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présentée par **Tommaso SCOGNAMIGLIO**

Character stacks for Riemann surfaces and multiplicities for characters of $\mathrm{GL}_n(\mathbb{F}_q)$

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Resumé

Nous étudions les multiplicités du produit tensoriel des caractères irréductibles de $\mathrm{GL}_n(\mathbb{F}_q)$ et la cohomologie des champs de caractères pour les surfaces de Riemann trouées et pour les surfaces non orientables.

Nous donnons une formule pour la multiplicité $\langle \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle$ pour tout k-uplet de caractères semi-simples deployés $(\mathcal{X}_1, \cdots, \mathcal{X}_k)$. Une telle formule était déjà connue pour un k-uplet *générique* grâce à [45],[46].

Parmi nos résultats, nous prouvons que ces multiplicités sont polynomiales en q avec des coefficients entiers non négatifs et nous obtenons un critère de non-vanification. La formule de la thèse est donnée en reliant la multiplicité $\langle \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle$ au comptage des classes d'isomorphismes des représentations d'un certain carquois étoilé sur $\mathbb{F}_q$.

Les champs de caractères pour les surfaces de Riemann classifient les systèmes locaux sur la surface avec une monodromie locale prescrite. Pour un choix générique de la monodromie, leur cohomologie est bien comprise grâce à [46],[45],[72].

Nous calculons les E-séries de ces champs de caractères et donnons une formule conjecturale pour leurs séries de Poincaré mixtes pour tout choix de monodromie (pas nécessairement générique). Nous vérifions cette conjecture dans le cas de la ligne projective et de quatre points.

Le résultat concernant la E-série est obtenu en comptant les points sur les corps finis, en généralisant l'approche introduite dans [46],[45].

Ces résultats complètent et renforcent également les résultats récents de Davison, Hennecart, Schelegel-Mejia [24] concernant une version champ-être de la théorie de Hodge non abélienne.

Enfin, nous donnons un contre-exemple à une formule suggérée par le travail de Letellier et Rodriguez-Villegas [65] pour la série de Poincaré mixte des champs de caractères pour les surfaces non orientables. Le contre-exemple est obtenu par une description explicite de ces champs de caractères pour la somme connexe de deux copies du plan projectif réel.

Mots-clés : Représentations de groupes réductifs finis, représentations de carquois, champs de carquois multiplicatifs, champs de caractères, cohomologie à support compact.

Abstract

We study multiplicities for tensor product of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ and the cohomology of character stacks for punctured Riemann surfaces and for non-orientable surfaces.

We give a formula for the multiplicity $\langle \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle$ for any k-tuple of semisimple split characters $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ of $\mathrm{GL}_n(\mathbb{F}_q)$. Such a formula was previously known for a *generic* k-tuple thanks to [45],[46].

Among our results, we prove that these multiplicities are polynomial in q with non-negative integer coefficients and we obtain a criterion for their non-vanishing. The formula in the thesis is given relating the multiplicity $\langle \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle$ to the counting of the isomorphims classes of representations of a certain star-shaped quiver over $\mathbb{F}_q$.

Character stacks for punctured Riemann surfaces classify local systems on the Riemann surfaces with prescribed local monodromy. For a generic choice of the monodromy, their cohomology is well understood thanks to [46],[45],[72]. We compute the E-series of these character stacks and give a conjectural formula for their mixed Poincaré series for any choice of monodromy (not necessarily generic). We verify this conjecture in the case of the projective line and four punctures.

The result about the E-series is obtained by counting points over finite fields, generalizing the approach introduced in [46],[45].

These results also complement and reinforce the recent findings of Davison, Hennecart, Schelegel-Mejia [24] regarding a stacky version of non-Abelian Hodge theory.

Finally, we give a counterexample to a formula suggested by the work of Letellier and Rodriguez-Villegas [65] for the mixed Poincaré series of character stacks for non-orientable surfaces. The counterexample is obtained by an explicit description of these character stacks for the connected sum of two copies of the real projective plane.

Keywords: Representations of finite reductive groups, quiver representations, multiplicative quiver stacks, character stacks, compactly supported cohomology.

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Contents

1 Introduction en français

L'étude des relations entre la théorie des représentations, la combinatoire et la géométrie algébrique, en particulier la compréhension des espaces de moduli, est l'une des lignes de recherche les plus actives en mathématiques ces dernières années.
La compréhension des interactions entre les représentations de groupes et d'algèbres et la cohomologie des espaces de modules s'est avérée être un instrument puissant pour éclairer à la fois la géométrie de ces espaces et la structure de ces représentations.
Une riche source de ce type d'interactions provient de l'étude des espaces de moduli apparaissant dans ce que l'on appelle la correspondance de Hodge non abélienne pour une surface de Riemann, c'est-à-dire les empilements et variétés de caractères et les espaces de moduli des faisceaux de Higgs.
L'étude de la cohomologie de ces objets est liée à un large éventail d'arguments, allant de l'étude du programme de Langlands géométrique et de la symétrie miroir [10],[43] à la preuve du lemme fondamental de Ngô [77] et aux états BPS en physique et en théorie des cordes [15],[27].
Dans cette thèse, nous nous intéressons principalement à l'approche introduite dans [44],[46],[45], où les auteurs ont relié la cohomologie des champs de caractères pour une surface de Riemann au calcul des multiplicités dans l'anneau de caractères de $\mathrm{GL}_n(\mathbb{F}_q)$ et aux représentations de carquois.

1.1 État de l'art sur les multiplicités et les champs de caractères

1.1.1 Multiplicités pour les représentations des groupes linéaires généraux finis

La table de caractères de $\mathrm{GL}_n(\mathbb{F}_q)$ est connue depuis 1955 grâce aux travaux de Green [41], qui en a donné une description combinatoire. Ses formules pour les valeurs des caractères irréductibles sont de nature algorithmique.
Deligne et Lusztig [26] ont ensuite introduit les méthodes cohomologiques ℓ-adiques dans l'étude de la théorie des représentations des groupes réductifs finis. En utilisant cette approche, Lusztig a trouvé dans [66] un moyen géométrique de structurer les caractères irréductibles d'un groupe réductif fini. Dans le même livre, il a introduit la notion de caractère irréductible semisimple et unipotent par analogie avec la décomposition de Jordan pour les classes de conjugaison.
Pour le groupe linéaire général fini $\mathrm{GL}_n(\mathbb{F}_q)$, la construction de Lusztig a conduit à une interprétation géométrique de la table de caractères trouvée par Green, voir par exemple Lusztig et Srinivasan [69].

Etant donné $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ caractères irréductibles de $\mathrm{GL}_n(\mathbb{F}_q)$, la multiplicité $\langle \mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{X}_3 \rangle$ est donnée par la formule

$$\langle \mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{X}_3 \rangle = \frac{1}{|\,\mathrm{GL}_n(\mathbb{F}_q)|} \sum_{g \in \mathrm{GL}_n(\mathbb{F}_q)} \mathcal{X}_1(g)\mathcal{X}_2(g)\overline{\mathcal{X}_3(g)} \tag{1.1.1}$$

Bien que la table des caractères de $\mathrm{GL}_n(\mathbb{F}_q)$ soit connue depuis longtemps, il n'est pas facile d'extraire des informations générales de la formule (2.1.1) ci-dessus, en raison de la description inductive des valeurs des caractères.

Example 1.1.1. Rappelons que les caractères *unipotents* de $\mathrm{GL}_n(\mathbb{F}_q)$, qui sont les "briques" de la table de caractères, sont en bijection avec les représentations irréductibles de S_n et donc avec les partitions de n.
Pour une partition μ, on désigne par χ^μ le caractère associé de S_n et par $\mathcal{X}_\mu$ le caractère unipotent associé de $\mathrm{GL}_n(\mathbb{F}_q)$ (dans notre paramétrisation, nous associons à la partition (n) le caractère trivial 1).
D'après la formule (2.1.1), il est presque impossible d'obtenir directement une description combinatoire de l'ensemble $\{(\lambda,\mu,\nu)\in\mathcal{P}_n \mid \langle \mathcal{X}_\lambda\otimes\mathcal{X}_\mu,\mathcal{X}_\nu\rangle\neq 0\}$.
Déjà pour S_n, le problème de donner un critère combinatoire pour la non-vanification des *coefficients de Kronecker* $g^\nu_{\lambda,\mu} := \langle \chi^\mu\otimes\chi^\lambda,\chi^\nu\rangle$, est toujours ouvert et constitue un domaine de recherche très actif.
Il est intéressant de noter que Letellier a montré que les deux problèmes étaient liés : en particulier, Letellier a montré que si $g^\nu_{\lambda,\mu}\neq 0$ alors $\langle \mathcal{X}_\lambda\otimes\mathcal{X}_\mu,\mathcal{X}_\nu\rangle\neq 0$ aussi.

Rappelons que la multiplicité $\langle \mathcal{X}_1\otimes\mathcal{X}_2,\mathcal{X}_3\rangle$ est égale à $\langle \mathcal{X}_1\otimes\mathcal{X}_2\otimes\mathcal{X}_3^*,1\rangle$ où $\mathcal{X}_3^*$ est le caractère dual de $\mathcal{X}_3$. Un des objectifs de cette thèse est de contribuer à l'étude des multiplicités $\langle \mathcal{X}_1\otimes\cdots\otimes\mathcal{X}_k,1\rangle$ pour tout k-uplets de caractères irréductibles $(\mathcal{X}_1,\dots,\mathcal{X}_k)$.
La compréhension de ces quantités est encore un problème ouvert en général, mais des progrès substantiels ont été réalisés récemment. Les premiers cas étudiés dans la littérature concernaient les k-uplets $(\mathcal{X}_1,\dots,\mathcal{X}_k)$ où chaque $\mathcal{X}_i$ est un caractère unipotent.
Hiss, Lübeck et Mattig [49] ont calculé, par exemple, les multiplicités $\langle \mathcal{X}_1\otimes\mathcal{X}_2\otimes\mathcal{X}_3,1\rangle$ pour les caractères unipotents $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3$ et $n\leq 8$ en utilisant CHEVIE. Ils ont remarqué que ces quantités sont des polynômes en q, avec des coefficients positifs. Lusztig [68] a étudié les multiplicités pour les faisceaux caractères de PGL_2.
Les premiers résultats généraux ont été obtenus dans les articles [45, Theorem 1.4.1],[46, Theorem 3.2.7] par Hausel, Letellier, Rodriguez-Villegas. Les auteurs [45],[46] se sont limités à une certaine classe de k-uplets $(\mathcal{X}_1,\dots,\mathcal{X}_k)$, appelée *générique* (voir la définition 8.1.1). Remarquons qu'un k-uplet de caractères unipotents n'est jamais générique.
Pour des k-uplets génériques de caractères semi-simples deployés, les auteurs [46, Theorem 1.4.1] prouvent une formule combinatoire générale pour la multiplicité $\langle \mathcal{X}_1\otimes\cdots\otimes\mathcal{X}_k,1\rangle$ et relient cette dernière quantité à la cohomologie des variétés de caractères et des variétés de carquois (voir ci-dessous pour plus de détails).
Ces résultats ont ensuite été généralisés à tout k-uplet de caractères génériques par Letellier [63, Théorème 6.10.1, Théorème 7.4.1].
Le seul résultat général connu à ce jour dans le cas non générique est le travail de Letellier [64, Proposition 1.2.1], qui décrit la multiplicité pour les k-uplets de caractères unipotents en termes de multiplicité pour les k-uplets génériques de caractères unipotents tordus et l'action d'un certain groupe de Weyl sur une certaine variété de carquois.
Un des objectifs de cette thèse est de contribuer à la compréhension des multiplicités pour des k-uplets qui ne sont pas nécessairement génériques.

Nous reprenons ici rapidement et plus en détail les résultats de [45], [46], car ils sont un élément clé de notre travail.
Soit L le sous-groupe de Levi $L = \mathrm{GL}_{m_1}(\mathbb{F}_q) \times \cdots \times \mathrm{GL}_{m_s}(\mathbb{F}_q)$ plongé en bloc diagonalement dans $\mathrm{GL}_n(\mathbb{F}_q)$, où $m_1, \ldots, m_s$ sont des entiers non négatifs tels que $m_1 + \cdots + m_s = n$. Considérons un caractère linéaire $\gamma : L \to \mathbb{C}^*$ donné par

$$\gamma(M_1, \ldots, M_s) = \gamma_1(\det(M_1)) \cdots \gamma_s(\det(M_s))$$

pour $\gamma_1, \ldots, \gamma_s \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)$.
Nous désignons par $R_L^G(\gamma)$ le caractère induit de Harisha-Chandra de $\mathrm{GL}_n(\mathbb{F}_q)$. Rappelons que si $\gamma_i \neq \gamma_j$ pour chaque $i \neq j$ le caractère $R_L^G(\gamma)$ est irréductible. Les caractères irréductibles de cette forme sont appelés *semisimples deployés* (voir §5.1.1 pour plus de détails).

Considérons maintenant un k-uplet de caractères semisimples deployés $\mathcal{X} = (R_{L_1}^G(\delta_1), \ldots, R_{L_k}^G(\delta_k))$, où, pour $i = 1, \ldots, k$, nous avons $L_i = \mathrm{GL}_{m_{i,1}}(\mathbb{F}_q) \times \cdots \times \mathrm{GL}_{m_{i,s_i}}(\mathbb{F}_q)$ et

$$\delta_i(M_1, \ldots, M_{s_i}) = \delta_{i,1}(\det(M_1)) \cdots \delta_{i,s_i}(M_{s_i}).$$

Soit maintenant $\mathcal{P}$ l'ensemble des partitions. Dans [45], les auteurs ont introduit, pour chaque multipartition $\boldsymbol{\mu} \in \mathcal{P}^k$ et chaque entier $g \geq 0$, une fonction rationnelle $\mathbb{H}_{\boldsymbol{\mu}}(z, w) \in \mathbb{Q}(z, w)$, définie en termes de polynômes de Macdonald (pour une définition précise, voir §3.8).
Considérons maintenant la multipartition $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in \mathcal{P}^k$, où chaque μ^j est obtenu à partir de $(m_{j,1}, \ldots, m_{j,s_j})$ à une permutation près.
Les auteurs [46, Theorem 3.2.7] ont montré que si les δ_i sont choisis de telle sorte que $(R_{L_i}^G(\delta_i))_{i=1}^k$ soit générique (un tel choix est toujours possible si q est suffisamment grand), nous avons :

$$\langle \Lambda \otimes R_{L_1}^G(\delta_1) \otimes \cdots \otimes R_{L_k}^G(\delta_k), 1 \rangle = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}) \qquad (1.1.2)$$

où Λ est le caractère de l'action de conjugaison de $\mathrm{GL}_n(\mathbb{F}_q)$ sur l'espace vectoriel $\mathbb{C}[\mathfrak{gl}_n(\mathbb{F}_q)^g]$.

Un aspect intéressant des résultats de [45],[46] est que la formule (1.1.2) ci-dessus est prouvée en donnant une interprétation en terme des reprèsentations des carquois à la quantité $\langle R_{L_1}^G(\delta_1) \otimes \cdots \otimes R_{L_k}^G(\delta_k), 1 \rangle$.

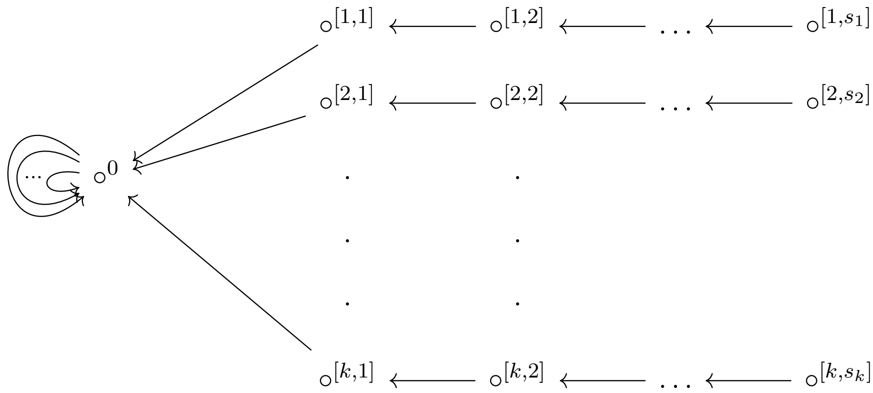
Rappelons que pour un carquois fini $\Gamma = (J, \Omega)$, où J est son ensemble de sommets et Ω son ensemble de flèches, dans [52], pour chaque vecteur de dimension $\beta \in \mathbb{N}^J$, Kac a introduit un polynôme à coefficients entiers $a_{\Gamma,\beta}(t)$, appelé *polynôme de Kac*, défini par le fait que $a_{\Gamma,\beta}(q)$ compte le nombre de classes d'isomorphisme des représentations *absolument indécomposables* de Γ (voir la définition 6.1.5) de dimension β sur $\mathbb{F}_q$, pour tout q.
Kac a montré que $a_{\Gamma,\beta}(t)$ est non nul si et seulement si β est une racine de Q et a conjecturé qu'il a des coefficients non négatifs. Cette dernière conjecture a d'abord été prouvée par Crawley-Boevey et Van der Bergh [21] dans le cas de β indivisible (c'est-à-dire $\gcd(\beta_j)_{j \in J} = 1$) et plus tard pour tout β par Hausel, Letellier, Rodriguez-Villegas dans [47].
Dans les deux cas, les auteurs ont obtenu la propriété de non-négativité en donnant une description des coefficients en termes de la cohomologie de certaines variétés de carquois.

Par exemple, si β est indivisible, dans [21, End of Proof 2.4], on montre qu'il existe une égalité

$$P_c(\mathcal{Q},t)=t^{d_{\mathcal{Q}}}a_{Q,\alpha}(t^2), \tag{1.1.3}$$

pour une certaine variété de carquois $\mathcal{Q}$, associée à Q,α, où $d_{\mathcal{Q}}$ est la dimension de $\mathcal{Q}$.

Considérons maintenant un k-uplet $\mathcal{X}=(R_{L_1}^G(\delta_1),\dots,R_{L_k}^G(\delta_k))$ comme ci-dessus et soit $Q=(I,\Omega)$ le carquois étoilé (voir la page suivante pour une image), avec k jambes de longueur $s_1,\dots,s_k$ respectivement et g boucles sur le sommet central.



Soit $\alpha_{\mathcal{X}}$ le vecteur de dimension $\alpha_{\mathcal{X}}\in\mathbb{N}^I$ défini par $(\alpha_{\mathcal{X}})_0=n$ et $(\alpha_{\mathcal{X}})_{[i,j]}=n-\sum_{h=1}^{j}m_{i,j}$. Remarquons que le carquois Q et le vecteur $\alpha_{\mathcal{X}}$ ne dépendent que des sous-groupes de Levi $L_1,\dots,L_k$ et non des caractères $\delta_1,\dots,\delta_k$.

Dans [46], on montre que, pour les sous-groupes de Levi $L_1,\dots,L_k$ introduits ci-dessus et un choix générique de $\delta_1,\dots,\delta_k$, la multiplicité $\langle\Lambda^g\otimes R_{L_1}^G(\delta_1)\otimes\cdots\otimes R_{L_k}^G(\delta_k),1\rangle$ est égal au nombre de classes d'isomorphisme de représentations absolument indécomposables de Q sur $\mathbb{F}_q$ de dimension $\alpha_{\mathcal{X}}$, i. e

$$\langle\Lambda\otimes R_{L_1}^G(\delta_1)\otimes\cdots\otimes R_{L_k}^G(\delta_k),1\rangle=a_{Q,\alpha_{\mathcal{X}}}(q). \tag{1.1.4}$$

Dans le même article, par un argument combinatoire, les auteurs trouvent une formule pour les polynômes de Kac pour les carquois étoilés et montrent en particulier que l'on a

$$a_{Q,\alpha_{\mathcal{X}}}(t)=\mathbb{H}_{\boldsymbol{\mu}}(0,\sqrt{t}) \tag{1.1.5}$$

et ils obtiennent ainsi la formule (1.1.2) citée ci-dessus.

L'interprétation en terme des reprèsentations de carquois des multiplicités rappelée ici a de nombreuses conséquences intéressantes. Par exemple, elle implique que la multiplicité

$$\langle\Lambda\otimes R_{L_1}^G(\delta_1)\otimes\cdots\otimes R_{L_k}^G(\delta_k),1\rangle\neq 0$$

si et seulement si $a_{Q,\alpha_{\mathcal{X}}}(t) \neq 0$, c'est-à-dire si et seulement si $\alpha_{\mathcal{X}}$ est une racine de Q (voir [46, Corollaire 1.4.2]).

Si $\alpha_{\mathcal{X}}$ est indivisible, la formule (1.1.3) donne en outre l'interprétation géométrique suivante des multiplicités

$$\langle \Lambda \otimes R_{L_1}^G(\delta_1) \otimes \cdots \otimes R_{L_k}^G(\delta_k), 1\rangle = q^{-d_{\mathcal{Q}}/2} P_c(\mathcal{Q}, \sqrt{q}). \tag{1.1.6}$$

La variété de carquois $\mathcal{Q}$ apparaissant dans le RHS de la formule (1.1.6) admet la description suivante. Fixez un k-uplet *générique* $\mathcal{O} = (\mathcal{O}_1, \dots, \mathcal{O}_k)$ (voir [45, Définition 2.2. 1]) d'orbites adjointes semisimples de $\mathfrak{gl}_n(\mathbb{C})$ telles que $\boldsymbol{\mu}$ est la multipartition donnée par les multiplicités des valeurs propres de $\mathcal{O}_1, \dots, \mathcal{O}_k$.

Dans [45], on montre que la variété $\mathcal{Q}$ est isomorphe à

$$\mathcal{Q}_{\mathcal{O}} \coloneqq \left\{(A_1, B_1, \dots, B_g, Y_1, \dots, Y_k) \in \mathfrak{gl}_n^{2g}(\mathbb{C}) \times \prod_{j=1}^k \mathcal{O}_j | \sum_{i=1}^g [A_i, B_i] + \sum_{j=1}^k Y_j = 0\right\} /\!/ \operatorname{GL}_n(\mathbb{C}).$$

Étudier si les variétés de la forme $\mathcal{Q}_{\mathcal{O}}$ pour $g = 0$ sont vides ou pas est généralement appelée le problème de Deligne-Simpson (voir par exemple [20]).

Comme mentionné au début, l'aspect le plus intéressant des résultats cités de [45] est que les fonctions $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$ (et donc les multiplicités pour le produit tensoriel des représentations de $\operatorname{GL}_n(\mathbb{F}_q)$ et les polynômes de Kac pour les carquois étoilés) sont ainsi liées à la cohomologie des champs de caractères génériques pour les surfaces de Riemann.

Cette relation entre des objets apparemment sans rapport s'est révélée être l'une des approches les plus efficaces pour calculer les invariants cohomologiques de ces espaces.

Nous passons en revue ces résultats et donnons des informations plus générales sur les champs de caractères dans le paragraphe ci-dessous.

1.1.2 Champs de caractères pour les surfaces de Riemann

Considérons une surface de Riemann Σ de genre $g \geq 0$, un sous-ensemble $D = \{p_1, \dots, p_k\} \subseteq \Sigma$ de k-points et un k-uplet $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_k)$ de classes de conjugaison semi-simples. Le champs de caractères associée est définie comme le champ quotient

$$\mathcal{M}_{\mathcal{C}} \coloneqq \left[\left\{\rho \in \operatorname{Hom}(\pi_1(\Sigma \setminus D), \operatorname{GL}_n(\mathbb{C})) \mid \rho(x_i) \in \mathcal{C}_i\right\}\right] \tag{1.1.7}$$

où chaque x_i est une petite boucle autour du point p_i. Ces champs classifient les systèmes locaux sur $\Sigma \setminus D$ tels que le monodromie autour du point p_i se situe dans $\mathcal{C}_i$, pour $i = 1, \dots, k$ et sont naturellement liés à certains espaces de moduli de fibrés de Higgs paraboliques sur Σ via la correspondance de Hodge non abélienne, voir par exemple les travaux de Simpson [88].

Le champ $\mathcal{M}_{\mathcal{C}}$ a la forme explicite suivante en termes d'équations matricielles :

$$\mathcal{M}_{\mathcal{C}} = \left[\left\{ (A_1, B_1, \dots, B_g, X_1, \dots, X_k) \in \mathrm{GL}_n^{2g}(\mathbb{C}) \times \prod_{j=1}^{k} \mathcal{C}_j | \prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{k} X_j = 1 \right\} \middle/ \mathrm{GL}_n(\mathbb{C}) \right]. \tag{1.1.8}$$

Dans ce qui suit, pour un champ complexe de type fini $\mathfrak{X}$, nous désignerons par $H_c^*(\mathfrak{X}) := H_c^*(\mathfrak{X}, \mathbb{C})$ sa cohomologie à support compact avec des coefficients $\mathbb{C}$ (ceci est bien défini grâce aux travaux de Laszlo et Olsson [59]).

Rappelons que chaque espace vectoriel $H_c^i(\mathfrak{X})$ est doté de la *filtration par le poids* $W_\bullet^i H_c^i(\mathfrak{X})$, à partir de laquelle on définit la série de Poincaré mixte $H_c(\mathfrak{X}, q, t)$

$$H_c(\mathfrak{X}, q, t) := \sum_{m,i} \dim(W_m^i / W_{m-1}^i) q^{\frac{m}{2}} t^i.$$

La E-série $E(\mathfrak{X}, q)$ est la spécialisation de $H_c(\mathfrak{X}, q, t)$ obtenue en branchant $t = -1$, la série de Poincaré $P_c(\mathfrak{X}, t)$ est la spécialisation de $H_c(\mathfrak{X}, q, t)$ obtenue en branchant $q = 1$ et la partie pure $PH_c(\mathfrak{X}, q, t)$ est définie comme

$$PH_c(\mathfrak{X}, q) = \sum_{m} \dim(W_m^{2m} / W_{m+1}^{2m}) q^{\frac{m}{2}}.$$

La géométrie et la cohomologie des champs de caractères ont été largement étudiées sous différents angles. La plupart des résultats ont été obtenus dans le cas où le k-uplet $\mathcal{C}$ est générique (voir la définition 9.1.1).

Pour un k-uplet générique $\mathcal{C}$, le champ $\mathcal{M}_{\mathcal{C}}$ est lisse et c'est un $\mathbb{G}_m$-gerbe sur le quotient GIT associé, que nous dénotons par $M_{\mathcal{C}}$. Par conséquent, la cohomologie de $\mathcal{M}_{\mathcal{C}}$ peut être facilement déduite de celle de la variété de caractères $M_{\mathcal{C}}$.

Nous commençons par une revue rapide des résultats connus obtenus sur la cohomologie des champs et variétés de caractères génériques, voir §9.1 pour plus de détails.

Les premiers résultats concernant ce sujet ont été obtenus dans le cas où $k = 1$ et $\mathcal{C}$ est une classe de conjugaison centrale. Pour $n \in \mathbb{N}$ et $d \in \mathbb{Z}$, posons que $\mathcal{M}_{n,d}$ soit le champ $\mathcal{M}_{\mathcal{C}}$ pour $k = 1$ et $\mathcal{C} = \{e^{\frac{2\pi i d}{n}} I_n\}$ c-a-d

$$\mathcal{M}_{n,d} = \left[\left\{ (A_1, B_1, \dots, A_g, B_g) \in \mathrm{GL}_n^{2g}(\mathbb{C}) \ \prod_{i=1}^{g} [A_i, B_i] = e^{\frac{2\pi i d}{n}} I_n \right\} \middle/ \mathrm{GL}_n(\mathbb{C}) \right].$$

L'orbite $\mathcal{C} = \{e^{\frac{2\pi i d}{n}}\}$ est générique si et seulement si $(n, d) = 1$.

Hitchin [48] a calculé le polynôme de Poincaré $P_c(M_{n,d}, t)$ dans le cas générique pour $n = 2$, en utilisant la correspondance de Hodge non abélienne et la théorie de Morse sur l'espace de moduli des faisceaux de Higgs. Gothen [40] a étendu son résultat pour $n = 3$.

Leur approche a ensuite été étendue pour calculer le polynôme de Poincaré $P_c(M_{\mathcal{C}}, t)$ dans le cas où $n = 2$, n'importe quel k et n'importe quel k-uplet générique $\mathcal{C}$ par Boden, Yogokawa [12] et lorsque $n = 3$, tout k et tout k-uplet générique $\mathcal{C}$ par García-Prada, Gothen et Munoz [37].

Cependant, les techniques de la théorie de Morse ne donnent pas d'information sur la filtration des poids sur la variété de caractères et sont difficiles à généraliser à n'importe quel n.

Hausel et Rodriguez-Villegas [44] ont été les premiers à obtenir un résultat général sur la filtration par le poids pour n quelconque.

Les auteurs ont calculé la E-série $E(\mathcal{M}_{n,d}, q)$ des champs $\mathcal{M}_{n,d}$ pour tout n, d tels que $(n, d) = 1$, en comptant les points sur les corps finis et ont proposé une formule conjecturale pour la série de Poincaré mixte $H_c(\mathcal{M}_{n,d}, q, t)$.

Schiffmann [84] a trouvé une expression pour la série de Poincaré $P_c(\mathcal{M}_{n,d}, t)$ dans le cas générique et Mellit [73] a vérifié plus tard que la formule de Schiffmann est en accord avec la spécialisation de Hausel et la conjecture de Rodriguez-Villegas à $q = 1$.

Hausel, Letellier et Rodriguez-Villegas ont ensuite généralisé les résultats de [44] et ont calculé [45, Théorème 1.2.3] la E-série $E(\mathcal{M}_{\mathcal{C}}, q)$ des champs $\mathcal{M}_{\mathcal{C}}$ pour tout k-uplet générique $\mathcal{C}$. Nous expliquons rapidement leurs résultats, car il s'agit du point de départ fondamental pour le développement de ce travail.

Les auteurs [45, Theorem 1.2.3] ont montré qu'il existe une égalité

$$E(\mathcal{M}_{\mathcal{C}}, q) = \frac{q^{\frac{d_{\boldsymbol{\mu}}}{2}}}{q-1}\mathbb{H}_{\boldsymbol{\mu}}\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right) \tag{1.1.9}$$

où $2d_{\boldsymbol{\mu}} = \dim(\mathcal{M}_{\mathcal{C}}) + 1$ et $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k)$ est la multipartition donnée par les multiplicités des valeurs propres de $\mathcal{C}_1, \ldots, \mathcal{C}_k$ respectivement.

Dans le même article, les auteurs [45, Conjecture 1.2.1] ont proposé la formule conjecturale suivante pour la série de Poincaré mixte $H_c(\mathcal{M}_{\mathcal{C}}, q, t)$, qui généralise la conjecture de Hausel et Rodriguez-Villegas énoncée dans [44] et qui déforme naturellement l' Identité (1.1.9) :

$$H_c(\mathcal{M}_{\mathcal{C}}, q, t) = \frac{(qt^2)^{\frac{d_\mu}{2}}}{qt^2-1}\mathbb{H}_{\boldsymbol{\mu}}\left(t\sqrt{q}, -\frac{1}{\sqrt{q}},\right). \tag{1.1.10}$$

Mellit [72, Theorem 7.12] a ensuite calculé la série de Poincaré $P_c(\mathcal{M}_{\mathcal{C}}, t)$ en utilisant la correspondance de Hodge non abélienne. Sa formule correspond à la spécialisation à $q = 1$ de la formule conjecturale (1.1.10) pour la série de Poincaré mixte.

Remarquons que, comme mentionné précédemment, la formule (1.1.2), la formule (1.1.9) et la conjecture (1.1.10) relient étroitement la compréhension de la cohomologie des champs de caractères génériques à la compréhension des multiplicités génériques pour les représentations de $\mathrm{GL}_n(\mathbb{F}_q)$ et les carquois étoilés.

Par exemple, la conjecture (1.1.10) implique que nous avons

$$PH_c(\mathcal{M}_{\mathcal{C}},q)=\frac{q^{\frac{d_{\boldsymbol{\mu}}}{2}}}{q-1}\langle \Lambda\otimes R^G_{L_1}(\delta_1)\otimes\cdots\otimes R^G_{L_k}(\delta_k),1\rangle \tag{1.1.11}$$

pour tout k-uplet générique $(R^G_{L_1}(\delta_1),\dots,R^G_{L_k}(\delta_k))$ tel que la multipartition associée est $\boldsymbol{\mu}$. De plus, lorsque le vecteur de dimension $\alpha_{\mathcal{X}}$ associé au k-uplet $\mathcal{X}=(R^G_{L_1}(\delta_1),\dots,R^G_{L_k}(\delta_k))$ est indivisible, la conjecture (1.1.10) implique que nous avons

$$PH_c(\mathcal{M}_{\mathcal{C}},q)=\frac{P_c(\mathcal{Q}_{\mathcal{O}},q)}{q-1} \tag{1.1.12}$$

c'est-à-dire que la partie pure de la série de Poincaré mixte de $\mathcal{M}_{\mathcal{C}}$ est égale (à un facteur $q-1$ près) au polynôme de Poincaré à support compact de sa contrepartie additive $\mathcal{Q}_{\mathcal{O}}$. Ceci est généralement connu sous le nom de "conjecture de pureté".

Alors que dans le cas générique les travaux cités donnent une description assez complète de la cohomologie des champs de caractères, la cohomologie des champs $\mathcal{M}_{\mathcal{C}}$ pour les k-uplets $\mathcal{C}$ non génériques a été peu étudiée jusqu'à récemment.
Les résultats les plus explicites et les plus généraux ont été obtenus dans le cas des champs $\mathcal{M}_{n,d}$.
Hausel et Rodriguez-Villegas ont été les premiers à obtenir un résultat général dans cette direction. Les auteurs [44, Theorem 3.8.1] ont exprimé les E-séries pour les champs $\mathcal{M}_{n,0}$ en termes des E-séries pour les champs de caractères génériques $\mathcal{M}_{n,1}$ par la formule suivante :

$$\operatorname{Exp}\left(\sum_{n\in\mathbb{N}}\frac{E(\mathcal{M}_{n,1},q)}{q^{n^2(2g-2)}}T^n\right)=\sum_{n\in\mathbb{N}}\frac{E(\mathcal{M}_{n,0},q)}{q^{n^2(2g-2)}}T^n \tag{1.1.13}$$

où Exp est l'exponentielle pléthystique dans l'anneau des séries formelles $\mathbb{Q}(q)[[T]]$ (voir §3.7 pour plus de détails sur les opérations pléthystiques). Le résultat des auteurs est obtenu en comptant les points sur les corps finis.

Fixons maintenant $r\in\mathbb{Q}$. Récemment, Davison, Hennercart et Schelegel-Mejia [Théorème 14.3, Corollaire 14.7]davison-hennecart ont prouvé la formule suivante exprimant la série de Poincaré à support compact de $\mathcal{M}_{n,d}$ pour tout n,d, en termes de série de Poincaré pour les champs de caractères génériques $\mathcal{M}_{n,1}$:

$$\sum_{\substack{(n,d)\in\mathbb{N}_{>0}\times\mathbb{Z}\\ \text{t.q } d=rn}}\frac{P_c(\mathcal{M}_{n,d},-t)}{t^{n^2(2g-2)}}z^nw^d=\operatorname{Exp}\left(\sum_{\substack{(n,d)\in\mathbb{N}_{>0}\times\mathbb{Z}\\ \text{t.q } d=rn}}\frac{P_c(\mathcal{M}_{n,1},-t)}{t^{n^2(2g-2)}}z^nw^d\right) \tag{1.1.14}$$

et ont formulé une conjecture similaire pour la série de Poincaré mixte de $H_c(\mathcal{M}_{n,d},q,t)$ pour tout n,d (voir la discussion après [24, Théorème 14.10]).
Ils ont obtenu cette formule en reliant la cohomologie des champ de caractères à la cohomologie de ce que l'on appelle les faisceaux BPS. Ces dernières sont des faisceaux perverses définies sur

les variétés de caractères et leur cohomologie est bien comprise pour les champs $\mathcal{M}_{n,d}$. Plus précisément, la correspondance de Hodge non abélienne pour les champs, prouvée dans [24], et le travail récent de Koseki et Kinjo [55] sur les faisceaux BPS pour le champ de module des fibrés de Higgs, donnent un moyen de calculer la cohomologie des faisceaux BPS pour un champ $\mathcal{M}_{n,d}$.

Remarquons en outre que, puisque les auteurs utilisent la correspondance de Hodge non abélienne qui ne préserve pas la filtration par le poids sur la cohomologie, leur méthode ne permet pas de prouver une formule analogue pour les E-séries ou les séries de Poincaré mixtes de $\mathcal{M}_{n,d}$.

Enfin, la cohomologie des faisceaux BPS pour les champs de caractères $\mathcal{M}_{\mathcal{C}}$ n'est pas comprise pour un $\mathcal{C}$ arbitraire et donc une généralisation de la formule (2.1.14) pour un $\mathcal{C}$ arbitraire n'est toujours pas prouvée.

Un des buts principaux de cette thèse est de contribuer à la compréhension de la cohomologie des champs de caractères $\mathcal{M}_{\mathcal{C}}$ pour des k-uplets non nécessairement génériques, pour tout k et $\mathcal{C}$.

1.1.3 Champs de caractères pour les surfaces non orientables

Une autre généralisation des résultats de [45] qui nous intéressera dans cette thèse est l'étude des champs de caractères pour les surfaces réelles non orientables plutôt que pour les surfaces de Riemann.

Notre point de vue sur la géométrie réelle est celui introduit par Atiyah [3], c'est-à-dire qu'une surface non orientable dans ce qui suit sera une paire (Σ, σ), où Σ est une surface de Riemann et $\sigma : \Sigma \to \Sigma$ est une involution anti-holomorphe telle que $\Sigma^{\sigma} = \varnothing$.

Remarquons que dans ce cas, le quotient $S = \Sigma/\langle\sigma\rangle$ est une surface réelle non orientable. Dénotons par $p : \Sigma \to S$ le morphisme quotient.

Fixons maintenant un sous-ensemble $E = \{y_1, \dots, y_k\} \subseteq S$ et un k-uplet $\mathcal{C}$ de classes de conjugaison semi-simples de $\mathrm{GL}_n(\mathbb{C})$. Remarquons que puisque l'action de σ est libre, l'involution σ définit un morphisme $\epsilon : \pi_1(S \setminus E) \to \mathbb{Z}/(2)$ ayant noyau $p_*(\pi_1(\Sigma/p^{-1}(E)))$.

Considérons maintenant le groupe $\mathrm{GL}_n(\mathbb{C})^+ \coloneqq \mathrm{GL}_n(\mathbb{C}) \rtimes_{\theta} \mathbb{Z}/(2)$, où $\theta : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ est l'involution de Cartan $\theta(M) = (M^t)^{-1}$ et dénotons par $\pi : \mathrm{GL}_n(\mathbb{C})^+ \to \mathbb{Z}/(2)$ la projection associée.

Le champ de caractères associée $\mathcal{M}^{\epsilon}_{\mathcal{C}}$ est définie comme

$$\mathcal{M}^{\epsilon}_{\mathcal{C}} = \left[\left\{\rho : \pi_1(S \setminus E) \to \mathrm{GL}_n(\mathbb{C})^+ \ \rho(z_i) \in \mathcal{C}_i \text{ and } \pi(\rho(g)) = \epsilon(g)\right\}/\,\mathrm{GL}_n(\mathbb{C})\right]$$

où chaque z_i est une boucle autour du point y_i. Le champ $\mathcal{M}^{\epsilon}_{\mathcal{C}}$ a la forme explicite suivante en termes d'équation matricielle :

$$\mathcal{M}_{\mathcal{C}}^{\epsilon} = \left[\left\{(D_1,\ldots,D_r,Z_1,\ldots,Z_k) \in \mathrm{GL}_n^r(\mathbb{C})\times\prod_{j=1}^{k}\mathcal{C}_j \mid D_1\theta(D_1)\ldots D_r\theta(D_r)Z_1\cdots Z_k=1\right\}/\,\mathrm{GL}_n(\mathbb{C})\right] \tag{1.1.15}$$

où $r=g+1$. Nous désignons par $M_{\mathcal{C}}^{\epsilon}$ le quotient GIT associé. Lorsque $k=1$ et $\mathcal{C}=\{e^{\frac{\pi id}{n}}I_n\}$, nous désignons le champ $\mathcal{M}_{\mathcal{C}}^{\epsilon}$ par $\mathcal{M}_{n,d}^{\epsilon}$.
Des définitions similaires peuvent être données lorsque σ a des points fixes, en utilisant le *groupe fondamental orbifold* du quotient $\Sigma/\langle\sigma\rangle$, voir par exemple [11].

Les champs $\mathcal{M}_{\mathcal{C}}^{\epsilon}$ sont profondément liées à ce que l'on appelle les *branes* à l'intérieur des espaces de moduli des fibrés de Higgs. Le calcul de la cohomologie des branes est un élément clé dans la compréhension de la symétrie miroir pour le système de Hitchin.
Des références sur le sujet peuvent être trouvées par exemple dans [6],[7],[11],[9].

Peu de résultats ont été montrés dans la littérature concernant la cohomologie des champs $\mathcal{M}_{\mathcal{C}}^{\epsilon}$. Récemment, Letellier et Rodriguez-Villegas [65, Theorem 1.4] ont calculé la E-série $E(\mathcal{M}_{\mathcal{C}^{\epsilon}},q)$ lorsque $\mathcal{C}$ est générique, en comptant les points sur les corps finis.

Baird et Wong [4] ont calculé le E-polynôme de variétés analogues $M_{n,d}^{\epsilon}$ lorsque l'involution anti-holomorphe σ a des points fixes. Leurs formules sont assez différentes de celles de [65].

Dans cette thèse, nous nous concentrerons sur le cas des champs $\mathcal{M}_{n,d}^{\epsilon}$ lorsque $(n,d)=1$. Dans ce cas, le champ $\mathcal{M}_{n,d}^{\epsilon}$ est un μ_2-gerbe sur la variété de caractères $M_{n,d}^{\epsilon}$. Cette dernière variété est profondément liée à l'espace de moduli des fibrés de Higgs réels et quaternioniques de rang n et de degré d sur Σ.

1.2 Aperçu de la thèse

Nous donnons ici un résumé rapide de nos principaux résultats.

1.2.1 Multiplicités

En ce qui concerne les multiplicités, nous étudions les multiplicités $\langle \mathcal{X}_1\otimes\cdots\otimes\mathcal{X}_k,1\rangle$ pour des k-uplets non nécessairement génériques k de caractères irréductibles $(\mathcal{X}_1,\cdots,\mathcal{X}_k)$.
Dans cette thèse, nous donnons une formule pour la quantité $\langle R_{L_1}^G(\gamma_1)\otimes\cdots\otimes R_{L_k}^G(\gamma_k),1\rangle$ pour tout choix de $\gamma_1,\ldots,\gamma_k$ (pas nécessairement générique), en termes de polynômes de Kac du carquois Q introduit précédemment.
Cette formule est obtenue en donnant une interprétation en terme des représentations de carquois de la multiplicité $\langle R_{L_1}^G(\gamma_1)\otimes\cdots\otimes R_{L_k}^G(\gamma_k),1\rangle$ pour tout k-uplet $(R_{L_1}^G(\gamma_1),\ldots,R_{L_k}^G(\gamma_k))$.
En conséquence de notre résultat, nous montrons que $\langle R_{L_1}^G(\gamma_1)\otimes\cdots\otimes R_{L_k}^G(\gamma_k),1\rangle$ est un polynôme en q avec des coefficients non-négatifs et nous montrons un critère pour verifier si c'est 0 ou pas en termes du système de racines de Q. Ces résultats sont contenus dans [85].

1.2.2 Cohomologie des champs de caractères

En ce qui concerne les champs de caractères pour les surfaces de Riemann, nous étudions la cohomologie des champs de caractères $\mathcal{M}_{\mathcal{C}}$ pour k-uplets qui ne sont pas nécessairement génériques. L'un des principaux résultats de cette thèse est une généralisation de la formule (2.1.14) à des $\mathcal{C}$ arbitraires pour la E-series $E(\mathcal{M}_{\mathcal{C}}, q)$ au lieu de la série de Poincaré $P_c(\mathcal{M}_{\mathcal{C}}, t)$. Nous obtenons ainsi une formule explicite pour $E(\mathcal{M}_{\mathcal{C}}, q)$ pour tout k-uplet $\mathcal{C}$, voir le Théorème 2.3.10 ci-dessous.
Nous donnons aussi une formule conjecturale (voir Conjecture 2.3.12) pour la série de Poincaré mixte $H_c(\mathcal{M}_{\mathcal{C}}, q, t)$, que nous vérifions dans le cas de $\Sigma = \mathbb{P}^1_{\mathbb{C}}$, $k = 4$ et une certaine famille de quadruples non-génériques. Ces résultats font partie de [86].
La conjecture 2.3.12 pour les champs $\mathcal{M}_{n,d}$ est déjà apparue dans [24], voir la discussion dans *loc. cit* après le Théorème 14.10. Remarquons que notre approche est très différente de celle de [24] car nous n'utilisons pas la théorie de Hodge non-abélienne ni les faisceaux BPS.

1.2.3 Surfaces non orientables

En ce qui concerne les champs de caractères génériques pour les surfaces réelles compactes non orientables, nous donnons une description explicite des champs $\mathcal{M}^{\epsilon}_{n,d}$ lorsque $(n, d) = 1$ et $r = 2$, c'est-à-dire pour une courbe elliptique (réelle).
Cette description donne un contre-exemple à une formule proposée par [65] pour les séries de Poincaré mixtes des champs $\mathcal{M}^{\epsilon}_{\mathcal{C}}$. La description de ce contre-exemple est le résultat principal de [87].

2 Introduction

The study of the relationships between representation theory, combinatorics and algebraic geometry, in particular the understanding of moduli spaces, is one of the most active lines of research in mathematics in recent years.
The understanding of the interactions between representations of groups and algebras and the cohomology of moduli spaces has proved itself to be a powerful instrument for shedding light both on the geometry of these spaces and the structure of these representations.
A rich source of this kind of interactions comes from the study of moduli spaces appearing in the so-called non-abelian Hodge correspondence for a Riemann surface, i.e character stacks and varieties and moduli spaces of Higgs bundles.
The study of the cohomology of these objects is related to a wide range of arguments, ranging from the study of Geometric Langlands program and mirror symmetry [10],[43] to Ngô's proof of fundamental lemma [77] and to BPS states in physics and string theory [15],[27].
In this thesis, we are mostly interested in the approach introduced in [44],[46],[45], where the authors related the cohomology of character stacks for a Riemann surface to the computation of multiplicities in the character ring of $\mathrm{GL}_n(\mathbb{F}_q)$ and quiver representations.

2.1 State of the art on multiplicities and character stacks

2.1.1 Multiplicities for representations of finite general linear groups

The character table of $\mathrm{GL}_n(\mathbb{F}_q)$ is known since 1955 by the work of Green [41], who gave a combinatorial description of it. His formulae for the values of the irreducible characters have an algorithmic nature.
Deligne and Lusztig [26] later introduced ℓ-adic cohomological methods to the study of the representation theory of finite reductive groups. Using this approach, in [66] Lusztig found a geometric way to costruct the irreducible characters of a finite reductive group. In the same book, he introduced the notion of a semisimple and unipotent irreducible character by analogy with the Jordan decomposition for the conjugacy classes.
For the finite general linear group $\mathrm{GL}_n(\mathbb{F}_q)$, Lusztig's construction led to a geometric interpretation of the character table found by Green, see for example Lusztig and Srinivasan [69].

Given $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$, the multiplicity $\langle \mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{X}_3 \rangle$ is given by the formula

$$\langle \mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{X}_3 \rangle = \frac{1}{|\mathrm{GL}_n(\mathbb{F}_q)|} \sum_{g \in \mathrm{GL}_n(\mathbb{F}_q)} \mathcal{X}_1(g)\mathcal{X}_2(g)\overline{\mathcal{X}_3(g)} \tag{2.1.1}$$

Altough the character table of $\mathrm{GL}_n(\mathbb{F}_q)$ is known for a long time, it is not easy to extract general information from Formula (2.1.1) above, due to the inductive description of the values of the characters.

Example 2.1.1. Recall that the *unipotent* characters of $\mathrm{GL}_n(\mathbb{F}_q)$, which are the "building blocks" of the character table, are in bijection with the irreducible representations of S_n and

so with the partitions of n.
For a partition μ, we denote by χ^μ the associated character of S_n and by $\mathcal{X}_\mu$ the associated unipotent character of $\mathrm{GL}_n(\mathbb{F}_q)$ (in our parametrization, we associate to the partition (n) the trivial character 1).
From Formula (2.1.1), it is nearly impossible to obtain directly a combinatorial description of the set $\{(\lambda,\mu,\nu)\in\mathcal{P}_n \mid \langle \mathcal{X}_\lambda\otimes\mathcal{X}_\mu,\mathcal{X}_\nu\rangle\neq 0\}$, where $\mathcal{P}_n$ is the set of the partitions of n.
Already for S_n, the problem of giving a combinatorial criterion for the non-vanishing of the *Kronecker coefficients* $g^\nu_{\lambda,\mu}:=\langle\chi^\mu\otimes\chi^\lambda,\chi^\nu\rangle$, is still open and is a very active area of research. Interestingly, the two problems were shown to be related by Letellier [64]: in particular, Letellier [64, Proposition 1.2.4], showed that if $g^\nu_{\lambda,\mu}\neq 0$ then $\langle \mathcal{X}_\lambda\otimes\mathcal{X}_\mu,\mathcal{X}_\nu\rangle\neq 0$ too.

Recall that the multiplicity $\langle\mathcal{X}_1\otimes\mathcal{X}_2,\mathcal{X}_3\rangle$ is equal to $\langle\mathcal{X}_1\otimes\mathcal{X}_2\otimes\mathcal{X}_3^*,1\rangle$ where $\mathcal{X}_3^*$ is the dual character of $\mathcal{X}_3$. One of the aims of this thesis is to contribute to the study of the multiplicites $\langle\mathcal{X}_1\otimes\cdots\otimes\mathcal{X}_k,1\rangle$ for any k-tuple of irreducible characters $(\mathcal{X}_1,\dots,\mathcal{X}_k)$.
The understanding of these quantites is still an open problem in general but substantial progress were made recently. The first cases studied in the literature concerned k-tuples $(\mathcal{X}_1,\dots,\mathcal{X}_k)$ where each $\mathcal{X}_i$ is an unipotent character.
Hiss, Lübeck and Mattig [49] computed, for example, the multiplicities $\langle\mathcal{X}_1\otimes\mathcal{X}_2\otimes\mathcal{X}_3,1\rangle$ for unipotent characters $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3$ and $n\leq 8$ using CHEVIE. They noticed that these quantities are polynomials in q, with positive coefficients. Lusztig [68] studied multiplicities for unipotent character sheaves of PGL_2.
The first general results were obtained in the papers [45, Theorem 1.4.1],[46, Theorem 3.2.7] by Hausel, Letellier, Rodriguez-Villegas. The authors [45],[46] restricted themselves to a certain class of k-tuples $(\mathcal{X}_1,\dots,\mathcal{X}_k)$, called *generic* (see Definition 8.1.1). Notice that a k-tuple of unipotent characters is never generic.
For generic k-tuples of semisimple split characters, the authors [46, Theorem 1.4.1] prove a general combinatorial formula for the multiplicity $\langle\mathcal{X}_1\otimes\cdots\otimes\mathcal{X}_k,1\rangle$ and relate the latter quantity to the cohomology of character varieties and quiver varieties (see below for more details).
These results were later generalised to any k-tuple of generic characters by Letellier [63, Theorem 6.10.1,Theorem 7.4.1].
The only general result known so far in the non-generic case is Letellier's work [64, Proposition 1.2.1], which describes the multiplicity for k-uples of unipotent characters in terms of the multiplicity for generic k-uples of twisted unipotent characters and the action of a certain Weyl group on a certain quiver variety.
One of the aim of this thesis is to to contribute to the understanding of multiplicities for k-tuples which are not necessarily generic.

We quickly resume here in more detail the results of [45], [46], since they are a key element for our work.
Let L be the Levi subgroup $L=\mathrm{GL}_{m_1}(\mathbb{F}_q)\times\cdots\times\mathrm{GL}_{m_s}(\mathbb{F}_q)$ embedded block diagonally in $\mathrm{GL}_n(\mathbb{F}_q)$, where $m_1,\dots,m_s$ are nonnegative integers such that $m_1+\cdots+m_s=n$. Consider

a linear character $\gamma : L \to \mathbb{C}^*$ given by

$$\gamma(M_1, \ldots, M_s) = \gamma_1(\det(M_1)) \cdots \gamma_s(\det(M_s))$$

for $\gamma_1, \ldots, \gamma_s \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)$.
We denote by $R_L^G(\gamma)$ the Harisha-Chandra induced character of $\mathrm{GL}_n(\mathbb{F}_q)$. Recall that if $\gamma_i \neq \gamma_j$ for each $i \neq j$ the character $R_L^G(\gamma)$ is irreducible. The irreducible characters of this form are called *semisimple split* (see §5.1.1 for more details).

Consider now a k-tuple of semisimple split characters $\mathcal{X} = (R_{L_1}^G(\delta_1), \ldots, R_{L_k}^G(\delta_k))$, where, for $i = 1, \ldots, k$, we have $L_i = \mathrm{GL}_{m_{i,1}}(\mathbb{F}_q) \times \cdots \times \mathrm{GL}_{m_{i,s_i}}(\mathbb{F}_q)$ and

$$\delta_i(M_1, \ldots, M_{s_i}) = \delta_{i,1}(\det(M_1)) \cdots \delta_{i,s_i}(M_{s_i}).$$

Let now $\mathcal{P}$ be the set of partitions. In [45], the authors introduced, for each multipartition $\boldsymbol{\mu} \in \mathcal{P}^k$ and each integer $g \geq 0$, a rational function $\mathbb{H}_{\boldsymbol{\mu}}(z, w) \in \mathbb{Q}(z, w)$, defined in terms of Macdonald polynomials (for a precise definition see §3.8).
Consider now the multipartition $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in \mathcal{P}^k$, where each μ^j is obtained from $(m_{j,1}, \ldots, m_{j,s_j})$ up to reordering.
The authors [46, Theorem 3.2.7] showed that if the δ_i's are chosen so that $(R_{L_i}^G(\delta_i))_{i=1}^k$ is generic (such a choice is always possible if q is big enough), we have:

$$\langle \Lambda \otimes R_{L_1}^G(\delta_1) \otimes \cdots \otimes R_{L_k}^G(\delta_k), 1 \rangle = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}) \tag{2.1.2}$$

where Λ is the character of the conjugation action of $\mathrm{GL}_n(\mathbb{F}_q)$ on the vector space $\mathbb{C}[\mathfrak{gl}_n(\mathbb{F}_q)^g]$.

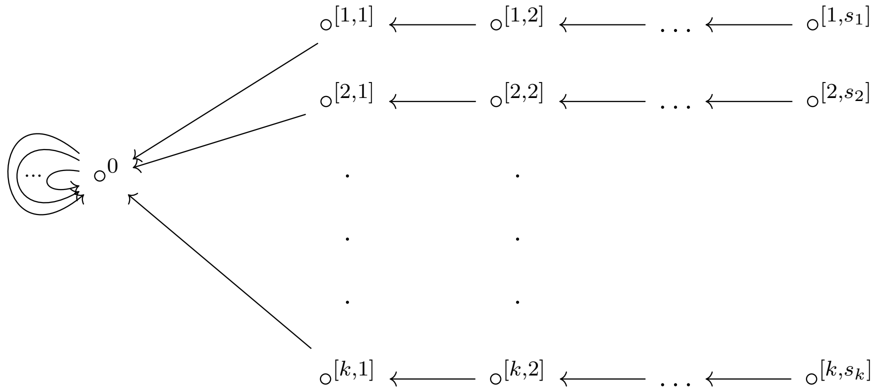
An interesting side of the results of [45],[46] is that Formula (2.1.2) above is proved by giving a quiver-theoretic interpretation to the quantity $\langle R_{L_1}^G(\delta_1) \otimes \cdots \otimes R_{L_k}^G(\delta_k), 1 \rangle$.

Recall that for a finite quiver $\Gamma = (J, \Omega)$, where J is its set of vertices and Ω its set of arrows, in [52], for each dimension vector $\beta \in \mathbb{N}^J$, Kac introduced a polynomial with integer coefficients $a_{\Gamma,\beta}(t)$, called *Kac polynomial*, defined by the fact that $a_{\Gamma,\beta}(q)$ counts the number of isomorphism classes of *absolutely indecomposable* representations of Γ (see Definition 6.1.5) of dimension β over $\mathbb{F}_q$, for any q.
Kac showed that $a_{\Gamma,\beta}(t)$ is non-zero if and only if β is a root of Q and conjectured that it has non-negative coefficients. The latter conjecture was first proved by Crawley-Boevey and Van der Bergh [21] in the case of indivisible β (i.e $\gcd(\beta_j)_{j \in J} = 1$) and later for any β by Hausel, Letellier, Rodriguez-Villegas in [47].
In both cases, the authors obtained the non-negativity property by giving a description of the coefficients in terms of the cohomology of certain quiver varieties.
For instance, if β is indivisible, in [21, End of Proof 2.4], it is shown, that there is an equality

$$P_c(\mathcal{Q}, t) = t^{d_{\mathcal{Q}}} a_{Q,\alpha}(t^2), \tag{2.1.3}$$

for a certain quiver variety $\mathcal{Q}$, associated to Q, α, where $d_{\mathcal{Q}}$ is the dimension of $\mathcal{Q}$.

Consider now a k-tuple $\mathcal{X} = (R_{L_1}^G(\delta_1), \ldots, R_{L_k}^G(\delta_k))$ as above and let $Q = (I, \Omega)$ be the star-shaped quiver (see the next page for a picture), with k legs of length $s_1, \ldots, s_k$ respectively and g loops on the central vertex.



Let $\alpha_{\mathcal{X}}$ be the dimension vector $\alpha_{\mathcal{X}} \in \mathbb{N}^I$ defined as $(\alpha_{\mathcal{X}})_0 = n$ and $(\alpha_{\mathcal{X}})_{[i,j]} = n - \sum_{h=1}^{j} m_{i,j}$. Notice that the quiver Q and the vector $\alpha_{\mathcal{X}}$ depend only on the Levi subgroups $L_1, \ldots, L_k$ and not on the characters $\delta_1, \ldots, \delta_k$.
In [46], it is shown that, for the Levi subgroups $L_1, \ldots, L_k$ introduced above and a generic choice of $\delta_1, \ldots, \delta_k$ the multiplicity $\langle \Lambda^g \otimes R_{L_1}^G(\delta_1) \otimes \cdots \otimes R_{L_k}^G(\delta_k), 1 \rangle$ is equal to the number of isomorphism classes of absolutely indecomposable representations of Q over $\mathbb{F}_q$ of dimension $\alpha_{\mathcal{X}}$, i.e

$$\langle \Lambda \otimes R_{L_1}^G(\delta_1) \otimes \cdots \otimes R_{L_k}^G(\delta_k), 1 \rangle = a_{Q,\alpha_{\mathcal{X}}}(q). \tag{2.1.4}$$

In the same paper, via a combinatorial argument, the authors find a formula for Kac polynomials for star-shaped quivers and show in particular that we have

$$a_{Q,\alpha_{\mathcal{X}}}(t) = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{t}) \tag{2.1.5}$$

and thus they obtain Formula (2.1.2) cited above.
The quiver-theoretic interpretation of multiplicities recalled here has many interesting consequences. For instance, it implies that the multiplicity

$$\langle \Lambda \otimes R_{L_1}^G(\delta_1) \otimes \cdots \otimes R_{L_k}^G(\delta_k), 1 \rangle \neq 0$$

if and only if $a_{Q,\alpha_{\mathcal{X}}}(t) \neq 0$, i.e if and only if $\alpha_{\mathcal{X}}$ is a root of Q (see [46, Corollary 1.4.2]).
If $\alpha_{\mathcal{X}}$ is indivisible, Formula (2.1.3) gives moreover the following geometric interpretation of multiplicities

$$\langle \Lambda \otimes R_{L_1}^G(\delta_1) \otimes \cdots \otimes R_{L_k}^G(\delta_k), 1 \rangle = q^{-d_{\mathcal{Q}}/2} P_c(\mathcal{Q}, \sqrt{q}). \tag{2.1.6}$$

The quiver variety $\mathcal{Q}$ appearing in the RHS of Formula (2.1.6) admits the following description. Fix a *generic* k-tuple $\mathcal{O} = (\mathcal{O}_1, \ldots, \mathcal{O}_k)$ (see [45, Definition 2.2.1]) of semisimple adjoint

orbits of $\mathfrak{gl}_n(\mathbb{C})$ such that $\boldsymbol{\mu}$ is the multipartition given by the multiplicities of eigenvalues of $\mathcal{O}_1, \dots, \mathcal{O}_k$.
In [45], it is shown that the variety $\mathcal{Q}$ is isomorphic to

$$\mathcal{Q}_{\mathcal{O}} \coloneqq \left\{(A_1, B_1, \dots, B_g, Y_1, \dots, Y_k) \in \mathfrak{gl}_n^{2g}(\mathbb{C}) \times \prod_{j=1}^{k} \mathcal{O}_j \mid \sum_{i=1}^{g} [A_i, B_i] + \sum_{j=1}^{k} Y_j = 0\right\} /\!/ \mathrm{GL}_n(\mathbb{C}).$$

The study of the non-emptiness of the varieties of the form $\mathcal{Q}_{\mathcal{O}}$ for $g = 0$ is usually called the Deligne-Simpson problem (see for example [20]).

As mentioned at the beginning, the most interesting aspect of the results just cited of [45] is that the functions $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$ (and thus the multiplicities for tensor product of representations of $\mathrm{GL}_n(\mathbb{F}_q)$ and Kac polynomials for star-shaped quivers) are thereby related to the cohomology of generic character stacks for Riemann surfaces.
This relationship between seemingly unrelated objects has proved itself to be one of the most effective approach to compute cohomological invariants of these spaces.
We review these results and give a more general background about character stacks in the paragraph below.

2.1.2 Character stacks for Riemann surfaces

Consider a Riemann surface Σ of genus $g \geq 0$, a subset $D = \{p_1, \dots, p_k\} \subseteq \Sigma$ of k-points and a k-tuple $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_k)$ of semisimple conjugacy classes. The associated character stack is defined as the quotient stack

$$\mathcal{M}_{\mathcal{C}} \coloneqq \left[\left\{\rho \in \mathrm{Hom}(\pi_1(\Sigma \setminus D), \mathrm{GL}_n(\mathbb{C})) \mid \rho(x_i) \in \mathcal{C}_i \text{ for } i = 1, \dots, k\right\} \Big/ \mathrm{GL}_n(\mathbb{C})\right] \tag{2.1.7}$$

where each x_i is a small loop around the point p_i. These stacks classify local systems on $\Sigma \setminus D$ such that the monodromy around the point p_i lies in $\mathcal{C}_i$, for $i = 1, \dots, k$ and are naturally related to certain moduli spaces of (strongly) parabolic Higgs bundles on Σ via the non-abelian Hodge correspondence, see for example the work of Simpson [88].

The stack $\mathcal{M}_{\mathcal{C}}$ has the following explicit form in terms of matrix equations:

$$\mathcal{M}_{\mathcal{C}} = \left[\left\{(A_1, B_1, \dots, B_g, X_1, \dots, X_k) \in \mathrm{GL}_n^{2g}(\mathbb{C}) \times \prod_{j=1}^{k} \mathcal{C}_j \mid \prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{k} X_j = 1\right\} \Big/ \mathrm{GL}_n(\mathbb{C})\right]. \tag{2.1.8}$$

In what follows, for a complex stack of finite type $\mathfrak{X}$, we will denote by $H_c^*(\mathfrak{X}) \coloneqq H_c^*(\mathfrak{X}, \mathbb{C})$ its compactly supported cohomology with $\mathbb{C}$-coefficients (this is well defined thanks to the work of Laszlo and Olsson [59]).
Recall that each vector space $H_c^i(\mathfrak{X})$ is endowed with the *weight filtration* $W_\bullet^i H_c^i(\mathfrak{X})$, from

which we define the mixed Poincaré series $H_c(\mathfrak{X},q,t)$

$$H_c(\mathfrak{X},q,t) := \sum_{m,i} \dim(W^i_m/W^i_{m-1}) q^{\frac{m}{2}} t^i.$$

The E-series $E(\mathfrak{X},q)$ is the specialization of $H_c(\mathfrak{X},q,t)$ obtained by plugging $t=-1$, the Poincaré series $P_c(\mathfrak{X},t)$ is the specialization of $H_c(\mathfrak{X},q,t)$ obtained by plugging $q=1$ and the pure part $PH_c(\mathfrak{X},q,t)$ is defined as

$$PH_c(\mathfrak{X},q) = \sum_{m} \dim(W^{2m}_m/W^{2m}_{m+1}) q^{\frac{m}{2}}.$$

The geometry and cohomology of character stacks have been extensively studied from different perspectives. Most of the results have been obtained in the case where the k-tuple $\mathcal{C}$ is generic (see Definition 9.1.1).
For a generic k-tuple $\mathcal{C}$, the stack $\mathcal{M}_{\mathcal{C}}$ is smooth and it is a $\mathbb{G}_m$-gerbe over the associated GIT quotient, which we denote by $M_{\mathcal{C}}$. Therefore, the cohomology of $\mathcal{M}_{\mathcal{C}}$ can be easily deduced from that of the character variety $M_{\mathcal{C}}$.
We start by giving a quick review of the known results obtained about the cohomology of generic character stacks and varieties, see §9.1 for more details.

