.$$

This is an example of a differential 1-form.

Definition 3.1.1. A *differential p-form* or a *form of degree p* on X is a map assigning to each point $a \in X$ an alternating p-form on T_aX.

Then, a differential 1-form is a map assigning to each point $a \in X$ an element of T_a^*X, i.e. a linear form on T_aX. It can be decomposed uniquely as

$$\alpha = \sum_{i=1}^{N} \alpha_i dx_i.$$

where the coefficients α_i are functions on X.

Differential p-forms can be obtained as combinations of exterior products of p forms of degree 1. Let u_i, $1 \leq i \leq p$, be differential 1-forms. *The exterior product* or *wedge product* of u_i is denoted by

$$u_1 \wedge \ldots \wedge u_p$$

Hence any p-form can be written as

$$\alpha := \sum_I \alpha_I dx_I$$

where $I = (i_1, \ldots, i_p)$, $1 \leq i_1 < \cdots < i_p \leq N$, $dx_I := dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ and α_I are functions on X. When the α_I are bounded, continuous or smooth functions we say that the form α is bounded, continuous or smooth.

In general, we can consider the exterior product of a p-form α and a q-form β. Their product is a $(p+q)$-form and enjoys the following properties

- associativity: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$
- distributivity: $(\alpha + \alpha') \wedge \beta = \alpha \wedge \beta + \alpha' \wedge \beta'$
- anti-commutativity: $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$.

For a function f we have $f \wedge \alpha = f\alpha$ and $(f\alpha) \wedge \beta = f(\alpha \wedge \beta)$. The anti-commutativity implies that $dx_i \wedge dx_i = 0$, $\alpha \wedge (\beta + \beta') = \alpha \wedge \beta + \alpha \wedge \beta'$ and every p-form with $p > N$ is zero. Then *forms of maximal degree* are forms of degree N.

We define the exterior derivative of forms. Let α be a p-form as above, of class $\mathscr{C}^1$. *The derivative of* α is the $(p+1)$-form:

$$d\alpha := \sum_I d\alpha_I \wedge dx_I.$$

If $d\alpha = 0$, we say that α *is closed.* It is clear that forms of maximal degree are always closed. Forms that can be written as $d\alpha$ are said to be *exact.*

Let $\alpha := \varphi dx_1 \wedge \ldots \wedge dx_N$ be a form of maximal degree where φ is a function in $\mathscr{L}^1(X)$. Define *the integral of* α by

$$\int_X \alpha = \int_X \varphi dx_1 \wedge \ldots \wedge dx_N := \int_X \varphi d\mathscr{L}^N.$$

Proposition 3.1.2. *Let α, α' be p-forms of class $\mathscr{C}^1$ and let β be a q-form of class $\mathscr{C}^1$. Then*

$$d(\alpha+\alpha') = d\alpha + d\alpha' \quad \textit{and} \quad d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p\alpha\wedge d\beta.$$

Moreover, if α is of class $\mathscr{C}^2$, then $d\alpha$ is closed, i.e. $d(d\alpha)=0$.

Let $f : X \to X'$ be a map of class $\mathscr{C}^1$. If $\alpha' = \sum \alpha'_I dx'_I$ is a p-form, define the *pull-back* of α' by f by

$$f^*(\alpha') := \sum_I (\alpha'_I \circ f) d(x'_I \circ f).$$

This is a p-form on X. It depends linearly on α'.

We endow the spaces $\mathscr{D}^p_{[k]}(X)$, $\mathscr{D}^p(X)$, $\mathscr{E}^p_{[k]}(X)$, $\mathscr{E}^p(X)$ of p-forms on X with the semi-norms and the topology induced by the ones on the coefficients of forms.

Definition 3.1.3. *A current of degree p and of dimension $N-p$* on X is a linear continuous form $T : \mathscr{D}^{N-p}(X) \to \mathbb{C}$. If α is a form in $\mathscr{D}^{N-p}(X)$, the value of T at α is denoted by $T(\alpha)$ or $\langle T, \alpha\rangle$. The form α is called *test form*. We say that *a sequence (T_n) of p-currents on X converges weakly* (or *converges in the sense of currents*) *to a current T* if $\langle T_n, \alpha\rangle \to \langle T, \alpha\rangle$ for every $\alpha \in \mathscr{D}^{N-p}(X)$.

So distributions are currents of maximal degree. If T is a current of degree 0, one can associate to T a distribution T' by

$$\langle T', \varphi\rangle := \langle T, \varphi dx_1 \wedge \ldots \wedge dx_N\rangle.$$

This correspondence between distributions and 0-currents is 1:1. Hence we often identify distributions and currents of degree 0.

Example 3.1.4. If T is a p-form with coefficients in $\mathscr{L}^1_{loc}(X)$ then T defines a current of degree p

$$\langle T, \alpha\rangle := \int_X T\wedge\alpha \quad \text{for } \alpha \in \mathscr{D}^{N-p}(X).$$

Example 3.1.5. Let $Y \subset \mathbb{R}^{N-p}$ be an open set and let $f : Y \to X$ be a proper injective map of class $\mathscr{C}^1$. Let $Z \subset Y$ be a Borel set and $Z' := f(Z)$. Define the current $[Z']$ of integration on Z' by

$$\langle [Z'], \alpha\rangle = \int_{Z'} f^*(\alpha) \quad \text{for } \alpha \in \mathscr{D}^{N-p}(X).$$

In particular, when $X = Y = Z = Z'$ and $f = \mathrm{id}$ we get the 0-current $[X]$ of integration on X.

Proposition 3.1.6. *Let T be a p-current on X and let $K \subset X$ be a compact set. Then there exist a positive integer k and a positive constant c such that*

$$|\langle T, \alpha \rangle| \leq c\|\alpha\|_{k,K} \quad \textit{for every } \alpha \in \mathscr{D}^{N-p}(X) \textit{ with } \operatorname{supp}(\alpha) \subset K.$$

Definition 3.1.7. If the integer k in Proposition 3.1.6 can be chosen independently of K we say that T is *of finite order* and the smallest integer k satisfying this property is called *order of T*.

One can check easily that currents in the examples 3.1.4 and 3.1.5 are of order zero. Currents of compact support are of finite order. We also have the following result.

Proposition 3.1.8. *Let T be a p-current of order k on X. Then we can extend T to a continuous linear form on $\mathscr{D}^{N-p}_{[k]}(X)$. Moreover, the extension is unique.*

Definition 3.1.9. *The support* of a p-current T is the smallest closed subset $\operatorname{supp}(T)$ of X such that T vanishes on $X \setminus \operatorname{supp}(T)$. That is, $\langle T, \alpha \rangle = 0$ for every test $(N-p)$-form $\alpha \in \mathscr{D}^{N-p}(X)$ with compact support in $X \setminus \operatorname{supp}(T)$.

The support of the current $[Z']$ in the example 3.1.5 is equal to $\overline{Z}'$.

Theorem 3.1.10. *Let T be a p-current with compact support in X. Then T can be extended to a continuous linear form on $\mathscr{E}^{N-p}(X)$. Moreover, the extension is unique.*

Exercise 3.1.11. *Let α be a 1-form on X. Show that $\alpha \wedge \alpha = 0$.*

Exercise 3.1.12. *Let f be a real-valued function of class $\mathscr{C}^1$ on X. Let $P : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^1$ function. Show that $d(P \circ f) = P'(f)df$.*

Exercise 3.1.13. *Let $T = \sum_I T_I dx_I$ be a p-current on X. Show that $\operatorname{supp}(T) = \bigcup \operatorname{supp}(T_I)$.*

Exercise 3.1.14. *Find a non-uniformly bounded sequence of smooth p-forms (T_n) with support in a fixed compact set $K \subset X$ such that $T_n \to 0$ in the sense of currents.*

3.2 Operations on currents and Poincaré's lemma

Let T be a p-current on X and let X' be an open subset of X. Then we can define the restriction of T to X' as follows:

$$\langle T_{|X'}, \alpha \rangle := \langle T, \alpha \rangle$$

for test forms α with compact support in X'. It is easy to check that $T_{|X'}$ is a p-current on X' and that the order of T' is smaller or equal to the order of T. In particular if T is a measure or a distribution then $T_{|X'}$ is also a measure or a distribution.

Let β be a q-form in $\mathscr{E}^q(X)$. Define *the wedge-product* $T \wedge \beta$ by

$$\langle T \wedge \beta, \alpha \rangle := \langle T, \beta \wedge \alpha \rangle \quad \text{for } \alpha \in \mathscr{D}^{N-p-q}(X).$$

This is a $(p+q)$-current on X. The $(p+q)$-current $\beta \wedge T$ is by definition equal to $(-1)^{pq} T \wedge \beta$. We will drop the sign $\wedge$ if p or q is equal to zero since the wedge product becomes the usual multiplication.

Proposition 3.2.1. *Let T be a p-current on X. Then T can be written in a unique way*

$$T = \sum_{|I|=p} T_I dx_I$$

where T_I are currents of degree 0 *on X.*

We have seen that 0-currents can be identified to distributions. One can consider that currents are forms with distribution coefficients.

We want to extend the operation of exterior derivative from forms to currents. When $\beta \in \mathscr{E}^p(X)$ and $\alpha \in \mathscr{D}^{N-p-1}(X)$ the classical Stokes formula implies that $\langle d\beta, \alpha \rangle = (-1)^{p+1} \langle \beta, d\alpha \rangle$. Hence we define *the exterior derivative* of currents as follows. Let T be a current of degree p on X. Define the $(p+1)$-current dT by

$$\langle dT, \alpha \rangle := (-1)^{p+1} \langle T, d\alpha \rangle \quad \text{for } \alpha \in \mathscr{D}^{N-p-1}(X).$$

The maps $T \mapsto dT$ is continuous for the topology on currents. When $dT = 0$ we say that T is *closed*. Currents of maximal degree (distributions) are always closed.

