REALIZING UNIFORMLY RECURRENT SUBGROUPS

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Abstract. We show that every uniformly recurrent subgroup of a locally compact group is the family of stabilizers of a minimal action on a compact space. More generally, every closed invariant subset of the Chabauty space is the family of stabilizers of an action on a compact space on which the stabilizer map is continuous everywhere. This answers a question of Glasner and Weiss. We also introduce the notion of a universal minimal flow relative to a uniformly recurrent subgroup and prove its existence and uniqueness.

1. Introduction

Let G be a locally compact group. Consider the space of subgroups $\mathrm{Sub}(G)$ endowed with the Chabauty topology [C] and recall that a subbasis of open sets for this topology is given by sets of the form

$$\mathcal{U}_C = \{H \in \mathrm{Sub}(G) : H \cap C = \varnothing\} \quad \text{and} \quad \mathcal{U}_V = \{H \in \mathrm{Sub}(G) : H \cap V \neq \varnothing\},$$

where C varies among the compact subsets and V among the open subsets of G. This topology makes $\mathrm{Sub}(G)$ a compact Hausdorff space on which G acts continuously by conjugation.

Glasner and Weiss [GW] initiated the study of *uniformly recurrent subgroups* (URS for short), i.e., closed, invariant, minimal subsets of $\mathrm{Sub}(G)$. This notion can be seen as a topological analogue of the measure-theoretic one of *invariant random subgroup* [AGV]. URSs have recently attracted some attention as it turned out that this notion is a convenient tool to study *boundary actions*, which for discrete groups are connected to C^**-simplicity* [K2, LBMB].

As was shown by Glasner and Weiss [GW], a URS is naturally associated to every minimal action $G \curvearrowright X$ on a compact space. Namely, consider the stabilizer map $\mathrm{Stab}\colon X \to \mathrm{Sub}(G)$. This map is usually not continuous. However it is upper semi-continuous, in the sense that for every net (x_i) converging to $x \in X$, every cluster point of $\mathrm{Stab}(x_i)$ in $\mathrm{Sub}(G)$ is contained in $\mathrm{Stab}(x)$. This property is enough to ensure that the closure of the image of Stab contains a unique URS. (This result is proved in [GW, Proposition 1.2]. See also the argument of [AG, Lemma I.1] to avoid the assumption, made throughout [GW], that X is metrizable.) The unique URS contained in $\overline{\mathrm{Stab}(X)}$ is called the *stabilizer URS* of $G \curvearrowright X$ and will be denoted by $\mathcal{S}_G(X)$.

In analogy with what is known for IRSs, Glasner and Weiss ask whether every URS arises in this way. In this paper, we answer this question in the affirmative.

Theorem 1.1. *Let G be a locally compact group and let $\mathcal{H} \subseteq \mathrm{Sub}(G)$ be a closed, invariant subset. Then there exists a continuous action of G on a compact space X such that the stabilizer map* $\mathrm{Stab}\colon X \to \mathrm{Sub}(G)$ *is everywhere continuous and its image is equal to $\mathcal{H}$. If G is second countable, X can be chosen to be metrizable.*

In the above result, $\mathcal{H}$ is not assumed to be a URS, and therefore the action $G \curvearrowright X$ cannot always be minimal. However if $\mathcal{H}$ is a URS, the continuity of Stab

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implies that for every minimal invariant subset Y of X, the image $\mathrm{Stab}(Y)$, being a closed minimal invariant subset of $\mathcal{H}$, is equal to $\mathcal{H}$. Restricting the action to Y yields the following.

Corollary 1.2. *Every URS of a locally compact group arises as the stabilizer URS of a minimal action on a compact space.*

If $\mathcal{H} = \{\{1_G\}\}$, Theorem 1.1 recovers a classical theorem in topological dynamics, due to Veech [V] (and previously to Ellis [E2] for discrete groups), stating that every locally compact group admits a free action on a compact space. The proof of Theorem 1.1 is largely inspired by the proof of this result.