The first results concerning this subject were obtained in the case where $k=1$ and $\mathcal{C}$ is a central conjugacy class. For $n \in \mathbb{N}$ and $d \in \mathbb{Z}$, let $\mathcal{M}_{n,d}$ be the stack $\mathcal{M}_{\mathcal{C}}$ for $k=1$ and $\mathcal{C} = \{e^{\frac{2\pi i d}{n}} I_n\}$ i.e

$$\mathcal{M}_{n,d} = \left[\left\{ (A_1,B_1,\dots,A_g,B_g) \in \mathrm{GL}_n^{2g}(\mathbb{C}) \mid \prod_{i=1}^{g} [A_i,B_i] = e^{\frac{2\pi i d}{n}} I_n \right\} \Big/ \mathrm{GL}_n(\mathbb{C}) \right].$$

The orbit $\mathcal{C} = \{e^{\frac{2\pi i d}{n}}\}$ is generic if and only if $(n,d)=1$.

Hitchin [48] computed the Poincaré polynomial $P_c(M_{n,d},t)$ in the generic case for $n=2$, using non abelian Hodge correspondence and Morse theory on the moduli space of Higgs bundles. Gothen [40] extended his result for $n=3$.
Their approach was later extended to compute the Poincaré polynomial $P_c(M_{\mathcal{C}},t)$ in the case where $n=2$, any k and any generic k-tuple $\mathcal{C}$ by Boden, Yogokawa [12] and where $n=3$, any k and any generic k-tuple $\mathcal{C}$ by García-Prada, Gothen and Munoz [37].
However, Morse theoretic techniques do not give information about weight filtration on the character variety and were hard to generalize to any n.

Hausel and Rodriguez-Villegas [44] were the first to obtain a general result about the weight filtration for any n.
The authors computed the E-series $E(\mathcal{M}_{n,d},q)$ of the stacks $\mathcal{M}_{n,d}$ for any coprime n,d, by counting points over finite fields and proposed a conjectural formula for the mixed Poincaré series $H_c(\mathcal{M}_{n,d},q,t)$.
Schiffmann [84] found an expression for the Poincaré series $P_c(\mathcal{M}_{n,d},t)$ in the generic case and

Mellit [73] later checked that Schiffmann's formula agrees with the specialization of Hausel and Rodriguez-Villegas conjecture at $q = 1$.

Hausel, Letellier and Rodriguez-Villegas afterwards generalized the results of [44] and computed [45, Theorem 1.2.3] the E-series $E(\mathcal{M}_{\mathcal{C}}, q)$ of the stacks $\mathcal{M}_{\mathcal{C}}$ for any generic k-tuple $\mathcal{C}$. We quickly explain in more detail their results, as it is the fundamental starting point for the development of this work.

The authors [45, Theorem 1.2.3] showed that there is an equality

$$E(\mathcal{M}_{\mathcal{C}}, q) = \frac{q^{\frac{d_{\boldsymbol{\mu}}}{2}}}{q-1} \mathbb{H}_{\boldsymbol{\mu}} \left(\sqrt{q}, \frac{1}{\sqrt{q}} \right) \tag{2.1.9}$$

where $2d_{\boldsymbol{\mu}} = \dim(\mathcal{M}_{\mathcal{C}}) + 1$ and $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k)$ is the multipartition given by the multiplicities of the eigenvalues of $\mathcal{C}_1, \dots, \mathcal{C}_k$ respectively.

In the same paper, the authors [45, Conjecture 1.2.1] proposed the following conjectural formula for the mixed Poincaré series $H_c(\mathcal{M}_{\mathcal{C}}, q, t)$, which generalizes Hausel and Rodriguez-Villegas conjecture stated in [44] and naturally deformes Identity (2.1.9):

$$H_c(\mathcal{M}_{\mathcal{C}}, q, t) = \frac{(qt^2)^{\frac{d_{\mu}}{2}}}{qt^2 - 1} \mathbb{H}_{\boldsymbol{\mu}} \left(t\sqrt{q}, -\frac{1}{\sqrt{q}}, \right). \tag{2.1.10}$$

Mellit [72, Theorem 7.12] later computed the Poincaré series $P_c(\mathcal{M}_{\mathcal{C}}, t)$ using non-abelian Hodge correspondence. His formula matches with the specialization at $q = 1$ of the conjectural formula (2.1.10) for the mixed Poincaré series.

Notice that, as mentioned before, Formula (2.1.2), Formula (2.1.9) and Conjecture (2.1.10) closely relate the understanding of cohomology of generic character stacks with the understanding of generic multiplicities for representations of $\mathrm{GL}_n(\mathbb{F}_q)$ and star-shaped quivers.

For instance, Conjecture (2.1.10) implies that we have

$$PH_c(\mathcal{M}_{\mathcal{C}}, q) = \frac{q^{\frac{d_{\boldsymbol{\mu}}}{2}}}{q-1} \langle \Lambda \otimes R_{L_1}^G(\delta_1) \otimes \cdots \otimes R_{L_k}^G(\delta_k), 1 \rangle \tag{2.1.11}$$

for any generic k-tuple $(R_{L_1}^G(\delta_1), \dots, R_{L_k}^G(\delta_k))$ such that the associated multipartition is $\boldsymbol{\mu}$. Moreover, when the dimension vector $\alpha_{\mathcal{X}}$ associated to the k-tuple $\mathcal{X} = (R_{L_1}^G(\delta_1), \dots, R_{L_k}^G(\delta_k))$ is indivisible, Conjecture (2.1.10) implies that we have

$$PH_c(\mathcal{M}_{\mathcal{C}}, q) = \frac{P_c(\mathcal{Q}_{\mathcal{O}}, q)}{q-1} \tag{2.1.12}$$

i.e that the pure part of the mixed Poincaré series of $\mathcal{M}_{\mathcal{C}}$ is equal (up to a $q-1$ factor) to the compactly supported Poincaré polynomial of its additive counterpart $\mathcal{Q}_{\mathcal{O}}$. This is usually known as the "purity conjecture".

While in the generic case the works just cited give a fairly complete description of the cohomology of character stacks, the cohomology of the stacks $\mathcal{M}_{\mathcal{C}}$ for non-generic k-tuples $\mathcal{C}$ has been little studied until recently.
The most explicit and general results have mostly been obtained in the case of the stacks $\mathcal{M}_{n,d}$.
Hausel and Rodriguez-Villegas were the first to obtain a general result in this direction. The authors [44, Theorem 3.8.1] expressed the E-series for the stacks $\mathcal{M}_{n,0}$ in terms of the E-series for the generic character stacks $\mathcal{M}_{n,1}$ by the following formula:

$$\operatorname{Exp}\left(\sum_{n\in\mathbb{N}} \frac{E(\mathcal{M}_{n,1},q)}{q^{n^2(2g-2)}}T^n\right) = \sum_{n\in\mathbb{N}} \frac{E(\mathcal{M}_{n,0},q)}{q^{n^2(2g-2)}}T^n \tag{2.1.13}$$

where Exp is the plethystic exponential in the ring of formal power series $\mathbb{Q}(q)[[T]]$ (see §3.7 for details about plethystic operations). The authors' result is obtained by counting points over finite fields.

Fix now $r \in \mathbb{Q}$. Recently, Davison, Hennercart and Schelegel-Mejia [24, Theorem 14.3, Corollary 14.7] proved the following formula expressing the compactly supported Poincaré series of $\mathcal{M}_{n,d}$ for any n, d, in terms of the Poincaré series for the generic character stacks $\mathcal{M}_{n,1}$:

$$\sum_{\substack{(n,d)\in\mathbb{N}_{>0}\times\mathbb{Z}\\ d=rn}} \frac{P_c(\mathcal{M}_{n,d},-t)}{t^{n^2(2g-2)}}z^n w^d = \operatorname{Exp}\left(\sum_{\substack{(n,d)\in\mathbb{N}_{>0}\times\mathbb{Z}\\ d=rn}} \frac{P_c(\mathcal{M}_{n,1},-t)}{t^{n^2(2g-2)}}z^n w^d\right) \tag{2.1.14}$$

and formulated a similar conjecture for the mixed Poincaré series of $H_c(\mathcal{M}_{n,d},q,t)$ for any n, d (see the discussion after [24, Theorem 14.10]).
They obtained this formula by relating the cohomology of a character stack with the cohomology of the so-called BPS sheaves. The latter are certain perverse sheaves defined on character varieties and their cohomology is well understood for the stacks $\mathcal{M}_{n,d}$. More precisely the non-abelian Hodge correspondence for stacks, proved in [24], and the recent work of Koseki and Kinjo [55] about BPS sheaves for the moduli stack of Higgs bundles, give a way to compute the cohomology of BPS sheaves for a stack $\mathcal{M}_{n,d}$.
Notice moreover that, since the authors use non abelian Hodge correspondence which does not preserve weight filtration on cohomology, their method does not allow to prove an analogous formula for the E-series or the mixed Poincaré series of $\mathcal{M}_{n,d}$.
Finally, the cohomology of BPS sheaves for character stacks $\mathcal{M}_{\mathcal{C}}$ is not understood for an arbitrary $\mathcal{C}$ and so a generalization of Formula (2.1.14) for an arbitrary $\mathcal{C}$ is still unproved.

One of the main aim of this thesis is to contribute to the understanding of the cohomology of character stacks $\mathcal{M}_{\mathcal{C}}$ for not-necessarily generic k-tuples, for any k and $\mathcal{C}$.

2.1.3 Character stacks for non-orientable surfaces

Another generalization of the results of [45] that will interest us in this thesis is the study of character stacks for real non-orientable surfaces rather than Riemann surfaces.

Our point of view on real geometry is that introduced by Atiyah [3], i.e a non-orientable surface in the following will be a pair (Σ, σ), where Σ is a Riemann surface and $\sigma : \Sigma \to \Sigma$ is an anti-holomorphic involution such that $\Sigma^{\sigma} = \varnothing$.
Notice that in this case indeed, the quotient $S = \Sigma/\langle \sigma \rangle$ is a non-orientable real surface. Denote by $p : \Sigma \to S$ the quotient map.

Fix now a subset $E = \{y_1, \dots, y_k\} \subseteq S$ and a k-tuple $\mathcal{C}$ of semisimple conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$. Notice that since the action of σ is free, the involution σ defines a morphism $\epsilon : \pi_1(S \setminus E) \to \mathbb{Z}/(2)$ with kernel $p_*(\pi_1(\Sigma/p^{-1}(E)))$
Consider now the group $\mathrm{GL}_n(\mathbb{C})^+ := \mathrm{GL}_n(\mathbb{C}) \rtimes_{\theta} \mathbb{Z}/(2)$, where $\theta : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ is the Cartan involution $\theta(M) = (M^t)^{-1}$ and denote by $\pi : \mathrm{GL}_n(\mathbb{C})^+ \to \mathbb{Z}/(2)$ the associated projection.
The associated character stack $\mathcal{M}^{\epsilon}_{\mathcal{C}}$ is defined as

$$\mathcal{M}^{\epsilon}_{\mathcal{C}} = \left[\left\{\rho : \pi_1(S \setminus E) \to \mathrm{GL}_n(\mathbb{C})^+ \mid \rho(z_i) \in \mathcal{C}_i \text{ and } \pi(\rho(g)) = \epsilon(g)\right\} / \mathrm{GL}_n(\mathbb{C})\right]$$

where each z_i is a loop around the point y_i. The stack $\mathcal{M}^{\epsilon}_{\mathcal{C}}$ has the following explicit form in terms of matrix equation:

$$\mathcal{M}^{\epsilon}_{\mathcal{C}} = \left[\left\{(D_1, \dots, D_r, Z_1, \dots, Z_k) \in \mathrm{GL}^r_n(\mathbb{C}) \times \prod_{j=1}^{k} \mathcal{C}_j \mid D_1\theta(D_1) \cdots D_r\theta(D_r) Z_1 \cdots Z_k = 1\right\} / \mathrm{GL}_n(\mathbb{C})\right] \tag{2.1.15}$$

where $r = g+1$. We denote by $M^{\epsilon}_{\mathcal{C}}$ the associated GIT quotient. When $k = 1$ and $\mathcal{C} = \{e^{\frac{\pi i d}{n}} I_n\}$, we denote the stack $\mathcal{M}^{\epsilon}_{\mathcal{C}}$ by $\mathcal{M}^{\epsilon}_{n,d}$.
Similar definitions can be given when σ has fixed points, using the *orbifold fundamental group* of the quotient $\Sigma/\langle \sigma \rangle$, see for example [11].

The stacks $\mathcal{M}^{\epsilon}_{\mathcal{C}}$ are deeply related to the so-called *branes* inside the moduli spaces of Higgs bundles. The computation of the cohomology of branes is a key part in understanding mirror symmetry for the Hitchin system.
References about the subject can be found for example in [6],[7],[11],[9].

Few results have been shown in the literature concerning the cohomology of the stacks $\mathcal{M}^{\epsilon}_{\mathcal{C}}$. Recently, Letellier and Rodriguez-Villegas [65, Theorem 1.4] computed the E-series $E(\mathcal{M}^{\epsilon}_{\mathcal{C}}, q)$ when $\mathcal{C}$ is generic, by counting points over finite fields.

Baird and Wong [4] computed the E-polynomial of analogous varieties $M^{\epsilon}_{n,d}$ when the anti-holomorphic involution σ has fixed points. Their formulas are quite different from the ones of [65].

In this thesis, we will focus on the case of the stacks $\mathcal{M}^{\epsilon}_{n,d}$ when $(n, d) = 1$. In this case, the stack $\mathcal{M}^{\epsilon}_{n,d}$ is a μ_2-gerbe over the character variety $M^{\epsilon}_{n,d}$. The latter variety is deeply related to the moduli space of real and quaternionic Higgs bundles of rank n and degree d over Σ.

2.2 Overview of the thesis

We give here a quick summary of our main results, which we will explain more precisely in the next Section.

2.2.1 Multiplicities

With regard to multiplicities, we study multiplicities $\langle \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle$ for not necessarily generic k-tuples of irreducible characters $(\mathcal{X}_1, \dots, \mathcal{X}_k)$.
In this thesis, we give a formula for $\langle R_{L_1}^G(\gamma_1) \otimes \cdots \otimes R_{L_k}^G(\gamma_k), 1\rangle$ for any choice of $\gamma_1, \dots, \gamma_k$ (not necessarily generic), in terms of Kac polynomials of the quiver Q introduced before.
This formula is obtained by giving a quiver theoretic interpretation of the multiplicity $\langle R_{L_1}^G(\gamma_1) \otimes \cdots \otimes R_{L_k}^G(\gamma_k), 1\rangle$ for any k-tuple $(R_{L_1}^G(\gamma_1), \dots, R_{L_k}^G(\gamma_k))$.
As a consequence of our result, we show that $\langle R_{L_1}^G(\gamma_1) \otimes \cdots \otimes R_{L_k}^G(\gamma_k), 1\rangle$ is a polynomial in q with non-negative coefficients and show a criterion for its non-vanishing in terms of the root system of Q. These results are contained in [85].

2.2.2 Cohomology of character stacks

With regard to character stacks for Riemann surfaces, we study cohomology of character stacks $\mathcal{M}_{\mathcal{C}}$ for k-tuples which are not necessarily generic. One of the main results of this thesis is a generalization of Formula (2.1.14) to arbitrary $\mathcal{C}$ for the E-series $E(\mathcal{M}_{\mathcal{C}}, q)$ instead of the Poincaré series $P_c(\mathcal{M}_{\mathcal{C}}, t)$.
As a result we get an explicit formula for $E(\mathcal{M}_{\mathcal{C}}, q)$ for any k-tuple $\mathcal{C}$, see Theorem 2.3.10 below.
We also give a conjectural formula (see Conjecture 2.3.12) for the mixed Poincaré series $H_c(\mathcal{M}_{\mathcal{C}}, q, t)$, which we verify in the case of $\Sigma = \mathbb{P}^1_{\mathbb{C}}$, $k = 4$ and a certain family of non-generic quadruples. These results are part of [86].
Conjecture 2.3.12 for the stacks $\mathcal{M}_{n,d}$ has already appeared in [24], see the discussion in *loc. cit* after Theorem 14.10. Let us notice that our approach is very different from that of [24] as we do not use non-abelian Hodge theory nor BPS sheaves.

2.2.3 Non-orientable surfaces

With regard to generic character stacks for non-orientable compact real surfaces, we give an explicit description of the stacks $\mathcal{M}^{\epsilon}_{n,d}$ when $(n, d) = 1$ and $r = 2$, i.e for a (real) elliptic curve. This description gives a counterexample to a formula suggested by [65] for the mixed Poincaré series of the stacks $\mathcal{M}^{\epsilon}_{\mathcal{C}}$. The description of this counterexample is the main result of the article [87].

2.3 Main results

2.3.1 Main results about multiplicities

Fix Levi subgroups $L_1, \dots, L_k$ of $\mathrm{GL}_n(\mathbb{F}_q)$ as in §2.1.1. To study the non-generic case, we will start by defining a stratification both on the set of k-tuples of semisimple split characters

$\mathcal{X} = (R^G_{L_1}(\gamma_1), \dots, R^G_{L_k}(\gamma_k))$ and on the set of representations of Q of dimension $\alpha_{\mathcal{X}}$. This stratification will be indexed by subsets $V \subseteq \mathbb{N}^I$.
The level of the stratification associated to $V = \{\alpha_{\mathcal{X}}\}$ will correspond to the case of generic k-tuples/absolutely indecomposable representations respectively.

Consider more generally any finite quiver $\Gamma = (J, \Omega)$. To a representation M of Γ, we associate the following subset $\mathcal{H}_M \subseteq \mathbb{N}^I$. Given the decomposition into indecomposable components

$$M \otimes_K \overline{K} \cong M_1^{n_1} \oplus \cdots \oplus M_h^{n_h},$$

we define

$$\mathcal{H}_M := \{\dim(M_1), \dots, \dim(M_h)\}.$$

For any $V \subseteq \mathbb{N}^J$, we give the following definition of the representations of Γ of level V (see Definition 6.2.2).

Definition 2.3.1. A representation M of Γ is said to be of level V if $\mathcal{H}_M = \{0 < \beta \leq \dim M \mid \beta \in V\}$.

Remark 2.3.2. For $\alpha \in \mathbb{N}^J$, denote by $\mathbb{N}^J_{\leq \alpha} = \{\delta \in \mathbb{N}^J \mid \delta \leq \alpha\}$. Consider a representation M such that $\dim M = \alpha$. Notice that, given $V, V' \subseteq \mathbb{N}^J$ such that $V \cap \mathbb{N}^J_\alpha = V' \cap \mathbb{N}^J_\alpha$, we have that M is of level V if and only if it is of level V'.

Notice that, for $\beta \in \mathbb{N}^J$, a representation of dimension β is of level $\{\beta\}$ if and only if is absolutely indecomposable . In particular, the number of isomorphism classes of representations of level $\{\beta\}$ and dimension β over finite fields is counted by the Kac polynomial $a_{\Gamma,\beta}(t)$.

However, for a general $V \subseteq \mathbb{N}^J$, the counting of the isomorphism classes of the representations of Γ of prescribed dimension and of level V over $\mathbb{F}_q$ does not seem to give an interesting generalization of Kac polynomials.
In this direction, to obtain such a generalization, we introduce the following definition of a representation of level **at most** V (see Definition 6.2.4).

Definition 2.3.3. For a subset $V \subseteq \mathbb{N}^J$, a representation M is said to be of level at most V if it is of level V' for some $V' \subseteq V$, i.e if and only if $\mathcal{H}_M \subseteq V$.

For any $\beta \in \mathbb{N}^J$ and any $V \subseteq \mathbb{N}^J$, we show that the number of isomorphism classes of representations of Γ of level at most V of dimension β over $\mathbb{F}_q$ is equal to the evaluation of a polynomial $M_{\Gamma,\beta,V}(t) \in \mathbb{Z}[t]$ at $t = q$.
Moreover, we prove a formula for the generating function of the polynomials $M_{\Gamma,\beta,V}(t)$ of the following type (see Lemma 6.2.2):

$$\operatorname{Exp}\left(\sum_{\gamma \in V} a_{\Gamma,\gamma}(t) y^\gamma\right) = \sum_{\beta \in \mathbb{N}^I} M_{\Gamma,\beta,V}(t) y^\beta. \tag{2.3.1}$$

where Exp is the plethystic exponential.

Remark 2.3.4. For $V=\mathbb{N}^J$, Formula (2.3.1) gives

$$\operatorname{Exp}\left(\sum_{\gamma\in\mathbb{N}^J} a_{\Gamma,\gamma}(t)y^\gamma\right)=\sum_{\beta\in\mathbb{N}^I} M_{\Gamma,\beta}(t)y^\beta \tag{2.3.2}$$

where $M_{\Gamma,\beta}(t)$ is the polynomial counting the number of isomorphism classes of representations of Γ of $\dim=\beta$ over $\mathbb{F}_q$. This was already proved in [50].

Consider now the quiver Q and the dimension vector $\alpha_{\mathcal{X}}$ introduced above. Let $(\mathbb{N}^I)^*\subseteq\mathbb{N}^I$ be the subset of vectors with non-increasing coordinates along the legs.
In §8.2, to any k-tuple $\mathcal{X}=(R^G_{L_1}(\delta_1),\dots,R^G_{L_k}(\delta_k))$ we associate an element $\sigma_{\mathcal{X}}\in\operatorname{Hom}(\mathbb{F}_q^*,\mathbb{C}^*)^I$ (see §8.2.1). Let $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}\subseteq(\mathbb{N}^I)^*$ be the subset defined by

$$\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}\coloneqq\{0<\beta\leq\alpha_{\mathcal{X}}\mid\sigma_{\mathcal{X}}^\beta=1\}$$

where $\sigma_{\mathcal{X}}^\beta\coloneqq\prod_{i\in I}((\sigma_{\mathcal{X}})_i)^{\beta_i}$. For any $V\subseteq(\mathbb{N}^I)^*$, we give the following definition of a k-tuple $(R^G_{L_i}(\delta_i))_{i=1}^k$ of level V (see Definition 8.2.1).

Definition 2.3.5. A k-tuple $\mathcal{X}=(R^G_{L_1}(\delta_1),\dots,R^G_{L_k}(\delta_k))$ is said of level V if $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}=\{0<\beta\leq\alpha\mid\beta\in V\}$.

For $V=\{\alpha_{\mathcal{X}}\}$, we show that if a k-tuple $(R^G_{L_1}(\delta_1),\dots,R^G_{L_k}(\delta_k))$ is of level $\{\alpha_{\mathcal{X}}\}$ it is generic. Conversely, if the k-tuple $\mathcal{X}$ is generic, there are no elements $\delta,\epsilon\in\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha}\setminus\{\alpha_{\mathcal{X}}\}$ such that $\delta+\epsilon=\alpha$, see Lemma 8.2.4.

The main result of this paper extends Formula (2.1.4) by relating the multiplicity for k-tuples of level V and representations of level at most V in the following way.

Theorem 2.3.6. *Let $V\subseteq(\mathbb{N}^I)^*$ and $\mathcal{X}=(R^G_{L_1}(\delta_1),\dots,R^G_{L_k}(\delta_k))$ be a k-tuple of level V. The following equality holds:*

$$\langle\Lambda\otimes R^G_{L_1}(\delta_1)\otimes\cdots\otimes R^G_{L_k}(\delta_k),1\rangle=M_{Q,\alpha_{\mathcal{X}},V}(q) \tag{2.3.3}$$

From Theorem 2.3.6 and Formula (2.3.1), in Proposition 8.2.12 we obtain the following criterion for the non-vanishing of the multiplicity $\langle\Lambda\otimes R^G_{L_1}(\delta_1)\otimes\cdots\otimes R^G_{L_k}(\delta_k),1\rangle$, generalizing the criterion for generic k-tuples of [46, Corollary 1.4.2] .

Proposition 2.3.7. *For a k-tuple $(R^G_{L_1}(\delta_1),\dots,R^G_{L_k}(\delta_k))$ of level V, the multiplicity*

$$\left\langle\Lambda\otimes R^G_{L_1}(\delta_1)\otimes\cdots\otimes R^G_{L_k}(\delta_k),1\right\rangle$$

is non-zero if and only there exist

- $\beta_1,\dots,\beta_r\in\Phi^+(Q)\cap V$
- $m_1,\dots,m_r\in\mathbb{N}$

such that $m_1\beta_1+\cdots+m_r\beta_r=\alpha_{\mathcal{X}}$

2.3.2 Main results about non-generic character stacks

An important tool to formulate and prove the main results of this thesis concerning non-generic character stacks is the construnction of character stacks as multiplicative quiver stacks, as first introduced by Crawley-Boevey and Shaw [18],[19], which we quickly recall here (see §7.2 for more details).
Notice that this construction is not needed for studying generic character stacks and does not appear for example in the articles [45], [72]. However, it is a key point in our paper, as it allows to distinguish between different levels of non-genericity for non-generic character stacks.

Let $s_1, \dots, s_k \in \mathbb{N}$ be such that, for each $i = 1, \dots, k$, the conjugacy class $\mathcal{C}_i$ has $s_i + 1$ distinct eigenvalues $\gamma_{i,0}, \dots, \gamma_{i,s_i}$ with multiplicities $m_{i,0}, \dots, m_{i,s_i}$ respectively. Let $Q = (I, \Omega)$ be the star-shaped quiver with g loops on the central vertex and k legs of length $s_1, \dots, s_k$.

Recall that for any $\beta \in \mathbb{N}^I$, there is a representation variety $R(\overline{Q}, \beta)^{\circ,*}$ and a multiplicative moment map

$$\Phi^*_\beta : R(\overline{Q}, \beta)^{\circ,*} \to \mathrm{GL}_\beta(\mathbb{C}) := \prod_{i \in I} \mathrm{GL}_{\beta_i}(\mathbb{C}).$$

For any $s \in (\mathbb{C}^*)^I$, we denote by s the central element $s := (s_i I_{\beta_i})_{i \in I} \in \mathrm{GL}_\beta$. The multiplicative quiver stack with parameters β, s is the quotient stack

$$\mathcal{M}^*_{s,\beta} := [(\Phi^*_\beta)^{-1}(s)/\,\mathrm{GL}_\beta].$$

Consider now the dimension vector $\alpha_{\mathcal{C}} \in \mathbb{N}^I$ defined as

$$(\alpha_{\mathcal{C}})_{[i,j]} = \sum_{h=j}^{s_i} m_{i,h}$$

for every $j = 0, \dots, s_i$, where we are identifiying $[i, 0] = 0$ for each $i = 1, \dots, k$. Notice that $(\alpha_{\mathcal{C}})_0 = n$. Let moreover $\gamma_{\mathcal{C}} \in (\mathbb{C}^*)^I$ be defined as follows

$$(\gamma_{\mathcal{C}})_{[i,j]} = \begin{cases} \prod_{i=1}^{k} \gamma_{i,0}^{-1} \text{ if } j = 0 \\ \gamma_{i,j}^{-1}\gamma_{i,j-1} \text{ otherwise} \end{cases} .$$

The results of Crawley-Boevey [18, Proposition 2] impliy that, for the elements $\alpha_{\mathcal{C}}, \gamma_{\mathcal{C}}$, there is an isomorphism of stacks

$$\mathcal{M}_{\mathcal{C}} \cong \mathcal{M}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}.$$

Let now $(\mathbb{N}^I)^* \subseteq \mathbb{N}^I$ be the subset of vectors with non-increasing coordinates along the legs and denote by $\mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}} \subseteq (\mathbb{N}^I)^*$ the subset defined as

$$\mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}} = \{\delta \in (\mathbb{N}^I)^* \mid \gamma^\delta_{\mathcal{C}} = 1 \;\text{ and } \delta \leq \alpha_{\mathcal{C}}\}$$

where $\gamma_{\mathcal{C}}^{\delta}=\prod_{i\in I}((\gamma_{\mathcal{C}})_i)^{\delta_i}$.

Example 2.3.8. It can be checked that a k-tuple $\mathcal{C}$ is *generic* if $\mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}=\{\alpha_{\mathcal{C}}\}$, see Lemma 9.1.4 for more details.

The subsets $\mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}$ define a natural stratification on the set of k-tuples $\mathcal{C}$ and so of character stacks $\mathcal{M}_{\mathcal{C}}$.
The introduction of this stratification is one of the key ingredients to study the cohomology of $\mathcal{M}_{\mathcal{C}}$ in the non-generic case. Notice that altough not explicitly defined, the subsets $\mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}$ appear implicitly in [24].

For any $\beta\in(\mathbb{N}^I)^*$ and for any $j=1,\dots,k$, the integers $(\beta_{[j,0]}-\beta_{[j,1]},\dots,\beta_{[j,s_j-1]}-\beta_{[j,s_j]},\beta_{[j,s_j]})$ up to reordering form a partition $\mu^j_\beta\in\mathcal{P}$. Denote by $\boldsymbol{\mu}_\beta\in\mathcal{P}^k$ the multipartition $\boldsymbol{\mu}_\beta=(\mu^1_\beta,\dots,\mu^k_\beta)$ and by $\mathbb{H}_\beta(z,w)$ the function $\mathbb{H}_{\boldsymbol{\mu}_\beta}(z,w)$.

Remark 2.3.9. Notice that for a k-tuple $\mathcal{C}$ of semisimple conjugacy classes, the multipartition $\boldsymbol{\mu}_{\alpha_{\mathcal{C}}}\in\mathcal{P}^k$ is the multipartition given by the multiplicities of the orbits $\mathcal{C}_1,\dots,\mathcal{C}_k$ respectively. Moreover, it can be checked that

$$\dim(\mathcal{M}_{\mathcal{C}})=2(\alpha_{\mathcal{C}},\alpha_{\mathcal{C}})+1,$$

where $(,)$ is the Euler form of Q. The result [45, Theorem 1.2.3] of Hausel, Letellier, Rodriguez-Villegas for a generic k-tuple $\mathcal{C}$ can thus be rewritten as follows:

$$\frac{E(\mathcal{M}_{\mathcal{C}},q)}{q^{(\alpha_{\mathcal{C}},\alpha_{\mathcal{C}})}}=\frac{q\mathbb{H}_{\alpha_{\mathcal{C}}}\left(\sqrt{q},\frac{1}{\sqrt{q}}\right)}{q-1}. \tag{2.3.4}$$

The main result about character stacks of this paper (see Theorem 9.3.2) is the following :

Theorem 2.3.10. *For any k-tuple of semisimple conjugacy classes $\mathcal{C}$, we have:*

$$\mathrm{Coeff}_{\alpha_{\mathcal{C}}}\left(\mathrm{Exp}\left(\sum_{\beta\in\mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}}\frac{q\mathbb{H}_\beta\left(\sqrt{q},\frac{1}{\sqrt{q}}\right)}{q-1}y^\beta\right)\right)=\frac{E(\mathcal{M}_{\mathcal{C}},q)}{q^{(\alpha_{\mathcal{C}},\alpha_{\mathcal{C}})}}. \tag{2.3.5}$$

We compute the E-series of the complex character stacks $\mathcal{M}_{\mathcal{C}}$ through the approach introduced in [44],[45],[65], i.e by reduction over finite fields and point counting. Namely, recall that if there exists a rational function $Q(t)\in\mathbb{Q}(t)$ such that, for any $\mathbb{F}_q$-stack $\mathcal{M}_{\mathcal{C},\mathbb{F}_q}$ obtained from $\mathcal{M}_{\mathcal{C}}$ by base change and any m, it holds

$$\#\mathcal{M}_{\mathcal{C},\mathbb{F}_q}(\mathbb{F}_{q^m})=Q(q^m),$$

we have an equality

$$E(\mathcal{M}_{\mathcal{C}},q)=Q(q),$$

see §3.4.1 for more details.
However, the way we count rational points of character stacks in this paper is quite different from that of [45]. The description of the rational functions $Q(t)$ for non-generic character stacks is given through the results of Chapter §5.8 and §7.3, where we show how to compute the rational points of a multiplicative quiver stack for a star-shaped quiver over $\mathbb{F}_q$.
Notice that in the articles [46],[45], the authors did not need to introduce multiplicative quiver stacks to compute $\mathbb{F}_q$-points of generic character stacks.

The results of §5.8 and §7.3 about the rational functions $Q(t)$ will be obtained as a consequence of one of the main technical results of this thesis which is Theorem 4.5.2. The latter theorem is very general and works for certain families of rational functions called *Log compatible*.
Therefore, to prove Theorem 2.3.10 we will have to prove that the rational functions involved in it satisfy this Log compatibility property.
The proof of Theorem 4.5.2 will be one of the main technical points of the first part of the thesis. We will have to use combinatorial objects different from the ones used for the generic case in [46],[45]. The definition of these objects and the study of their properties do not seem to have been given before in the literature and constitute the main topic of the sections §4.3,§4.5.

Remark 2.3.11. Theorem 4.5.2 about Log compatible functions can be used to give another proof of Theorem 2.3.6 about multiplicities for k-tuples of Harisha-Chandra characters, as shown in section §8.3. In the case of multiplicities, we could avoid the technical result by interpreting multiplicities in terms of the counting of isomorphism classes in the category of representations of a quiver.
For the E-series of character stacks, we lack such a categorical interpretation and we don't know an alternative way to Theorem 4.5.2.

Hausel, Letellier, Rodriguez Villegas conjectural formula (2.1.10) for the mixed Poincaré series of character stacks for generic k-tuples and Theorem 2.3.10 suggest the following conjecture for the mixed Poincaré series of character stacks:

Conjecture 2.3.12. *For any k-tuple of semisimple orbits $\mathcal{C}$, we have:*

$$\mathrm{Coeff}_{\alpha_{\mathcal{C}}}\left(\mathrm{Exp}\left(\sum_{\beta\in\mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}}\frac{(qt^2)\mathbb{H}_\beta\left(t\sqrt{q},\frac{1}{\sqrt{q}}\right)}{qt^2-1}y^\beta\right)\right)=\frac{H_c(\mathcal{M}_{\mathcal{C}},q,t)}{(qt^2)^{(\alpha_{\mathcal{C}},\alpha_{\mathcal{C}})}}. \tag{2.3.6}$$

In §9.4, we verify that Conjecture 2.3.12 holds in the case where $\Sigma=\mathbb{P}^1_{\mathbb{C}}$, $|D|=4$ and the following family of non-generic quadruples.
Pick $\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{C}^*\setminus\{1,-1\}$ and denote by $\mathcal{C}_j$ the conjugacy class of the diagonal matrix

$$\begin{pmatrix}\lambda_j & 0\\ 0 & \lambda_j^{-1}\end{pmatrix}.$$

Assume moreover that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ have the following property. Given $\epsilon_1, \ldots, \epsilon_4 \in \{1, -1\}$ such that $\lambda_1^{\epsilon_1} \cdots \lambda_4^{\epsilon_4} = 1$, then either $\epsilon_1 = \cdots = \epsilon_4 = 1$ or $\epsilon_1 = \cdots = \epsilon_4 = -1$.

2.3.3 A common approach: non-generic to generic

One of the interesting aspects of our results about non-generic character stacks and non-generic multiplicities is that in both cases E-series for any k-tuple $\mathcal{C}$ of semisimple conjugacy classes and multiplicities for any k-tuple $(R_{L_1}^G(\gamma_1), \ldots, R_{L_k}^G(\gamma_k))$ are expressed in terms of the E-series (respectively multiplicities) for generic k-tuples.
Similar type of results, relating non-generic to generic, have already appeared elsewhere, see for example the results of [64] recalled above for unipotent characters or the discussion at the end of §2.1.2.
We can also cite Davison's work [22, Theorem B] which recently showed that the cohomology of non-generic quiver stacks can be expressed in terms of Kac polynomials and Letellier's work [61] where he computed the E-series for character stacks with unipotent local monodromies in terms of the generic case.

2.4 Main results about character stacks for non-orientable surfaces

In [65], the authors obtained a combinatorial formula for the E-series $E(\mathcal{M}_{\mathcal{C}}^{\epsilon}, q)$ for any generic k-tuple $\mathcal{C}$ of semisimple conjugacy classes. Surprinsingly, the formulas found by the authors for the E-series $E(\mathcal{M}_{\mathcal{C}}^{\epsilon}, q)$ strongly resemble the ones computing E-series for character stacks for Riemann surfaces found in [45].
For instance, for $r = 2h$ we have an equality $E(\mathcal{M}_{\mathcal{C}}^{\epsilon}, q) = E(\mathcal{M}_{\mathcal{C}}, q)$ where $\mathcal{M}_{\mathcal{C}}$ is associated to a Riemann surface of genus h, see [65, Remark 1.5].
The authors' [65, Theorem 4.8] verified that a completely analogous formula to that of Conjecture 2.3.12 holds for $\mathcal{M}_{\mathcal{C}}^{\epsilon}$ in the case of $r = 1$ and $k = 1$. Therefore, it would have been natural to expect a similar formula to hold for any r. We give an explicit description of some of these stacks in the case $r = 2$, giving a counterexample to the expected formula. More precisely, we show the following.

Theorem 2.4.1. *Put $r = 2$, $k = 1$ and consider $\mathcal{M}_{n,d}^{\epsilon}$ for $(n, d) = 1$. Then $\mathcal{M}_{\mathcal{C}}^{\epsilon}$ is a μ_2-gerbe over $\mathbb{C}^*$. In particular, its mixed Poincaré series is*

$$H_c(\mathcal{M}_{\mathcal{C}}^{\epsilon}, q, t) = qt^2 + t.$$

To prove Theorem 2.4.1 we need some results of independent interest concerning the geometry of the spaces $\mathcal{M}_{n,d}^{\epsilon}$.
To summarize these results, let $M_{n,d}$ be the character variety associated to the Riemann surface Σ (of genus $r - 1$) and $\sigma : \Sigma \to \Sigma$ be the involution which sends a representation $\tilde{\rho} \in M_{n,d}$ to $\sigma(\tilde{\rho}) = \theta(\tilde{\rho})(\sigma_*)$ (for more details and a definition of σ_* see chapter §10).
In section §10.4, we show the following Theorem:

Theorem 2.4.2. *If r is odd, the fixed point locus $M_{n,d}^{\sigma}$ is isomorphic to $M_{n,d}^{\epsilon}$. If r is even, there is an open-closed decomposition $M_{n,d}^{\sigma} = M_{n,d}^{\sigma,+} \bigsqcup M_{n,d}^{\sigma,-}$ such that $M_{n,d}^{\sigma,+} \cong M_{n,d}^{\sigma,-} \cong M_{n,d}^{\epsilon}$.*

Theorem 2.4.2 (and the others in §10.4) are probably known to the experts but we could not locate a reference in the literature. We review them here for the sake of completeness.

Theorem 2.4.1 is obtained by studying the action of σ_* on $M_{n,d}$ for elliptic curves and finding an explicit isomorphism

$$M_{n,d}^{\sigma,+} \cong \mathbb{C}^*.$$

3 Geometric and combinatoric background

This chapter recalls the geometric and combinatorics tools needed in the rest of the thesis. The base field is $K=\mathbb{C}$ or $K=\mathbb{F}_q$ where $\mathbb{F}_q$ is the finite field with q elements.
In section §3.1 we review some generalities about the complex represention theory of finite groups and fix some notations about the product and the convolution of class functions.
In section §3.2 we introduce some notations for varieties and algebraic groups over $\mathbb{F}_q$ and their twisted Frobenius structures.
In section §3.3 we fix some notations and recall some generalities about algebraic stacks and in particular about quotient stacks $[X/G]$, where X is a variety and G a reductive group.
In section §3.4 we recall the definition and some basic properties of compactly supported cohomology $H^*_c(\mathfrak{X})$ and weight filtration for a stack $\mathfrak{X}$ of finite type over K.
In sections §3.5 and §3.6 we recall some notations and properties of partitions and multitypes. The latter are of one the main combinatorial objects used in the thesis as they parametrize, for example, the conjugacy classes of the groups of the type $\mathrm{GL}_\alpha(\mathbb{F}_q)$ for $\alpha\in\mathbb{N}^I$.
In section §3.7, we review the definition and properties of λ-rings and, in particular, of the *plethystic exponential* Exp and of the *plethystic logarithm* Log. These operations will be the main tool through which to express the non-generic case in terms of the generic ones, both for multiplicities and character stacks.
In section §3.8, we recall the definition of the HLRV kernels $\mathbb{H}_{n,g}(z,w)$, which are rational functions of fundamental importance for the description of generic multiplicities and generic character stacks.

3.1 Finite groups, irreducible characters and convolution

Let H be a finite group. We denote by $\mathcal{C}(H)$ the set of complex valued class functions, i.e the functions $f:H\to\mathbb{C}$ which are constant on the conjugacy classes of H. The constant function equal to 1 is going to be denoted by 1.
For $f,g\in\mathcal{C}(H)$, we denote by $\langle f,g\rangle$ the quantity

$$\langle f,g\rangle=\frac{1}{|H|}\sum_{h\in H}f(h)\overline{g(h)}.$$

Recall that an orthonormal basis of $\mathcal{C}(H)$ is given by the irreducible characters of H. We will denote the set of irreducible characters of H by $H^\vee$.
The vector space $\mathcal{C}(H)$ is endowed with a ring structure $(\mathcal{C}(H),\otimes)$ induced by tensor product of representations, i.e for $f,g\in\mathcal{C}(H)$ we define the class function $f\otimes g\in\mathcal{C}(H)$ as

$$f\otimes g(h)=f(h)g(h).$$

The ring $(\mathcal{C}(H),\otimes)$ is usually called the character ring of H.

Given two class functions $f,g\in\mathcal{C}(H)$ and an irreducible character $\chi\in H^\vee$, the quantity $\langle f\otimes g,\chi\rangle$ is usually called the *multiplicity* of χ in the product $f\otimes g$.

The computation of multiplicities is of fundamental importance for the full understanding of the representation theory of the finite group H.

The vector space $\mathcal{C}(H)$ can be endowed with another ring structure $(\mathcal{C}(H), *)$ given by the *convolution product*.

Given two class functions $f_1, f_2 \in \mathcal{C}(H)$, the convolution $f_1 * f_2$ is the class function defined as

$$f_1 * f_2(g) = \sum_{h \in H} f_1(gh) f_2(h^{-1}).$$

Denote by $\mathrm{Cl}(H)$ the set of conjugacy classes of H. For any $\mathcal{O} \in \mathrm{Cl}(H)$, we denote by $1_{\mathcal{O}} \in \mathcal{C}(H)$ the characteristic function of $\mathcal{O}$. For a central element $\eta \in Z_H$, we denote by 1_η the characteristic function of the conjugacy class $\{\eta\}$.

Notice that, for any central element $\eta \in H$ and any class function f, there is an equality

$$\langle f * 1_\eta, 1_e \rangle = \frac{f(\eta)}{|H|}.$$

Recall now that

$$1_e = \sum_{\chi \in H^\vee} \frac{\chi(e)}{|H|} \chi. \tag{3.1.1}$$

We have therefore

$$\frac{f(\eta)}{|H|} = \sum_{\chi \in H^\vee} \langle f * 1_\eta, \chi \rangle \frac{\chi(1)}{|H|}. \tag{3.1.2}$$

For any two class functions $f_1, f_2 : H \to \mathbb{C}$ and an irreducible character $\chi \in H^\vee$, there is an equality

$$\langle f_1 * f_2, \chi \rangle = \langle f_1, \chi \rangle \langle f_2, \chi \rangle \frac{|H|}{\chi(1)} \tag{3.1.3}$$

(see for example [51, Theorem 2.13]). In particular, from Formula (3.1.2), we deduce the identity:

$$\frac{f(\eta)}{|H|} = \sum_{\chi \in H^\vee} \langle f, \chi \rangle \langle 1_\eta, \chi \rangle = \sum_{\chi \in H^\vee} \langle f, \chi \rangle \frac{\chi(\eta)}{\chi(1)} \frac{\chi(1)}{|H|}. \tag{3.1.4}$$

Remark 3.1.1. If H is abelian, we can build an isomorphism $\psi : (\mathcal{C}(H), \otimes) \to (\mathcal{C}(H), *)$ in the following way.

Since H is abelian, the set of irreducible characters $H^\vee$ is a group and we can find an isomorphism

$$\theta : H \cong H^\vee$$

$$g \to \chi_g.$$

We define $\psi : \mathcal{C}(H) \to \mathcal{C}(H)$ by extending by linearity $\psi(1_g) = \chi_g$. Notice that

$$\psi(1_{g_1} * 1_{g_2}) = \psi(1_{g_1 g_2}) = \chi_{g_1 g_2} = \chi_{g_1} \otimes \chi_{g_2},$$

i.e ψ is an isomorphism $(\mathcal{C}(H), *) \to (\mathcal{C}(H), \otimes)$.

However, notice that in general, i.e if H is not abelian, the rings $(\mathcal{C}(H), \otimes)$ and $(\mathcal{C}(H), *)$ are not isomorphic.

3.2 Varieties over finite fields and twisted Frobenius

Let $q = p^r$ where p is a prime number, $\mathbb{F}_q$ the field with q elements and $\overline{\mathbb{F}}_q$ its algebraic closure. In this paragraph, we review some properties of varieties over $\mathbb{F}_q$. We follow Milne's book [74] and Digne and Michel's book [28]. We denote by $F : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$ the Frobenius morphism $F(x) = x^q$.

Consider a variety X over $\overline{\mathbb{F}}_q$, i.e a reduced and separated scheme of finite type over $\overline{\mathbb{F}}_q$. We say that X is defined over $\mathbb{F}_q$ or equivalently that it admits an $\mathbb{F}_q$-structure if there is an $\mathbb{F}_q$-variety X_0 and an isomorphism

$$X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q) \cong X.$$

Via the isomorphism above, the morphism

$$F_{X_0} \times Id : X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q) \to X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$$

defines a corresponding morphism on X, usually called geometric Frobenius, and denoted by $F_X : X \to X$. Here F_{X_0} is the Frobenius morphism of the scheme X_0, i.e the morphism of schemes given by the identity on the topological space and by taking the q-th power of the elements of the structure sheaf.

Remark 3.2.1. Notice that a variety X can be endowed with different $\mathbb{F}_q$-structures (and so different associated Frobenius morphisms). Consider for example the 1-dimensional torus $\mathbb{G}_m = \mathrm{Spec}(\overline{\mathbb{F}}_q[t, t^{-1}])$.
The canonical $\mathbb{F}_q$-structure of $\mathbb{G}_m$ is the $\mathbb{F}_q$-variety $X_0 = \mathrm{Spec}(\mathbb{F}_q[t, t^{-1}])$ and the corresponding Frobenius is the morphism $F : \mathbb{G}_m \to \mathbb{G}_m$ defined as $t \to t^q$.
However, $\mathbb{G}_m$ can be endowed with another $\mathbb{F}_q$-structure. Consider the $\mathbb{F}_q$-variety

$$X_0' := \mathrm{Spec}(\mathbb{F}_q[x, y]/(x^2 + y^2 - 1)).$$

It is possible to show that if -1 is not a square in $\mathbb{F}_q$, the variety X_0' is not isomorphic to X_0, however we have

$$\mathbb{F}_q[x, y]/(x^2 + y^2 - 1) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} \cong \overline{\mathbb{F}}_q[t, t^{-1}].$$

In particular, X_0' is another $\mathbb{F}_q$-structure of $\mathbb{G}_m$. The corresponding Frobenius morphism $F_{X_0'} : \mathbb{G}_m \to \mathbb{G}_m$ sends t to t^{-q}.

Consider an affine variety $X = \mathrm{Spec}(A)$ with A a finitely generately $\overline{\mathbb{F}}_q$-algebra. In this case, we have the following description of the $\mathbb{F}_q$-structures of X, see [28, Proposition 4.18].

Proposition 3.2.2. *A morphism $F : X \to X$ is the Frobenius morphism attached to an $\mathbb{F}_q$-structure of X if and only if the corresponding map of algebras $F : A \to A$ has the following properties:*

- *The map F has image equal to $A^{[q]} := \{a^q \mid a \in A\}$.*
- *For any $a \in A$, there exists $n \in \mathbb{N}$ such that $F^n(a) = a^{q^n}$.*

In this case, the corresponding variety X_0 is $\mathrm{Spec}(A^0)$, where $A_0 = \{a \in A \mid a^q = F(a)\}$.

Because of Proposition 3.2.2, for an affine variety X over $\overline{\mathbb{F}}_q$, we call an $\mathbb{F}_q$-Frobenius morphism (or simply a Frobenius morphism), a map $F : X \to X$ respecting the two properties of Proposition 3.2.2.
More generally, for any X we give the following definition.

Definition 3.2.3. A Frobenius morphism $F : X \to X$ is a bijective morphism such that there exists an affine covering $(U_j)_{j\in J}$ of X such that, for each j, we have $F(U_j) \subseteq U_j$ and $F|_{U_j} : U_j \to U_j$ is a Frobenius morphism of an affine variety, i.e respects the properties of Proposition 3.2.2.

Notice that, because of Proposition 3.2.2, given a variety X and a Frobenius morphism $F : X \to X$, glueing the associated $\mathbb{F}_q$-structures of the affine covering $(U_j)_{j\in J}$, we obtain an $\mathbb{F}_q$-structure X_0 such that $F_X = F$.

For this reason, hereafter we use the following terminology.

Definition 3.2.4. A variety over $\mathbb{F}_q$ is a couple (X, F) where X is a variety over $\overline{\mathbb{F}}_q$ and F is an $\mathbb{F}_q$- Frobenius morphism $F : X \to X$.

For an $\mathbb{F}_q$-variety (X, F), for any $m \geq 1$, we denote by $X(\mathbb{F}_{q^m})$ the set $X_0(\mathbb{F}_{q^m})$, i.e

$$X(\mathbb{F}_{q^m}) = X(\overline{\mathbb{F}}_q)^{F^m}.$$

Whenever the $\mathbb{F}_q$-structure of X is clear, we will often drop the Frobenius morphism in the notation and we will simply use the terminology "the $\mathbb{F}_q$-variety X".

Example 3.2.5. For the $\mathbb{F}_q$-variety $\mathbb{G}_m$, with its canonical Frobenius $F : \mathbb{G}_m \to \mathbb{G}_m$, we have

$$\mathbb{G}_m(\mathbb{F}_{q^m}) = (\overline{\mathbb{F}}_q^*)^{F^m} = \mathbb{F}_{q^m}^*.$$

Denote now by $F' : \mathbb{G}_m \to \mathbb{G}_m$ the Frobenius morphism attached to the $\mathbb{F}_q$-structure X_0' introduced in Remark 3.2.1. Notice that, in this case, we have for instance

$$\mathbb{G}_m(\mathbb{F}_q) = (\overline{\mathbb{F}}_q^*)^{F'} = \{x \in \overline{\mathbb{F}}_q^* \mid xx^q = 1\}.$$

With this terminology, a morphism of $\mathbb{F}_q$-varieties $f : (X, F) \to (Y, F)$ is a morphism of $\overline{\mathbb{F}}_q$-schemes $f : X \to Y$ which commutes with the corresponding Frobenius maps, i.e such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow F & & \downarrow F \\ X & \xrightarrow{f} & Y \end{array}$$

Given an affine variety (X, F) over $\mathbb{F}_q$ with Frobenius $F : X \to X$, consider the variety X^d and the twisted Frobenius

$$F_d : X^d \to X^d$$

defined as

$$F_d(x_1, \dots x_d) = (F(x_d), F(x_1), \dots, F(x_{d-1})).$$

In the following, we will usually denote the $\mathbb{F}_q$-variety (X^d, F_d) by X_d. Notice that there is a bijection $X_d(\mathbb{F}_q) = X(\mathbb{F}_{q^d})$. Indeed , we have

$$X_d(\mathbb{F}_q) = X^d(\overline{\mathbb{F}}_q)^{F_d} = \{(x_1, \dots, x_d) \in X(\overline{\mathbb{F}}_q)^d \mid x_1 = F(x_d)\ x_2 = F(x_1) \dots\ , x_d = F(x_{d-1})\} =$$

$$= \{(F^2(x), F^3(x), \dots, F^{d-1}(x), x, F(x)) \in X(\overline{\mathbb{F}}_q)^d \mid x \in X(\overline{\mathbb{F}}_q) \text{ and } F^d(x) = x\}$$

which is in bijection with $X(\mathbb{F}_{q^d})$. In particular, since $X^d(\mathbb{F}_q) = X(\mathbb{F}_q)^d$, we find that in general the varieties (X^d, F) and X_d are not isomorphic over $\mathbb{F}_q$.

Example 3.2.6. Consider the $\mathbb{F}_q$-variety $\mathbb{G}_m$. The $\mathbb{F}_q$-variety $(\mathbb{G}_m)_2$ is the 2-dimensional torus over $\overline{\mathbb{F}}_q$ equipped with the non-split Frobenius

$$F_2 : \mathbb{G}_m^2 \to \mathbb{G}_m^2$$

$$(z, w) \to (w^q, z^q).$$

In particular, we have

$$(\mathbb{G}_m)_2(\mathbb{F}_q) = (\mathbb{G}_m^2(\overline{\mathbb{F}}_q))^{F_2} =$$

$$\{(z, w) \in \overline{\mathbb{F}}_q^* \times \overline{\mathbb{F}}_q^* \mid z = w^q \text{ and } w = z^q\} = \{(z, z^q) \in \overline{\mathbb{F}}_q^* \times \overline{\mathbb{F}}_q^* \mid z \in \mathbb{F}_{q^2}^*\}$$

and the latter set is in bijection with $\mathbb{F}_{q^2}^*$. However, notice that $F_2^2(z, w) = (z^{q^2}, w^{q^2})$ and therefore we have

$$(\mathbb{G}_m)_2(\mathbb{F}_{q^2}) = \mathbb{F}_{q^2}^* \times \mathbb{F}_{q^2}^*.$$

In particular, in general it is not true that $X_d(\mathbb{F}_{q^m})$ is in bijection with $X(\mathbb{F}_{q^{dm}})$ for any m.

Notice that for any $\mathbb{F}_q$-variety (X, F) and any $d \geq 1$, the diagonal embedding $\Delta : X \to X^d$ is an $\mathbb{F}_q$-morphism

$$\Delta : X \to X_d.$$

3.2.1 General linear groups over finite fields

For $n \in \mathbb{N}$, we denote by GL_n the general linear group over $\overline{\mathbb{F}}_q$. The group GL_n is endowed with a canonical Frobenius morphism $F((a_{i,j})_{i,j}) = (a^q_{i,j})_{i,j}$ for a matrix $(a_{i,j})_{i,j} \in \mathrm{GL}_n$.
For $\alpha \in \mathbb{N}^I$, we denote by GL_α the $\mathbb{F}_q$-linear algebraic group

$$\mathrm{GL}_\alpha \coloneqq \prod_{i \in I} \mathrm{GL}_{\alpha_i}$$

endowed with the canonical Frobenius.

Remark 3.2.7. For each $n, d \geq 1$, we define an embedding $(\mathrm{GL}_n)_d \subseteq \mathrm{GL}_{nd}$ over $\mathbb{F}_q$ in the following way. Let $\Delta : \mathrm{GL}_n^d \to \mathrm{GL}_{nd}$ be the block diagonal embedding.
Notice that while Δ induces a morphism over $\overline{\mathbb{F}}_q$, it does not define a morphism over $\mathbb{F}_q$ from (GL_n^d, F_d) to (GL_{nd}, F)
Consider then the partition $\sigma \in S_{nd}$ given by

$$\sigma = (1,\ (n+1), \cdots, (n(d-1)+1)) \cdots (n,\ 2n, \cdots, dn)$$

and the associated partition matrix $J_\sigma \in \mathrm{GL}_{nd}$.
Fix an element $g_\sigma \in \mathrm{GL}_{nd}$ such that $g_\sigma^{-1} F(g_\sigma) = J_\sigma$ (such an element exists because of the surjectivity of the Lang map see for example [28, Theorem 4.29]). The embedding

$$g_\sigma \Delta g_\sigma^{-1} : (\mathrm{GL}_n^d, F_d) \to (\mathrm{GL}_{nd}, F)$$

is defined over $\mathbb{F}_q$.
Similarly, we can define an $\mathbb{F}_q$-embedding of $(\mathrm{GL}_\alpha)_d$ inside $\mathrm{GL}_{\alpha d}$ for any $d \geq 1$ and any $\alpha \in \mathbb{N}^I$.

Example 3.2.8. Let $n = 2$. Fix $x \in \mathbb{F}^*_{q^2} \setminus \mathbb{F}^*_q$ and let T_ϵ be the torus

$$T_\epsilon \coloneqq \left\{ \frac{1}{x^q - x} \begin{pmatrix} ax^q - bx & -a+b \\ (a-b)xx^q & -ax+bx^q \end{pmatrix} \mid a, b \in \overline{\mathbb{F}_q}^* \right\}.$$

The torus T_ϵ is F-stable and is $\mathrm{GL}_2(\mathbb{F}_q)$-conjugated to the torus $(\mathbb{G}_m)_2$ embedded inside GL_2 as in Remark 3.2.7 above.

Finally, we give the definition of *weight* for an embedding $\eta : \mathbb{G}_m \to \mathrm{GL}_n$. The weight $|\eta|$ is defined as the integer such that

$$\det \circ \eta : \mathbb{G}_m \to \mathbb{G}_m$$
$$z \to z^{|\eta|}.$$

This definition can be extended to a morphism $\eta : (\mathbb{G}_m)_d \to \mathrm{GL}_n$ defined over $\mathbb{F}_q$ in the following way. In this case, we define

$$|\eta| \coloneqq \frac{|\eta \circ \Delta|}{d},$$

where $\Delta : \mathbb{G}_m \to (\mathbb{G}_m)_d$ is the diagonal embedding.

Example 3.2.9. For each $d \geq 1$, the weight of the embedding $\eta : (\mathbb{G}_m)_d \to \mathrm{GL}_d$ defined above is

$$|\eta| = 1.$$

3.3 Notations on stacks and quotient stacks

We follow [78] for notations and basic properties of Artin stacks. For us a stack $\mathfrak{X}$over the field K will be a category $\mathfrak{X}$ fibered in groupoid over the category of K-schemes Sch_K with the following properties:

- The diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times_{\mathrm{Spec}(K)} \mathfrak{X}$ is representable by algebraic spaces
- There exists a smooth surjection $\pi : X \to \mathfrak{X}$, where X is a scheme.

For a K-scheme T, we denote by $\mathfrak{X}(T)$ the fiber groupoid of $\mathfrak{X}$ over T. Recall that $\mathfrak{X}$ is said of finite type if we can pick a smooth surjection $\pi : X \to \mathfrak{X}$ such that X is of finite type.

When $K = \mathbb{F}_q$ and $\mathfrak{X}$ is a stack of finite type, for any $m \in \mathbb{N}_{>0}$, the number of $\mathbb{F}_q$-points $\#\mathfrak{X}(\mathbb{F}_{q^m})$ is defined as

$$\#\mathfrak{X}(\mathbb{F}_{q^m}) = \sum_{x \in \mathfrak{X}(\mathbb{F}_{q^m})} \frac{1}{|\operatorname{Aut}(x)(\mathbb{F}_{q^m})|}. \tag{3.3.1}$$

This quantity is well defined since $\mathfrak{X}(\mathbb{F}_{q^m})$ is an essentially finite groupoid for any m (see for example [8, Lemma 3.2.2]).

The stacks that will appear in this thesis are mainly going to be quotient stacks . We review the definitions and some properties of these objects.
Consider an algebraic variety X over K and a K-algebraic group G acting on X. The quotient stack $[X/G]$ is the category fibered over the category of K-schemes defined as follows:

- An object over a K-scheme T is a pair (P, f), where $P \to T$ in a principal G-bundle and $f : P \to X$ is a G-equivariant map.
- For two objects (P, f),(P, f') over schemes T, T', a morphism $\alpha : (P, f) \to (P', f')$ over a morphism $h : T \to T'$ is a bundle map $\alpha : P \to P'$ such that $f' \circ \alpha = f$.

If $X = \mathrm{Spec}(K)$, the stack $[\mathrm{Spec}(K)/G]$ is called the *classifying stack* of G and is usually denoted by BG. Notice that BG is the moduli stack of principal G-bundles, i.e for any K-scheme T, giving a morphism $T \to BG$ is equivalent to giving a principal G-bundle over T.

3.3.1 Quotient stacks and GIT quotient

Consider the case where G is a (geometrically) reductive group (see [29]) and $X = \mathrm{Spec}(A)$ is an affine variety.

It is a known result that the K-algebra A^G is finitely generated. This was shown for semisimple complex groups by Weil and later generalised to any geometrically reductive group by Nagata. The associated algebraic variety $\mathrm{Spec}(A^G)$ is denoted by $X//G$ and it is usually called the GIT quotient.
The variety $X//G$ is indeed a categorical quotient in the category Sch_K and, if K is algebraically closed, the K-points of $X//G$ are in bijection with the closed orbits of the action of G on X.
For such X, G, there is a canonical morphism of stacks

$$f : [X/G] \to X//G.$$

The map f in general it is not an isomorphism. This is true in the case of schematically free action. Consider more generally an algebraic variety (not necessarily affine) X and a linear algebraic group G (not necessarily reductive).
Recall that the action of G on X is said to be schematically free if it is set-theoretically free and the action map $G \times X \to X$ is proper. For schematically free action we have the following standard lemma.

Lemma 3.3.1. *If the action of G on X is set-theoretically free, the algebraic stack $[X/G]$ is an algebraic space. Moreover, if the action is scheme-theoretically free, the algebraic space $[X/G]$ is a scheme, denoted by X/G, and the map $X \to X/G$ is a principal G-bundle.*

Remark 3.3.2. In the case where X is affine, G is reductive, the scheme X/G is the GIT quotient $X//G$ and Lemma 3.3.1 above is equivalent to state that the map $f : [X/G] \to X/G$ is an isomorphism.

Remark 3.3.3. The map f carries a great deal of information about the geometry of the action of G on X, but it is also extremely complicated to describe in general. Recently, Alper [2, Remark 4.8] showed that if $K = \mathbb{C}$, the map f is a *good moduli space.*
Davison [23, Theorem 6.1] showed that, under some conditions on the action of G, the map f altough far from being proper, admits a sort of Decomposition Theorem

We will also need the following two Lemmas about isomorphism classes of quotient stacks. Both Lemma 3.3.4, 3.3.5 are probably well known to the experts but we were not able to find a reference in the literature.

Lemma 3.3.4. *Let $K = \mathbb{C}$ and G be a K-linear algebraic group acting on the left on a K-scheme X. Let $H \leq G$ be a closed subgroup. Suppose that there exists a G-equivariant map $q : X \to G/H$, where G acts on G/H by left multiplication.*
Denote by $X_H = q^{-1}(eH)$. The group H acts on X_H and there is an isomorphism of quotient stacks

$$[X/G] \cong [X_H/H]. \tag{3.3.2}$$

Moreover, if X is an affine variety and G, H are reductive, there is an isomorphism of varieties:

$$X//G \cong X_H//H. \tag{3.3.3}$$

Proof. Notice firstly that, if X is affine and G, H are reductive, the isomorphism (3.3.3) is implied by the isomorphism (3.3.2) as the varieties $X_H//H, X//G$ are good moduli spaces for the stacks $[X_H/H], [X/G]$ respectively, as recalled above. We are thus reduced to show isomorphism (3.3.2).

Notice that in general there is always a canonical map $\alpha : [X_H/H] \to [X/G]$, obtained by extension of the structure group of a principal bundle from H to G. We must construct an inverse $\beta : [X_H/H] \to [X/G]$.
Fix a scheme S and a pair $(P, f) \in [X/G](S)$, where $P \to S$ is a principal G-bundle and $f : P \to X$ is a G-equivariant map. Consider the variety P_H and the morphism f_H such that all square diagrams are cartesian:

$$\begin{array}{ccccc} P_H & \xrightarrow{f_H} & X_H & \longrightarrow & eH \\ \downarrow & & \downarrow & & \downarrow \\ P & \xrightarrow{f} & X & \xrightarrow{q} & G/H. \end{array}$$

We verify that P_H is a principal H-bundle over S. Notice that $P_H \to P$ is a closed embedding and in the following we identify P_H with its image in P. Notice that f_H is H-equivariant, being the restriction of f.

Denote by $\widetilde{f} \coloneqq q \circ f$. If $p \in P_H$, then for each $h \in H$, we have

$$\widetilde{f}(h \cdot p) = h \cdot \widetilde{f}(p) = h \cdot eH = eH,$$

i.e $h \cdot p \in P_H$.
The closed subspace $P_H \subseteq P$ is thus H-stable and therefore we get an H action on it. Let us show that this action equips P_H with the structure of a principal H-bundle.
Denote by $\pi : G \to G/H$ the map defined as $\pi(g) = [g^{-1}]$, where $[g] \in G/H$ is the class of g in the quotient. Notice that π is a left principal bundle, while the usual quotient map is a right one.
Consider an fpqc open covering $(S_j)_{j \in J}$ of S such that

$$P_j \coloneqq P \times_S S_j \cong G \times S_j$$

and, similarly, denote by $(P_H)_j = P_H \times_S S_j$. By pullback, $\widetilde{f}$ defines G-equivariant morphisms $\widetilde{f}_j : G \times S_j \to G/H$, i.e morphisms $\overline{f}_j : S_j \to G/H$ such that

$$\widetilde{f}_j((g, s)) = g \cdot \overline{f}_j(s).$$

Consider an étale open covering $(V_l)_{l \in L}$ of G/H such that we have cartesian diagrams

$$\begin{array}{ccc} H \times V_l & \longrightarrow & G \\ \downarrow & & \downarrow \pi \\ V_l & \xrightarrow{\overline{r}_l} & G/H \end{array}$$

and denote by $r_l : V_l \to G$ the morphism such that the morphism $H \times V_l \to G$ sends (h, v) to $h \cdot r_l(v)$.
Denote by $S_{j,l} = \overline{f}_j^{-1}(V_l)$ and by $P_{j,l} = P_j \times_{S_j} S_{j,l}$ and $(P_H)_{j,l} = P_H \times_{S_j} S_{j,l}$ the associated pullback, by $f_{j,l} : S_{j,l} \to V_l$ the associated morphisms. Notice that, for each j, l, we have

$$(P_H)_{j,l} = \{(g, s) \in G \times S_{j,l} \mid \pi(g^{-1}) = \overline{r}_j(f_{j,l}(s))\}.$$

Because of the definition of the schemes V_l, there is an isomorphism $(P_H)_{j,l} \cong H \times S_{j,l}$ given by

$$H \times S_{j,l} \to (P_H)_{j,l}$$

$$(h, s) \to (h \cdot r_l(f_{j,l}(s)), s).$$

Notice that $(S_{j,l})_{j \in J, l \in L}$ is an fpqc covering of S and therefore P_H is an H-principal bundle over S. We define thus

$$\beta : [X/G] \to [X_H/H]$$

as

$$\beta(P, f) = (P_H, f_H).$$

It is not hard to check that the morphism β is an inverse to α. □

Lemma 3.3.5. *Consider linear algebraic groups G, H and a variety X with an action of $G \times H$ such that the action of H on X obtained by restriction is schematically free. The quotient X/H is a scheme equipped with a G-action and there is an isomorphism of quotient stacks*

$$[X/G \times H] \cong [(X/H)/G].$$

Notice that the quotient X/H is a scheme by Lemma 3.3.1. The G-action on it is defined as

$$g \cdot \pi_H(x) = \pi_H(g \cdot x),$$

where $\pi_H : X \to X/H$ is the quotient map. Notice that π_H is G-equivariant.