Consider a $\mathscr{C}^1$ map $f : X \to X'$ which is proper on the support of T. Assume that $N' \geq \dim T = N - p$. Then we can define *the push-forward* $f_*(T)$ as follows

$$\langle f_*(T), \alpha \rangle := \langle T, f^*(\alpha) \rangle \quad \text{for } \alpha \in \mathscr{D}^{N-p}(X').$$

The current $f_*(T)$ is of degree $N' - N + p$ and of dimension $N - p$. Hence the operator f_* preserves the dimension of currents.

Example 3.2.2. Let $U \subset X$ be a domain with $\mathscr{C}^1$ boundary. Define

$$\langle [U], \alpha \rangle := \int_U \alpha \quad \text{for } \alpha \in \mathscr{D}^N(X).$$

The boundary bU of U has the canonical orientation induced by the orientation on X. The classical Stokes formula implies that $d[U] = (-1)^N [bU]$.

Proposition 3.2.3. *The push-forward operator is continuous and commutes with the exterior derivative, i.e. we have*

$$f_*(dT) = (-1)^{N'-N} d(f_* T).$$

Now consider a p-current $T = \sum T_I dx_I$ on X and a p'-current $T' = \sum T'_{I'} dx_{I'}$ on X'. Define the $(p+p')$-current $T \otimes T$ on $X \times X'$ by

$$T \otimes T' := \sum_{I,I'} (T_I \otimes T'_{I'}) dx_I \wedge dx'_{I'}$$

where $T_I \otimes T'_{I'}$ is defined in the sense of distributions. The current $T \otimes T'$ is called *tensor product* of T and T'.

Proposition 3.2.4. *The tensor product $T \otimes T'$ depends continuously on T and T'. Moreover, we have*

$$d(T \otimes T') = (dT) \otimes T' + (-1)^p T \otimes (dT').$$

Let $\pi : X \times X' \to X$ be the canonical projection. When $T' = [X']$ the current $T \otimes [X']$ is called *the pull-back of T by π* and is denoted by $\pi^*(T)$. More generally, consider a submersion $f : Y \to X$. Locally we can identify Y with a product $X \times X'$ and f with the projection on X. Then we can define $f^*(T)$ using a partition of unity.

Proposition 3.2.5. *Let f be as above. Then $f^*(T)$ depends continuously on T and commute with the extrerior derivative, i.e. $f^*(dT) = df^*(T)$.*

Using the above result we can prove the following result which is called *Poincaré lemma.*

Theorem 3.2.6. *Let X be a starshaped domain in $\mathbb{R}^N$. Then every closed p-current T on X, $p \geq 1$, is exact, i.e. there is a $(p-1)$-current S such that $dS = T$.*

Exercise 3.2.7. *Let T be a p-current on X and let β be q-form in $\mathscr{E}^q(X)$. Show that* $\operatorname{supp}(T \wedge \beta) \subset \operatorname{supp}(T) \cap \operatorname{supp}(\beta)$. *Find an example such that* $\operatorname{supp}(T \wedge \beta) \neq \operatorname{supp}(T) \cap \operatorname{supp}(\beta)$.

Exercise 3.2.8. *Let T be a p-current of order k on X. Let β be q-form of class $\mathscr{C}^k$. Show that $T \wedge \beta$ is well defined and is a current of order $\leq k$.*

Exercise 3.2.9. *Let T be a p-current and β be a q-form in $\mathscr{E}^q(X)$. Show that*

$$d(T \wedge \beta) = dT \wedge \beta + (-1)^p T \wedge d\beta.$$

Exercise 3.2.10. *Let $f : X' \to X$ be a bijective $\mathscr{C}^1$ map and T be a p-current on X. Show that*

$$f_*(f^*(T)) = T.$$

Deduce that $f^* = (f^{-1})_*$ and $f_* = (f^{-1})^*$.

Exercise 3.2.11. *Let $f : X' \to X$ be a $\mathscr{C}^1$ submersion. Let T and β be as above. Show that*

$$f^*(T \wedge \beta) = f^*(T) \wedge f^*(\beta).$$

Is there an analogous formula for the push-forward operator?

Exercise 3.2.12. *Let T be a p-current on X and let $f : X \to X'$ be a $\mathscr{C}^1$ map proper on the support of T. If β is a q-form in $\mathscr{E}^q(X')$ show that*

$$f_*(T \wedge f^*(\beta)) = f_*(T) \wedge \beta.$$

Exercise 3.2.13. *Is the Poincaré's lemma valid for an arbitrary domain X in $\mathbb{R}^N$?*

3.3 Convolution and regularization

Let $T = \sum T_I dx_I$ be a p-current on $\mathbb{R}^N$. Let φ be a function in $\mathscr{E}(\mathbb{R}^N)$. Assume that either T or φ has compact support. Recall that we can identify T_I with a distribution on $\mathbb{R}^N$. Define *the convolution $T * \varphi$ of T and φ* by

$$T * \varphi := \sum (T_I * \varphi) dx_I.$$

This is a p-form in $\mathscr{E}^p(\mathbb{R}^N)$.

Proposition 3.3.1. *Let T and φ be as above. Then*

$$\operatorname{supp}(T * \varphi) \subset \operatorname{supp}(T) + \operatorname{supp}(\varphi)$$

and

$$d(T * \varphi) = (dT) * \varphi.$$

If ψ be a function in $\mathscr{D}(\mathbb{R}^N)$ then

$$(T * \varphi) * \psi = T * (\varphi * \psi).$$

Theorem 3.3.2. *Let T be a p-current on $\mathbb{R}^N$ and let χ_ϵ be as in Section 2.3. Then $T * \chi_\epsilon \to T$ in the sense of currents and $\operatorname{supp}(T * \chi_\epsilon) \to \operatorname{supp}(T)$ in the Hausdorff sense when $\epsilon \to 0$.*

We can approximate currents in the general case.

Theorem 3.3.3. *Let T be a p-current on X. Then there exist smooth p-forms T_n on X such that $T_n \to T$ in the sense of currents and $\operatorname{supp}(T_n) \to \operatorname{supp}(T)$ in the Hausdorff sense. If T is closed then the forms T_n can be chosen to be closed.*

Exercise 3.3.4. *Let* τ_x *be as in Section 2.3. Show that*

$$\langle T * \varphi, \alpha \rangle = \int_{x \in \mathbb{R}^N} \langle (h_x)_* T, \alpha \rangle \varphi(x) d\mathscr{L}^N(x)$$

where $h_x(y) := y + x$.

Exercise 3.3.5. *Show that in the first part of Theorem 3.3.3 we can choose* T_n *with compact support.*

Chapter 4

Currents on manifolds

After introducing differentiable manifolds, their tangent bundles and differental forms on manifolds we state the regularization theorem for currents on manifolds and some properties of de Rham cohomologies in the case of compact manifolds.

4.1 Differentiable manifolds

We define differentiable manifolds of dimension N and of class $\mathscr{C}^k$, $k \in \mathbb{N}^* \cup \{\infty\}$. Let X be a separated topological space which is a finite or countable union of compact sets. Assume that X can be covered by a collection of opens sets $(U_i)_{i\in I}$ such that there are continuous bijective maps $\varphi_i : U_i \to V_i$ where V_i are open subsets of $\mathbb{R}^N$. Assume also that the *transition maps*

$$\varphi_{ji} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j).$$

are of class $\mathscr{C}^k$. Define $V_{ji} := \varphi_i(U_i \cap U_j)$.

Definition 4.1.1. A family of (U_i, φ_i) satisfying the previous property is an *atlas of class* $\mathscr{C}^k$ of X. The topological space X equipped with such an atlas is a *manifold of class* $\mathscr{C}^k$ *and of dimension* N. The components of $\varphi_i = (\varphi_{i,1}, \dots, \varphi_{i,N})$ are *local coordinates* on the *chart* (U_i, φ_i) of X.

A space X may have different structures of $\mathscr{C}^k$ manifold. Two $\mathscr{C}^k$ atlas (U_i, φ_i) and (W_j, ψ_j) of X define the same structure if $\psi_j \circ \varphi_i^{-1}$ are $\mathscr{C}^k$ maps on $\varphi_i(U_i \cap W_j)$ for all i, j. In practice, we often say that X is a manifold of class $\mathscr{C}^k$ when it has a $\mathscr{C}^k$ structure. If $(U_i, \varphi_i)_{i\in I}$ is an atlas of X there is a finite ou countable sub-atlas $(U_j, \varphi_j)_{j\in J}$ with $J \subset I$.

Example 4.1.2. Euclidian spaces $\mathbb{R}^N$, spheres in $\mathbb{R}^N$, tori, projective spaces $\mathbb{PR}^N$, and their open sets are examples of manifolds. If X and X' are manifolds then $X \times X'$ is a manifold.

Definition 4.1.3. Let X be a manifold as in Definition 4.1.1. We say that X is *orientable* if it admits an atlas (U_i, φ_i) as above such that the transition maps φ_{ji} have positive jacobian.

Definition 4.1.4. Let Y be a closed subset of a manifold X of dimension N. We say that Y is a *submanifold of dimension* n of X if for every point $y \in Y$ there are local coordinates $(x_1, \ldots, x_N)$ on a neighbourhood W of y such that $Y \cap W = \{x_1 = \cdots = x_{N-n} = 0\}$.

Definition 4.1.5. Let X and X' be manifolds of class $\mathscr{C}^k$ equipped with atlases (U_i, φ_i) and (U'_j, φ'_j). A continuous map $f : X \to X'$ is said to be *of class* $\mathscr{C}^k$ if $\varphi'_j \circ f \circ \varphi_i^{-1}$ are of class $\mathscr{C}^k$ on their domains of definition for all i and j. The map f is an *imbedding* if it is injective and if $f(X')$ is a submanifold of X. A $\mathscr{C}^k$ bijective map $f : X \to X'$ is called *diffeomorphism* between X and X', in this case, we say that X and X' are *diffeomorphic*.