In addition, given a locally compact group G and a URS $\mathcal{H} \subseteq \mathrm{Sub}(G)$, we define and study a relative universal minimal flow $M(G, \mathcal{H})$ as the largest minimal compact G-space in which every subgroup in $\mathcal{H}$ fixes a point. For $\mathcal{H} = \{\{1_G\}\}$, this is the usual universal minimal flow of G. We prove that $M(G, \mathcal{H})$ is unique up to isomorphism and characterize under what conditions it is metrizable.

Related work. In an independent work [E1] that appeared while this paper was being completed, G. Elek proves Corollary 1.2 for finitely generated groups using a different method. In another recent preprint, T. Kawabe [K1] has obtained a proof of Corollary 1.2 for countable discrete groups when the URS consists of amenable subgroups.

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2. Proof for discrete groups

If G is a discrete group, Theorem 1.1 is substantially simpler, therefore we prove this first.

Let Z be a discrete set. We denote by βZ its Stone–Čech compactification. Given a subset $W \subseteq Z$, the notation $\overline{W}$ refers to the closure in βZ. Given a group action $G \curvearrowright Z$ and $g \in G$, we denote by $\mathrm{Fix}_g(Z)$ the set of points fixed by g, and by $\mathrm{Mov}_g(Z)$ its complement. The proof of the following lemma is inspired by Ellis's theorem [E2] that the action $G \curvearrowright \beta G$ is free.

Lemma 2.1. *Let $G \curvearrowright Z$ be a group action on a discrete set Z. Then for every $g \in G$, we have $\mathrm{Fix}_g(\beta Z) = \overline{\mathrm{Fix}_g(Z)}$ and $\mathrm{Mov}_g(\beta Z) = \overline{\mathrm{Mov}_g(Z)}$. In particular, the stabilizer map $\mathrm{Stab}\colon \beta Z \to \mathrm{Sub}(G)$ is continuous.*

Proof. Clearly $\mathrm{Fix}_g(\beta Z) \supseteq \overline{\mathrm{Fix}_g(Z)}$. Moreover, $\beta Z = \mathrm{Fix}_g(\beta Z) \sqcup \mathrm{Mov}_g(\beta Z) = \overline{\mathrm{Fix}_g(Z)} \cup \overline{\mathrm{Mov}_g(Z)}$ (the second equality follows from the density of Z), and therefore $\mathrm{Mov}_g(\beta Z) \subseteq \overline{\mathrm{Mov}_g(Z)}$. Let us check the reverse inclusions. We can find a function $f\colon Z \to \{0, 1, 2\}$ with the property that for every $x \in \mathrm{Mov}_g(Z)$, we have $|f(gx) - f(x)| \geq 1$ (such a function can be easily defined separately on every g-orbit). The function f extends to a function on βZ that we still denote by f. It follows that for every $\omega \in \overline{\mathrm{Mov}_g(Z)}$, we have $|f(g\omega) - f(\omega)| \geq 1$, and therefore $\omega \in \mathrm{Mov}_g(\beta Z)$, showing that $\overline{\mathrm{Mov}_g(Z)} = \mathrm{Mov}_g(\beta Z)$. This implies in particular that $\overline{\mathrm{Mov}_g(Z)}$ and $\overline{\mathrm{Fix}_g(Z)}$ are disjoint. The inclusion $\mathrm{Fix}_g(\beta Z) \subseteq \overline{\mathrm{Fix}_g(Z)}$ then also follows from the fact that $\beta Z = \overline{\mathrm{Fix}_g(Z)} \cup \overline{\mathrm{Mov}_g(Z)}$.

Finally, it is well-known that for discrete groups the continuity of the stabilizer map is equivalent to the fact that for every $g \in G$ the set $\mathrm{Fix}_g(\beta Z)$ is clopen (see, e.g., [LBMB, Lemma 2.2]), which is a consequence of the first statement. $\square$

Given a collection of subgroups $A \subseteq \mathrm{Sub}(G)$, we write $Z_A = \sqcup_{H\in A} G/H$ and endow it with the discrete topology. There is an obvious action $G \curvearrowright Z_A$, by letting G act separately on each coset space.