Proof. We define a map $\alpha : [X/G \times H] \to [(X/H)/G]$ as follows. Consider $S \in \mathrm{Sch}_K$ and a pair $(P, f) \in [X/G \times H](S)$. Since $P \to S$ is a principal $G \times H$-bundle, the quotient P/H has an induced structure of G-bundle. Moreover, as the map $f : P \to X$ is $G \times H$-equivariant, it descends to a G-equivariant map

$$\overline{f} : P/H \to X/H.$$

We define then

$$\alpha((P, f)) = (P/H, \overline{f}).$$

We define now a map $\beta : [(X/H)/G] \to [X/G \times H]$ as follows. Consider a pair $(Q, d) \in [(X/H)/G](S)$ and let $P \in \mathrm{Sch}_K$ and $f : P \to X$ such that the following diagram is cartesian.

$$\begin{array}{ccc} P & \xrightarrow{f} & X \\ \downarrow & & \downarrow \pi_H \\ Q & \xrightarrow{d} & X/H \end{array}$$

Notice that

$$P = \{(x,q) \in Q \times X \mid \pi_H(x) = d(q)\}.$$

We define a $G \times H$-action on P as follows:

$$(g,h) \cdot (x,q) = ((g,h) \cdot x, h \cdot q).$$

Notice that $f : P \to X$ is $G \times H$-equivariant. We will show that $P \to S$ is a $G \times H$-principal bundle. Consider an fpqc covering $(S_j)_{j \in J}$ of S such that $Q \times_S S_j \cong G \times S_j$. The morphism d defines by pullback, G-equivariant morphisms $\tilde{d_j} : G \times S_j \to X/H$, i.e morphisms $\overline{d_j} : S_j \to X/H$ such that

$$\tilde{d_j}((g,s)) = g \cdot \overline{d_j}(s).$$

Consider an fpqc cover $(Y_l)_{l \in L}$ of X/H such that we have cartesian diagrams

$$\begin{array}{ccc} H \times Y_l & \longrightarrow & X \\ \downarrow & & \downarrow \pi_H \\ Y_l & \xrightarrow{\overline{r_l}} & X/H \end{array}$$

and denote by $r_l : Y_l \to X$ the corresponding morphism such that the morphism $H \times Y_l \to G$ sends (h,y) to $h \cdot r_l(y)$.
Denote by $S_{j,l} = \overline{d_j}^{-1}(Y_l)$ and by $P_{j,l} = P_j \times_{S_j} S_{j,l}$ and by $f_{j,l} : S_{j,l} \to V_l$ the associated morphisms. Notice that we have the following commutative diagram, where both squares are cartesian:

$$\begin{array}{ccccc} G \times H \times S_{j,l} & \xrightarrow{\mathrm{id} \times \mathrm{id} \times f_{j,l}} & G \times H \times Y_l & \xrightarrow{\psi_l} & X \\ \downarrow & & \downarrow & & \downarrow \pi_H \\ G \times S_{j,l} & \xrightarrow{\mathrm{id} \times f_{j,l}} & G \times Y_l & \xrightarrow{\tilde{r_l}} & X/H \end{array}$$

where $\tilde{r_l}(g,y) = g \cdot \overline{r_l}(y)$ and $\psi_l((g,h),y) = (g,h) \cdot r_l(y)$. We deduce therefore that the following square is cartesian:

$$\begin{array}{ccc} G \times H \times S_{j,l} & \xrightarrow{\psi_l \circ \mathrm{id} \times \mathrm{id} \times f_{j,l}} & X \\ \downarrow & & \downarrow \pi_H \\ G \times S_{j,l} & \xrightarrow{\tilde{r_l} \circ \mathrm{id} \times f_{j,l}} & X/H \end{array}$$

Notice that

$$\tilde{r_l} \circ f_{j,l} = \tilde{d_j}|_{G \times S_{j,l}},$$

from which we deduce that

$$P_{j,l} \cong G \times H \times S_{j,l}.$$

We define therefore

$$\beta((Q,d)) = (P,f).$$

It is not hard to check that α and β are inverse one to each other.

□

We end this paragraph by recalling that the point counting of quotient stacks over finite fields is well understood. More precisely, for $K = \mathbb{F}_q$, in [8, Lemma 2.5.1], it is shown that for a quotient stack $\mathfrak{X} = [X/G]$ where G is a connected linear algebraic group, we have

$$\#\mathfrak{X}(\mathbb{F}_{q^m}) = \frac{\#X(\mathbb{F}_{q^m})}{\#G(\mathbb{F}_{q^m})} \tag{3.3.4}$$

for any $m \in \mathbb{N}$.

3.4 Compactly supported cohomology of stacks and weight filtration

Let $\mathfrak{X}$ be an algebraic stack of finite type over an algebraically closed field K. For $K = \mathbb{C}$, we denote by $H_c^*(\mathfrak{X})$ the compactly-supported cohomology groups with compact support with coefficients in $\mathbb{C}$.
For $K = \overline{\mathbb{F}}_q$, we denote by $H_c^*(\mathfrak{X})$ the compactly supported étale cohomology groups with coefficients in $\overline{\mathbb{Q}}_\ell$. Both cohomology theories are well defined thanks to the recent work [59] of Laszlo and Olsoon.
When $K = \mathbb{C}$, each vector space $H_c^k(\mathfrak{X})$ is endowed with the weight filtration $W_\bullet^k$, by the work of Deligne, see [25, Chapter 8].
For stacks over $\overline{\mathbb{F}}_q$ we have an analogous definition of a weight filtration induced by the Frobenius action on cohomology, see for example [65, Section 2.2].

Remark 3.4.1. For a linear algebraic group G acting on the left on a scheme X of finite type over $\mathbb{C}$, the compactly supported cohomology $H_c^*([X/G])$ and its weight filtration have the following more concrete description.
Consider an embedding $G \subseteq \mathrm{GL}_r(\mathbb{C})$. For any $m \in \mathbb{N}$, denote by $V_m = \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^r)$. Notice that G acts on the right on the vector space V_m and acts freely on the open dense subset $U_m = \mathrm{Hom}^{surj}(\mathbb{C}^m, \mathbb{C}^r)$, given by surjective homomorphisms.
Consider the left action of G on $X \times V_m$ defined as $g \cdot (x,u) = (g \cdot x, u \cdot g^{-1})$. It is a known fact that the action of G on U_m is schematically free and the quotient stack $[X \times U_m/G]$ is thus a scheme, which is usually denoted by $X \times_G U_m$.
The projection $p : X \times V_m \to X$ induces a quotient morphism $\overline{p} : [X \times V_m/G] \to [X/G]$ such that the following diagram is 2-cartesian

$$\begin{array}{ccc} X \times V_m & \longrightarrow & X \\ \downarrow & & \downarrow \\ [X \times V_m/G] & \longrightarrow & [X/G] \end{array}$$

and so the morphism $\overline{p}$ is a vector bundle. In particular, we have an isomorphism

$$W_j H_c^i([X/G]) = W_{j+\dim(V_m)} H_c^{i+2\dim(V_m)}([X \times V_m/G]) .$$

Let $Z_m = V_m \setminus U_m$. The codimension $\mathrm{codim}_{V_m}(Z_m)$ goes to ∞ for $m \to +\infty$. In particular, for any $i \in \mathbb{Z}$, there exists $m \in \mathbb{N}$ such that

$$i + 2\dim(V_m) \geq 2\dim(Z_m) + 2(\dim(X) - \dim(G)) + 2.$$

Notice that in this case, we have

$$H_c^{i+2\dim(V_m)}([X \times Z_m/G]) = H_c^{i+2\dim(V_m)-1}([X \times Z_m/G]) = \{0\}.$$

From the long exact sequence in compactly supported cohomology for the open-closed decomposition

$$[X \times V_m/G] = X \times_G U_m \bigsqcup [X \times Z_m/G],$$

we deduce therefore that

$$H_c^{i+2\dim(V_m)}([X \times V_m/G]) = H_c^{i+\dim(V_m)}(X \times_G U_m)$$

and thus that, for any j, we have

$$W_j H_c^i([X/G]) = W_{j+\dim(V_m)} H_c^{i+2\dim(V_m)}(X \times_G U_m).$$

In particular, the (compactly supported) cohomology of the stack $[X/G]$ can be computed from that of the varieties $X \times_G U_m$.

This costrunction is an algebro-geometric version of the Borel construnction of equivariant cohomology in differential geometry and was initially taken as a definition of G-equivariant compactly supported cohomology or G-equivariant Borel-Moore homology of X, see [30].

3.4.1 Mixed Poincaré series

For $\mathfrak{X}$ over $\mathbb{C}$, we define the mixed Poincaré series $H_c(\mathfrak{X}, q, t)$ as

$$H_c(\mathfrak{X}; q, t) := \sum_{k,m} \dim(W_m^k/W_{m-1}^k) q^{\frac{m}{2}} t^k. \tag{3.4.1}$$

Notice that the specialization $H_c(\mathfrak{X}, 1, t)$ of $H_c(\mathfrak{X}, q, t)$ at $q = 1$ is equal to the Poincaré series $P_c(\mathfrak{X}, t)$ of the stack $\mathfrak{X}$.

When $\sum_k (-1)^k \dim(W_m^k/W_{m-1}^k)$ is finite for each m, we define the E-series:

$$E(\mathfrak{X}, q) := H_c(\mathfrak{X}; q, -1) = \sum_{m,k} \dim(W_m^k/W_{m-1}^k)(-1)^k q^{\frac{m}{2}}. \tag{3.4.2}$$

For the E-series of quotient stacks, moreover, we have the following Theorem (see [65, Theorem 2.5]).

Theorem 3.4.2. *Let G be a connected linear algebraic group acting on a separated scheme of finite type over K. The E-series of $[X/G]$ is well-defined and*

$$E([X/G], q) = E(X, q)E(BG, q).$$

An efficient approach to compute E-series for complex stacks is counting points over $\mathbb{F}_q$. More precisely, we give the following definition of a (strongly) *rational count* stack.
Let E be a $\mathbb{Z}$-finitely generated algebra and $\mathfrak{Y}$ be an E-stack. Assume that there exists $\psi : E \to \mathbb{C}$ such that

$$\mathfrak{Y} \times_{\mathrm{Spec}(R),\psi} \mathrm{Spec}(\mathbb{C}) \cong \mathfrak{X}.$$

The stack $\mathfrak{Y}$ is called a *spreading out* of $\mathfrak{X}$. For any $\varphi : E \to \mathbb{F}_q$, denote by

$$\mathfrak{X}^{\varphi} := \mathfrak{Y} \times_{\mathrm{Spec}(R),\varphi} \mathrm{Spec}(\mathbb{F}_q).$$

Definition 3.4.3. We say that the stack $\mathfrak{X}$ is (strongly) *rational count* if there exists an open $U \subseteq \mathrm{Spec}(E)$ and a rational function $Q(t)$ such that for any $\varphi : E \to \mathbb{F}_q$ with $\varphi(\mathrm{Spec}(\mathbb{F}_q)) \subseteq U$, it holds

$$\#\mathfrak{X}^{\varphi}(\mathbb{F}_{q^n}) = Q(q^n)$$

for every $n \geq 1$.

For strongly rational count stacks, the authors [65, Theorem 2.8] show the following result.

Theorem 3.4.4. *If a quotient stack $\mathfrak{X} = [X/G]$ is (strongly) rational count, we have:*

$$E([X/G], q) = Q(q). \tag{3.4.3}$$

Remark 3.4.5. Let $\mathfrak{X} = [X/G]$ be a quotient stack over $\mathbb{C}$. Consider a finitely generated $\mathbb{Z}$-algebra E and E-schemes Y_1, Y_2 which are spreading out of X, G respectively. The E-stack $\mathfrak{Y}$ is a spreading out $\mathfrak{X}$ and, for any $\varphi : E \to \mathbb{F}_q$, we have

$$\mathfrak{X}^{\varphi} = [X^{\varphi}/G^{\varphi}].$$

Notice moreover that by Formula (3.3.4), we have therefore an equality

$$\#\mathfrak{X}^{\varphi}(\mathbb{F}_{q^m}) = \frac{\#X^{\varphi}(\mathbb{F}_{q^m})}{\#G^{\varphi}(\mathbb{F}_{q^m})}.$$

Therefore, we deduce that the stack $\mathfrak{X}$ is (strongly) rational count if and only if X, G are (strongly) polynomial count.

Consider a reductive algebraic group G (i.e $G = \mathrm{GL}_n, \mathrm{PGL}_n$). In [25] it is shown that each cohomology group $H_c^m(BG)$ is pure of weight m. In [25] this is stated for cohomology rather

than cohomology with compact support. The latter case is an immediate consequence thanks to Poincaré duality.
From Theorem 3.4.4, we deduce the following Lemma:

Lemma 3.4.6. *Suppose that G is (strongly) polynomial count. The classifying stack BG is strongly polynomial count and we have*

$$H_c(BG, q, t) = E(BG, qt^2) = \frac{1}{E(G, q)}. \tag{3.4.4}$$

Example 3.4.7. We deduce, for instance, that, for each $n \in \mathbb{N}$, we have

$$H_c(B\,\mathrm{GL}_n, q, t) = \frac{1}{(qt^2)^{\frac{n(n-1)}{2}}} \frac{1}{(qt^2 - 1)\cdots((qt^2)^n - 1)} \tag{3.4.5}$$

and

$$H_c(B\,\mathrm{PGL}_n, q, t) = \frac{1}{(qt^2)^{\frac{n(n-1)}{2}}} \frac{1}{((qt^2)^2 - 1)\cdots((qt^2)^n - 1)} \tag{3.4.6}$$

Lastly, we will need the following Proposition about the cohomology of a quotient stack $[X/G]$. Assume that $G = \mathrm{GL}_n$ and the center $\mathbb{G}_m \subseteq \mathrm{GL}_n$ acts trivially on the scheme X. There is thus an induced action of PGL_n on X and a canonical morphism $h : [X/\mathrm{GL}_n] \to [X/\mathrm{PGL}_n]$.

Proposition 3.4.8. *Let X be a $\mathbb{C}$-variety with a GL_n-action trivial on the center. We have a natural isomorphism of mixed Hodge structures:*

$$H_c^*([X/\mathrm{GL}_n]) = H_c^*([X/\mathrm{PGL}_n]) \otimes H_c^*(B\mathbb{G}_m) \tag{3.4.7}$$

Proof. We start by the case in which $X = \mathrm{Spec}(\mathbb{C})$ and the canonical morphism $\pi : B\,\mathrm{GL}_n \to B\,\mathrm{PGL}_n$. In this case, eq.(3.4.7) is a direct consequence of eq.(3.4.5), eq.(3.4.6).
Notice now that there is a cartesian diagram:

$$\begin{array}{ccc} B\mathbb{G}_m & \longrightarrow & B\,\mathrm{GL}_n \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\psi} & B\,\mathrm{PGL}_n \end{array}$$

where $\psi : \mathrm{Spec}(\mathbb{C}) \to B\,\mathrm{PGL}_n$ is the canonical projection. Since ψ is a smooth covering, for each $q \in \mathbb{Z}$, the sheaf $R^q\pi_!\mathbb{C}$ is a local system with fiber $H_c^p(B\mathbb{G}_m)$.
Moreover, as PGL_n is connected, each local system is trivial over $B\,\mathrm{PGL}_n$, see for example [1, Proposition 6.13]. In particular, the Leray spectral sequence for compactly supported cohomology and the morphism π in second page is

$$E_2^{p,q} : H_c^p(B\,\mathrm{PGL}_n) \otimes H_c^q(B\mathbb{G}_m) \Rightarrow H_c^{p+q}(B\,\mathrm{GL}_n).$$

From eq.(3.4.7), we deduce that the spectral sequence collapses at page 2, i.e that the canonical morphism

$$H_c^p(B\,\mathrm{GL}_n) \to H_c^p(B\mathbb{G}_m)$$

is surjective for every p.

Consider now a general X. In this case too, we have a Leray spectral sequence for compactly supported cohomology with second page

$$E_2^{p,q} = H_c^p\left([X/\operatorname{PGL}_n], R^q h_!\mathbb{C}\right) \Rightarrow H_c^{p+q}([X/\operatorname{GL}_n]).$$

Notice that there is 2-Cartesian diagram

$$\begin{array}{ccc} [X/\operatorname{GL}_n] & \longrightarrow & B\operatorname{GL}_n \\ \downarrow & & \downarrow \\ [X/\operatorname{PGL}_n] & \longrightarrow & B\operatorname{PGL}_n \end{array}$$

where the morphism $\pi : B\operatorname{GL}_n \to B\operatorname{PGL}_n$ on the right is the canonical morphism between classifying spaces. In particular, we have

$$R^q h_!\mathbb{C} \cong r^* R^q \pi_!\mathbb{C}$$

where $r : [X/\operatorname{PGL}_n] \to B\operatorname{PGL}_n$.
We deduce thus that each $R^q h_!\mathbb{C}$ is trivial. Moreover, the associated map

$$H_c^p([X/\operatorname{GL}_n]) \to H_c^p(B\mathbb{G}_m)$$

is surjective, since the map $H_c^p(B\operatorname{GL}_n) \to H_c^p(B\mathbb{G}_m)$ is surjective, as remarked above. Therefore, the spectral sequence $E_2^{p,q}$ collapses at page 2 and we obtain an isomorphism

$$H_c^*\left([X/\operatorname{GL}_n]\right) = H_c^*\left([X/\operatorname{PGL}_n]\right) \otimes H_c^*(B\mathbb{G}_m).$$

□

Remark 3.4.9. Notice that under the hypothesis of Proposition 3.4.8, we have

$$H_c([X/\operatorname{GL}_n], q, t) = \frac{H_c([X/\operatorname{PGL}_n], q, t)}{qt^2 - 1} \tag{3.4.8}$$

3.5 Partitions and multipartitions

We follow the classical book by Macdonald [70]. Recall that a partition λ is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h, \dots)$ with finite non-zero terms. We denote by $\mathcal{P}$ the set of all partitions and by $\mathcal{P}^* \subseteq \mathcal{P}$ the subset of nonzero partitions.
A partition λ will be denoted either by $\lambda = (\lambda_1, \lambda_2 \dots, \lambda_h)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h$ or by $\lambda = (1^{m_{1,\lambda}}, 2^{m_{2,\lambda}}, \dots)$ where $m_{k,\lambda}$ is the number of occurrencies of the number k in the partition λ. We will denote by λ' the partition conjugate to λ.
The *size* of λ is

$$|\lambda| = \sum_i \lambda_i$$

and its length $l(\lambda)$ is the biggest i such that $\lambda_i \neq 0$. For each $n \in \mathbb{N}$, we denote by $\mathcal{P}_n$ the subset of partitions of size n.
For two partitions λ, μ, we denote by $\langle \lambda, \mu \rangle$ the quantity

$$\langle \lambda, \mu \rangle = \sum_i \lambda_i' \mu_i'.$$

The set $\mathcal{P}$ admits different possible orderings. In the following, we will denote by $\lambda \leq \mu$ the ordering induced by the *lexicographic* order, i.e $\lambda \leq \mu$ if and only if $\lambda_i \leq \mu_i$ for any i.

Recall that the conjugacy classes of the symmetric group S_n are in bijection with $\mathcal{P}_n$. Indeed, each element σ can be written as a product of disjoint cycles $\sigma = \sigma_1 \cdots \sigma_k$ and, up to reordering, the lengths of the cycles $\sigma_1, \dots, \sigma_k$ give the associated partition.
For each $\lambda \in \mathcal{P}_n$, denote by z_λ the cardinality of the centralizer of an element of the conjugacy class associated to λ. If $\lambda = (1^{m_{1,\lambda}}, 2^{m_{2,\lambda}}, \dots,)$, we have

$$z_\lambda = \prod_j m_{j,\lambda}! j^{m_{j,\lambda}}.$$

Recall moreover that the set of irreducible characters of S_n is in bijection with $\mathcal{P}_n$. In our bijection we associate to the partition (n) the trivial character of S_n. We denote the irreducible character of S_n associated to λ by χ^λ and its value at the conjugacy class associated to $\mu \in \mathcal{P}_n$ by χ^λ_μ.

Fix now a finite set I. We consider the set of multipartitions $\mathcal{P}^I$. The elements of $\mathcal{P}^I$ will be usually be denoted in bold letters $\boldsymbol{\lambda} \in \mathcal{P}^I$. To avoid confusion with the notation used for partition, we will use the notation $\boldsymbol{\lambda} = (\lambda^i)_{i \in I}$. For $\boldsymbol{\lambda} \in \mathcal{P}^I$, the *size* $|\boldsymbol{\lambda}| \in \mathbb{N}^I$ of $\boldsymbol{\lambda}$ is defined as

$$|\boldsymbol{\lambda}|_i \coloneqq |\lambda^i|.$$

For an element $\alpha \in \mathbb{N}^I$, we will denote by $(1^\alpha) \in \mathcal{P}^I$ the multipartition $((1^{\alpha_i}))_{i \in I}$ and by $(\alpha) \in \mathcal{P}^I$ the multipartition $((\alpha_i))_{i \in I}$.
The order $\leq$ induces the corresponding lexicographical ordering on $\mathcal{P}^I$, which we still denote by $\leq$.

3.5.1 Partitions and symmetric functions

Let $\mathbf{x} = \{x_1, x_2, \dots, \}$ be an infinite set of variables. Denote by $\Lambda(\mathbf{x})$ the ring of symmetric functions over $\mathbb{Q}$ in the variables $x_1, \dots, x_n, \dots$. Notice that the ring $\Lambda(\mathbf{x})$ admits a grading given by the degree of a symmetric function. We denote by $\Lambda(\mathbf{x})_n$ the degree n part.
We can show that the ring $\Lambda(\mathbf{x})$ is a polynomial ring in an infite set of variables. More precisely, for each $n \in \mathbb{N}$, consider the elementary symmetric function $e_n(\mathbf{x}) \in \Lambda(\mathbf{x})_n$, the complete symmetric function $h_n(\mathbf{x}) \in \Lambda(\mathbf{x})_n$ and power sum $p_n(\mathbf{x}) \in \Lambda(\mathbf{x})_n$ defined as

$$e_n(\mathbf{x}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$h_n(\mathbf{x}) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$
$$p_n(\mathbf{x}) = x_1^n + x_2^n + \cdots .$$

For each $\lambda = (\lambda_1, \ldots, \lambda_h) \in \mathcal{P}$, we introduce the associated elements

$$e_\lambda(\mathbf{x}) = e_{\lambda_1}(\mathbf{x}) \cdots e_{\lambda_h}(\mathbf{x})$$

$$h_\lambda(\mathbf{x}) = h_{\lambda_1}(\mathbf{x}) \cdots h_{\lambda_h}(\mathbf{x})$$

$$p_\lambda(\mathbf{x}) = p_{\lambda_1}(\mathbf{x}) \cdots p_{\lambda_h}(\mathbf{x}).$$

The families of functions $\{e_\lambda(\mathbf{x})\}_{\lambda \in \mathcal{P}}, \{h_\lambda(\mathbf{x})\}_{\lambda \in \mathcal{P}}, \{p_\lambda(\mathbf{x})\}_{\lambda \in \mathcal{P}}$ are basis of the $\mathbb{Q}$-vector space $\Lambda(\mathbf{x})$, or equivalently, the families of functions $\{e_n(\mathbf{x})\}_{n \in \mathbb{N}}, \{h_n(\mathbf{x})\}_{n \in \mathbb{N}}, \{p_n(\mathbf{x})\}_{n \in \mathbb{N}}$ freely generate the ring $\Lambda(\mathbf{x})$.

Another important basis of the ring $\Lambda(\mathbf{x})$ is given by Schur functions $\{s_\lambda(\mathbf{x})\}_{\lambda \in \mathcal{P}}$. There are multiple ways to define these functions. One possible way is by the following formula:

$$s_\lambda(\mathbf{x}) = \sum_{\mu \in \mathcal{P}_n} \frac{\chi_\mu^\lambda}{z_\mu} p_\mu(\mathbf{x}).$$

Recall moreover that on the ring $\Lambda(\mathbf{x})$ it is defined a canonical bilinear product $\langle , \rangle$ making the Schur functions orthonormal, i.e

$$\langle s_\lambda(\mathbf{x}), s_\mu(\mathbf{x}) \rangle = \delta_{\lambda,\mu}.$$

3.6 Multitypes

A multitype is a function $\omega : \mathbb{N} \times \mathcal{P}^I \to \mathbb{N}$ such that its support (i.e the elements $(d, \boldsymbol{\mu})$ such that $\omega(d, \boldsymbol{\mu}) \neq 0$) is finite and $\omega(0, \boldsymbol{\lambda}) = \omega(d, 0) = 0$ for any $\boldsymbol{\lambda} \in \mathcal{P}^I$ and $d \in \mathbb{N}$.
On the set $\mathbb{N} \times \mathcal{P}^I$ put the total order defined by the following rules.

- If $d > d'$ then $(d, \boldsymbol{\lambda}) > (d', \boldsymbol{\mu})$.
- If $d = d'$ and $|\boldsymbol{\lambda}| > |\boldsymbol{\mu}|$ then $(d, \boldsymbol{\lambda}) > (d, \boldsymbol{\mu})$
- If $|\boldsymbol{\lambda}| = |\boldsymbol{\mu}|$ then $(d, \boldsymbol{\lambda}) > (d', \boldsymbol{\mu})$ if $\boldsymbol{\lambda} > \boldsymbol{\mu}$.

We can alternately think of a multitype ω as a non-decreasing sequence $\omega = (d_1, \boldsymbol{\lambda}_1) \ldots (d_r, \boldsymbol{\lambda}_r)$, where the value $\omega(d, \boldsymbol{\lambda})$ corresponds to the number of times the element $(d, \boldsymbol{\lambda})$ appear in the sequence $(d_1, \boldsymbol{\lambda}_1) \ldots (d_r, \boldsymbol{\lambda}_r)$.

We will denote by $\mathbb{T}_I$ the set of multitypes. If $|I| = 1$, we will call multytipes simply types.
For $\omega \in \mathbb{T}_I$ where $\omega = (d_1, \boldsymbol{\lambda}_1) \ldots (d_r, \boldsymbol{\lambda}_r)$ and $d \in \mathbb{N}_{>0}$, we denote by $\psi_d(\omega)$ the multitype

$$\psi_d(\omega) \coloneqq (dd_1, \boldsymbol{\lambda}_1) \ldots (dd_r, \boldsymbol{\lambda}_r).$$

Define the *size* $|\omega|$ of a multitype ω as the following element of $\mathbb{N}^I$

$$|\omega| := \sum_{(d,\boldsymbol{\mu})\in\mathbb{N}\times\mathcal{P}^I} d\omega(d,\boldsymbol{\mu})|\boldsymbol{\mu}|$$

and the integer $w(\omega) \in \mathbb{N}$ as the quantity

$$w(\omega) := \prod_{(d,\boldsymbol{\mu})\in\mathbb{N}\times\mathcal{P}^I} d^{\omega(d,\boldsymbol{\mu})}\omega(d,\boldsymbol{\mu})!.$$

For $\alpha \in \mathbb{N}^I$, we denote by $\mathbb{T}_\alpha \subseteq \mathbb{T}_I$ the subset of multitypes of size α.

Example 3.6.1. Assume $|I| = 1$. For each $\lambda = (\lambda_1, \dots, \lambda_h) \in \mathcal{P}_n$, there is an associated type $\omega_\lambda = (1,(1^{\lambda_1}))\dots(1,(1^{\lambda_h})) \in \mathbb{T}_n$. Whenever the context is clear, we will denote ω_λ simply by $\lambda \in \mathbb{T}_n$.
Notice that, if $\lambda = (1^{m_{1,\lambda}}, 2^{m_{2,\lambda}}, \dots)$, for the type ω_λ we have

$$\omega_\lambda((1,(1^{\lambda_i}))) = m_{\lambda,\lambda_i}$$

and therefore

$$w_{\omega_\lambda} = \prod_{j\geq 1} m_{\lambda,j}!$$

The sum of multitypes endows the set $\mathbb{T}_I$ with an associative operation

$$* : \mathbb{T}_I \times \mathbb{T}_I \to \mathbb{T}_I.$$

More precisely, for ω_1, ω_2 multitypes , we define $\omega_1 * \omega_2$ as the multitype such that

$$\omega_1 * \omega_2(d,\boldsymbol{\lambda}) := \omega_1(d,\boldsymbol{\lambda}) + \omega_2(d,\boldsymbol{\lambda}).$$

The reason for this choice of notation will be clear later. Notice that if $|\omega_1| = \alpha$ and $|\omega_2| = \beta$, we have $|\omega_1 * \omega_2| = \alpha + \beta$.

We view $\mathbb{N}_{>0} \times \mathbb{N}^I$ as a subset of $\mathbb{N}_{>0} \times \mathcal{P}^I$ by associating to (d,α) the element $(d,(1^\alpha))$. We call a multitype ω *semisimple* if its support is contained in $\mathbb{N}_{>0} \times \mathbb{N}^I$. Given a semisimple ω we will think of it as a function $\mathbb{N}_{>0} \times \mathbb{N}^I \to \mathbb{N}$ which we still denote by ω with

$$\omega(d,\alpha) := \omega(d,(1^\alpha)).$$

Whenever the context is clear, we will frequently switch between the two notations for semisimple multitypes.
For each $\alpha \in \mathbb{N}^I$, we denote by ω_α the semisimple type defined as

$$\begin{cases} \omega_\alpha(1,\alpha) = 1 \\ \omega_\alpha(d,\beta) = 0 \text{ if } (d,\beta) \neq (1,\alpha) \end{cases}.$$

We denote by $\mathbb{T}_I^{ss} \subseteq \mathbb{T}$ the subset of semisimple types. Notice that for any semisimple multitype $\omega \in \mathbb{T}_I^{ss}$, there exists $d_1, \dots, d_r \in \mathbb{N}_{>0}$ and $\alpha_1, \dots, \alpha_r \in \mathbb{N}^I$ such that

$$\omega = \psi_{d_1}(\omega_{\alpha_1}) * \cdots * \psi_{d_r}(\omega_{\alpha_r}).$$

For a type $\omega = (d_1, \boldsymbol{\lambda}_1) \dots (d_r, \boldsymbol{\lambda}_r) \in \mathbb{T}_I$, we define its semisimplification $\omega^{ss} \in \mathbb{T}_I^{ss}$ as the semisimple type

$$\omega^{ss} := (d_1, (1^{|\boldsymbol{\lambda}_1|})) \dots (d_r, (1^{|\boldsymbol{\lambda}_r|})),$$

i.e $\omega^{ss} = \psi_{d_1}(\omega_{|\boldsymbol{\lambda}_1|}) * \cdots * \psi_{d_r}(\omega_{|\boldsymbol{\lambda}_r|})$.

To a semisimple type $\omega = (d_1, \alpha_1) \dots (d_r, \alpha_r)$, we associate the following polynomial $P_\omega(t) \in \mathbb{Z}[t]$

$$P_\omega(t) := \prod_{j=1}^{r} (t^{d_j} - 1).$$

Notice that for any $\alpha \in \mathbb{N}^I$, it holds

$$P_{\omega_\alpha}(t) = t - 1.$$

For a semisimple multitype $\omega = (d_1, \beta_1) \cdots (d_r, \beta_r) \in \mathbb{T}_\alpha^{ss}$ put

$$C_\omega^o := \begin{cases} \mu(d) d^{r-1} (-1)^{r-1} (r-1)! & \text{if } d_1 = d_2 = \cdots = d_r = d \\ 0 & \text{otherwise.} \end{cases}$$

where μ denotes the ordinary Möbius function.

Lastly, we introduce the notion of *levels* for semisimple multitypes.

Definition 3.6.2. For a subset $V \subseteq \mathbb{N}^I$ and a semisimple multitype ω with

$$\omega = \psi_{d_1}(\omega_{\alpha_1}) * \cdots * \psi_{d_r}(\omega_{\alpha_r}),$$

we say that ω is of level V if $\alpha_j \in V$ for each $j = 1, \dots, r$.

Example 3.6.3. Notice that, for any $\alpha \in \mathbb{N}^I$, the multitype ω_α is of level $\{\alpha\}$. Conversely, the only multitype $\omega \in \mathbb{T}_\alpha$ of level $\{\alpha\}$ is ω_α.

3.6.1 Multitypes and conjugacy classes

For any $\alpha \in \mathbb{N}^I$, the multitypes $\mathbb{T}_\alpha$ parametrize the conjugacy classes of $\mathrm{GL}_\alpha(\mathbb{F}_q)$ in the following way.
Firstly, recall that the conjugacy classes of $\mathrm{GL}_\alpha(\mathbb{F}_q)$ are in bijection with the F-stable conjugacy classes of $\mathrm{GL}_\alpha(\overline{\mathbb{F}}_q)$, i.e for each element $O \in \mathrm{Cl}(\mathrm{GL}_\alpha(\mathbb{F}_q))$ there exists an F-stable conjugacy class $\mathcal{O} \subseteq \mathrm{GL}_\alpha(\overline{\mathbb{F}}_q)$ such that

$$O = \mathcal{O}^F,$$

see for example [28, Proposition 4.2.14].

We fix the following notations. For each element $x \in \overline{\mathbb{F}}_q^*$ and each multipartition $\boldsymbol{\lambda} \in \mathcal{P}^I$, denote by $J(x, \boldsymbol{\lambda}) \in \mathrm{GL}_\alpha(\overline{\mathbb{F}}_q)$ the element such that $J(x, \boldsymbol{\lambda})_i$ is the matrix with upper triangular Jordan blocks of eigenvalue x and sizes indexed by the partition λ^i.

Consider now an orbit θ of the action of F on $\overline{\mathbb{F}}_q^*$ and let $d = |\theta|$. For each $\boldsymbol{\lambda} \in \mathcal{P}^I$, the Frobenius acts on the element

$$J'(\theta, \boldsymbol{\lambda}) = \bigoplus_{x \in \theta} J(x, \boldsymbol{\lambda}) \in \mathrm{GL}_{d|\boldsymbol{\lambda}|}(\overline{\mathbb{F}}_q)$$

(where we are taking the componentwise direct sum) by permuting the factors $J(x, \boldsymbol{\lambda})$ and so the conjugacy class of $J'(\theta, \boldsymbol{\lambda})$ is F-stable. Therefore, there exists an element $J(\theta, \boldsymbol{\lambda}) \in \mathrm{GL}_{d\boldsymbol{\lambda}}(\mathbb{F}_q)$ conjugated to $J'(\theta, \boldsymbol{\lambda})$.

Fix now a conjugacy class $O \in \mathrm{Cl}_{\mathrm{GL}_\alpha(\mathbb{F}_q)}$ and an element $g = (g_i)_{i \in I} \in \mathcal{O}^F$. For each $i \in I$, denote by $E_{g_i} \subseteq \overline{\mathbb{F}}_q^*$ the set of eigenvalues of g_i and denote by $E_O \subseteq \overline{\mathbb{F}}_q^*$ the subset

$$E_O = \bigcup_{i \in I} E_{g_i}.$$

Since the conjugacy class $\mathcal{O}$ is F-stable, the subset E_O is F-stable too. There exist therefore $\theta_1, \dots, \theta_r$ orbits for action of F on $\overline{\mathbb{F}}_q^*$ such that $\theta_i \neq \theta_j$ for each $i \neq j$ and $E_O = \bigsqcup_{j=1}^r \theta_j$. Looking at the Jordan decomposition of the elements g_i, we find multipartition $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_r$ such that g is conjugated to

$$\prod_{j=1}^r J(\theta_j, \boldsymbol{\lambda}_j).$$

For each $j = 1, \dots, r$, denote by d_j the cardinality of θ_j. The type associated to O is

$$\omega_O = (d_1, \boldsymbol{\lambda}_1) \dots (d_r, \boldsymbol{\lambda}_r).$$

Example 3.6.4. Let $I = \{1, 2, 3, 4\}$ and $\alpha = (2, 1, 1, 1)$. Consider the conjugacy class O given by

$$O = \left\{ \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \lambda, \lambda, \lambda \right) \mid \lambda \in \mathbb{F}_q^* \right\}.$$

The associated type is $\omega_O = (1, (1^\alpha))$.

Remark 3.6.5. Notice that multitype ω_O of a conjugacy class O is semisimple if and only if the Jordan block of the elements of O all have size 1, i.e if and only if the conjugacy class O is semisimple.

For any $O \in \mathrm{Cl}(\mathrm{GL}_\alpha(\mathbb{F}_q))$ and any $\omega \in \mathbb{T}_\alpha$, we write $O \sim \omega$ if $\omega_O = \omega$. Similarly, for any $g \in \mathrm{GL}_\alpha(\mathbb{F}_q)$, we write $g \sim \omega$ if the type of the conjugacy class of g is ω.

Recall that the cardinality of the centralizer $C_{\mathrm{GL}_\alpha(\mathbb{F}_q)}(g)$ for $g \in \mathrm{GL}_\alpha(\mathbb{F}_q)$ depends only on the type of g. More precisely, for each $\lambda = (1^{m_{1,\lambda}}, 2^{m_{2,\lambda}}, \dots,) \in \mathcal{P}$, let

$$Z_\lambda(t) = t^{\langle \lambda, \lambda \rangle} \prod_{j \geq 1} \varphi_{m_{j,\lambda}}(t^{-1})$$

where $\varphi_m(t) = (1-t)(1-t^2)\cdots(1-t^m)$. For a multipartition $\boldsymbol{\lambda} \in \mathcal{P}^I$, define

$$Z_{\boldsymbol{\lambda}}(t) := \prod_{i \in I} Z_{\lambda^i}(t)$$

and for a multitype $\omega \in \mathbb{T}$ with $\omega = (d_1, \boldsymbol{\lambda}_1) \dots (d_r, \boldsymbol{\lambda}_r)$ let

$$Z_\omega(t) = \prod_{j \geq 1}^{r} Z_{\boldsymbol{\lambda}_j}(t^{d_j}). \tag{3.6.1}$$

We have the following Lemma, see for example [70, II, (1.6)].

Lemma 3.6.6. *For any $\omega \in \mathbb{T}_\alpha$ and $g \in \mathrm{GL}_\alpha(\mathbb{F}_q)$ such that $g \sim \omega$ we have*

$$|C_{\mathrm{GL}_\alpha(\mathbb{F}_q)}(g)| = Z_\omega(q). \tag{3.6.2}$$

3.7 Lambda rings and plethystic operations

In this paragraph we recall the definition and some properties of λ-rings. We follow [39].

Definition 3.7.1.
- A λ-ring R is a commutative $\mathbb{Q}$-algebra with homomorphisms $\psi_d : R \to R$ for any $d \geq 1$ such that $\psi_{d'}(\psi_d(r)) = \psi_{dd'}(r)$ for every $d, d' \in \mathbb{N}_{>0}$ and $r \in R$. The morphisms ψ_d are called Adams operations.
- A morphism $f : R \to R'$ of λ-rings is a morphism of $\mathbb{Q}$-algebras commuting with the Adams operations.

For any partition $\mu = (\mu_1, \dots, \mu_h)$, we denote by $\psi_\mu : R \to R$ the homomorphism defined by $\psi_\mu(r) = \psi_{\mu_1}(r) \cdots \psi_{\mu_h}(r)$.

Example 3.7.2. The ring $\mathbb{Q}(t)$ is a λ-ring , with Adams operations $\psi_d(f(t)) = f(t^d)$. Notice that we have

$$\psi_d(t^n) = t^{nd}.$$

Remark 3.7.3. The ring $\Lambda(\mathbf{x})$ is a λ-ring, with Adams operation $\psi_d(f(\mathbf{x})) = f(\mathbf{x}^d)$. We remark that we have

$$\psi_d(p_n(\mathbf{x})) = p_{nd}(\mathbf{x}).$$

Notice that, for any λ-ring R and any element $r \in R$, we can define a unique λ-ring homomorphism $\Lambda(\mathbf{x}) \to R$ as

$$p_\lambda(\mathbf{x}) \to \psi_\lambda(r).$$

More precisely, for any λ-ring R, we can define an operation called *plethysm* linear on the first component

$$\circ : \Lambda(\mathbf{x}) \times R \to R$$

such that we have

- For any $r \in R$, the map $- \circ r : \Lambda(\mathbf{x}) \to R$ is a ring homomorphism.
- $p_d(\mathbf{x}) \circ a = \psi_d(a)$.

For every integer $n \in \mathbb{N}$, denote by $\sigma_n(f)$ the element

$$\sigma_n(f) \coloneqq \sum_{\lambda \in \mathcal{P}_n} \frac{\psi_\lambda(f)}{z_\lambda} \tag{3.7.1}$$

Notice that, since

$$h_n(\mathbf{x}) = \sum_{\lambda \in \mathcal{P}_n} \frac{p_\lambda(\mathbf{x})}{z_\lambda},$$

(see for example [70, I, (2.14)], we have

$$\sigma_n(f) = h_n(\mathbf{x}) \circ f,$$

for any $f \in R$. We have the following Lemma.

Lemma 3.7.4. *For any $f, g \in R$, we have the following identity:*

$$\sigma_n(f+g) = \sum_{n_1+n_2=n} \sigma_{n_1}(f)\sigma_{n_2}(g). \tag{3.7.2}$$

Proof. Notice indeed that we can rewrite

$$\sigma_n(f) = \sum_{\lambda \in \mathcal{P}_n} \frac{\psi_\lambda(f)}{z_\lambda} = \sum_{\lambda \in \mathcal{P}_n} \prod_{j \geq 1} \frac{1}{m_{j,\lambda}!} \left(\frac{\psi_j(f)}{j}\right)^{m_{j,\lambda}} = \sum_{\substack{(m_j)_{j\geq 1} \\ \sum m_j j = n}} \prod_{j \geq 1} \frac{1}{m_j!} \left(\frac{\psi_j(f)}{j}\right)^{m_j}.$$

We have therefore

$$\sum_{n_1+n_2=n} \sigma_{n_1}(f)\sigma_{n_2}(g) = \sum_{n_1+n_2=n} \left(\sum_{\substack{(m^1_j)_{j\geq 1} \\ \sum m^1_j j = n_1}} \prod_{j \geq 1} \frac{1}{m^1_j!} \left(\frac{\psi_j(f)}{j}\right)^{m^1_j} \right) \left(\sum_{\substack{(m^2_j)_{j\geq 1} \\ \sum m^2_j j = n_2}} \prod_{j \geq 1} \frac{1}{m^2_j!} \left(\frac{\psi_j(g)}{j}\right)^{m^2_j} \right) = \tag{3.7.3}$$

$$\sum_{n_1+n_2=n} \sum_{\substack{(m^1_j, m^2_j)_{j\geq 1} \\ \sum m^1_j j = n_1 \\ \sum m^2_j j = n_2}} \prod_{j \geq 1} \frac{1}{m^1_j! m^2_j!} \left(\frac{\psi_j(f)}{j}\right)^{m^1_j} \left(\frac{\psi_j(g)}{j}\right)^{m^2_j} \tag{3.7.4}$$

and

$$\sigma_n(f+g) = \sum_{\substack{(m_j)_{j\geq 1}\\ \sum m_j j = n}} \prod_{j\geq 1} \frac{1}{m_j!}\left(\frac{\psi_j(f+g)}{j}\right)^{m_j} = \sum_{\substack{(m_j)_{j\geq 1}\\ \sum m_j j = n}} \prod_{j\geq 1} \frac{1}{m_j!}\left(\sum_{m_j^1+m_j^2=m_j} \frac{\psi_j(f)^{m_j^1}}{j^{m_j^1}}\frac{\psi_j(g)^{m_j^2}}{j^{m_j^2}}\frac{m_j!}{m_j^1!m_j^2!}\right). \tag{3.7.5}$$

Rearranging the terms, we see that the RHS of eq.(3.7.4) and eq.(3.7.5) are equal. □

For a λ-ring R, consider now the ring $R[[y_i]]_{i\in I}$. For $\alpha \in \mathbb{N}^I$ we will denote by y^α the monomial

$$y^\alpha := \prod_{i\in I} y_i^{\alpha_i}.$$

We endow the ring $R[[y_i]]_{i\in I}$ with the λ-ring structure given by the Adams operations defined as

$$\psi_d(ry^\alpha) := \psi_d(r)y^{d\alpha}$$

for $r \in R$ and $\alpha \in \mathbb{N}^I$. Denote by $R[[y_i]]^+_{i\in I}$ the ideal generated by the y_i's.

The *plethystic exponential* is the following map

$$\mathrm{Exp} : R[[y_i]]^+_{i\in I} \to 1 + R[[y_i]]^+_{i\in I}$$

$$\mathrm{Exp}(f) = \exp\left(\sum_{n\geq 1}\frac{\psi_n(f)}{n}\right). \tag{3.7.6}$$

Notice that for $f, g \in R[[y_i]]^+_{i\in I}$, we have

$$\mathrm{Exp}(f+g) = \mathrm{Exp}(f)\,\mathrm{Exp}(g). \tag{3.7.7}$$

We have the following Lemma, which gives another formula to compute Exp.

Lemma 3.7.5. *For any $f \in R[[y_i]]^+_{i\in I}$, we have*

$$\mathrm{Exp}(f) = \sum_{n\geq 1}\sigma_n(f) \tag{3.7.8}$$

Proof. By Formula (3.7.7) and Formula (3.7.2), it is enough to show eq.(3.7.8) in the case of $f = ry^\alpha$ for any $r \in R$ and for any $\alpha \in \mathbb{N}^I$, i.e we can assume $|I| = 1$. Recall that, in the ring $\Lambda(\mathbf{x})[[T]]$, we have the following identity (see for example [70, I, (2.10)]):

$$\exp\left(\sum_{n\geq 1}\frac{p_n(\mathbf{x})}{n}T^n\right) = \sum_{n\geq 1} h_n(\mathbf{x})T^n. \tag{3.7.9}$$

We can rewrite eq.(3.7.9) as the identity

$$\mathrm{Exp}(p_1(\mathbf{x})T) = \sum_{n\geq 1} h_n(\mathbf{x})T^n \tag{3.7.10}$$

For any $r \in R$, the plethysm $-\circ r$: is a λ-ring homomorphism $\Lambda(\mathbf{x}) \to R$. We have therefore

$$\mathrm{Exp}(rT) = \mathrm{Exp}(p_1(\mathbf{x}) \circ rT) = (-\circ r)(\mathrm{Exp}(p_1(\mathbf{x})T)) = (-\circ r)\left(\sum_{n\geq 1} h_n(\mathbf{x})T^n\right) \tag{3.7.11}$$

$$= \sum_{n\geq 1} \sigma_n(r)T^n = \sum_{n\geq 1} \sigma_n(rT). \tag{3.7.12}$$

□

Example 3.7.6. Consider $R = \mathbb{Q}$ and $|I| = 1$. In the ring $\mathbb{Q}[[T]]$ we have:

$$\mathrm{Exp}(T) = \exp\left(\sum_{n\geq 1} \frac{T^n}{n}\right) = \exp\left(\log\left(\frac{1}{1-T}\right)\right) = \frac{1}{1-T} \tag{3.7.13}$$

Consider the λ-ring $R = \mathbb{Q}[q]$, equipped with the Adams operations

$$\psi_d(f(q)) = f(q^d)$$

for $f(q) \in \mathbb{Q}[q]$. Notice that $\psi_d(\mathbb{Z}[q]) \subseteq \mathbb{Z}[q]$. We have the following Lemma:

Lemma 3.7.7. *Given* $f \in \mathbb{Z}[q][[y_i]]^+_{i\in I}$ *we have* $\mathrm{Exp}(f) \in 1 + \mathbb{Z}[q][[y_i]]^+_{i\in I}$

Proof. Notice that by Formula (3.7.7), it is enough to show that $\mathrm{Exp}(q^m y^\alpha) \in 1+\mathbb{Z}[q][[y_i]]^+_{i\in I}$ for any $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}^I$, i.e we can assume $|I| = 1$.
Similarly to Example 3.7.6 above, we can show that, for any $m \in \mathbb{N}$, in the ring $\mathbb{Q}[q][[T]]$, we have

$$\mathrm{Exp}(q^m T) = \frac{1}{1-q^m T} = \sum_{n\in\mathbb{N}} q^{mn}T^n \in 1 + \mathbb{Z}[q][[T]] \tag{3.7.14}$$

□

The plethystic exponential admits an inverse operation $\mathrm{Log} : 1 + R[[y_i]]^+_{i\in I} \to R[[y_i]]^+_{i\in I}$ known as *plethystic logarithm.* The plethystic logarithm can either be defined by the property $\mathrm{Log}(\mathrm{Exp}(f)) = f$ or by the following explicit rule. For $\alpha \in \mathbb{N}^I$ we put

$$\overline{\alpha} \coloneqq \gcd(\alpha_i)_{i\in I}$$

and we define $U_\alpha \in R$ by:

$$\log(f) = \sum_{\alpha \in \mathbb{N}^I} \frac{U_\alpha}{\overline{\alpha}} y^\alpha. \tag{3.7.15}$$

Then put

$$\operatorname{Log}(f) \coloneqq \sum_{\alpha \in \mathbb{N}^I} V_\alpha y^\alpha \tag{3.7.16}$$

where

$$V_\alpha \coloneqq \frac{1}{\overline{\alpha}} \sum_{d \mid \overline{\alpha}} \mu(d) \psi_d(U_{\frac{\alpha}{d}}) \tag{3.7.17}$$

Notice that, for $A, B \in 1 + R[[y_i]]^+_{i \in I}$, we have

$$\operatorname{Log}(AB) = \operatorname{Log}(A) + \operatorname{Log}(B).$$

We have the following Lemma, which is often useful to compute plethystic logarithm (for a proof see [75, Lemma 22]). For $g \in R$ let

$$g_n \coloneqq \frac{1}{n} \sum_{s \mid n} \mu\left(\frac{n}{s}\right) \psi_s(g).$$

Lemma 3.7.8. *Given $f_1, f_2 \in 1 + R[[y_i]]^+_{i \in I}$ such that*

$$\log(f_1) = \sum_{d \geq 1} g_d \log(\psi_d(f_2)) \tag{3.7.18}$$

the following equality holds

$$\operatorname{Log}(f_1) = g \operatorname{Log}(f_2). \tag{3.7.19}$$

3.7.1 Plethysm and multitypes

Let $\mathcal{K}_I^{ss}$ be the $\mathbb{Q}$-vector space having as a base the semisimple multitypes $\mathbb{T}_I^{ss}$. The size of the types endows $\mathcal{K}_I^{ss}$ with an $\mathbb{N}^I$-grading and we denote by $\mathcal{K}_\alpha^{ss}$ the elements of grade α.
The operation $*$ endows $\mathcal{K}_I^{ss}$ with the structure of a $\mathbb{Q}$-algebra in the following way. For $x = x_1\omega_1 + \cdots + x_r\omega_r$ and $y = x'_1\omega'_1 + \cdots + x'_h\omega'_h$ with $x_s, x'_t \in \mathbb{Q}$ and $\omega_s, \omega'_t \in \mathbb{T}_I^{ss}$, we define

$$x * y \coloneqq \sum_{\substack{1 \leq s \leq r \\ 1 \leq t \leq h}} x_s x'_t \omega_s * \omega'_t.$$

Notice that $(\mathcal{K}_I^{ss}, *)$ is an $\mathbb{N}^I$-graded algebra, i.e $\mathcal{K}_\alpha^{ss} * \mathcal{K}_\beta^{ss} \subseteq \mathcal{K}_{\alpha+\beta}^{ss}$.

The functions $\psi_d : \mathbb{T}_I \to \mathbb{T}_I$ endow the $\mathbb{Q}$-algebra $\mathcal{K}_I^{ss}$ with the structure of a λ-ring with Adams operations

$$\psi_d(q_1\omega_1 + \cdots + q_r\omega_r) = q_1\psi_d(\omega_1) + \cdots + q_r\psi_d(\omega_r)$$

for any element $x = q_1\omega_1 + \cdots + q_r\omega_r \in \mathcal{K}_I^{ss}$ wth $q_1, \dots, q_r \in \mathbb{Q}$ and $\omega_1, \dots, \omega_r \in \mathbb{T}_I^{ss}$.
Notice that given a semisimple multitype $\omega = (d_1, (1^{\alpha_1}))\dots(d_r, (1^{\alpha_r}))$, we have an equality

$$\omega = \psi_{d_1}(\omega_{\alpha_1}) * \cdots * \psi_{d_r}(\omega_{\alpha_r}).$$

Therefore, we deduca that $\mathcal{K}_I^{ss}$ is isomorphic to the ring of polynomials in the variables $\psi_d(\omega_\alpha)$ for $(d, \alpha) \in \mathbb{N}_{>0} \times \mathbb{N}^I$.

Consider now the ring $\hat{\mathcal{K}}_I^{ss} := \mathcal{K}_I^{ss}[[y_i]]_{i \in I}$. For semisimple multitypes of level V, we have the following lemma:

Lemma 3.7.9. *For any $V \subseteq \mathbb{N}^I$, in the ring $\hat{\mathcal{K}}_I^{ss}$ we have:*

$$\operatorname{Exp}\left(\sum_{\alpha \in V} \omega_\alpha y^\alpha\right) = \sum_{\substack{\omega \in \mathbb{T}_I^{ss} \\ \text{of level } V}} \frac{\omega}{w(\omega)} y^{|\omega|} \tag{3.7.20}$$

Proof. By eq.(3.7.1), there is an equality

$$\operatorname{Exp}\left(\sum_{\alpha \in V} \omega_\alpha y^\alpha\right) = \prod_{\alpha \in V}\left(\sum_{n \in \mathbb{N}} \sigma_n(\omega_\alpha) y^{n\alpha}\right) = \prod_{\alpha \in V}\left(\sum_{\lambda \in \mathcal{P}} \frac{\psi_\lambda(\omega_\alpha)}{z_\lambda} y^{|\lambda|\alpha}\right). \tag{3.7.21}$$

For each semisimple type ω of level V, there exist unique $\beta_1 \neq \beta_2 \neq \ldots \neq \beta_h \in V$ and integers $d_{1,1}, \dots d_{1,l_1}, d_{2,1}, \dots, d_{h,l_h}$ such that $\omega = (d_{1,1}, (1^{\beta_1}))(d_{1,2}, (1^{\beta_1}))\cdots(d_{h,l_h}, (1^{\beta_h}))$ i.e

$$\omega = \psi_{d_{1,1}}(\omega_{\beta_1}) * \cdots * \psi_{d_{h,l_h}}(\omega_{\beta_h}).$$

Up to reordering, we can assume that for each $j = 1, \dots, h$, the sequence of integers $(d_{j,1}, \dots, d_{j,l_j})$ forms a partition λ_j. Therefore, we have

$$\omega = \prod_{j=1}^{h} \psi_{\lambda_j}(\omega_{\beta_j}).$$

Notice moreover that $z_{\lambda_1} \cdots z_{\lambda_h} = w(\omega)$. This implies that the RHS of eq.(3.7.21) is equal to the RHS of eq.(3.7.20). □

3.8 HLRV kernels

Let $\mathbf{x_1} = \{x_{1,1}, x_{1,2} \dots\}, \dots, \mathbf{x_k} = \{x_{k,1}, \dots\}$ be k sets of infinitely many variables and put

$$\Lambda_k := \mathbb{Q}(z, w) \otimes \Lambda(\mathbf{x_1}) \otimes \cdots \otimes \Lambda(\mathbf{x_k}),$$

i.e Λ_k is the ring of functions over $\mathbb{Q}(z, w)$ separetely symmetric in each set of variables.
The λ-ring structures on each $\Lambda(\mathbf{x}_i)$ define a natural λ-ring structure on Λ_k, with Adams operations $\psi_d : \Lambda_k \to \Lambda_k$ given by

$$\psi_d(f(\mathbf{x}_1, \dots, \mathbf{x}_k)) = f(\mathbf{x}_1^d, \dots, \mathbf{x}_k^d)$$

Moreover, on Λ_k it is defined a natural bilinear form obtained by extending by linearity

$$\langle f_1(\mathbf{x_1})\cdots f_k(\mathbf{x_k}), g_1(\mathbf{x_1})\cdots g_k(\mathbf{x_k})\rangle = \prod_{i=1}^{k} \langle f_i, g_i\rangle .$$

For any $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k) \in \mathbb{T}_n^k$, denote by $s_{\boldsymbol{\omega}} \in \Lambda_k$ the function

$$s_{\boldsymbol{\omega}} \coloneqq s_{\omega_1}(\mathbf{x_1})\cdots s_{\omega_k}(\mathbf{x_k}),$$

where, for each $i = 1, \dots, k$, given $\omega_i = (d_{i,1}, \lambda_{i,1})\dots(d_{i,r_i}, \lambda_{i,r_i})$, we define

$$s_{\omega_i}(\mathbf{x_i}) = \prod_{j=1}^{r_i} s_{\lambda_{i,j}}(\mathbf{x_i}^{d_{i,j}}).$$

Example 3.8.1. Consider a multipartition $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k)$ with $|\mu^i| = |\mu^j| = n$ for each i, j. We denote by $\boldsymbol{\mu}$ also the corresponding element in $\mathbb{T}_n^k$, with the notation introduced in Example 3.6.1. Notice that, for each $j = 1, \dots, s_i$, we have $s_{(\mu^i_j)}(\mathbf{x_i}) = h_{\mu^i_j}(\mathbf{x_i})$ and therefore

$$s_{\omega_i}(\mathbf{x_i}) = \prod_{j=1}^{s_i} h_{\mu^i_j}(\mathbf{x_i}) = h_{\mu^i}(\mathbf{x_i}).$$

For $r \in \mathbb{N}$ and any $\lambda \in \mathcal{P}$, let $\mathcal{H}_{r,\lambda}(z,w)$ be the hook function:

$$\mathcal{H}_{r,\lambda}(z,w) \coloneqq \prod_{s\in\lambda} \frac{(z^{2a(s)+1} - w^{2l(s)+1})^r}{(z^{2a(s)+2} - w^{2l(s)})(z^{2a(s)} - w^{2l(s)+2})} \tag{3.8.1}$$

and the associated series $\Omega_r(z,w) \in \Lambda_k[[T]]$

$$\Omega_r(z,w) \coloneqq \sum_{\lambda\in\mathcal{P}} \mathcal{H}_{r,\lambda}(z,w) \prod_{i=1}^{k} H_\lambda(\mathbf{x_i}, z^2, w^2) T^{|\lambda|} \tag{3.8.2}$$

where $H_\lambda(\mathbf{x_i}, q, t)$ are the (modified) Macdonald symmetric polynomials (for a definition see [38, I.11]).

For any $n \in \mathbb{N}$, Hausel, Letellier and Rodriguez-Villegas [45] introduced the following function $\mathbb{H}_{n,r}(z,w) \in \Lambda_k$, defined as

$$\mathbb{H}_{n,r}(z,w) \coloneqq (z^2-1)(1-w^2)\,\mathrm{Coeff}_{T^n}(\mathrm{Log}(\Omega_r(z,w))). \tag{3.8.3}$$

These functions are known as HLRV kernel and are of fundamental importance in the description of the cohomology of generic character varieties and generic multiplicities, see §8 and §9.

Lastly, consider an element $\boldsymbol{\omega} \in \mathbb{T}_n^k$. We denote by $\mathbb{H}_{\boldsymbol{\omega},r}(z,w) \in \mathbb{Q}(z,w)$ the rational function defined as

$$\mathbb{H}_{\boldsymbol{\omega},r}(z,w) \coloneqq \langle \mathbb{H}_{n,r}(z,w), s_{\boldsymbol{\omega}} \rangle.$$

4 General linear groups and admissible subtori

In this chapter, we study the properties of certain subtori of the groups GL_α, which we call *admissible*. Put $n_\alpha = \sum_{i\in I}\alpha_i$ and embed GL_α inside GL_{n_α} through the block diagonal embedding. The admissible subtori are the subtori of GL_α which are center of a Levi subgroup of GL_{n_α}.
These subtori will play a central role in the thesis. They are the center of the stabilizers of the action of GL_α on $R(Q,\alpha)$ and of GL_α on a multiplicative moment fiber $(\Phi^*_\alpha)^{-1}(\sigma)$.
In addition, they are a key part of the classification of the irreducible characters of $\mathrm{GL}_\alpha(\mathbb{F}_q)$.
The aim of this chapter is to study the combinatorial properties of these tori and how they are related to multitypes and plethystic operations.

In section §4.1, we review some properties of reductive groups over a field K, of their Levi and parabolic subgroups and recall the definition of flag varieties for GL_n.
In section §4.2 we consider $K = \mathbb{F}_q$, we describe the maximal tori of G defined over $\mathbb{F}_q$ in terms of its Weyl group and of its root system and give a more explicit description of these objects in the case of general linear groups.

In section §4.3, we give the definition of an admissible subtorus of GL_α and we show how to associate a semisimple multitype $[S] \in \mathbb{T}^{ss}_\alpha$ to each admissible subtorus $S \subseteq \mathrm{GL}_\alpha$.
In section §4.4, we associate a graph Γ_S to each admissible subtorus S. This association is a key technical point of the proof of Theorem 4.5.2 regarding *Log compatible families*, whose proof is the main content of section §4.5.
Theorem 4.5.2 is the main technical result of the thesis, from which we will deduce Theorem 9.3.2 about E-series of character stacks for Riemann surfaces.

4.1 Reductive groups, maximal tori and Levi subgroups

Let K be an algebraically closed field. In this paragraph, G is a connected reductive group over K. We denote by $\mathrm{rank}(G)$ the dimension of a maximal torus of G.
For a maximal torus $T \subseteq G$, we denote by

$$X_*(T) := \mathrm{Hom}(T, \mathbb{G}_m)$$

and

$$Y_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$$

the group of *characters* and *cocharacters* of T respectively. Recall that these are free abelian groups of rank $\mathrm{rank}(G)$ and that there is a pairing

$$\langle , \rangle : Y_*(T) \times X_*(T) \to \mathbb{Z},$$

where, for $\beta \in Y_*(T), \alpha \in X_*(T)$, we have

$$\alpha \circ \beta(z) = z^{\langle \beta,\alpha\rangle}$$

for any $z \in \mathbb{G}_m$.
Denote by $W_G(T)$ the Weyl group of G with respect to T, i.e $W_G(T) = N_G(T)/T$. Notice that $W_G(T)$ acts on $X_*(T)$ as follows

$$w \cdot \alpha(t) = \alpha(w \cdot t)$$

for each $w \in W_G(T)$, $t \in T$ and $\alpha \in X_*(T)$. For each $w \in W_G(T)$, we denote by

$$w : X_*(T) \to X_*(T)$$

the corresponding endomorphism.
Recall that inside $X_*(T)$ there is the *root system* $\Phi(T) \subseteq X_*(T)$ given by the characters appearing in the weight space decomposition of the adjoint action of T on $\mathfrak{g} = \mathrm{Lie}(G)$.
For any $\epsilon \in \Phi(T)$, there is an injective homomorphism $u_\epsilon : \mathbb{G}_a \to G$ such that for any $x \in \overline{\mathbb{F}}_q$ and any $t \in T$, we have

$$tu_\epsilon(x)t^{-1} = u_\epsilon(\epsilon(t)x).$$

We denote by $U_\epsilon \subseteq G$ the subgroup $U_\epsilon := \mathrm{Im}(u_\epsilon)$.

Moreover, inside $Y_*(T)$ there is a dual root system $\Phi^\vee(T)$, provided with a canonical bijection

$$\Phi(T) \leftrightarrow \Phi^\vee(T)$$

$$\epsilon \leftrightarrow \epsilon^\vee$$

such that $\langle \epsilon^\vee, \epsilon \rangle = 2$ for every $\epsilon \in \Phi(T)$.

4.1.1 Levi subgroups and parabolic subgroups

Recall that a parabolic subgroup $P \subseteq G$ is a closed, connected subgroup containing a Borel subgroup and denote by U_P its unipotent radical. A Levi factor of P is a reductive subgroup $L \subseteq P$ such that $P = LU_P$, where U_P is the unipotent radical of P. A Levi factor of a parabolic subgroup is called a *Levi subgroup* of G.

Recall that, for any Levi subgroup $L \subseteq G$, there exists a maximal torus $T \subseteq G$ such that $T \subseteq L$. Moreover, the Levi subgroup L can be described in terms of the root systems $\Phi(T)$ as follows.
Consider the subset $\Phi_L(T) \subseteq \Phi(T)$ defined as

$$\Phi_L(T) := \{\epsilon \in \Phi(T) \mid \mathrm{Ker}(\epsilon) \supseteq Z_L^\circ\},$$

where Z_L° is the connected componenent containing the identity of the center $Z_L \subseteq L$.

We have the following Lemma, see [89, Lemma 8.4.2].

Lemma 4.1.1. *The subset $\Phi_L(T)$ is a root subsystem of $\Phi(T)$ and we have :*

1. $C_G(Z_L^\circ) = L$.