One checks easily that the previous properties of f depend only on the $\mathscr{C}^k$ structures on X and X' but not on the atlases (U_i, φ_i) and (U'_j, φ'_j). Using the open sets V_i, V_{ji} of $\mathbb{R}^N$ and transition maps φ_{ji} we can construct an abstract manifold diffeomorphic to X as follows.

Proposition 4.1.6. *Let V_i, V_{ji} and φ_{ji} be as above. Consider the disjoint union $\widetilde{X}$ of the V_i and the following relation $\sim$. For x and y in $\widetilde{X}$, we have $x \sim y$ if $y = \varphi_{ji}(x)$ for some i, j. Then $\sim$ is a relation of equivalence and the quotient $\widetilde{X}/\sim$ is a manifold diffeomorphic to X.*

The proof of this proposition uses the fact that (φ_{ji}) is a cocycle, i.e. $\varphi_{ii} = \mathrm{id}$, $\varphi_{ij} \circ \varphi_{ji} = \mathrm{id}$ and $\varphi_{ik} \circ \varphi_{kj} \circ \varphi_{ji} = \mathrm{id}$.

The following theorem shows that every manifold admits a partition of unity.

Theorem 4.1.7. *Let X be a $\mathscr{C}^k$ manifold and let (W_i, φ_i) be an open cover of X. Then there exist positive $\mathscr{C}^k$ functions g_i with support in W_i such that for each point $x \in X$ we have $g_i(x) = 0$ except for a finite number of g_i and $\sum g_i = 1$ on X.*

Manifolds and maps of class $\mathscr{C}^\infty$ are called *smooth*. In what follows we assume that the manifolds are smooth in order to simplify the notation. We also define manifolds X *with boundary* in the same way as in Definition 4.1.1 but we replace V_i by open sets in the closed half-space $\mathbb{R}^N_- := \{(x_1, \ldots, x_N) \in \mathbb{R}^N,\ x_1 \leq 0\}$. The union of $\varphi_i^{-1}(V_i \cap \{x_1 = 0\})$ is called *the boundary* of X and is denoted by bX. Observe that $X \setminus bX$ is a manifold without boundary.

Complex manifolds are defined as in Definition 4.1.1 but we replace V_i by open sets of $\mathbb{C}^N$ and we assume that the maps φ_i are holomorphic. *A complex manifold with boundary* is a real manifold X with boundary bX such that $X \setminus bX$ is a complex manifold.

Exercise 4.1.8. *Let X be a connected manifold of dimension 1. Show that X is either diffeomorphic to a circle or to the real line.*
Hint: show that X can be covered by a finite or countable family of open sets U_n, $n \in \mathbb{Z}$, such that $U_n \cap U_{n+1} \cap U_{n+2} = \varnothing$.

Exercise 4.1.9. *Show that there is no smooth injective map from the sphere $\mathbb{S}^2$ onto $\mathbb{R}^2$.*

Exercise 4.1.10. *Let X be a smooth manifold. Show that there exists a proper smooth function $f : X \to \mathbb{R}$. Such a function is called an exhaustion function.*

Exercise 4.1.11. *Let X be a manifold. Show that any smooth function f on X can be written as $f = \sum f_i$ where f_i are smooth with compact support such that for every x we have $f_i(x) = 0$ except for a finite number of f_i.*

4.2 Vector bundles

We will consider an important class of manifolds: the vector bundles which are locally product of $\mathbb{R}^d$ with a manifold. Let X, E be manifolds of dimension N and $N+d$ respectively, and let $\pi : E \to X$ be a smooth map. Assume that (U_i) is a covering of X by open sets and that $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^d$ are diffeomorphisms such that $\Phi(\pi^{-1}(x)) = \{x\} \times \mathbb{R}^d$. Assume also that for all i, j and $x \in U_i \cap U_j$ the transition maps $\Phi_{ji} := \Phi_j \circ \Phi_i^{-1}$ define linear automorphisms on $\{x\} \times \mathbb{R}^d$. The last property implies the existence of smooth maps $H_{ji} : U_i \cap U_j \to \mathrm{GL}(d, \mathbb{R})$ such that $\Phi_{ji}(x, v) = \big(x, H_{ji}(x)(v)\big)$.

Definition 4.2.1. Under the previous assumptions, we say that $\pi : E \to X$ (or E for short) is a *vector bundle of rank d over the base* X and $\pi^{-1}(x)$ is *a fiber* of E. When $d = 1$ we say that $\pi : E \to X$ is a *line bundle over* X. The map Φ_i is *the trivialisation* of E over U_i. *A section* of E is a continuous map $s : X \to E$ such that $\pi \circ s = \mathrm{id}$.

One can check that the union of $\Phi_i^{-1}(U_i \times \{0\})$ is a section of E. This is the *zero section*. Observe that the set of of all the sections of E is a vector space. If $Y \subset X$ is a manifold then $F := \pi^{-1}(Y)$ is a vector bundle of rank d over Y.

Definition 4.2.2. Let $\pi' : E' \to X'$ be another vector bundle. A smooth map $\tau : E \to E'$ which sends linearly fibers to fibers is called *morphism* between vector bundles. If τ is bijective then E and E' are said to be *isomorphic*. The bundle E is *trivial* if it is isomorphic to the product $X \times \mathbb{R}^d$.

Proposition 4.2.3. *Let $\pi : E \to X$ be a vector bundle. Then there exists a riemannian metric on E, i.e. a scalar product $p_x(\cdot, \cdot)$ on each fiber $\pi^{-1}(x)$ which depends smoothly on x.*

Consider a manifold X of dimension N and a family of open sets U_i which covers X. Let $H_{ji} : U_i \cap U_j \to \mathrm{GL}(d, \mathbb{R})$ be smooth maps. Assume that (H_{ji}) is a *cocycle*, i.e. $H_{ii}(x) = \mathrm{id}$ for $x \in U_i$, $H_{ij} \circ H_{ji}(x) = \mathrm{id}$ for $x \in U_i \cap U_j$ and $H_{ik} \circ H_{kj} \circ H_{ji}(x) = \mathrm{id}$ for $x \in U_i \cap U_j \cap U_k$ for all i, j, k. Let $\widetilde{E}$ be the disjoint union of $U_i \times \mathbb{R}^d$. Define the relation $\sim$ on $\widetilde{E}$ as follows. For two points $(x, v) \in U_i$ and $(x', v') \in U_j$ we have $(x, v) \sim (x', v')$ if $x = x'$ and $v' = H_{ji}(x)(v)$.

Theorem 4.2.4. *Let X, U_i, H_{ji} and $\widetilde{E}$ be as above. Then $\sim$ is an equivalence relation and $\widetilde{E}/\sim$ admits a structure of vector bundle of rank d over X.*

If H_{ji} are as in Definition 4.2.1 then $\widetilde{E}/\sim$ is isomorphic to E. Let $K_{ji} : U_i \cap U_j \to \mathrm{GL}(d, \mathbb{R})$ be defined by $K_{ji}(x) := H_{ji}(x)^{-1}$. Then (K_{ji}) is a cocycle. Using these maps we can define a vector bundle E^* of rank d over X. This bundle is called *the dual* of E.

We can consider a vector bundle E as a collection of vector spaces parametrized by a manifold X. Then some operations on vector spaces can be extended to vector bundles. More precisely, we can define $E \oplus E'$, $E \otimes E'$, $\Lambda^p E$ when E and E' are bundles over the same base X.

Riemannian metrics on E and E' induce riemannian metrics on E^*, $E \oplus E'$, $E \otimes E'$ and $\Lambda^p E$.

Complex vector bundles over complex manifolds are defined analogously but we replace $\mathbb{R}^d$ by $\mathbb{C}^d$ and the maps Φ_i are assumed to be holomorphic.

Exercise 4.2.5. *Show that a line bundle is trivial if and only if it admits a non-vanishing smooth section. Find an analogous criterion for vector bundles.*

Exercise 4.2.6. *Show that any line bundle over $\mathbb{R}$ is trivial. Find a non-trivial line bundle L over $\mathbb{S}^1$. Show that $L \otimes L$ is trivial.*

Exercise 4.2.7. *Let E be a vector bundle of rank d over X. Show that $\Lambda^d E$ is a line bundle over X.*

4.3 Tangent bundle and differential forms

If X is an open subset of $\mathbb{R}^N$, a tangent vector at a is simply a derivation $\sum \lambda_n \frac{\partial}{\partial x_n}$. The set of tangent vectors at a is isomorphic to $\mathbb{R}^N$. The tangent bundle of X is (isomorphic to) $X \times \mathbb{R}^N$.

The tangent bundle of a general manifold X is not always trivial and can be defined using an atlas (U_i, φ_i) as in Definition 4.1.1. Define $H_{ji} : U_i \cap U_j \to \mathrm{GL}(N, \mathbb{R})$ the maps such that $H_{ji}(x)$ is the jacobian of φ_{ji} at the point $\varphi_i(x)$. One can check that these maps satisfy the assumptions of Theorem 4.2.4. We then obtain a vector bundle of rank N over X which is the *tangent bundle of* X that one denotes by $\pi : TX \to X$. The fiber of x is denoted by T_xX. A section of TX is called *vector field* on X.

The dual T^*X of TX is called *cotangent bundle* of X. A section of $\Lambda^p(T^*X)$ is a *p-form* on X. A non-vanishing real form of maximal degree N on X is called *volume form.* If α is a p-form and α' is a p'-form on X then $\alpha \wedge \alpha'$ is a $(p+q)$-form on X. It is defined using local coordinates. If $f : X' \to X$ is a smooth map then one can also define the pull-back $f^*(\alpha)$ of α which is a p-form on X'. Using local coordinates one shows that $f^*(\alpha \wedge \alpha') = f^*(\alpha) \wedge f^*(\alpha')$.