Proposition 2.2. *Let G be a discrete group and $\mathcal{H} \subseteq \mathrm{Sub}(G)$ be a closed invariant subset. Let $A \subseteq \mathcal{H}$ be a subset such that the set of all conjugates of subgroups in A is dense in $\mathcal{H}$. Then the compact G-space $X = \beta Z_A$ verifies the conclusion of Theorem 1.1.*

Remark 2.3. Of course, one can choose $A = \mathcal{H}$. However, if $\mathcal{H}$ is assumed to be a URS, then one can simply choose $A = \{H\}$ for any $H \in \mathcal{H}$, so that $X = \beta(G/H)$.

Proof. Continuity of the stabilizer map was already proved in Lemma 2.1. Moreover, the image of $Z_A \subseteq \beta Z_A$ is a dense subset of $\mathcal{H}$ by the assumption on A. Since Z_A is dense in βZ_A, it follows that the image of βZ_A is precisely $\mathcal{H}$. □

For the reduction to a metrizable space when G is countable discrete, we refer directly to the general case of a second countable locally compact group (cf. Proposition 3.6). However, we note that in this case, one can always choose a metrizable realization of the URS that is zero-dimensional.

3. Proof for locally compact groups

Let G be a locally compact group. We will always see G as a uniform space endowed with the *right uniformity* whose entourages are

$$\mathcal{U}_V = \{(g_1, g_2) : \exists v \in V \; vg_1 = g_2\},$$

where V varies over symmetric neighborhoods of 1_G. (Note that some authors call this the *left* uniformity instead.)

A pseudometric d on G is called *right-invariant* if $d(g_1h, g_2h) = d(g_1, g_2)$ for all $g_1, g_2, h \in G$, and is said to be (right) uniformly continuous if it is uniformly continuous as a function $d \colon G \times G \to \mathbf{R}$. Note that every continuous, right-invariant pseudometric is uniformly continuous. Our first step is to extract from the proof of the Birkhoff–Kakutani metrization theorem what we need for the sequel.

Lemma 3.1. *Let $g \in G$, $g \neq 1_G$ and let U be a neighborhood of 1_G. Then there exists a right-invariant, continuous pseudometric d on G such that:*

(i) $d \leq 8$;
(ii) $1/2 \leq d(1_G, g) \leq 1$;
(iii) *the d-ball of radius 4 around 1_G is relatively compact;*
(iv) $\{x \in G : d(1_G, x) < 1/2\} \subseteq U$.

Proof. We follow the proof of the Birkhoff–Kakutani theorem from [B]. Without loss of generality, we may assume that U is *symmetric* ($U = U^{-1}$), relatively compact and that $g \notin U$. Let $V_0 = U \cup gU \cup (gU)^{-1}$, $V_{-1} = V_0^3$, $V_{-2} = V_{-1}^3$, $V_{-3} = G$; let V_1 be a symmetric neighborhood of 1_G such that $V_1^3 \subseteq U$, and for each $n \geq 1$, let V_{n+1} be a symmetric neighborhood of 1_G such that $V_{n+1}^3 \subseteq V_n$. Thus for all $n \geq -3$, V_n is symmetric and $V_{n+1}^3 \subseteq V_n$. Define $\rho \colon G^2 \to \mathbf{R}$ by

$$\rho(x, y) = \inf\{2^{-n} : xy^{-1} \in V_n\}$$

and $d \colon G^2 \to G$ by

$$d(x, y) = \inf\{\sum_{i=0}^{k-1} \rho(x_i, x_{i+1}) : x_0 = x, x_k = y, x_1, \ldots, x_{k-1} \in G\}.$$

