Approximate isomorphism of randomizations with a distinguished small substructure

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Based on joint work with James Hanson.

$\aleph_0$-categorical, $\aleph_0$-stable theories

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Let $T \in \mathcal{T}$. Then:

- *T is one-based, i.e., for every algebraically closed sets A, B in a model of T^{eq}, we have $A \perp_{A \cap B} B$.*

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- *If D is a strongly minimal set interpretable in T, then its associated geometry is either disintegrated, or the geometry of an affine or projective space over a finite field.*

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- The most fundamental and well-behaved examples of $\aleph_0$-categorical, $\aleph_0$-stable metric theories (Hilbert spaces, probability algebras) are not one-based.
- Moreover, they do not contain anything resembling a strongly minimal set.
- No structure theory for $\aleph_0$-categorical, $\aleph_0$-stable metric theories is currently available.

Reminders on beautiful pairs

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If T is $\aleph_0$-categorical and stable, then the common L_P-theory of all beautiful pairs is well-behaved, and denoted by T_P.

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Question

When is T_P $\aleph_0$-categorical?

A characterization of the class $\mathcal{T}$

Theorem (folklore)

Let T be an $\aleph_0$-categorical, stable classical theory. TFAE:

1. *T is $\aleph_0$-stable (i.e., $T \in \mathcal{T}$).*
2. *T is one-based.*
3. *T_P is $\aleph_0$-categorical.*

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If T is the theory of atomless L^1 Banach lattices, then T_P has exactly two models up to isomorphism.

Nevertheless, these two models are approximately isomorphic!

Approximate isomorphism for pairs of models

Let (M, N) and (M', N') be two elementary pairs of models of a theory T.

Definition

Given $\epsilon > 0$, an ϵ-isomorphism between (M, N) and (M', N') is an isomorphism $\sigma\colon M \to M'$ such that:

$$d_H(\sigma(N), N') \leq \epsilon.$$

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The theory T_P is approximately $\aleph_0$-categorical if every two separable models of T_P are approximately isomorphic.

A conjecture

Conjecture (Ben Yaacov–Berenstein–Henson '08)

If T is an $\aleph_0$-categorical, $\aleph_0$-stable metric theory, then T_P is approximately $\aleph_0$-categorical.

Randomizations

Let $\Omega = ([0,1], \mu)$ be the Lebesgue space. If M is a countable set, we denote by $M^\Omega = L^0(\Omega, M)$ the space of M-valued random variables on Ω, with metric:

$$d(f, f') = \mu\{\omega \in \Omega : f(\omega) \neq f'(\omega)\}.$$

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Let M be a countable structure in a classical language L. Then M^Ω is an L^R-structure for $L^R = \{\mathbb{P}[\varphi] : \varphi \text{ an } L\text{-formula}\}$, where given $f \in (M^\Omega)^n$,

$$\mathbb{P}[\varphi]^{M^\Omega}(f) = \mu\{\omega \in \Omega : M \models \varphi(f(\omega))\}.$$

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If T is the L-theory of M, we denote $T^R = \mathrm{Th}(M^\Omega)$.
If T is $\aleph_0$-categorical (resp., $\aleph_0$-stable), then so is T^R.

Pairs of randomizations

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Theorem (I. '17)

Let T be an $\aleph_0$-categorical, stable, classical theory, and let (M, N) be a beautiful pair of countable models of T. Then:

1. *(M^{Ω^2}, N^Ω) is a beautiful pair of models of T^R.*

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1. (M^{Ω^2}, N^Ω) *is a beautiful pair of models of* T^R.
2. $(M^{\Omega^2}, M^\Omega) \models (T^R)_P$.
3. *In fact, the separable models of* $(T^R)_P$ *are precisely those of the form* $(M^{\Omega^2}, \hat{h}(M^\Omega))$ *for* $\hat{h} \in \mathrm{End}(M)^{\Omega^2}$ *(up to isomorphism).*

Here, $\mathrm{End}(M)^{\Omega^2} = L^0(\Omega^2, \mathrm{End}(M))$.

A nice test case for the conjecture

In particular, given $T \in \mathcal{T}$ with infinite models, the metric theory T^R is $\aleph_0$-categorical, $\aleph_0$-stable, but $(T^R)_P$ is not $\aleph_0$-categorical.

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Question

Is $(T^R)_P$ approximately $\aleph_0$-categorical?

A sufficient condition

Definition

Let M be the countable model of a theory $T \in \mathcal{T}$.
We say T has approximable random endomorphisms if for every $\hat{h} \in \mathrm{End}(M)^{\Omega^2}$ and $\epsilon > 0$ there is $\hat{g} \in \mathrm{Aut}(M)^{\Omega^2}$ such that, for every $f \in M^{\Omega}$, we have $d(\hat{g}(f), \hat{h}(f)) \leq \epsilon$.

Note: the distance $d(\hat{g}(f), \hat{h}(f))$ is given by

$$\mu\{(\omega_1, \omega_2) \in \Omega^2 : g(\omega_1, \omega_2) f(\omega_1) \neq h(\omega_1, \omega_2) f(\omega_1)\}.$$

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Lemma

If T has approximable random endomorphisms, then $(T^R)_P$ is approximately $\aleph_0$-categorical.

Proof.

We have $d_H(\hat{g}(M^\Omega), \hat{h}(M^\Omega)) \leq \epsilon$, so $\hat{g}$ is an ϵ-isomorphism between (M^{Ω^2}, M^Ω) and $(M^{\Omega^2}, \hat{h}(M^\Omega))$. □

Pairs of randomized infinite sets

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Let M be a countable infinite set and let $\hat{h} \in \mathrm{End}(M)^{\Omega^2}$.