An orientation on X is an orientation on each fiber T_xX of TX which depends continuously on x. This induces an orientation on the fibers of $\Lambda^p(T^*X)$.

Proposition 4.3.1. *A manifold X of dimension N is orientable if and only if the line bundle $\Lambda^N(T^*X)$ is trivial, i.e. there is a volume form on X.*

An orientable connected manifold X admits two orientations. If Ω and Ω' are volume forms on X then there is a non-vanishing function f such that $\Omega' = f\Omega$. This function is either positive or negative. Then we can divide the family of volume forms into two classes such that the two volume forms Ω_1 and Ω_2 are in the same class if and only if $\Omega_1 = \lambda\Omega_2$ with λ a strictly positive function. Fixing an orientation on X is deciding that the forms in one class are positive, and negative in the other class. If X is a manifold with boundary then an orientation of $X \setminus bX$ induces an orientation on bX. Indeed this is the case for open sets of $\mathbb{R}^N_-$.

A riemannian metric on TX is called *riemannian metric* on X. It induces riemannian metrics on $\Lambda^p(T^*X)$ and a distance on X. Then we can talk about volumes of varieties, subvarieties, Hausdorff measures ...

Exercise 4.3.2. *Show that every smooth vector field on the sphere $\mathbb{S}^2$ vanishes at some point.*
Hint: $\mathbb{S}^2 \setminus \{1$ point$\}$ is diffeomorphic to $\mathbb{C}$.

Exercise 4.3.3. *Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus. Show that the tangent bundle of $\mathbb{T}$ is trivial.*

Exercise 4.3.4. *Let X be an oriented manifold and let $\sigma : X \to X$ be an involution, i.e. a diffeomorphism without fixed point such that $\sigma \circ \sigma = \mathrm{id}$. Let $\widetilde{X}$ be the space obtained from X by identifying the points x and $\sigma(x)$. Show that $\sigma \setminus X$ is a manifold which is orientable if and only if σ preserves the orientation.*

4.4 Currents and de Rham theorems

In what follows we consider only riemannian oriented manifolds. The spaces of forms and currents on a manifold X are defined exactly as in the case of an open set in $\mathbb{R}^N$. A p-form on X is of course a p-current. We give some other classes of examples.

Example 4.4.1. (Integration on manifolds) Let Y be a submanifold of codimension p of X. We can define a p-current of integration on Y by

$$\langle [Y], \alpha \rangle := \int_Y \alpha \quad \text{for } \alpha \in \mathscr{D}^{N-p}(X).$$

In order to integrate a form on Y one can use the partition of unity and local coordinates. If $\alpha_{|Y}$ is supported on a chart $\varphi_i : U_i \to V_i$ of Y then $\int_Y \alpha := \int_{V_i} (\varphi_i^{-1})^*(\alpha)$. More generally consider a manifold X' of dimension $N-p$ with or without boundary and a smooth proper injective map $f : X' \to X$. Then we can define a p-current $[f(X')]$ by

$$\langle [f(X')], \alpha \rangle := \int_{X'} f^*(\alpha) \quad \text{for } \alpha \in \mathscr{D}^p(X).$$

We have the following Stokes formula.

Theorem 4.4.2. *Let $f : X' \to X$ be as above. Then*

$$d[f(X')] = (-1)^{N-p}[f(bX')].$$

In particular if Y is a submanifold of X (without boundary) then the current $[Y]$ is closed.

The following fundamental theorem is due to de Rham.

Theorem 4.4.3. *Let T be a p-current on a manifold X. Then there are smooth p-forms T_n such that $T_n \to T$ in the sense of currents and* $\operatorname{supp}(T_n) \to \operatorname{supp}(T)$. *Moreover if T is closed (resp. exact) then T_n can be chosen to be closed (resp. exact).*

In what follows we assume that X is a compact manifold without boundary. However the following results can be extended to non-compact manifolds but the statement is more complicated.

We have seen that the set of exact p-forms (resp. p-currents) is a subspace of the space of closed p-forms (resp. p-currents). The de Rham cohomology groups (with real coefficients) are the quotient spaces

$$H^p(X, \mathbb{R}) := \frac{\{\text{real-valued closed } p\text{-forms on } X\}}{\{\text{real-valued exact } p\text{-forms on } X\}}.$$

Using currents we can define in the same way the following groups

$$H^p_{cur}(X,\mathbb{R}) := \frac{\{\text{real-valued closed } p\text{-currents on } X\}}{\{\text{real-valued exact } p\text{-currents on } X\}}.$$

Since p-forms are also p-currents, we obtain a natural linear map

$$\pi : H^p(X,\mathbb{R}) \to H^p_{cur}(X,\mathbb{R}).$$

The class of a closed p-form or p-current T in these cohomology groups will be denoted by $[T]$. Forms or currents in the same class are said to be *cohomologous.*

A more precise version of theorem 4.4.3 implies the following result.

Theorem 4.4.4. *The linear map π is an isomorphism. In particular every closed p-current is cohomologous to a closed smooth p-form.*

The dimension b_p of $H^p(X,\mathbb{R})$ is called *Betti number.* We always have $b_0 = b_N = 1$. De Rham and Poincaré's results imply the following theorem.

Theorem 4.4.5. *The group $H^p(X,\mathbb{R})$ has finite dimension and the map*

$$P : H^p(X,\mathbb{R}) \times H^{N-p}(X,\mathbb{R}) \to \mathbb{R}, \qquad ([\alpha],[\beta]) \mapsto \int_X \alpha\wedge\beta$$

is well-defined and is a non-degenerate bilinear form. In particular $H^{N-p}(X,\mathbb{R})$ is isomorphic to the dual of $H^p(X,\mathbb{R})$.

The map P is called *the Poincaré duality.*

Exercise 4.4.6. *Show that the current of integration on the unit circle is not exact in $\mathbb{R}^2\setminus\{0\}$.*

Exercise 4.4.7. *Let $T=\mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus. Determine the dimension and a basis of $H^p(X,\mathbb{R})$ for $p=0,1,2$. Same question for the sphere $\mathbb{S}^2$.*

Chapter 5

Slicing theory

In this chapter we consider the spaces of locally normal currents and locally flat currents. We describe the Federer slicing theory for these currents.

5.1 Normal and flat currents

Definition 5.1.1. A current T on X is said to be *locally normal* if the currents T and dT are of order zero. Locally normal currents with compact support are called *normal*.

Let $\mathscr{N}^p_{loc}(X)$ denote the space of locally normal p-currents on X, $\mathscr{N}^p_K(X)$ the space of normal p-currents with support in a compact set K and $\mathscr{N}^p(X) := \bigcup_K \mathscr{N}^p(K)$ the space of normal p-currents. Define also the semi-norms

$$\mathscr{N}_K(T) := \|T\|_K + \|dT\|_K.$$

The space $\mathscr{N}^p_{loc}(X)$ is a Fréchet space with respect to the family of semi-norms $\mathscr{N}_K(\cdot)$ and $\mathscr{N}^p(K)$ is a Banach space with respect to the norm $\mathscr{N}_K(\cdot)$. Example 4.4.1 gives us a family of locally normal currents. The space of normal currents is stable under exterior differentiation. However for many purpose one considers a larger space which is also stable under this operation.

We are going to define the flat currents which are less regular than the normal currents. They are obtained by taking the closure of the normal currents with respect to a family of semi-norms.

Define for a compact subset K of X and smooth form φ with compact support in X the flat semi-norm

$$\mathscr{F}_K(\varphi) := \max\big(\sup_K \|\varphi(x)\|, \sup_K \|d\varphi(x)\|\big)$$

and for a p-current T on X

$$\mathscr{F}_K(T) := \sup\big\{|\langle T, \varphi\rangle|, \quad \varphi \in \mathscr{D}^{N-p}(X) \text{ and } \mathscr{F}_K(\varphi) \leq 1\big\}.$$

Proposition 5.1.2. *Let T be a p-current on X. Then $\mathscr{F}_K(T) < \infty$ if and only if there are a p-current R and a $(p-1)$-current S of order zero supported on K such that $T = R + dS$. Moreover*

$$\mathscr{F}_K(T) = \min\left\{\|R\| + \|S\|, \quad R, S \text{ as above}\right\}.$$

In particular T has support in K.

Definition 5.1.3. A p-current T is *flat on K* if it belongs to the closure of $\mathscr{N}_K(X)$ for the flat semi-norm $\mathscr{F}_K$. A current T is *flat* if it is flat on compact subsets of X and is *locally flat* if φT is flat for any smooth function φ with compact support in X.

Observe that not all the currents T with $\mathscr{F}_K(T) < \infty$ are flat on K. The space of flat currents is a subspace of such currents. Let $\mathscr{F}^p_K(X)$ denote the space of p-currents flat on K, $\mathscr{F}^p(X)$ the space of flat p-currents and $\mathscr{F}^p_{loc}(X)$ is the space of locally flat p-currents on X.

Proposition 5.1.4. *The space of p-currents flat on K of finite mass is equal to the closure of $\mathscr{N}^p_K(X)$ for the mass norm on X.*

The following theorem gives a very useful characterization of flat currents.