We have that ρ is symmetric, right-invariant and

$$\rho(x_0, x_1) \leq \epsilon \text{ and } \rho(x_1, x_2) \leq \epsilon \text{ and } \rho(x_2, x_3) \leq \epsilon \implies \rho(x_0, x_3) \leq 2\epsilon.$$

By [B, Lemma 6.2], d is a right-invariant pseudometric on G that satisfies
$$\frac{1}{2}\rho(x,y) \leq d(x,y) \leq \rho(x,y), \quad \text{for all } x,y \in G.$$
By the triangle inequality and right invariance, we have
$$|d(ux,vy) - d(x,y)| \leq d(ux,x) + d(vy,y) \leq \rho(u,1_G) + \rho(v,1_G),$$
showing that d is right uniformly continuous. Observe that ρ may not separate points and that is why we obtain a pseudometric rather than a metric.

We note that as $g \in V_0 \setminus V_1$, we have that $\rho(1_G,g) = 1$ and thus $1/2 \leq d(1_G,g) \leq 1$. Moreover, $\{x \in G : d(1_G,x) < 4\} \subseteq V_{-2}$ is relatively compact. Finally, if $x \notin V_1$, we have $\rho(1_G,x) \geq 1$ and thus $d(1_G,x) \geq 1/2$, proving that $\{x \in G : d(1_G,x) < 1/2\} \subseteq V_1 \subseteq U$. □

Remark 3.2. If G is second countable, then by a result of Struble [S], there always exists a proper, right-invariant metric on G and in that case, one can use this metric instead of the pseudometric provided by Lemma 3.1 in what follows (with small modifications of the proof).

Given a closed subgroup $H \leq G$, we equip the homogeneous space G/H with the quotient of the right uniformity of G. Explicitly, its entourages are
$$\mathcal{U}_V = \{(g_1H, g_2H) : \exists v \in V \ vg_1H = g_2H\},$$
where V varies over symmetric neighborhoods of 1_G. If d is a right-invariant, continuous pseudometric on G, define d_H on G/H by
$$d_H(g_1H, g_2H) = \inf_{h \in H} d(g_1h, g_2). \tag{3.1}$$
Note that by right invariance, d_H is a pseudometric on G/H. Moreover, for every $g \in G$, we have $d_H(gxH, xH) \leq d(g, 1_G)$ which implies that d_H is uniformly continuous.

Given $g \in G$ and $V \ni 1_G$, we denote
$$\mathrm{Mov}_g^V(G/H) = \{xH \in G/H : gxH \notin VxH\}.$$

The following lemma is adapted from the proof of Veech's theorem by Kechris, Pestov, and Todorčević [KPT, Appendix A].

Lemma 3.3. *Let $g \in G$ and $V \ni 1_G$ be open. Let $H \leq G$ be a closed subgroup. Then there exists $n \in \mathbf{N}$ and a uniformly continuous function $F\colon G/H \to \mathbf{R}^n$ with $\|F\|_\infty \leq 8$ such that*
$$\|F(gxH) - F(xH)\|_\infty \geq 1/4 \quad \textit{for every } xH \in \mathrm{Mov}_g^V(G/H).$$
Moreover, the dimension n of the target $\mathbf{R}^n$ can be chosen to depend only on g and V but not on H.

Proof. Choose a metric d as in Lemma 3.1 (with $U = V$). Define d_H as in (3.1). Using Zorn's lemma, choose a subset $A \subseteq G/H$ which is maximal with the property
$$aH, bH \in A \text{ and } aH \neq bH \implies d_H(aH, bH) \geq 1/8.$$
Define a graph Γ with vertex set A where aH and bH are connected by an edge if and only if $d_H(aH, bH) < 3$.

