# Separably categorical structures and Banach representations

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# Setting

Let $G$ be a Polish group, and fix a compatible left-invariant metric $d_L$. The corresponding completion $M = \widehat{G}_L$ can be seen as a metric first-order structure with automorphism group $G$.

A compactification of $G$ (equivalently, of $M$) will be a uniformly continuous $G$-map $\nu : G \to X$ with dense image, where $X$ is a compact $G$-space.

The continuous functions $f : G \to \mathbb{R}$ that factor through a compactification of $G$ are exactly the Roelcke uniformly continuous functions (i.e. functions uniformly continuous with respect to both the left and right uniformities of $G$). They form an algebra, $\mathrm{UC}(G)$.

# Banach representations of compact $G$-spaces

If $X$ is a compact $G$-space and $V$ is a Banach space, a representation of $X$ on $V$ is given by a pair

$$\alpha : X \to V^*,$$
$$h : G \to \mathrm{Iso}(V),$$

where $h$ is a continuous homomorphism and $\alpha$ is a weak$^*$-continuous $G$-map with respect to the dual action $G \times V^* \to V^*$, $(g\phi)(v) = \phi(h(g)^{-1}(v))$.

If $\mathcal{K}$ is a class of Banach spaces, the $G$-space $X$ is said $\mathcal{K}$-approximable if the family of its representations on Banach spaces $V \in \mathcal{K}$ separates points of $X$.

# The dynamical hierarchy after Glasner and Megrelishvili

**Theorem**

*Good dynamical properties of a continuous function $f : G \to \mathbb{R}$ are related to good classes $\mathcal{K}$ and the possibility of factoring $f$ through a $\mathcal{K}$-approximable compactification $X$, as follows.*

| *Name: $f$ is* | *The orbit $Gf \subset \mathcal{C}(X) \subset \mathcal{C}(G)$* | *$\mathcal{K}$ is the class of* |
|---|---|---|
| *AP* | *is precompact* | *Euclidean* |
| *WAP* | *is weakly precompact* | *reflexive* |
| *Asplund* | *has metrizable closure in $\mathbb{R}^X$* | *Asplund* |
| *$Tame_u$* | *is precompact in $\mathcal{B}_1(X)$* | *Rosenthal* |
| *UC* | *$(-)$* | *Banach spaces* |

$$AP(G) \subset WAP(G) \subset Asp(G) \subset Tame_u(G) \subset UC(G)$$

# Roelcke precompact Polish groups after Ben Yaacov and Tsankov

$G$ is Roelcke precompact if for every open $U \subset G$ there is a finite $F \subset G$ such that $UFU = G$.

**Theorem**

*Equivalenlty, $G$ acts approximately oligomorphically on $M$: $M^n /\!\!/ G$ is compact for every $n$; in other words, $M$ is an $\aleph_0$-categorical structure.*

Examples: $S_\infty$, $\mathrm{Aut}(\mathbb{Q}, <)$, $\mathrm{Aut}(RG)$, $\mathrm{Homeo}(2^\omega)$, $\mathrm{Iso}(\mathbb{U}_1)$, $\mathrm{Aut}(\mu)$, $\mathrm{Aut}^*(\mu)$, $\mathcal{U}(H)$, $\mathrm{Homeo}_+([0,1])$, etc.

# Functions as formulas

## Proposition

*Take $f \in UC(G)$ and define $\varphi : G^2 \to \mathbb{R}$ by $\varphi(h, g) = f(h^{-1}g)$. Then $\varphi$ extends to an invariant continuous function $\varphi : M^2 \to \mathbb{R}$.*

*If $G$ is Roelcke precompact, then $\varphi(x, y)$ is a $\emptyset$-definable predicate on $M$ (and so $f = \varphi_1 = \varphi(1, \cdot)$ is an $M$-definable predicate on $M$). Moreover, dynamical properties of $f$ correspond to classical model-theoretic properties of $\varphi$, as follows.*

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# Examples

- For the groups $S_\infty$, $\mathrm{Aut}(\mu)$, $\mathrm{Aut}^*(\mu)$ and $\mathcal{U}(H)$ we have $\mathrm{WAP}(G) = \mathrm{UC}(G)$.
- For $\mathrm{Aut}(\mathbb{Q}, <)$ we have $\mathrm{WAP}(G) \subsetneq \mathrm{Tame}_u(G) = \mathrm{UC}(G)$.
- For $\mathrm{Aut}(RG)$, $\mathrm{WAP}(G) = \mathrm{Tame}_u(G) \subsetneq \mathrm{UC}(G)$.
- For $\mathrm{Homeo}(2^\omega)$, $\mathrm{WAP}(G) = \mathrm{Tame}_u(G) \subsetneq \mathrm{UC}(G)$.
- For $\mathrm{Iso}(\mathbb{U}_1)$, $\mathrm{Tame}_u(G)$ is trivial.

The group $\mathrm{Homeo}_+([0, 1])$, for which it is known that $\mathrm{WAP}(G)$ is trivial but $\mathrm{Tame}_u(G) = \mathrm{UC}(G)$, offers an example of a completely unstable NIP structure.

# Functions as formulas, back

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# Strongly uniformly continuous functions

A continuous $f : G \to \mathbb{R}$ is called strongly uniformly continuous if it factors through a compactification $X$ such that, for all $x \in X$, the map

$$g \in G \mapsto gx \in X$$

is left uniformly continuous.

The algebra $\mathrm{SUC}(G)$ is the greatest subalgebra of $\mathrm{UC}(G)$ whose associated compactification has the structure of a right topological semigroup.

We have $\mathrm{Asp}(G) \subset \mathrm{SUC}(G)$. Glasner and Megrelishvili showed that $\mathrm{SUC}(\mathrm{Homeo}_+([0,1]))$ is trivial.

# $\mathrm{WAP}(G) = \mathrm{Asp}(G) = \mathrm{SUC}(G)$

## Theorem (I.)

*If $M$ is $\aleph_0$-categorical and $f \in SUC(M)$, then the associated formula is stable.*

## Corollary

*Let $G$ be a Roelcke precompact Polish group. Then $WAP(G) = Asp(G) = SUC(G)$.*

# An easy crucial example

Consider the function $f$ on $G = \mathrm{Aut}(\mathbb{Q}, <)$ given by

$$f(g) = \begin{cases} 1 & \text{if } 0 \leq g(0) \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $f(g^{-1}h) = 1$ means $g(0) \leq h(0)$.

$f$ is $\mathrm{Tame}_u$ but not WAP.

If it was SUC (sketch):

- we factor $f$ through an SUC compactification $X$;
- an irrational $r \in \mathbb{R} \setminus \mathbb{Q}$ can be seen as an element $x_r \in X$;
- since $f$ factors through $X$ and $g \mapsto gx_r$ is left uniformly continuous, there is a neighborhood $U$ of the identity such that $h(a) < r$ for every $a < r$ ($a \in \mathbb{Q}$) and $h \in U$;
- a neighborhood of the identity of $G$ is the stabilizer of a finite tuple of rationals.

Contradiction.

Merci beaucoup.

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