Theorem 5.1.5. *If R is a p-current and S is a $(p-1)$-current both supported on K with coefficients in $\mathscr{L}^1(X)$ then $T = R + dS$ is in $\mathscr{F}^p_L(X)$ for every compact set L such that $K \Subset L$. Conversely if T is a p-current flat on K then there are a p-current R and a $(p-1)$-current S supported in L with coefficients in $\mathscr{L}^1(X)$ such that $T = R + dS$.*

Corollary 5.1.6. *A p-current T is locally flat on X if and only if for every smooth function φ with compact support on X there are currents R and S with coefficients in $\mathscr{L}^1(X)$ and with compact support such that $\varphi T = R + dS$.*

Exercise 5.1.7. *Let T be a locally normal p-current on X. Show that φT is normal for every smooth function φ with compact support in X.*

Exercise 5.1.8. *Let φ be a subharmonic function on an open subset X of $\mathbb{R}^N$. Show that φ defines a normal 0-current on X.*

Exercise 5.1.9. *Let T be a locally flat p-current on X. Show that dT is locally flat.*

Exercise 5.1.10. *Find all the flat currents supported on a point $a \in X$.*

Exercise 5.1.11. *Let L be a closed subset of X and let T be a locally normal current on $X \setminus L$. Assume that T has locally finite mass near L. Show that the trivial extension of T is locally flat on X. Recall that the trivial extension of T is the extension by zero on L.*
Hint: let $0 \leq u_n \leq 1$ be smooth functions vanishing near L and such that $u_n \to \mathbf{1}_{X\setminus L}$ uniformly on compact sets. Then the trivial extension $\widetilde{T}$ of T satisfies $\widetilde{T} = \lim u_n T$.

Exercise 5.1.12. *Let $D(a_n, r_n)$ be a sequence of disjoint discs in $\mathbb{R}^2$ converging to 0. Assume that $\sum r_n^2 < \infty$. Define the current T of dimension 0 by*

$$\langle T, \alpha \rangle := \sum_n \int_{D(a_n,r_n)} \alpha, \qquad \alpha \in \mathscr{D}^2(\mathbb{R}^2).$$

Show that T is of order zero. Compute dT. What is the order of dT?

5.2 Federer's support theorems

In this section we give some results of Federer which describe flat currents with small support.

Theorem 5.2.1. *Let T be a locally flat p-current on X. If $\mathscr{H}^{N-p}(\mathrm{supp}(T)) = 0$ then $T = 0$.*

Theorem 5.2.2. *Let T be a locally flat p-current on X. Assume that T is supported in a submanifold Y. If Y has codimension p, then $T = f[Y]$ with f a function in $\mathscr{L}^1_{loc}(Y)$. If Y has codimension $\leq p$ then T is a current on Y, i.e. there is a current T' on Y such that $T = i_*(T')$ where $i : Y \to X$ is the canonical inclusion map.*

The following result is more general.

Theorem 5.2.3. *Let $f, g : X \to X'$ be smooth maps and let T be a flat p-current on X. If $f = g$ on $\mathrm{supp}(T)$ then $f_*(T) = g_*(T)$.*

Exercise 5.2.4. *Let Y be a submanifold of X. Find a current T on X with support on Y which is not a current on Y.*

Exercise 5.2.5. *Let Y_1, Y_2 be submanifolds of codimension p of X. Assume that $Y_1 \cap Y_2$ is a submanifold of codimension $> p$ of X. Find all the locally flat p-currents of finite mass supported on $Y_1 \cup Y_2$.*

Exercise 5.2.6. *Let Y be a submanifold of codimension p of X. Find all the closed p-currents with locally finite mass supported on Y.*

Exercise 5.2.7. *Let A be a closed set in X such that $\mathscr{H}^{p-1}(A) = 0$. Let M be a closed oriented submanifold of dimension p in $X \setminus A$. Assume M has bounded volume in X. Show that $[M]$ is a flat closed current of order zero.*

5.3 Slicing of flat currents

Let $\pi : X \to X'$ be a submersion. Then the fibers of π are submanifolds of dimension $N - N'$ of X. Let T be a p-current on X with $N - p \geq N'$. We are going to define the slice $\langle T, \pi, y\rangle$ for $y \in X'$. This is a p-current on $\pi^{-1}(y)$. If T is smooth, $\langle T, \pi, y\rangle$ is the restriction of T to $\pi^{-1}(y)$. If T is the current of integration on a submanifold Y of X then $\langle T, \pi, y\rangle$ is the current of integration on $Y \cap \pi^{-1}(y)$ for y generic. We will extend this to locally flat currents. Namely we have the following theorem.

Theorem 5.3.1. *Let π be a submersion as above and let T be a locally flat p-current on X with $N - p \geq N'$. Then for almost every $y \in X'$ there is a locally flat p-current T_y on $\pi^{-1}(y)$ such that for every N'-form Ω with compact support on X' and every $(N - N' - p)$-form α with compact support on X we have*

$$\big\langle T, \alpha \wedge \pi^*(\Omega)\big\rangle = \int_{X'} \big\langle T_y, \alpha_{|\pi^{-1}(y)}\big\rangle \Omega(y).$$

Moreover two families of locally flat currents (T_y) and (T'_y) satisfying the previous property are equal almost everywhere on X'.

The currents T_y are called *the slices of T by π* and are denoted by $\langle T, \pi, y\rangle$. They can be considered as $(N' + p)$-currents on X. Slicing satisfies the following properties.

Proposition 5.3.2. *For almost every $y \in X'$ we have*

1. $\operatorname{supp}\langle T, \pi, y\rangle \subset \operatorname{supp}(T) \cap \pi^{-1}(y)$.
2. *If ψ is a smooth form on X then $\langle \psi \wedge T, \pi, y\rangle = \psi \wedge \langle T, \pi, y\rangle$.*
3. $\langle dT, \pi, y\rangle = d\langle T, \pi, y\rangle$.
4. *If T has locally finite mass then $\langle T, \pi, y\rangle$ has locally finite mass.*
5. *If T is locally normal the $\langle T, \pi, y\rangle$ is locally normal.*

The slices of T can be computed as follows. Let $(y_1, \ldots, y_{N'})$ be local coordinates of X'. Let $\Omega := dy_1 \wedge \ldots \wedge dy_{N'}$ be the canonical volume form and let $\Omega_{y,r}$ be its restriction to the ball of center y and of radius r.

Proposition 5.3.3. *For almost every $y \in X'$ we have*

$$\big\langle \langle T, \pi, y\rangle, \alpha\big\rangle = \lim_{r\to 0} \frac{\big\langle T \wedge \pi^*(\Omega_{y,r}), \alpha\big\rangle}{c(N')r^{N'}}, \quad \alpha \in \mathscr{D}^{N-N'-p}(X)$$

where $c(N')$ is the volume of the unit ball in $\mathbb{R}^{N'}$.

Exercise 5.3.4. *Let $\pi : X \to X'$ be as above. Find a locally flat p-current T on X such that $T \neq 0$ but $\langle T, \pi, y\rangle = 0$ for almost every y.*

Chapter 6

Currents in complex analysis

In this chapter X denotes a complex manifold of dimension N, for example an open set in $\mathbb{C}^N$. The complex coordinates in a chart of X or in $\mathbb{C}^N$ are denoted by $z = (z_1, \ldots, z_N)$. The main object of this chapter is to study positive closed currents interoduced by Pierre Lelong. This notion generalizes analytic subsets in complex manifolds and has many applications in complex analysis and in the theory of dynamical systems.

6.1 Positive forms and positive currents

If we identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ we can write $z_n := x_n + ix_{n+N}$ with x_n real. Hence linear forms dx_n, dx_{n+N} can be written in a unique way as linear combinations with complex coefficients of $dz_n := dx_n + idx_{n+N}$ and of $d\overline{z}_n := dx_n - idx_{n+N}$. If $I = (i_1, \ldots, i_r) \subset \{1, \ldots, N\}$ is a multi-index define $dz_I := dz_{i_1} \wedge \ldots \wedge dz_{i_r}$ and $d\overline{z}_I := d\overline{z}_{i_1} \wedge \ldots \wedge d\overline{z}_{i_r}$. Then any r-form α on X can be written locally in a unique way as

$$\alpha = \sum_{|I|+|J|=r} \alpha_{IJ} dz_I \wedge d\overline{z}_J$$

where α_{IJ} are functions with complex values. We say that α is *a form of bidegree* (p, q) if $\alpha_{IJ} = 0$ when either $|I| \neq p$ or $|J| \neq q$. We have the following decomposition of spaces of r-forms as direct sums of spaces of (p, q)-forms

$$\mathscr{E}^r(X) = \bigoplus_{p+q=r} \mathscr{E}^{p,q}(X) \quad \text{and} \quad \mathscr{D}^r(X) = \bigoplus_{p+q=r} \mathscr{D}^{p,q}(X)$$

and by duality we get

$$\mathscr{D}'_r(X) = \bigoplus_{p+q=r} \mathscr{D}'_{p,q}(X) \quad \text{and} \quad \mathscr{E}'_r(X) = \bigoplus_{p+q=r} \mathscr{E}'_{p,q}(X).$$

A current in $\mathscr{D}'_{p,q}(X)$ or in $\mathscr{E}'_{p,q}(X)$ are said to be *current of bidegree* (p, q) *and of bidimension* $(N-p, N-q)$. They act trivially on forms of bidegree $(N-p', N-q')$ when $(p', q') \neq (p, q)$.

Proposition 6.1.1. *Let $f : X \to X'$ be a holomorphic map. If α is a (p, q)-form on X' then $f^*(\alpha)$ is a (p, q)-form on X. If T is a current of bidimension (p, q) on X and if f is proper on* $\text{supp}(T)$ *then $f_*(T)$ is a current of bidimension (p, q). If T is a current of bidegree (p, q) on X' and if f is a submersion then $f^*(T)$ is a current of bidegree (p, q) on X.*

If T is a (p, q)-current on X then we can decompose dT as the sum of a $(p + 1, q)$-current ∂T and of a $(p, q + 1)$-current $\overline{\partial}T$. Since $d(dT) = 0$ we have $\partial(\partial T) = 0$, $\overline{\partial}(\overline{\partial}T) = 0$ and $\partial\overline{\partial}T = -\overline{\partial}\partial T$. If α is a test form we have

$$\langle \partial T, \alpha\rangle = (-1)^{p+q+1}\langle T, \partial\alpha\rangle \quad \text{and} \quad \langle \overline{\partial}T, \alpha\rangle = (-1)^{p+q+1}\langle T, \overline{\partial}\alpha\rangle.$$

It is easily to check that f^*, f_* commute with ∂ and $\overline{\partial}$. Define $d^c := \frac{1}{2i\pi}(\partial - \overline{\partial})$, then $dd^c = \frac{i}{\pi}\partial\overline{\partial}$ is a real operator. We define also *the conjugate* of a form or a current by the formulas

$$\overline{\alpha} := \sum \overline{\alpha}_{IJ} d\overline{z}_I \wedge dz_J \quad \text{and} \quad \langle \overline{T}, \alpha\rangle := \overline{\langle T, \overline{\alpha}\rangle}$$

where $\alpha = \sum \alpha_{IJ} dz_I \wedge d\overline{z}_J$.

