Eberlein oligomorphic groups

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When topological dynamics meets model theory

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Based on joint work with Itaï Ben Yaacov and Todor Tsankov.
The Fourier–Stieltjes algebra

Let $G$ be a topological group. We denote by $C(G)$ the algebra of complex-valued bounded continuous functions on $G$.

A function $f \in C(G)$ is positive definite if

$$\sum_{ij} c_i \overline{c_j} f(g_j^{-1}g_i) \geq 0$$

for every $g_1, \ldots, g_n \in G$ and $c_1, \ldots, c_n \in \mathbb{C}$.

The linear span of the family of positive definite functions on $G$ is denoted by $B(G)$. It is actually a subalgebra of $C(G)$, the Fourier–Stieltjes algebra of $G$. 
Fact (Gelfand–Naimark–Segal construction)

The following are equivalent:

1. $f \in B(G)$.

2. There is a continuous unitary representation $\pi : G \to \mathcal{U}(H)$ and vectors $v, w \in H$ such that, for every $g \in G$,

   $$f(g) = \langle v, \pi(g)w \rangle.$$
The WAP algebra

A function $f \in C(G)$ is **weakly almost periodic** if the orbit $Gf$ is weakly precompact in $C(G)$. They form an algebra, $\text{WAP}(G)$.

**Fact (Grothendieck’s double limit criterion, Megrelishvili’s reflexive representation theorem)**

The following are equivalent:

1. $f \in \text{WAP}(G)$.
2. For any sequences $g_i, h_j \in G$ we have (whenever both limits exist)
   \[
   \lim_i \lim_j f(g_ih_j) = \lim_j \lim_i f(g_ih_j).
   \]
3. There exists a continuous reflexive representation $\pi : G \to \text{Iso}(V)$ and vectors $v \in V$, $w \in V^*$ such that
   \[
   f(g) = \langle v, \pi(g)w \rangle \text{ for all } g \in G.
   \]
Eberlein groups

It follows that $B(G) \subset WAP(G)$. However, $WAP(G)$ is always closed in the norm topology of $C(G)$, whereas $B(G)$ is almost never closed.

Definition
A topological group $G$ is Eberlein if $\overline{B(G)} = WAP(G)$.

Examples
- Compact groups are Eberlein (Peter–Weyl).
- The group $\mathbb{Z}$ is not Eberlein (Rudin). Neither is any locally compact noncompact nilpotent group (Chou).
- Eberlein groups include $SL_n(\mathbb{R})$ (Veech), $U(\ell^2)$ (Megrelishvili), Aut$([0,1], \mu)$ (Glasner) or $S(\mathbb{N})$ (Glasner–Megrelishvili).
The algebra $\text{UC}(G)$

A function $f \in C(G)$ is $\text{UC}$ if for every $\epsilon > 0$ there is a neighborhood $1 \in U \subset G$ such that

$$|f(ugu') - f(g)| < \epsilon$$

for every $g \in G$ and $u, u' \in U$. We have $\text{WAP}(G) \subset \text{UC}(G)$.

**Definition**

We say that $G$ is a $\text{WAP}$ group if $\text{WAP}(G) = \text{UC}(G)$, and that it is strongly Eberlein if $\text{B}(G) = \text{UC}(G)$.

**Problem** (Glasner–Megrelishvili)

Show a WAP group that is not Eberlein.
Oligomorphic groups

A topological group $G$ is **oligomorphic** if it can be presented as a closed permutation group $G \leq S(X)$ of a countable set whose orbit spaces $X^n/G$ are finite for every $n$.

*Equivalently:* $G = \text{Aut}(M)$ for some $\aleph_0$-categorical classical structure $M$ (Ryll-Nardzewski).

Generalization: closed groups of isometries $G \leq \text{Iso}(X)$ of Polish metric spaces with compact closed-orbit spaces $X^n // G$ are exactly the **Roelcke precompact** Polish groups (Ben Yaacov–Tsankov, Rosendal).

*Equivalently:* $G = \text{Aut}(M)$ for some $\aleph_0$-categorical metric structure $M$. 
Motivation

A number of tools are available for oligomorphic groups.

