«Stably free» actions of the free group

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Motivation

- Finding a non-abelian free group inside a given group is of interest in many contexts.
- Given an action of a group, one may wish to find two or more elements of the group that act freely.

(Recall: an action $G \curvearrowright X$ is *free* if $gx \neq x$ for every $g \in G \setminus \{1\}$ and every $x \in X$.)

Motivation

Let M be a countably infinite saturated structure.
One can use Ehrenfeucht–Mostowski models to prove:

Fact

$\mathrm{Aut}(\mathbb{Q}, <)$ *embeds into* $\mathrm{Aut}(M)$.
In particular, $\mathrm{Aut}(M)$ *contains non-abelian free groups.*

(The free group is left-orderable.)

This holds as well in the metric setting, i.e., for non-compact separable (approximately) saturated structures.

Motivation

Let M be a countably infinite (classical) saturated structure.

Theorem (Evans–Tsankov, 2016)

There exists a free subgroup $F \leq \mathrm{Aut}(M)$ *in two generators such that the induced action* $F \curvearrowright M^{eq} \setminus \mathrm{acl}(\emptyset)$ *is free.*

Remark

If $F \curvearrowright X$ is a free action, then the unitary representation $F \curvearrowright \ell^2(X)$ is a multiple of the left regular rep., $F \curvearrowright \ell^2(F)$.

Corollary

Suppose $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset) = Y$. *Then* $\ell^2(M^{eq})$ *splits as*

$$\ell^2(Y) \oplus \ell^2(M^{eq} \setminus Y) = \lambda \cdot 1 \oplus \mu \cdot \ell^2(F).$$

Hence $\mathrm{Aut}(M) \curvearrowright \ell^2(M^{eq})$ *has spectral gap.*

Motivation

Theorem (Tsankov, 2012)

Suppose M is $\aleph_0$-categorical. Then every separable unitary representation of $\mathrm{Aut}(M)$ is a subrepresentation of $\mathrm{Aut}(M) \curvearrowright \ell^2(M^{eq})$.

Theorem (Evans–Tsankov, 2016)

There exists a free subgroup $F \leq \mathrm{Aut}(M)$ in two generators such that the induced action $F \curvearrowright M^{eq} \setminus \mathrm{acl}(\emptyset)$ is free.

Corollary (Evans–Tsankov, 2016)

Let M be classical and $\aleph_0$-categorical. Then every unitary representation of $\mathrm{Aut}(M)$ has spectral gap.
In other words, $\mathrm{Aut}(M)$ has Kazhdan's Property (T).

Motivation

Now let M be a *metric* $\aleph_0$-categorical structure.

Question (Tsankov, 2012, reformulated)

Does $\mathrm{Aut}(M)$ have Property (T)?

None of the previous steps work!

Assume M is non-compact.
Denote $\mathrm{Aut}(M)^\circ = \mathrm{Aut}(M/\operatorname{acl}(\emptyset))$.

Theorem (I., 2020)

There exists a free subgroup $F \leq \mathrm{Aut}(M)^\circ$ in two generators such that every unitary representation of $\mathrm{Aut}(M)^\circ$ splits as $\lambda \cdot 1 \oplus \mu \cdot \ell^2(F)$.
Hence $\mathrm{Aut}(M)$ has Property (T).

Producing a free action of $\mathbb{Z}$

Let M be a countable saturated structure.

How does one produce $\tau \in \mathrm{Aut}(M)$ such that

$$\tau^n(x) \neq x$$

for every $x \in M \setminus \mathrm{acl}(\emptyset)$ and every $n \in \mathbb{Z} \setminus \{0\}$?

Possible answer: by back-and-forth.

Indeed, one can achieve it by an easy back-and-forth using:

Lemma (Neumann)

Let G be a group acting on a set X such that all orbits are infinite. Then for every finite $A, B \subseteq X$ there is $g \in G$ such that $gA \cap B = \emptyset$.