Definition 6.1.2. A (p, p)-form α is said to be *positive* if at each point it is equal to a finite combination of forms $(i\beta_1 \wedge \overline{\beta}_1) \wedge \ldots \wedge (i\beta_p \wedge \overline{\beta}_p)$ where β_i are $(1, 0)$-forms which might depend on the point. The form α is said to be *weakly positive* if $\alpha \wedge \beta$ is positive for any positive $(k - p, k - p)$-form β. A (p, p)-current T is called *positive* (resp. *weakly positive*) if $\langle T, \alpha\rangle \geq 0$ for every weakly positive (resp. positive) test $(k - p, k - p)$-form α.

The operators f^* and f_* preserve the positivity. Positive currents of maximal bidegree are positive measures.

Proposition 6.1.3. *Let $T = \sum i^{p^2} T_{IJ} dz_I \wedge d\overline{z}_J$ be a positive current. Then T is of order zero. In particular T_{IJ} are distributions of order zero, i.e. complex measures.*

Define

$$\beta := i\partial\overline{\partial}\|z\|^2 \quad \text{and} \quad \sigma_T := \frac{1}{2^{N-p}(N - p)!} T \wedge \beta^{N-p}.$$

Then σ_T is a positive measure that we call *trace measure* of T. The mass of σ_T on a Borel set B is called *mass of T on B* and is also denoted by $\|T\|_B$. Recall that we are in $\mathbb{C}^N$ or in a chart of X.

Proposition 6.1.4. *Let $T = \sum i^{p^2} T_{IJ} dz_I \wedge d\overline{z}_J$ be a positive current and let σ_T be the trace measure of T. Then the total variations $|T_{IJ}|$ of T_{IJ} verify $|T_{IJ}| \leq 2^N \sigma_T$.*

The following theorem is due to Wirtinger.

Theorem 6.1.5. *Let $V \subset \mathbb{C}^N$ be a manifold of pure dimension p. Then the volume form on V associated to the standard metric in $\mathbb{C}^N$ is equal to $\frac{1}{2^p p!}\beta^p_{|V}$. In particular we have*

$$\text{volume}(V) = \frac{1}{2^p p!}\int_V \beta^p.$$

A closed set V in X is an *analytic subset* of X if for each point $a \in V$ there is a neighbourhood U of a such that $V \cap U = \{f_1 = \cdots = f_n = 0\}$ where f_i are holomorphic functions on U. A point $a \in V$ is *regular* if $V \cap U$ is a manifold for U small enough. One says that V has *pure dimension* p if $\dim(V \cap U) = p$ for every a. The following Lelong's theorem gives us an important class of positive closed currents.

Theorem 6.1.6. *Let V be an analytic subset of pure dimension p of X and let $\text{reg}(V)$ be the regular part of V. Define the current $[V]$ of bidimension (p,p) by*

$$\langle [V], \alpha\rangle := \int_{\text{reg}(V)} \alpha, \qquad \alpha \in \mathscr{D}^{p,p}(X).$$

Then the current $[V]$ is well defined and is positive and closed.

This theorem says in particular that the volume of $\text{reg}(V)$ near singular points is locally bounded.

Exercise 6.1.7. *Let α be a postive (N,N)-form on X. Show that α can be written as $\alpha = \varphi(idz_1 \wedge d\bar{z}_1) \wedge \ldots \wedge (idz_N \wedge d\bar{z}_N)$ where φ a positive function.*

Exercise 6.1.8. *Let T be a weakly positive (p,p)-current and let α be weakly positive (q,q)-form on X. Show that T is positive if $p = 0, 1, N-1$ or N. Find an example of a weakly positive form α which is not positive. Show that $T \wedge \alpha$ is weakly positive if either T or α is positive, and $T \wedge \alpha$ is positive if both T and α are positive. Find an example so that $T \wedge \alpha$ is not weakly positive.*

Exercise 6.1.9. *Let T be a (weakly) positive (p,p)-current on an open set X of $\mathbb{C}^N$. Let χ_ϵ be an approximation of identity in $\mathbb{C}^N$ as in Section 2.3. Then $T * \chi_\epsilon$ is a (weakly) positive (p,p)-form on the open set where it is defined. If T is closed then $T * \chi_\epsilon$ is closed. Moreover we have $T * \chi_\epsilon \to T$ in the sense of currents as $\epsilon \to 0$.*

Exercise 6.1.10. *Let T and T' be two positive closed (p,p)-currents on an open subset X of $\mathbb{C}^N$ with $p \le N-1$. Let K be a compact subset of X. Assume that $T = T'$ outside K. Show that $\|T\|_K = \|T'\|_K$.*

Exercise 6.1.11. *Let $T = i^{p^2}\sum T_{IJ} dz_I \wedge d\bar{z}_J$ be a positive closed (p,p)-current on an open subset X of $\mathbb{C}^N$. Show that*

$$\sigma_T = \frac{1}{2^{N-p}}\left(\sum_{|I|=p} T_{II}\right) idz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge idz_N \wedge d\bar{z}_N.$$

6.2 Plurisubharmonic functions

Plurisubharmonic functions were introduced by Lelong and Oka.

Definition 6.2.1. An u.s.c function $u : X \to \mathbb{R} \cup \{-\infty\}$ in $\mathscr{L}^1_{loc}(X)$ is *plurisubharmonic* (psh for short) if $dd^c u$ is a positive $(1,1)$-current. We say that u is *pluriharmonic* if u and $-u$ are psh.

Let ρ be a smooth positive function with compact support in $\mathbb{C}^N$ depending only on $\|z\|$ such that $\int \rho d\mathscr{L}^{2N} = 1$. Define $\rho_\epsilon(z) := \epsilon^{-2N}\rho(\epsilon^{-1}z)$ (see also Section 2.3).

Theorem 6.2.2. *If u is a psh function in an open set $X \subset \mathbb{C}^N$ then $u * \rho_\epsilon$ is smooth psh on $X_\epsilon := \{z \in X,\ \mathrm{dist}(z, \partial X) > \epsilon\}$ and decrease to u when ϵ decreases to* 0.

Theorem 6.2.3. *Let u be an u.s.c. function on an open set $X \subset \mathbb{C}^N$ which is not identically $-\infty$. Then u is psh if and only if its restriction to any complex line L is either subharmonic or identically $-\infty$ on each compenent of $X \cap L$.*

Example 6.2.4. If a is a point in the unit ball B of $\mathbb{C}^N$ then $\log\|z - a\|$ is psh on $\mathbb{C}^N$. If (λ_n) is a sequence of positive numbers and $(a_n) \subset$ B such that $\sum \lambda_n \log\|a_n\| > -\infty$, then $\sum \lambda_n \log\|z - a_n\|$ defines a psh function on B. The sequence (a_n) can be dense in B.

Proposition 6.2.5. *The set* $\mathrm{PSH}(X)$ *of psh functions on X is a convex cone with the following properties.*

1. *If a function $\chi : (\mathbb{R} \cup \{-\infty\})^p \to \mathbb{R} \cup \{-\infty\}$ is convex and increasing in each variable and if $u_1, \ldots, u_p$ are psh functions on X then $\chi(u_1, \ldots, u_p)$ is psh or is identically $-\infty$ on X. In particular* $\max(u_1, \ldots, u_p)$ *is psh.*

2. *If u_n are psh functions decreasing to a function u then either u is identically $-\infty$ or is psh.*

3. *If (u_n) is a sequence of psh functions locally bounded from above then $(\sup_n u_n)^*$ is a psh function and $(\sup_n u_n)^* = \sup_n u_n$ almost everywhere.*

4. *If $f : X' \to X$ is a holomorphic map and if u is psh on X then on each component of X', $u \circ f$ is either p.s.h. or identically $-\infty$.*

The following result is called the *Lelong-Poincaré formula.*

Theorem 6.2.6. *Let f be a holomorphic function on X which is not identically zero. Then* $\log|f|$ *is a psh function which is pluriharmonic outside the zero set $f^{-1}(0)$ of f. Moreover we have*

$$dd^c \log|f| = \sum m_i [V_i]$$

where V_i are irreducible components of $f^{-1}(0)$ and m_i are their multiplicities.