- Unlike many other cases, $B(G)$ is separable. ($\text{UC}(G)$ is separable.)
- We have a **Classification Theorem** for unitary representations of oligomorphic groups (Tsankov).
- We have a model-theoretic interpretation of the WAP semigroup compactification (Ben Yaacov–Tsankov).
Motivation

Let \( M \) be an \( \aleph_0 \)-categorical metric structure, \( G = \text{Aut}(M) \).

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Formulas generating $B(G)$ for oligomorphic $G$

Let $M$ be a classical $\aleph_0$-categorical structure, $G = \text{Aut}(M)$. The first basic observation is the following.

**Lemma**

If a formula $\varphi(x, y)$ defines an equivalence relation on $M^n$ (more generally, if $\varphi(x, b)$ defines a weakly normal set), then $g \mapsto \varphi(a, gb)$ is in $B(G)$.

**Proof.**

We have $\varphi(a, gb) = \langle e_{[a]_\varphi}, \pi(g)e_{[b]_\varphi} \rangle$ for the natural map $\pi : G \to \mathcal{U}(\ell^2(M^n/\varphi))$. 

Before we give a converse to this statement we recall the general form of the unitary representations of $G$. 

\[\square\]
Classification theorem for unitary representations of oligomorphic groups

Fact (Tsankov)

Let $G$ be an oligomorphic group.

- Every unitary representation of $G$ is a direct sum of irreducible representations.
- Every irreducible unitary representation is a subrepresentation of the quasi-regular representation $\pi_V : G \to \mathcal{U}(\ell^2(G/V))$ for some open subgroup $V \leq G$.

Remark: the matrix coefficients induced by $\pi_V$ are generated by the basic ones

$$g \mapsto \langle e_{h_0 V}, \pi_V(g)e_{h_1 V} \rangle \quad (= \langle e_{h_0 V}, e_{gh_1 V} \rangle).$$
Formulas generating $B(G)$ for oligomorphic $G$

Now, every open subgroup $V' \leq G$ is the stabilizer of an imaginary element of $M$: there is a definable equivalence relation $\varphi(x, y)$ and a tuple $b \in M^n$ such that $V' = \{ g \in G : M \models \varphi(b, gb) \}$.

Applying this to $V' = h_1 V h_1^{-1}$ and taking $a = h_0 h_1^{-1} b$, we have

$$\langle e_{h_0} V, \pi_V(g) e_{h_1} V \rangle = \varphi(a, gb).$$

We obtain the following:

**Proposition**

$B(G)$ is the closed algebra generated by the functions $g \mapsto \varphi(a, gb)$ where $\varphi(x, y)$ is a definable equivalence relation on $M$. 
Semitopological semigroup compactifications

A semitopological semigroup compactification of $G$ is a compact semitopological semigroup $S$ together with a continuous homomorphism $\alpha : G \to S$ with dense image.

There is a one-to-one correspondence:

- closed $G$-bi-invariant subalgebras of $WAP(G)$ $\leftrightarrow$ semitopological semigroup compactifications of $G$
- $A \subset WAP(G)$ $\mapsto$ maximal ideal space of $A$
- functions $f \in C(G)$ that factor through $\alpha$ $\leftrightarrow$ $\alpha : G \to S$
- inclusions $\leftrightarrow$ quotients
The WAP and Hilbert compactifications

In particular, the compactifications $G \to W$ and $G \to H$ corresponding to $\text{WAP}(G)$ and $\text{B}(G)$ have the structure of semitopological semigroups. We have a continuous surjective commuting homomorphism $W \to H$.

Moreover, they are semitopological *-semigroup compactifications, that is, they admit continuous involutions

$$* : W \to W \text{ and } * : H \to H$$

extending the inverse function on the image of $G$. 
Representations of semigroups

If $V$ is a reflexive Banach space, we denote by $\Theta(V)$ the compact semitopological semigroup of linear contractions of $V$:

$$\Theta(V) = \{ T \in L(V) : \|T\| \leq 1 \}.$$

**Fact (Shtern)**

*Every compact semitopological semigroup can be embedded in $\Theta(V)$ for some reflexive Banach space $V$.*

**Definition**

A semitopological semigroup $S$ is **Hilbert representable** if it can be embedded in $\Theta(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. 
Representations of semigroup compactifications

Fact

- $H$ is Hilbert representable.
- If $S$ is a Hilbert representable semitopological semigroup compactification of $G$, then it is a quotient of $H$.
- $G$ is Eberlein if and only if $W$ is Hilbert representable.