Producing a free action of $\mathbb{Z}$

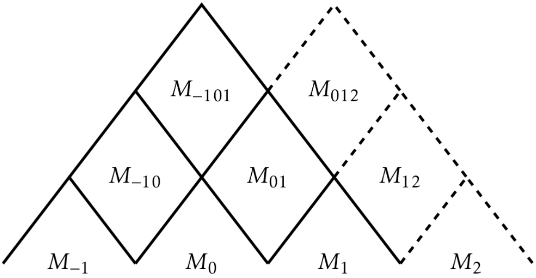
Alternative (learnt from a talk by P. Simon): an *external* back-and-forth pilling up copies of M in an *independent* manner.

Suppose that the theory of M is stable. Now:

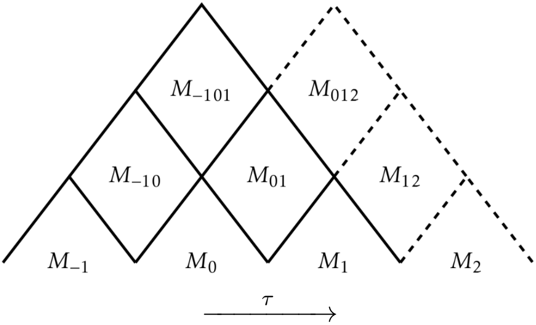
- Take M_0 a copie of M (enumerated).
- Let M_1 be another copie with $M_0 \equiv M_1$ and $M_0 \downarrow M_1$.
- Take a copie $M_{0,1}$ containing M_0 and M_1.
- Choose copies $M_{-1,0}$, M_{-1} with $M_{-1,0}M_{-1}M_0 \equiv M_{0,1}M_0M_1$ and $M_{-1,0} \downarrow_{M_0} M_{0,1}$.

Producing a free action of $\mathbb{Z}$

- Choose $M_{-1,0,1}$ containing all previous copies, then perform a step forward to get $M_{0,1,2}$, $M_{1,2}$, M_2 ...

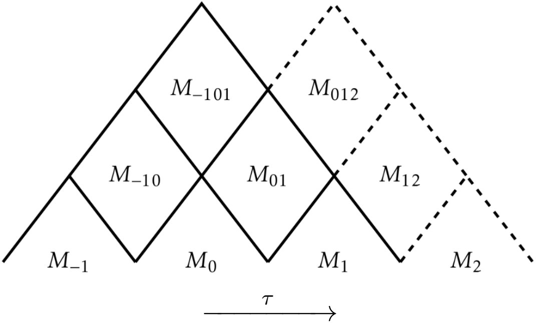


Producing a free action of $\mathbb{Z}$



- At the end we get a family $\{M_I\}$ indexed by the finite intervals of $\mathbb{Z}$. We identify M with its union, and define $\tau \in \mathrm{Aut}(M)$ by $\tau M_I = M_{I+1}$.

Producing a free action of $\mathbb{Z}$



- If $\tau(x) = x$ and $x \in M_I$, take any $n \geq |I|$. Then $x = \tau^n(x) \in M_I \cap M_{I+n}$ and $M_I \mathop{\smile\!\!\!\!\!\mid}\, M_{I+n}$, so $x \in \mathrm{acl}(\emptyset)$.
- Thus τ yields a free action of $\mathbb{Z}$ on $M \setminus \mathrm{acl}(\emptyset)$.

Semi-global stable independence

What if the theory is not stable?

Solution: Use any available independence relation with the properties needed for the construction.

- Algebraic independence will suffice (in classical logic):

$$M \underset{B}{\downarrow} N \quad \text{iff} \quad M \cap N = \operatorname{acl}(B)$$

(to simplify, say $T = T^{\mathrm{eq}}$, $M, N \models T$, $B \subseteq M \cap N$)

This is local stable independence relative to the equality.