We claim that Γ has bounded degree and that the bound on the degree does not depend on H. To see this, let $b_1H, \ldots, b_nH$ be distinct neighbors of aH. This means that there exist $h_1, \ldots, h_n \in H$ such that $d(a, b_ih_i) < 3$ for every $i = 1, \ldots, n$. Furthermore, by the definition of A, we have $d(b_ih_i, b_jh_j) \geq 1/8$ for every $i \neq j$. Since d is right-invariant, this implies that the elements $x_i = b_ih_ia^{-1}$ lie in the ball of radius 3 around 1_G and have distance at least $1/8$ between each other. It follows that their cardinality does not exceed the size ℓ of a finite cover by balls of radius $1/16$ of the ball of radius 3 (which is relatively compact by Lemma 3.1). Therefore Γ has degree bounded by ℓ.

It follows that Γ can be colored with at most $n = \ell + 1$ colors in such a way that no two adjacent vertices have the same color. Let $A = A_1 \sqcup \cdots \sqcup A_n$ be the resulting partition of the vertices. For every $i = 1, \ldots, n$, let $f_i \colon G/H \to \mathbf{R}$ be given by $f_i(xH) = d_H(xH, A_i)$. Set $F = (f_1, \ldots, f_n)$.

Consider $xH \in \mathrm{Mov}_g^V(G/H)$ and note that condition (iv) in Lemma 3.1 implies that $d_H(xH, gxH) \geq 1/2$. By the definition of A, there exists a point $aH \in A$ such that $d_H(xH, aH) < 1/8$. Let i be such that $aH \in A_i$. Then $f_i(xH) < 1/8$.

Next we examine $f_i(gxH)$. Observe that

$$\begin{aligned} d_H(gxH, aH) &\leq d_H(gxH, xH) + d_H(xH, aH) \\ &\leq d(gx, x) + 1/8 \leq 9/8. \end{aligned}$$

We claim that aH is the closest point in A_i to gxH. Indeed, if another point in A_i were closer to xH, it would have to lie at a distance less than $18/8$ from aH, which is not possible because two points in A_i lie at distance at least 3. Therefore

$$\begin{aligned} f_i(gxH) &= d_H(gxH, aH) \\ &\geq d_H(gxH, xH) - d_H(xH, aH) \\ &\geq 1/2 - 1/8 = 3/8. \end{aligned}$$

We deduce that

$$\|F(gxH) - F(xH)\|_\infty \geq |f_i(gxH) - f_i(xH)| \geq 3/8 - 1/8 = 1/4. \qquad \square$$

We are now ready to prove Theorem 1.1. Let $\mathcal{H} \subseteq \mathrm{Sub}(G)$ be a closed invariant subset. Let $A \subseteq \mathcal{H}$ be such that the union of the orbits of elements of A is dense in $\mathcal{H}$. Let $Z = \sqcup_{H \in A} G/H$, endowed with the disjoint union topology and uniform structure. For $g \in G$ and $V \ni 1_G$ open, we denote

$$\mathrm{Mov}_g^V(Z) = \{z \in Z : gz \notin Vz\} = \sqcup_{H \in A} \mathrm{Mov}_g^V(G/H).$$

As a consequence of the last sentence in Lemma 3.3 (stating that the dimension n of the codomain of F is uniform in H), if one is given g and V, the functions F obtained in Lemma 3.3 can be coalesced together to obtain a uniformly continuous function on Z, and therefore Lemma 3.3 remains valid for the uniform space Z. We record this in the next lemma.

Lemma 3.4. *Let $g \in G$ and $V \ni 1_G$ be open. Then there exists $n \in \mathbf{N}$ and a bounded, uniformly continuous function $F \colon Z \to \mathbf{R}^n$ such that*

$$\|F(gz) - F(z)\|_\infty \geq 1/4 \quad \text{for every } z \in \mathrm{Mov}_g^V(Z).$$

Let $\mathrm{C_{ub}}(Z)$ be the commutative C^*-algebra of bounded, uniformly continuous functions on Z and let $S(Z)$ be its Gelfand spectrum (this is often called the *Samuel compactification of the uniform space* Z).