2. $Z_L^\circ = \bigcap_{\epsilon \in \Phi_L(T)} (\mathrm{Ker}(\epsilon)^\circ)$

3. $L = T \prod_{\epsilon \in \Phi_L(T)} U_\epsilon.$

Example 4.1.2. For $G = \mathrm{GL}_n$, Levi subgroups and parabolic subgroups can be explicitly described as follows. For any $n_0, \ldots, n_s \in \mathbb{N}$ such that $n_0 + \cdots + n_s = n$, the subgroup $L_{n_0,\ldots,n_s}$ defined as

$$L_{n_0,\ldots,n_s} = \begin{pmatrix} \mathrm{GL}_{n_s} & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & \mathrm{GL}_{n_{s-1}} & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \mathrm{GL}_{n_{s-2}} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{GL}_{n_0} \end{pmatrix}$$

is a Levi subgroup of G. We will denote the group $L_{n_0,\ldots,n_s}$ simply by

$$\mathrm{GL}_{n_s} \times \cdots \times \mathrm{GL}_{n_0} \subseteq \mathrm{GL}_n .$$

Notice that

$$Z_L = \begin{pmatrix} \lambda_s I_{n_s} & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_{s-1} I_{n_{s-1}} & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_{s-2} I_{n_{s-2}} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_0 I_{n_0} \end{pmatrix},$$

for $\lambda_0, \ldots, \lambda_s \in \overline{\mathbb{F}}_q^*$ and in particular that Z_L is connected.

A parabolic subgroup $P \supseteq L$ containing L as Levi factor is, for instance, given by the upper block triangular matrices

$$P = \begin{pmatrix} \mathrm{GL}_{n_s} & * & * & * & * & \ldots & * \\ 0 & \mathrm{GL}_{n_{s-1}} & * & * & * & \ldots & * \\ 0 & 0 & \mathrm{GL}_{n_{s-2}} & * & * & \ldots & * \\ \vdots & \vdots & \vdots & \ddots & \ldots & \ldots & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{GL}_{n_0} \end{pmatrix}.$$

It is not difficult to verify that, for any Levi subgroup $L \subseteq \mathrm{GL}_n$, there exist $n_0, \ldots, n_r$ such that $n_0 + \cdots + n_r = n$ and L is conjugated to

$$\mathrm{GL}_{n_r} \times \cdots \times \mathrm{GL}_{n_0} .$$

4.1.2 Flag varieties

For $G = \mathrm{GL}_n$ and P the parabolic subgroup containing $\mathrm{GL}_{n_0} \times \cdots \times \mathrm{GL}_{n_s}$ as a Levi factor introduced above, the quotient variety GL_n / P is usually called *partial flag variety* and it has the following geometric description.

Proposition 4.1.3. *The variety GL_n / P is isomorphic to the following variety of partial flags of K^n:*

$$\mathrm{GL}_n / P \cong \{\mathcal{F} = (\mathcal{F}_s \subseteq \mathcal{F}_{s-1} \subseteq \cdots \subseteq \mathcal{F}_0 = K^n) \mid \dim(\mathcal{F}_i) = \sum_{j=i}^{s} n_j\}. \tag{4.1.1}$$

The isomorphism is obtained by associating to gP the flag $\mathcal{F}$ such that $\mathcal{F}_i$ is the image via g of the span of the first $\sum_{j=i}^{s} n_j$ vector of the canonical basis of K^n.

4.2 Finite reductive groups, rational tori and Levi subgroups

In this section and in the rest of the chapter, G is a reductive group defined over $\mathbb{F}_q$ with a fixed Frobenius morphism $F : G \to G$. In the cases that interest us in this article, G will always be taken to be a product of factors of type $(\mathrm{GL}_n)_d$'s. Recall that we always have an F-stable maximal subtorus $T \subseteq G$.
We denote by ϵ_G the rank of a maximal split F-stable subtorus of G. Notice that in general $\epsilon_G \neq \mathrm{rank}(G)$. If $\mathrm{rank}(G) = \epsilon_G$, we say that G is *split.*

Example 4.2.1. Consider the group $(\mathrm{GL}_n)_d$. Let $T \subseteq G$ be the maximal torus $T_n \times \cdots \times T_n \subseteq (\mathrm{GL}_n)_d$, where we denote by $T_n \subseteq \mathrm{GL}_n$ the torus of diagonal matrices. Notice that T is F-stable and $\dim(T) = \mathrm{rank}(G) = nd$. However, it is possible to verify that $\epsilon_G = n$, i.e $(\mathrm{GL}_n)_d$ is split if and only if $d = 1$.

Consider an F-stable maximal torus T. Notice that since T is F-stable, the Frobenius acts on the groups $X_*(T), Y_*(T)$ as follows

$$F : X_*(T) \to X_*(T)$$

$$\alpha \to \alpha \circ F$$

and

$$F : Y_*(T) \to Y_*(T)$$
$$\beta \to F \circ \beta.$$

4.2.1 Twisted Frobenius of maximal tori

Fix now a F-stable maximal torus $T \subseteq G$. As T is F-stable, the Frobenius acts on the Weyl group too. Given two elements $h_1, h_2 \in W_G(T)$, we say that they are F-conjugated if there exists $w \in W_G(T)$ such that $h_1 = w h_2 F(w)^{-1}$.

The set of F-conjugacy classes of $W_G(T)$, usually denoted by $H^1(F, W_G(T))$, parametrize the G^F-conjugacy classes of F-stable maximal tori in the following way.
Given a F-stable maximal torus T' there exists $g \in G$ such that $gTg^{-1} = T'$. As $F(T') = T'$ we see that $\dot{w} := g^{-1}F(g)$ belongs to $N_G(T)$ and so determines an associated element $w \in W_G(T)$.

We can reformulate this correspondence in terms of the twisted $\mathbb{F}_q$-structures of the torus T. While the conjugation by g provides an isomorphism $T' \cong T$ over $\overline{\mathbb{F}}_q$, this isomorphism is not in general an $\mathbb{F}_q$-morphism $(T', F) \to (T, F)$.
However, endowing T with the $\mathbb{F}_q$ structure coming from the twisted Frobenius $\dot{w}F : T \to T$, the conjugation by g is an $\mathbb{F}_q$ -isomorphism

$$(T', F) \cong_{\mathbb{F}_q} (T, \dot{w}F).$$

In the following, we assume to have fixed, for each $w \in W_G(T)$, a corresponding F-stable maximal torus $T_w \subseteq G$.

Example 4.2.2. Consider the case of $G = \mathrm{GL}_n$ and $T = T_n$ the torus of diagonal matrices. In this case, we have $W_G(T_n) = S_n$ and the F-action on S_n is trivial. In particular, the F-conjugacy classes of S_n are the conjugacy classes of S_n and are therefore indexed by the partitions $\mathcal{P}_n$ of size n.
For any $\lambda = (\lambda_1, \dots, \lambda_h) \in \mathcal{P}_n$, any associated F-stable maximal torus T' is $\mathrm{GL}_n(\mathbb{F}_q)$-conjugated to

$$(\mathbb{G}_m)_{\lambda_1} \times \cdots \times (\mathbb{G}_m)_{\lambda_h},$$

i.e

$$(T', F) \cong (\mathbb{G}_m)_{\lambda_1} \times \cdots \times (\mathbb{G}_m)_{\lambda_h}.$$

Example 4.2.3. Consider the group $G = (\mathrm{GL}_n)_d$ and the F-stable maximal torus T introduced above. The Weyl group $W_G(T)$ is isomorphic to S_n^d and the corresponding Frobenius action $F : S_n^d \to S_n^d$ is given by

$$F(\sigma_1, \dots, \sigma_d) = (\sigma_d, \sigma_1, \dots, \sigma_{d-1}).$$

The F-conjugacy classes of S_n^d are in bijection with the conjugacy classes of S_n in the following way. Consider $\tau = (\tau_1, \dots, \tau_d), \sigma = (\sigma_1, \dots, \sigma_d) \in S_n^d$. The element $\tau\sigma F(\tau)^{-1}$ is equal to

$$\tau\sigma F(\tau)^{-1} = (\tau_1\sigma_1\tau_d^{-1}, \tau_2\sigma_2\tau_1^{-1}, \dots, \tau_d\sigma_d\tau_{d-1}^{-1}).$$

Notice that we have

$$\prod_{i=0}^{d-1} (\tau\sigma F(\tau)^{-1})_{d-i} = \tau_d(\sigma_d\sigma_{d-1}\cdots\sigma_1)\tau_d^{-1} = \tau_d \left(\prod_{i=0}^{d-1} \sigma_{d-i} \right) \tau_d^{-1}.$$

We deduce therefore that $\sigma, \sigma' \in S_n^d$ are F-conjugated if and only if $\prod_{i=0}^{d-1} \sigma_{d-i}, \prod_{i=0}^{d} \sigma'_{d-i}$ are conjugated in S_n.

Consider a pair of F-stable maximal tori T, T' with $gT'g^{-1} = T$ and $\dot{w} = g^{-1}F(g) \in N_G(T')$ and $w \in W_G(T')$ as above. There is an isomorphism of abelian groups

$$\Psi_g : X_*(T') \to X_*(T)$$

$$\alpha \to \alpha(g^{-1} - g)$$

such that $\Psi_g(\Phi(T')) = \Phi(T)$. Notice that in general, Ψ_g does not commute with the respective Frobenius morphisms on $X_*(T), X_*(T')$ and indeed we have

$$\Psi_g^{-1} F \Psi_g = w \circ F : X_*(T') \to X_*(T'). \tag{4.2.1}$$

4.2.2 The case of finite general linear groups

For $m \in \mathbb{N}$, let GL_m be the general linear group over $\mathbb{F}_q$, with the canonical $\mathbb{F}_q$-structure $F : \mathrm{GL}_m \to \mathrm{GL}_m$. Consider the maximal torus of diagonal matrices $T_m \subseteq \mathrm{GL}_m$. Notice that in this case $W_{\mathrm{GL}_m}(T_m) = S_m$ and the F-action on $W_{\mathrm{GL}_m}(T_m)$ is trivial.

Let $\epsilon_i \in X_*(T_m)$ be the homomorphism

$$\epsilon_i \left(\begin{pmatrix} z_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & z_2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & z_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_m \end{pmatrix} \right) = z_i.$$

Notice that the subset $\{\epsilon_1, \dots, \epsilon_m\} \subset X_*(T_m)$ is a basis of the free abelian group $X_*(T_m)$, which we denote by $\mathcal{B}(T_m)$. Notice moreover that, for each $i = 1, \dots, m$, we have that

$$F(\epsilon_i) = q\epsilon_i.$$

Moreover, for such a basis, we have that

$$\Phi(T_m) = \{\pm\epsilon_i \mp \epsilon_j \mid i \neq j \in \{1, \dots, m\}\}.$$

For $h, j \in \{1, \dots, m\}$, denote by $\epsilon_{h,j} := \epsilon_h - \epsilon_h$. We denote by $\Phi^+(T_m)$ the set of positive roots with respect to the Borel subgroup of upper triangular matrices, i.e

$$\Phi^+(T_m) = \{\epsilon_{i,j} \mid i < j\}.$$

For any other F-stable maximal torus $T \subseteq \mathrm{GL}_m$, fix g such that $gT_mg^{-1} = T$ and the corresponding permutation $w \in W_{\mathrm{GL}_m}(T_m) = S_m$ as the end of paragraph above. Denote by $\mathcal{B}(T) = \Psi_g(\mathcal{B}(T_m))$.
Whenever the torus T is fixed and the context is clear we will denote by ϵ_i also the element $\Psi_g(\epsilon_i) \in \mathcal{B}(T)$ and by $\epsilon_{i,j}$ the element $\Psi_g(\epsilon_{i,j}) \in \Phi(T)$. We denote by $\Phi^+(T) = \Psi_g(\Phi^+(T_m))$.

Notice that, by eq.(4.2.1), in the character group $X_*(T)$ we have

$$F(\epsilon_i) = q\epsilon_{w(i)}.$$

Consider now $\alpha \in \mathbb{N}^I$, put $n_\alpha := \sum_{i \in I} \alpha_i$ and consider GL_α as a subgroup of GL_{n_α} through the block diagonal embedding. Fix an F-stable maximal torus $T \subseteq \mathrm{GL}_\alpha \subseteq \mathrm{GL}_{n_\alpha}$. Notice that T is a maximal torus for GL_α and GL_{n_α}. Consider then the basis $\mathcal{B}(T)$ as above.

Remark 4.2.4. For $i \in I$, let $\pi_i : \mathrm{GL}_\alpha \to \mathrm{GL}_{\alpha_i}$ the canonical projection. For a maximal torus $T \subseteq \mathrm{GL}_\alpha$, denote by $T_i := \pi_i(T)$. Notice that

$$T \subseteq \prod_{i \in I} T_i \subseteq \prod_{i \in I} \mathrm{GL}_{\alpha_i}.$$

As T is a maximal torus, we have thus an equality $T = \prod_{i \in I} T_i$. For dimension reasons, we deduce that, for each $i \in I$, T_i is a maximal torus of GL_{α_i}. From the identity $T = \prod_{i \in I} T_i$, we deduce that there is an isomorphism $X_*(T) = \bigoplus_{i \in I} X_*(T_i)$.

Notice that we can choose $g \in \mathrm{GL}_\alpha$ such that $gT_m g^{-1} = T$. We deduce therefore that putting $\mathcal{B}_i(T) = \mathcal{B}(T) \cap X_*(T_i)$, we obtain a partition

$$\mathcal{B}(T) = \bigsqcup_{i \in I} \mathcal{B}_i(T)$$

such that each $\mathcal{B}_i(T)$ is a basis of $X_*(T_i)$ and $\mathcal{B}_i(T)$ is w-stable for every $i \in I$.

4.2.3 F-stable Levi subgroups

Consider a Levi subgroup $L \subseteq \mathrm{GL}_m$ and assume that L is F-stable. Similarly to what we said about F-stable tori in Example 4.2.2, we can show that there exist $d_0, \dots, d_r \in \mathbb{N}$ and $m_0, \dots, m_r$ such that L is conjugated by an element of $\mathrm{GL}_n(\mathbb{F}_q)$ to the group

$$(\mathrm{GL}_{m_0})_{d_0} \times \cdots \times (\mathrm{GL}_{m_r})_{d_r},$$

i.e there is an $\mathbb{F}_q$-isomorphism

$$(L, F) \cong (\mathrm{GL}_{m_0})_{d_0} \times \cdots \times (\mathrm{GL}_{m_r})_{d_r}.$$

Notice that in this case we have an isomorphism

$$(Z_L, F) \cong (\mathbb{G}_m)_{d_0} \times \cdots \times (\mathbb{G}_m)_{d_r}.$$

4.3 Subtori and multitypes

For $\alpha \in \mathbb{N}^I$, consider GL_α as a subgroup of GL_{n_α} via the block diagonal embedding. Recall that I can be thought of as the set of vertices of a star-shaped quiver. We introduce here the definition of the admissible subtori of GL_α.

Admissible subtori will play a significant role in this thesis. For instance, they appear in the classification of the irreducible characters of the finite group $\mathrm{GL}_\alpha(\mathbb{F}_q)$, see §5.6. They are also a key part of the proof of Theorem 4.5.2, which will be the main technical result needed to study the cohomology of non-generic character stacks.

Definition 4.3.1. An subtorus S of GL_α is said *admissible* if there exists a Levi subgroup $L_S \subseteq \mathrm{GL}_{n_\alpha}$ such that $Z_{L_S} = S$.

Example 4.3.2. For any $\alpha \in \mathbb{N}^I$, there is an admissible subtorus $Z_\alpha \subseteq \mathrm{GL}_\alpha$, given by $Z_\alpha := Z_{\mathrm{GL}_{n_\alpha}} \subseteq \mathrm{GL}_\alpha$, i.e the elements of Z_α are of the form $(\lambda I_{\alpha_i})_{i\in I}$, for $\lambda \in \overline{\mathbb{F}}_q^*$.

We have the following Lemma (see [28, Proposition 3.4.6])

Lemma 4.3.3. *For an admissible S and a Levi subgroup L_S such that $Z_{L_S} = S$, we have $C_{\mathrm{GL}_{n_\alpha}}(S) = L_S$. In particular, the group L_S is unique.*

Remark 4.3.4. Notice that from Lemma 4.3.3 above, we have that S is F-stable if and only if L_S is F-stable.

Example 4.3.5. Put $|I| = 1$ and let $S \subseteq \mathrm{GL}_2$ be the torus

$$S = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{pmatrix}, \quad \lambda \in \overline{\mathbb{F}}_q^* \right\}.$$

Notice that $C_{\mathrm{GL}_2}(S) = T_2$, where T_2 is the torus of diagonal matrices. However, $Z_{T_2} = T_2 \neq S$. We deduce thus that the torus S is not admissible.

Consider an admissible subtorus $S \subseteq \mathrm{GL}_\alpha$ and the associated Levi subgroup $L_S \subseteq \mathrm{GL}_{n_\alpha}$. The group $C_{\mathrm{GL}_\alpha}(S)$ is a Levi subgroup of GL_α (see [28, Proposition 3.4.7]) which we will denote by $\widetilde{L_S}$.
Notice that $\widetilde{L_S}$ is equal to $L_S \cap \mathrm{GL}_\alpha$ as $C_{\mathrm{GL}_{n_\alpha}}(S) \cap \mathrm{GL}_\alpha = C_{\mathrm{GL}_\alpha}(S)$. In particular, there exists a maximal torus $T \subseteq \mathrm{GL}_\alpha$ such that $S \subseteq T \subseteq \widetilde{L_S}$.
Conversely, consider an F-stable Levi subgroup $L \subseteq \mathrm{GL}_{n_\alpha}$ such that there exists a maximal torus $T \subseteq L \cap \mathrm{GL}_\alpha$. As $Z_L \subseteq T$, the center Z_L is an admissible subtorus of GL_α.

Example 4.3.6. Notice that even if two admissible tori S, S' are different, we can have $\widetilde{L_S} = \widetilde{L_{S'}}$. Consider for example $S = Z_\alpha$ and S' defined as

$$S' = \{(\lambda_i I_{\alpha_i})_{i\in I} \mid (\lambda_i)_{i\in I} \in (\overline{\mathbb{F}}_q^*)^I\}.$$

In general, we have $S \neq S'$. However, for any $\alpha \in \mathbb{N}^I$, for both tori we have

$$\widetilde{L_S} = \widetilde{L_{S'}} = \mathrm{GL}_\alpha \,.$$

For each $\alpha \in \mathbb{N}^I$, denote by $\mathcal{Z}_\alpha$ the subset of F-stable admissible subtori of GL_α and denote by $\mathcal{Z}$ the set defined as

$$\mathcal{Z} := \bigsqcup_{\alpha \in \mathbb{N}^I} \mathcal{Z}_\alpha.$$

Example 4.3.7. Consider $I = \{1,2,3,4\}$ and $\alpha = (2,1,1,1) \in \mathbb{N}^I$. Notice that $n_\alpha = 5$. Consider for example the admissible subtori $S_1, S_2, S_3 \subseteq \mathrm{GL}_\alpha$ given by

$$S_1 = \left\{ \left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \lambda, \lambda, \lambda \right) \mid \lambda, \mu \in \overline{\mathbb{F}}_q^* \right\}$$

$$S_2 = \left\{ \left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \lambda, \mu, \lambda \right) \mid \lambda, \mu \in \overline{\mathbb{F}}_q^* \right\}$$

and

$$S_3 = \left\{ \left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \gamma, \delta, \eta \right) \mid \lambda, \mu, \gamma, \delta, \eta \in \overline{\mathbb{F}}_q^* \right\}.$$

In this case, L_{S_1} is $\mathrm{GL}_5(\mathbb{F}_q)$-conjugated to $\mathrm{GL}_4 \times \mathrm{GL}_1$ and $\widetilde{L_{S_1}} = T_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1$, where $T_2 \subseteq \mathrm{GL}_2$ is the torus of diagonal matrices. Moreover, we have that L_{S_2} is $\mathrm{GL}_5(\mathbb{F}_q)$-conjugated to $\mathrm{GL}_2 \times \mathrm{GL}_3$ and $\widetilde{L_{S_1}} = \widetilde{L_{S_2}}$.
Lastly, notice that L_{S_3} is the maximal torus of diagonal matrices $T_5 \subseteq \mathrm{GL}_5$, and $\widetilde{L_{S_3}} = \widetilde{L_{S_1}}$ too.

For a multitype $\omega = (d_1, \boldsymbol{\lambda}_1) \dots (d_r, \boldsymbol{\lambda}_r)$ of size α, we denote by $S_\omega \in \mathcal{Z}_\alpha$ the torus defined as

$$(Z_{|\boldsymbol{\lambda}_1|})_{d_1} \times \cdots \times (Z_{|\boldsymbol{\lambda}_r|})_{d_r} \subseteq \mathrm{GL}_\alpha$$

where $(Z_{|\boldsymbol{\lambda}_1|})_{d_1} \times \cdots \times (Z_{|\boldsymbol{\lambda}_r|})_{d_r}$ is considered a subtorus of GL_α via the componentwise block diagonal embedding. Put $\beta_j = |\boldsymbol{\lambda}_j| \in \mathbb{N}^I$, for each $j = 1, \dots, r$. For the Levi subgroup $L_\omega \subseteq \mathrm{GL}_{n_\alpha}$ defined as

$$L_\omega = (\mathrm{GL}_{|\beta_1|})_{d_1} \times \cdots \times (\mathrm{GL}_{|\beta_r|})_{d_r}$$

embedded block diagonally, we have $Z_{L_\omega} = S_\omega$, i.e S_ω is admissible.
We will denote by $\widetilde{L_\omega}$ the Levi subgroup of GL_α defined as $\widetilde{L_\omega} := L_\omega \cap \mathrm{GL}_\alpha$. Notice that the groups $L_\omega, S_\omega, \widetilde{L_\omega}$ depend only on the semisimplification ω^{ss} of ω.

Remark 4.3.8. Let $\omega \in \mathbb{T}_\alpha$ and $d_1, \dots, d_r \in \mathbb{N}$ and $\beta_1, \dots, \beta_r \in \mathbb{N}^I$ with

$$\omega^{ss} = \psi_{d_1}(\omega_{\beta_1}) * \cdots * \psi_{d_r}(\omega_{\beta_r}).$$

Notice that, for each $i \in I$, we have a Levi subgroup

$$(\mathrm{GL}_{(\beta_1)_i})_{d_1} \times \cdots \times (\mathrm{GL}_{(\beta_r)_i})_{d_r} \subseteq \mathrm{GL}_{\alpha_i}$$

embedded block diagonally. The Levi subgroup $\widetilde{L_\omega}$ is given by

$$\widetilde{L_\omega} = \prod_{i \in I} (\mathrm{GL}_{(\beta_1)_i})_{d_1} \times \cdots \times (\mathrm{GL}_{(\beta_r)_i})_{d_r}.$$

From the description of the Levi subgroups of GL_{n_α} of §4.2.3, we deduce the following Lemma.

Lemma 4.3.9. *For any F-stable admissible $E \in \mathcal{Z}_\alpha$, there exists a unique semisimple type, which we denote by $[E]$, such that E is $\mathrm{GL}_\alpha(\mathbb{F}_q)$-conjugated to $S_{[E]}$.*

Example 4.3.10. For any $\alpha \in \mathbb{N}^I$, we have that

$$[Z_\alpha] = \omega_\alpha.$$

Let $\sim$ be the equivalence relation on $\mathcal{Z}_\alpha$, induced by the conjugation by $\mathrm{GL}_\alpha(\mathbb{F}_q)$ and let $\overline{\mathcal{Z}} := \mathcal{Z}/\sim$ be the quotient set. The map $\mathcal{Z}_\alpha \to \mathbb{T}^{ss}_\alpha$ given by $S \to [S]$, induces thus a bijection

$$\overline{\mathcal{Z}} \cong \mathbb{T}^{ss}_I.$$

For an admissible torus S and $(d, \alpha) \in \mathbb{N} \times \mathbb{N}^I$, we denote by $m_{((d,\alpha),S)}$ the value $[S]((d, \alpha))$ i.e the number of appearances of (d, α) in the writing

$$[S] = \psi_{d_1}(\omega_{\alpha_1}) * \cdots * \psi_{d_r}(\omega_{\alpha_r}).$$

Lastly, we give the following definition of levels for the admissible subtori.,

Definition 4.3.11. Given $S \in \mathcal{Z}$ and $V \subseteq \mathbb{N}^I$, we say that S is of level V if $[S]$ is of level V (see Definition 3.6.2).

Example 4.3.12. Consider the tori S_1, S_2, S_3 introduced in Example 4.3.7. The torus S_1 is the product $Z_{(1,1,1,1)} \times Z_{(1,0,0,0)}$ embedded componentwise block diagonally into GL_α. The type $[S_1]$ is therefore the semisimple type

$$[S_1] = (1, (1^{(1,1,1,1)}))(1, (1^{(1,0,0,0)})) = \omega_{(1,1,1,1)} * \omega_{(1,0,0,0)}.$$

Similarly, we have

$$[S_2] = \omega_{(1,0,1,0)} * \omega_{(0,1,0,1)}$$

and

$$[S_3] = \omega_{(1,0,0,0)} * \omega_{(1,0,0,0)} * \omega_{(0,1,0,0)} * \omega_{(0,0,1,0)} * \omega_{(0,0,0,1)}.$$

Notice that for $V = \{(1, 1, 1, 1), (1, 0, 0, 0)\}$, we have that S_1 if of level V, while S_2, S_3 are not.

4.3.1 Regular elements and Möbius function for admissible tori

In this paragraph we give to $\mathcal{Z}$ the structure of a locally finite poset, with the ordering induced by inclusion and we introduce the associated Möbius function

$$\mu : \mathcal{Z} \times \mathcal{Z} \to \mathbb{Z}$$

and we recall more generally some properties of the Möbius function of a locally finite poset. The Möbius function μ is going to be one the main technical ingredient in the proof of Theorem 4.5.2.

4.3.2 Poset of F-stable admissible subtori

For any two elements S, S' of $\mathcal{Z}_\alpha$, we say that $S \leq S'$ if $S \subseteq S'$. Notice that $Z_\alpha \leq S$ for any admissible $S \subseteq \mathrm{GL}_\alpha$. For any $S \in \mathcal{Z}$, we denote by S^{reg} the subset of *regular elements* of S defined as

$$S^{reg} := \{s \in S \mid s \notin S' \text{ for any } S' \lneq S,\ S' \in \mathcal{Z}\}. \tag{4.3.1}$$

We have the following disjoint union

$$S = \bigsqcup_{S' \leq S} (S')^{reg} \tag{4.3.2}$$

and so, taking F-fixed points,

$$S^F = \bigsqcup_{S' \leq S} ((S')^{reg})^F. \tag{4.3.3}$$

In particular, we have an equality $|S^F| = \sum\limits_{S' \leq S} |((S')^{reg})^F|$. Notice that, if

$$[S] = \psi_{d_1}(\omega_{\beta_1}) * \cdots * \psi_{d_r}(\omega_{\beta_r}),$$

we have

$$S^F = \prod_{j=1}^{r} (\mathbb{G}_m)_{d_j}(\mathbb{F}_q) = \prod_{j=1}^{r} \mathbb{F}^*_{q^{d_j}}$$

and therefore we have

$$|S^F| = P_{[S]}(q). \tag{4.3.4}$$

4.3.3 Möbius functions of locally finite posets

For a finite poset $(X, \leq)$ denote by

$$\mu_X : X \times X \to \mathbb{Z}$$

its associated Möbius functions. Recall that μ_X is defined by the following two properties:

- $\mu(x, x) = 1$ for each $x \in X$

- For each $x \lneq x'$, we have

$$\sum_{x \leq x'' \lneq x'} \mu(x, x'') = -\mu(x, x') \tag{4.3.5}$$

The Möbius function has the following property.

Proposition 4.3.13. *Given $f_1, f_2 : X \to \mathbb{C}$ such that*

$$f_1(x) = \sum_{x' \leq x} f_2(x'),$$

we have an equality

$$f_2(x) = \sum_{x' \leq x} f_1(x')\mu_X(x', x). \tag{4.3.6}$$

Lastly, we recall the following standard Lemma about Möbius functions.

Lemma 4.3.14. *Let $(X, \leq), (Y, \leq)$ be two locally finite posets and equip $X \times Y$ with the ordering defined as $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$.*
For the locally finite poset $(X \times Y, \leq)$, we have

$$\mu_{X\times Y}((x, y), (x', y')) = \mu_X((x, x'))\mu_Y((y, y')). \tag{4.3.7}$$

Proof. By induction we can assume that

$$\mu_{X\times Y}((x, y), (x'', y'')) = \mu_X((x, x''))\mu_Y((y, y'')) \tag{4.3.8}$$

for all $(x, y) < (x'', y'') < (x', y')$. From eq.(4.3.5) we have therefore

$$\mu_{X\times Y}((x, y)(x', y')) = -\sum_{(x,y)\leq(x'',y'')<(x',y')} \mu_X(x, x'')\mu_Y(y, y'') = \tag{4.3.9}$$

$$\sum_{x\leq x''<x'} \mu_X(x, x'') \sum_{y\leq y''\leq y'} \mu_Y(y, y'') + \mu_X(x, x') \sum_{y\leq y''<y'} \mu_Y(y, y''). \tag{4.3.10}$$

By eq.(4.3.5), $\sum_{y\leq y''\leq y'} \mu_Y(y, y'')$ and $\sum_{y\leq y''<y'} \mu_Y(y, y'') = -\mu_Y(y, y')$ and therefore

$$\mu_{X\times Y}((x, y)(x', y')) = \mu_X(x, x') \sum_{y\leq y''<y'} \mu_Y(y, y'') = -\mu_X(x, x')\mu_Y(y, y'). \tag{4.3.11}$$

□

For $x \in X$, denote by $[x, \infty]_X \subseteq X$ the poset

$$[x, \infty] = \{x' \in X \mid x' \geq x\}.$$

Notice that, for each $x' \in [x, \infty]_X$, from eq.(4.3.5), we deduce that we have:

$$\mu_X(x, x') = \mu_{[x,\infty]_X}(x, x') \tag{4.3.12}$$

Example 4.3.15. Consider $\mathbb{N}_{>0}$ with the ordering $\leq$ defined as $d \leq n$ if and only if $d|n$. In this case, for any $r \leq r'$, we have

$$\mu_{\mathbb{N}>0}(r, r') = \mu\left(\frac{r'}{r}\right),$$

where $\mu : \mathbb{N} \to \mathbb{Z}$ us the usual Möbius function.

4.3.4 Möbius function for admissible subtori

Notice that the ordering $\leq$ equips the set $\mathcal{Z}$ with the structure of a locally finite poset. In the following, we denote simply by

$$\mu(-,-) : \mathcal{Z} \times \mathcal{Z} \to \mathbb{Z}$$

the associated Möbius function.

Example 4.3.16. Let $f_1, f_2 : \mathcal{Z} \to \mathbb{C}$ be the functions defined as

$$f_1(S) = |S^F|$$

and

$$f_2(S) = |(S^{reg})^F|.$$

By eq.(4.3.6) and eq.(4.3.3), we have the following identity:

$$|(S^{reg})^F| = \sum_{S' \leq S} |(S')^F| \mu(S', S) = \sum_{S' \leq S} P_{[S']}(q)\mu(S', S). \tag{4.3.13}$$

Consider more generally a complex valued function $f : S^F \to \mathbb{C}$. We define a function $f : \mathcal{Z} \to \mathbb{C}$ as follows:

$$f(S') = \begin{cases} 0 \text{ if } S' \not\leq S \\ \sum_{s \in (S')^F} f(s) \text{ otherwise} \end{cases}.$$

We define similarly $g : \mathcal{Z} \to \mathbb{C}$ by

$$g(S') = \begin{cases} 0 \text{ if } S' \not\leq S \\ \sum_{s \in ((S')^{reg})^F} f(s) \text{ otherwise} \end{cases}.$$

From Identity (4.3.3) we deduce that

$$f(S) = \sum_{S' \leq S} g(S')$$

and so that, from eq.(4.3.6), we have:

$$\sum_{s\in(S^{reg})^F} f(s) = g(S) = \sum_{S'\leq S} f(S')\mu(S',S) = \sum_{S'\leq S}\left(\sum_{t\in(S')^F} f(t)\right)\mu(S',S). \tag{4.3.14}$$

4.3.5 Multitype of a conjugacy class and admissible subtori

The correspondence between $\mathbb{T}_\alpha$ and the conjugacy classes of $\mathrm{GL}_\alpha(\mathbb{F}_q)$ of §3.6.1 can be explained using the notion of admissible subtori in the following way.
Consider a conjugacy class O of $\mathrm{GL}_\alpha(\mathbb{F}_q)$, an element $g \in O$ and its decomposition into a semisimple and unipotent element $g = g_{ss}g_u$. The centralizers $C_{\mathrm{GL}_\alpha}(g_{ss}), C_{\mathrm{GL}_{n_\alpha}}(g_{ss})$ are Levi subgroups of $\mathrm{GL}_\alpha, \mathrm{GL}_{n_\alpha}$ respectively.
Notice that $C_{\mathrm{GL}_\alpha}(g_{ss}) \subseteq C_{\mathrm{GL}_{n_\alpha}}(g_{ss})$ and therefore there exists a maximal torus $T \subseteq \mathrm{GL}_\alpha$ such that

$$T \subseteq C_{\mathrm{GL}_\alpha}(g_{ss}) \subseteq C_{\mathrm{GL}_{n_\alpha}}(g_{ss}).$$

The center of the Levi subgroup $C_{\mathrm{GL}_{n_\alpha}}(g_{ss})$ is thus an admissible subtorus $S \subseteq \mathrm{GL}_\alpha$. Let

$$[S] = (d_1,(1^{\beta_1}))\dots(d_r,(1^{\beta_r}))$$

be the associated semisimple type. Up to conjugacy we can assume that

$$S = (Z_{\beta_1})_{d_1} \times \cdots \times (Z_{\beta_r})_{d_r}$$

embedded block diagonally.
Since $g_{ss} \in S^{reg}$ and $[g_{ss}, g_u] = 1$, we have that g_u belongs to $C_{\mathrm{GL}_\alpha}(S)$, i.e

$$g_u \in \widetilde{L_S} = \prod_{i\in I}(\mathrm{GL}_{(\beta_1)_i})_{d_1} \times \cdots \times (\mathrm{GL}_{(\beta_r)_i})_{d_r}$$

and therefore g_u determines, for each $i \in I$ and $j = 1,\dots,r$ unipotent elements $g_{u,j,i} \in \mathrm{GL}_{(\beta_j)_i}(\mathbb{F}_{q^{d_j}})$ for each $i \in I$. For each $j = 1,\dots,r$, the Jordan forms of $(g_{u,j,i})_{i\in I}$ determine a multipartition $\boldsymbol{\lambda}_j \in \mathcal{P}^I$ such that $|\boldsymbol{\lambda}_j| = \beta_j$.
The type ω_O associated to O is thus

$$\omega_O = (d_1,\boldsymbol{\lambda}_1)\dots(d_r,\boldsymbol{\lambda}_r).$$

Remark 4.3.17. Let $\omega \in \mathbb{T}_\alpha$ and $g \in \mathrm{GL}_\alpha(\mathbb{F}_q)$ such that $g \sim \omega$. Consider the Jordan decomposition $g_{ss}g_u = g$. Let S be the admissible torus given by the center of $C_{\mathrm{GL}_{n_\alpha}}(g_{ss})$, introduced above. Notice that $g_{ss} \in (S^{reg})^F$. Conversely, for every $s \in (S^{reg})^F$, the element $sg_u \in \mathrm{GL}_\alpha(\mathbb{F}_q)$ is of type ω.
The map $(S^{reg})^F \to \{\mathrm{GL}_\alpha(\mathbb{F}_q)-$ orbits of type $\omega\}$ which sends s to the orbit of sg_u is surjective. Two elements $s, s' \in (S^{reg})^F$ have the same image if and only if there exists $g \in \mathrm{GL}_\alpha(\mathbb{F}_q)$

such that $gsg^{-1} = s'$ and $gug^{-1} = u$. Notice that, since $gsg^{-1} = s'$, we have

$$gC_{\mathrm{GL}_{n_\alpha}}(s)g^{-1} = gSg^{-1} = C_{\mathrm{GL}_{n_\alpha}}(s') = S$$

and therefore

$$gC_{\mathrm{GL}_{n_\alpha}}(S)g^{-1} = gL_Sg^{-1} = C_{\mathrm{GL}_{n_\alpha}}(S) = L_S.$$

The fibers of the map are therefore identified with the group

$$\{g \in \mathrm{GL}_\alpha(\mathbb{F}_q) \mid gL_Sg^{-1} = L_S \text{ and } gug^{-1} = u\}/\widetilde{L_S}^F$$

which has cardinality $w(\omega)$.

4.4 Admissible subtori, graphs and Möbius functions

In this section, we will associate certain graphs, said *admissible*, to the elements of $\mathcal{Z}$. This construction will be useful to understand the Möbius function $\mu : \mathcal{Z} \times \mathcal{Z} \to \mathbb{Z}$ and to develop the combinatorial arguments of §4.4.4, both of which will be key parts of the proof of Theorem 4.5.2 about Log compatible functions.

4.4.1 Notations about graphs

We fix some notations about graphs. Let Γ be a finite graph with set of vertices M and $m = |M|$.

Definition 4.4.1. We say that Γ is of type K_m if it is the complete graph associated to M, i.e each pair of distinct vertices is connected exactly by one edge.
We say that Γ is *admissible* if each of its connected components is of type K_d for some d.

Remark 4.4.2. Notice that the property of being admissible for a graph Γ can be stated in the following equivalent way.
For any two $m, m' \in M$, there is at most one edge of Γ joining m to m' and, if $m_1, m_2, m_3 \in M$ are such that there is an edge of Γ between m_1 and m_2 and an edge of Γ between m_2 and m_3, there is an edge of Γ between m_1 and m_3.
In particular, an admissible graph Γ is totally determined by the partition of M given by the sets of vertices of the connected components of Γ.

4.4.2 Root system and graphs

Let now $\alpha \in \mathbb{N}^I$ and fix an F-stable maximal torus $T \subseteq \mathrm{GL}_\alpha \subseteq \mathrm{GL}_{n_\alpha}$.

Denote simply by $\mathcal{B}, \Phi, \Phi^+$ the sets $\mathcal{B}(T), \Phi(T), \Phi^+(T)$ and by $\sigma \in S_{n_\alpha}$ the permutation such that $F(\epsilon_i) = q\epsilon_{\sigma(i)}$ for each $\epsilon_i \in \mathcal{B}$.
For any two admissible graphs Γ, Γ' with set of vertices $\mathcal{B}$ and sets of edges $\Omega_\Gamma, \Omega_{\Gamma'}$ respectively, we say that $\Gamma \leq \Gamma'$ if $\Omega_\Gamma \supseteq \Omega_{\Gamma'}$.

We denote by $A(\mathcal{B},\sigma)$ be the poset of admissible and σ-stable graph with set of vertices $\mathcal{B}$ and by $\mu_{\mathcal{B},\sigma}(-,-)$ the associated Möbius function. Moreover, we will denote the complete graph with vertices $\mathcal{B}$ by $\Gamma_\alpha \in A(\mathcal{B},\sigma)$.

Remark 4.4.3. From Remark 4.4.2, we see that the poset $A(\mathcal{B},\sigma)$ is the the poset of σ-stable partitions of the set $\mathcal{B}$ with ordering given by the reversed inclusion, i.e the fixed point set lattice considered in [42].
In the latter article, the author computed certain values of the Möbius function $\mu_{\mathcal{B},\sigma}$ and in particular the values $\mu_{\mathcal{B},\sigma}(\Gamma_\alpha,\Gamma')$ for each Γ'. We will review this result in Proposition 4.4.4.
We prefer to introduce this graph theoretic description, as in our opinion this can ease the notations and give a more direct understanding of the results of this section about the relationship between admissible graphs and admissible tori.

Fix now an admissible σ-stable graph Γ with set of vertices $\mathcal{B}$. Notice that, as Γ is σ-stable, σ acts by permutation on the set of connected components of Γ. Assume that this action has r orbits of length $d_1,\dots,d_r$ respectively, which we denote by $O_1,\dots,O_r$.
For each $j=1,\dots,r$, denote by $\mathcal{B}_j^\Gamma \subseteq \mathcal{B}$ the set of vertices contained in the orbit O_j. Notice that each $\mathcal{B}_j^\Gamma$ is σ-stable and there is an equality

$$\mathcal{B} = \bigsqcup_{j=1}^{r} \mathcal{B}_j^\Gamma .$$

For each $j=1,\dots,r$, choose a partition of $\mathcal{B}_j^\Gamma$ into d_j subsets

$$\mathcal{B}_j^\Gamma = \mathcal{B}_{j,1}^\Gamma \bigsqcup \cdots \bigsqcup \mathcal{B}_{j,d_j}^\Gamma$$

such that:

- Each $\mathcal{B}_{j,h}^\Gamma$ is given by the vertices of a connected component belonging to the orbit O_j
- We have $\sigma(\mathcal{B}_{j,h}^\Gamma) = \mathcal{B}_{j,h+1}^\Gamma$ for each $h=1,\dots,d_j$ (here we consider the indices modulo d_j).

For each $j=1,\dots,r$, let $\beta_j \in \mathbb{N}^I$ be the element defined as

$$(\beta_j)_i := |\mathcal{B}_{j,1}^\Gamma \cap \mathcal{B}_i|$$

for $i \in I$. We denote by $\omega_\Gamma \in \mathbb{T}_\alpha^{ss}$ the semisimple multityped defined as

$$\omega_\Gamma := \psi_{d_1}(\omega_{\beta_1}) * \cdots * \psi_{d_r}(\omega_{\beta_r}).$$

In [42], it is shown the following Proposition.

Proposition 4.4.4. *For each $\Gamma \in A(\mathcal{B},\sigma)$, we have*

$$\mu_{A(\mathcal{B},\sigma)}(Z_\alpha,\Gamma) = C^o_{\omega_\Gamma}. \tag{4.4.1}$$

Denote now by $\Gamma_{j,h}$ the restriction of Γ to the set $\mathcal{B}^{\Gamma}_{j,h}$. Notice that $\Gamma_{j,h}$ is the complete graph with vertices $\mathcal{B}^{\Gamma}_{j,h}$ and so Γ is totally determined by the subsets $\{\mathcal{B}^{\Gamma}_{j,1}\}_{j=1,\dots,r}$. Notice, in addition, that for each $j = 1, \dots, r$, we have $\sigma^{d_j}(\mathcal{B}^{\Gamma}_{j,1}) = \mathcal{B}^{\Gamma}_{j,1}$.
We have the following Lemma.

Lemma 4.4.5. *There is an equivalence of posets*

$$[\Gamma, \infty]_{A(\mathcal{B},\sigma)} \cong \prod_{j=1}^{r} [\Gamma_{j,1}, \infty]_{A(\mathcal{B}^{\Gamma}_{j,1}, \sigma^{d_j})} \tag{4.4.2}$$

and, for each $\Gamma' \geq \Gamma$, *denoting by* $\Gamma'_{j,h}$ *the restriction of* Γ' *to* $\mathcal{B}^{\Gamma}_{j,h}$, *we have*

$$\mu_{\mathcal{B},\sigma}(\Gamma, \Gamma') = \prod_{j=1}^{r} \mu_{\mathcal{B}^{\Gamma}_{j,1}, \sigma^{d_j}}(\Gamma_{j,1}, \Gamma'_{j,1}). \tag{4.4.3}$$

Proof. Notice indeed that, given admissible graphs $\Gamma'_{j,1}$ with vertices $\mathcal{B}^{\Gamma}_{j,1}$ for each $j = 1, \dots, r$, there exist a unique σ-stable and admissible graph Γ' with vertices $\mathcal{B}$ containing as subgraphs $\Gamma'_{1,1}, \dots, \Gamma'_{r,1}$ and such that $\Gamma' \geq \Gamma$.
Eq.(4.4.3) is thus a consequence of eq.(4.4.2) and Lemma 4.3.7.

□

4.4.3 Admissible subtori and admissible graphs

Fix now an admissible torus $S \subseteq T$. Denote by $J_S \subseteq \Phi$ the subset

$$J_S := \{\epsilon \in \Phi \mid S \subseteq \mathrm{Ker}(\epsilon)\}.$$

From Lemma 4.1.1 we deduce that we have

$$S = \bigcap_{\epsilon \in J_S} \mathrm{Ker}(\epsilon)$$

and

$$L_S = T \prod_{\epsilon \in J_S} U_\epsilon.$$

Notice moreover that the subgroup S is F-stable if and only if J_S is σ stable.

We now associate the following graph Γ_S to the admissible torus S.

- The set of vertices of Γ_S is $\mathcal{B}$
- Γ_S has an edge between vertices ϵ_i and ϵ_j if and only if $\epsilon_{i,j} \in J_S \cap \Phi^+$.

We denote by Ω_{Γ_S} be the set of edges of Γ_S. The group S is F-stable if and only if Γ_S is σ-invariant.

Example 4.4.6. Let $I = \{\cdot\}$ and T be the torus of diagonal matrices $T \subseteq \mathrm{GL}_m$. In this case, σ is trivial.
The graph Γ_T is thus the graph with no edges and m vertices, while the graph $\Gamma_{Z_{\mathrm{GL}_m}}$ associated to Z_{GL_m} is the complete graph with m vertices K_m.

Example 4.4.7. For any I and any $\alpha \in \mathbb{N}^I$, notice that $\Gamma_{Z_\alpha} = \Gamma_\alpha$.

We can now state the following Lemma, relating admissible graphs and subtori.

Lemma 4.4.8. *For any admissible torus S, the graph Γ_S is admissible. Conversely, for any σ-stable admissible graph Γ with set of vertices $\mathcal{B}$, there is a unique F-stable admissible torus $S \subseteq T$ such that $\Gamma_S = \Gamma$.*

Here σ-stable means that Γ has an edge between ϵ_i and ϵ_j if and only if it has an edge between $\epsilon_{\sigma(i)}$ and $\epsilon_{\sigma(j)}$.

Proof. For an admissible subtorus S, notice that if $\epsilon_{i,j}, \epsilon_{j,h} \in J_S$ then $\epsilon_{i,h} \in J_S$. From Remark 4.4.2 we deduce that Γ_S is admissible.

Consider now an admissible Γ and the subset

$$J_\Gamma \coloneqq \{\epsilon_{j,h} \in \Phi \mid \text{there is an edge of } \Gamma \text{ which has vertices } \epsilon_j, \epsilon_h\}.$$

From [28, Corollary 3.3.4], the subset J_Γ is a root subsystem and, from [89, Lemma 8.4.2], we have that the torus

$$S_\Gamma \coloneqq \bigcap_{\epsilon \in J_\Gamma} \mathrm{Ker}(\epsilon) \tag{4.4.4}$$

is admissible and F-stable with

$$L_{S_\Gamma} = T \prod_{\epsilon \in J_\Gamma} U_\epsilon. \tag{4.4.5}$$

It is not difficult to check that the graph associated to S_Γ is Γ.

□

Let S, S' be two admissible subtori such that $S \supseteq T, S' \supseteq T$. From Lemma 4.4.8, we deduce the following Proposition

Proposition 4.4.9. *Given $S, S' \subseteq T$, we have that $S \leq S'$ if and only if $\Gamma_S \leq \Gamma_{S'}$.*

From eq.(4.3.5), Proposition 4.4.9 and Lemma 4.4.8, we deduce the following Lemma.

Lemma 4.4.10. *For any $S, S' \in \mathcal{Z}$ such that $S, S' \subseteq T$, we have an equality*

$$\mu_{A(\mathcal{B},\sigma)}(\Gamma_S, \Gamma_{S'}) = \mu(S, S') \tag{4.4.6}$$

Consider now an admissible graph $\Gamma \in A(\mathcal{B}, \sigma)$ and the admissible F-stable torus S_Γ associated to Γ. We have the following proposition.

Proposition 4.4.11. *With the notations introduced above, the multitype associated to* S_Γ *is*

$$[S_\Gamma] = \psi_{d_1}(\omega_{\beta_1}) * \cdots * \psi_{d_r}(\omega_{\beta_r}),$$

i.e we have

$$[S_\Gamma] = \omega_\Gamma.$$

Fix now $\Gamma', \Gamma \in A(\mathcal{B}, \sigma)$ such that $\Gamma \leq \Gamma'$, and denote by $S = S_\Gamma$ and $S' = S_{\Gamma'}$. Notice that we have $S \subseteq S'$. The torus S is $\mathrm{GL}_\alpha(\mathbb{F}_q)$ conjugated to $\prod_{j=1}^{r} (Z_{\beta_j})_{d_j}$. The admissible graphs $\Gamma'_{j,1}$ correspond to admissible tori $S'_j \subseteq \mathrm{GL}_{\beta_j}$, for each $j = 1, \ldots, r$.
From eq.(4.4.6) and eq.(4.4.3), we deduce the following equality

$$\mu(S, S') = \prod_{j=1}^{r} \mu(Z_{\beta_j}, S'_j) = \prod_{j=1}^{r} C^o_{[S'_j]}. \tag{4.4.7}$$

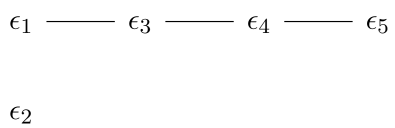
Example 4.4.12. Consider the set $I = \{1, 2, 3, 4\}$, the dimension vector $\alpha = (2, 1, 1, 1)$ and the admissible tori $S_1, S_2, S_3 \in \mathcal{Z}_\alpha$ of Example 4.3.7. Notice that S_1, S_2, S_3 are all contained in the maximal torus $T = T_2 \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m$, where $T_2 \subseteq \mathrm{GL}_2$ is the maximal torus of diagonal matrices and more precisely that $T = S_3$.
With the notations just introduced, we have $\mathcal{B} = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\}$, σ is the identity and

$$\mathcal{B}_1 = \{\epsilon_1, \epsilon_2\} \quad \mathcal{B}_2 = \{\epsilon_3\}$$

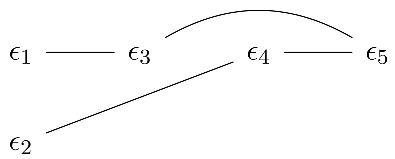
and

$$\mathcal{B}_3 = \{\epsilon_4\} \quad \mathcal{B}_4 = \{\epsilon_5\}.$$

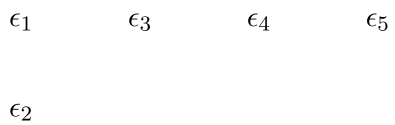
The graph Γ_{S_1} associated to the torus S_1 is



The graph Γ_{S_2} associated to the torus S_2 is



The graph Γ_{S_3} associated to the graph S_3 is



Notice that $\Gamma_{S_1}, \Gamma_{S_2} \leq \Gamma_{S_3}$ and we have corresponding inclusions $S_1, S_2 \subseteq S_3$

Example 4.4.13. Consider $I = \{1, 2\}$ and the dimension vector $\alpha = (2, 2) \in \mathbb{N}^I$, i.e $\mathrm{GL}_\alpha = \mathrm{GL}_2 \times \mathrm{GL}_2$ and let $T \subseteq \mathrm{GL}_\alpha$ be the torus

$$T = T_\epsilon \times T_\epsilon \subseteq \mathrm{GL}_2 \times \mathrm{GL}_2,$$

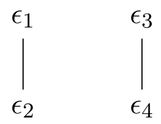
where $T_\epsilon \subseteq \mathrm{GL}_2$ is the torus of Example 3.2.8. In this case, we have $\mathcal{B} = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ with

$$\mathcal{B}_1 = \{\epsilon_1, \epsilon_2\} \quad \mathcal{B}_2 = \{\epsilon_3, \epsilon_4\},$$

and σ is the permutation $\sigma = (12)(34) \in S_4$.
Let $S \subseteq \mathrm{GL}_\beta$ be the admissible subtorus

$$S = \left\{ \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \right) \mid \lambda, \mu \in \overline{\mathbb{F}_q}^* \right\}.$$

Notice that $S \subseteq T$ and the graph Γ_S is given by



Notice that Γ_S has two connected components which are both stabilized by σ. With the notations introduced before, we have therefore two orbits O_1, O_2 with $\mathcal{B}_1^{\Gamma_S} = \mathcal{B}_1$ and $\mathcal{B}_2^{\Gamma_S} = \mathcal{B}_2$ and $d_1 = d_2 = 1$. Denote by $\Gamma_{S,1}, \Gamma_{S,2}$ the restriction of Γ to $\mathcal{B}_1, \mathcal{B}_2$ respectively.
Notice moreover that the associated elements β_1, β_2 are given by

$$\beta_1 = (2, 0) \quad \beta_2 = (0, 2)$$

and, from Lemma 4.4.11 we find

$$[S] = \omega_{\beta_1} * \omega_{\beta_2}.$$

Notice indeed that the torus S is $Z_{\beta_1} \times Z_{\beta_2}$ embedded block diagonally in GL_α.

Moreover, from eq.(4.4.7), we deduce that

$$\mu(S, T) = \mu_{\mathcal{B}_1,\sigma}(\Gamma_{S,1}, \Gamma_{T_\epsilon})\mu_{\mathcal{B}_2,\sigma}(\Gamma_{S,2}, \Gamma_{T_\epsilon}) = \mu(Z_2, T_\epsilon)^2, \tag{4.4.8}$$

where $Z_2 = Z_{\mathrm{GL}_2}$. Notice that eq.(4.4.8) can be checked directly from the definition of the Möbius function μ.
Notice indeed that we have

$$\{Z_{\beta_1} \times Z_{\beta_2}, Z_{\beta_1} \times T_\epsilon, T_\epsilon \times Z_{\beta_2}\} = \{S'' \in \mathcal{Z}_\alpha \mid Z_{\beta_1} \times Z_{\beta_2} \leq S'' \lneqq T_\epsilon \times T_\epsilon\}.$$

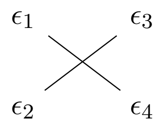
From. eq(4.3.5), we deduce that we have

$$\mu(Z_{\beta_1} \times Z_{\beta_2}, Z_{\beta_1} \times T_\epsilon) = \mu(Z_{\beta_1} \times Z_{\beta_2}, T_\epsilon \times Z_{\beta_2}) = \mu(Z_2, T_\epsilon) = -1$$

and thus from eq.(4.3.5) that we have

$$\mu(S, T_\epsilon \times T_\epsilon) = -(-1-1+1) = 1 = \mu(Z_2, T_\epsilon)^2.$$

Consider now the admissible and σ-stable graph Γ' given by



and denote by $S' = S_{\Gamma'}$. The torus S' is given by

$$S' = \left\{ \left(\frac{1}{x^q - x} \begin{pmatrix} ax^q - bx & -a+b \\ (a-b)xx^q & -ax+bx^q \end{pmatrix}, \frac{1}{x^q - x} \begin{pmatrix} ax^q - bx & -a+b \\ (a-b)xx^q & -ax+bx^q \end{pmatrix} \right) \mid a, b \in \overline{\mathbb{F}_q}^* \right\}. \tag{4.4.9}$$

Notice that in this case, the graph Γ' has 2 connected components, which are swapped by σ. We have therefore a single orbit O_1 of length $d_1 = 2$ with

$$\mathcal{B}^{\Gamma'}_{1,1} = \{\epsilon_1, \epsilon_3\} \quad \mathcal{B}^{\Gamma'}_{1,2} = \{\epsilon_2, \epsilon_4\}.$$

Notice that the associated dimension vector β'_1 is $\beta'_1 = (1,1)$. Proposition 4.4.11 states therefore that S' is $\mathrm{GL}_\alpha(\mathbb{F}_q)$-conjugated to the torus

$$(Z_{(1,1)})_2 \subseteq \mathrm{GL}_\alpha,$$

which is also directly seen by the expression of S' in eq.(4.4.9).
Notice that

$$\sigma^2 = \mathrm{Id} : \mathcal{B}^{\Gamma'}_{1,1} \to \mathcal{B}^{\Gamma'}_{1,1}.$$

From eq.(4.4.3), we find therefore

$$\mu(S', T) = \mu_{\mathcal{B}^{\Gamma'}_{1,1}, \mathrm{Id}}(\Gamma'_{1,1}, \Gamma_{T_2}) = \mu(Z_{(1,1)}, Z_{(1,0)} \times Z_{(0,1)}).$$

4.4.4 Inclusion of admissible subtori

Let $\omega_1, \omega_2 \in \mathbb{T}^{ss}_\alpha$. Fix a maximal torus T such that $S_{\omega_2} \subseteq T \subseteq \mathrm{GL}_\alpha$. Define the set P_{ω_1,ω_2} as

$$P_{\omega_1,\omega_2} \coloneqq \{S \in \mathcal{Z}_\alpha \mid [S] = \omega_1 \ , S \leq S_{\omega_2}\}.$$

In this paragraph we give a combinatorial description of P_{ω_1,ω_2} which will be used in the proof of Theorem 4.5.2.

Assume that

$$\omega_1 = \psi_{d_1}(\omega_{\beta_1}) * \cdots * \psi_{d_r}(\omega_{\beta_r})$$

and

$$\omega_2 = \psi_{d'_1}(\omega_{\beta'_1}) * \cdots * \psi_{d'_t}(\omega_{\beta'_t})$$

respectively.
Up to reordering the factors in the product $\omega_1 = \prod_{j=1}^{r} \psi_{d_j}(\omega_{\beta_j})$, we can assume that there exists a strictly increasing sequence $i_1 < \cdots < i_k \in \{1, \ldots, r\}$ such that:

- $(d_j, \beta_j) = (d_1, \beta_1)$ for $j = 1, \ldots, i_1$
- $(d_j, \beta_j) = (d_{i_p}, \beta_{i_p})$ for all $i_{p-1} < j \leq i_p$ for $p \in \{2, \ldots, k\}$

Notice that $i_1 = m_{((d_1,\beta_1),\omega_1)}$ and

$$i_h - \sum_{p=1}^{h-1} i_p = m_{((d_{i_h},\beta_{i_h}),\omega_1)}.$$

Let M_{ω_1,ω_2} be the set of partitions of $\{1, \ldots, t\}$ into r non-empty disjoint subsets $X_1, \ldots, X_r$ with the following properties:

- If h belongs to X_i, then $d_i | d'_h$
- For every $j = 1, \ldots, r$, it holds $\displaystyle\sum_{h \in X_j} \frac{d'_h}{d_j} \beta'_h = \beta_j$.

We will denote the element of M_{ω_1,ω_2} associated to the subsets $X_1, \ldots, X_r$ by $(X_1, \ldots, X_r)$.

Consider now the group W'_{ω_1} defined by

$$W'_{\omega_1} := S_{m_{((d_{i_1},\beta_{i_1}),\omega_1)}} \times \cdots \times S_{m_{((d_{i_k},\beta_{i_k}),\omega_1)}}.$$

The set M_{ω_1,ω_2} is endowed with an action of the group W'_{ω_1} defined by the following rule. Consider elements $\sigma = (\sigma_1, \ldots, \sigma_k) \in W'_{\omega_1}$ and $(X_1, \ldots, X_r) \in M_{\omega_1,\omega_2}$. We define

$$\sigma \cdot (X_1, \ldots, X_r) := (X_{\sigma_1(1)}, \ldots, X_{\sigma_1(i_1)}, X_{\sigma_2(i_1+1)}, \ldots, X_{\sigma_k(r)}).$$

Notice that W'_{ω_1} acts freely on M_{ω_1,ω_2}. We denote by $\overline{M_{\omega_1,\omega_2}}$ the quotient set $M_{\omega_1,\omega_2}/W'_{\omega_1}$. We will now define the following morphism

$$\pi_{\omega_1,\omega_2} : P_{\omega_1,\omega_2} \to \overline{M_{\omega_1,\omega_2}}.$$

We denote by Γ' the graph associated to S_{ω_2} with respect to the torus T. Let $\{\mathcal{B}^{\Gamma'}_{j,h}\}_{\substack{j=1,\ldots,t \\ h=1,\ldots,d'_j}}$ be the partition of the set $\mathcal{B}$ introduced in Paragraph §4.4 above for the graph Γ'.
Consider an F-stable admissible torus $S \subseteq S_{\omega_2}$ with $[S] = \omega_1$ and the corresponding σ-stable graph $\Gamma \leq \Gamma'$, i.e $\Omega_\Gamma \supseteq \Omega_{\Gamma'}$.

Let $O_1, \dots, O_r$ be the r orbits for the action of σ on the connected components of Γ of length $d_1, \dots, d_r$ respectively and assume to have fixed representatives $\Gamma_{1,1}, \dots, \Gamma_{r,1}$ for each of the orbits.
For each $j = 1, \dots, r$, there exists a subset $X_j \subseteq \{1, \dots, t\}$ and, for each $l \in X_j$, a subset $Z_l \subseteq \{1, \dots, d'_l\}$, such that $\Gamma_{j,1}$ is the complete graph with vertices

$$\bigsqcup_{l \in X_j} \bigsqcup_{z \in Z_l} \mathcal{B}^{\Gamma'}_{l,z}.$$

Notice that the subsets X_j do not depend on the choice of the representatives $\Gamma_{1,1}, \dots, \Gamma_{r,1}$ and form a partition of the set $\{1, \dots, t\}$. The partition $(X_1, \dots, X_r)$ belongs to M_{ω_1,ω_2}.
Indeed, since the orbit O_j has length d_j, we must have that

$$\sigma^s \left(\bigsqcup_{l \in X_j} \bigsqcup_{z \in Z_l} \mathcal{B}^{\Gamma'}_{l,z} \right) \cap \bigsqcup_{l \in X_j} \bigsqcup_{z \in Z_l} \mathcal{B}^{\Gamma'}_{l,z} = \varnothing$$

for any $0 < s \leq d_j - 1$ and

$$\sigma^{d_j} \left(\bigsqcup_{l \in X_j} \bigsqcup_{z \in Z_l} \mathcal{B}^{\Gamma'}_{l,z} \right) = \bigsqcup_{l \in X_j} \bigsqcup_{z \in Z_l} \mathcal{B}^{\Gamma'}_{l,z}.$$

Recall that $\sigma(\mathcal{B}^{\Gamma'}_{l,z}) = \mathcal{B}^{\Gamma'}_{l,z+1}$, where the index z of $\mathcal{B}^{\Gamma'}_{l,z}$ is always considered modulo d'_l. We deduce therefore that Z_l is such that

$$(Z_l + s) \cap Z_s = \varnothing \mod d_j$$

for each $0 < s \leq d_j - 1$ and

$$Z_l + d_j = Z_j \mod d_j.$$

This implies that $d_j | d'_l$ and that there exists $a_l \in Z_l$ such that

$$Z_l = \left\{ a_l + d_j k \mid k = 1, \dots, \frac{d'_l}{d_j} \right\}.$$

In particular, it holds that $|Z_l| = \dfrac{d'_l}{d_j}$, from which we deduce that

$$\sum_{l \in X_j} \frac{d'_l}{d_j} \beta'_l = \beta_j.$$

We define then

$$\pi_{\omega_1,\omega_2}(U) \coloneqq [(X_1, \dots, X_r)]$$

where $[(X_1, \dots, X_r)]$ is the class of the element $(X_1, \dots, X_r)$ in the quotient $\overline{M_{\omega_1,\omega_2}}$.
Notice that the morphism π_{ω_1,ω_2} is well-defined, i.e does not depend on the choice of the

ordering of the orbits $O_1, \dots, O_r$ and it is surjective as we are taking the class $[(X_1, \dots, X_r)]$ in the quotient $\overline{M_{\omega_1,\omega_2}}$ for the action of W'_{ω_1}.

From the description of the subsets Z_l given above, we deduce that, for each $[(X_1, \dots, X_r)]$, the fiber $\pi^{-1}_{\omega_1,\omega_2}([(X_1, \dots, X_r)])$ has cardinality

$$|\pi^{-1}_{\omega_1,\omega_2}([(X_1, \dots, X_r)])| = \prod_{j=1}^{r} d_j^{|X_j|-1}. \tag{4.4.10}$$

4.5 Log-compatible functions and plethystic identities

In this section, we recall the definition of a Log compatible family, first introduced by Letellier [62] and we prove our main Theorem 4.5.2 about these families pf rational functions, which will be the key tool of this paper to compute the E-series of the cohomology of non-generic character stacks.

4.5.1 Log-compatible families

Consider a family of rational functions $\{F_\omega(t)\}_{\omega \in \mathbb{T}_I} \subseteq \mathbb{Q}(t)$. For any $V \subseteq \mathbb{N}^I$, we define the rational function $F_{\alpha,V}(t) \in \mathbb{Q}(t)$ as follows:

$$F_{\alpha,V}(t) \coloneqq \sum_{\omega \in \mathbb{T}_\alpha} \frac{F_\omega(t)}{w(\omega)} \left(\sum_{\substack{S' \leq S_\omega \\ S' \text{ of level } V}} P_{[S']}(t)\mu(S', S_\omega) \right). \tag{4.5.1}$$

For $V = \{\alpha\}$ we will use the notation $F_{\alpha,gen}(t) \coloneqq F_{\alpha,\{\alpha\}}(t)$. Notice that

$$F_{\alpha,gen}(t) = \sum_{\omega \in \mathbb{T}_\alpha} \frac{F_\omega(t)}{w(\omega)}(t-1)\mu(Z_\alpha, S_\omega) = \sum_{\omega \in \mathbb{T}_\alpha} \frac{F_\omega(t)}{w(\omega)}(t-1)C^o_{\omega^{ss}}.$$

We give the following definition of a Log-compatible family $\{F_\omega(t)\}_{\omega \in \mathbb{T}_I}$.