Semi-global stable independence

- We may just as well use "full" local stable independence:

$$M \mathop{\downarrow}_{B} N \quad \text{iff} \quad \begin{array}{l} \forall \varphi(x,y) \text{ stable}, \ \forall m \in M^x, \\ \mathrm{tp}_\varphi(m/N) \text{ is } \mathrm{acl}(B)\text{-definable} \end{array}$$

(to simplify, say $T = T^{\mathrm{eq}}$, $M, N \models T$, $B \subseteq N$)

It satisfies:

(1) *(Invariance) If $M \downarrow_B N$ and $MBN \equiv M'B'N'$, then $M' \downarrow_{B'} N'$.*
(2) *(Monotonicity) If $M \subseteq M'$, $N \subseteq N'$ and $M' \downarrow_B N'$, then $M \downarrow_B N$.*
(3) *(Transitivity) Suppose $N \subseteq N'$. Then $M \downarrow_B N'$ if and only if $M \downarrow_B N$ and $M \downarrow_N N'$.*
(4) *(Existence) For every M, B, N there exists M' such that $M' \equiv_B M$ and $M' \downarrow_B N$.*
(5) *(Weak symmetry) Suppose $B \subseteq M$. If $M \downarrow_B N$, then $N \downarrow_B M$.*
(6) *(Stable anti-reflexivity) If $M \downarrow_B N$ and S is a sort with a stable metric, then $M \cap N \cap S \subseteq \mathrm{acl}(B)$.*

Special actions of $\mathbb{F}_2$

Let $\mathbb{F}_2$ be the free group in the generators a, b.

How does one produce a free action $\mathbb{F}_2 \curvearrowright M$?

Evans–Tsankov: via a back-and-forth argument using Neumann's Lemma, similar to the one mentioned earlier but more involved.

Question

Can one construct a "cairn" over $\mathbb{F}_2$, similarly as for $\mathbb{Z}$, to obtain a "stably free" action $\mathbb{F}_2 \curvearrowright M$?

Yes!

Intervals in $\mathbb{F}_2$

First: we define a family of finite subsets of $\mathbb{F}_2$ that will play the role of the intervals of $\mathbb{Z}$.

Given $k \in \mathbb{N}$, $0 \leq i < 4$ and $n = 4k + i$, let $\ell_n \in \mathbb{F}_2$ be

$$\ell_n = \begin{cases} a & \text{if } i = 0, \\ a^{-1} & \text{if } i = 1, \\ b & \text{if } i = 2, \\ b^{-1} & \text{if } i = 3. \end{cases}$$

Let $[\mathbb{F}_2]^{<\omega}$ be the set of finite subsets of $\mathbb{F}_2$.
We have the left action $\mathbb{F}_2 \curvearrowright [\mathbb{F}_2]^{<\omega}$, $wS = \{ws : s \in S\}$.

Intervals in $\mathbb{F}_2$

Let $I_0 = \{e\}$ and then, recursively,

$$I_{n+1} = I_n \cup \ell_n I_n.$$

This gives us a chain $\mathbb{I}_0 = \{I_n\}_{n\in\mathbb{N}} \subseteq [\mathbb{F}_2]^{<\omega}$:

$$\{e\} \subseteq \{e, a\} \subseteq \{a^{-1}, e, a\} \subseteq \left\{\begin{matrix} a^{-1}, e, a \\ ba^{-1}, b, ba \end{matrix}\right\} \subseteq \left\{\begin{matrix} b^{-1}a^{-1}, b^{-1}, b^{-1}a \\ a^{-1}, e, a \\ ba^{-1}, b, ba \end{matrix}\right\}$$

$$\subseteq \left\{\begin{matrix} b^{-1}a^{-1}, b^{-1}, b^{-1}a, ab^{-1}a^{-1}, ab^{-1}, ab^{-1}a \\ a^{-1}, e, a, a^2 \\ ba^{-1}, b, ba, aba^{-1}, ab, aba \end{matrix}\right\} \subseteq \ldots$$

The *intervals* of $\mathbb{F}_2$ will be all the translates of sets of the chain, plus the empty set:

$$\mathbb{I} = \mathbb{F}_2\mathbb{I}_0 \cup \{\emptyset\}.$$

Intervals in $\mathbb{F}_2$

We have:

Lemma

- *If n is even, then $I_n \cap \ell_n I_n = I_{n-3}$.*
- *If n is odd, then $I_n \cap \ell_n I_n = I_{n-1}$.*

Lemma

Every proper subinterval of I_{n+1} is a subinterval of I_n or of $\ell_n I_n$.