Proposition 3.5. *The G-space $X = S(Z)$ verifies the conclusion of Theorem 1.1.*

Proof. Since Z is dense in X, it is enough to prove that for every $\omega \in X$ and every net $(z_i)_i \subseteq Z$ converging to ω, the stabilizers G_{z_i} converge to G_ω. Fix $\omega \in X$ and a net $(z_i) \subseteq Z$ with $z_i \to \omega$. By compactness, we may assume that G_{z_i} converges to some $K \in \mathcal{H}$. We have $K \leq G_\omega$.

Towards a contradiction, suppose that the inclusion is strict and let $g \in G_\omega \setminus K$. Let $V \ni 1_G$ be a compact, symmetric neighborhood of 1_G such that $Vg \cap K = \varnothing$. This defines an open condition for the Chabauty topology, so $g \notin VG_{z_i}$ for i large enough, i.e., $z_i \in \mathrm{Mov}_g^V(Z)$. By Lemma 3.4, we can find a function $F \colon Z \to \mathbf{R}^n$

with the property that $\|F(gz_i) - F(z_i)\|_\infty \geq 1/4$ for all i large enough. Since F extends to X, passing to the limit, we get $\|F(g\omega) - F(\omega)\|_\infty \geq 1/4$, contradicting the fact that $g \in G_\omega$. Therefore $K = G_\omega$ and the stabilizer map is continuous as claimed.

That the image of Stab is equal to $\mathcal{H}$ now follows from the fact that $\mathrm{Stab}(Z)$ is a dense subset of $\mathcal{H}$. □

It remains to prove the claim of the last sentence in the statement of Theorem 1.1.

Proposition 3.6. *Let X be the G-space constructed above. If G is second countable, then there exists a metrizable quotient Y of X such that* Stab *is continuous on Y and* $\mathrm{Stab}(Y) = \mathcal{H}$.

Proof. Fix a countable basis $\mathcal{B}$ at 1_G. Let $Z = \sqcup_{H \in A} G/H$ as before. We will define the quotient Y as the Gelfand space of a separable G-invariant subalgebra $\mathcal{A}$ of $C_{\mathrm{ub}}(Z)$. Note that for the Stab map to be continuous on Y, we only need that Lemma 3.4 hold for $\mathcal{A}$, i.e.:

$$\forall V \in \mathcal{B}\ \forall g \in G\ \exists F \in \mathcal{A} \quad \|F(gz) - F(z)\|_\infty > 1/8 \quad \text{for all } z \in \mathrm{Mov}_g^V(Z),$$

that is, the function $F = (f_1, \dots, f_n) \colon Z \to \mathbf{R}^n$ can be chosen in such a way that $f_1, \dots, f_n \in \mathcal{A}$. Thus all we need to show is that for a fixed $V \in \mathcal{B}$, there is a countable collection $\mathcal{A}_V$ of functions $F \colon Z \to \mathbf{R}^n$ such that

$$\forall g \in G \exists F \in \mathcal{A}_V\ \forall z \in Z \quad gz \in Vz \quad \text{or} \quad \|F(gz) - F(z)\|_\infty > 1/8.$$

Provided that this is done, we can take $\mathcal{A}$ to be the smallest G-invariant, closed subalgebra that contains $\bigcup_{V \in \mathcal{B}} \mathcal{A}_V$, which is still separable.

Lemma 3.4 and uniform continuity imply that for every $g \in G$, there exist $F \colon Z \to \mathbf{R}^n$ and an open $U \ni g$ such that

$$\forall g' \in U\ \forall z \in Z \quad g'z \in Vz \quad \text{or} \quad \|F(g'z) - F(z)\|_\infty > 1/8.$$

Now the fact that G is Lindelöf implies that we can find a countable collection of functions F that works for all g. □

4. Universal minimal flow relative to a URS

4.1. **Existence and uniqueness.** If $\mathcal{H}$ and $\mathcal{K}$ are URSs of G, we write $\mathcal{H} \preceq \mathcal{K}$ if for all $H \in \mathcal{H}$, there exists $K \in \mathcal{K}$ such that $H \leq K$. This relation is a partial order on the set of URSs of G (see [LBMB, Corollary 2.15]; the proof given there for countable groups extends easily to locally compact groups), however we shall not use this fact.

