Coalescence of strongly ergodic distal systems

Tomás Ibarlucía

IMJ–PRG, Université Paris Diderot

Christmas Tree Meeting

Orsay, November 30, 2017.

Joint work with Todor Tsankov.

A rigidity result

Theorem (Ioana–Tucker-Drob '16)

Let Γ be a countable group and let X, Y be two probability-measure-preserving Γ-systems. Suppose:

- *X is strongly ergodic.*
- *Y is distal ("generalized discret spectrum").*
- *Y is weakly contained in X.*

Then, Y is a factor of X.

Recall

Fix a countable group Γ. Let $\Gamma \curvearrowright X$, $\Gamma \curvearrowright Y$ be p.m.p. systems.

- X is strongly ergodic if every asymptotically invariant sequence of sets is asymptotically trivial:

  $$\text{if } \mu(\gamma a_n \triangle a_n) \to 0 \ \forall \gamma \in \Gamma, \text{ then } \mu(a_n)(1 - \mu(a_n)) \to 0.$$

- Y is weakly contained in X if for every $\epsilon > 0$, every $b_1, \ldots, b_n \subset Y$ and $\gamma_1, \ldots, \gamma_k \in \Gamma$, there are $a_1, \ldots, a_n \subset X$ such that

  $$|\mu(b_i \cap \gamma_l b_j) - \mu(a_i \cap \gamma_l a_j)| < \epsilon$$

  for every $i, j \leq n$ and $l \leq k$.

Recall

- X has discrete spectrum if $L^2(X) = \bigoplus E_i$ for some finite-dimensional Γ-invariant subspaces E_i.
- An extension $X \to Y$ is a compact extension if $L^2(X) = \bigoplus \langle F_i \rangle_{L^\infty(Y)}$, where each F_i is finite and $E_i = \langle F_i \rangle_{L^\infty(Y)}$ is Γ-invariant.
  For ergodic X, this is equivalent to being a skew product of Y with a homogeneous space of a compact group:
  $$X = Y \times_\rho K/L.$$
- X is distal if it can be obtained as a tower of compact extensions starting with a compact system.

A rigidity result

Theorem (Ioana–Tucker-Drob '16)

Let Γ be a countable group and let X, Y be two probability-measure-preserving Γ-systems. Suppose:

- *X is strongly ergodic.*
- *Y is distal ("generalized discret spectrum").*
- *Y is weakly contained in X.*

Then, Y is a factor of X.

This generalizes a result of Abért–Elek ('12) (same, for finite Y).

A rigidity result

Corollary (Ioana–Tucker-Drob '16)

Let X, Y be strongly ergodic systems with discrete spectrum. If each is weakly contained in the other, then they are isomorphic.

Proof.

We have factors $X \to Y \to X$, and *every self-factor of a discrete spectrum ergodic system is an isomorphism* (i.e., discrete spectrum ergodic systems are coalescent). □

Note that distal ergodic systems need not be coalescent: one can construct a rank 2 ergodic $\mathbb{Z}$-system with a proper self-factor. However, this is not strongly ergodic!

A coalescence result

Theorem (I.–Tsankov)

Strongly ergodic distal systems are coalescent. More generally: if $X \to Z$ is a distal extension, then every self-factor $X \to X$ which induces the identity on Z is an isomorphism.

Question

Are there strongly ergodic distal systems other that than those with discrete spectrum?

Apparently yes: Glasner and Weiss (private communication) claim they can construct an example.

Distal systems of property (T) groups

We are able to use coalescence to show the following.

Theorem

If Γ has property (T), then every ergodic distal Γ-system has discrete spectrum.

This was previously known to Chifan and Peterson by methods of operator algebras (unpublished).

Distal systems of property (T) groups

We fix a countable group Γ with property (T).

Theorem (I.–Tsankov)

Let Z be an ergodic Γ-system. Then there exists a unique universal ergodic compact extension $\widehat{Z} \to Z$.

(I.e., every ergodic compact extension of Z is a factor of $\widehat{Z}$.)

Proof.

- Existence follows from transfinite amalgamation.
- Uniqueness follows from coalescence. □

Remark

We have $\widehat{1} = b\Gamma$ (the Bohr compactification).

Distal systems of property (T) groups

Corollary

Every automorphism of Z extends to an automorphism of $\widehat{Z}$.

Theorem

For every ergodic system Z we have $\widehat{\widehat{Z}} = \widehat{Z}$.

From this (applied to $Z = 1$) we deduce that distal ergodic Γ-systems actually have discrete spectrum.

Γ-systems as metric structures

We come back to a general countable group Γ.

Given a p.m.p. system $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$, we consider the associated measure algebra

$$M_X = (\mathcal{B}/{\sim_\mu}, d, \mu, \cap, \cup, \cdot^c, 0, 1, \gamma\cdot)_{\gamma\in\Gamma}$$

augmented with symbols for each of the elements of Γ, interpreted by the corresponding automorphisms of the measure algebra.

This is a metric structure in the sense of continuous logic.

A nice language for ergodic theory

Some easy translations:

1. Y is a factor of X iff M_Y embeds in M_X as a substructure.
2. Y is weakly contained in X iff $\mathrm{Th}_\exists(M_Y) \subseteq \mathrm{Th}_\exists(M_X)$, iff M_Y embeds in an elementary extension of M_X (e.g., an ultrapower).
3. X is strongly ergodic iff every model of $\mathrm{Th}(M_X)$ is ergodic, iff every elementary extension of M_X is ergodic.

A model-theoretic invariant

Let M be a metric structure and $a \in M$.

Definition

We say that a is algebraic if it belongs to a compact definable subset of M. We denote $a \in \mathrm{acl}^M(\emptyset)$.

E.g.: the roots of a (1-variable, integer) polynomial in a field (since they form a *finite* definable set).

A model-theoretic invariant

In particular, if $M = M_X$ is the structure associated to a Γ-system X, we can consider the set of algebraic elements of M_X, which forms a substructure:

$$\mathrm{acl}^{M_X}(\emptyset) \subseteq M_X.$$

Hence there is a corresponding associated factor:

$$X \to A.$$

What is this factor A?

A rigidity result: a strengthening

For free actions of $\mathbb{Z}$, A is trivial.

On the other hand, we have the following.

Theorem (I.–Tsankov)

Let X be a strongly ergodic system and let A be the factor corresponding to $\mathrm{acl}^{M_X}(\emptyset)$. Then, the extension $X \to A$ is weakly mixing. In particular, every distal factor of X is a factor of A.

This immediately implies the result of Ioana–Tucker-Drob.

A finer model-theoretic invariant

Let M be a metric structure and $a \in M$.

Definition

We say that a is existentially algebraic if it belongs to a compact $\exists$-definable subset of M. We denote $a \in \mathrm{acl}_{\exists}^{M}(\emptyset)$.

This is a stronger condition than algebraicity, so we have

$$\mathrm{acl}_{\exists}^{M}(\emptyset) \subseteq \mathrm{acl}^{M}(\emptyset).$$

A rigidity result: a further strengthening

Theorem

Let X be a strongly ergodic system and let $A_\exists$ be the factor corresponding to $\mathrm{acl}_\exists^{M_X}(\emptyset)$. Then, every distal factor of X is a factor of $A_\exists$.

This immediately implies that strongly ergodic distal systems are coalescent.

Question

For which systems X does $A_\exists$ coincide with the maximal distal factor of X?

Merci de votre attention.