Definition 4.5.1. We say that $\{F_\omega(t)\}_{\omega \in \mathbb{T}_I}$ is *Log compatible* if for any $\alpha \in \mathbb{N}^I$, $\omega \in \mathbb{T}_\alpha$ and for every multitypes $\nu_1, \dots, \nu_r$ and integers $d_1, \dots, d_r$ such that $\psi_{d_1}(\nu_1) * \cdots * \psi_{d_r}(\nu_r) = \omega$, we have

$$\prod_{j=1}^{r} F_{\nu_j}(t^{d_j}) = F_\omega(t).$$

4.5.2 Plethysm and Log compatibility: main result

Fix now an element $\alpha \in \mathbb{N}^I$ and a subset $V \subseteq \mathbb{N}^I$. We have the following theorem:

Theorem 4.5.2. *For a Log compatible family $\{F_\omega(t)\}_{\omega \in \mathbb{T}_I} \subseteq \mathbb{Q}(t)$, we have:*

$$\mathrm{Coeff}_\alpha\left(\mathrm{Exp}\left(\sum_{\beta\in V}F_{\beta,gen}(t)y^\beta\right)\right)=F_{\alpha,V}(t) \tag{4.5.2}$$

Proof. There is a unique morphism Θ of λ-rings

$$\Theta:\mathcal{K}_I^{ss}\to\mathbb{Q}(t)$$

obtained by extending

$$\Theta(\omega_\alpha)=F_{\alpha,gen}(t).$$

By Lemma 3.7.9, Identity (4.5.2) is equivalent to the following Identity

$$\sum_{\substack{\nu\in\mathbb{T}_\alpha^{ss}\\ \text{level } V}}\frac{\Theta(\nu)}{w(\nu)}=F_{\alpha,V}(t) \tag{4.5.3}$$

The RHS of eq.(4.5.3), is given by

$$F_{\alpha,V}(t)=\sum_{\omega\in\mathbb{T}_\alpha}\frac{F_\omega(t)}{w(\omega)}\left(\sum_{\substack{S\leq S_\omega\\ \text{level } V}}P_{[S]}(t)\mu(S,S_\omega)\right). \tag{4.5.4}$$

Consider now $\beta_1,\dots,\beta_r\in\mathbb{N}^I$ such that $\nu=\psi_{d_1}(\omega_{\beta_1})*\cdots*\psi_{d_r}(\omega_{\beta_r})$. We have therefore that

$$\frac{\Theta(\nu)}{w(\nu)}=\frac{1}{w(\nu)}\prod_{j=1}^r\left(\sum_{\omega_j\in\mathbb{T}_{\beta_j}}\frac{F_{\omega_{\beta_j}}(t^{d_j})}{w(\omega_j)}(t^{d_j}-1)\mu(Z_{\beta_j},S_{\omega_j})\right)= \tag{4.5.5}$$

$$\sum_{\omega\in\mathbb{T}_\alpha}\frac{F_\omega(t)P_\nu(t)}{w(\nu)}\left(\sum_{\substack{\omega_1\in\mathbb{T}_{\beta_1},\dots,\omega_r\in\mathbb{T}_{\beta_r}\\ \psi_{d_1}(\omega_1)*\cdots*\psi_{d_r}(\omega_r)=\omega}}\frac{1}{w(\omega_1)\cdots w(\omega_r)}\prod_{j=1}^r\mu(Z_{\beta_j},S_{\omega_j})\right) \tag{4.5.6}$$

Fix now a multitype $\omega\in\mathbb{T}_\alpha$ with

$$\$omega=(d_1',\boldsymbol{\lambda}_1)\cdots(d_t',\boldsymbol{\lambda}_t)$$

for multipartitions $\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_t$ and its associated admissible torus $S_\omega\subseteq\mathrm{GL}_\alpha$.

Denote by $H_{\nu,\omega}$ the set defined by

$$H_{\nu,\omega}\coloneqq\{(\omega_1,\dots,\omega_r)\in\mathbb{T}_{\beta_1}\times\cdots\times\mathbb{T}_{\beta_r}\mid\psi_{d_1}(\omega_1)*\cdots*\psi_{d_r}(\omega_r)=\omega\}$$

and by $\delta_\nu : H_{\nu,\omega} \to \mathbb{Z}$ the function defined as

$$\delta_\nu((\omega_1, \dots, \omega_r)) := \prod_{j=1}^{r} \mu(Z_{\beta_j}, S_{\omega_j}).$$

Let $M_{\nu,\omega^{ss}}$ be the set introduced in eq. (4.4.7). Consider the following function

$$f_{\nu,\omega} : M_{\nu,\omega^{ss}} \to H_{\nu,\omega}$$

defined as:

$$f_{\nu,\omega}((X_1, \dots, X_r)) = (\omega_1, \dots, \omega_r)$$

where

$$\omega_i(d, \boldsymbol{\lambda}) = \#\{h \in X_i \text{ such that } \left(\frac{d'_h}{d_i}, \boldsymbol{\lambda}_h\right) = (d, \boldsymbol{\lambda})\}$$

for every $(d, \boldsymbol{\lambda}) \in \mathbb{N} \times \mathcal{P}^I$.

The function $f_{\nu,\omega}$ is surjective and for each $(\omega_1, \dots, \omega_r)$, the cardinality of the fiber is given by

$$|f_{\nu,\omega}^{-1}(\omega_1, \dots, \omega_r)| = \prod_{(d,\boldsymbol{\lambda}) \in \mathbb{N} \times \mathcal{P}^I} \frac{\omega(d, \boldsymbol{\lambda})!}{\omega_1(d, \boldsymbol{\lambda})! \cdots \omega_r(d, \boldsymbol{\lambda})!} \tag{4.5.7}$$

Notice that for any $(\omega_1, \dots, \omega_r) \in H_{\nu,\omega}$, we have the following equality:

$$\prod_{(d,\boldsymbol{\lambda}) \in \mathbb{N} \times \mathcal{P}^I} \frac{\omega(d, \boldsymbol{\lambda})!}{\omega_1(d, \boldsymbol{\lambda})! \cdots \omega_r(d, \boldsymbol{\lambda})!} = \frac{w(\omega)}{w(\omega_1) \cdots w(\omega_r)} \frac{\prod_{j=1}^r \prod_{h \in X_j} \left(\frac{d'_h}{d_j}\right)}{\prod_{l=1}^t d'_l}. \tag{4.5.8}$$

As $\sum_{j=1}^r |X_j| = t$, the right hand side of eq.(4.5.8) is equal to

$$\frac{w(\omega)}{w(\omega_1) \cdots w(\omega_r)} \frac{1}{\prod_{j=1}^r d_j^{|X_j|}}. \tag{4.5.9}$$

For an element $m = (X_1, \dots, X_r) \in M_{\nu,\omega^{ss}}$, put

$$d_m := \prod_{j=1}^{r} d_j^{|X_j|}.$$

We can thus rewrite the RHS of eq.(4.5.6) as:

$$\frac{F_\omega(t)}{w(\nu)} \left(\sum_{(\omega_1,\dots,\omega_r) \in H_{\nu,\omega}} \frac{1}{w(\omega_1) \cdots w(\omega_r)} \delta_\nu((\omega_1, \dots, \omega_r)) \right) = \tag{4.5.10}$$

$$\frac{F_\omega(t)}{w(\nu)w(\omega)}\left(\sum_{m\in M_{\nu,\omega}}\frac{w(\omega)}{w(\omega_1)\cdots w(\omega_r)}\frac{\delta_\nu(f_{\nu,\omega}(m))}{|f_{\nu,\omega}^{-1}(m)|}\right)=\frac{F_\omega(t)}{w(\nu)w(\omega)}\left(\sum_{m\in M_{\nu,\omega}}\delta_\nu(f_{\nu,\omega}(m))d_m\right). \tag{4.5.11}$$

The set $H_{\nu,\omega}$ is endowed with the following action of W'_ν. An element $\sigma=(\sigma_1,\ldots,\sigma_k)\in W'_\nu$ acts on $(\omega_1,\ldots,\omega_r)\in H_{\nu,\omega}$ by

$$\sigma\cdot(\omega_1,\ldots,\omega_r)=(\omega_{\sigma_1(1)},\ldots,\omega_{\sigma_1(i_1)},\omega_{\sigma_2(1)},\ldots,\omega_{\sigma_k(k)}).$$

Notice that the function δ_ν is W'_ν invariant and $f_{\nu,\omega}$ is W'_ν equivariant. The function

$$\delta_\nu\circ f_{S,\omega}:M_{\nu,\omega^{ss}}\to\mathbb{Z}$$

is therefore W'_ν invariant and descends to a function $\overline{M_{\nu,\omega^{ss}}}\to\mathbb{Z}$ which we still denote by $\delta_\nu\circ f_{\nu,\omega}$. Notice moreover that the quantity d_m is W'_ν invariant too and so d_m is well defined for an element $m\in\overline{M_{\nu,\omega^{ss}}}$ too.

The right hand side of eq.(4.5.10) is therefore equal to

$$\frac{F_\omega(t)P_\nu(t)}{w(\omega)}\left(\sum_{m\in\overline{M_{\nu,\omega^{ss}}}}\delta_\nu(f_{\nu,\omega}(m))\frac{d_m|W'_\nu|}{w(\nu)}\right). \tag{4.5.12}$$

Notice that, for any $m\in\overline{M}_{\nu,\omega^{ss}}$, we have

$$\frac{d_m|W'_\nu|}{w(\nu)}=\prod_{j=1}^{r}\frac{d_j^{|X_j|}}{d_j}=\prod_{j=1}^{r}d_j^{|X_j|-1}$$

By eq.(4.4.10), we can thus rewrite the sum of eq.(4.5.12) as

$$\frac{F_\omega(t)P_\nu(t)}{w(\omega)}\left(\sum_{S\in P_{\nu,\omega^{ss}}}\delta_\nu(f_{\nu,\omega}(\pi_{\nu,\omega^{ss}}(S)))\right). \tag{4.5.13}$$

From the remarks made in §4.4 and eq. (4.4.7), we see that we have an equality

$$\delta_\nu(f_{\nu,\omega}(\pi_{\nu,\omega^{ss}}(S)))=\mu(S,S_\omega)$$

and so, from eq.(4.5.10), we deduce that

$$\frac{\Theta(\nu)}{w(\nu)}=\sum_{\omega\in\mathbb{T}_\alpha}\frac{F_\omega(t)P_\nu(t)}{w(\omega)}\sum_{S\in P_{\nu,\omega^{ss}}}\mu(S,S_\omega) \tag{4.5.14}$$

Summing over the $\nu \in \mathbb{T}_\alpha^{ss}$ of level V, we have therefore:

$$\mathrm{Coeff}_\alpha\left(\mathrm{Exp}\left(\sum_{\beta\in V} F_{\beta,gen}(t)y^\beta\right)\right) = \sum_{\substack{\nu\in\mathbb{T}_\alpha^{ss}\\ \text{of level } V}} \frac{\Theta(\nu)}{w(\nu)} = \tag{4.5.15}$$

$$\sum_{\substack{\nu\in\mathbb{T}_\alpha^{ss}\\ \text{of level } V}} \sum_{\omega\in\mathbb{T}_\alpha} \frac{F_\omega(t)P_\nu(t)}{w(\omega)} \sum_{S\in P_{\nu,\omega^{ss}}} \mu(S,S_\omega) = \sum_{\omega\in\mathbb{T}_\alpha} \frac{F_\omega(t)}{w(\omega)} \left(\sum_{\substack{S\leq S_\omega\\ \text{of level } V}} P_{[S]}(t)\mu(S,S_\omega)\right) \tag{4.5.16}$$

The right hand side is equal to $F_{\alpha,V}(t)$ by eq.(4.5.4). □

Remark 4.5.3. The notion of a Log compatible family and the polynomials $F_{\alpha,gen}(t)$ had already been introduced in [62, Paragraph 2.1.2]
Letellier [62, Theorem 2.2] used these notions to show the case where $V = \mathbb{N}^I$ of Theorem 4.5.2 above. His proof is different from ours as it uses symmetric funtions and does not seem to extend immediately to the case of any V.

5 Representation theory of finite reductive groups and Log compatibility

In this chapter, G is a connected reductive group over $\mathbb{F}_q$ with Frobenius morphism $F : G \to G$. We review the construnction of the irreducible characters of $G(\mathbb{F}_q)$, with a focus on the case of $\mathrm{GL}_n(\mathbb{F}_q), \mathrm{GL}_\alpha(\mathbb{F}_q)$ for $\alpha \in \mathbb{N}^I$. We will mostly follow the book by Digne and Michel [28]. Fix a prime ℓ such that $(\ell, q) = 1$ and an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. In the following, we identify vector spaces over $\overline{\mathbb{Q}}_\ell$ with vector spaces over $\mathbb{C}$, through this isomorphism.

Section §5.1 and §5.2 address the definition of Deligne-Lusztig induction and of the unipotent characters of a finite reductive group.
For GL_α, given an F-stable maximal torus $T \subseteq \mathrm{GL}_\alpha$ and a character $\theta : T^F \to \mathbb{C}^*$, in section §5.3 we define an associated graph, in a sort of dual way to what has been done in §4.4.
This construction is used in section §5.4 to give the definition of a reduced character of a Levi subgroup and, using the latter, to build the irreducible characters of the groups $\mathrm{GL}_\alpha(\mathbb{F}_q)$.
Moreover, in section §5.6, the construction of Section §5.3 is used to show how to associate a multitype to an irreducible character of $\mathrm{GL}_\alpha(\mathbb{F}_q)$.
These results are preliminary to sections §5.7 and §5.8, where we show the main technical results of the chapter, Theorem 5.7.5 and Theorem 5.8.4.
The latter theorems are consequences of Theorem 4.5.2 about Log compatible families and show how to compute certain invariants in the rings $(\mathcal{C}(\mathrm{GL}_\alpha(\mathbb{F}_q)), \otimes), (\mathcal{C}(\mathrm{GL}_\alpha(\mathbb{F}_q)), *)$ respectively.
These results are going to be used in chapter §8 and chapter §9 to compute multiplicities for k-tuples of Harisha-Chandra characters/E-series of character stacks for Riemann surfaces respectively.

5.1 Deligne-Lusztig induction

Consider an F-stable Levi subgroup L of G, a parabolic subgroup P having L as Levi factor and denote by U_P the unipotent radical of P. Recall that there is an isomorphism $P/U_P \cong L$ and denote by $\pi_L : P \to L$ the associated quotient map.

Remark 5.1.1. Notice that in general P can not be taken to be F-stable. We can find an F-stable parabolic subgroup $P \supseteq L$ if and only if $\epsilon_L = \epsilon_G$.

Denote by $\mathcal{L}$ the Lang map $\mathcal{L} : G \to G$ given by $\mathcal{L}(g) = g^{-1}F(g)$. The variety $X_L \coloneqq \mathcal{L}^{-1}(U_P)$ has a left G^F-action and a right L^F-action by multiplication on the left/right respectively.
These actions induce actions on the compactly supported étale cohomology groups $H^i_c(X_L, \overline{\mathbb{Q}}_\ell)$ and so endow the virtual vector space

$$H^*_c(X_L, \overline{\mathbb{Q}}_\ell) \coloneqq \bigoplus_{i \geq 0} (-1)^i H^i_c(X_L, \overline{\mathbb{Q}}_\ell)$$

with the structure of a virtual G^F-representation-L^F.

For an L^F-representation M, we define the Deligne-Lusztig induction $R_L^G(M)$ as the virtual G^F-representation given by

$$R_L^G(M) = H_c^*(X_L, \overline{\mathbb{Q}}_\ell) \otimes_{\mathbb{C}[L^F]} M.$$

We will denote by R_L^G the induced linear map

$$R_L^G : \mathcal{C}(L^F) \to \mathcal{C}(G^F).$$

Remark 5.1.2. In the case that will interest us in the thesis, it will always be true that the functor R_L^G does not depend on the choice of the parabolic subgroup $P \supseteq L$ (see for example [13]).

5.1.1 Harisha-Chandra characters

Consider the case where L is split, i.e there exist $n_0, \dots, n_s$ such that

$$(L, F) \cong \mathrm{GL}_{n_0} \times \cdots \times \mathrm{GL}_{n_s}.$$

In this case, we can take P, U_P to be F-stable too. The variety X_L is a U_P-principal bundle over the finite variety G^F/U^F, see the discussion before [28, Lemma 9.1.5].
Since U_P is isomorphic to an affine space, the cohomology $H_c^*(X_L, \overline{\mathbb{Q}}_\ell)$ is concentrated in degree $2\dim(U_P)$ and we have an equality

$$R_L^G(M) = \mathbb{C}[G^F/U^F] \otimes_{\mathbb{C}[L^F]} M$$

for every L^F-representation M.
In the split case, we can give the following equivalent description of this functor. For an L^F-representation M, denote by $\mathrm{Infl}_{L^F}^{P^F}(M)$ the natural lift to a P^F-representation via the quotient map π_L.
In [28, Proposition 5.18 (1)], it is shown the following Lemma: .

Lemma 5.1.3. *We have an isomorphis of functors:*

$$R_L^G \cong \mathrm{Ind}_{P^F}^{G^F}(\mathrm{Infl}_{L^F}^{P^F}).$$

The functor on the right hand side is usually called Harisha-Chandra induction. For any character $\gamma : L^F \to \mathbb{C}^*$, we call $R_L^G(\gamma)$ an *Harisha-Chandra character.* If an Harisha-Chandra character is irreducible, we will call it *semisimple split.*

The Harisha-Chandra induction can be explicitly described on class functions. More precisely, consider a class function $f \in \mathcal{C}(L^F)$ and $g \in G^F$. By inflation, we get a class function $f \in \mathcal{C}(P^F)$ and we have:

$$R_L^G(f) = \sum_{\substack{hP^F \in G^F/P^F \\ h^{-1}gh \in P^F}} f(h^{-1}gh). \tag{5.1.1}$$

In the case of $G = \mathrm{GL}_n$, the formula above has the following reformulation in terms of flag varieties. Let $n_0, \dots, n_s$ be integers such that $L \subseteq \mathrm{GL}_n$ is the split Levi subgroup $\mathrm{GL}_{n_0} \times \cdots \times \mathrm{GL}_{n_s}$ embedded diagonally into GL_n i.e

$$L = \begin{pmatrix} \mathrm{GL}_{n_s} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \mathrm{GL}_{n_{s-1}} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \mathrm{GL}_{n_{s-2}} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{GL}_{n_0} \end{pmatrix}.$$

The finite group L^F is therefore isomorphic to

$$L^F = \mathrm{GL}_{n_0}(\mathbb{F}_q) \times \cdots \times \mathrm{GL}_{n_s}(\mathbb{F}_q).$$

Let P be the F-stable parabolic subgroup containing L and the upper triangular matrices. Recall that the quotient G/P is identified with the variety of partial flags inside $\overline{\mathbb{F}}_q$:

$$\mathrm{GL}_n / P = \{\mathcal{F} = (\mathcal{F}_s \subseteq \mathcal{F}_{s-1} \subseteq \cdots \subseteq \mathcal{F}_0 = \overline{\mathbb{F}}_q^n) \mid \dim(F_i) = \sum_{j=i}^{s} n_j\}. \tag{5.1.2}$$

The $\mathbb{F}_q$-rational points $\mathrm{GL}_n(\mathbb{F}_q)/P(\mathbb{F}_q) = (\mathrm{GL}_n / P)(\mathbb{F}_q)$ are thus identied with partial flags of vector subspaces of $\mathbb{F}_q^n$

$$(\mathrm{GL}_n / P)(\mathbb{F}_q) = \{\mathcal{F} = (\mathcal{F}_s \subseteq \mathcal{F}_{s-1} \subseteq \cdots \subseteq \mathcal{F}_0 = \mathbb{F}_q^n) \mid \dim(F_i) = \sum_{j=i}^{s} n_j\}. \tag{5.1.3}$$

Notice that for $g \in \mathrm{GL}_n(\mathbb{F}_q)$ and $hP(\mathbb{F}_q) \in \mathrm{GL}_n(\mathbb{F}_q)/P(\mathbb{F}_q)$, we have $h^{-1}gh \in P(\mathbb{F}_q)$, if and only if $g \cdot hP(\mathbb{F}_q) = hP(\mathbb{F}_q)$, i.e if and only if g stabilizes the flag associated to $hP(\mathbb{F}_q)$.

We end by recalling the following properties of Deligne-Lusztig induction.

Lemma 5.1.4. *For a reductive group G and a Levi subgroup L, the following holds:*

1. *Given an F-stable Levi subgroup $L' \supseteq L$, there is an isomorphism of functors: $R_{L'}^G(R_L^{L'}) \cong R_L^G$.*
2. *Assume there exist reductive groups G_1, G_2 and Levi subgroups L_1, L_2 such that $G = G_1 \times G_2$ and $L = L_1 \times L_2$. For an L_1^F-representation M_1 and an L_2^F-representation M_2, there is a natural isomorphism $R_L^G(M_1 \boxtimes M_2) = R_{L_1}^{G_1}(M_1) \boxtimes R_{L_2}^{G_2}(M_2)$.*

Proof. The first point is shown in [28, Proposition 9.1.8]. For the second point, we can choose parabolic subgroups $G_1 \supseteq P_1 \supseteq L_1$ and $G_2 \supseteq P_2 \supseteq L_2$ such that P_1, P_2 have as Levi factor L_1, L_2 respectively. Notice thus that $P = P_1 \times P_2 \subseteq G$ is a parabolic subgroup having L as Levi factor and $U_P = U_{P_1} \times U_{P_2}$. We have therefore

$$\mathcal{L}^{-1}(U_P) = \mathcal{L}^{-1}(U_{P_1}) \times \mathcal{L}^{-1}(U_{P_1})$$

and therefore

$$H_c^*(\mathcal{L}^{-1}(U_P),\overline{\mathbb{Q}_\ell}) \cong H_c^*(\mathcal{L}^{-1}(U_{P_1}),\overline{\mathbb{Q}_\ell}) \otimes H_c^*(\mathcal{L}^{-1}(U_{P_2}),\overline{\mathbb{Q}_\ell}),$$

from which we deduce that

$$R_L^G(M_1 \boxtimes M_2) = R_{L_1}^{G_1}(M_1) \boxtimes R_{L_2}^{G_2}(M_2).$$

□

Remark 5.1.5. Consider an F-stable Levi subgroup $L' \supseteq L$ and a linear character $\theta : (L')^F \to \mathbb{C}^*$. By restriction, we can consider it as a character $\theta : L^F \to \mathbb{C}^*$.
For any $f \in \mathcal{C}(L^F)$, we have an identity $R_L^{L'}(\theta f) = \theta R_L^{L'}(f)$ and therefore, by Lemma 5.1.4(1), an equality

$$R_L^G(\theta f) = R_{L'}^G(\theta R_L^{L'}(f)).$$

5.2 Unipotent characters

5.2.1 Frobenius actions on Weyl groups of F-stable maximal tori

Fix a group G and an F-stable maximal torus $T \subseteq G$ as above. We follow the notations of [28, Chapter 11].
Denote by W the Weyl group $W_G(T)$ and by $\widetilde{W}$ the semidirect group $W \rtimes \langle F \rangle$, where $\langle F \rangle$ is the group generated by the finite order automorphism induced by F on W.
Denote by $\mathcal{C}(WF)$ the vector space of function $f : W \to \mathbb{C}$ constant on F-conjugacy classes. Equivalenty, a function $f \in \mathcal{C}(WF)$ can be seen as a function on the coset $WF \subseteq \widetilde{W}$, invariant under W-conjugation.
The vector space $\mathcal{C}(WF)$ is endowed with the Hermitian product defined by

$$\langle f, g \rangle_{WF} = \frac{1}{|W|} \sum_{w \in WF} f(w)\overline{g(w)},$$

for $f, g \in \mathcal{C}(WF)$.

Example 5.2.1. Let $G = (\mathrm{GL}_n)_d$ and T be the torus of Example 4.2.3. By the remarks thereby made, the morphism

$$\psi_d : \mathcal{C}(S_n) \to \mathcal{C}(WF)$$

defined as

$$\psi_d(f)(\sigma_1, \dots, \sigma_d) = f(\sigma_d \cdots \sigma_1)$$

is an isomorphism.
Notice that ψ_d is an isometry too. Indeed, for each $f, f' \in \mathcal{C}(S_n)$, we have:

$$\langle \psi_d(f), \psi_d(f') \rangle = \frac{1}{|S_n^d|} \sum_{(\sigma_1,\dots,\sigma_d) \in S_n^d} f(\sigma_d \cdots \sigma_1)\overline{f'(\sigma_d \cdots \sigma_1)} = \tag{5.2.1}$$

$$= \frac{1}{|S_n^d|} \sum_{\sigma \in S_n} f(\sigma)\overline{f'(\sigma)} \#\{(\sigma_1, \dots, \sigma_d) \in S_n^d \mid \sigma_d \cdots \sigma_1 = \sigma\} = \langle f, f' \rangle \tag{5.2.2}$$

where the last equality comes from the fact that, for each $\sigma \in S_n$, we have

$$\#\{(\sigma_1, \dots, \sigma_d) \in S_n^d \mid \sigma_d \cdots \sigma_1 = \sigma\} = |S_n^{d-1}|.$$

In [28, Chapter 11.6], it is shown the following Lemma.

Lemma 5.2.2. *For each $\chi \in (W^\vee)^F$, there exists an extension $\widetilde{\chi} \in (\widetilde{W^\vee})$. The restrictions of these irreducible characters to the coset $WF \subseteq \widetilde{W}$, i.e the elements $\{\widetilde{\chi} \in \mathcal{C}(WF)\}_{\chi \in (W^\vee)^F}$ are an orthonormal basis of $\mathcal{C}(WF)$.*

5.2.2 The case of finite general linear groups

Consider the group G and the torus T of Example 5.2.1 above. In this case, the constructions of Lemma 5.2.2 can be explicitly described as follows.
An irreducible character $\chi \in W^\vee$ is determined by partitions $\lambda^1, \dots, \lambda^d \in \mathcal{P}_n$ such that

$$\chi = \chi^{\lambda^1} \boxtimes \cdots \boxtimes \chi^{\lambda^d}.$$

The character χ is F-stable if and only if $\lambda^1 = \cdots = \lambda^d = \lambda$, i.e $\chi = (\chi^\lambda)^{\boxtimes d}$, so that $(W^\vee)^F$ is in bijection with $\mathcal{P}_n$.
Consider now a representation $\rho : W \to \mathrm{GL}(V_\lambda)$ affording the irreducible character χ^λ.
Let $\tau_d \in S_d$ be the permutation $\tau_d = (1\ 2 \cdots\ d)$. For $((\sigma_1, \dots, \sigma_d), F^i) \in \widetilde{W}$, we define $\widetilde{\rho}(((\sigma_1, \dots, \sigma_d), F^i)) \in \mathrm{GL}(V_\lambda^d)$ extending by linearity

$$\widetilde{\rho}(((\sigma_1, \dots, \sigma_d), F^i))(v_1 \otimes \cdots \otimes v_d) = \rho(\sigma_1)v_{\tau_d^i(1)} \otimes \cdots \otimes \rho(\sigma_d)v_{\tau_d^i(d)}$$

for each $v_1, \dots, v_d \in V_\lambda$.
Notice tha, for any $((\sigma_1, \dots, \sigma_d), F^i)((\sigma'_1, \dots, \sigma'_d), F^j) \in \widetilde{W}$, we have

$$\widetilde{\rho}(((\sigma_1, \dots, \sigma_d), F^i)((\sigma'_1, \dots, \sigma'_d), F^j))(v_1 \otimes \cdots \otimes v_d) = \widetilde{\rho}(((\sigma_1\sigma'_{\tau_d^i(1)}, \dots, \sigma_d\sigma'_{\tau_d^i(d)}), F^{i+j}))(v_1 \otimes \cdots \otimes v_d) = \tag{5.2.3}$$

$$= \rho(\sigma_1\sigma'_{\tau_d^i(1)})v_{\tau_d^{i+j}(1)} \otimes \cdots \otimes \rho(\sigma_d\sigma'_{\tau_d^i(d)}) \cdot v_{\tau_d^{i+j}(d)} = \rho(\sigma_1)\rho(\sigma'_{\tau_d^i(1)})v_{\tau_d^{i+j}(1)} \otimes \cdots \otimes \rho(\sigma_d)\rho(\sigma'_{\tau_d^i(d)}) \cdot v_{\tau_d^{i+j}(d)} = \tag{5.2.4}$$

$$\widetilde{\rho}((\sigma_1, \dots, \sigma_d), F^i)(\widetilde{\rho}((\sigma'_1, \dots, \sigma'_d), F^j)(v_1 \otimes \cdots \otimes v_d)) \tag{5.2.5}$$

i.e $\widetilde{\rho}$ defines a representation $\widetilde{\rho} : \widetilde{W} \to \mathrm{GL}(V_\lambda^{\otimes d})$.
The character of the representation $\widetilde{\rho}$ is the extension $\widetilde{(\chi^\lambda)^{\boxtimes d}}$ of Lemma 5.2.2 above. Consider the restriction of $\widetilde{(\chi^\lambda)^{\boxtimes d}}$ to the coset $WF \subseteq \widetilde{W}$ and the corresponding function $\widetilde{(\chi^\lambda)^{\boxtimes d}} \in$

$\mathcal{C}(WF)$.
We now verify that we have

$$\psi^{-1}(\widetilde{(\chi^\lambda)^{\boxtimes d}}) = \chi^\lambda \in \mathcal{C}(S_n).$$

We have indeed, for each $\sigma \in S_n$,

$$\psi^{-1}(\widetilde{(\chi^\lambda)^{\boxtimes d}})(\sigma) = \mathrm{tr}(\widetilde{\rho}((\sigma, 1, \dots, 1), F)). \tag{5.2.6}$$

Fix now a basis $\mathcal{B} = \{e_1, \dots, e_h\}$ of V_λ. Recall that a basis of $V_\lambda^{\otimes d}$ is given by

$$\mathcal{B}^{\otimes d} := \{e_{j_1} \otimes \dots \otimes e_{j_d} \mid (j_1, \dots, j_d) \in \{1, \dots, h\}^d\}.$$

For any $(j_1, \dots, j_d) \in \{1, \dots, h\}^d$, we have

$$\widetilde{\rho}((\sigma, 1, \dots, 1), F)(e_{j_1} \otimes \dots \otimes e_{j_d}) = e_{j_d} \otimes \rho(\sigma) e_{j_1} \otimes \dots \otimes e_{j_{d-1}} \tag{5.2.7}$$

We deduce therefore that the coefficient of the element $e_{j_1} \otimes \dots \otimes e_{j_d}$ in the writing of $\widetilde{\rho}((\sigma, 1, \dots, 1), F)(e_{j_1} \otimes \dots \otimes e_{j_d})$ in the basis $\mathcal{B}^{\otimes d}$ is given by

- the coefficient of the element e_j in the writing of $\rho(e_j)$ in the basis $\mathcal{B}$ if $j_1 = \dots = j_d = j$.
- 0 otherwise.

We deduce therefore that

$$\mathrm{tr}(\widetilde{\rho}((\sigma, 1, \dots, 1), F)) = \mathrm{tr}(\rho(\sigma)),$$

i.e that we have

$$\psi^{-1}(\widetilde{(\chi^\lambda)^{\boxtimes d}}) = \chi^\lambda.$$

5.2.3 Definition of unipotent characters

In [28, Chapter 11.6] for $f \in \mathcal{C}(WF)$, it is defined a class function $R_f : G^F \to \mathbb{C}$ as

$$R_f := \frac{1}{|W|} \sum_{w \in W} f(w) R_{T_w}^G(1). \tag{5.2.8}$$

In *locus cit*, it shown that the map $f \to R_f$ induces an isometry $\mathcal{C}(WF) \to \mathcal{C}(G^F)$. In particular, the elements $\{R_{\tilde{\chi}}\}_{\chi \in (W^\vee)^F}$ have norm 1 and are pairwise orthogonal in $\mathcal{C}(G^F)$.

Consider now an F-stable Levi subgroup $L \supseteq T$ and the corresponding Weyl group $W_L := W_L(T)$ which is an F-stable subgroup of W. Define the induction map

$$\mathrm{Ind}_{W_L F}^{WF} : \mathcal{C}(W_L F) \to \mathcal{C}(WF)$$

as

$$\mathrm{Ind}_{W_L F}^{WF}(f)(w) = \frac{1}{|W_L|} \sum_{\substack{h \in W_L \\ h^{-1} w F(h) \in W_L}} f(h^{-1} w F(h)).$$

In [28, Proposition 11.6.6] it is shown the following Lemma.

Lemma 5.2.3. *For any $f \in \mathcal{C}(W_L F)$, we have:*

$$R_L^G(R_f) = R_{\mathrm{Ind}_{W_L F}^{WF}(f)} \tag{5.2.9}$$

Let $G = (\mathrm{GL}_n)_d$ and T be the torus considered above. In [28, Chapter 11.7], it is shown the following Lemma.

Lemma 5.2.4. *For each $\chi \in (W^\vee)^F$, the class function $R_{\tilde{\chi}}$ is an irreducible character of G^F.*

The irreducible characters of this form are called *unipotent characters.* In particular, for every $\lambda \in \mathcal{P}_n$, there is a corresponding irreducible character $R_{\widetilde{\chi_\lambda^{\boxtimes d}}}$, which we will denote by R_λ.

Notice that Lemma 5.2.4 is true for any group G of the form $(\mathrm{GL}_{n_1})_{d_1} \times \cdots (\mathrm{GL}_{n_r})_{d_r}$. In this case, the unipotent characters, by Lemma 5.1.4(2), are in bijection with the multipartitions $\boldsymbol{\lambda} \in \mathcal{P}_{n_1} \times \cdots \times \mathcal{P}_{n_r}$ and we denote by $R_{\boldsymbol{\lambda}}$ the associated irreducible unipotent character.
For a such a group G, all F-stable Levi subgroups are still of the form $(\mathrm{GL}_{n'_1})_{d'_1} \times \cdots (\mathrm{GL}_{n'_s})_{d'_s}$. By eq.(5.2.9), we deduce the following Proposition.

Proposition 5.2.5. *Let $G = (\mathrm{GL}_{n_1})_{d_1} \times \cdots (\mathrm{GL}_{n_r})_{d_r}$ and $L \subseteq G$ an F-stable Levi subgroup such that*

$$(L, F) \cong (\mathrm{GL}_{n'_1})_{d'_1} \times \cdots (\mathrm{GL}_{n'_s})_{d'_s}.$$

For any $\boldsymbol{\mu} \in \mathcal{P}_{n'_1} \times \cdots \times \mathcal{P}_{n'_s}$, the character $R_L^G(R_{\boldsymbol{\mu}})$ belongs to the vector space spanned by the unipotent characters of G^F.

Example 5.2.6. Consider $G = \mathrm{GL}_n$ and $T = T_n$ the torus of diagonal matrices. In this case, $W = S_n$ and the action of F on S_n is trivial and therefore the functions $\mathcal{C}(S_n F)$ are the class functions $\mathcal{C}(S_n)$.
Notice that $W_T(T)$ is the trivial group. In particular, from Lemma 5.2.3, we deduce that we have an equality

$$R_T^G(1) = R_{\mathrm{Ind}_{\{e\}}^{S_n}} \tag{5.2.10}$$

The character $\mathrm{Ind}_{\{e\}}^{S_n}$ is the character of the group algebra $\mathbb{C}[S_n]$, i.e we have

$$\mathrm{Ind}_{\{e\}}^{S_n} = \sum_{\lambda \in \mathcal{P}_n} \chi_{(n)}^\lambda \chi^\lambda.$$

We deduce therefore that we have an equality

$$R_T^G(1) = \sum_{\lambda \in \mathcal{P}_n} \chi_{(n)}^\lambda R_\lambda. \tag{5.2.11}$$

In particular, the Harisha-Chandra character $R_T^G(1)$ is not irreductible and decomposes as a direct sum of unipotent characters. This is the reason why we avoid using the term "semisimple split" for an Harisha-Chandra character $R_L^G(\gamma)$ which is not necessarily irreducible.

Consider an F-stable Levi subgroup $L \subseteq G$, a unipotent character $R_f \in \mathcal{C}(L^F)$ and a linear character $\theta : L^F \to \mathbb{C}^*$. Fix a central element $\gamma \in G^F$. Notice that $\gamma \in L^F$ too. Mackey's formula [28, Proposition 10.1.2] implies the following Proposition.

Proposition 5.2.7. *We have an equality:*

$$R_L^G(\theta R_f)(\gamma) = R_L^G(R_f)(e)\theta(\gamma). \tag{5.2.12}$$

5.3 Characters of tori and graphs

Fix $\alpha \in \mathbb{N}^I$, consider the group GL_α and fix an F-stable maximal torus $T \subseteq \mathrm{GL}_\alpha$. We follow the notations of section §4.4.
Recall that inside $Y_*(T)$ there is the dual root system $\Phi^\vee \subseteq Y_*(T)$ which is provided with a canonical bijection $\Phi \leftrightarrow \Phi^\vee$.

Consider now a character $\theta : T^F \to \mathbb{C}^*$. In the following, we will show how to associate an admissible graph Γ_θ with vertices $\mathcal{B}$ to the character θ.
In [28, Proposition 11.7.1] it is shown the following Lemma:

Lemma 5.3.1. *Fix an isomorphism $\overline{\mathbb{F}}_q^* \cong (\mathbb{Q}/\mathbb{Z})_{p'}$ and let $n \in \mathbb{N}$ be such that T is split over $\mathbb{F}_{q^n}$. For any $m \in \mathbb{N}$, denote by $N_{F^m} : T \to T$ the map $N_{F^m}(x) = \prod_{j=0}^{m-1} F^j(x)$.*
There is a short exact sequence:

$$1 \longrightarrow Y_*(T) \xrightarrow{F-1} Y_*(T) \xrightarrow{\delta_T} T^F \longrightarrow 1.$$

where $\delta_T(\beta) = N_{F^n}\left(\beta\left(\frac{1}{q^n-1}\right)\right)$, where we are identifying $q^n - 1$ with an element of $\overline{\mathbb{F}}_q^*$, via the isomorphism $\overline{\mathbb{F}}_q^* \cong (\mathbb{Q}/\mathbb{Z})_{p'}$

In particular, the character $\theta : T^F \to \mathbb{C}^*$ induces by restriction a morphism

$$\widetilde{\theta} := \theta \circ \delta_T : Y_*(T) \to \mathbb{C}^*.$$

The graph Γ_θ is defined as follows.

- The set of vertices of Γ_θ is $\mathcal{B}$.
- For each $h > j$, there is an edge between ϵ_h and ϵ_j if and only if $\epsilon_{h,j}^\vee \in \mathrm{Ker}(\widetilde{\theta})$.

From Remark 4.4.2, we deduce the following Lemma.

Lemma 5.3.2. *For any $\theta : T^F \to \mathbb{C}^*$, the graph Γ_θ is admissible.*

In particular, as remarked in Paragraph §4.4, there exists a unique admissible subtorus $S_\theta \subseteq \mathrm{GL}_\alpha$ such that $\Gamma_{S_\theta} = \Gamma_\theta$.
We will denote by $L_\theta = C_{\mathrm{GL}_{n_\alpha}}(S_\theta)$ and by $\widetilde{L_\theta} = L_\theta \cap \mathrm{GL}_\alpha$. The Levi subgroup L_θ is the *connected centralizer* of θ, as introduced in [26, Definition 5.19].

Example 5.3.3. Consider the subset $I = \{1,2,3,4\}$, the dimension vector $\alpha = (2,1,1,1)$ and the torus T of diagonal matrices of Example 4.4.12. Notice that

$$T^F = \left\{ \left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \gamma, \delta, \eta \right) \mid \lambda, \mu, \gamma, \delta, \eta \in \mathbb{F}_q^* \right\}.$$

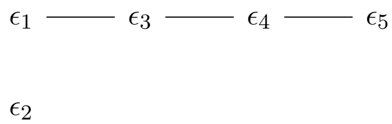
Consider $(\beta_1, \beta_2) \in \mathrm{Hom}(\mathbb{F}_q, \mathbb{C}^*)^2$ and let

$$\theta_{\beta_1,\beta_2} : T^F \to \mathbb{C}^*$$

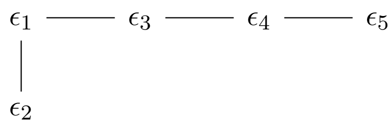
defined as

$$\theta_{\beta_1,\beta_2}(\lambda, \mu, \gamma, \delta, \eta) = \beta_1(\lambda\gamma\delta\eta)\beta_2(\mu).$$

If $\beta_1 \neq \beta_2$, the graph $\Gamma_{\theta_{\beta_1,\beta_2}}$ is



and the admissible torus $S_{\theta_{\beta_1,\beta_2}}$ is therefore the torus S_1 of Example 4.3.7.
If $\beta_1 = \beta_2$, the graph $\Gamma_{\theta_{\beta_1,\beta_1}}$ is



and the admissible torus $S_{\theta_{\beta_1,\beta_1}}$ is Z_α.

5.4 Irreducible characters of finite general linear group

5.4.1 Reduced characters and connected centralizers

Let now $G = \mathrm{GL}_n$ (i.e $|I| = 1$) and consider an F-stable Levi subgroup $L \subseteq \mathrm{GL}_n$ and a linear character $\theta : L^F \to \mathbb{C}^*$. The Levi subgroup L contains an F-stable maximal torus T. From θ, T, we determine the connected centralizer $L_\theta \supseteq L$ as above. The character θ is called *reduced* if $L_\theta = L$.

Remark 5.4.1. While the connected centralizer L_θ does depend on the choice of torus T, from [26, Proposition 5.11(ii), Proposition 5.20] we deduce that for two F-stable maximal tori $T, T' \subseteq L$ and corresponding connected centralizers L_θ, L'_θ, there exists an element $g \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $gL_\theta g^{-1} = L'_\theta$.
In particular, the choice of the torus is not relevant to determine whether the character θ is reduced or not.

For any two positive integers r, d such that $r|d$, the norm map $N_{\mathbb{F}^*_{q^d}/\mathbb{F}^*_{q^r}} : \mathbb{F}^*_{q^d} \to \mathbb{F}^*_{q^r}$ induces by precomposition an injective homomorphism

$$\Gamma_{r,d} \coloneqq \mathrm{Hom}(\mathbb{F}^*_{q^r}, \mathbb{C}^*) \to \mathrm{Hom}(\mathbb{F}^*_{q^d}, \mathbb{C}^*).$$

We denote by Γ the inductive limit via these maps

$$\Gamma := \varinjlim \mathrm{Hom}(\mathbb{F}_{q^d}^*, \mathbb{C}^*).$$

Notice that, for any $d \geq 1$, we can view $\mathrm{Hom}(\mathbb{F}_{q^d}^*, \mathbb{C}^*)$ as a subgroup of Γ through the universal maps of the limit. The Frobenius morphism acts by precomposition on each term $\mathrm{Hom}(\mathbb{F}_{q^d}^*, \mathbb{C}^*)$ (i.e $F(\gamma) = \gamma \circ F$) and so defines a morphism $F : \Gamma \to \Gamma$.
Consider the Levi subgroup

$$L = (\mathrm{GL}_{n_1})_{d_1} \times \cdots \times (\mathrm{GL}_{n_r})_{d_r}$$

with $n_1, \ldots, n_r$, $d_1, \ldots, d_r$ positive integers such that $d_1 n_1 + \cdots + d_r n_r = n$ and let T be the maximal torus

$$(T_{n_1})_{d_1} \times \cdots (T_{n_r})_{d_r}.$$

The group L^F is isomorphic to $\mathrm{GL}_{n_1}(\mathbb{F}_{q^{d_1}}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{F}_{q^{d_r}})$. A character $\theta : L^F \to \mathbb{C}^*$ is therefore given by an element $(\theta_1, \ldots, \theta_r) \in \mathrm{Hom}(\mathbb{F}_{q^{d_1}}^*, \mathbb{C}^*) \times \cdots \times \mathrm{Hom}(\mathbb{F}_{q^{d_r}}^*, \mathbb{C}^*)$ such that

$$\theta(M_1, \ldots, M_r) = \prod_{j=1}^{r} \theta_j(\det(M_j))$$

with $M_j \in \mathrm{GL}_{n_j}(\mathbb{F}_{q^{d_j}})$. We have the following Lemma

Lemma 5.4.2. *The character θ is reduced if and only if the F-orbits of $\theta_1, \ldots, \theta_r$ inside Γ have length $d_1, \ldots, d_r$ respectively and are pairwise disjoint.*

Proof. Notice that, for any $h \in \{1, \ldots, n\}$ there exist unique $i_h \in \{1, \ldots, r\}$ and $j_h \in \{1, \ldots, d_h\}$ such that

$$\sum_{s=1}^{i_h - 1} d_s n_s < h \leq \sum_{s=1}^{i_h} d_s n_s$$

and

$$\sum_{s=1}^{i_h - 1} d_s n_s + n_{i_h}(j_h - 1) < h \leq \sum_{s=1}^{i_h - 1} d_s n_s + n_{i_h} j_h.$$

From the definition of $\widetilde{\theta}$, we deduce that, for $h_1, h_2 \in \{1, \ldots, n\}$, we have that

$$\widetilde{\theta}(\epsilon_{h_1,h_2}^{\vee}) = 1$$

if and only if

$$\theta_{i_{h_1}}^{q^{j_{h_1}}} \theta_{i_{h_2}}^{-q^{j_{h_2}}} = 1$$

as elements of Γ, from which we deduce the Lemma above.

$\square$

Example 5.4.3. Consider a split Levi subgroup $L \subseteq G$ and $n_0, \dots, n_s$ such that

$$(L, F) \cong \mathrm{GL}_{n_0} \times \cdots \times \mathrm{GL}_{n_s}.$$

A character $\gamma : L^F \to \mathbb{C}^*$ corresponds thus to $\gamma_0, \dots, \gamma_s \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)$ and γ is reduced if and only if $\gamma_j \neq \gamma_h$ for each $h \neq j$.

Consider now two Levi subgroups $L \subseteq \mathrm{GL}_n$ and $L' \subseteq \mathrm{GL}_{n'}$ and the product Levi subgroup $M = L \times L' \subseteq \mathrm{GL}_m$ embedded block diagonally, where $m = n + n'$.
Assume that $L = (\mathrm{GL}_{n_1})_{d_1} \times \cdots \times (\mathrm{GL}_{n_r})_{d_r}$ and $L' = (\mathrm{GL}_{n'_1})_{d'_1} \times \cdots \times (\mathrm{GL}_{n'_s})_{d'_s}$ and consider two reduced characters $\theta : L^F \to \mathbb{C}^*$ and $\theta' : (L')^F \to \mathbb{C}^*$ corresponding to $(\theta_1, \dots, \theta_r), (\theta'_1, \dots, \theta'_s)$ where $\theta_i \in \mathrm{Hom}(\mathbb{F}^*_{q^{d_i}}, \mathbb{C}^*), \theta'_j \in \mathrm{Hom}(\mathbb{F}^*_{q^{d'_j}}, \mathbb{C}^*)$ for $i = 1, \dots, r, j = 1, \dots, s$.
Consider the character

$$\gamma = \theta \times \theta' : M^F \to \mathbb{C}^*.$$

Its connected centralizer M_γ admits the following description. For $i \in \{1, \dots, r\}$, consider the subset $J_i \subseteq \{1, \dots, s\}$ defined by

$$J_i := \big\{ j \in \{1, \dots, s\} \mid d'_j = d_i \text{ and the } F\text{-orbits inside } \Gamma \text{ of } \theta_i, \theta'_j \text{ have nonempty intersection} \big\}.$$

Notice that either $J_i = \varnothing$ or $J_i = \{j_i\}$ for an element $j_i \in \{1, \dots, s\}$, as the characters θ, θ' are both reduced. Denote by $I' \subseteq \{1, \dots, r\}$ the subset $I' := \{i \mid J_i = \varnothing\}$ and by $J' \subseteq \{1, \dots, s\}$ the subset $J' = \{1, \dots, s\} - \bigsqcup_{i=1}^{r} J_i$.

A similar argument to the one used to prove Lemma 5.4.2 shows that a connected centralizer M_γ is $\mathrm{GL}_n(\mathbb{F}_q)$- conjugated to the Levi subgroup

$$\prod_{i \in I'} (\mathrm{GL}_{n_i})_{d_i} \prod_{j \in J'} (\mathrm{GL}_{n'_j})_{d'_j} \prod_{i \in (I')^c} (\mathrm{GL}_{n_i + n'_{j_i}})_{d_i}.$$

Via this conjugation, the character γ corresponds to the character associated to

$$((\theta_i)_{i \in I'}, (\theta'_j)_{j \in J'}, (\theta_i)_{i \in (I')^c}).$$

5.4.2 Characters of Levi subgroups

Consider $L, T \subseteq \mathrm{GL}_m$ and $\theta : L^F \to \mathbb{C}^*$ as before. As mentioned at the end of §5.3, the character θ can be extended to the connected centralizer $\theta : L_\theta^F \to \mathbb{C}^*$ and $L_\theta \supseteq L$.
Conversely, for each character $\gamma : T^F \to \mathbb{C}^*$ such that $L_\gamma \supseteq L$, the character γ can be first extended to $\gamma : L_\gamma^F \to \mathbb{C}^*$ and then restricted to obtain a linear character $\gamma : L^F \to \mathbb{C}^*$.
We deduce therefore the following Proposition.

Proposition 5.4.4. *There is a bijection:*

$$\mathrm{Hom}(L^F, \mathbb{C}^*) \leftrightarrow \{\gamma \in \mathrm{Hom}(T^F, \mathbb{C}^*) \mid L_\gamma \supseteq L\}. \tag{5.4.1}$$

Consider now $G = \mathrm{GL}_\alpha$, an admissible torus S, the associated Levi subgroups $L_S = Z_{\mathrm{GL}_{m_\alpha}}(S) \subseteq \mathrm{GL}_{m_\alpha}$ and $\widetilde{L_S} = L_S \cap \mathrm{GL}_\alpha$. Consider an F-stable maximal torus T with $L_S \supseteq T \supseteq S$. The correspondence (5.4.1) can be rewritten as the following partition of $\mathrm{Hom}(L_S^F, \mathbb{C}^*)$.

Proposition 5.4.5. *There are bijections:*

$$\{\theta \in \mathrm{Hom}(\widetilde{L_S}^F, \mathbb{C}^*) \mid \Gamma_\theta \leq \Gamma_S\} \leftrightarrow \bigsqcup_{U \leq S} \{\theta \in \mathrm{Hom}(T^F, \mathbb{C}^*) \mid \Gamma_\theta = \Gamma_U\} \leftrightarrow \mathrm{Hom}(L_S^F, \mathbb{C}^*). \tag{5.4.2}$$

5.5 Construction of irreducible characters

In this paragraph, we quickly recall how to build the character table of the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$. We start from the following Lemma, which will also be needed later.

Lemma 5.5.1. *Consider $G = \mathrm{GL}_n$, a Levi subgroup $L \subseteq G$ and two characters $R_{\varphi_1}, R_{\varphi_2}$ for $\varphi_1, \varphi_2 \in \mathcal{C}(W_L F)$. Let $\theta : L^F \to \mathbb{C}^*$ be a reduced character. We have the following equality:*

$$\langle R_L^G(\theta R_{\varphi_1}), R_L^G(\theta R_{\varphi_2}) \rangle_{G^F} = \langle R_{\varphi_1}, R_{\varphi_2} \rangle_{L^F}.$$

Notice in particular that if $\varphi_1 = \varphi_2 = \tilde{\psi}$ with $\psi \in (W_L^\vee)^F$, we obtain that

$$\langle R_L^G(\theta R_{\tilde{\psi}}), R_L^G(\theta R_{\tilde{\psi}}) \rangle = \langle R_{\tilde{\psi}}, R_{\tilde{\psi}} \rangle = 1.$$

In particular, the character $R_L^G(\theta R_{\tilde{\psi}})$ is a virtual irreducible character, i.e an irreducible character up to a sign.
From these remarks, in [69, Theorem 3.2], it is shown the following Theorem.

Theorem 5.5.2. *For an irreducible character $\chi \in \mathrm{GL}_n(\mathbb{F}_q)^\vee$, we have*

$$\chi = \epsilon_G \epsilon_L R_L^G(\theta R_{\tilde{\varphi}}),$$

where L is an F-stable Levi subgroup, $\varphi \in (W_L^\vee)^F$ and $\theta : L^F \to \mathbb{C}^*$ is a reduced character. Two characters χ_1, χ_2 with associated data $(L_1, \theta_1, \varphi_1)$ and $(L_2, \theta_2, \varphi_2)$ are equal if and only if the triple $(L_1, \theta_1, \varphi_1), (L_2, \theta_2, \varphi_2)$ are $\mathrm{GL}_n(\mathbb{F}_q)$-conjugated.

For an irreducible character χ with associated datum (L, θ, φ), we will refer to the couple (L, θ) as the semisimple part of χ. This is well defined up to $\mathrm{GL}_n(\mathbb{F}_q)$-conjugacy.

Example 5.5.3. For a split Levi subgroup $L \subseteq \mathrm{GL}_n$, with

$$L \cong_{\mathbb{F}_q} \mathrm{GL}_{n_0} \times \cdots \times \mathrm{GL}_{n_s},$$

from Theorem 5.5.2 above, we deduce that, for any $\gamma = (\gamma_0, \ldots, \gamma_s) : L^F \to \mathbb{C}^*$, the Harisha-Chandra character $R_L^G(\gamma)$ is irreducible if and only if γ is reduced, i.e if and only if $\gamma_h \neq \gamma_j$ for each $h \neq j$.
In §8.4.1, we show how to prove the latter result in an alternative way, using quiver representations and our main result about multiplicities, Theorem 8.2.8.

Remark 5.5.4. Let $G = \mathrm{GL}_n$ and consider now an F-stable Levi subgroup L, a character $\gamma : L^F \to \mathbb{C}^*$ (not necessarily reduced) and a unipotent irreducible character $R_{\tilde{\psi}}$ for $\psi \in (W_L^\vee)^F$. Let L_γ be the connected centralizer of γ. By Remark 5.1.5, we have an equality

$$R_L^G(\gamma R_{\tilde{\psi}}) = R_{L_\gamma}^G(\gamma R_L^{L_\gamma}(R_{\tilde{\psi}})). \tag{5.5.1}$$

Notice that from Proposition 5.2.5, we have that $R_L^{L_\gamma}(R_{\tilde{\psi}})$ belongs to the vector space spanned by the unipotent characters of L_γ. We deduce thus the following Proposition.

Proposition 5.5.5. *For any $\gamma : L^F \to \mathbb{C}^*$ and any $\psi \in (W_L^\vee)^F$, the character $R_L^G(\gamma R_{\tilde{\psi}})$ belongs to the vector space spanned by the irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ with semisimple part (L_γ, γ).*

For any $\alpha \in \mathbb{N}^I$, the irreducible characters $\mathrm{GL}_\alpha(\mathbb{F}_q)^\vee$ have a similar description to that of Theorem 5.5.2. Namely, as each $\chi \in \mathrm{GL}_\alpha(\mathbb{F}_q)^\vee$ is of the form

$$\chi = \boxtimes_{i \in I} \chi_i,$$

with $\chi_i \in \mathrm{GL}_{\alpha_i}^\vee(\mathbb{F}_q)$, from Lemma 5.1.4, we deduce that there exist an F-stable Levi subgroup $L \subseteq \mathrm{GL}_\alpha$, a unipotent character $R_{\tilde{\psi}}$ with $\psi \in (W_L^\vee)^F$ and a reduced character $\theta : L^F \to \mathbb{C}^*$ such that $\chi = \epsilon_G \epsilon_L R_L^G(\theta R_{\tilde{\psi}})$.

5.6 Type of an irreducible character

Let $\chi \in \mathrm{GL}_n(\mathbb{F}_q)^\vee$ with associated datum (L, θ, φ). Up to conjugacy, L is equal to $(\mathrm{GL}_{n_1})_{d_1} \times \cdots \times (\mathrm{GL}_{n_r})_{d_r}$ and $\varphi = R_{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}_{n_1} \times \cdots \times \mathcal{P}_{n_r}$ a multipartition. The type

$$\omega = (d_1, \lambda_1) \ldots (d_r, \lambda_r) \in \mathbb{T}_n$$

is called the type of the irreducible character χ.

Example 5.6.1. Consider a split Levi subgroup L, a reduced $\theta : L^F \to \mathbb{C}^*$ and the irreducible character $R_L^G(\theta)$. There exists a partition $\mu = (\mu_1, \ldots, \mu_h) \in \mathcal{P}_n$ such that L is $\mathrm{GL}_n(\mathbb{F}_q)$-conjugated to $L_\mu = \mathrm{GL}_{\mu_1} \times \mathrm{GL}_{\mu_2} \times \cdots \times \mathrm{GL}_{\mu_h} \subseteq \mathrm{GL}_n$. The type of the character $R_L^G(\theta)$ is

$$(1, (\mu_1)) \cdots (1, (\mu_h)).$$

In a similar way, for any finite set I and any $\alpha \in \mathbb{N}^I$, to each irreducible character $\chi \in \mathrm{GL}_\alpha(\mathbb{F}_q)^\vee$, we can associate a multitype $\omega_\chi \in \mathbb{T}_\alpha$. Let $\chi = \epsilon_{\mathrm{GL}_\alpha} \epsilon_L R_L^{\mathrm{GL}_\alpha}(R_{\tilde{\varphi}} \theta)$ with $\theta : L^F \to \mathbb{C}^*$ a reduced character and $R_{\tilde{\varphi}}$ a unipotent character of L^F with $\varphi \in (W_L^\vee)^F$.
Consider an F-stable torus $T \subseteq L$ and the restriction of $\theta : T^F \to \mathbb{C}^*$. As explained in §5.3, this determines a Levi subgroup $L_\theta \subseteq \mathrm{GL}_{|\alpha|}$ with admissible center $S_\theta \subseteq T$ and such that

$$L_\theta \cap \mathrm{GL}_\alpha = \widetilde{L_\theta} = L.$$

Consider the semisimple multitype

$$[S_\theta] = (d_1, (1^{\beta_1})) \dots (d_r, (1^{\beta_r})).$$

By Remark 4.3.8, we see that $\widetilde{L_\theta}$ is $\mathrm{GL}_\alpha(\mathbb{F}_q)$-conjugated to

$$\prod_{i \in I} \prod_{j=1}^{r} (\mathrm{GL}_{(\beta_j)_i})_{d_j} \subseteq \prod_{i \in I} \mathrm{GL}_{\alpha_i}.$$

The set $(W_L^\vee)^F$ is thus in bijection with

$$\prod_{i \in I} \prod_{j=1}^{r} S^\vee_{(\beta_j)_i} = \prod_{i \in I} \prod_{j=1}^{r} \mathcal{P}_{(\beta_j)_i}.$$

The element φ determines multipartition $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_r \in \mathcal{P}^I$ such that $|\boldsymbol{\lambda}_j| = \beta_j$. The type ω_χ associated to χ is given by

$$\omega_\chi = (d_1, \boldsymbol{\lambda}_1) \dots (d_r, \boldsymbol{\lambda}_r).$$

Example 5.6.2. Let $I = \{1, 2, 3, 4\}$ and $\alpha = (2, 1, 1, 1)$. Let $T \subseteq \mathrm{GL}_2$ be the F-stable torus of diagonal matrices, consider $\beta \neq \gamma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)$ and the associated character $(\beta, \gamma) : T^F \to \mathbb{C}^*$. Let χ be the character $\chi \in \mathrm{GL}_\alpha(\mathbb{F}_q)^\vee$

$$\chi = R_T^G((\beta, \gamma)) \boxtimes \gamma(\det) \boxtimes \gamma(\det) \boxtimes \gamma(\det).$$

Let $\beta_1 = (1, 1, 1, 1)$ and $\beta_2 = (1, 0, 0, 0)$. The associated multitype is

$$\omega_\chi = (1, (\beta_1))(1, (\beta_2)).$$

Remark 5.6.3. Given $\omega \in \mathbb{T}_\alpha$, consider an irreducible character $\chi \in \mathrm{GL}_\alpha(\mathbb{F}_q)^\vee$ of type ω. Fix $S \in \mathcal{Z}_\alpha$ such that $[S] = \omega^{ss}$ (for example $S = S_\omega$). We can assume then that

$$\chi = \epsilon_{\widetilde{L_S}} \epsilon_{\mathrm{GL}_\alpha} R^{\mathrm{GL}_\alpha}_{\widetilde{L_S}}(\theta R_{\widetilde{\varphi}})$$

with $\theta : \widetilde{L_S}^F \to \mathbb{C}^*$ such that $S_\theta = S$ and $R_{\widetilde{\varphi}}$ a unipotent character of $\widetilde{L_S}^F$. For any $\gamma : \widetilde{L_S}^F \to \mathbb{C}^*$ such that $S_\gamma = S$, the character $\epsilon_{\widetilde{L_S}} \epsilon_{\mathrm{GL}_\alpha} R^{\mathrm{GL}_\alpha}_{\widetilde{L_S}}(\gamma R_{\widetilde{\varphi}})$ is irreducible and of type ω.

The map from

$$\{\gamma : \widetilde{L_S}^F \to \mathbb{C}^* \mid S_\gamma = S\}$$

to

$$\{\chi \in \mathrm{GL}_\alpha(\mathbb{F}_q)^\vee \text{ of type } \omega\}$$

which sends γ to $\epsilon_{\widetilde{L_S}} \epsilon_{\mathrm{GL}_\alpha} R^{\mathrm{GL}_\alpha}_{\widetilde{L_S}}(\gamma R_{\widetilde{\varphi}})$ is surjective. A similar argument to the one used to define the multitype of a conjugacy class of $\mathrm{GL}_\alpha(\mathbb{F}_q)$ in §4.3.5 shows that its fibers have cardinality $w(\omega)$.

Recall that the value $\frac{\chi(1)}{|\operatorname{GL}_\alpha(\mathbb{F}_q)|}$ for $\chi \in \operatorname{GL}_\alpha(\mathbb{F}_q)^\vee$ depends only on the type of χ. More precisely, for a partition $\lambda \in \mathcal{P}$, let $H_\lambda(t)$ be the *hook polynomial*

$$H_\lambda(t) = \prod_{s\in\lambda}(1-t^{h(s)}).$$

For a multipartition $\boldsymbol{\lambda} = (\lambda^i)_{i\in I} \in \mathcal{P}^I$, we define $H_{\boldsymbol{\lambda}}(t) \coloneqq \prod_{i\in I} H_{\lambda^i}(t)$. Given a multitype $\omega = (d_1, \boldsymbol{\lambda}_1)\dots(d_r, \boldsymbol{\lambda}_r)$, define $H^\vee_\omega(t)$ as

$$H^\vee_\omega(t) \coloneqq \frac{(-1)^{f(\omega)}}{q^{\left(\sum_{i\in I}\frac{\alpha_i(\alpha_i-1)}{2}-n(\omega)\right)}\prod_{j=1}^r H_{\boldsymbol{\lambda}_j}(t^{d_j})} \tag{5.6.1}$$

where if $|\boldsymbol{\lambda}_1| = \beta_1, \dots, |\boldsymbol{\lambda}_r| = \beta_r$, then $f(\omega) = \sum_{j=1}^r |\beta_j|$. We have the following Proposition (see for example [70, IV, 6.7]).

Proposition 5.6.4. *For any $\chi \in \operatorname{GL}_\alpha(\mathbb{F}_q)$, we have:*

$$\frac{\chi(1)}{|\operatorname{GL}_\alpha(\mathbb{F}_q)|} = H^\vee_{\omega_\chi}(q). \tag{5.6.2}$$

5.7 Log compatibility for family of class functions

5.7.1 Multiplicative parameters

Given $\sigma = (\sigma_i)_{i\in I} \in \operatorname{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$ and $\delta \in \mathbb{N}^I$, we denote by σ^δ the element of $\operatorname{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)$ defined as

$$\sigma^\delta \coloneqq \prod_{i\in I}\sigma_i^{\delta_i}.$$

We denote by $\mathcal{H}_\sigma$ the subset of $\mathbb{N}^I$ defined as

$$\mathcal{H}_\sigma \coloneqq \{\delta \in \mathbb{N}^I \mid \sigma^\delta = 1\}$$

and, for any $\alpha \in \mathbb{N}^I$, by $\mathcal{H}_{\sigma,\alpha}$ the intersection $\mathcal{H}_{\sigma,\alpha} \coloneqq \mathcal{H}_\sigma \cap \mathbb{N}^I_{\leq\alpha}$.

Definition 5.7.1. For an admissible torus $S \in \mathcal{Z}_\alpha$, we say that S is of level σ if it is of level $\mathcal{H}_{\sigma,\alpha}$.

Let $S \subseteq \operatorname{GL}_\alpha$ be an admissible torus and $\gamma = (\gamma_i)_{i\in I} \in \operatorname{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$. Let $\det_I$ be the morphism $\det_I : \operatorname{GL}_\alpha(\mathbb{F}_q) \to (\mathbb{F}_q^*)^I$ defined as

$$\det{}_I((g_i)_{i\in I}) \coloneqq (\det(g_i))_{i\in I}.$$

For an element $\gamma = (\gamma_i)_{i\in I} \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$ denote by ρ_γ the character

$$\rho_\gamma((g_i)_{i\in I}) \coloneqq \prod_{i\in I} \gamma_i(\det(g_i)).$$

By eq.(4.3.14), there is an equality

$$\sum_{s\in(S^{\mathrm{reg}})^F} \rho_\gamma(s) = \sum_{S'\leq S} \left(\sum_{s'\in(S')^F} \rho_\gamma(s') \right) \mu(S',S). \tag{5.7.1}$$

As $\det_I$ is invariant up to conjugation, to evaluate $\sum_{s'\in(S')^F} \rho_\gamma(s')$, we can assume S' to be equal to

$$S' = (Z_{\beta_1})_{d_1} \times \cdots \times (Z_{\beta_r})_{d_r}$$

embedded block diagonally, for certain $\beta_1, \dots, \beta_r \in \mathbb{N}^I$ and $d_1, \dots, d_r \in \mathbb{N}_{>0}$. The finite group $(S')^F$ is identified with $\mathbb{F}^*_{q^{d_1}} \times \cdots \times \mathbb{F}^*_{q^{d_r}}$ and, via this identification, we see that

$$\rho_\gamma(z_1, \dots, z_r) = \gamma^{\beta_1}(N_{\mathbb{F}^*_{q^{d_1}}/\mathbb{F}_q^*}(z_1)) \cdots \gamma^{\beta_r}(N_{\mathbb{F}^*_{q^{d_r}}/\mathbb{F}_q^*}(z_r)).$$

Therefore, we have that $\rho_\gamma|_{(S')^F} = 1$ if and only $\beta_j \in \mathcal{H}_{\gamma,\alpha}$ for each $j = 1, \dots, r$, i.e if and only if S' is of level γ. We deduce the following Proposition.