Definition 4.1. Let G be a group, $G \curvearrowright X$ be a minimal action on a compact space X, and let $\mathcal{H}$ be a URS of G. We will say that X *is subordinate to* $\mathcal{H}$ if $\mathcal{H} \preceq \mathcal{S}_G(X)$.

Recall that given two compact G-spaces X and Y, we say that X *factors onto* Y if there exists a continuous, surjective, G-equivariant map $X \to Y$. Given a collection $\mathcal{E}$ of compact G-spaces, we say that $X \in \mathcal{E}$ is *universal* for $\mathcal{E}$ if it factors onto all elements of $\mathcal{E}$.

The goal of this section is to establish the following theorem.

Theorem 4.2. *For every URS $\mathcal{H}$ of G, there exists a minimal G-space $M(G, \mathcal{H})$, unique up to isomorphism, which is subordinate to $\mathcal{H}$ and is universal for minimal G-spaces subordinate to $\mathcal{H}$.*

Definition 4.3. The space $M(G, \mathcal{H})$ will be called the *universal minimal flow of G relative to $\mathcal{H}$*.

If $\mathcal{H} = \{\{1_G\}\}$, then $M(G, \mathcal{H})$ is just the usual universal minimal flow of G.

The existence is easy. Let $H \in \mathcal{H}$ be arbitrary and recall that $S(G/H)$ denotes the Samuel compactification of G/H. Let $M \subseteq S(G/H)$ be a minimal subset. Then M verifies the universal property. Indeed, let $G \curvearrowright X$ be a minimal G space subordinate to $\mathcal{H}$. Then there exists a point $x \in X$ such that H stabilizes x. The orbital map $G \to X, g \mapsto g \cdot x$ descends to a uniformly continuous map $G/H \to X$, which extends to a G-map $S(G/H) \to X$, and taking the restriction to M shows that M factors onto X. We have already proven that the collection of stabilizers of $G \curvearrowright M$ is equal to $\mathcal{H}$; in particular, M is subordinate to $\mathcal{H}$.

Our next goal is to check uniqueness. For this, it is enough to prove that M is *coalescent*, i.e., that every continuous G-equivariant map $M \to M$ is a homeomorphism. For the usual (non-relative) universal minimal flow of G, this is a result of Ellis [E2]. Our proof is close to the exposition by Uspenskij [U] of Ellis's theorem. The proof of this result is based on the fact that $S(G)$ carries a natural semigroup structure. The main difference is that in our case, $S(G/H)$ does not carry such a structure; however, we can find a semigroup inside $S(G/H)$ that is sufficient for our purposes.

Let $\mathrm{Fix}_H(M)$ be the set of points in M fixed by H. Observe that for every $\omega \in \mathrm{Fix}_H(M)$, the orbital map $G/H \to G \cdot \omega$ extends to a continuous equivariant map $r_\omega \colon S(G/H) \to M$, which is moreover the unique G-map $S(G/H) \to S(G/H)$ sending H to ω. Hence, we get a map $S(G/H) \times \mathrm{Fix}_H(M) \to M$ continuous in the first variable.

Lemma 4.4. *For every $\omega \in \mathrm{Fix}_H(M)$, we have $r_\omega(\mathrm{Fix}_H(M)) \subseteq \mathrm{Fix}_H(M)$. In particular, $\mathrm{Fix}_H(M)$ is a right-topological semigroup under the operation $\mathrm{Fix}_H(M) \times \mathrm{Fix}_H(M) \to \mathrm{Fix}_H(M), (\eta, \omega) \mapsto \eta\omega := r_\omega(\eta)$.*

Proof. This is obvious because the map r_ω is G-equivariant. □

Since $\mathrm{Fix}_H(M)$ is a compact, right-topological semigroup, by a well-known theorem of Ellis, $\mathrm{Fix}_H(M)$ contains idempotent elements.