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Note that the quotient $\mathrm{Aut}(M) \backslash \mathrm{End}(M)$ is countable.

Thus up to multiplying by an element of $\mathrm{Aut}(M)^{\Omega^2}$, we may assume that $\hat{h}$ only takes countable many values from $\mathrm{End}(M)$.

That is, there are $h_k \in \mathrm{End}(M)$ and a partition $\{A_k\}_{k \in \mathbb{N}}$ of Ω^2 such that $\hat{h}|_{A_k}$ is constantly h_k.

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Sketch of proof (continued).

Fix $h = h_k$ for some $k \in \mathbb{N}$.

Claim

For every $n \in \mathbb{N}$ *there are* $g_0, \ldots, g_{n-1} \in \mathrm{Aut}(M)$ *such that, for every* $a \in M$, $|\{i < n : g_i(a) \neq h(a)\}| \leq 1$.

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Then take a "horizontal" equipartition $\{A_k^i\}_{i<n}$ of A_k, i.e.,

$$\mu((A_k^i)_\omega) = \frac{1}{n}\mu((A_k)_\omega) \text{ at each vertical slice } \{\omega\} \times \Omega.$$

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Putting the $\hat{g}_k$ together, we get $\hat{g} \in \mathrm{Aut}(M)^{\Omega^2}$ such that $d(\hat{g}(f), \hat{h}(f)) \leq 1/n$ for every $f \in M^\Omega$. □

Preservation results

An $\aleph_0$-categorical structure M can be identified with the oligomorphic permutation group $\mathrm{Aut}(M) \curvearrowright M$.

Lemma

Let $G \curvearrowright M$ and $H \curvearrowright N$ be oligomorphic permutation groups.

- *If G and H have approximable random endomorphisms, then so do $G \times H \curvearrowright M \sqcup N$ and $G^N \rtimes H \curvearrowright M \times N$.*
- *If G is a finite index supergroup of H and H has approximable random endomorphisms, then so does G.*

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Lemma

Having approximable random endomorphisms is preserved by bi-interpretability.

We can deduce that $(T^R)_P$ is approximately $\aleph_0$-categorical for many $T \in \mathcal{T}$, e.g., monadically stable theories, theories of crossing equivalences relations, etc.

A family of counterexamples

Theorem (Hanson–I. '21)

Let $T \in \mathcal{T}$ interpret a non-disintegrated strongly minimal set. Then $(T^R)_P$ is not approximately $\aleph_0$-categorical.

Thus, the conjecture of Ben Yaacov–Berenstein–Henson fails.

Randomized vector spaces

Example

Let M be an infinite vector spaces over a finite field k, and let $N \preceq M$ have infinite codimension. If $0 < \epsilon < 1$, then there is no ϵ-isomorphism between $(M^{\Omega^2}, N^{\Omega})$ and $(M^{\Omega^2}, M^{\Omega})$.

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Sketch of proof.

Suppose there is such an ϵ-isomorphism σ, and let $R = \sigma(N^{\Omega})$. So M^{Ω^2} is $\aleph_0$-saturated over R, and R is ϵ-close to M^{Ω}.

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Given $\alpha \in \Omega^2$, let $R(\alpha) \subseteq M$ be "the induced subspace" at α (to be defined carefully).

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Given $\alpha \in \Omega^2$, let $R(\alpha) \subseteq M$ be "the induced subspace" at α (to be defined carefully).

By saturation of M^{Ω^2} over R, the subspace $R(\alpha)$ has infinite codimension almost everywhere.

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Sketch of proof (continued).

Write $M = \bigcup_{i \in \mathbb{N}} M_i$ where $M_i \subseteq M_{i+1}$ and $\dim(M_i) < \infty$.

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Thus $d_i(\alpha) := \dim(M_i / M_i \cap R(\alpha)) \to \infty$ almost everywhere.

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Hence:

$$q_i := \int_{\Omega^2} \frac{|M_i \cap R(\alpha)|}{|M_i|} \, d\alpha = \int_{\Omega^2} \frac{1}{|k|^{d_i(\alpha)}} \, d\alpha \to 0.$$

But also:

$$q_i = \frac{1}{|M_i|} \sum_{v \in M_i} \mu\{\alpha \in \Omega^2 : v \in R(\alpha)\},$$

so there is $v \in M$ such that $\mu\{\alpha : v \in R(\alpha)\} < \delta < 1 - \epsilon$.

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so there is $v \in M$ such that $\mu\{\alpha : v \in R(\alpha)\} < \delta < 1 - \epsilon$.

Thus the constant function v on Ω^2 is in M^Ω, but $d(v, f) > 1 - \delta > \epsilon$ for all $f \in R$, a contradiction. □

Guesswork and questions

Let $\mathcal{T}_0$ be the class of theories $T \in \mathcal{T}$ such that $(T^R)_P$ is approximately $\aleph_0$-categorical.

Conjecture

For $T \in \mathcal{T}$, we have $T \in \mathcal{T}_0$ iff all strongly minimal sets of T^{eq} are disintegrated (equivalently, iff T is interpretable in $(\mathbb{Q}, <)$).

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Question

1. Is the class $\mathcal{T}_0$ closed under interpretations?
2. Is the class $\mathcal{T}_0$ closed under finite covers?

Same questions for the property of having approximable random endomorphisms.

Thank you.