Proposition 5.7.2. *For any $\gamma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$ and any $S \in \mathcal{Z}_\alpha$, we have:*

$$\sum_{s\in(S^{\mathrm{reg}})^F} \rho_\gamma(s) = \sum_{\substack{S'\leq S \\ \textit{of level } \gamma}} |(S')^F| \mu(S',S) = \sum_{\substack{S'\leq S \\ \textit{of level } \gamma}} P_{[S']}(q)\mu(S',S) \tag{5.7.2}$$

5.7.2 Multiplicities for Log compatible families

Assume now to have been given a family $\{r_\alpha\}_{\alpha\in\mathbb{N}^I}$ with $r_\alpha \in \mathcal{C}(\mathrm{GL}_\alpha(\mathbb{F}_q))$.

Definition 5.7.3. We say that $\{r_\alpha\}_{\alpha\in\mathbb{N}^I}$ is Log compatible if, for each $\alpha \in \mathbb{N}^I$, the value of r_α is constant on conjugacy classes of the same type and its value at a type $\omega \in \mathbb{T}_\alpha$ is of the form $R_\omega(q)$ where $R_\omega(t) \in \mathbb{Q}(t)$ and the family $\{R_\omega(t)\}_{\omega\in\mathbb{T}_I}$ is Log compatible.

Remark 5.7.4. Notice that given two Log compatible families $\{r_\alpha\}_{\alpha\in\mathbb{N}^I}, \{r'_\alpha\}_{\alpha\in\mathbb{N}^I}$ the family $\{r_\alpha r'_\alpha\}_{\alpha\in\mathbb{N}^I}$ is Log compatible too.

For each $\omega \in \mathbb{T}$, denote by $\widetilde{R}_\omega(t) \coloneqq \dfrac{R_\omega(t)}{Z_\omega(t)}$. From eq.(3.6.1), we deduce tht $\{Z_\omega(t)\}_{\omega\in\mathbb{T}_I}$ is Log compatible and therefore the family $\{\widetilde{R}_\omega(t)\}_{\omega\in\mathbb{T}_I}$ is Log compatible too.
For each $\alpha \in \mathbb{N}^I$ and for each $\sigma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$, we will denote the polynomial $\widetilde{R}_{\alpha,\mathcal{H}_{\sigma,\alpha}}(t)$ introduced in eq.(4.5.1) by $\widetilde{R}_{\alpha,\sigma}(t)$.

Notice that for each $\alpha \in \mathbb{N}^I$ and any character $\gamma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$, there is an equality

$$\langle r_\alpha \otimes \rho_\gamma, 1\rangle = \frac{1}{|\operatorname{GL}_\alpha(\mathbb{F}_q)|} \sum_{g \in \operatorname{GL}_\alpha(\mathbb{F}_q)} r_\alpha(g)\rho_\gamma(g) = \frac{1}{|\operatorname{GL}_\alpha(\mathbb{F}_q)|} \sum_{\omega \in \mathbb{T}_\alpha} \sum_{g \sim \omega} r_\alpha(g)\rho_\gamma(g) = \tag{5.7.3}$$

$$\sum_{\omega \in \mathbb{T}_\alpha} \frac{R_\omega(q)}{|\operatorname{GL}_\alpha(\mathbb{F}_q)|} \left(\sum_{g \sim \omega} \rho_\gamma(g) \right) = \sum_{\omega \in \mathbb{T}_\alpha} \frac{R_\omega(q)}{Z_\omega(q)} \sum_{\substack{O \in \operatorname{Cl}(\operatorname{GL}_\alpha(\mathbb{F}_q)) \\ O \sim \omega}} \rho_\gamma(O). \tag{5.7.4}$$

Using the parametrization of the set $\{O \sim \omega\}$ introduced in Remark 4.3.17, we deduce that the RHS of eq.(5.7.4) is equal to

$$\sum_{\omega \in \mathbb{T}_\alpha} \frac{R_\omega(q)}{Z_\omega(q) w(\omega)} \left(\sum_{s \in (S_\omega^{\mathrm{reg}})^F} \rho_\gamma(s) \right) = \sum_{\omega \in \mathbb{T}_\alpha} \frac{\widetilde{R}_\omega(q)}{w(\omega)} \left(\sum_{\substack{S' \leq S_\omega \\ \text{of level } \gamma}} P_{[S']}(q) \mu(S', S_\omega) \right) = \widetilde{R}_{\alpha,\gamma}(q) \tag{5.7.5}$$

where the equality at the middle is a consequence of eq.(5.7.2).

From Theorem 4.5.2 we thus get the following result:

Theorem 5.7.5. *For any Log compatible family $\{r_\alpha \in \mathcal{C}(\operatorname{GL}_\alpha(\mathbb{F}_q))\}_{\alpha \in \mathbb{N}^I}$, for any $\alpha \in \mathbb{N}^I$ and any $\gamma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$, there is an equality:*

$$\langle r_\alpha \otimes \rho_\gamma, 1\rangle = \operatorname{Coeff}_\alpha \left(\operatorname{Exp} \left(\sum_{\beta \in \mathcal{H}_\gamma} \tilde{R}_{\beta,gen}(q) y^\beta \right) \right) \tag{5.7.6}$$

Remark 5.7.6. The definition of a Log compatible family of class functions had already been introduced by Letellier in [62, Paragraph 2.1.2].
The author [62, Theorem 2.2] thereby showed the case where $\gamma_i = 1$ for each $i \in I$ of Theorem 5.7.5. However, his method is different from ours and does not seem to extend to the case of a general $\gamma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$.

Remark 5.7.7. For $\sigma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$ and $\beta \in \mathbb{N}^I$, we say that σ is generic with respect to β if $\mathcal{H}_{\sigma,\beta} = \{\beta\}$. If q is big enough, for any β there exists a character σ generic with respect to β.

Fix now $\alpha \in \mathbb{N}^I$. Assume that q is sufficiently big and for any $0 < \beta \leq \alpha$, choose $\gamma_{\beta,gen} \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$ generic with respect to β. The argument preceding Theorem 5.7.5, shows that for any $0 < \beta \leq \alpha$, there is an equality

$$\langle r_\beta \otimes \rho_{\gamma_{\beta,gen}}, 1\rangle = \widetilde{R}_{\beta,gen}(q). \tag{5.7.7}$$

Notice that the multiplicity $\langle r_\beta \otimes \rho_{\gamma_{\beta,gen}}, 1\rangle$ is therefore given by a polynomial in q which does

not depend on $\gamma_{\beta,gen}$ but only on the dimension vector β. Eq.(5.7.7) had already been proved in [62, Theorem 2.2].
Eq.(5.7.6) and eq.(5.7.7) give therefore a way to express the multiplicity $\langle r_\alpha \otimes \rho_\gamma, 1\rangle$ for any $\gamma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$ in terms of the analogous multiplicities for generic parameters.

5.8 Dual Log compatibility for families of class functions

5.8.1 Multiplicative parameters

Given an element $\eta = (\eta_i)_{i\in I} \in (\mathbb{F}_q^*)^I$ and $\delta \in \mathbb{N}^I$, we define

$$\eta^\delta = \prod_{i\in I} \eta_i^{\delta_i}.$$

We denote by $\mathcal{H}_\eta$ the subset of $\mathbb{N}^I$ defined as

$$\mathcal{H}_\eta := \{\delta \in \mathbb{N}^I \mid \eta^\delta = 1\}$$

and, for any $\alpha \in \mathbb{N}^I$, by $\mathcal{H}_{\eta,\alpha}$ the intersection $\mathcal{H}_{\eta,\alpha} := \mathcal{H}_\eta \cap \mathbb{N}^I_{\leq\alpha}$.

Definition 5.8.1. For an admissible torus $S \in \mathcal{Z}_\alpha$, we say that S is of level η if it is of level $\mathcal{H}_{\eta,\alpha}$.

Assume now to have fixed, for each $S \in \mathcal{Z}_\alpha$, an F-stable maximal torus $\mathrm{GL}_\alpha \supseteq T_S \supseteq S$ in such a way that if $S \leq S'$ then $T_S = T_{S'}$. Define then the functions $g_\eta, f_\eta : \mathcal{Z}_\alpha \to \mathbb{C}$ as

$$g_\eta(S) := \sum_{\substack{\theta:T_S^F \to \mathbb{C}^* \\ \Gamma_\theta = \Gamma_S}} \theta(\eta)$$

and

$$f_\eta(S) := \sum_{\substack{\theta:T_S^F \to \mathbb{C}^* \\ \Gamma_\theta \leq \Gamma_S}} \theta(\eta).$$

By Identity (4.3.6), we have

$$g_\eta(S) = \sum_{S'\leq S} \mu(S', S) f_\eta(S') \tag{5.8.1}$$

Notice that by the bijection of eq.(5.4.2), for each $S \in \mathcal{Z}_\alpha$, we have

$$f_\eta(S) = \sum_{\theta:L_S^F \to \mathbb{C}^*} \theta(\eta).$$

Fix now S with type $[S] = (d_1, \beta_1) \dots (d_r, \beta_r)$, with $\beta_1, \dots, \beta_r \in \mathbb{N}^I$. Notice that there exists $h \in \mathrm{GL}_\alpha(\mathbb{F}_q)$ such that

$$hSh^{-1} = \prod_{j=1}^r (Z_{\beta_j})_{d_j}$$

and so

$$hL_Sh^{-1} = \prod_{j=1}^{r} (\mathrm{GL}_{n_{\beta_j}})_{d_j}.$$

In particular, a character $\theta : L_S^F \to \mathbb{C}^*$, via the conjugation above, corresponds to an element $(\theta_1, \dots, \theta_r) \in \mathrm{Hom}(\mathbb{F}^*_{q^{d_1}}, \mathbb{C}^*) \times \cdots \times \mathrm{Hom}(\mathbb{F}^*_{q^{d_r}}, \mathbb{C}^*)$ such that

$$\theta(M_1, \dots, M_r) = \prod_{j=1}^{r} \theta_j(\det(M_j))$$

with $M_j \in \mathrm{GL}_{n_j}(\mathbb{F}_{q^{d_j}})$.
As the element $\eta \in \mathrm{GL}_\alpha(\mathbb{F}_q)$ is central, we have the following equality

$$\theta(\eta) = \prod_{j=1}^{r} \theta_j(\eta^{\beta_j}). \tag{5.8.2}$$

In particular, eq.(5.8.2) implies that $f_\eta(S) \neq 0$ if and only if $\eta^{\beta_j} = 1$ for each $j = 1, \dots, r$, i.e if and only if S is of level η. We deduce the following Proposition.

Proposition 5.8.2. *For each $S \in \mathcal{Z}_\alpha$, there is an equality*

$$g_\eta(S) = \sum_{\substack{S' \leq S \\ \text{of level } \eta}} |\,\mathrm{Hom}(L_{S'}^F, \mathbb{C}^*)| \mu(S', S) = \sum_{\substack{S' \leq S \\ \text{of level } \eta}} P_{[S']}(q) \mu(S', S). \tag{5.8.3}$$

5.8.2 Convolution for dual Log compatible families

Assume to have been given a family $\{c_\alpha\}_{\alpha \in \mathbb{N}^I}$ with $c_\alpha \in \mathcal{C}(\mathrm{GL}_\alpha(\mathbb{F}_q))$.

Definition 5.8.3. The family $\{c_\alpha\}_{\alpha \in \mathbb{N}^I}$ is said to be *dual Log compatible* if, for any $\chi \in \mathrm{GL}_\alpha(\mathbb{F}_q)$, the product $\langle c_\alpha, \chi \rangle$ depends only on the type of χ and the value of $\langle c_\alpha, \chi \rangle$ for $\chi \sim \omega$ is of the form $C_\omega(q)$ where $\{C_\omega(t) \in \mathbb{Q}(t)\}_{\omega \in \mathbb{T}}$ is a family of rational functions such that for any $d_1, \dots, d_r \in \mathbb{N}$ and $\omega_1 \in \mathbb{T}_{\beta_1}, \dots, \omega_r \in \mathbb{T}_{\beta_r}$ such that $\psi_{d_1}(\omega_1) * \cdots * \psi_{d_r}(\omega_r) = \omega$, we have:

$$C_{\omega_1}(t^{d_1}) \cdots C_{\omega_r}(t^{d_r}) \prod_{j=1}^{r} H^\vee_{\omega_j}(t^{d_j}) = C_\omega(t) H^\vee_\omega(t). \tag{5.8.4}$$

i.e the family $\{C_\omega(t) H^\vee_\omega(t)\}_{\omega \in \mathbb{T}}$ is Log compatible.

For each $\omega \in \mathbb{T}$, denote by $\widetilde{C}_\omega(t) := C_\omega(t) H^\vee_\omega(t)$. The family $\{\widetilde{C}_\omega\}_{\omega \in \mathbb{T}}$ is therefore Log compatible. For each $\alpha \in \mathbb{N}^I$ and for each $\eta \in (\mathbb{F}_q^*)^I$, we will denote the polynomial $\widetilde{C}_{\alpha, \mathcal{H}_{\eta,\alpha}}(t)$ introduced in eq.(4.5.1) by $\widetilde{C}_{\alpha,\eta}(t)$.

For each $\alpha \in \mathbb{N}^I$ and any parameter $\eta \in (\mathbb{F}_q^*)^I$, by eq.(3.1.4), we deduce the following chain of equalities:

$$\frac{c_\alpha(\eta)}{|\operatorname{GL}_\alpha(\mathbb{F}_q)|} = \sum_{\chi\in\operatorname{GL}_\alpha(\mathbb{F}_q)^\vee} \langle c_\alpha,\chi\rangle \frac{\chi(\eta)}{\chi(1)}\frac{\chi(1)}{|\operatorname{GL}_\alpha(\mathbb{F}_q)|} = \tag{5.8.5}$$

$$= \sum_{\omega\in\mathbb{T}_\alpha}\sum_{\substack{\chi\in\operatorname{GL}_\alpha(\mathbb{F}_q)^\vee\\ \chi\sim\omega}} \langle c_\alpha,\chi\rangle\frac{\chi(\eta)}{\chi(1)}H^\vee_\omega(q) = \sum_{\omega\in\mathbb{T}_\alpha}\widetilde{C}_\omega(q)\left(\sum_{\substack{\chi\in\operatorname{GL}_\alpha(\mathbb{F}_q)^\vee\\ \chi\sim\omega}}\frac{\chi(\eta)}{\chi(1)}\right). \tag{5.8.6}$$

By Remark 5.2.7, the RHS of eq.(5.8.6) is equal to

$$\sum_{\omega\in\mathbb{T}_\alpha}\frac{\widetilde{C}_\omega(q)}{w(\omega)}\left(\sum_{\substack{\theta:\widetilde{L_{S_\omega}}\to\mathbb{C}^*\\ \Gamma_\theta=\Gamma_{S_\omega}}}\theta(\eta)\right) = \sum_{\omega\in\mathbb{T}_\alpha}\frac{\widetilde{C}_\omega(q)}{w(\omega)}\left(\sum_{\substack{S'\leq S_\omega\\ \text{of level }\eta}}P_{[S']}(q)\mu(S',S_\omega)\right) = \widetilde{C}_{\alpha,\eta}(q) \tag{5.8.7}$$

where the equality at the middle is a consequence of eq.(5.8.3). We deduce therefore the following Theorem:

Theorem 5.8.4. *For any dual Log compatible family $\{c_\alpha\}_{\alpha\in\mathbb{N}^I}$ and any $\eta\in(\mathbb{F}_q^*)^I$, there is an equality*

$$\frac{c_\alpha(\eta)}{|\operatorname{GL}_\alpha(\mathbb{F}_q)|} = \operatorname{Coeff}_\alpha\left(\operatorname{Exp}\left(\sum_{\beta\in\mathcal{H}_\eta}\widetilde{C}_{\beta,gen}(q)y^\beta\right)\right) \tag{5.8.8}$$

Remark 5.8.5. For $\beta\in\mathbb{N}^I$ and $\eta\in(\mathbb{F}_q^*)^I$, say that η is generic with respect to β if $\mathcal{H}_{\eta,\beta}=\{\beta\}$. For any β, if q is sufficienty big, there exists an element of $(\mathbb{F}_q^*)^I$ generic with respect to it.

Fix now $\alpha\in\mathbb{N}^I$. Assume that q is sufficiently big and for any $0<\beta\leq\alpha$, choose $\eta_{\beta,gen}\in(\mathbb{F}_q^*)^I$ generic with respect to β. The reasoning used to prove Theorem 5.8.4, shows that for any $0<\beta\leq\alpha$, there is an equality

$$\frac{c_\beta(\eta_{\beta,gen})}{|\operatorname{GL}_\beta(\mathbb{F}_q)|} = \widetilde{C}_{\beta,gen}(q). \tag{5.8.9}$$

Notice therefore that the quantity $\dfrac{c_\beta(\eta_{\beta,gen})}{|\operatorname{GL}_\beta(\mathbb{F}_q)|}$ is given by a rational function in q which does not depend on $\eta_{\beta,gen}$ but only on the dimension vector β.
Eq.(5.8.8) and eq.(5.8.9) give therefore a way to express the quantity $\dfrac{c_\alpha(\eta)}{|\operatorname{GL}_\alpha(\mathbb{F}_q)|}$ for any central element $\eta\in(\mathbb{F}_q^*)^I$ in terms of the analogous values for generic parameters.

Remark 5.8.6. While the notion of a Log compatible family of class functions already appeared in Letellier's article [62], the definiton of a dual Log compatible family seems to not have been given before in the literature.

The latter notion is going to be the key technical point to show our Theorem 9.3.2 about E-series of character stacks for Riemann surfaces.

Example 5.8.7. Let $I = \{\cdot\}$ and, for any $n \in \mathbb{N}$, let $f_n : \mathrm{GL}_n(\mathbb{F}_q) \to \mathbb{C}$ be the class function

$$f_n(h) := \#\{(x, y) \in \mathrm{GL}_n(\mathbb{F}_q) \times \mathrm{GL}_n(\mathbb{F}_q) \mid [x, y] = h\}$$

for $h \in \mathrm{GL}_n(\mathbb{F}_q)$. For any $\chi \in \mathrm{GL}_n(\mathbb{F}_q)^\vee$ of type ω, it holds $\langle f_n, \chi\rangle = \frac{1}{H^\vee_\omega(q)}$. More generally for any finite group G and any irreducible character $\chi \in G^\vee$ it holds:

$$\sum_{(a,b)\in G^2} \chi([a, b]) = \frac{|G|}{\chi(1)}. \tag{5.8.10}$$

This equality is obtained by applying Schur's lemma in a classical way as explained in [44, Paragraph 2.3]. Notice that, from the identity $\langle f_n, \chi\rangle = \frac{1}{H^\vee_\omega(q)}$, we immediately deduce that the family $\{f_n\}_{n\in\mathbb{N}}$ is dual Log compatible.

The notion of dual Log compatibility is well behaved with respect to convolution, as explained by the following Lemma.

Lemma 5.8.8. *Let $\{f_\alpha\}_{\alpha\in\mathbb{N}^I}, \{f'_\alpha\}_{\alpha\in\mathbb{N}^I}$ be two families of dual Log compatible functions. The family $\{k_\alpha\}_{\alpha\in\mathbb{N}^I}$, defined as $k_\alpha := \frac{f_\alpha * f'_\alpha}{q^{\sum_{i\in I}\alpha_i^2}}$ is dual Log compatible.*

Proof. Let $F_{\omega,\alpha}(t), F'_{\omega,\alpha}(t)$ be the polynomials such that $\langle f_\alpha, \chi\rangle = F_{\omega,\alpha}(q)$ and $\langle f'_\alpha, \chi\rangle = F'_{\omega,\alpha}(q)$ for every $\chi \in \mathrm{GL}_\alpha(\mathbb{F}_q)^\vee$ of multitype $\omega \in \mathbb{T}_\alpha$. By eq.(3.1.3), we see that

$$\langle k_\alpha, \chi\rangle = \frac{F_{\omega,\alpha}(q)F'_{\omega,\alpha}(q)}{H^\vee_\omega(q)q^{\sum_{i\in I}\alpha_i^2}}.$$

Fix $d_1, \dots, d_r \in \mathbb{N}$ and $\omega_1 \in \mathbb{T}_{\beta_1}, \dots, \omega_r \in \mathbb{T}_{\beta_r}$ such that $\psi_{d_1}(\omega_1) * \cdots * \psi_{d_r}(\omega_r) = \omega$. To check eq.(5.8.4) for the functions k_α, we need to verify that

$$\prod_{j=1}^r \frac{F_{\omega_j,\beta_j}(q^{d_j})F'_{\omega_j,\beta_j}(q^{d_j})}{H^\vee_{\omega_j}(q^{d_j})q^{d_j\sum_{i\in I}(\beta_j)_i^2}} \prod_{j=1}^r H^\vee_{\omega_j}(q^{d_j}) = \frac{F_{\omega,\alpha}(q)F'_{\omega,\alpha}(q)}{H^\vee_\omega(q)q^{\sum_{i\in I}\alpha_i^2}} H^\vee_\omega(q). \tag{5.8.11}$$

By dual Log compatibiliy for f_α, f'_α, this is equivalent to verify the equality

$$\left(\frac{\prod_{j=1}^r H^\vee_{\omega_j}(t^{d_j})}{H^\vee_\omega(t)}\right)^2 = \frac{t^{\sum_{i\in I}\alpha_i^2}}{\prod_{j=1}^r t^{\sum_{j=1}^r d_j\sum_{i\in I}(\beta_j)_i^2}} \tag{5.8.12}$$

which is a direct consequence of the Identity (5.6.1).

□

Remark 5.8.9. From Lemma 5.8.8 above and Example 5.8.7, we deduce that for any $g \geq 1$, the family of class function $\{f_n^g : \mathrm{GL}_n(\mathbb{F}_q) \to \mathbb{C}\}_{n\in\mathbb{N}}$, where

$$f_n^g(h) = \frac{\#\{(x_1, y_1, \ldots, x_g, y_g) \mid \prod_{i=1}^g [x_i, y_i] = h\}}{q^{(n^2(g-1))}}$$

is dual Log compatible.

In [44, Section 2.3], it is shown that $\langle f_n^g, \chi\rangle = \left(\frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{\chi(1)}\right)^{2g-1}$, from which it is possible to check Dual log compatibility directly from eq.(5.8.4).

6 Quiver representations and a generalization of Kac polynomials

In this chapter, we study representations of a finite quiver $Q = (I, \Omega)$.

In section §6.1, we recall the definition of representations of Q over a field K and we review some properties of the representation theory of a quiver $Q = (I, \Omega)$.

For $K = \mathbb{F}_q$, we recall the results of Kac [52] and Hua [50], regarding the counting of the isomorphism classes of $\mathbb{F}_q$-representations.

More precisely, in [52], the author showed that, for any $\beta \in \mathbb{N}^I$, there exists a polynomial $a_{Q,\beta}(t) \in \mathbb{Z}[t]$, called *Kac polynomial*, such that $a_{Q,\beta}(q)$ counts the number of isomorphism classes of absolutely indecomposable representations of Q over $\mathbb{F}_q$ of dimension β.

These polynomials have a geometric interpretation in terms of the cohomology of certain quiver varieties, see for example [47].

In [50], the author uses Kac polynomials to give formulas for the number of isomorphism classes of representations of Q over $\mathbb{F}_q$ of fixed dimension $\alpha \in \mathbb{N}^I$.

In section §6.2, we introduce the notion of levels for representations of a quiver Γ. For $V \subseteq \mathbb{N}^I$, a representation M over K is called *of level at most* V if the dimension vector of the indecomposable components of $M \otimes \overline{K}$ belong to V.

We show that, for any V, α, there exists a polynomial $M_{Q,\alpha,V}(t) \in \mathbb{N}[t]$ such that $M_{Q,\alpha,V}(q)$ counts the number of isomorphism classes of representations of level at most V of dimension α over $\mathbb{F}_q$. Notice that for $V = \{\alpha\}$, we have an equality $M_{Q,\alpha,\{\alpha\}}(t) = a_{Q,\alpha}(t)$.

In general, in Lemma 6.2.2 we show a formula expressing $M_{Q,\alpha,V}(t)$ in terms of the Kac polynomials $a_{Q,\beta}(t)$ for $\beta \in V$. The polynomials $M_{Q,\alpha,V}(t)$ will be used in Chapter §8 to compute the multiplicities for k-tuples of Harisha-Chandra characters of $\mathrm{GL}_n(\mathbb{F}_q)$.

In §6.2.2, we show how to express the polynomials $M_{Q,\alpha,V}(t)$ in terms of certain representations of the finite group $\mathrm{GL}_\alpha(\mathbb{F}_q)$, generalizing the results of Letellier [62] about Kac polynomials and DT invariants.

6.1 Quiver representations

A quiver Q is an oriented graph $Q = (I, \Omega)$, where I is its set of vertices and Ω is its set of arrows. We will always assume that I, Ω are finite sets. For an arrow $a : i \to j$ in Ω we denote by $i = t(a)$ its *tail* and by $j = h(a)$ its *head*.

Fix a field K. A representation M of Q over K is given by a (finite dimensional) K-vector space V_i for each vertex $i \in I$ and by linear maps $M_a : M_{t(a)} \to M_{h(a)}$ for each $a \in \Omega$.

Given two representations M, M' of Q, a morphism $f : M \to M'$ is given by linear maps $\{f_i : M_i \to M'_i\}_{i \in I}$ such that, for all $a \in \Omega$, we have:

$$f_{h(a)} M_a = M'_a f_{t(a)}.$$

The category of representations of Q over K is denoted by $\mathrm{Rep}_K(Q)$. For a representation

M, the *dimension vector* $\dim M \in \mathbb{N}^I$ is defined as

$$\dim M \coloneqq (\dim M_i)_{i \in I}.$$

It is an isomorphism invariant of the category $\operatorname{Rep}_k(Q)$.

For a representation M of dimension α, up to fixing a basis of the vector spaces M_i for each $i \in I$, we can assume that $M_i = K^{\alpha_i}$. For $a \in \Omega$, the linear map $M_a : K^{t(a)} \to K^{h(a)}$ can be therefore identified with a matrix in $\operatorname{Mat}(\alpha_{h(a)}, \alpha_{t(a)}, K)$.
Consider then the affine space

$$R(Q,\alpha) \coloneqq \bigoplus_{a \in \Omega} \operatorname{Mat}(\alpha_{h(a)}, \alpha_{t(a)}, K).$$

We can endow $R(Q,\alpha)$ with the action of the group $\mathrm{GL}_\alpha = \prod_{i \in I} \mathrm{GL}_{\alpha_i}$ defined by

$$g \cdot (M_a)_{a \in \Omega} = (g_{h(a)} M_a g_{t(a)}^{-1})_{a \in \Omega}.$$

The orbits of this action are exactly the isomorphism classes of representations of Q of $\dim = \alpha$.

Denote by $(-,-) : \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$ the Euler form of Q, defined by

$$(\alpha,\beta) = \sum_{a \in \Omega} \alpha_{t(a)} \beta_{h(a)} - \sum_{i \in I} \alpha_i \beta_i.$$

We briefly recall also the definition of the *moment map* of Q. Denote by $\overline{Q}$ the double quiver $\overline{Q} = (I, \overline{\Omega})$ with the same vertices of Q and as set of arrows $\overline{\Omega} = \{a, a^* \mid a \in \Omega\}$ where $a^* : j \to i$ for $a : i \to j$. For $\alpha \in \mathbb{N}^I$, the moment map μ_α is the morphism

$$\mu_\alpha : R(\overline{Q},\alpha) \to \operatorname{End}_0(\alpha)$$

$$\overline{x} = (x_a, x_{a^*})_{a \in \Omega} \to \sum_{a \in \Omega} [x_a, x_{a^*}]$$

where

$$\operatorname{End}_0(\alpha) = \{(M_i) \in \operatorname{End}(\alpha) \mid \sum_{i \in I} \operatorname{tr}(M_i) = 0\}.$$

Given $\lambda \in K^I$ such that $\lambda \cdot \alpha = 0$, the element $(\lambda_i I_{\alpha_i})_{i \in I}$ (which we still denote by λ) is a central element of $\operatorname{End}_0(\alpha)$ and the fiber $\mu_\alpha^{-1}(\lambda)$ is GL_α invariant. We denote by $\mathcal{Q}_{\lambda,\alpha}$ the quiver variety associated to λ, which is defined as the GIT quotient

$$\mathcal{Q}_{\lambda,\alpha} \coloneqq \mu_\alpha^{-1}(\lambda) /\!/ \mathrm{GL}_\alpha.$$

The quiver stack $\mathcal{M}_{\lambda,\alpha}$ is defined as the quotient stack

$$\mathcal{M}_{\lambda,\alpha} \coloneqq [\mu_\alpha^{-1}(\lambda) / \mathrm{GL}_\alpha].$$

6.1.1 Multiplicative quiver stacks

There is also a multiplicative version of the moment map μ_α, usually called multiplicative moment map.
More precisely, for $\alpha \in \mathbb{N}^I$, let $R(\overline{Q},\alpha)^\circ$ be the open subvariety of $R(\overline{Q},\alpha)$ such that $(1+x_bx_{b^*}),(1+x_{b^*}x_b)$ is invertible for every $b \in \Omega$.
Assume to have fixed an ordering $<$ on Ω. The multiplicative moment map Φ_α is the GL_α-equivariant morphism

$$\begin{aligned} \Phi_\alpha : R(\overline{Q},\alpha)^\circ &\to \mathrm{GL}_\alpha \\ (x_a,x_{a^*}) &\to \prod_{a\in\Omega}(1+x_ax_{a^*})(1+x_{a^*}x_a)^{-1} \end{aligned} \tag{6.1.1}$$

where we are taking the ordered product with respect to $<$.
For $\sigma \in (K^*)^I$, the element $(\sigma_i I_{\alpha_i})_{i\in I} \in \mathrm{GL}_\alpha$, which we still denote by σ, is central and the fiber $\Phi_\alpha^{-1}(\sigma)$ is GL_α-invariant.
We define the multiplicative quiver stack $\mathcal{M}_{\sigma,\alpha}$ of parameters σ,α as the quotient stack

$$\mathcal{M}_{\sigma,\alpha} \coloneqq [\Phi_\alpha^{-1}(\sigma)/\operatorname{GL}_\alpha].$$

Remark 6.1.1. While the morphism Φ_α depends on the ordering $<$, the isomorphism class of the multiplicative quiver stack $\mathcal{M}_{\sigma,\alpha}$ does not depend on it, see for example [19, Proposition 1.4].

6.1.2 Krull-Schmidt decomposition and endomorphism rings

Recall that the category $\mathrm{Rep}_K(Q)$ is abelian and Krull-Schimdt, i.e we have the following Theorem, see for example [56, Theorem 1.11].

Theorem 6.1.2. *Each object $M \in \mathrm{Rep}_K(Q)$ admits a decomposition into a direct sum of indecomposable ones*

$$M = \bigoplus_{j\in J} M_j^{n_j}$$

and such a decomposition is unique up to permuting the factors.

Recall that a representation M is indecomposable if and only if $\mathrm{End}(M)$ is a local algebra. Its maximal ideal is denoted by $\mathrm{Rad}(M) \subseteq \mathrm{End}(M)$. It is the set of nilpotent endomorphisms of M. The quotient $\mathrm{End}(M)/\operatorname{Rad}(M)$ is thus a division algebra, which is usually denoted by $\mathrm{top}(M)$.
More generally, given two representations M, N it is possible to define a subset $\mathrm{Rad}(M,N)$ of $\mathrm{Hom}(M,N)$ as

$$\mathrm{Rad}(M,N) = \{g \in \mathrm{Hom}(M,N) \mid 1+gf \in \mathrm{Aut}(N)\ ,\forall f \in \mathrm{Hom}(N,M)\}.$$

If $M = N$ and M is indecomposable, $\mathrm{Rad}(M) = \mathrm{Rad}(M, M)$. In general, $\mathrm{Rad}(N, N)$ is still an ideal of $\mathrm{End}(N)$ and if M, N are non-isomorphic indecomposable representations, $\mathrm{Rad}(M, N) = \mathrm{Hom}(M, N)$. The radical is additive i.e

$$\mathrm{Rad}(M \oplus M', N) = \mathrm{Rad}(M, N) \oplus \mathrm{Rad}(M', N).$$

We have the following proposition (see [36, Section 3.2]):

Proposition 6.1.3. *Given a representation X of Q and an endomorphism $\varphi \in \mathrm{End}(X)$, φ is invertible if and only if its class $\overline{\varphi}$ in $\mathrm{End}(X)/\mathrm{Rad}(X)$ is invertible.*

Remark 6.1.4. Fix a representation X and its Krull-Schmidt decomposition $X = \bigoplus_{j \in J} X_j^{r_j}$. For an endomorphism $\varphi \in \mathrm{End}(X)$, we denote by φ_j the associated element in $\mathrm{End}(X_j^{r_j})$ and by $\overline{\varphi}_j$ the associated element in

$$\mathrm{End}(X_j^{r_j})/\mathrm{Rad}(X_j^{r_j}) \cong \mathrm{Mat}(r_j, \mathrm{top}(X_j)).$$

As $\mathrm{Rad}(X_i, X_j) = \mathrm{Hom}(X_i, X_j)$ for every $i \neq j$, the following isomorphism of K-algebras holds:

$$\mathrm{End}(X)/\mathrm{Rad}(X) \cong \bigoplus_{j \in J} \mathrm{Mat}(r_j, \mathrm{top}(X_j)). \tag{6.1.2}$$

$$\varphi \longrightarrow (\overline{\varphi}_j)_{j \in J} \tag{6.1.3}$$

Proposition 6.1.3 can therefore be rephrased as: φ is an isomorphism if and only if $\overline{\varphi_j}$ is invertible for each $j \in J$.

6.1.3 Indecomposable over finite fields and Kac polynomials

In this paragraph, unless explicitly specified, we assume $K = \mathbb{F}_q$. Wedderburn's theorem implies that every finite dimensional division algebra over $\mathbb{F}_q$ is a finite field. For an indecomposable representation $M \in \mathrm{Rep}_{\mathbb{F}_q}(Q)$, we have therefore $\mathrm{top}(M) = \mathbb{F}_{q^d}$ for some $d \geq 1$.

Fix a representation X with Krull-Schimdt's decomposition

$$X = \bigoplus_{j \in J} X_j^{r_j}$$

and integers d_j such that $\mathrm{top}(X_j) = \mathbb{F}_{q^{d_j}}$ for each $j \in J$. From Remark 6.1.4, there is a morphism of finite groups

$$p_X : \mathrm{Aut}(X) \to \prod_{j \in J} \mathrm{GL}_{r_j}(\mathbb{F}_{q^{d_j}}) \tag{6.1.4}$$

$$\varphi \longrightarrow (\overline{\varphi})_{j \in J}. \tag{6.1.5}$$

and its kernel $\mathrm{Ker}(p_X)$ is the subset $U_X := \{1 + f \mid f \in \mathrm{Rad}(X)\} \subseteq \mathrm{Aut}(X)$. In particular, all the elements inside $\mathrm{Ker}(p_X)$ are unipotent.

The integers d_j admit the following description in terms of *absolutely indecomposable* representations (see Definition 6.1.5 below).

Definition 6.1.5. A representation V over a field K is absolutely indecomposable if $V \otimes_K \overline{K}$ is indecomposable.

To relate absolutely indecomposable and indecomposable representations over $\mathbb{F}_q$, we introduce the action of the Frobenius.
Denote by $(\overline{\mathbb{F}}_q)_{\mathrm{Fr}^i}$ the set $\overline{\mathbb{F}}_q$ with the structrure of $\overline{\mathbb{F}}_q$ vector space given by

$$\lambda \cdot v = F^i(\lambda)v.$$

Consider an $\overline{\mathbb{F}}_q$-representation N. We define $\mathrm{Fr}^i(N)$ to be the representation $N \otimes_{\overline{\mathbb{F}}_q} (\overline{\mathbb{F}}_q)_{\mathrm{Fr}^i}$. Over the parameter space $R(Q,\alpha)(\overline{\mathbb{F}}_q)$, the Frobenius action corresponds to the usual (geometric) Frobenius, i.e for $x \in R(Q,\alpha)(\overline{\mathbb{F}}_q)$ the representation $\mathrm{Fr}^i(x)$ is given by the element $F^i(x) \in R(Q,\alpha)(\overline{\mathbb{F}}_q)$, where $F : R(Q,\alpha) \to R(Q,\alpha)$ is the canonical Frobenius.

Consider a representation M of Q over $\mathbb{F}_q$ and $d \in \mathbb{N}_{>0}$ such that $\mathrm{Fr}^d(M) \cong M$. Since the stabilizers of the action of GL_α on $R(Q,\alpha)$ are connected, by [28, Proposition 4.2.14] there exists an $\mathbb{F}_{q^d}$-representation M_0 of Q such that

$$M_0 \otimes_{\mathbb{F}_{q^d}} \overline{\mathbb{F}_q} \cong M.$$

In this case, we say thus that M is *defined over* $\mathbb{F}_{q^d}$. Define $\mathrm{size}(M)$ to be the smallest d such that $\mathrm{Fr}^d(M) \cong M$, i.e such that M is defined over $\mathbb{F}_{q^d}$. Notice that, for $M \in R(Q,\alpha)(\overline{\mathbb{F}}_q)$, the size $\mathrm{size}(M)$ is the cardinality of the orbit of M for the action of the Frobenius on $R(Q,\alpha)(\overline{\mathbb{F}}_q)$. Notice moreover that $\mathrm{size}(M)$ is an isomorphism invariant and we can therefore talk about the size of an isomorphism class.
By what we just said, we deduce that, for any $\beta \in \mathbb{N}^I$ and $d > 0$, we have the following equality.

$$a_{Q,\beta}(q^d) = \#\{ \text{ Isomorphism classes of indecomposable representations } M \text{ over } \overline{\mathbb{F}_q} \mid \tag{6.1.6}$$

$$\dim(M) = \beta \text{ and } \mathrm{Fr}^d(M) \cong M\}\} = \tag{6.1.7}$$

$$= \bigsqcup_{r \mid d} \#\{ \text{ Isomorphism classes of indecomposable representations } M \text{ over } \overline{\mathbb{F}_q} \mid \tag{6.1.8}$$

$$\dim(M) = \beta \text{ and } \mathrm{size}(M) = r\} \tag{6.1.9}$$

Indecomposable representations are described in terms of absolutely indecomposable ones by the following Lemma, see [52].

Lemma 6.1.6. *For every indecomposable representation W of Q of dimension α over $\overline{\mathbb{F}_q}$ such that* $\mathrm{size}(W) = d$*, there exists an indecomposable representation M over $\mathbb{F}_q$ such that* $\mathrm{top}(M) = d$ *and*

$$M \otimes_{\mathbb{F}_q} \mathbb{F}_{q^d} \cong \bigoplus_{i=0}^{d-1} \mathrm{Fr}^i(W).$$

Conversely, for an indecomposable representation M over $\mathbb{F}_q$, such that $\mathrm{top}(M) = \mathbb{F}_{q^d}$, *there exists an absolutely indecomposable representation W of Q over $\mathbb{F}_{q^d}$ such that* $\dim(W) = \frac{\alpha}{d}$, $\mathrm{size}(W) = d$ *and we have :*

$$M \otimes_{\mathbb{F}_q} \mathbb{F}_{q^d} \cong \bigoplus_{i=0}^{d-1} \mathrm{Fr}^i(W). \tag{6.1.10}$$

Remark 6.1.7. Notice that Lemma 6.1.6 implies in particular that, given an indecomposable representation M of Q of dimension α over $\mathbb{F}_q$ with $\mathrm{top}(M) = \mathbb{F}_{q^d}$, we have $d|\overline{\alpha} := \gcd(\alpha_i)_{i\in I}$.

Moreover, absolutely indecomposables of Q over finite fields are described by the following result, see [52, Theorem A].

Theorem 6.1.8. *(i) There exists a polynomial with integer coefficients $a_{Q,\alpha}(t) \in \mathbb{Z}[t]$ such that, for each q, $a_{Q,\alpha}(q)$ is equal to the number of isomorphism classes of absolutely indecomposable representations of Q over $\mathbb{F}_q$ of dimension vector α.*

(ii) The Kac polynomial $a_{Q,\alpha}(t) \neq 0$ if and only if $\alpha \in \Phi(Q)^+$. Moreover, $a_{Q,\alpha}(t) \equiv 1$ if and only if α is a real root (see [52, Section (a)] for a definition). Otherwise $a_{Q,\alpha}(t)$ is monic of degree $2 - \alpha^t C \alpha$

Here $\Phi^+(Q) \subseteq \mathbb{N}^I$ is the root system of Q, introduced in [52, Section (a)] and $C = (C_{i,j})_{i,j\in I}$ is the Cartan matrix of the quiver given by

$$C_{ij} = \begin{cases} 2 - 2(\text{the number of edges joining } i \text{ to itself}) & \text{if } i = j \\ -(\text{the number of edges joining } i \text{ to } j) & \text{otherwise.} \end{cases}$$

Thanks to the isomorphism (6.1.10), in [52, Section 1.14] it is shown the following Proposition.

Proposition 6.1.9. *There exists a polynomial $I_{Q,\alpha,d}(t)$ such that, for any q, $I_{Q,\alpha,d}(q)$ is equal to the number of isomorphism classes of indecomposable representations M such that* $\dim M = \alpha$ *and* $\mathrm{top}(M) = \mathbb{F}_{q^d}$ *and we have*

$$I_{Q,\alpha,d}(t) = \frac{1}{d} \sum_{r|d} \mu\left(\frac{d}{r}\right) a_{Q,\frac{\alpha}{d}}(t^r) \tag{6.1.11}$$

Proof. Let $X_d = \{r \in \mathbb{N} \mid r \text{ divides } d\}$. Notice that X_d is a poset with the ordering given by $r' \leq r$ if and only if $r'|r$. Consider the functions $f_1, f_2 : X_d \to \mathbb{N}$ defined as

$$f_1(r) = a_{Q,\frac{\alpha}{d}}(q^r)$$

$$f_2(r) = \#\{ \text{ Isomorphism classes of indecomposable representations } M \text{ over } \overline{\mathbb{F}_q} \mid$$

$$\dim(M) = \frac{\alpha}{d} \text{ and } \mathrm{size}(M) = r\}.$$

By eq.(6.1.8), we have that $f_1(r) = \sum_{r' \leq r} f_2(r)$ and, therefore, by Proposition 4.3.13, we have

$$f_2(d) = \sum_{r|d} \mu\left(\frac{d}{r}\right) f_1(r) = \sum_{r|d} \mu\left(\frac{d}{r}\right) a_{Q,\frac{\alpha}{d}}(q^r). \tag{6.1.12}$$

Notice that, by Lemma 6.1.6, we have that

$$df_2(d) = \#\{\text{Isomorphism classes of indecomposable representations of } Q \text{ over } \mathbb{F}_q \mid$$
$$\dim(M) = \alpha, \operatorname{top}(M) = \mathbb{F}_{q^d}\},$$

from which we deduce Proposition 6.1.9.

□

We finish the section by recalling the Kac conjecture which was proved by Hausel, Letellier, Rodriguez-Villegas in [47, Corollary 1.5]

Theorem 6.1.10. *For any $\alpha \in \mathbb{N}^I$, the Kac polynomial $a_{Q.\alpha}(t)$ has nonnegative integer coefficients.*

6.2 Quiver representations of level V

Let $Q = (I, \Omega)$ be a finite quiver and let V be a subset of $\mathbb{N}^I$. For $\alpha \in \mathbb{N}^I$ we denote by $\mathbb{N}^I_{\leq\alpha}$ the subset $\mathbb{N}^I_{\leq\alpha} \coloneqq \{0 \leq \beta \leq \alpha \mid \beta \in \mathbb{N}^I\}$ and similarly $V_{\leq\alpha} \coloneqq \{0 \leq \beta \leq \alpha \mid \beta \in V\}$.

Example 6.2.1. Given $\lambda \in \mathbb{C}^I$, we denote by V_λ the subset $V_\lambda \coloneqq \{\beta \in \mathbb{N}^I \mid \beta \cdot \lambda = 0\}$, where $\cdot$ is the canonical orthogonal product on $\mathbb{N}^I$. Notice that for $\lambda = 0$ we have $V_\lambda = \mathbb{N}^I$.

To a representation X of dimension α, we associate the following subset $\mathcal{H}_X \subseteq \mathbb{N}^I_{\leq\alpha}$. Given the decomposition of $X \otimes_K \overline{K}$ into indecomposable components

$$X \otimes_K \overline{K} = \bigoplus_{j \in J} Y_j^{r_j},$$

we define

$$\mathcal{H}_X \coloneqq \{\dim Y_j\}_{j \in J}. \tag{6.2.1}$$

For any $V \subseteq \mathbb{N}^I$, we give the following definition of the representations of the quiver Q of level V.

Definition 6.2.2. A representation X of dimension α is said to be of level V if we have $\mathcal{H}_X = V_{\leq\alpha}$. For $V = V_\lambda$ with $\lambda \in \mathbb{C}^I$, we say that X is of level λ.

Example 6.2.3. Let $V = \{\alpha\}$ for a vector $\alpha \in \mathbb{N}^I$. A representation $X \in R(Q, \alpha)$ is of level $\{\alpha\}$ if and only if $X \otimes_K \overline{K}$ is indecomposable, i.e if and only if X is absolutely indecomposable.

The notion of being of level V induces a stratification (indexed by the subsets of $\mathbb{N}^I_{\leq\alpha}$) on the representations of Q of dimension α.

Since on the set of subsets of $\mathbb{N}^I_{\leq\alpha}$ there is a natural order relation, induced by inclusion, we can also consider the filtration associated to such a stratification. In particular, we give the following definition of a representation of level at most V.

Definition 6.2.4. For a subset $V \subseteq \mathbb{N}^I$, a representation X is said to be of level at most V if it is of level V' for some $V' \subseteq V$. For $V = V_\lambda$ we say that a representation is of level at most λ.

Remark 6.2.5. Notice that a representation X of dimension α is of level at most V if and only if $\mathcal{H}_X \subseteq V_{\leq\alpha}$. In particular, for $V = \mathbb{N}^I$, any representation of Q is of level at most $\mathbb{N}^I$.

If $K = \mathbb{F}_q$ and $\lambda \in \mathbb{C}^I$, Definition 6.2.4 for representations of level at most λ is equivalent to the following one:

Definition 6.2.6. A representation X is of level at most λ if given its Krull-Schimdt decomposition $X = \bigoplus_{j\in J} X_j^{r_j}$ we have $\dim X_j \cdot \lambda = 0$ for each $j \in J$.

Proof. Given the Krull-Schidmt decomposition $X = \bigoplus_{j\in J} X_j^{r_j}$, from Lemma 6.1.6, for each $j \in J$ there is an isomorphism

$$X_j \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} = Y_j \oplus \mathrm{Fr}(Y_j) \oplus \cdots \oplus \mathrm{Fr}^{d_j-1}(y_j)$$

where $\mathrm{top}(X_j) = \mathbb{F}_{q^{d_j}}$. The decomposition in indecomposable factors of $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ is thus

$$X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} = \bigoplus_{j\in J} (Y_j \oplus \mathrm{Fr}(Y_j) \oplus \cdots \oplus \mathrm{Fr}^{d_j-1}(Y_j))$$

with $\dim Y_j = \frac{\dim x_j}{d_j}$. We have then $\dim Y_j \cdot \lambda = 0$ if and only if $\dim X_j \cdot \lambda = 0$, for each $j \in J$. □

For $\alpha \in \mathbb{N}^I$, the representations of Q of level at most V form a constructible subset of $R(Q, \alpha)$, as explained by the following proposition:

Proposition 6.2.7. *There exists a constructible subset $R(Q, \alpha, V) \subseteq R(Q, \alpha)$ such that ,for each extension $K \subseteq L$, the set of L-points $R(Q, \alpha, V)(L)$ is the subset of representations of level at most V of Q over L. If $V = V_\lambda$ for $\lambda \in \mathbb{C}^I$, we denote $R(Q, \alpha, V_\lambda)$ by $R(Q, \alpha, \lambda)$.*

Proof. In [52, Section 1,8], Kac showed that, for any $\alpha \in \mathbb{N}^I$, there exists a constructible subset $A(Q, \alpha) \subseteq R(Q, \alpha)$, such that, for any field extension $K \subseteq L$, the set of L-points $A(Q, \alpha)(L)$ is the subset of absolutely indecomposable representationsover L.
Let $\Psi_{V,\alpha}$ be the constructible subset defined by

$$\Psi_{V,\alpha} = \bigoplus_{\substack{\alpha_1,\ldots,\alpha_r \in V \\ \text{s.t } \alpha_1+\cdots+\alpha_r=\alpha}} A(Q, \alpha_1) \times \cdots \times A(Q, \alpha_r).$$

Consider the morphism

$$\Phi_{V,\alpha} : \mathrm{GL}_\alpha \times \Psi_{V,\alpha} \to R(Q,\alpha)$$

defined by

$$\Phi_{V,\alpha}(g, M_1, \dots, M_r) := g \cdot (M_1 \oplus \cdots \oplus M_r).$$

for $g \in \mathrm{GL}_\alpha$, $M_1 \in A(Q,\alpha_1), \dots, M_r \in A(Q,\alpha_r)$ and $\alpha_1 + \cdots + \alpha_r = \alpha$. We define $R(Q,\alpha,V)$ to be the image of $\Phi_{V,\alpha}$, which is a constructible subset by Chevalley's theorem.

□

Example 6.2.8. Consider the case where Q is the Jordan quiver (the quiver with one vertex and one arrow), i.e $|I| = |\Omega| = 1$.

For $n \in \mathbb{N}$ and a field K, the representation space $R(Q,n)$ is given by the $n \times n$ matrices $\mathrm{Mat}(n,K)$. The isomorphism classes of $R(Q,n)$ correspond to the conjugacy classes of $\mathrm{Mat}(n,K)$.

If $\overline{K} = K$, the indecomposable representations correspond to matrices conjugated to a single Jordan block and the decomposition into indecomposable components of a representation $M \in \mathrm{Mat}(n,K)$ corresponds to the writing of M into its Jordan form.

Consider now the subset $V = \{1\} \subseteq \mathbb{N}$. Notice that in this case, a representation is of level $\{1\}$ if and only if is of level at most $\{1\}$.

A matrix $M \in \mathrm{Mat}(n,K)$ is of level $\{1\}$ if and only if its Jordan form over $\overline{K}$ has only blocks of size 1, i.e if and only if M is diagonalizable over $\overline{K}$.

The subset $R(Q,n,\{1\}) \subseteq R(Q,n) = \mathrm{Mat}(n,K)$ is therefore given by the semisimple matrices of size n.

Remark 6.2.9. Let k be an algebraically closed field of char $= 0$ and let λ be an element of k^I. We denote by π the projection map $\pi_\lambda : \mu_\alpha^{-1}(\lambda) \to R(Q,\alpha)$ sending $(x_a, x_{a^*})_{a\in\Omega}$ to $\pi((x_a, X_{a^*})_{a\in\Omega}) = (X_a)_{a\in\Omega}$.

From the result of Crawley-Boevey [17, Lemma 3.2], we deduce that a representation $x \in R(Q,\alpha)$ belongs to $\mathrm{Im}(\pi_\lambda)$ if and only if, given its Krull-Schimdt decomposition $x = \bigoplus_{j\in J} x_j$, we have $\dim x_j \cdot \lambda = 0$ for each $j \in J$.

From Remark 6.2.9, we deduce therefore the following Proposition.

Proposition 6.2.10. *For $K = \mathbb{C}$, for any $\alpha \in \mathbb{N}^I$ and any $\lambda \in \mathbb{C}^I$, we have*

$$R(Q,\alpha,\lambda) = \mathrm{Im}(\pi_\lambda).$$

Remark 6.2.11. Altough the result of Crawley-Boevey is there stated only for algebraically closed field of char $= 0$, Remark 6.2.9 above can be extended over a finite field $\mathbb{F}_q$ of sufficiently big characteristic. More precisely, consider $\lambda \in \mathbb{Z}^I$ and still denote by λ the corresponding element of $\mathbb{F}_q^I$.

Using an argument similar to the one of Proof 6.2, we deduce that, if $q >> 0$, we have

$$\mathrm{Im}(\pi_\lambda) = R(Q, \alpha, \lambda).$$

The following Lemma provides a way to compute the number of isomorphism classes of representations of level at most V of dimension α over a finite field $\mathbb{F}_q$.

Lemma 6.2.12. *For each $V \subseteq \mathbb{N}^I$ and $\alpha \in \mathbb{N}^I$ there exists a polynomial $M_{Q,\alpha,V}(t) \in \mathbb{Z}[t]$ such that, for any q, $M_{Q,\alpha,V}(q)$ is equal to the number of isomorphism classes of representations of level at most V of dimension vector α over $\mathbb{F}_q$. Moreover, the following identity holds:*

$$\mathrm{Exp}\left(\sum_{\beta \in V} a_{Q,\beta}(t) y^\beta\right) = \sum_{\alpha \in \mathbb{N}^I} M_{Q,\alpha,V}(t) y^\alpha \tag{6.2.2}$$

Proof. For $\beta \in \mathbb{N}^I$ denote by $a_{Q,\beta,V}(t)$ the polynomial defined by:

$$a_{Q,\beta,V}(t) = \begin{cases} 0 & \text{if } \beta \notin V \\ a_{Q,\beta}(t) & \text{if } \beta \in V \end{cases} .$$

With an argument similar to that of Proposition 6.1.9, the number of isomorphism classes of indecomposable representations M of level at most V of dimension β over $\mathbb{F}_q$ such that $\mathrm{top}(M) = \mathbb{F}_{q^d}$ is equal to

$$\frac{1}{d} \sum_{r|d} \mu\left(\frac{d}{r}\right) a_{Q\frac{\alpha}{d},V}(q^r).$$

Let then $I_{Q,\beta,V}(t)$ be the polynomial defined by:

$$I_{Q,\alpha,V}(t) = \sum_{d|\overline{\alpha}} \frac{1}{d} \sum_{r|d} \mu\left(\frac{d}{r}\right) a_{Q,\frac{\alpha}{d},V}(t^r) \tag{6.2.3}$$

Notice that for any q, $I_{Q,\beta,V}(q)$ is equal to the number of isomorphism classes of indecomposable representations of level at most V and of dimension β over $\mathbb{F}_q$. For each $\gamma \in \mathbb{N}^I$, denote by $M_{Q,\gamma,V}(t)$ the polynomials defined by the following identity:

$$\sum_{\gamma \in \mathbb{N}^I} M_{Q,\gamma,V}(t) y^\gamma = \prod_{\beta \in \mathbb{N}^I} (1 - y^\beta)^{-I_{Q,\beta,V}(t)}. \tag{6.2.4}$$

As $\mathrm{Rep}_{\mathbb{F}_q}(Q)$ is a Krull-Schimdt category, we deduce that, for any q, $M_{Q,\gamma,V}(q)$ is equal to the number of isomorphism classes of representations of level at most V of dimension γ over

$\mathbb{F}_q$. Specializing the identity (6.2.4) at $t = q$ we can rewrite it as the following identity :

$$\sum_{\alpha \in \mathbb{N}^I} M_{Q,\alpha,V}(q) y^\alpha = \prod_{\alpha \in \mathbb{N}^I} \prod_{\substack{M \in Ind(Q,\alpha,V)/\cong \\ \mathrm{top}(M)=\mathbb{F}_{q^d}}} \frac{1}{1 - y^{\frac{\alpha}{d}(d)}} = \prod_{\alpha \in \mathbb{N}^I} \prod_{d | \overline{\alpha}} \psi_d \left(\frac{1}{1 - y^{\frac{\alpha}{d}}} \right)^{\frac{1}{d} \sum_{r|d} \mu\left(\frac{d}{r}\right) a_{Q,\frac{\alpha}{d},V}(q^r)} \tag{6.2.5}$$

The right hand side of Equation (6.2.5) can be rewritten as

$$\prod_{\alpha \in \mathbb{N}^I} \prod_{d | \overline{\alpha}} \psi_d \left(\frac{1}{1 - y^{\frac{\alpha}{d}}} \right)^{\frac{1}{d} \sum_{r|d} \mu\left(\frac{d}{r}\right) a_{Q_V,\frac{\alpha}{d}}(q^r)} = \prod_{\gamma \in \mathbb{N}^I} \prod_{d \geq 1} \psi_d \left(\frac{1}{1 - y^{\gamma}} \right)^{\frac{1}{d} \sum_{r|d} \mu\left(\frac{d}{r}\right) a_{Q,\gamma,V}(q^r)} \tag{6.2.6}$$

As Equation (6.2.6) holds for any q, we deduce that the following identity holds

$$\log \left(\sum_{\alpha \in \mathbb{N}^I} M_{Q,\alpha,V}(t) y^\alpha \right) = \sum_{\substack{\gamma \in \mathbb{N}^I \\ d \geq 1}} \psi_d \left(\frac{1}{1 - y^\gamma} \right) (a_{Q,\gamma,V}(t))_d \tag{6.2.7}$$

where $(a_{Q,\gamma,V}(t))_d = \dfrac{1}{d} \sum_{r|d} \mu\left(\dfrac{d}{r}\right) a_{Q,\gamma,V}(t^r)$. By Lemma 3.7.8 and Equation (6.2.7), we deduce finally:

$$\mathrm{Log} \left(\sum_{\alpha \in \mathbb{N}^I} M_{Q,\alpha,V}(t) y^\alpha \right) = \sum_{\alpha \in \mathbb{N}^I} a_{Q,\alpha,V}(t) \,\mathrm{Log} \left(\frac{1}{1 - y^\alpha} \right) = \sum_{\alpha \in V} a_{Q,\alpha}(t) y^\alpha. \tag{6.2.8}$$

□

Example 6.2.13. Consider the Jordan quiver Q of Example 6.2.8, $V = \{1\}$ and apply Formula (6.2.2). As $a_{Q,1}(t) = t$, we find

$$\sum_{n \in \mathbb{N}} M_{Q,n,\{1\}}(t) y^n = \mathrm{Exp}\,(ty) = \sum_{n \in \mathbb{N}} t^n y^n \tag{6.2.9}$$

where the last equality comes from Example 3.7.13. We obtain therefore

$$M_{Q,n,\{1\}}(t) = t^n.$$

Evaluating $M_{Q,n,\{1\}}(t)$ at $t = q$, by Example 6.2.8, we find that the number of semisimple conjugacy classes of $\mathrm{Mat}(n, \mathbb{F}_q)$ is q^n. This is a classical combinatorial result (see for example [58]) which comes from the observation that the semisimple conjugacy classes are in bijection with the monic polynomials of $\mathbb{F}_q[t]$ of degree n.

From Lemma 6.2.12, we deduce the following proposition.

Proposition 6.2.14. *1. For any subset $V \subseteq \mathbb{N}^I$ and $\alpha \in \mathbb{N}^I$, the polynomial $M_{Q,\alpha,V}(t)$ has nonnegative integers coefficients.*

2. The polynomial $M_{Q,\alpha,V}(t)$ is non-zero if and only there exist

- $\beta_1, \dots, \beta_r \in (\Phi^+(Q) \cap V)$
- $h_1, \dots, h_r \in \mathbb{N}$

such that $h_1\beta_1 + \dots + h_r\beta_r = \alpha$

Proof. By Lemma 3.7.7 the polynomials $M_{Q,\alpha,V}(t)$ have integer coefficients. By the definition of Exp and Lemma 6.2.12, $M_{Q,\alpha,V}(t)$ is a sum of products of the form

$$\frac{a_{Q,\beta_1}(t^{n_1})^{m_1}}{k_1}\frac{a_{Q,\beta_2}(t^{n_2})^{m_2}}{k_2}\cdots\frac{a_{Q,\beta_l}(t^{n_l})^{m_l}}{k_l} \tag{6.2.10}$$

with $k_1, \dots k_l, m_1, \dots, m_l, n_1, \dots, n_l$ positive integers such that $m_1n_1\beta_1 + \dots + n_lm_l\beta_l = \alpha$. Kac conjecture (see Theorem 6.1.10) implies that these products have nonnegative coefficients. By Proposition 6.1.8, we see that a product as in Equation (6.2.10) is different from 0 if and only if $\beta_1, \dots, \beta_l \in \Phi^+(Q)$ and so we deduce property (2). □

6.2.1 Quiver stacks

Let $K = \mathbb{C}$ and fix $\lambda \in \mathbb{C}^I$. Let $\alpha \in \mathbb{N}^I$ and consider the associated quiver stack $\mathcal{M}_{\lambda,\alpha}$. Davison [22, Theorem B] showed that we have:

$$P_c(\mathcal{M}_{\lambda,\alpha}, t) = t^{-2(\alpha,\alpha)}\,\mathrm{Coeff}_\alpha\left(\mathrm{Exp}\left(\sum_{\beta\cdot\lambda=0}\frac{t^2}{t^2-1}a_{Q,\beta}(t^2)y^\beta\right)\right) \tag{6.2.11}$$

where (α,α) is the Euler form of Q. From Identity (6.2.11) and Proposition 6.2.12 in the case where $V = V_\lambda$, we deduce the following Proposition.

Proposition 6.2.15. *1. The quiver stack $\mathcal{M}_{\lambda,\alpha}$ (and so the quiver variety $\mathcal{Q}_{\lambda,\alpha}$) is non empty if and only $M_{Q,\alpha,\lambda}(t) \neq 0$.*

2. The number of irreducible components of top dimension of the stack $\mathcal{M}_{\lambda,\alpha}$ is equal to the top degree coefficient of $M_{Q,\alpha,\lambda}(t)$.

6.2.2 Counting representations of level at most V

In the paper [62, Theorem 1.1], Letellier linked the Kac polynomial $a_{Q,\alpha}(t)$ to the representation theory of the finite group $\mathrm{GL}_\alpha(\mathbb{F}_q)$. In this paragraph, we will explain how to generalize his results to the case of representations of level at most V for certain subsets $V \subseteq \mathbb{N}^I$.

The finite group $\mathrm{GL}_\alpha(\mathbb{F}_q)$ acts on the finite set $R(Q,\alpha)(\mathbb{F}_q)$. We denote the associated complex character of $\mathrm{GL}_\alpha(\mathbb{F}_q)$ by r_α.

Remark 6.2.16. Fix a dimension vector $\beta \in \mathbb{N}^I$, an integer $r \in \mathbb{N}$, an indecomposable representation $M \in R(Q,\beta)(\mathbb{F}_q)$ and denote by N the representation $N = M^{\oplus r}$. As seen at the beginning of paragraph §6.1.3, there is an isomorphism $\mathrm{Aut}(N)/U_N \cong \mathrm{GL}_r(\mathbb{F}_{q^d})$ where $\mathrm{top}(M) = \mathbb{F}_{q^d}$.
As U_N is a unipotent subgroup, the morphism $\det_I$ passes to the quotient $\mathrm{Aut}(N)/U_N$ and induces thus a morphism $\det_I : \mathrm{GL}_r(\mathbb{F}_{q^d}) \to (\mathbb{F}_q^*)^I$. Its value at a matrix $A \in \mathrm{GL}_r(\mathbb{F}_{q^d})$ is given by

$$\det{}_I(A) = (N_{\mathbb{F}^*_{q^d}/\mathbb{F}_q^*}(\det(A)^{\frac{\beta_i}{d}}))_{i \in I} \tag{6.2.12}$$

Consider now a subset $V \subseteq \mathbb{N}^I$. The main result of this paragraph is the following Theorem:

Theorem 6.2.17. *If there exists an element $\sigma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$ such that $V_{\leq \alpha} = \mathcal{H}_{\sigma,\alpha}$, we have:*

$$\langle r_\alpha \otimes \rho_\sigma, 1 \rangle = M_{Q,\alpha,V}(q) \tag{6.2.13}$$

Proof. Fix a representation x inside $R(Q,\alpha)(\mathbb{F}_q)$. We start by showing that $\rho_\sigma|_{\mathrm{Stab}(x)} \cong 1$ if and only if $x \in R(Q,\alpha,V)(\mathbb{F}_q)$. Consider the Krull-Schmidt decomposition

$$x = \bigoplus_{j \in J} x_j^{r_j}.$$

Let β_j be the dimension vector $\dim x_j$ and d_j the integer such that $\mathbb{F}_{q^{d_j}} = \mathrm{top}(x_j)$ for $j \in J$. As explained at the beginning of §6.1.3, quotienting by the subgroup $U_x \subseteq \mathrm{Stab}(x)$ there is an isomorphism

$$\mathrm{Stab}(x)/U_x \cong \prod_{j \in J} \mathrm{GL}_{r_j}(\mathbb{F}_{q^{d_j}}).$$

The character ρ_σ is trivial over U_x and induces therefore a character $\rho_\sigma : \prod_{j \in J} \mathrm{GL}_{r_j}(\mathbb{F}_{q^{d_j}}) \to \mathbb{C}^*$ which by Remark 6.2.16 is given by

$$\rho_\sigma((A_j)_{j \in J}) = \prod_{j \in J} \sigma^{\frac{\beta_j}{d_j}}(N_{\mathbb{F}^*_{q^{d_j}}/\mathbb{F}_q^*}(A_j)).$$

Therefore, we deduce that $\rho_\sigma|_{\mathrm{Stab}(x)} \equiv 1$ if and only if $\dfrac{\beta_j}{d_j} \in \mathcal{H}_{\sigma,\alpha} = V_{\leq \alpha}$ for each $j \in J$. This is exactly the condition that must hold for x to be of level at most V.

From the discussion above, we deduce therefore that we have:

$$\langle r_\alpha \otimes \rho_\sigma, 1 \rangle = \frac{1}{|\,\mathrm{GL}_\alpha(\mathbb{F}_q)|} \sum_{x \in R(Q,\alpha)(\mathbb{F}_q)} \sum_{g \in \mathrm{Stab}(x)} \rho_\sigma(g) = \frac{1}{|\,\mathrm{GL}_\alpha(\mathbb{F}_q)|} \sum_{x \in R(Q,\alpha,V)(\mathbb{F}_q)} |\,\mathrm{Stab}(x)|. \tag{6.2.14}$$

Applying the Burnside formula to the RHS of eq.(6.2.14) we obtain thus the equality

$$\langle r_\alpha \otimes \rho_\sigma, 1 \rangle = M_{Q,\alpha,V}(q).$$

□

Theorem 6.2.17 can also be proved using the results of Section §5.7. Hua [50, Proof of Theorem 4.3] showed indeed the following formula for the class functions r_α.