Lemma 4.5. *Let $\iota \in \mathrm{Fix}_H(M)$ be an idempotent. Then the map $r_\iota \colon S(G/H) \to M$ is a retraction of $S(G/H)$ onto M.*

Proof. We need to prove that $r_\iota|_M = \mathrm{id}$. Since $r_\iota(\iota) = \iota^2 = \iota$, by G-equivariance, r_ι is the identity on the orbit of ι, which is dense in M by minimality, whence the conclusion. □

Lemma 4.6. *Every continuous G-map $M \to M$ is of the form r_ω for some $\omega \in \mathrm{Fix}_H(M)$.*

Proof. Let $f \colon M \to M$ be a continuous G-map. Let $\iota \in \mathrm{Fix}_H(M)$ be an idempotent. Consider $f \circ r_\iota \colon S(G/H) \to M$. As this map is continuous and equivariant, we have $f \circ r_\iota = r_\omega$ for $\omega = f(\iota)$. Since $r_\iota|_M = \mathrm{id}$, this implies that $f = r_\omega$. □

Proposition 4.7. *M is coalescent.*

Proof. Let $f \colon M \to M$ be a continuous G-map. We need to show that f is injective. By equivariance, we have $f(\mathrm{Fix}_H(M)) \subseteq \mathrm{Fix}_H(M)$. By Lemma 4.6, there exists $\omega \in \mathrm{Fix}_H(M)$ such that $f = r_\omega$. Therefore $f(\mathrm{Fix}_H(M)) = \mathrm{Fix}_H(M)\omega$ is a compact left ideal of $\mathrm{Fix}_H(M)$ and thus a compact subsemigroup of $\mathrm{Fix}_H(M)$. By Ellis's theorem, there exists an idempotent $\iota \in \mathrm{Fix}_H(M)\omega$. Let $\eta \in \mathrm{Fix}_H(M)$ be such that $f(\eta) = \eta\omega = \iota$. Let $g = r_\eta$. Now by Lemma 4.5, for all $x \in M$,

$$(f \circ g)(x) = r_\omega(r_\eta(x)) = x\eta\omega = x\iota = r_\iota(x) = x.$$

The map g, being continuous and equivariant, is surjective by minimality. Since $f \circ g = \mathrm{id}$, it follows that f is injective. □

4.2. **Metrizability of $M(G, \mathcal{H})$.** It is a natural question for which pairs $(G, \mathcal{H})$ the relative universal minimal flow $M(G, \mathcal{H})$ can be identified with a more familiar, concrete G-space. A case in which this can be done is when the URS $\mathcal{H}$ contains a cocompact subgroup H (and thus is necessarily a single compact conjugacy class). In this case, $M(G, \mathcal{H})$ can be identified with the homogeneous space G/H. The following proposition says that there is little hope beyond this case.

Proposition 4.8. *Let G be a locally compact second countable group and let $\mathcal{H}$ be a URS of G. Then $M(G, \mathcal{H})$ is metrizable iff $\mathcal{H}$ contains a cocompact subgroup.*

Proof. The "if" direction is clear. For the other, suppose that $M(G, \mathcal{H})$ is metrizable. Following the argument for the proof of Theorem 1.2 in [BYMT], we conclude that $M(G, \mathcal{H})$ contains a G_δ orbit $G \cdot x_0$. As G is σ-compact, the orbit $G \cdot x_0$ is also F_σ, implying that its complement is G_δ. If the complement is non-empty, it must be dense by minimality, contradicting the Baire category theorem. Thus the action $G \curvearrowright M(G, \mathcal{H})$ is transitive and if we put $H = G_{x_0}$, Effros's theorem (see, e.g., [H, Theorem 7.12]) implies that H is cocompact. As a consequence of Theorem 1.1, the point stabilizers of $G \curvearrowright M(G, \mathcal{H})$ are precisely the elements of $\mathcal{H}$. Therefore $\mathcal{H}$ contains a cocompact subgroup as claimed. □

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