Definition 6.2.7. A subset E of X is *locally pluripolar* if for every point $a \in X$ there is a neighbourhood U of a and a psh function u on U such that $E \cap U \subset \{u = -\infty\}$. The set E is said to be *locally complete pluripolar* if u can be chosen so that $E \cap U = \{u = -\infty\}$. A subset E is *pluripolar* (resp. *complete pluripolar*) if there is a psh function u on X such that $E \subset \{u = -\infty\}$ (resp. $E = \{u = -\infty\}$)

Proposition 6.2.8. *Any proper analytic subset of X is complete pluripolar. A countable union of complete pluripolar (resp. pluripolar) sets is complete pluripolar (resp. pluripolar). The Hausdorff dimension of a pluripolar set is smaller or equal to $2N - 2$.*

Theorem 6.2.9. *Let E be a closed locally pluripolar subset of X and let u be a psh function on $X \setminus E$. Assume that u is locally bounded from above near E. Then there is a psh function $\widetilde{u}$ on X such that $\widetilde{u} = u$ outside E.*

Exercise 6.2.10. *Construct a psh function on $\mathbb{C}^N$ which is nowhere continuous.*

Exercise 6.2.11. *Construct two psh functions u, v on $\mathbb{C}^N$ which are equal on a ball* B *but are not equal. Show that there are two positive closed $(1,1)$-currents T, S which are equal on* B *but are not equal and it is possible to choose* $\mathrm{supp}(T)$ *and* $\mathrm{supp}(S)$ *connected.*

Exercise 6.2.12. *Let $z_n = x_n + ix_{n+N}$ be the coordinates of $\mathbb{C}^N$ and let $\pi(z) := (x_1, \ldots, x_N)$ denote the projection on $\mathbb{R}^N$. Let v be a real-valued function on an open set U of $\mathbb{R}^N$ such that $u := v \circ \pi$ is psh on $\pi^{-1}(U)$. Show that v is convex in U.*
Hint: assume first that v is smooth.

Exercise 6.2.13. *Let u be a real-valued function on $\mathbb{C}^N$. We say that u is log-homogeneous if*

$$u(\lambda z) = \log|\lambda| + u(z) \quad \text{for } \lambda \in \mathbb{C}^* \text{ and } z \in \mathbb{C}^N.$$

1. *Prove that if u is such a function then there is a constant $c > 0$ such that $u(z) \leq \log|z| + c$. If P is a polynomial of degree $d \geq 1$ find a positive constant α such that $\alpha \log|P|$ is log-homogeneous.*

2. *Let u be a psh function on $\mathbb{C}^N$ such that $u(z) \leq \log|z| + c$ with $c > 0$. Define*

$$v(z_0, z_1, \ldots, z_N) := u\left(\frac{z_1}{z_0}, \ldots, \frac{z_N}{z_0}\right) + \log|z_0| \quad \text{if } z_0 \neq 0.$$

   Show that v can be extended to a psh log-homogenenous function on $\mathbb{C}^{N+1}$.

Exercise 6.2.14. *Let $\pi : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ be the canonical projection. A holomorphic section of π over an open set $U \subset \mathbb{P}^N$ is a holomorphic map $s : U \to \mathbb{C}^{N+1} \setminus \{0\}$ such that $\pi \circ s = \mathrm{id}$.*

1. *If s and s' be two holomorphic sections of π over U, show that there is a non-vanishing holomorphic function h on U such that $s' = hs$.*

2. *If u is a psh log-homogeneous function on $\mathbb{C}^{N+1}$ define $T_U := dd^c(u \circ s)$. Show that T_U is a positive closed $(1,1)$-current independent of s. Show that there is a unique positive closed $(1,1)$-current T on $\mathbb{P}^N$ such that $\pi^*(T) = dd^c u$.*

3. *Let $\omega_{\rm FS}$ be the positive closed $(1,1)$-current on $\mathbb{P}^N$ such that $\pi^*(\omega_{\rm FS}) = dd^c \log \|z\|$. Show that $\omega_{\rm FS}$ is smooth and $\int_{\mathbb{P}^N} \omega_{\rm FS}^N = 1$.*

4. *Let P be a homogeneous polynomial on $\mathbb{C}^{N+1}$ of degree $d \geq 1$. Show that there is a positive closed $(1,1)$-current T on $\mathbb{P}^N$ such that $\pi^*(T) = dd^c \log |P|$. Describe geometrically $\pi^*(T)$ when $N = 1$ and compute $\langle T, \omega_{\rm FS}^{N-1}\rangle$.*

The form $\omega_{\rm FS}$ is called Fubini-Study form on $\mathbb{P}^N$. It is invariant under the action of the unitary group $U(N+1)$.

Exercise 6.2.15. *Let u be a strictly negative psh function on X. Show that $v := -\log(-u)$ is psh. Show that ∇v is in $\mathscr{L}^2_{loc}$.*

Exercise 6.2.16. *Let u be a strictly psh function on X. Show that ∇u is in $\mathscr{L}^{2-\epsilon}_{loc}$ for $0 < \epsilon \leq 1$. Show that ∇u is in $\mathscr{L}^2_{loc}$ if u is locally bounded.*

Exercise 6.2.17. *Let u be a pluriharmonic function in a ball B in $\mathbb{C}^N$. Show that there is a holomorphic function f on B such that $u = \Re(f)$. Deduce that u is real analytic.*

Exercise 6.2.18. *Let u_n be sequence of pluriharmonic functions which are locally uniformly bounded on X. Show that there is a subsequence converging locally uniformly to a pluriharmonic function.*

Exercise 6.2.19. *In $\mathbb{C}^2$ describe geometrically the currents*

$$dd^c \log^+ |z_1| \quad \text{and} \quad dd^c \log \max(|z_1|, |z_2|).$$

Compute the mass of $dd^c \log^+ \|z\|$ on the unit sphere of $\mathbb{C}^2$.

Exercise 6.2.20. *Let E be a complete pluripolar set in $\mathbb{C}^N$ and let V be a connected submanifold of $\mathbb{C}^N$. Show that either $E \cap V$ is complete pluripolar or $V \subset E$.*

Exercise 6.2.21. *Construct a polar subset of $\mathbb{C}^N$ which is not pluripolar and a pluripolar subset of $\mathbb{C}^N$ which is not complete pluripolar.*

6.3 Intersection of currents and slicing

Let T be a positive (p,p)-current on X with $p \leq N-1$. Consider a $\mathscr{C}^2$ psh function u on X. Then $dd^c u$ is a positive continuous form and it makes sense to consider the $(p+1,p+1)$-current $dd^c u \wedge T$ which is positive. We want to extend this to psh functions which are locally σ_T-integrable in particular to u continuous or locally bounded.

Assume that T is **positive** and **closed**. Define

$$dd^c u \wedge T := dd^c(uT).$$

Theorem 6.3.1. *Let T and u be as above. Then*

1. *The current $dd^c u \wedge T$ is well defined and is positive and closed. If (u_n) is a sequence of psh functions decreasing to u then $dd^c u_n \wedge T \to dd^c u \wedge T$ in the sense of currents.*

2. *If u_n are psh continuous functions converging locally uniformly toward u and if T_n are positive closed (p,p)-currents converging to T then $dd^c u_n \wedge T_n \to dd^c u \wedge T$.*

We have the following Chern-Levine-Nirenberg inequalities.

Theorem 6.3.2. *Let $L \Subset K$ be two compact subsets of X. Let T, u be as above and let $u_1, \ldots, u_m$, v be psh functions with $m+p \leq N$. Assume that u_n are locally bounded functions and v is locally σ_T-integrable. Then there is a constant c_{LK} independent of T, u, v and u_n such that*

1. $\|dd^c u \wedge T\|_L \leq c_{LK} \|T\|_K \|u \mathbf{1}_K\|_{\mathscr{L}^1(\sigma_T)}$.

2. $\|dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge T\|_L \leq c_{LK} \|T\|_K \|u_1\|_{\mathscr{L}^\infty(K)} \ldots \|u_m\|_{\mathscr{L}^\infty(K)}$.

3. $\|v dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge T\|_L \leq c_{LK} \|vT\|_K \|u_1\|_{\mathscr{L}^\infty(K)} \ldots \|u_m\|_{\mathscr{L}^\infty(K)}$.

We have seen that the assumption that u is σ_T-integrable is verified when u is continuous or locally bounded. The following result gives another situation where this assumption is easy to check.

Theorem 6.3.3. *Let X be an open set in $\mathbb{C}^N$ and let K be a compact subset of X. Let T be a positive closed (p,p)-current on X with $p \leq N-1$. If a psh function u on X is locally bounded on $X \setminus K$ then it is locally σ_T-integrable. If $u_1, \ldots, u_m$ are psh on X and locally bounded on $X \setminus K$ with $m+p \leq N$ then*

1. *$dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge T$ is a positive closed $(m+p, m+p)$-current on X.*

2. *If u_j^n are psh on X and decreasing to u_j then*

$$dd^c u_1^n \wedge \ldots \wedge dd^c u_m^n \wedge T \to dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge T.$$

In particular $dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge T$ is symmetric with respect to $u_1, \ldots, u_m$.

Theorem 6.3.4. *Let $\pi : X \to X'$ be a holomorphic submersion between two open sets in $\mathbb{C}^N$ and $\mathbb{C}^{N-p}$. Let T be a positive closed (p,p)-current on X. Assume that π is proper on the support of T. Then the slice $\langle T, \pi, a\rangle$ exists for every $a \in X'$. Moreover it is a positive measure with mass independent of a.*

Exercise 6.3.5. *Show that the current $dd^c \log|z_1| \wedge dd^c \log|z_2|$ is well defined and compute it.*

Exercise 6.3.6. *Show that the measure $(dd^c \log\|z\|)^N$ is well defined and compute it.*

Exercise 6.3.7. *Let $L := \{(z_1, z_2) \in \mathbb{C}^2,\ |z_1| = |z_2|\}$. Define for $n \geq 1$*

$$v_n(z) := |z_1^{2n} + z_2^{2n}|^{1/2n} \quad \textit{and} \quad v(z) := \max(|z_1|, |z_2|).$$

1. *Show that $0 \leq v_n(z) \leq 2^{1/n} v(z)$ and that $v_n \to v$ locally uniformly on $\mathbb{C}^2 \setminus L$.*
2. *Prove that $(dd^c v_n)^2 = 0$ and $(dd^c v)^2 \neq 0$.*
3. *Show that $v_n \to v$ pointwise except on a set of Hausdorff dimension 2.*

Hint:

2. *Prove it first out of the origin, and show that $(dd^c v_n)^2$ has no mass at 0.*
3. *Write $z_2 = z_1 e^{i\theta}$ for $z \in L$. Show that the following set has Hausdorff dimension zero*

$$E := \big\{\theta,\ \liminf_{n\to\infty} \big(\cos(n\theta)\big)^{1/n} < 1\big\}.$$

Exercise 6.3.8. *Let u_1 and u_2 be two psh functions in $\mathbb{C}^2$. Define $T_i := dd^c u_i$. Show that u_1 is locally σ_{T_2}-integrable if and only if u_2 is locally σ_{T_1}-integrable.*

Exercise 6.3.9. *Let X, L, K and T be as in Theorem 6.3.2. Let u be a smooth psh function on X. Show that there is a constant c_{LK} independent of u and T such that*

$$\|i\partial u \wedge \overline{\partial} u \wedge T\|_L \leq c_{LK} \|u\|^2_{\mathscr{L}^\infty(K)} \|T\|_K.$$

Show that if u is a continuous psh function then the currents $\partial u \wedge T$ and $\overline{\partial} u \wedge T$ are well-defined. If u_n are continuous psh functions converging locally uniformly to u show that $\partial u_n \wedge T \to \partial u \wedge T$ and $\overline{\partial} u_n \wedge T \to \overline{\partial} u \wedge T$.