Lemma 6.2.18. *For $\alpha \in \mathbb{N}^I$ and $g \in \mathrm{GL}_\alpha(\mathbb{F}_q)$ such that $g \sim \omega$ and $\omega = (d_1, \boldsymbol{\lambda}_1) \dots (d_r, \boldsymbol{\lambda}_r)$, we have*

$$r_\alpha(g) = \prod_{j=1}^{r} \prod_{a \in \Omega} (q^{d_j})^{\langle \lambda_j^{s(a)}, \lambda_j^{t(a)} \rangle}. \tag{6.2.15}$$

Notice that Formula (6.2.15) implies in particular that the family of class functions $\{r_\alpha\}_{\alpha \in \mathbb{N}^I}$ is Log compatible. This was already remarked by Letellier [62], where the author in addition [62, Proposition 2.4] showed that

$$\widetilde{R}_{\alpha,gen}(t) = a_{Q,\alpha}(t) \tag{6.2.16}$$

for any $\alpha \in \mathbb{N}^I$ and used the latter equality to show the case of $V = \mathbb{N}^I$ of Lemma 6.2.12. However, Letellier's approach is different from ours as it involves symmetric functions and does not seem to extend immediately to the case of any V.
Notice that Theorem 5.7.5, Formula (6.2.16) and Lemma 6.2.12 give an alternative way to show Theorem 6.2.17.

7 Star-shaped quivers, multiplicative quiver stacks and character stacks for Riemann surfaces

In this chapter, we review some properties of *star-shaped* quivers and of the associated multiplicative quiver stacks. This type of quivers will be of fundamental importance for showing both the results of Chapter §8 regarding multiplicities for k-tuples of Harisha-Chandra characters of $\mathrm{GL}_n(\mathbb{F}_q)$ and the results of Chapter §9 about the cohomology of character stacks $\mathcal{M}_{\mathcal{C}}$ for a Riemann surface.

In section §7.1, we introduce star-shaped quivers, fix some notations about their representations and recall the results of [46] expressing the Kac polynomials for these quivers in terms of the HLRV kernel, see §3.8.

In section §7.2, we review the definition of the multiplicative moment map Φ^*_α for star-shaped quivers and of the associated multiplicative quiver stacks $\mathcal{M}^*_{\sigma,\alpha}$. The map Φ^*_α is the restriction of the morphism Φ_α introduced in §6.1.1 to a certain open subset $R(\overline{Q},\alpha)^{\circ,*} \subseteq R(\overline{Q},\alpha)^{\circ}$.

In section §7.3, we show that, for each $\sigma \in (\mathbb{C}^*)$ and $\beta \in \mathbb{N}^I$, the multiplicative quiver stack $\mathcal{M}^*_{\sigma,\beta}$ is isomorphic to a certain stack $\mathcal{M}_{\mathbf{L},\mathbf{P},\sigma}$, defined in terms of partial Springer resolutions of conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$. The latter stacks are the stacky versions of the varieties considered by Letellier [61] in the generic case.

In section §7.4, we review the definition of character stacks for a punctured Riemann surface Σ and a k-tuple $\mathcal{C}$ of conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$ and show how they are related to multiplicative quiver stacks for star-shaped quivers.

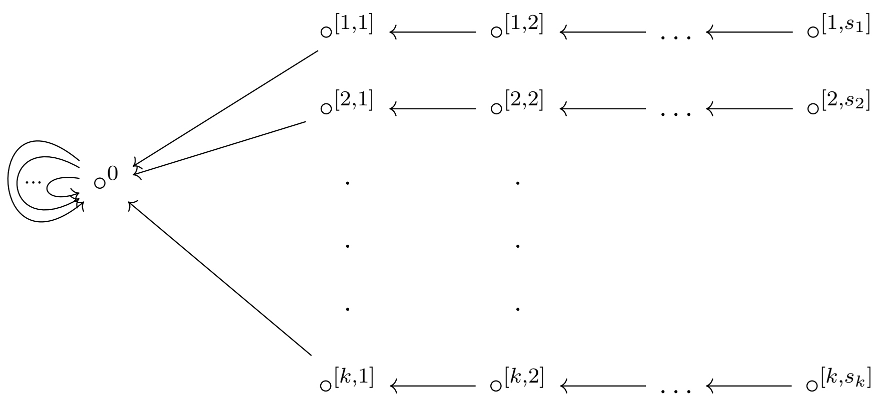
In particular, for any k-tuple $\mathcal{C}$ of semisimple conjugacy classes, we show that there exists $\gamma_{\mathcal{C}} \in (\mathbb{C}^*)^I, \alpha_{\mathcal{C}} \in \mathbb{N}^I$ such that we have an isomorphism

$$\mathcal{M}_{\mathcal{C}} \cong \mathcal{M}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}.$$

The latter isomorphism is going to be one of the key ingredients of our proof of Theorem 9.3.2 about E-series of character stacks.

7.1 Star-shaped quivers

Fix $g, k \geq 0$, integers $s_1, \dots, s_k \in \mathbb{N}$ and let $Q = (I, \Omega)$ be the following *star-shaped quiver* with g loops on the central vertex:



We will denote the vertex 0 also by $[i,0]$ for $i = 1,\dots,k$.

For a representation $x \in R(Q,\alpha)$, for each $h = 1,\dots,k$ and $j = 0,\dots,s_h$, we denote by $x_{h,j} \in \mathrm{Mat}(\alpha_{[h,j]},\alpha_{[h,j+1]},K)$ the matrix associated to the arrow a having $s(a) = [h,j+1]$ and $t(a) = [h,j]$, where we put $x_{h,s_h} = 0$.
Similarly, for an element $\overline{x} \in R(\overline{Q},\alpha)$ we denote by $x^*_{h,j} \in \mathrm{Mat}(\alpha_{[h,j+1]},\alpha_{[h,j]},K)$ the matrix associated to the arrow $a^* \in \overline{\Omega}$.
Lastly, for $i = 0,\dots,g$ and a representation $\overline{x} \in R(\overline{Q},\alpha)$, we denote by $e_1,\dots,e_g,e_1^*,\dots,e_g^* \in \mathrm{Mat}(\alpha_0,K)$ the matrix associated to the g loops of Q and the corresponding reversed arrows of $\overline{Q}$ respectively.

We denote by $(\mathbb{N}^I)^*$ the subset of dimension vectors that are non increasing along the legs and for a subset $V \subseteq \mathbb{N}^I$, by $V^* = V \cap (\mathbb{N}^I)^*$.
For any $\beta \in \mathbb{N}^I$, denote by $R(Q,\beta)^* \subseteq R(Q,\beta)$ the representations which have injective maps along the legs. Notice that if $\beta \notin (\mathbb{N}^I)^*$, we have $R(Q,\beta)^* = \varnothing$.

Remark 7.1.1. Consider $\beta \in (\mathbb{N}^I)^*$ and denote by

$$\mathrm{GL}'_\beta = \prod_{i \in I\setminus\{0\}} \mathrm{GL}_{\beta_i}\,.$$

Notice that $\mathrm{GL}_\beta = \mathrm{GL}_{\beta_0} \times \mathrm{GL}'_\beta$ and the action of GL'_β obtained by restriction of the action of GL_β on $R(Q,\beta)^*$ is free. Indeed, consider an element $x \in R(Q,\beta)^*$ and $g = (g_i)_{i\in I\setminus\{0\}} \in \mathrm{GL}'_\beta$ such that

$$g \cdot x = x.$$

For each $j = 0,\dots,k$, we have

$$x_{j,0}g^{-1}_{[j,1]} = x_{j,0}$$

and, since $x_{j,0}$ is injective, we deduce that $g_{[j,1]} = I_{\alpha_{[j,1]}}$. Similarly, for each $j = 0,\dots,k$, we have

$$g_{[j,1]}x_{j,1}g^{-1}_{[j,2]} = x_{j,1}g^{-1}_{[j,2]} = x_{j,1}$$

and therefore $g_{[j,2]} = I_{\alpha_{[j,2]}}$. By recurrence, we deduce that $g = (I_{\beta_i})_{i\in I\setminus\{0\}}$.

For any such Q, we denote by $Q_0 = (I,\Omega_0)$ the quiver with same vertices of Q, where we eliminate the g loops on the central vertex. Notice that we have

$$R(Q,\alpha) = \mathfrak{gl}_{\alpha_0}^{\oplus g} \bigoplus R(Q_0,\alpha).$$

7.1.1 Indecomposable of star-shaped quivers

The indecomposable representations of the quiver Q have the following property (for a proof see [46, Lemma 3.2.1]).

Lemma 7.1.2. *If $M \in \mathrm{Rep}_K(Q)$ is an indecomposable representation such that $(\dim M)_0 \neq 0$, then all the maps of M along the legs are injective. In particular $\dim M \in (\mathbb{N}^I)^*$.*

Using Lemma 7.1.2, Kac polynomials of star-shaped quivers were computed in [46]. More precisely, for any $\beta \in (\mathbb{N}^I)^*$ and for any $j = 1, \dots, k$, the integers

$$(\beta_{[j,0]} - \beta_{[j,1]}, \dots, \beta_{[j,s_j-1]} - \beta_{[j,s_j]}, \beta_{[j,s_j]})$$

up to reordering form a partition $\mu_\beta^j \in \mathcal{P}$.
Denote by $\boldsymbol{\mu}_\beta \in \mathcal{P}^k$ the multipartition

$$\boldsymbol{\mu}_\beta = (\mu_\beta^1, \dots, \mu_\beta^k)$$

and the associated element $\boldsymbol{\mu}_\beta \in \mathbb{T}_n^k$, as in Example 3.6.1. Moreover, denote by $\mathbb{H}_\beta(z, w)$ the function

$$\mathbb{H}_\beta(z, w) = \mathbb{H}_{\boldsymbol{\mu}_\beta, 2g}(z, w).$$

In [46], the authors show the following Theorem.

Theorem 7.1.3. *For any $\beta \in (\mathbb{N}^I)^*$, we have*

$$a_{Q,\beta}(t) = \mathbb{H}_\beta(0, \sqrt{t}). \tag{7.1.1}$$

7.2 Multiplicative quiver stacks for star-shaped quiver stacks

For a star-shaped quiver $Q = (I, \Omega)$, we introduce two variants of the multiplicative moment map, which are going to be the key objects for our study of character stacks.

Let $R(\overline{Q}, \alpha)^{\circ,l} \subseteq R(\overline{Q}, \alpha)^\circ$ be the open subset of representations

$$R(\overline{Q}, \alpha)^{\circ,l} := \{(x_a, x_{a^*})_{a \in \Omega} \in R(\overline{Q}, \alpha) \mid x_a \text{ is invertible for every loop } a \text{ around } 0\}.$$

Notice that $R(\overline{Q}, \alpha)^{\circ,l} = \varnothing$ if $\alpha_0 = 0$. We denote by

$$\Phi_\alpha^l = \Phi_\alpha|_{R(\overline{Q},\alpha)^{\circ,l}}$$

the restriction of the multiplicative moment map, i.e

$$\Phi_\alpha^l : R(\overline{Q}, \alpha)^{\circ,l} \to \mathrm{GL}_\alpha$$

$$(e_1, \dots, e_g, e_1^*, \dots, e_g^*, x_{1,1}, \dots, x_{k,s_k}, x_{k,s_k}^*) \to \prod_{i=1}^{g} (1+e_i e_i^*)(1+e_i^* e_i)^{-1} \prod_{h=1}^{k} \prod_{j=1}^{s_h} (1+x_{h,j} x_{h,j}^*)(1+x_{h,j+1}^* x_{h,j+1})^{-1}.$$

For $\sigma \in (K^*)^I$, we define the multiplicative quiver stack $\mathcal{M}_{\sigma,\alpha}^l$ as the quotient stack

$$\mathcal{M}_{\sigma,\alpha}^l := [(\Phi_\alpha^l)^{-1}(\sigma) / \mathrm{GL}_\alpha].$$

Notice that, for a point $\overline{x} \in (\Phi_\alpha^l)^{-1}(\sigma)$, we have the following relationships. At the central vertex, we have:

$$\prod_{h=1}^{k}(1+x_{i,0}x_{i,0}^*)=\sigma_0 I_{\alpha_0} \tag{7.2.1}$$

For any $h=1,\dots,k$ and $j=1,\dots,s_h$, we have

$$(1+x_{h,j}x_{h,j}^*)(1+x_{h,j-1}^*x_{h,j-1})^{-1}=\sigma_{[h,j]}I_{\alpha_{[h,j]}} \tag{7.2.2}$$

which can be rewritten as

$$x_{h,j}x_{h,j}^*-\sigma_{[h,j]}x_{h,j-1}^*x_{h,j-1}=(\sigma_{[h,j]}-1)I_{\alpha_{[h,j]}}. \tag{7.2.3}$$

Notice that, for $j=s_h$, we have

$$\sigma_{[h,s_h]}x_{h,s_h-1}^*x_{h,s_h-1}=(1-\sigma_{[h,s_h]})I_{\alpha_{[h,s_h]}}. \tag{7.2.4}$$

Example 7.2.1. Let $Q=(I,\Omega)$ be the Jordan quiver, i.e the quiver with 1 vertex and one arrow. For $n\in\mathbb{N}$, the variety $R(\overline{Q},n)$ is $\mathfrak{gl}_n(K)\times\mathfrak{gl}_n(K)$ and the variety $R(\overline{Q},n)^{\circ,l}$ is given by

$$R(\overline{Q},n)^{\circ,l}=\{(e,e^*)\in\mathfrak{gl}_n(K)\times\mathfrak{gl}_n(K)\mid e,1+ee^*,1+e^*e\in\mathrm{GL}_n(K)\}.$$

Notice that the variety $R(\overline{Q},n)^{\circ,l}$ is isomorphic to $\mathrm{GL}_n(K)\times\mathrm{GL}_n(K)$ via the isomorphism

$$R(\overline{Q},n)^{\circ,l}\to\mathrm{GL}_n(K)\times\mathrm{GL}_n(K)$$

$$(e,e^*)\to(e,e^{-1}+e^*).$$

Via this identification, the multiplicative moment map Φ_n^l corresponds to the morphism

$$\Phi_n^l:\mathrm{GL}_n(K)\times\mathrm{GL}_n(K)\to\mathrm{GL}_n(K)$$

given by

$$\mu_n^{\circ,*}(A,B)=[A,B].$$

Consider now the open subset $R(\overline{Q},\alpha)^{\circ,*}\subseteq R(\overline{Q},\alpha)^{\circ,l}$ defined as

$$R(\overline{Q},\alpha)^{\circ,*}=\{(x_a,x_{a^*})_{a\in\Omega}\in R(\overline{Q},\alpha)\mid x_a\text{ is injective for each }a\in\Omega\}.$$

Notice that $R(\overline{Q},\alpha)^{\circ,*}=\varnothing$ if $\alpha\notin(\mathbb{N}^I)^*$.
The multiplicative moment map which will interest most in the thesis is the restriction of Φ_α to $R(\overline{Q},\alpha)^{\circ,*}$, which we will denote by

$$\Phi_\alpha^*\coloneqq\Phi_\alpha|_{R(\overline{Q},\alpha)^{\circ,*}}.$$

For $\sigma\in(K^*)^I$, we define the multiplicative quiver stack $\mathcal{M}_{\sigma,\alpha}^*$ of parameter σ,α as the quotient stack

$$\mathcal{M}^*_{\sigma,\alpha} \coloneqq [(\Phi^*_\alpha)^{-1}(\sigma)/\operatorname{GL}_\alpha].$$

7.3 Geometric description of multiplicative quiver stacks

In the rest of the chapter, we assume $K = \mathbb{C}$. In this section, for a star-shaped quiver $Q = (I, \Omega)$, we describe an isomorphism between the multiplicative quiver stacks $\mathcal{M}^*_{\sigma,\alpha}$ and certain quotient stacks defined in terms of Springer resolutions of conjugacy classes of $\operatorname{GL}_n(\mathbb{C})$.

7.3.1 Springer resolutions of conjugacy classes

Consider a Levi subgroup $L \subseteq \operatorname{GL}_n(\mathbb{C})$ and a parabolic subgroup $P \supseteq L$ having L as Levi factor. Let $U_P \subseteq P$ be the unipotent radical. Fix an element $z \in Z_L$ and let Y_z be the variety

$$Y_z \coloneqq \{(X, gP) \in \operatorname{GL}_n(\mathbb{C}) \times \operatorname{GL}_n(\mathbb{C})/P \mid g^{-1}Xg \in zU\}.$$

Let $\pi_z : Y_z \to \operatorname{GL}_n(\mathbb{C})$ be the projection $\pi_z((gP, X)) = X$. The following proposition is well known (see for instance [35] and the reference thereby for unipotent orbits).

Proposition 7.3.1. *The image of π_z is the Zariski closure $\overline{C}$ of a conjugacy class $C \subseteq \operatorname{GL}_n(\mathbb{C})$ and the morphism π_z is a resolution of singularities.*
If $z \in (Z_L)^{\mathrm{reg}}$, the map π_z is an isomorphism between Y_z and the conjugacy class of z in $\operatorname{GL}_n(\mathbb{C})$.

The morphism $\pi_z : Y_z \to \overline{C} \subseteq \operatorname{GL}_n(\mathbb{C})$ is sometimes called a *partial Springer resolution.*

Remark 7.3.2. The variety Y_z can be described in the following equivalent way. Consider $n_0, \dots, n_s$ such that $L = \operatorname{GL}_{n_0} \times \cdots \times \operatorname{GL}_{n_s}$. The element $z \in Z_L$ corresponds therefore to $z_0, \dots, z_s \in \mathbb{C}^*$ such that

$$z = (z_s I_{n_s}, \dots, z_0 I_{n_s}).$$

Identify GL_n/P with the corresponding partial flag variety, as in §4.1.2. We have

$$Y_z = \Bigg\{ (X, \mathcal{F}) \in \operatorname{GL}_n(\mathbb{C}) \times \operatorname{GL}_n(\mathbb{C})/P \mid X(\mathcal{F}_j) \subseteq \mathcal{F}_j \text{ for each } j = 0, \dots, s$$

$$\text{and the morphism induced by } X \text{ on } \mathcal{F}_j/\mathcal{F}_{j+1} \text{ is } z_j I_{n_j} \Bigg\}$$

7.3.2 Multiplicative quiver stacks for star-shaped quivers and resolution of conjugacy classes

Consider now a star-shaped quiver $Q = (I, \Omega)$, a dimension vector $\beta \in (\mathbb{N}^I)^*$ and a parameter $\sigma \in (\mathbb{C}^*)^I$. For each $h = 1, \dots, k$ and $j = 0, \dots, s_h$, define

$$n_{h,j} \coloneqq \beta_{[h,j]} - \beta_{[h,j+1]} \tag{7.3.1}$$

where we are identifying $\beta_{[h,0]} = \beta_0$ and $\beta_{[h,s_h+1]} = 0$. For $h = 1, \ldots, k$ let then L_h be the Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$

$$L_h = \prod_{j=0}^{s_h} \mathrm{GL}_{n_{h,j}}(\mathbb{C}).$$

Fix parabolic subgroups $P_h \supseteq L_h$ such that the Levi factor of P_h is L_h and let $U_h \subseteq P_h$ the associated unipotent subgroup.
The element $\sigma \in (\mathbb{C}^*)^I$ determines, for each $h = 1, \ldots, k$, the following element $z_h \in Z_{L_h}$. Choose elements $z_{1,0}, \ldots, z_{k,0} \in \mathbb{C}^*$ such that

$$z_{1,0} \cdots z_{k,0} = \sigma_0.$$

Let then z_h be the central element

$$z_h := (z_{1,0} I_{n_{h,0}}, z_{1,0}\sigma_{[h,1]} I_{n_{1,1}}, \ldots, z_{h,0}\sigma_{[h,1]} \cdots \sigma_{[h,s_h]} I_{n_{h,s_h}}) \in \prod_{j=0}^{s_h} \mathrm{GL}_{n_{h,j}}(\mathbb{C}).$$

Denote by $\mathbf{L}$ the k-tuple of Levi subgroups $(L_1, \ldots, L_k)$ and by $\mathbf{P}$ the k-tuple of parabolic subgroups $(P_1, \ldots, P_k)$.
Let now $X_{\mathbf{L},\mathbf{P},\sigma}$ be the variety defined as

$$X_{\mathbf{L},\mathbf{P},\sigma} := \left\{ (A_1, B_1, \ldots, A_g, B_g, y_1 P_1, X_1, \ldots) \in \mathrm{GL}_n^{2g}(\mathbb{C}) \times \prod_{h=1}^{k} Y_{z_h} \mid \prod_{i=1}^{g} [A_i, B_i] X_1 \cdots X_k = 1 \right\}$$

and $\mathcal{M}_{\mathbf{L},\mathbf{P},\sigma}$ the quotient stack

$$\mathcal{M}_{\mathbf{L},\mathbf{P},\sigma} := [X_{\mathbf{L},\mathbf{P},\sigma} / \mathrm{GL}_n(\mathbb{C})].$$

We have the following Theorem for the stack $\mathcal{M}_{\mathbf{L},\mathbf{P},\sigma}$.

Theorem 7.3.3. *For any $\beta \in (\mathbb{N}^I)^*$ and any $\sigma \in (\mathbb{C}^*)^I$, there is an isomorphism of stacks*

$$\mathcal{M}^*_{\beta,\sigma} \cong \mathcal{M}_{\mathbf{L},\mathbf{P},\sigma}$$

for $\mathbf{L}, \mathbf{P}$ as above.

In the proof, we suppose $g = 0$ to simplify the notations. The case of higher genus is an immediate generalization.

Proof. We define the following morphism

$$f : (\Phi^*_\beta)^{-1}(\sigma) \to X_{\mathbf{L},\mathbf{P},\sigma}.$$

For an element $\overline{x} \in (\Phi^*_\beta)^{-1}(\sigma)$, consider the flag

$$\mathcal{F}_{j,\overline{x}} = (\mathbb{C}^n \supseteq \mathrm{Im}(x_{j,0}) \supseteq \mathrm{Im}(x_{j,0}x_{j,1}) \supseteq \cdots \supseteq \mathrm{Im}(x_{j,0} \cdots x_{j,s_j-1})).$$

Notice that, for each $h = 0, \dots, s_j - 1$, we have

$$\dim(\operatorname{Im}(x_{j,0} \cdots x_{j,h})) = \beta_{[j,h]},$$

since $x_{j,r}$ is injective for each j and r. In particular, $\mathcal{F}_{j,\overline{x}}$ belongs to the partial flag variety GL_n / P_j. We define therefore

$$f(\overline{x}) = (\mathcal{F}_{1,\overline{x}}, z_{1,0} + z_{1,0}x_{1,0}x_{1,0}^*, \mathcal{F}_{2,\overline{x}}, \dots, z_{k,0} + z_{k,0}x_{k,0}x_{k,0}^*).$$

For each $h = 1, \dots, k$, put $X_h \coloneqq z_{h,0} + z_{h,0}x_{h,0}x_{h,0}^*$. Notice that from eq.(7.2.1), we have that

$$X_1 \cdots X_k = 1.$$

To check that the morphism f is well defined we need to check thus the following two conditions.

1. The flag $\mathcal{F}_{h,\overline{x}}$ is X_h invariant for each h.
2. The morphism that X_h induces on the the quotient space

$$\operatorname{Im}(x_{h,0} \cdots x_{h,j}) / \operatorname{Im}(x_{h,0} \cdots x_{h,j+1})$$

   and which we denote by $\overline{X}_{h,j}$ is equal to $z_{h,j} I_{\beta_{[h,j]}}$.

From eq.(7.2.3), by recurrence, we deduce that for each $h = 1, \dots, k$ and each $j = 0, \dots, s_h - 1$ and $v \in \mathbb{C}^{\alpha_{[h,j]}}$, we have

$$X_j(x_{h,0} \cdots x_{h,j}(v)) = z_{h,0}x_{h,0} \cdots x_{h,j}(v) + z_{h,0}x_{h,0}x_{h,0}^* x_{h,0} \cdots x_{h,j}(v) = \tag{7.3.2}$$

$$= \frac{z_{h,0}}{\sigma_{[h,1]} \cdots \sigma_{[h,j+1]}} x_{h,0} \cdots x_{h,j}(v) + x_{h,0} \cdots x_{h,j} x_{h,j+1} x_{h,j+1}^*(v) \tag{7.3.3}$$

where we are putting $x_{h,s_h} = x_{h,s_h}^* = 0$.
Notice that

$$\frac{z_{h,0}}{\sigma_{[h,1]} \cdots \sigma_{[h,j+1]}} = z_{h,j}.$$

From eq.(7.3.3), we deduce therefore that properties $1), 2)$ above are respected for each h, j and therefore f is well defined.

We use the notations of Remark 7.1.1. Notice that from the aforementioned remark, the action of GL'_β on $(\Phi_\beta^*)^{-1}(\sigma)$ is free.
In addition, notice that the map f is GL'_β-invariant. Denote by

$$\widetilde{f} : (\Phi_\beta^*)^{-1}(\sigma) / \mathrm{GL}'_\beta \to X_{\mathbf{L},\mathbf{P},\sigma}$$

the associated morphism. By Lemma 3.3.5, to show that

$$\mathcal{M}_{\beta,\sigma}^* \cong \mathcal{M}_{\mathbf{L},\mathbf{P},\sigma},$$

it is sufficient to show that $\widetilde{f}$ is an isomorphism.

We define the following morphism $\theta : X_{\mathbf{L},\mathbf{P},\sigma} \to (\Phi^*_\beta)^{-1}(\sigma)/\operatorname{GL}'_\beta$. Consider an element

$$(\mathcal{F}_1, X_1, \ldots, \mathcal{F}_k, X_k) \in X_{\mathbf{L},\mathbf{P},\sigma}.$$

For each $h = 1, \ldots, k$ and $j = 1, \ldots, s_h$, fix a basis of the vector space $\mathcal{F}_{h,j}$ and denote by

$$x_{h,j-1} : \mathbb{C}^{\beta_{[h,j]}} \to \mathbb{C}^{\beta_{[h,j-1]}}$$

the morphism such that $z_{h,j-1}x_{h,j-1}$ corresponds to the writing of the inclusion $\mathcal{F}_{h,j} \subseteq \mathcal{F}_{h,j-1}$ in the respective fixed basis.
By definition of $X_{\mathbf{L},\mathbf{P},\sigma}$, we have that

$$(X_h - z_{h,j}I_n)(\mathcal{F}_{h,j}) \subseteq \mathcal{F}_{h,j+1},$$

i.e $X_h - z_{h,j}I_n$ defines a morphism $\mathcal{F}_{h,j} \to \mathcal{F}_{h,j+1}$ and we denote by

$$x^*_{h,j} : \mathbb{C}^{\beta_{[h,j]}} \to \mathbb{C}^{\beta_{[h,j+1]}}$$

its associated matrix in the fixed basis.
Notice that, by definition, for each $h = 1, \ldots, k$, we have

$$X_h = z_{h,0} + z_{h,0}x_{h,0}x^*_{h,0} \tag{7.3.4}$$

Notice moreover, that, for each $j = 1, \ldots, s_h$, we have that $z_{h,j}x_{j,h}x^*_{h,j}$ is the matrix associated to the morphism

$$X_j - z_{h,j}I_n : \mathcal{F}_{h,j} \to \mathcal{F}_{h,j}$$

and $x^*_{h,j-1}z_{h,j-1}x_{h,j-1}$ is the matrix associated to the morphism

$$X_j - z_{h,j-1}I_n : \mathcal{F}_{h,j} \to \mathcal{F}_{h,j}$$

in the respective basis.
In particular, we have that

$$z_{h,j}x_{j,h}x^*_{h,j} - z_{h,j-1}x^*_{h,j-1}x_{h,j-1} = (z_{h,j} - z_{h,j-1})I_{\beta_{[h,j]}} \tag{7.3.5}$$

and, since $\dfrac{z_{h,j-1}}{z_{h,j}} = \sigma_{[h,j]}$, we find

$$x_{j,h}x^*_{h,j} - \sigma_{[h,j]}x^*_{h,j-1}x_{h,j-1} = (1 - \sigma_{[h,j]})I_{\beta_{[h,j]}} \tag{7.3.6}$$

By eq.(7.2.3), we deduce that $(x_{h,j}, x^*_{h,j})_{\substack{h=1,\ldots,k \\ j=0,\ldots,s_h-1}}$ defines a point $\overline{x} \in (\Phi^*_\beta)^{-1}(\sigma)$ and we put

$$\theta(\mathcal{F}_1, X_1, \ldots, \mathcal{F}_k, X_k) = \overline{x}.$$

From eq.(7.3.4) and the definition of $\overline{x}$, we deduce that θ and $\widetilde{f}$ are inverse one to each other, i.e that $\widetilde{f}$ is an isomorphism. $\square$

Remark 7.3.4. Notice that Theorem 7.3.3 shows in particular that the isomorphism class of the stack $\mathcal{M}_{\mathbf{L},\mathbf{P},\sigma}$ does not depend on the choice of the central elements $z_1,\dots,z_k$.

7.4 Character stacks for Riemann surfaces and multiplicative quiver stacks

Fix integers $g,k\in\mathbb{N}$, a Riemann surface Σ of genus g and a subset $D=\{d_1,\dots,d_k\}\subseteq\Sigma$. In this paragraph we recall the definition of character stacks for the Riemann surface Σ with punctures at the points of D and their relationship with multiplicative quiver stacks for star-shaped quivers.

Let $\mathcal{C}$ be a k-tuple of adjoint orbits $\mathcal{C}=(\mathcal{C}_1,\dots,\mathcal{C}_k)$. Denote by $X_{\mathcal{C}}$ the following affine variety

$$X_{\mathcal{C}}\coloneqq\{\rho\in\operatorname{Hom}(\pi_1(\Sigma\setminus D),\mathrm{GL}_n(\mathbb{C}))\mid\rho(\delta_h)\in\overline{\mathcal{C}_h}\ \text{ for } h=1,\dots,k\}$$

where, for each $h=1,\dots,k$, we denote by δ_h a small loop around the point d_h.
The variety $X_{\mathcal{C}}$ is the variety of representations of the fundamental group of $\Sigma\setminus D$ with image lying in $\overline{\mathcal{C}_h}$ around the points of D or, equivalently, the variety of local systems on $X\setminus D$ with prescribed monodromy around D.

Recall that the fundamental group $\pi_1(\Sigma\setminus D)$ admits the following explicit presentation

$$\pi_1(\Sigma\setminus D)=\langle a_1,b_1,\dots,a_g,b_g,\delta_1,\dots,\delta_k\mid[a_1,b_1]\cdots[a_g,b_g]\delta_1\cdots\delta_k=1\rangle$$

where each δ_i is a loop around the puncture x_i.
The variety $X_{\mathcal{C}}$ can therefore be written down in the following explicit way:

$$X_{\mathcal{C}}=\left\{(A_1,B_1,\dots,A_g,B_g,X_1,\dots,X_k)\in\mathrm{GL}_n(\mathbb{C})^{2g}\times\overline{\mathcal{C}_1}\times\cdots\times\overline{\mathcal{C}_k}\mid[A_1,B_1]\cdots[A_g,B_g]X_1\cdots X_k=1\right\}.$$

The character stack $\mathcal{M}_{\mathcal{C}}$ associated to $(\Sigma,D,\mathcal{C})$ is defined as the quotient stack

$$\mathcal{M}_{\mathcal{C}}\coloneqq[X_{\mathcal{C}}/\,\mathrm{GL}_n(\mathbb{C})].$$

We define also the character variety $M_{\mathcal{C}}$, given by the GIT quotient,

$$M_{\mathcal{C}}\coloneqq X_{\mathcal{C}}/\!/\,\mathrm{GL}_n(\mathbb{C}).$$

Consider now $\mathbf{L},\mathbf{P},\sigma$ as before and let $\mathcal{C}=(\mathcal{C}_1,\dots,\mathcal{C}_k)$ be the k-tuple such that $\overline{\mathcal{C}_j}$ is the image of the projection $Y_{z_j}\to\mathrm{GL}_n$.
Notice that the projections $\pi_{z_1},\dots,\pi_{z_h}$ induce a morphism

$$\pi_{\mathcal{C}}:X_{\mathbf{L},\mathbf{P},\sigma}\to X_{\mathcal{C}}$$

$$(A_1,B_1,\dots,A_g,B_g,g_1P_1,X_1,\dots,g_kP_k,X_k)\to(A_1,B_1,\dots,A_g,B_g,X_1,\dots,X_k).$$

As π is $\mathrm{GL}_n(\mathbb{C})$-equivariant, it descends to a morphism of quotient stacks , which we still denote by $\pi : \mathcal{M}_{\mathbf{L},\mathbf{P},\sigma} \to \mathcal{M}_{\mathcal{C}}$ and so, by Theorem 7.3.3, to a morphism,

$$\pi : \mathcal{M}^*_{\beta,\sigma} \to \mathcal{M}_{\mathcal{C}}.$$

Remark 7.4.1. Notice that, if $z_1 \in (Z_{L_1})^{\mathrm{reg}}, \dots, z_k \in (Z_{L_k})^{\mathrm{reg}}$, the morphism π is actually an isomorphism.

Remark 7.4.2. Notice that the morphism $\pi_{\mathcal{C}}$ is obtained by restricting the product of the partial Springer resolution $Y_{z_h} \to \mathcal{C}_h$ and then quotienting by GL_n. The decomposition theorem (and its equivariant version) for partial Springer resolutions are well understood in terms of the representation theory of Weyl groups.
Altough we will not cover this in the thesis, it is natural to expect that the cohomological properties of the morphism $\pi_{\mathcal{C}}$ could have a similar description.

Example 7.4.3. Consider the case where $g = 0, k = 2$, where $\sigma_i = 1$ for each $i \in I$ and $\beta_{[h,j]} = n - j$ for each $h = 1, 2$.
In this case, we have $L_1 = L_2 = T$ where $T \subseteq \mathrm{GL}_n(\mathbb{C})$ is the maximal torus of diagonal matrices. We can therefore take $P_1 = P_2 = B$, where B is the Borel subgroup of upper triangular matrices. Denote by U the unipotent radical of B, i.e the upper triangular matrices having only 1's on the diagonal.
It is not difficult to see that for the corresponding orbits $\mathcal{C}_1 = \mathcal{C}_2$ and $\overline{\mathcal{C}_1} = N$, where $N \subseteq \mathrm{GL}_n(\mathbb{C})$ is the subvariety of unipotent matrices. The variety $X_{\mathcal{C}}$ is therefore given by

$$X_{\mathcal{C}} = \{(X_1, X_2) \in N^2 \mid X_1 X_2 = 1\}.$$

The morphism which sends (X_1, X_2) to X_1 is an isomorphism between $X_{\mathcal{C}} \cong N$. There is thus an isomorphism $\mathcal{M}_{\mathcal{C}} \cong [N/\mathrm{GL}_n(\mathbb{C})]$.
Similarly, the variety $X_{\mathbf{L},\mathbf{P},\sigma}$ is isomorphic to the variety

$$\{(X, g_1 B, g_2 B) \in N \times \mathrm{GL}_n(\mathbb{C})/B \times \mathrm{GL}_n(\mathbb{C})/B \mid g_1^{-1} X g_1 \in U \text{ and } g_2^{-1} X g_2 \in U\} \cong Y_e \times_N Y_e.$$

The variety $Y_e \times_N Y_e$ is the so-called *Steinberg variety*, well studied in geometric representation theory of reductive groups and Weyl groups (see for example [16]).
There is thus an isomorphism

$$\mathcal{M}^*_{\beta,\sigma} \cong [Y_e \times_N Y_e/\mathrm{GL}_n(\mathbb{C})].$$

Via these identifications, the morphism $\pi : \mathcal{M}^*_{\beta,\sigma} \to \mathcal{M}_{\mathcal{C}}$ corresponds to the morphism

$$\pi : [Y_e \times_N Y_e/\mathrm{GL}_n(\mathbb{C})] \to [N/\mathrm{GL}_n(\mathbb{C})]$$

obtained by taking the quotient by $\mathrm{GL}_n(\mathbb{C})$ of the canonical morphism $Y_e \times_N \times Y_e \to N$.

Conversely, to a k-tuple $\mathcal{C}$, we start by associating the following star-shaped quiver $Q = (I, \Omega)$, the following $\alpha_{\mathcal{C}} \in \mathbb{N}^I$ and $\gamma_{\mathcal{C}} \in (\mathbb{C}^*)^I$.

Assume that each $\mathcal{C}_h$ has eigenvalues $\gamma_{h,0}, \dots, \gamma_{h,s_h}$ with Jordan forms associated to partitions $\lambda_{h,0}, \dots, \lambda_{h,s_h} \in \mathcal{P}$.

Put

$$\widehat{s_h} = \sum_{r=0}^{s_h} l(\lambda'_{h,r})$$

and $Q = (I, \Omega)$ be the star-shaped quiver with k legs of length $\widehat{s_1}, \dots, \widehat{s_k}$ respectively.

Consider the dimension vector $\alpha_{\mathcal{C}} \in (\mathbb{N}^I)^*$ defined as follows. For each $h = 1, \dots, k$ and each $j \in \{1, \dots, \widehat{s_h}\}$ there exists unique $p_j \in \{0, \dots, s_h\}$ such that $\sum_{r=0}^{p_j-1} l(\lambda'_{h,r}) < j \leq \sum_{r=0}^{p_j} l(\lambda'_{h,r})$. Put

$$r_j = j - \sum_{r=0}^{p_j-1} l(\lambda'_{h,r}).$$

We define then

$$(\alpha_{\mathcal{C}})_{[h,j]} = \begin{cases} n \text{ if } j = 0 \\ \sum_{r=0}^{p_j-1} |\lambda_{h,r}| + \sum_{r=1}^{r_j} (\lambda'_{h,p_j})_r \text{ otherwise.} \end{cases}$$

For each $h = 1, \dots, k$, denote by $\widehat{\gamma}_h \in (\mathbb{C}^*)^{\{0, \dots, \widehat{s_h}\}}$ the element such that $(\widehat{\gamma}_h)_0 = \gamma_{h,0}$ and

$$(\widehat{\gamma}_h)_j = \gamma_{h,p_j}.$$

We define then the element $\gamma_{\mathcal{C}} \in (\mathbb{C}^*)^I$ as follows:

$$(\gamma_{\mathcal{C}})_{[h,j]} := \begin{cases} \prod_{h=1}^{k} \gamma_{h,0}^{-1} \text{ if } j = 0 \\ \widehat{\gamma_h}_j^{-1} \widehat{\gamma_h}_{j-1} \text{ otherwise .} \end{cases}$$

Example 7.4.4. Assume now that each $\mathcal{C}_h$ is semisimple (i.e $\overline{\mathcal{C}_h} = \mathcal{C}_h$) and it is the conjugacy class of a diagonal matrix C_h with distinct eigenvalues $\gamma_{h,0}, \dots, \gamma_{h,s_h} \in \mathbb{C}^*$ and multiplicities $m_{h,0}, \dots, m_{h,s_h}$ respectively.

Notice that for each $h = 1, \dots, k$ and each $j = 0, \dots, s_h$, we have $\lambda_{h,j} = (1^{m_{h,j}})$ and therefore $\widehat{s_h} = s_h$. The quiver $Q = (I, \Omega)$ is therefore the star-shaped quiver with k legs of length $s_1, \dots, s_k$ respectively.

The dimension vector $\alpha_{\mathcal{C}} \in (\mathbb{N}^I)^*$ is given by

$$(\alpha_{\mathcal{C}})_{[h,j]} = \sum_{l=j}^{s_h} m_{h,l}$$

and $\gamma_{\mathcal{C}} \in (\mathbb{C}^*)^I$ is given by

$$(\gamma_{\mathcal{C}})_{[h,j]} = \begin{cases} \prod_{h=1}^{k} \gamma_{i,0}^{-1} \text{ if } j = 0 \\ \gamma_{h,j}^{-1}\gamma_{h,j-1} \text{ otherwise} \end{cases} .$$

From $\alpha_{\mathcal{C}}, \gamma_{\mathcal{C}}$, define $\mathbf{L} = (L_1, \dots, L_k)$, $\mathbf{P} = (P_1, \dots, P_k)$ and $z_1 \in Z_{L_1}, \dots, z_k \in Z_{L_k}$ as above. Notice that, for each $h = 1, \dots, k$, we have that the image of $\pi_h : Y_{z_h} \to \mathrm{GL}_n$ is $\overline{\mathcal{C}_j}$.

From the reasoning above, we have therefore a morphism

$$\pi : \mathcal{M}^*_{\alpha_{\mathcal{C}},\gamma_{\mathcal{C}}} \to \mathcal{M}_{\mathcal{C}}.$$

Notice that if $\mathcal{C}_h$ is semisimple for each $h = 1, \dots, k$, we have that $z_1 \in (Z_{L_1}^{reg}), \dots, z_k \in (Z_{L_k})^{reg}$. In particular, from Remark 7.4.1, we deduce the following Theorem.

Theorem 7.4.5. *For any k-tuple of semisimple conjugacy classes $\mathcal{C}$, we have an isomorphism of stacks*

$$\mathcal{M}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}} \cong \mathcal{M}_{\mathcal{C}}.$$

7.4.1 Remarks on character stacks for k-tuples of not necessarily semisimple conjugacy classes

In paragraph §7.2, we introduced two versions of the multiplicative moment map for a star-shaped quiver: the morphism Φ^*_β and the morphism Φ^l_β.
While we described the relationship between multiplicative quiver stacks $\mathcal{M}^*_{\sigma,\beta}$ and character stacks $\mathcal{M}_{\mathcal{C}}$, we did not give such a description for the map Φ^l_β.
In this paragraph, we make a few remarks on the behaviour of the multiplicative quiver stack $\mathcal{M}^l_{\beta,\sigma}$, which in general is far more complicated than that of $\mathcal{M}^*_{\beta,\sigma}$.

Firstly, we show that for k-tuples of semisimple conjugacy classes, which are the objects which interest us in this thesis, the two maps give the same result as explained by the following Lemma.

Lemma 7.4.6. *For a k-tuple $\mathcal{C}$ such that $\mathcal{C}_h$ is semisimple for each $h = 1, \dots, k$, we have $\mathcal{M}^l_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}} = \mathcal{M}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}$, i.e*

$$(\Phi^l_{\alpha_{\mathcal{C}}})^{-1}(\gamma_{\mathcal{C}}) = (\Phi^*_{\alpha_{\mathcal{C}}})^{-1}(\gamma_{\mathcal{C}}).$$

Proof. For an element $\overline{x} \in (\Phi^l_{\alpha_{\mathcal{C}}})^{-1}(\gamma_{\mathcal{C}})$ we show that $x_{h,j}$ is injective for every $h = 1, \dots, k$ and $j = 1, \dots, s_h$ in the following way.

Firstly, consider the case where $j = s_h$ and let $v \in \mathbb{C}^{(\alpha_{\mathcal{C}})_{[h,s_h]}}$ such that $x_{h,s_h-1}(v) = 0$.
By eq.(7.3.2), we have

$$v + x^*_{h,s_h-1}x_{h,s_h-1}(v) = (\gamma_{\mathcal{C}})^{-1}_{[h,s_h]}v \tag{7.4.1}$$

and therefore $v = (\gamma_{\mathcal{C}})^{-1}_{[h,s_h]}v$. Since $(\gamma_{\mathcal{C}})_{[h,s_h]} = \gamma^{-1}_{h,s_h}\gamma_{h,s_h-1} \neq 1$, we deduce that $v = 0$. Consider now $j < s_h$ and let $v \in \mathbb{C}^{(\alpha_{\mathcal{C}})_{[h,j]}}$ such that $x_{h,j-1}(v) = 0$. By eq.(7.2.3), we have

$$x_{h,j}x^*_{h,j}(v) = ((\gamma_{\mathcal{C}})_{[h,j]} - 1)v. \tag{7.4.2}$$

and using eq.(7.2.3), we show in a similar way that

$$x_{h,r}x^*_{h,r}\cdots x^*_{h,j}(v) = (\gamma_{\mathcal{C}})_{[h,r]} - 1) = (\prod_{s=j}^{r-1}(\gamma_{\mathcal{C}})_{[h,j]} - 1)x^*_{h,r-1}\cdots x^*_{h,j}(v). \tag{7.4.3}$$

Notice that $\prod_{s=j}^{r-1}(\gamma_{\mathcal{C}})_{[h,j]} = \gamma_{h,j-1}\gamma^{-1}_{h,r-1} \neq 1$ and, for each r, we have thus

$$x_{h,j+1}\cdots x_{h,r}x^*_{h,r}\cdots x^*_{h,j+1}(v) = \frac{x_{h,j+1}\cdots x_{h,r+1}x^*_{h,r+1}\cdots x^*_{h,j+1}(v)}{(\prod_{s=j}^{r-1}(\gamma_{\mathcal{C}})_{[h,j]} - 1)}. \tag{7.4.4}$$

We deduce that we have an equality

$$v = \frac{x_{h,j+1}\cdots x_{h,s_h}x^*_{h,s_h}\cdots x^*_{h,j+1}(v)}{\prod_{r=j+1}^{s_h}(\prod_{s=j}^{r-1}(\gamma_{\mathcal{C}})_{[h,j]} - 1)}. \tag{7.4.5}$$

At the same time, eq.(7.4.3), for $r = s_h - 1$, gives the equality:

$$x_{h,s_h}x^*_{h,s_h}\cdots x^*_{h,j+1}(v) = (\prod_{s=j}^{s_h-1}(\gamma_{\mathcal{C}})_{[h,j]} - 1)x^*_{h,s_h-1}\cdots x^*_{h,j+1}(v) \tag{7.4.6}$$

From eq.(7.3.2), we deduce that we have

$$x^*_{h,s_h}\cdots x^*_{h,j+1}(v) = (\gamma_{\mathcal{C}})_{[h,s_h]}(x^*_{h,s_h}\cdots x^*_{h,j+1}(v) + x^*_{h,s_h}x_{h,s_h}x^*_{h,s_h}\cdots x^*_{h,j+1}(v)) = \tag{7.4.7}$$

$$= \prod_{s=j}^{s_h}(\gamma_{\mathcal{C}})_{[h,j]}x^*_{h,s_h}\cdots x^*_{h,j+1}(v). \tag{7.4.8}$$

Notice that $(\gamma_{\mathcal{C}})_{[h,j]} = \gamma_{h,j-1}\gamma^{-1}_{h,s_h} \neq 1$ and therefore $x^*_{h,s_h}\cdots x^*_{h,j+1}(v) = 0$. From eq.(7.4.5), we see that $v = 0$.

□

For a general k-tuple $\mathcal{C}$ (i.e of not necessarily semisimple conjugacy classes), we can have that $\Phi^l_{\alpha_{\mathcal{C}}}(\gamma_{\mathcal{C}}) \neq \Phi^*_{\alpha_{\mathcal{C}}}(\gamma_{\mathcal{C}})$. However, we can relate the character variety $M_{\mathcal{C}}$ (rather than the character stack) to the multiplicative quiver variety $M^l_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}$, as explained by the following Theorem, shown in [90].

Theorem 7.4.7. *For any k-tuple $\mathcal{C}$ of conjugacy classes, we have an isomorphism*

$$M_{\mathcal{C}} \cong M^l_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}.$$

Notice that the analogous of Theorem 7.4.7 does not hold in general for character stacks and multiplicative quiver stacks, i.e in general

$$\mathcal{M}^l_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}} \ncong \mathcal{M}_{\mathcal{C}}$$

as explained by the following Example.

Example 7.4.8. Consider as in Example 7.4.3, $n = k = 2$, $g = 0$ and the pair $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$, with $\mathcal{C}_2 = \mathcal{C}_2$ and $\mathcal{C}_1$ is the regular unipotent conjugacy class of GL_2, i.e the conjugacy class of

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

As remarked in Example 7.4.3, we have an isomorphism

$$\mathcal{M}_{\mathcal{C}} \cong [N/\,\mathrm{GL}_2].$$

Notice in particular that the stack $\mathcal{M}_{\mathcal{C}}$ is irreducible.

In this case, the associated quiver $Q = (I, \Omega)$ and the dimension vector $\alpha_{\mathcal{C}}$ are depicted below

$$\begin{array}{ccc} 1 & & \\ \downarrow & & \\ 2 & \longleftarrow & 1 \end{array}$$

The associated element $\gamma_{\mathcal{C}}$ is given by

$$(\gamma_{\mathcal{C}})_i = 1 \quad \text{for each } i \in I.$$

As explained in [53, Example 2.3], in this case we have an isomorphism

$$\mathcal{M}^l_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}} \cong \mathcal{M}_{0,\alpha_{\mathcal{C}}},$$

where we denote by $0 = (0)_{i\in I} \in \mathbb{C}^I$ and $\mathcal{M}_{0,\alpha_{\mathcal{C}}}$ is the associated quiver stack, see §6.1.
By Lemma 6.2.15, we have therefore that the number of irreducible components of maximal dimension is the top degree coefficient of the polynomial

$$M_{Q,0,\alpha_{\mathcal{C}}}(t) = M_{Q,\alpha_{\mathcal{C}}}(t).$$

A direct computation shows that

$$M_{Q,\alpha_{\mathcal{C}}}(t) = 5$$

and, in particular, the stack $\mathcal{M}^l_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}$ is not irreducible and therefore not isomorphic to the character stack $\mathcal{M}_{\mathcal{C}}$.

8 Multiplicities for tensor product of representations of finite general linear group

In this chapter, we study multiplicities $\langle \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle$ for k-tuples $\mathcal{X} = (\mathcal{X}_1, \dots, \mathcal{X}_k)$ of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$.
These multiplicities are well understood when the k-tuple is *generic*, thanks to Hausel, Letellier and Rodriguez-Villegas' works [45],[46] and their later generalization by Letellier [63]. These results will be reviewed in section §8.1.

Our main result is a formula for the multiplicity $\langle R^G_{L_1}(\gamma_1) \otimes \cdots \otimes R^G_{L_k}(\gamma_k), 1\rangle$ for any k-tuple of Harisha-Chandra characters $(R^G_{L_1}(\gamma_1), \dots, R^G_{L_k}(\gamma_k))$ (not necessarily generic). We will show this formula in two different ways.

At first istance, we follow a more algebraic approach. In section §8.2 we show how to relate the multiplicity $\langle R^G_{L_1}(\gamma_1) \otimes \cdots \otimes R^G_{L_k}(\gamma_k), 1\rangle$ to the counting of representations of at most a certain level of a certain star-shaped quiver $Q = (I, \Omega)$. This quiver-theoretic interpretation gives a way to express $\langle R^G_{L_1}(\gamma_1) \otimes \cdots \otimes R^G_{L_k}(\gamma_k), 1\rangle$ in terms of the Kac polynomials $a_{Q,\beta}(t)$.

In section §8.3 we show the same formula by following a more combinatorial approach. In particular, we apply the results of §5.7 to the Log compatible family of class functions $\{r^*_\alpha\}_{\alpha \in \mathbb{N}^I}$, where r^*_α is the character of the $\mathrm{GL}_\alpha(\mathbb{F}_q)$-representation $\mathbb{C}[R(Q,\alpha)^*(\mathbb{F}_q)]$.
For both approaches, we use in a key way Lemma 8.2.11, which relates Deligne-Lusztig induction to quiver representations. Such a result does not appear to have been previously reported in the literature.
Lasty, in section §8.4, we show some concrete applications of our results. In particular, we show through quiver representations the classical criterion for the irreducibility of an Harisha-Chandra character $R^G_L(\gamma)$ of $\mathrm{GL}_n(\mathbb{F}_q)$ and we compute explicitly the multiplicity $\langle \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle$ for any k-tuple of semisimple split characters of $\mathrm{GL}_2(\mathbb{F}_q)$.

8.1 Multiplicities in the generic case

Hausel, Letellier, Rodriguez-Villegas [46, Definition 2.2.5] gave the following definition of genericity for k-tuples of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$.

Definition 8.1.1. We say that the a k-tuple $\mathcal{X} = (\mathcal{X}_1, \dots, \mathcal{X}_k)$ is generic, where $\mathcal{X}_i = \epsilon_{L_i}\epsilon_{\mathrm{GL}_n} R^G_{L_i}(\gamma_i R_{\varphi_i})$, if for any F-stable Levi subgroup $M \subseteq \mathrm{GL}_n$ and $g_1, \dots, g_k \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $Z_M \subseteq g_i L_i g_i^{-1}$, the character Γ_M of Z_M^F defined as

$$\Gamma_M(z) = \prod_{i=1}^k \gamma_i(g_i z g_i^{-1})$$

for $z \in Z_M^F$, is a generic linear character of Z_M^F.
By this, it is meant that $\Gamma_M|_{Z_G^F}$ is trivial and for any F-stable $M \subseteq M' \subsetneq G$ the restriction $\Gamma_M|_{Z^F_{M'}}$ is non trivial.

Example 8.1.2. Consider a k-tuple of irreducible characters $(\alpha_1 \circ \det, \ldots, \alpha_k \circ \det)$, with $\alpha_i \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)$. This k-tuple is generic if and only if the element $\alpha_1 \cdots \alpha_k$ has order n.

Consider now a k-tuple $\mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_k)$ of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ and the element $\boldsymbol{\omega}_{\mathcal{X}} := (\omega_{\mathcal{X}_1}, \ldots, \omega_{\mathcal{X}_k}) \in \mathbb{T}_n^k$. Letellier [63, Theorem 6.10.1] showed the following result.

Theorem 8.1.3. *Fix $g \in \mathbb{N}$ and let Λ be the character of the representation of $\mathrm{GL}_n(\mathbb{F}_q)$ on $\mathbb{C}[\mathfrak{gl}_n(\mathbb{F}_q)^g]$, where $\mathrm{GL}_n(\mathbb{F}_q)$ acts by conjugation. For any generic k-tuple $\mathcal{X}$, we have*

$$\mathbb{H}_{\boldsymbol{\omega}_{\mathcal{X}},2g}(0, \sqrt{q}) = \langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle. \tag{8.1.1}$$

Assume that each $\mathcal{X}_i$ is semisimple split, i.e $\mathcal{X}_i = R_{L_i}^G(\gamma_i)$ and let $\mu^i = (\mu_1^i, \ldots, \mu_{s_i}^i)$ be the partition such that $\omega_{\mathcal{X}_i} = (1, (\mu_1^i)) \ldots (1, (\mu_{s_i}^i))$. We have therefore $\boldsymbol{\omega}_{\mathcal{X}} = \boldsymbol{\mu} = (\mu^1, \ldots, \mu^k)$ and

$$\mathbb{H}_{\boldsymbol{\mu},2g}(0, \sqrt{q}) = \langle \Lambda \otimes R_{L_1}^G(\gamma_1) \otimes \cdots \otimes R_{L_k}^G(\gamma_k), 1 \rangle \tag{8.1.2}$$

Formula (8.1.2) was already proved in [46, Theorem 1.4.1].

8.1.1 Star-shaped quivers and Harisha-Chandra characters

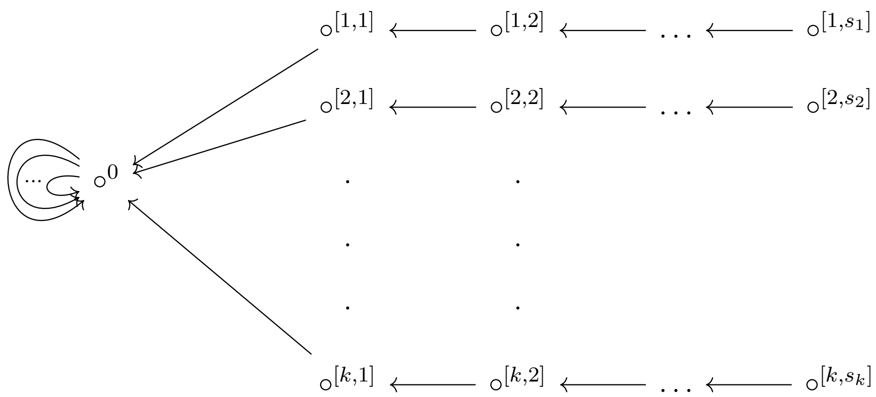
Fix an integer $g \geq 0$ and let $\mathcal{X}$ be a k-tuple of Harisha-Chandra characters

$$\mathcal{X} = (R_{L_1}^G(\gamma_1), \ldots, R_{L_k}^G(\gamma_k))$$

with

$$L_j = \mathrm{GL}_{n_{j,0}} \times \cdots \times \mathrm{GL}_{n_{j,s_j}}.$$

Let $Q = (I, \Omega)$ be the following *star-shaped quiver*:



We will denote the vertex 0 also by $[i, 0]$ for $i = 1, \ldots, k$. Let $\alpha_{\mathcal{X}} \in \mathbb{N}^I$ be the dimension vector defined as

$$(\alpha_{\mathcal{X}})_{[i,j]} = n - \sum_{h=j}^{s_i} n_{i,j} \text{ otherwise.}$$

Notice that Q and $\alpha_{\mathcal{X}}$ depend only on $L_1, \dots, L_k$ and not on the characters $\gamma_1, \dots, \gamma_k$.

If $\mathcal{X}$ is generic, notice that Formula (8.1.2) can be rewritten as

$$\mathbb{H}_{\alpha_{\mathcal{X}}}(0, \sqrt{q}) = \langle \Lambda \otimes R^G_{L_1}(\gamma_1) \otimes \cdots \otimes R^G_{L_k}(\gamma_k), 1 \rangle \tag{8.1.3}$$

i.e, from Theorem 7.1.3, that we have

$$a_{Q,\alpha_{\mathcal{X}}}(q) = \langle \Lambda \otimes R^G_{L_1}(\gamma_1) \otimes \cdots \otimes R^G_{L_k}(\gamma_k), 1 \rangle \tag{8.1.4}$$

8.2 Multiplicities for Harisha-Chandra characters and quiver representations

8.2.1 Levels for k-tuples of characters

Consider now, for each $i = 1, \dots, k$, a character $\gamma_i = (\gamma_{i,0}, \dots, \gamma_{i,s_i}) : L_i^F \to \mathbb{C}^*$. To the k-tuple of characters $\mathcal{X} = (R^G_{L_i}(\gamma_i))_{i=1}^k$ we associate an element $\sigma_{\mathcal{X}} \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$ defined as:

$$(\sigma_{\mathcal{X}})_{[i,j]} \coloneqq \begin{cases} \prod_{i=1}^k \gamma_{i,0} \text{ if } j = 0 \\ \gamma_{i,j}\gamma_{i,j-1}^{-1} \text{ otherwise} \end{cases} . \tag{8.2.1}$$

Recall that the subset $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}} \subseteq (\mathbb{N}^I)^*$ is defined as follows

$$\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha} \coloneqq \{\delta \in (\mathbb{N}^I)^* \mid 0 < \delta \leq \alpha_{\mathcal{X}} \ , \ \sigma_\gamma^\delta = 1\}.$$

For a subset $V \subseteq (\mathbb{N}^I)^*$, we give the following definition of a k-tuple $(R^G_{L_1}(\gamma_1), \dots, R^G_{L_k}(\gamma_k))$ of level V.

Definition 8.2.1. The k-tuple $(R^G_{L_1}(\gamma_1), \dots, R^G_{L_k}(\gamma_k))$ is said to be of level V if $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}} = V_{\leq \alpha_{\mathcal{X}}}$. For $\lambda \in \mathbb{C}^I$, we say that $(R^G_{L_1}(\gamma_1), \dots, R^G_{L_k}(\gamma_k))$ is of level λ if it is of level V_λ.

Remark 8.2.2. Notice that any k-tuple $\mathcal{X} = (R^G_{L_1}(\gamma_1), \dots, R^G_{L_k}(\gamma_k))$ is automatically of level $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}$.

Example 8.2.3. Notice the k-tuple of split Levi characters $(R^G_{L_i}(1))_{i=1}^k$ is of level $(\mathbb{N}^I)^*$. In this case indeed $\sigma_{\mathcal{X}} = 1$ and so $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}} = (\mathbb{N}^I_{\leq \alpha_{\mathcal{X}}})^*$.

We have the following Lemma for generic k-tuples of Harisha-Chandra characters.

Lemma 8.2.4. *For a k-tuple of Harisha-Chandra characters $\mathcal{X} = (R^G_{L_1}(\gamma_1), \dots, R^G_{L_k}(\gamma_k))$, if $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}} = \{\alpha\}$, then $\mathcal{X}$ is generic as in Definition 8.1.1. On the other side, if the k-tuple $\mathcal{X}$ is generic, there are no elements $\delta, \epsilon \in \mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}} \setminus \{\alpha_{\mathcal{X}}\}$ such that $\delta + \epsilon = \alpha_{\mathcal{X}}$.*

Proof. Pick $M \subseteq G$ an F-stable Levi subgroup and let $m_1 \dots, m_r, d_1, \dots, d_r$ be the non-negative integer associated to M such that

$$(M, F) \cong (\mathrm{GL}_{m_1})_{d_1} \times \cdots \times (\mathrm{GL}_{m_r})_{d_r}.$$

There is therefore an isomorphism

$$(Z_M, F) \cong (\mathbb{G}_m)_{d_1} \times \cdots \times (\mathbb{G}_m)_{d_r}.$$

There exist elements $g_1, \dots, g_k \in G^F$ such that $g_i Z_M g_i^{-1} \subseteq L_i$ if and only if there exist k embeddings

$$\lambda_i : (\mathbb{G}_m)_{d_1} \times \cdots \times (\mathbb{G}_m)_{d_r} \hookrightarrow (L_i, F)$$

respecting the condition about *weights* we will explicitate in Equation (8.2.2) below. For $j = 1, \dots, r$, we denote by $\lambda_i^j : (\mathbb{G}_m)_{d_j} \to (L_i, F)$ the restriction of λ_i to the subgroup $\{1\} \times \cdots \times (\mathbb{G}_m)_{d_j} \times \{1\} \times \cdots$ so that

$$\lambda_i = \prod_{j=1}^{r} \lambda_i^j.$$

The composition of λ_i^j and the inclusion $L_i \subseteq G$ defines a morphism which we still denote by $\lambda_i^j : (\mathbb{G}_m)_{d_j} \to \mathrm{GL}_n$ that must respect the following equality:

$$|\lambda_i^j| = m_j \tag{8.2.2}$$

For $i = 1, \dots, k$ and $l = 0, \dots, s_i$, denote by $p_{i,l}$ the projection $p_{i,l} : L_i \to \mathrm{GL}_{n_{i,l}}$ and by $\lambda_{i,l}^j$ the morphism

$$\lambda_{i,l}^j \coloneqq p_{i,l} \circ \lambda_i^j : (\mathbb{G}_m)_{d_j} \to \mathrm{GL}_{n_{i,l}}.$$

Denote by $\gamma_i^{g_i} : Z_M^F \to \mathbb{C}^*$ the morphism given by $\gamma_i^{g_i}(z) = \gamma_i(g_i z g_i^{-1})$. Via the identifications above, the character $\gamma_i^{g_i}$ corresponds to the character

$$\gamma_i \circ \lambda_i : (\mathbb{G}_m)_{d_1}(\mathbb{F}_q) \times \cdots \times (\mathbb{G}_m)_{d_r}(\mathbb{F}_q) = \mathbb{F}_{q^{d_1}}^* \times \cdots \times \mathbb{F}_{q^{d_r}}^* \to \mathbb{C}^*$$

given by

$$(x_1, \dots, x_r) \longrightarrow \prod_{j,l} \gamma_{i,l}(N_{\mathbb{F}^*_{q^{d_j}}/\mathbb{F}_q^*}(x_j))^{|\lambda_{i,l}^j|} \tag{8.2.3}$$

The equality

$$1 = \prod_{i=1}^{k} \gamma_i^{g_i} : Z_M^F \to \mathbb{C}^*$$

therefore holds if and only if for every $j = 1, \dots, r$

$$\prod_{i,l} \gamma_{i,l}(N_{\mathbb{F}^*_{q^{d_j}}/\mathbb{F}_q^*}(x_j))^{|\lambda_{i,l}^j|} = 1 \tag{8.2.4}$$

Put $N_{\mathbb{F}_{q^{d_j}}/\mathbb{F}_q}(x_j)=y$. The following equality holds:

$$\prod_{i,l}\gamma_{i,l}(y)^{|\lambda^j_{i,l}|}=\left(\prod_i\gamma_{i,0}(y)\right)^{m_j}\prod_{i,l}(\gamma_{i,l}\gamma^{-1}_{i,l-1}(y))^{m_j-\sum_{s=1}^{l-1}|\lambda_{i,s}|}=\sigma_{\mathcal{X}}^{\delta_j}(y) \tag{8.2.5}$$

where δ_j is the element of $\mathbb{N}^I$ given by $\delta_0=m_j$ and

$$\delta_{[i,j]}=m_j-\sum_{s=1}^{l-1}|\lambda_{i,s}|.$$

Therefore, from Equation (8.2.5), we deduce that if $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}=\{\alpha_{\mathcal{X}}\}$ the k-tuple of characters $\mathcal{X}=(R^G_{L_1}(\gamma_1),\dots,R^G_{L_k}(\gamma_k))$ is generic.