Question (Glasner–Megrelishvili)
Conversely, if a $S$ is a semigroup quotient of $H$, is it Hilbert representable?

Theorem
Yes if $G$ is oligomorphic.
Regular elements, inverse semigroups

Let $S$ be a semigroup. An element $p \in S$ is regular if there is $q \in S$ such that $p = pqp$. If moreover $q = qpq$, then $q$ is an inverse of $p$. $S$ is an inverse semigroup if every element has a unique inverse.

E.g. the inverse semigroup of partial bijections of a set.

Fact

- $S$ is an inverse semigroup if and only if every element is regular and the idempotents commute.
- Let $G \to S$ be a semitopological *-semigroup compactification of $G$. The following are equivalent for any element $p \in S$.
  1. $p$ is regular.
  2. $p$ has a unique inverse.
Stable independence, one-based structures

Let $M$ be a saturated structure. A formula $\varphi(x, y)$ is stable if for every type $t \in S(M)$, the function $d_t \varphi : M^n \to \mathbb{C}$,

$$d_t \varphi(b) = \varphi(x, b)^t,$$

is $M$-definable.

Given sets $A, B, C \subset M^{eq}$ (we fix an enumeration of $A$), we say that $A$ is stably independent from $C$ over $B$,

$$A \downarrow C,$$

if for every stable formula $\varphi$ the type $tp_\varphi(A/BC)$ extends to a type $t \in S(M)$ such that $d_t \varphi(y)$ is definable over $acl^{eq}(B)$.

We say that $M$ is one-based for stable independence if for any algebraically closed sets $A, B \subset M^{eq}$ we have

$$A \downarrow B.$$
Theorem

Let $G$ be an oligomorphic group, say $G = \text{Aut}(M)$ for an $\aleph_0$-categorical classical structure $M$.

- $H$ is the semigroup of partial elementary maps of $M^{eq}$ with algebraically closed domain. Equivalently, $H$ is the closure of $G$ in $\Theta(\ell^2(M^{eq}))$. In particular, $H$ is an inverse semigroup (and so are all of its semigroup quotients).

- The following are equivalent:
  1. $W$ is an inverse semigroup.
  2. The idempotents of $W$ commute.
  3. $M$ is one-based for stable independence.
  4. $G$ is Eberlein.

- $G$ is strongly Eberlein if and only if $M$ is $\aleph_0$-stable.
Remark: $\Theta(\ell^2)$ is not an inverse semigroup (but it is the WAP compactification of the Eberlein Roelcke precompact group $\mathcal{U}(\ell^2)$).

Examples

- The groups $S(\mathbb{N})$, $\text{Aut}(\mathbb{Q}, <)$, $\text{Homeo}(2^\omega)$ and $\text{Aut}(RG)$ are Eberlein oligomorphic groups.
- The automorphism group of Hrushovski’s $\aleph_0$-categorical stable pseudoplane is a WAP group that is not Eberlein.
A model-theoretic description of $W$

Let $G = \text{Aut}(M)$ where $M$ is an $\aleph_0$-categorical metric structure. The left-completion $E = \hat{G}_L$ is the semigroup of elementary embeddings $M \to M$. The UC-compactification coincides with $R = (E \times E) \parallel G$. Then $R$ can be seen as the space of types $[x, y]_R$ of pairs of embeddings.

The WAP-compactification is the quotient formed by the types $[x, y]$ restricted to stable formulas.

The $*$-semigroup structure of $W$ is as follows:

- $[x, y]^* = [y, x]$.
- $[x, y][y, z] = [x, z]$ if $x \downarrow_y z$. 
Restriction to equivalence relations

By our characterization of $B(G)$ we have that $H$ is the quotient formed by the types $[x, y]_H$ restricted to definable equivalence relations.

Then the map

$$[x, y]_H \mapsto x^{-1} \circ y$$

gives the identification of $H$ with the semigroup of partial elementary maps $M^{eq} \to M^{eq}$ with algebraically closed domain.
The key to the equivalences of the main theorem is the following description of idempotents and regular elements.

**Lemma**

Let $p = [x, y] \in W$.

- $p$ is an idempotent if and only if $x \equiv_{x \cap y} y$ and $x \downarrow_{x \cap y} y$.
- $p$ is regular if and only if $x \downarrow_{x \cap y} y$. 
Merci beaucoup.