Exercise 6.3.10. *Let π and T be as in Theorem 6.3.4. Assume that $p = 1$ and u is a potential of T, i.e. $T = dd^c u$. Show that the restriction of u to $\pi^{-1}(a)$ is a potential of $\langle T, \pi, a\rangle$.*

Exercise 6.3.11. *Let π be as in Theorem 6.3.4. Assume T is a positive closed (p,p)-current on X. Let* $\mathrm{supp}^*(T)$ *be the smallest closed subset of X such that T is a continuous form outside* $\mathrm{supp}^*(T)$. *Assume that π is proper on* $\mathrm{supp}^*(T)$. *Show that the slice $\langle T, \pi, a\rangle$ exists for every $a \in X'$.*

6.4 Skoda's extension theorem

Let E be a closed subset of X. Let T be a current on $X \setminus E$. Assume that the mass of T is locally bounded near E. More precisely we assume that at every point $z \in E$ there is a neighbourhood U of z such that

$$\|T\|_{U\setminus E} < \infty.$$

Then one can consider the trivial extension $\widetilde{T}$ of T in X. Basically since T has measure coefficients we extend the measure by putting the mass zero on E. One can also define $\widetilde{T}$ as follows. Let $0 \leq \chi_n \leq 1$ be a sequence of smooth functions vanishing near E and such that χ_n increase to $\mathbf{1}_{X\setminus E}$ locally uniformly on $X \setminus E$. Then if α is a test form with compact support in X we define

$$\langle \widetilde{T}, \alpha\rangle := \lim_{n\to\infty} \langle T, \chi_n \alpha\rangle.$$

The following theorem was proved by Skoda [18].

Theorem 6.4.1. *Let E be an analytic subset of X. Let T be a positive closed (p,p)-current on $X \setminus E$. Assume that the mass of T is locally bounded near E. Then the trivial extension $\widetilde{T}$ of T is a* **positive closed** *(p,p)-current on X.*

This implies the following result due to Bishop.

Theorem 6.4.2. *Let E and X be as above. Let V be an analytic subset of pure dimension p of $X \setminus E$. Assume that V has locally finite mass near E. Then $\overline{V}$ is an analytic subset of X.*

Remark 6.4.3. The previous results still hold when E is a locally complete pluripolar set of X and when T is a positive current such that $dd^cT \leq S$ on $X \setminus E$ where S is a current of order zero on X, see [1, 17, 4, 8].

When E is an analytic subset of small dimension the assumption on the mass of T is not necessary. The following theorem is due to Harvey-Polking.

Theorem 6.4.4. *Let E be a closed subset of X such that $\mathscr{H}^{2N-2p-1}(E) = 0$. Let T be a positive closed (p,p)-current on $X \setminus E$. Then T has finite mass near E and the trivial extension $\widetilde{T}$ of T is positive closed on X.*

This gives in particular the Remmert-Stein theorem. Let A be an analytic subset of X of dimension less or equal to $N-p-1$. If V is an analytic subset of pure dimension $N-p$ of $X \setminus A$ then $\overline{V}$ is an analytic subset of X.

Exercise 6.4.5. *Find a (non-complete) pluripolar set E on a manifold X and a positive closed current T such that T has finite mass near E but $\widetilde{T}$ is not closed.*

Exercise 6.4.6. *Let T be a positive closed $(1,1)$-current in $\mathbb{C}^2 \setminus \{z_1 = 0\}$. Assume that T has finite mass near $\{z_1 = 0,\ |z_2| > 1\}$. Show that T has locally finite mass near $\{z_1 = 0\}$. Deduce that the trivial extension of T is positive closed on $\mathbb{C}^2$.*

Exercise 6.4.7. *Find a positive dd^c-closed $(1,1)$-current on $\mathbb{C}^2 \setminus \{z_1 = 0\}$ with mass locally bounded near $\{z_1 = 0\}$ so that its trivial extension is not dd^c-closed.*

6.5 Lelong number and Siu's theorem

Let σ be a positive measure on X. It is possible to define the *upper* and *lower* m*-densities* of σ at $a \in X$. More precisely

$$\Theta^*(m,a) := \limsup_{r\to 0} \frac{\sigma(\mathrm{B}(a,r))}{c_m r^m} \quad \text{and} \quad \Theta_*(m,a) := \liminf_{r\to 0} \frac{\sigma(\mathrm{B}(a,r))}{c_m r^m}$$

where c_m is the volume of the unit ball in $\mathbb{R}^m$. We have $c_{2k} = \pi^k/k!$. In general the limit does not exist. When it exists we call it simply m*-density* of σ at a. It turns out that the trace measure σ_T of a positive closed (p,p)-current T has this remarkable property.

Definition 6.5.1. The $(2N-2p)$-density of σ_T at a is called *Lelong number* of T at a and is denoted by $\nu(T,a)$.

The following results were proved by Siu.

Theorem 6.5.2. *Let X and T be as above. Then the Lelong number $\nu(T,a)$ is a non-negative finite number which does not depend on the local coordinates on X.*

Theorem 6.5.3. *Let X and T be as above. Then the level set $\{\nu(T,a) > c\}$ is an analytic subset of dimension $\leq N-p$ of X for every $c > 0$.*

Corollary 6.5.4. *Let X and T be as above. Then there is a finite or countable family of irreducible analytic sets V_i of dimension $N-p$ of X and positive constants λ_i such that $T' := T - \sum \lambda_i[V_i]$ is a positive closed current such that $\{\nu(T',a) > 0\}$ is a finite or countable union of analytic subsets of dimension $< N-p$.*

Exercise 6.5.5. *Let T be a positive closed (p,p)-current on X. Assume that $T = dd^cU$ where U is a locally bounded $(p-1,p-1)$-form. Show that the Lelong number of T is zero at every point of X.*

Exercise 6.5.6. *Let V be an analytic subset of pure dimension $N-p$ of X. Compute the Lelong number of $[V]$ at a point a of X.*

Exercise 6.5.7. *Compute the Lelong number of $(dd^c \log\|z\|)^p$ at all points of $\mathbb{C}^N$.*

Bibliography

[1] Alessandrini, L. and Bassanelli, G. Plurisubharmonic currents and their extension across analytic subsets, *Forum Math.*, **5** (1993), no. 6, 577–602.

[2] Chemin, J.-Y. *Théorie des distributions et analyse de Fourier*, cours à l'école polytechnique de Paris, 2003.

[3] Demailly, J.-P. Monge-Ampère Operators, Lelong numbers and Intersection theory in Complex Analysis and Geometry, *Plemum Press* (1993), 115-193, *(V. Ancona and A. Silva editors).*

[4] Dabbek, K., Elkhadhra, F. and El Mir, H. Extension of plurisubharmonic currents, *Math. Z.*, **245** (2003), 455-481.

[5] Demailly, J.-P. Regularization of closed positive currents and intersection theory, *J. Algebraic Geometry*, **1** (1992), 361-409.

[6] de Rham, G. *Variétés Différentiables*, Paris, Hermann, 1960. (3rd Ed., 1973).

[7] Dinh, T.-C. and Sibony, N. Regularization of currents and entropy, *Ann. Sci. Ecole Norm. Sup.*, **37** (2004), 959-971.

[8] Dinh, T.-C. and Sibony, N. Theory of currents and dynamics of correspondences, *to appear.*

[9] Duval, J. and Sibony, N. Polynomial convexity, rational convexity, and currents, *Duke Math. J.*, **79**, No.2 (1995), 487-513.

[10] Federer, H. *Geometric Measure Theory*, New York, Springer-Verlag, 1969.

[11] Harvey, R. and Polking, J. Extending analytic objects, *Comm. Pure Appl. Math.*, **28** (1975), 701-727.

[12] Harvey, R. and Shiffman, B. A characterization of holomorphic chains, *Ann. of Math. (2)*, **99** (1974), 553-587.

[13] Hörmander, L. *The analysis of Linear partial differential operators I*, Springer-Verlag, 1983.

[14] Hörmander, L. *An introduction to complex Analysis in Several Variables*, Third Edition, North-Holland, 1990.

[15] Lelong, P. *Fonctions plurisousharmoniques et formes différentielles positives*, Dunod, Paris, 1968.

[16] Méo, M. Image inverse d'un courant positif fermé par une application surjective, *C.R.A.S.*, **322** (1996), 1141-1144.

[17] Sibony, N. Quelques problèmes de prolongement de courants en analyse complexe, *Duke Math. J.*, **52** (1985), no.1, 157–197.

[18] Skoda, H. Prolongement des courants positifs, fermés de masse finie, *Invent. Math.*, **66** (1982), 361-376.

[19] Schwartz, L. *Théorie des distributions*, Hermann, Paris, 1966.

[20] Warner, F.W. *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, 1971.