Proposition

The set of intervals is closed under intersections.

A cairn of models over $\mathbb{F}_2$

Let M be a separable (approx.) saturated structure.

We build a cairn $(M_I)_{I \in \mathbb{I}}$ of copies of M (except for $M_\emptyset = \emptyset$) inside some large homogeneous extension $\widehat{M}$.

Recursively, suppose we have defined $(M_I)_{I \leq I_n}$ such that:

- If $J \leq I \leq I_n$, $J \neq \emptyset$, then $M_J \preceq M_I$.
- For every $m < n$, $(M_I)_{I \leq I_m} \equiv (M_{\ell_m I})_{I \leq I_m}$.
- If $I, J \leq I_n$, then $M_I \mathop{\smile\!\!\!\!\!\!|}_{M_{I \cap J}} M_J$.

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- If $I, J \leq I_n$, then $M_I \mathop{\smile\!\!\!\!\!\!\,|}_{M_{I \cap J}} M_J$.

Say $I_n \cap \ell_n I_n = I_k$ ($k = n-1$ or $n-3$, and in any case $\ell_k = \ell_n^{-1}$). There is $\tau \in \mathrm{Aut}(\widehat{M})$ sending $(M_{\ell_k I})_{I \leq I_k}$ to $(M_I)_{I \leq I_k}$

By *existence*, there is $(N_I)_{I \leq I_n}$ such that

$$(N_I)_{I \leq I_n} \equiv_{M_{I_k}} (\tau M_I)_{I \leq I_n} \quad \text{and} \quad N_{I_n} \mathop{\smile\!\!\!\!\!\!\,|}_{M_{I_k}} M_{I_n}.$$

A cairn of models over $\mathbb{F}_2$

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For every $I \leq I_n$ with $\ell_n I \leq I_n$, it follows that $M_{\ell_n I} = N_I$. For every other $I \leq I_n$, define $M_{\ell_n I} = N_I$.

Finally, set $M_{I_{n+1}}$ to be any copie of M containing both M_{I_n} and $M_{\ell_n I_n}$.

One checks that the inductive hypotheses now hold for $(M_I)_{I \leq I_{n+1}}$.

A cairn of models over $\mathbb{F}_2$

At the end we identify $M = \overline{\bigcup_{I \in \mathbb{I}} M_I}$.

We have $(M_I)_{I \in \mathbb{I}} \equiv (M_{aI})_{I \in \mathbb{I}}$ and $(M_I)_{I \in \mathbb{I}} \equiv (M_{bI})_{I \in \mathbb{I}}$, thus there are $\tau_a, \tau_b \in \mathrm{Aut}(\widehat{M})$ such that $\tau_a M_I = M_{aI}$ and $\tau_b M_I = M_{bI}$ for every $I \in \mathbb{I}$.

Theorem

The structure M admits an action $\mathbb{F}_2 \curvearrowright^{\tau} M$ by automorphisms and a direct system of submodels $(M_I)_{I \in \mathbb{I}}$ (with base $M_\emptyset = \emptyset$) such that:

- *M is the direct limit of $(M_I)_{I \in \mathbb{I}}$.*
- *For every $w \in \mathbb{F}_2$ and $I \in \mathbb{I}$, $\tau_w M_I = M_{wI}$.*
- *If $I, J \in \mathbb{I}$, then $M_I \mathop{\smile}\limits_{M_{I \cap J}} M_J$.*

Back to the application

If M is $\aleph_0$-categorical (say, with $\mathrm{dcl}(\emptyset) = \mathrm{acl}(\emptyset)$) and $\mathrm{Aut}(M) \curvearrowright H$ is a unitary representation, one can show that the submodels M_I induce subspaces $H_I \subseteq H$ that are permuted by $\mathbb{F}_2$ and have the property that

$$H_I \underset{H_{I \cap J}}{\perp} H_J$$

for every $I, J \in \mathbb{I}$.

This yields the desired splitting

$$H = \lambda \cdot 1 \oplus \mu \cdot \ell^2(\mathbb{F}_2).$$

Thank you.