Conversely, assume that the k-tuple $\mathcal{X}$ is generic and assume the existence of $\delta\in\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}$ such that $\delta\neq\alpha_{\mathcal{X}}$ and $\epsilon:=\alpha_{\mathcal{X}}-\delta$ belongs to $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha}$ too. Consider the F-stable split Levi subgroup $M=\mathrm{GL}_{\delta_0}\times\mathrm{GL}_{\epsilon_0}\subseteq\mathrm{GL}_n$ embedded block diagonally. Notice that in particular $Z_M\cong\mathbb{G}_m\times\mathbb{G}_m$.
For each $i=1,\dots,k$, there exist embeddings $\lambda^1_i,\lambda^2_i:\mathbb{G}_m\to(L_i,F)$ such that, with the notations used before,

$$|\lambda^1_{i,l}|=\delta_{[i,l]}-\delta_{[i,l+1]}$$

and

$$|\lambda^2_{i,l}|=\epsilon_{[i,l]}-\epsilon_{[i,l+1]}.$$

The associated embeddings $\lambda^1_i\times\lambda^2_i:(Z_M,F)\to(L_i,F)$ correspond to elements $g_1,\dots,g_k\in G^F$ such that $g_iZ_Mg_i^{-1}\subseteq L_i$ and for $(x_1,x_2)\in Z_M^F=\mathbb{F}_q^*\times\mathbb{F}_q^*$ we have

$$\Gamma_M(z)=\prod_{i=1}^k\gamma_i^{g_i}(x_1,x_2)=\sigma^{\delta}_{\mathcal{X}}(x_1)\sigma^{\epsilon}_{\mathcal{X}}(x_2)=1.$$

□

Remark 8.2.5. For any $\beta\in(\mathbb{N}^I)^*$, there exists a generic k-tuple $\mathcal{X}$ of Harisha-Chandra characters such that $\alpha_{\mathcal{X}}=\beta$ if q is sufficiently big (see for example the discussion after [46, Proposition 2.2.4]).

8.2.2 Star-shaped quiver of type A and Harisha-Chandra characters

Consider the case in which $k=1$, $g=0$ i.e $\mathcal{X}=(R^G_L(\gamma))$ and the associated type A quiver $Q=(I,\Omega)$. In this paragraph, we show how to express the character $R^G_L(\gamma)$ in terms of the representations of Q.
This relationship is going to be the main ingredient to prove Theorem 8.2.8, which is the main result of this chapter.

Consider $n_0, \dots, n_l \in \mathbb{N}$ such that

$$(L, F) \cong \mathrm{GL}_{n_0} \times \cdots \times \mathrm{GL}_{n_l}.$$

Denote by $I = \{0, \dots, l\}$ the set of vertices of Q and by $\alpha = \alpha_{\mathcal{X}}$ the dimension vector associated to $R_L^G(\gamma)$. Notice that

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_l)$$

with

$$\alpha_j = \sum_{h=j}^{l} n_h.$$

Let $\gamma_0, \dots, \gamma_l \in \mathrm{Hom}(\mathbb{F}_q, \mathbb{C}^*)$ such that $\gamma = (\gamma_0, \dots, \gamma_l)$, with the notations of §5.4.1. In this case, denote by $\sigma_\gamma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$ the associated element and put

$$\rho_\gamma = \rho_{\sigma_\gamma}.$$

Lastly, denote by r_α^* the character of $\mathrm{GL}_\alpha(\mathbb{F}_q)$ given by its action on the finite set $R(Q, \alpha)^*(\mathbb{F}_q)$. We have the following Lemma.

Lemma 8.2.6. *For any $g_0 \in \mathrm{GL}_n(\mathbb{F}_q)$, we have:*

$$R_L^G(\gamma)(g_0) = \sum_{\substack{j=1,\dots,l \\ g_j \in \mathrm{GL}_{\alpha_j}(\mathbb{F}_q)}} r_\alpha^*(g_0, g_1, \dots, g_l) \frac{\rho_\gamma(g_0, g_1, \dots, g_l)}{|\mathrm{GL}_{\alpha_1}(\mathbb{F}_q)| \cdots |\mathrm{GL}_{\alpha_l}(\mathbb{F}_q)|} \tag{8.2.6}$$

Proof. Let P be the unique parabolic subgroup of GL_n containing the upper triangular matrices and L. Recall that by Formula (5.1.1) we have:

$$R_L^G(\gamma)(g_0) = \sum_{\substack{h \in \mathrm{GL}_n(\mathbb{F}_q)/P(\mathbb{F}_q) \\ g_0 \cdot hP(\mathbb{F}_q) = hP(\mathbb{F}_q)}} \gamma(h^{-1} g_0 h).$$

The character r_α^* satisfies:

$$r_\alpha^*(g_0, g_1, \dots, g_l) = \#\Bigg\{ f_l \in \mathrm{Hom}^{inj}(\mathbb{F}_q^{\alpha_l}, \mathbb{F}_q^{\alpha_{l-1}}), \dots,$$

$$f_1 \in \mathrm{Hom}^{inj}(\mathbb{F}_q^{\alpha_1}, \mathbb{F}_q^{n}) \ \text{ s.t } \ g_0 f_1 g_1^{-1} = f_1, \dots, \ g_{l-1} f_l g_l^{-1} = f_l \Bigg\}.$$

We denote by $X(g_0)$ the set defined as:

$$X(g_0) := \Bigg\{ g_1 \in \mathrm{GL}_{\alpha_1}(\mathbb{F}_q), \dots, g_l \in \mathrm{GL}_{\alpha_l}(\mathbb{F}_q), \ f_l \in \mathrm{Hom}^{inj}(\mathbb{F}_q^{\alpha_l}, \mathbb{F}_q^{\alpha_{l-1}}), \dots,$$

$$f_1 \in \mathrm{Hom}^{inj}(\mathbb{F}_q^{\alpha_1}, \mathbb{F}_q^n) \text{ s.t } g_0 f_1 g_1^{-1} = f_1, \dots, \ g_{l-1} f_l g_l^{-1} = f_l \Big\}$$

and for $x = (f_1, \dots, f_l, g_1, \dots, g_l) \in X(g)$ we put

$$\rho_\gamma(x) \coloneqq \rho_\gamma(g_0, g_1, \dots, g_l).$$

For a fixed $g_0 \in \mathrm{GL}_n(\mathbb{F}_q)$, we have thus the following equality :

$$\sum_{\substack{j=1,\\ g_j \in \mathrm{GL}_{\alpha_j}(\mathbb{F}_q)}}^{l} \frac{r_\alpha^*(g_0, g_1, \dots, g_l)\rho_\gamma(g_0, g_1, \dots, g_l)}{|\mathrm{GL}_{\alpha_1}(\mathbb{F}_q)| \cdots |\mathrm{GL}_{\alpha_l}(\mathbb{F}_q)|} = \sum_{x \in X(g_0)} \frac{\rho_\gamma(x)}{|\mathrm{GL}_{\alpha_1}(\mathbb{F}_q)| \cdots |\mathrm{GL}_{\alpha_l}(\mathbb{F}_q)|}.$$

There is a map $\psi : X(g_0) \to (\mathrm{GL}_n(\mathbb{F}_q)/P(\mathbb{F}_q))^{g_0}$ defined as

$$\psi((g_1, \dots, g_l, f_1, \dots, f_l)) = (\mathrm{Im}(f_1 \cdots f_l) \subseteq \mathrm{Im}(f_1 \cdots f_{l-1}) \subseteq \cdots \subseteq \mathrm{Im}(f_1) \subseteq \mathbb{F}_q^n).$$

To see that ψ is well defined we need to verify that the subspaces $\mathrm{Im}(f_1 \cdots f_l)$,$\mathrm{Im}(f_1 \cdots f_{l-1})$ $\dots, \mathrm{Im}(f_1)$ are all g_0-stable.

Start with $\mathrm{Im}(f_1)$. We have $g_0 f_1 = f_1 g_1$ and so $g_0(\mathrm{Im}(f_1)) \subseteq \mathrm{Im}(f_1)$. For a general $j \geq 1$ we see similarly

$$g - 0 f_1 \cdots f_j = f_1 g_1 f_2 \cdots f_j = \cdots\cdots = f_1 \cdots f_j g_j.$$

Let us show that the map ψ is surjective. Given a g_0-stable flag

$$(V_l \subseteq V_{l-1} \subseteq \cdots \subseteq V_1 \subseteq \mathbb{F}_q^n) = hP(\mathbb{F}_q),$$

we can choose for each $j = 1, \dots, l$ a basis $\mathfrak{B}_j$ of V_j such that $\mathfrak{B}_j \subseteq \mathfrak{B}_{j-1}$ as ordered sets. The choices of the $\mathfrak{B}_j$s define morphisms $f_j : \mathbb{F}_q^{\alpha_j} \hookrightarrow \mathbb{F}_q^{\alpha_{j-1}}$ such that $\mathrm{Im}(f_1 f_2 \cdots f_j) \subseteq \mathbb{F}_q^n$ is g_0-stable for any $j = 1, \dots, l$.

For each $j = 1, \dots, l$, the automorphism $g_0|_{\mathrm{Im}(f_1 f_2 \cdots f_j)}$ written in the basis $\mathfrak{B}_j$ define an element $g_j \in \mathrm{GL}_{\alpha_j}(\mathbb{F}_q)$ and the element $x_h \in X(g)$ defined as $x_h \coloneqq (g_1, \dots, g_l, f_1, \dots, f_l)$ is such that

$$\psi(x) = (V_l \subseteq V_{l-1} \subseteq \cdots \subseteq V_1 \subseteq \mathbb{F}_q^n),$$

i.e the morphism ψ is surjective.

There is an action of $\prod_{j=1}^{l} \mathrm{GL}_{\alpha_j}(\mathbb{F}_q)$ on $X(g)$ defined as

$$(m_1, \dots, m_l) \cdot (g_1, \dots, g_l, f_1, \dots, f_l) \coloneqq (m_1 g_1 m_1^{-1}, \dots, m_l g_l m_l^{-1}, f_1 m_1^{-1}, m_1 f_2 m_2^{-1}, \dots, f_l m_l^{-1}).$$

Notice that the latter action is free by Remark 7.1.1. The map ψ is $\prod_{j=1}^{l} \mathrm{GL}_{\alpha_j}(\mathbb{F}_q)$ invariant and, for each $h \in \mathrm{GL}_n(\mathbb{F}_q)/P(\mathbb{F}_q)$, the fiber $\psi^{-1}(h)$ is equal to the orbit $\left(\prod_{j=1}^{l} \mathrm{GL}_{\alpha_j}(q)\right) \cdot x_h$.

Therefore, as $\rho_\gamma((m_1, \dots, m_l) \cdot x) = \rho_\gamma(x)$ for any $(m_1, \dots, m_l) \in \prod_{j=1}^{l} \mathrm{GL}_{\alpha_j}(\mathbb{F}_q)$, we have

$$\sum_{x \in X(g_0)} \frac{\rho_\gamma(x)}{|\mathrm{GL}_{\alpha_1}(\mathbb{F}_q)| \cdots |\mathrm{GL}_{\alpha_l}(\mathbb{F}_q)|} = \sum_{h \in \mathrm{GL}_n(\mathbb{F}_q)/P(\mathbb{F}_q))^g} \rho_\gamma(\psi^{-1}(h)).$$

We are thus left to show that $\rho_\gamma(\psi^{-1}(h)) = \gamma(h^{-1} g_0 h)$. On the one side, by evaluating ρ_γ at the element $x_h \in \psi^{-1}(h)$ defined above, we see that:

$$\rho_\gamma(\psi^{-1}(h)) = ((\gamma_{l-1}^{-1}\gamma_l)(\det(g_0|_{V_l}))) \cdots (\gamma_0(\det(g_0))).$$

On the other side, the matrix $h^{-1} g_0 h$ is a block upper triangular matrix:

$$h^{-1} g h = \begin{pmatrix} g_l' & & & & & \\ 0 & g_{l-1}' & & & & \\ 0 & 0 & g_{l-2}' & & & \\ 0 & 0 & 0 & * & & \\ 0 & 0 & 0 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & 0 & g_0' \end{pmatrix}$$

where g_l' is $g_0|_{V_l}$ written in the basis $\mathfrak{B}_l$, the matrix $\begin{pmatrix} g_l' & * \\ 0 & g_{l-1}' \end{pmatrix}$ is equal to $g_0|_{V_{l-1}}$ written in the basis $\mathfrak{B}_{l-1}$ and so on. Thus, we have the following identity:

$$\gamma(h^{-1} g_0 h) = \prod_{j=0}^{l} \gamma_j(\det(g_j')) =$$

$$= \gamma_l(\det(g_0|_{V_l}))(\gamma_{l-1}(\det(g_0|_{V_{l-1}}))\gamma_{l-1}^{-1}(\det(g_0|_{V_l})))(\gamma_{l-2}(\det(g_0|_{V_{l-2}}))\gamma_{l-2}^{-1}(\det(g - 0|_{V_{l-1}}))))$$
$$\cdots (\gamma_0(\det(g_0))\gamma_0^{-1}(\det(g_0|_{V_1}))) =$$
$$= (\gamma_{l-1}^{-1}\gamma_l(\det(g_0|_{V_l}))) \cdots (\gamma_0(\det(g_0))).$$

□

Remark 8.2.7. Notice that Lemma 8.2.6 gives a way to express Harisha-Chandra induction (or split Deligne-Lusztig induction) in terms of quiver representation. A similar formula seems to not have been known before in the literature.

It would be interesting to find a way to relate quiver representations to Deligne-Lusztig characters associated to non-split Levi subgroups too.

8.2.3 Main result

The main result of this chapter is the following Theorem.

Theorem 8.2.8. *Let $V \subseteq (\mathbb{N}^I)^*$ and let $\mathcal{X} = (R^G_{L_1}(\gamma_1), \dots, R^G_{L_k}(\gamma_k))$ be a k-tuple of Harisha-Chandra characters of $\mathrm{GL}_n(\mathbb{F}_q)$ of level V. The following equality holds:*

$$\left\langle \Lambda \otimes \bigotimes_{i=1}^{k} R^G_{L_i}(\gamma_i), 1 \right\rangle = M_{Q,\alpha_{\mathcal{X}},V}(q) = \mathrm{Coeff}_{\alpha_{\mathcal{X}}}\left(\mathrm{Exp}\left(\sum_{\beta \in V} \mathbb{H}_{\beta}(0,\sqrt{q})y^{\beta}\right)\right). \tag{8.2.7}$$

Notice that the last equality of eq.(8.2.7) is a consequence of Lemma 6.2.2 and Theorem 7.1.3. Before giving the proof of Theorem 8.2.8, we make some examples of cases in which this result was already known.

Remark 8.2.9. Consider a generic k-tuple $\mathcal{X} = (R^G_{L_1}(\delta_1), \dots, R^G_{L_k}(\delta_k))$. By Lemma 8.2.4 and Lemma 6.2.12, we have $M_{Q,\alpha_{\mathcal{X}},\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}}(t) = a_{Q,\alpha_{\mathcal{X}}}(t)$ and therefore Formula (8.2.7) implies the following Identity:

$$\left\langle \Lambda \otimes \bigotimes_{i=1}^{k} R^G_{L_i}(\delta_i), 1 \right\rangle = a_{Q,\alpha_{\mathcal{X}}}(q) \tag{8.2.8}$$

which had already been proved in [46, Theorem 3.4.1.].

Example 8.2.10. Consider the case where $V = (\mathbb{N}^I)^*$. As remarked in Example 8.2.3, the k-tuple $(R^G_{L_i}(1))_{i=1}^k$ is of level $(\mathbb{N}^I)^*$.
From Lemma 7.1.2, we deduce that for a subset $V \subseteq (\mathbb{N}^I)^*$ and any $\beta \in (\mathbb{N}^I)^*$, the representations $R(Q,\beta,V)$ of level V are all contained in $R(Q,\beta)^*$.
In particular, for $V = (\mathbb{N}^I)^*$ we have an identity $R(Q,\beta,(\mathbb{N}^I)^*) = R(Q,\beta)^*$. Formula (6.2.2) implies thus the following identity:

$$\sum_{\beta \in (\mathbb{N}^I)^*} M^*_{Q,\beta}(t)y^{\beta} = \mathrm{Exp}\left(\sum_{\beta \in (\mathbb{N}^I)^*} a_{Q,\beta}(t)y^{\beta}\right) \tag{8.2.9}$$

where the polynomials $M^*_{Q,\beta}(t)$ are such that, for any q, $M^*_{Q,\beta}(q)$ is equal to the number of isomorphism classes of representations of dimension β with injective maps along the legs over $\mathbb{F}_q$.
By eq.(8.2.7) we obtain the identity

$$\langle R^G_{L_1}(1) \otimes \cdots \otimes R^G_{L_k}(1), 1 \rangle = M^*_{Q,\alpha_{\mathcal{X}}}(q) \tag{8.2.10}$$

The latter identity was already proved in [46, Proposition 3.2.5]. Roughly speaking, in this case Identity (8.2.10) comes from the fact that $\prod_{i=1}^{k} R^G_{L_i}(1)(h)$, for each $h \in \mathrm{GL}_n(\mathbb{F}_q)$, is the number of k-tuple of flags of $\mathbb{F}_q^n$ of type $\alpha_{\mathcal{X}}$ fixed by h and Burnside's formula (see [46, Lemma 2.1.1]).

Fix now a k-tuple $\mathcal{X} = (R_{L_1}^G(\gamma_1), \ldots, R_{L_k}^G(\gamma_k))$ and, to ease the notations, denote by $\rho_{\mathcal{X}}$ the character $\rho_{\sigma_{\mathcal{X}}}$ as in §6.2.2. For any $\alpha \in \mathbb{N}^I$, we denote by r_α^* the complex character of $\mathrm{GL}_\alpha(\mathbb{F}_q)$ given by its action on the finite set $R(Q,\alpha)^*(\mathbb{F}_q)$.

Notice that Lemma 8.2.6 implies the following Lemma:

Lemma 8.2.11. *We have the following identity:*

$$\left\langle r_{\alpha_{\mathcal{X}}}^* \otimes \rho_{\mathcal{X}}, 1\right\rangle = \left\langle \Lambda \otimes \bigotimes_{i=1}^{k} R_{L_i}^G(\gamma_i), 1\right\rangle \tag{8.2.11}$$

Proof of Theorem 7.2.8. The proof of Theorem 6.2.17 can be slightly modified to show that for $V \subseteq (\mathbb{N}^I)^*$ such that $V_{\leq \alpha_{\mathcal{X}}} = \mathcal{H}_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}^*$ we have

$$\langle r_{\alpha_{\mathcal{X}}}^* \otimes \rho_{\mathcal{X}}, 1\rangle = M_{Q,\alpha_{\mathcal{X}},V}(q). \tag{8.2.12}$$

From eq.(8.2.11) and eq.(8.2.12) we deduce directly Theorem 8.2.8. □

8.2.4 Non-vanishing of multiplicities

From Proposition 6.2.14, we deduce the following proposition.

Proposition 8.2.12. *1. For a k-tuple of Harisha-Chandra characters*

$$\mathcal{X} = (R_{L_1}^G(\gamma_1), \ldots, R_{L_k}^G(\gamma_k)),$$

the multiplicity $\langle \Lambda \otimes R_{L_1}^G(\gamma_1) \otimes \cdots \otimes R_{L_k}^G(\gamma_k), 1\rangle$ is the evaluation at q of a polynomial with non-negative coefficients.

2. Given $Q, \alpha_{\mathcal{X}}$ as above the multiplicity $\left\langle \Lambda \otimes R_{L_1}^G(\gamma_1) \otimes \cdots \otimes R_{L_k}^G(\gamma_k), 1\right\rangle$ is non-zero if and only there exist

- $\beta_1, \ldots, \beta_r \in (\Phi^+(Q) \cap V)$
- $m_1, \ldots, m_r \in \mathbb{N}$

such that $m_1\beta_1 + \cdots + m_r\beta_r = \alpha_{\mathcal{X}}$.

Notice that this implies that if $\Phi^+(Q) \cap V = \varnothing$ the multiplicity is 0. Similarly, if $\alpha_{\mathcal{X}} \notin V$ we have that the multiplicity is 0. Indeed, as $V_{\leq \alpha_{\mathcal{X}}} = \mathcal{H}_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}^*$, if $\beta_i \in V$ with $\beta_i \leq \alpha$ and $m_i\beta_i \leq \alpha$ too, we have $m_i\beta_i \in V$ and $\alpha_{\mathcal{X}} = m_1\beta_1 + \cdots + m_r\beta_r \in V$.

Remark 8.2.13. Identity (8.2.7) implies that the multiplicity $\left\langle \Lambda \otimes \bigotimes_{i=1}^{k} R_{L_i}^G(\gamma_i), 1\right\rangle$ does not depend on the characters $\gamma_1, \ldots, \gamma_k$ but only on the Levi subgroups $L_1, \ldots, L_k$ and the subset $\mathcal{H}_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}^*$.

As $a_{Q,\beta}(t) \neq 0$ if and only if $\beta \in \Phi^+(Q)$, we deduce more precisely that the multiplicity depends only on the intersection of $\mathcal{H}^*_{\sigma_\mathcal{X},\alpha_\mathcal{X}}$ and $\Phi^+(Q)$.

8.3 Multiplicities for Harisha-Chandra characters and Log compatibility

In this paragraph, we give a alternative proof of Theorem 8.2.8 which uses the results of Section §5.8.2, rather than the levels of representations of quivers.
We start by noticing that we have the following Lemma.

Lemma 8.3.1. *For $\alpha \in \mathbb{N}^I$ and $g \in \mathrm{GL}_\alpha(\mathbb{F}_q)$ such that $g \sim \omega$ and $\omega = (d_1, \boldsymbol{\lambda}_1)\dots(d_r, \boldsymbol{\lambda}_r)$, we have*

$$r^*_\alpha(g) = \begin{cases} \Lambda(g_0)\displaystyle\prod_{j=1}^r \prod_{a\in\Omega_0} (q^{d_j})^{\langle \lambda_j^{s(a)}, \lambda_j^{t(a)}\rangle - 1}(q^{d_j}-1) \text{ if } |\boldsymbol{\lambda}_j| \in (\mathbb{N}^I)^* \text{ for all } j \\ 0 \text{ otherwise} \end{cases} \tag{8.3.1}$$

In particular, the family of functions $\{r^*_\alpha\}_{\alpha\in\mathbb{N}^I}$ is Log compatible.

The proof follows closely the arguments of Hua's [50, Proof of Theorem 4.3] for the family $\{r_\alpha\}_{\alpha\in\mathbb{N}^I}$. We give a sketch of the proof for completeness.

Proof. Notice that we have:

$$r^*_\alpha(g) = \Lambda(g_0) \prod_{a\in\Omega_0} |\{M \in \mathrm{Hom}^{inj}(s(a), t(a), \mathbb{F}_q) \mid g_{s(a)} M g_{t(a)}^{-1} = M\}|. \tag{8.3.2}$$

Put

$$r^*_{\alpha,0}(g) = \prod_{a\in\Omega_0} |\{M \in \mathrm{Hom}^{inj}(s(a), t(a), \mathbb{F}_q) \mid g_{s(a)} M g_{t(a)}^{-1} = M\}|.$$

We use the notations of §3.6.1. Consider F-orbits $\theta_1, \dots, \theta_r$ of cardinality $d_1, \dots, d_r$ respectively and multipartitions $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_r$ of size $\beta_1, \dots, \beta_r$ respectively, such that g is conjugated to

$$\prod_{j=1}^r J(\theta_j, \boldsymbol{\lambda}_j).$$

Hua's arguments show that

$$r^*_{\alpha,0}(g) = \prod_{j=1}^r r^*_{d\beta_j,0}(J(\theta_j, \boldsymbol{\lambda}_j)).$$

Notice that if there exists $j \in \{1, \dots, r\}$ such that $\beta_j \notin (\mathbb{N}^I)^*$, we have $r^*_{d\beta_j,0}(J(\theta_j, \boldsymbol{\lambda}_j)) = 0$ and therefore $r^*_\alpha(g) = 0$.
Otherwise, we have

$$\prod_{a\in\Omega_0} |\{M \in \mathrm{Hom}^{inj}(s(a), t(a), \mathbb{F}_q) \mid J(\theta_j, \boldsymbol{\lambda}_j)_{s(a)} M (J(\theta_j, \boldsymbol{\lambda}_j))_{t(a)}^{-1} = M\}| = \prod_{a\in\Omega_0} (q^{d_j})^{\langle \lambda_j^{s(a)}, \lambda_j^{t(a)}\rangle - 1}(q^{d_j}-1) \tag{8.3.3}$$

We deduce therefore the equality of Lemma 8.3.1. □

By Theorem 5.7.5, for any $\sigma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)^I$, we have an identity

$$\langle r_\alpha^* \otimes \rho_\sigma, 1\rangle = \mathrm{Coeff}_\alpha\left(\mathrm{Exp}\left(\sum_{\beta\in\mathcal{H}_\sigma} \widetilde{R^*}_{\beta,gen}(q) y^\beta\right)\right) \tag{8.3.4}$$

where $\widetilde{R^*}_{\alpha,gen}(t)$ are the polynomials associated to the Log compatible family $\{r_\alpha^*\}_{\alpha\in\mathbb{N}^I}$. Notice that $\widetilde{R^*}_{\beta,gen}(t) = 0$ if $\beta \notin (\mathbb{N}^I)^*$. In particular, if σ is such that $\mathcal{H}^*_{\sigma,\beta} = \{\beta\}$, we have that

$$\widetilde{R}_{\beta,gen}(q) = \langle r_\beta^* \otimes \rho_\sigma, 1\rangle.$$

From Remark 5.7.7 and Theorem 7.1.3, we see that we have

$$\widetilde{R}_{\beta,gen}(t) = a_{Q,\beta}(t) = \mathbb{H}_\beta(0, \sqrt{t}) \tag{8.3.5}$$

if $\beta \in (\mathbb{N}^I)^*$.
Eq.(8.3.4) gives therefore another way to show Identity (8.2.7).

8.4 Computations

8.4.1 Irreducibility for semisimple split characters

Consider the case of $g = 0$ and $k = 2$. Consider a split F-stable Levi subgroup $L \subseteq \mathrm{GL}_n$ and $\gamma : L^F \to \mathbb{C}^*$. Let $\mathcal{X}$ be the couple of Harisha Chandra characters $\mathcal{X} = (R_L^G(\gamma), R_L^G(\gamma^{-1}))$. Notice that $R_L^G(\gamma^{-1})$ is the dual of $R_L^G(\gamma)$ and therefore we have

$$\langle R_L^G(\gamma) \otimes R_L^G(\gamma^{-1}), 1\rangle = \langle R_L^G(\gamma), R_L^G(\gamma)\rangle.$$

Using Theorem 4.5.2 we give an alternative proof of the classical result that

$$\langle R_L^G(\gamma), R_L^G(\gamma)\rangle = 1 \tag{8.4.1}$$

if and only if $\gamma_i \neq \gamma_j$ for all $i \neq j$. Notice that the Identity (8.4.1) holds if and only if the character $R_L^G(\gamma)$ is irreducible.
Let $L = \mathrm{GL}_{n_0} \times \cdots \times \mathrm{GL}_{n_l}$ and $g = 0$. The associated quiver Q is thus the following type A quiver.

$$\circ^{[1,l]} \longrightarrow \circ^{[1,l-1]} \longrightarrow \dots \longrightarrow \circ^{[1,1]} \longrightarrow \circ^{0} \longleftarrow \circ^{[2,1]} \longleftarrow \circ^{[2,2]} \longleftarrow \dots \longleftarrow \circ^{[2,l]}$$

The associated dimension vector $\alpha_{\mathcal{X}}$ is

$$\alpha_{\mathcal{X}} = (n_l, n_l + n_{l-1}, \dots, n_l + \cdots + n_1, n, n_1 + \cdots + n_l, \dots, n_{l-1} + n_l, n_l)$$

and the element $\sigma_{\mathcal{X}}$ is equal to

$$\sigma_{\mathcal{X}} = (\gamma_l\gamma_{l-1}^{-1}, \gamma_{l-1}\gamma_{l-2}^{-1}, \dots, \gamma_1\gamma_0^{-1}, 1, \gamma_0\gamma_1^{-1}, \dots, \gamma_{l-1}^{-1}\gamma_{l-2}, \gamma_l^{-1}\gamma_{l-1}).$$

As Q is a Dynkin quiver of type A, the subset $\Phi^+(Q)$ has an explicit description and each root $\beta \in \Phi^+(Q)$ is real, i.e $a_{Q,\beta}(t) = 1$ (see Proposition 6.1.8). For $j = 0, \dots, l$ and $h = 0, \dots, l$ define the dimension vector $\beta_{j,h}$ as

$$(\beta_{j,h})_i := \begin{cases} 0 \text{ if } i = [1, a] \text{ with } a > j \text{ or } i = [2, b] \text{ with } b > h \\ 1 \text{ otherwise} \end{cases}.$$

The set $\Phi^+(Q) \cap (\mathbb{N}^I)^*$ is given by $\{\beta_{j,h}\}_{j,h=0,\dots,l}$. Let us denote by $M_{j,h}$ the absolutely indecomposable representation of dimension vector $\beta_{j,h}$ over $\mathbb{F}_q$. Notice that $\sigma_{\mathcal{X}}^{\beta_{j,h}} = \gamma_j\gamma_h^{-1}$ and so we see that $\sigma_{\mathcal{X}}^{\beta_{i,i}} = 1$ for every $i = 0, \dots, l$. The representation

$$M = \bigoplus_{j=0}^{l} M_{j,j}^{\oplus n_j}$$

is thus of level at most $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}$ and the dimension vector of M is equal to α. If $\gamma_j = \gamma_h$ for $j \neq h$, we have $\beta_{j,h}, \beta_{h,j} \in \mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}$. The representation

$$N = \left(\bigoplus_{m \neq 0,j,h} M_{m,m}^{\oplus n_m} \right) \oplus M_{j,j}^{n_j-1} \oplus M_{h,h}^{n_h-1} \oplus M_{0,0}^{n_0-1} \oplus M_{j,h} \oplus M_{h,j}$$

is therefore of level at most $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}$ and of dimension vector $\dim N = \alpha_{\mathcal{X}}$. We deduce that $M_{Q,\alpha_{\mathcal{X}},\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}}(t) = 1$ if and only if $\gamma_j \neq \gamma_h$ for every $j \neq h$.

8.4.2 Explicit computation for $n = 2$

Let us look at the case where $\mathcal{X}$ is a k-tuple of semisimple split characters of $\mathrm{GL}_2(\mathbb{F}_q)$ with $\mathcal{X} = (R_T^G(\gamma_1), \dots, R_T^G(\gamma_k))$ where $T \subseteq \mathrm{GL}_2$ is the (split) maximal torus of diagonal matrices. Each character γ_i is thus of the form

$$\gamma_i = (\delta_i, \beta_i) : \mathbb{F}_q^* \times \mathbb{F}_q^* \to \mathbb{C}^*$$

with $\delta_i, \beta_i : \mathbb{F}_q^* \to \mathbb{C}^*$.

We fix $g = 0$ in the following. The associated quiver $Q = (I, \Omega)$ has thus a central vertex 0 and k other vertices $[1, 1], \dots, [k, 1]$. We will denote the vertex $[i, 1]$ simply by i for each $i = 1, \dots, k$.

The associated dimension vector α is given by $\alpha_0 = 2$ and $\alpha_i = 1$ for each $i = 1, \dots, k$. The quiver Q and the dimension vector $\alpha_{\mathcal{X}}$ for $k = 4$ are depicted below.

$$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow & & \\ 1 & \longrightarrow & 2 & \longleftarrow & 1 \\ & & \uparrow & & \\ & & 1 & & \end{array}$$

For the sake of simplicity, we assume that $\delta_i^{-1} = \beta_i$ and $\beta_i^2 \neq 1$, for each $i = 1, \dots, k$. The element $\sigma_{\mathcal{X}}$ associated to $\mathcal{X} = (R_T^G(\gamma_i))_{i=1}^k$ is thus given by $(\sigma_{\mathcal{X}})_0 = \prod_{i=1}^k \delta_i$ and $(\sigma_{\mathcal{X}})_i = \delta_i^{-2}$. Notice that $\sigma_{\mathcal{X}}^{\alpha_{\mathcal{X}}} = 1$.
We will explicitly verify that the following equality holds:

$$\left\langle \bigotimes_{i=1}^k R_T^G(\gamma_i), 1 \right\rangle = \mathrm{Coeff}_{\alpha_{\mathcal{X}}}\left(\mathrm{Exp}\left(\sum_{\eta \in \mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}} a_{Q,\eta}(q) y^{\eta}\right)\right) = M_{Q,\alpha_{\mathcal{X}},\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}}}(q) \tag{8.4.2}$$

Notice that if $\eta \leq \alpha$ and $\eta \in (\mathbb{N}^I)^*$ then either $(\eta)_0 = 1$ or $(\eta)_0 = 2$. For an element $\eta \in (\mathbb{N}^I)^*$ such that $\eta_0 = 1$, we have $a_{Q,\eta}(t) = 1$. An element $\eta \in (\mathbb{N}^I)^*$ such that $\eta_0 = 1$ is identified by the subset $A_\eta \subseteq \{1, \dots, k\}$ defined as $A_\eta \coloneqq \{1 \leq i \leq k \ \text{ s.t. } \ \eta_i = 1\}$.
For such an η, we have thus

$$\sigma_{\mathcal{X}}^{\eta} = \left(\prod_{i=1}^k \delta_i\right) \prod_{j \in A_\eta} \delta_j^{-2} = \prod_{j \in A_\eta} \delta_j^{-1} \prod_{h \in A_\eta^c} \delta_h.$$

For $\eta_1, \dots \eta_r \in (\mathbb{N}^I)^*$ and $m_1, \dots, m_r \in \mathbb{N}^*$ such that $m_1\eta_1 + \cdots + m_r\eta_r = \alpha$ we have that either $r = 1$, $m = 1$ and $\eta = \alpha$, either $r = 2$, $m_1 = m_2 = 1$ and $\eta_1 + \eta_2 = \alpha$ with $\eta_1 \neq \eta_2$ and $(\eta_1)_0 = (\eta_2)_0 = 1$. The right hand side of Equation (8.4.2) is thus equal to:

$$\mathrm{Coeff}_{\alpha_{\mathcal{X}}}\left(\mathrm{Exp}\left(\sum_{\eta \in \mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha}} a_{Q,\eta}(q) y^{\eta}\right)\right) = a_{Q,\alpha}(q) + \frac{1}{2} \sum_{\substack{\eta \in \mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}} \\ \text{s.t } \eta_0 = 1}} a_{Q,\eta}(q) a_{Q,\alpha-\eta}(q) = a_{Q,\alpha_{\mathcal{X}}}(q) + \frac{|\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}} - \{\alpha_{\mathcal{X}}\}}{2} \tag{8.4.3}$$

Notice that the cardinality $|\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha} - \{\alpha\}|$ is even as the set $\mathcal{H}^*_{\sigma_{\mathcal{X}},\alpha_{\mathcal{X}}} - \{\alpha\}$ admits the involution without fixed points which sends η to $\alpha_{\mathcal{X}} - \eta$.
The left hand side of Equation (8.4.2) can be computed explicitly using the character table of $\mathrm{GL}_2(\mathbb{F}_q)$, which can be found for example on [28, Page 194].
We have indeed four types for the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$.

1. We say that g is of type ω_1 (and we write $g \sim \omega_1$) if g is of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ for $\lambda \in \mathbb{F}_q^*$

2. We say that g is of type ω_2 (and we write $g \sim \omega_2$) if g is conjugated to a Jordan block $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ for $\lambda \in \mathbb{F}_q^*$. The centralizer of such a g has cardinality $q(q-1)$

3. We say that g is of type ω_3 (and we write $g \sim \omega_3$) if g is conjugated to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $\lambda \neq \mu \in \mathbb{F}_q^*$. The centralizer of such a g has cardinality $(q-1)^2$

4. We say that g is of type ω_4 (and we write $g \sim \omega_4$) if g is conjugated to a matrix of the form $\begin{pmatrix} 0 & -1 \\ xx^q & x+x^q \end{pmatrix}$ with $x \neq x^q \in \mathbb{F}_{q^2}^*$. The centralizer of such a g has cardinality q^2-1

For a semisimple split character of the form $R_T^G(\gamma)$ and $g \in \mathrm{GL}_2(\mathbb{F}_q)$, the value $R_T^G(\gamma)(g)$ depends on the type of g in the following way:

1. If $g \sim \omega_1$, then $R_T^G(\gamma)(g) = (q+1)\gamma(\lambda)$
2. If $g \sim \omega_2$, then $R_T^G(\gamma)(g) = \gamma(\lambda)$
3. If $g \sim \omega_3$ and g is conjugated to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, then $R_T^G(\gamma)(g) = \gamma(\lambda,\mu) + \gamma(\mu,\lambda)$
4. If $g \sim \omega_4$, then $R_T^G(\gamma)(g) = 0$.

To compute the left hand side of Equation (8.4.2), we will split the sum over the types of the conjugacy classes:

$$\left\langle \bigotimes_{i=1}^k R_T^G(\gamma_i), 1 \right\rangle = \frac{1}{|\mathrm{GL}_2(\mathbb{F}_q)|} \sum_{g \in \mathrm{GL}_2(\mathbb{F}_q)} \prod_{i=1}^k R_T^G(\gamma_i)(g) = \frac{1}{|\mathrm{GL}_2(\mathbb{F}_q)|} \sum_{\omega \in \mathbb{T}_2} \sum_{g \sim \omega} \prod_{i=1}^k R_T^G(\gamma_i)(g).$$

We see that

$$\frac{1}{|\mathrm{GL}_2(\mathbb{F}_q)|} \sum_{g \sim \omega_1} R_T^G(\gamma_i)(g) = \frac{(q+1)^k(q-1)}{(q-1)^2 q(q+1)} = \frac{(q+1)^{k-1}}{q(q-1)}$$

and

$$\frac{1}{|\mathrm{GL}_2(\mathbb{F}_q)|} \sum_{g \sim \omega_2} R_T^G(\gamma_i)(g) = \frac{(q-1)}{q(q-1)} = \frac{1}{q}.$$

In a similar way,

$$\frac{1}{|\mathrm{GL}_2(\mathbb{F}_q)|} \sum_{g \sim \omega_3} R_T^G(\gamma_i)(g) = \frac{1}{2(q-1)^2} \sum_{\substack{\lambda,\mu \in \mathbb{F}_q^* \\ \lambda \neq \mu}} \prod_{i=1}^k (\delta_i(\lambda\mu^{-1}) + \delta_i^{-1}(\lambda\mu^{-1})) =$$

$$\frac{1}{2(q-1)^2} \sum_{\lambda,\mu \in \mathbb{F}_q^*} \prod_{i=1}^k (\delta_i(\lambda\mu^{-1}) + \delta_i^{-1}(\lambda\mu^{-1})) - \frac{2^k(q-1)}{2(q-1)^2}.$$

As the homomorphism $\mathbb{F}_q^* \times \mathbb{F}_q^* \to \mathbb{F}_q^*$ which sends (λ, μ) to $\lambda\mu^{-1}$ is surjective, we can rewrite the sum above as:

$$-\frac{2^{k-1}}{(q-1)} + \frac{(q-1)}{2(q-1)^2} \sum_{\epsilon \in \mathbb{F}_q^*} \sum_{B \subseteq \{1,\dots,k\}} \prod_{i \in B} \delta_i(\epsilon) \prod_{i \in B^c} \delta_i^{-1}(\epsilon) =$$

$$-\frac{2^{k-1}}{(q-1)} + \frac{1}{2(q-1)} \sum_{\substack{\eta \in (\mathbb{N}^I)^* \\ \eta_0 = 1}} \sum_{\epsilon \in \mathbb{F}_q^*} \sigma_\gamma^\eta(\epsilon) = \frac{|\mathcal{H}^*_{\sigma_\gamma, \alpha_{\mathcal{X}}} - \{\alpha\}|}{2} - \frac{2^{k-1}}{(q-1)}.$$

and so

$$\left\langle \bigotimes_{i=1}^k R_T^G(\gamma_i), 1 \right\rangle = \frac{(q+1)^{k-1}}{q(q-1)} + \frac{1}{q} + \frac{|\mathcal{H}^*_{\sigma_\gamma, \alpha} - \{\alpha\}|}{2} - \frac{2^{k-1}}{(q-1)} \tag{8.4.4}$$

$$= \frac{(q+1)^{k-1} + (q-1) - 2^{k-1}q}{q(q-1)} + \frac{|\mathcal{H}_{\sigma_\gamma, \alpha} - \{\alpha\}|}{2}. \tag{8.4.5}$$

Recall that the Identity (8.2.8) gives an equality $a_{Q,\alpha_{\mathcal{X}}}(q) = \left\langle \bigotimes_{i=1}^k R_T^G(\mu_i), 1 \right\rangle$ for a generic k-tuple $(R_T^G(\mu_i))_{i=1}^k$. The multiplicity $\left\langle \bigotimes_{i=1}^k R_T^G(\mu_i), 1 \right\rangle$ can be computed in the same way as Identity (8.4.4) above and gives the identity:

$$a_{Q,\alpha}(q) = \frac{(q+1)^{k-1} + (q-1) - 2^{k-1}q}{q(q-1)}. \tag{8.4.6}$$

From Identity (8.4.3) and Identity (8.4.5), we deduce Identity (8.4.2).

9 Cohomology of non-generic character stacks for Riemann surfaces

In this chapter we study cohomology of character stacks for punctured Riemann surfaces $\mathcal{M}_{\mathcal{C}}$, when the conjugacy classes of the k-tuple $\mathcal{C}$ are semisimple. If the k-tuple $\mathcal{C}$ is *generic*, the cohomology admits an almost complete description due to the results of the articles [45],[72], which we review in Section §9.1.

Our main result is a formula for the E-series $E(\mathcal{M}_{\mathcal{C}}, q)$ for any k-tuple $\mathcal{C}$, not necessarily generic, and a conjecture for the mixed Poincaré series $H_c(\mathcal{M}_{\mathcal{C}}, q, t)$. The proof of this formula is obtained using the results of Section §5.8.

More precisely, in section §9.2, we show that the family of class functions associated to the multiplicative moment map for a star-shaped quiver over $\mathbb{F}_q$ is dual Log compatible.

The proof of this result is obtained by reducing the statement to the case of the Kronecker quiver through some convolution arguments and it is the main technical point of the chapter.

In section §9.3, we show how to apply this result to express the E-series $E(\mathcal{M}_{\mathcal{C}}, q)$ in terms of the E-series for generic character stacks and we propose our conjecture for the mixed Poincaré series.

Finally, in section §9.4, we verify that this conjecture holds in the case of $\Sigma = \mathbb{P}^1$, $k = 4$ and a certain family of non-generic k-tuples, by giving an explicit geometric description of the corresponding character stacks.

9.1 Generic character stacks

Let $\mathcal{C}$ be a k-tuple of semisimple conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$. Let $Q = (I, \Omega)$ be the associated quiver and $\gamma_{\mathcal{C}} \in (\mathbb{C}^*)^I, \alpha_{\mathcal{C}} \in (\mathbb{N}^I)^*$ the associated parameters, introduced in §7.4. With similar notations to §5.8.1, we put

$$\mathcal{H}^*_{\gamma_{\mathcal{C}}, \alpha_{\mathcal{C}}} = \{\delta \in (\mathbb{N}^I)^* \mid \gamma_{\mathcal{C}}^{\delta} = 1 \text{ and } \delta \leq \alpha_{\mathcal{C}}\}.$$

In [45], it is given the following definition of a generic $\mathcal{C}$.

Definition 9.1.1. The k-tuple $\mathcal{C}$ is *generic* if given a subspace W of $\mathbb{C}^n$ which is stabilized by some $X_i \in \mathcal{C}_i$, for each $i = 1, \dots, k$, such that

$$\prod_{i=1}^{k} \det(X_i|_W) = 1$$

then either $W = \{0\}$ or $W = \mathbb{C}^n$.

Example 9.1.2. Let $k = 1$ and $\mathcal{C} = \{e^{\frac{2\pi i d}{n}}\}$. In this case, the stack $\mathcal{M}_{\mathcal{C}}$ is denoted by $\mathcal{M}_{n,d}$. The conjugacy class $\mathcal{C}$ is generic if and only if $(n, d) = 1$. E-series for generic character stacks $\mathcal{M}_{n,d}$ were computed in [44].

Remark 9.1.3. For any $\beta \in (\mathbb{N}^I)^*$, there exists a generic k-tuple $\mathcal{C}'$ with associated dimension vector $\alpha_{\mathcal{C}'} = \beta$ (see for example [45, Lemma 2.1.2]).

If $\mathcal{C}$ is generic, the geometry of the character stack $\mathcal{M}_{\mathcal{C}}$ is particularly well behaved. In particular, the variety $X_{\mathcal{C}}$ is smooth and the action of PGL_n on $X_{\mathcal{C}}$ is schematically free, see [45, Theorem 2.15]. From Lemma 3.3.1, we deduce that we have $[X_{\mathcal{C}}/\mathrm{PGL}_n(\mathbb{C})] = M_{\mathcal{C}}$ and the variety $M_{\mathcal{C}}$ is smooth.
In addition, from Lemma 3.4.8, the canonical morphism $\mathcal{M}_{\mathcal{C}} \to M_{\mathcal{C}}$ is a $\mathbb{G}_m$-gerbe and we have an equality

$$H_c(\mathcal{M}_{\mathcal{C}}, q, t) = \frac{H_c(M_{\mathcal{C}}, q, t)}{qt^2 - 1}. \tag{9.1.1}$$

In particular, the cohomology of $\mathcal{M}_{\mathcal{C}}$ is determined from that of the smooth variety $M_{\mathcal{C}}$.

We have the following Lemma for generic k-tuples, whose analogous result for k-tuples of Harisha-Chandra characters is Lemma 8.2.4.

Lemma 9.1.4. *If $\mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}} = \{\alpha_{\mathcal{C}}\}$ the k-tuple $\mathcal{C}$ is generic. On the other side, if $\mathcal{C}$ is generic, there are no $\delta, \epsilon \in \mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}} \setminus \{\alpha_{\mathcal{C}}\}$ such that $\delta + \epsilon = \alpha_{\mathcal{C}}$.*

Proof. Suppose that for a k-tuple $\mathcal{C}$ there exists a proper subspace $0 \subset W \subset \mathbb{C}^n$ and $X_1 \in \mathcal{C}_1, \dots, X_k \in \mathcal{C}_k$ such that $X_i(W) \subseteq W$ for each i and

$$\prod_{i=1}^k \det(X_i|_W) = 1.$$

For $i = 1 \dots, k$ and $j = 0, \dots, s_i$, put $V_{\gamma_{i,j}} = \mathrm{Ker}(X_i - \gamma_{i,j} I_n)$ and $W_{\gamma_{i,j}} = W \cap V_{\gamma_{i,j}}$. Notice that, for each i, we have

$$W = \bigoplus_{j=0}^{s_i} W_{\gamma_{i,j}}$$

and

$$\det(X_i|_W) = \prod_{j=0}^{s_i} \gamma_{i,j}^{\dim W_{\gamma_{i,j}}}.$$

Consider now the dimension vector $\beta \in (\mathbb{N}^I)^*$ defined as

$$\beta_{[i,j]} = \sum_{h=j}^{s_i} \dim(W_{\gamma_{i,j}}).$$

Notice that $\beta < \alpha_{\mathcal{C}}$. Moreover, we have

$$\gamma_{\mathcal{C}}^{\beta} = \prod_{i=1}^k \gamma_{i,0}^{-\beta_0} \prod_{j=1}^{s_i} (\gamma_{i,j}^{-1}\gamma_{i,j-1})^{\beta_{[i,j]}} = \prod_{i=1}^k \gamma_{i,0}^{-\sum_{h=0}^{s_i} \dim W_{\gamma_{i,j}}} \prod_{j=1}^{s_i} \gamma_{i,j}^{-\sum_{h=j}^{s_i} \dim W_{\gamma_{i,j}}} \gamma_{i,j-1}^{\sum_{h=j}^{s_i} \dim W_{\gamma_{i,j}}} = \tag{9.1.2}$$

$$= \prod_{i=1}^{k} \prod_{j=0}^{s_i} \gamma_{i,j}^{-\dim W_{\gamma_{i,j}}} = \prod_{i=1}^{k} \det(X_i|_W)^{-1} = 1. \tag{9.1.3}$$

Conversely, suppose that there exists $\beta \in \mathcal{H}^*_{\gamma_\mathcal{C},\alpha_\mathcal{C}} \setminus \{\alpha_\mathcal{C}\}$ such that $\epsilon := \alpha_\mathcal{C} - \beta$ belongs to $\mathcal{H}^*_{\gamma_\mathcal{C},\alpha_\mathcal{C}} \setminus \{\alpha_\mathcal{C}\}$ too. Notice that, since $\epsilon \in (\mathbb{N}^I)^*$, for each j, h, we have

$$\beta_{[h,j]} - \beta_{[h,j+1]} \leq (\alpha_\mathcal{C})_{[h,j]} - (\alpha_\mathcal{C})_{[h,j+1]} = m_{h,j},$$

where $m_{h,j}$ is the multiplicity of the eigenvalue $\gamma_{h,j}$ in the orbit $\mathcal{C}_j$.
Put $m = \beta_0$ and let $W = \mathbb{C}^m \subseteq \mathbb{C}^n$ be the span of the first m vectors of the canonical basis. Notice that $m < n$, since $\epsilon \in (\mathbb{N}^I)^*$.
For each $i = 1, \ldots, k$, there exists a diagonal matrix $X_i \in \mathcal{C}_i$ such that its first m diagonal entries are given by $\beta_{[i,s_i]}$ times the element γ_{i,s_i}, then $\beta_{[i,s_i-1]} - \beta_{[i,s_i]}$ times the element γ_{i,s_i-1} and so on. Notice that W is X_i-stable for each $i = 1, \ldots, k$ and, moreover,

$$\det(X_i|_W) = \gamma_\mathcal{C}^\beta = 1,$$

from which we deduce that $\mathcal{C}$ is not generic.

□

Remark 9.1.5. It is not true that for a generic k-tuple $\mathcal{C}$ we have $\mathcal{H}^*_{\gamma_\mathcal{C},\alpha_\mathcal{C}} = \{\alpha_\mathcal{C}\}$. Consider for instance the case where $k = 3, n = 2$ and $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ where each $\mathcal{C}_i$ is the conjugacy class of the following matrix X_i:

$$X_1 = \begin{pmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, X_2 = \begin{pmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, X_3 = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}.$$

The associated quiver Q and dimension vector $\alpha_\mathcal{C}$ are:

$$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow & & \\ 1 & \longrightarrow & 2 & \longleftarrow & 1 \end{array}$$

The associated parameter $\gamma_\mathcal{C}$ is given by

$$\begin{array}{ccccc} & & 2 & & \\ & & \downarrow & & \\ 2 & \longrightarrow & -1 & \longleftarrow & \frac{1}{4} \end{array}$$

Notice in particular that $\mathcal{H}^*_{\gamma_\mathcal{C},\alpha_\mathcal{C}} = \{\alpha_\mathcal{C}, \beta\}$, where $\beta_0 = 2$ and $\beta_{[i,1]} = 0$ for each $i = 1, 2, 3$. By Lemma 9.1.4 we deduce that $\mathcal{C}$ is generic, while $\mathcal{H}^*_{\gamma_\mathcal{C},\alpha_\mathcal{C}} \neq \{\alpha_\mathcal{C}\}$.

Hausel, Letellier, Rodriguez-Villegas [45, Theorem 5.2.3] showed the following result:

Theorem 9.1.6. *For any generic k-tuple $\mathcal{C}$ of semisimple conjugacy classes, we have:*

$$\frac{E(\mathcal{M}_{\mathcal{C}},q)}{q^{(\alpha_{\mathcal{C}},\alpha_{\mathcal{C}})}}=\frac{q\mathbb{H}_{\alpha_{\mathcal{C}}}\left(\sqrt{q},\frac{1}{\sqrt{q}}\right)}{q-1}. \tag{9.1.4}$$

Remark 9.1.7. The quantity $(\alpha_{\mathcal{C}},\alpha_{\mathcal{C}})$ can be expressed in terms of the multiplicities of the eigenvalues of $\mathcal{C}_1,\dots,\mathcal{C}_k$ as follows. Notice indeed that $\boldsymbol{\mu}_{\alpha_{\mathcal{C}}}$ is the multipartition given by the multiplicities of the eigenvalues of $\mathcal{C}_1,\dots,\mathcal{C}_k$ and we have

$$(\alpha_{\mathcal{C}},\alpha_{\mathcal{C}})=n^2(2g-2+k)-\sum_{i,j}(\mu^j_{\alpha_{\mathcal{C}}})^2_i.$$

In [45, Theorem 2.15] it is also shown that we have

$$\dim(M_{\mathcal{C}})=2(\alpha_{\mathcal{C}},\alpha_{\mathcal{C}})+2. \tag{9.1.5}$$

In the same paper, the authors [45, Conjecture 1.2.1] proposed the following conjectural identity for the mixed Poincaré series of the character stack $\mathcal{M}_{\mathcal{C}}$, when $\mathcal{C}$ is generic, naturally deforming eq.(9.1.4):

Conjecture 9.1.8. *For any generic k-tuple $\mathcal{C}$ of semisimple conjugacy classes, we have*

$$\frac{H_c(\mathcal{M}_{\mathcal{C}},q,t)}{(qt^2)^{(\alpha_{\mathcal{C}},\alpha_{\mathcal{C}})}}=\frac{(qt^2)\mathbb{H}_{\alpha_{\mathcal{C}}}\left(t\sqrt{q},\frac{1}{\sqrt{q}}\right)}{qt^2-1}. \tag{9.1.6}$$

The specialization at $q=1$ of Conjecture 9.1.8, i.e the Poincaré series of generic character stacks, was verified by Mellit [72] by counting rational points over finite fields of the corresponding moduli spaces of parabolic Higgs bundles .

9.2 Dual log compatibility for multiplicative moment map

In this chapter, we show how to relate the results about dual Log compatible families of §5.8.2 to the study of multiplicative quiver stacks for star-shaped quivers.

Consider now the construction of §7.2 in the case in which $K=\mathbb{F}_q$. Fix $\beta\in\mathbb{N}^I$, consider the multiplicative moment map

$$\Phi^*_\beta : R(\overline{Q},\beta)^{\circ,*}\to \mathrm{GL}_\beta$$

over $\mathbb{F}_q$. We denote by $m_\beta:\mathrm{GL}_\beta(\mathbb{F}_q)\to\mathbb{C}$ the class function defined by

$$m_\beta(g):=\frac{|(\Phi^*_\beta)^{-1}(g)(\mathbb{F}_q)|}{q^{(\beta,\beta)}}.$$

Notice that, for $\sigma \in (\mathbb{F}_q^*)^I$, we have:

$$\langle m_\beta * 1_\sigma, 1_e \rangle = \frac{\#\mathcal{M}^*_{\sigma,\beta}(\mathbb{F}_q)}{q^{(\beta,\beta)}}.$$

For the family of class functions $\{m_\alpha\}_{\alpha \in \mathbb{N}^I}$ we have the following Theorem:

Theorem 9.2.1. *The family $\{m_\alpha\}_{a \in \mathbb{N}^I}$ is dual Log compatible*

The proof of Theorem 9.2.1 is the most technical part of the chapter and is going to be given through several steps.

We start by showing Proposition 9.2.1 in the case where $Q = (I, \Omega)$ is the star-shaped quiver with two vertices $I = \{0, 1\}$ and one arrow $a : 1 \to 0$ between them (i.e $g = 0$ and $k = 1$). This is usually called the Kronecker quiver, see below.

$$0 \xleftarrow{\ a\ } 1$$

A dimension vector α is thus a couple $\alpha = (\alpha_0, \alpha_1) \in \mathbb{N}^2$ and the function $m_\alpha(g_0, g_1)$ for a couple $(g_0, g_1) \in \mathrm{GL}_{\alpha_0}(\mathbb{F}_q) \times \mathrm{GL}_{\alpha_1}(\mathbb{F}_q)$ is given by

$$m_\alpha(g_0, g_1) = \frac{\#\{f \in \mathrm{Hom}^{inj}(\mathbb{F}_q^{\alpha_1}, \mathbb{F}_q^{\alpha_0}), f^* \in \mathrm{Hom}(\mathbb{F}_q^{\alpha_0}, \mathbb{F}_q^{\alpha_1}) \mid 1 + ff^* = g_0,\ 1 + f^*f = g_1^{-1}\}}{q^{\alpha_0\alpha_1 - \alpha_0^2 - \alpha_1^2}}.$$

Remark 9.2.2. Notice that given $f \in \mathrm{Hom}^{inj}(\mathbb{F}_q^{\alpha_1}, \mathbb{F}_q^{\alpha_0})$ and $f^* \in \mathrm{Hom}(\mathbb{F}_q^{\alpha_0}, \mathbb{F}_q^{\alpha_1})$ such that $1 + ff^* \in \mathrm{GL}_{\alpha_0}(\mathbb{F}_q)$ then $1 + f^*f \in \mathrm{GL}_{\alpha_1}(\mathbb{F}_q)$. It is enough to check indeed that $1 + f^*f$ is injective. Given $x, y \in \mathbb{F}_q^{\alpha_1}$ such that $(1 + f^*f)(x) = (1 + f^*f)(y)$ we have indeed

$$f \circ (1 + f^*f)(x) = f \circ (1 + f^*f)(y)$$

and, given that $f \circ (1 + f^*f) = (1 + ff^*) \circ f$ and $1 + ff^*$ is invertible, we deduce that $f(x) = f(y)$ and so that $x = y$.

Lemma 9.2.3. *In the case where Q is the Kronecker quiver, the family $\{m_\alpha\}_{\alpha \in \mathbb{N}^I}$ is Dual log compatible.*

Proof. Notice that $m_\alpha \equiv 0$ if $\alpha \notin (\mathbb{N}^I)^*$. Fix then $\alpha \in (\mathbb{N}^I)^*$ and denote by $\alpha_2 = \alpha_0 - \alpha_1$. Fix an irreducible character $\chi = \chi_0 \boxtimes \chi_1 \in \mathrm{GL}_\alpha(\mathbb{F}_q)^\vee$ with $\chi_i \in \mathrm{GL}_{\alpha_i}(\mathbb{F}_q)^\vee$ for $i = 0, 1$. We have:

$$\langle m_\alpha, \chi \rangle = \frac{1}{|\mathrm{GL}_\alpha(\mathbb{F}_q)| q^{(\alpha,\alpha)}} \sum_{\substack{f \in \mathrm{Hom}^{inj}(\mathbb{F}_q^{\alpha_1}, \mathbb{F}_q^{\alpha_0}) \\ f^* \in \mathrm{Hom}(\mathbb{F}_q^{\alpha_0}, \mathbb{F}_q^{\alpha_1}) \\ \text{s.t } 1 + ff^* \in \mathrm{GL}_{\alpha_0}(\mathbb{F}_q)}} \chi_0(1 + ff^*)\chi_1((1 + f^*f)^{-1}).$$

Let $J_\alpha \in \mathrm{Hom}^{inj}(\mathbb{F}_q^{\alpha_1}, \mathbb{F}_q^{\alpha_0})$ be the block matrix given by the identity on the first α_1 rows and 0 everywhere else, i.e

$$J_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline & & & \mathbf{0} & & & \end{pmatrix}$$

Let P_α be the centralizer of J_α inside GL_α,, i.e

$$P_\alpha = \{g \in \mathrm{GL}_\alpha \mid gJ_\alpha g^{-1} = J_\alpha\}.$$

Notice that if $g = (g_0, g_1) \in P_\alpha$, then g_0 preserves $\mathrm{Im}(J_\alpha)$ and $g_1 = g_0|_{\mathrm{Im}(J_\alpha)}$. Denote by $\pi_0 : \mathrm{GL}_\alpha \to \mathrm{GL}_{\alpha_0}$ the canonical projection. We deduce that π_0 is an isomorphism from P_α to the image $P \coloneqq \pi_0(P_\alpha)$.

Notice that $P \subseteq \mathrm{GL}_{\alpha_0}$ is the parabolic subgroups given by the matrices which preserve the image of J_α.

We denote by $L \subseteq P$ its Levi factor given by $\mathrm{GL}_{\alpha_1} \times \mathrm{GL}_{\alpha_2}$ embedded block diagonally.

The action of $\mathrm{GL}_\alpha(\mathbb{F}_q)$ on $\mathrm{Hom}^{inj}(\mathbb{F}_q^{\alpha_1}, \mathbb{F}_q^{\alpha_0})$ is transitive and we can therefore identity the latter set with $\mathrm{GL}_\alpha(\mathbb{F}_q)/P_\alpha(\mathbb{F}_q)$ via the map which sends $(g_0, g_1)P_\alpha(\mathbb{F}_q) \to g_0 J_\alpha g_1^{-1}$. We can thus rewrite the sum above as :

$$\frac{1}{|\mathrm{GL}_\alpha(\mathbb{F}_q)|q^{(\alpha,\alpha)}} \sum_{(g_0,g_1)P_\alpha(\mathbb{F}_q)\in \mathrm{GL}_\alpha(\mathbb{F}_q)/P_\alpha(\mathbb{F}_q)} \sum_{\substack{f^*\in \mathrm{Hom}(\mathbb{F}_q^{\alpha_0}, \mathbb{F}_q^{\alpha_1}) \\ \text{s.t. } 1+g_0J_\alpha g_1^{-1}f^* \in \mathrm{GL}_{\alpha_0}(\mathbb{F}_q)}} \chi_0(1+g_0J_\alpha g_1^{-1}f^*)\chi_1((1+f^*g_0J_\alpha g_1^{-1})^{-1}). \tag{9.2.1}$$

For each $(g_0, g_1)P_\alpha(\mathbb{F}_q)$ we can rewrite the last term of eq.(9.2.1) as

$$\sum_{\substack{f^*\in \mathrm{Hom}(\mathbb{F}_q^{\alpha_0}, \mathbb{F}_q^{\alpha_1}) \\ \text{s.t. } 1+J_\alpha g_1^{-1}f^*g_0 \in \mathrm{GL}_{\alpha_0}(\mathbb{F}_q)}} \chi_0(g_0(1 + J_\alpha g_1^{-1}f^*g_0)g_0^{-1})\chi_1(g_1(1 + g_1^{-1}f^*g_0J_\alpha)^{-1}g_1^{-1}). \tag{9.2.2}$$

Notice that for any $(g_0, g_1) \in \mathrm{GL}_\alpha(\mathbb{F}_q)$, we have a bijection

$$\{f^* \in \mathrm{Hom}(\mathbb{F}_q^{\alpha_0}, \mathbb{F}_q^{\alpha_1}) \text{ s.t. } 1+J_\alpha g_1^{-1}f^*g_0 \in \mathrm{GL}_{\alpha_0}(\mathbb{F}_q)\} \leftrightarrow \{f^* \in \mathrm{Hom}(\mathbb{F}_q^{\alpha_0}, \mathbb{F}_q^{\alpha_1}) \text{ s.t. } 1+J_\alpha f^* \in \mathrm{GL}_{\alpha_0}(\mathbb{F}_q)\}$$

$$g_1 f^* g_0^{-1} \longleftarrow f^*.$$

As χ_0, χ_1 are class functions, from eq.(9.2.2), we can rewrite the sum in eq.(9.2.1) as

$$\frac{1}{|P_\alpha(\mathbb{F}_q)|q^{(\alpha,\alpha)}} \sum_{\substack{f^*\in \mathrm{Hom}(\mathbb{F}_q^{\alpha_0}, \mathbb{F}_q^{\alpha_1}) \\ \text{s.t. } 1+J_\alpha f^* \in \mathrm{GL}_{\alpha_0}(q)}} \chi_0(1 + J_\alpha f^*)\chi_1((1 + f^*J_\alpha)^{-1}). \tag{9.2.3}$$

Writing f^* as a block matrix $(A|B)$ with $A \in \mathrm{Mat}(\alpha_1, \mathbb{F}_q)$, we have

$$1 + J_\alpha f^* = \begin{pmatrix} 1+A & B \\ 0 & 1 \end{pmatrix}$$

and $1 + f^* J_\alpha = 1 + A$. Let $\alpha_2 = \alpha_0 - \alpha_1$. We can rewrite the sum of eq.(9.2.3) as

$$\frac{1}{|P_\alpha(\mathbb{F}_q)|q^{(\alpha,\alpha)}} \sum_{\substack{M \in \mathrm{GL}_{\alpha_1}(\mathbb{F}_q) \\ B \in \mathrm{Mat}(\alpha_2,\alpha_1,\mathbb{F}_q)}} \chi_0\left(\begin{pmatrix} M & B \\ 0 & 1 \end{pmatrix}\right) \chi_1(M^{-1}). \tag{9.2.4}$$

By eq.(3.1.1), the latter sum can be rewritten as

$$\frac{1}{|P(\mathbb{F}_q)|q^{(\alpha,\alpha)}} \sum_{\chi_2 \in \mathrm{GL}_{\alpha_2}(\mathbb{F}_q)^\vee} \sum_{h \in P(\mathbb{F}_q)} \chi_0(h) \,\mathrm{Infl}_L^P(\chi_1^* \boxtimes \chi_2)(h) \frac{\chi_2(1)}{|\mathrm{GL}_{\alpha_2}(\mathbb{F}_q)|} = \tag{9.2.5}$$

$$= \frac{1}{q^{(\alpha,\alpha)}} \sum_{\chi_2 \in \mathrm{GL}_{\alpha_2}(\mathbb{F}_q)^\vee} \langle \mathrm{Res}_{P(\mathbb{F}_q)}(\chi_0), \mathrm{Infl}_L^P(\chi_1 \boxtimes \chi_2)\rangle \frac{\chi_2(1)}{|\mathrm{GL}_{\alpha_2}(\mathbb{F}_q)|} = \tag{9.2.6}$$

$$= \frac{1}{q^{(\alpha,\alpha)}} \sum_{\chi_2 \in \mathrm{GL}_{\alpha_2}(\mathbb{F}_q)^\vee} \langle \chi_0, R_L^G(\chi_1 \boxtimes \chi_2)\rangle \frac{\chi_2(1)}{|\mathrm{GL}_{\alpha_2}(\mathbb{F}_q)|}. \tag{9.2.7}$$

We start by the case where the type of $\chi = \chi_0 \boxtimes \chi_1$ is $(1, \boldsymbol{\lambda})$, where $\boldsymbol{\lambda} = (\lambda^0, \lambda^1)$ with $\lambda^0 \in \mathcal{P}_{\alpha_0}, \lambda^1 \in \mathcal{P}_{\alpha_1}$. We have then

$$\chi_0 = (\gamma \circ \det) R_{\lambda^0}$$

and

$$\chi_1 = (\gamma \circ \det) R_{\lambda^1},$$

with $\gamma \in \mathrm{Hom}(\mathbb{F}_q^*, \mathbb{C}^*)$.
Let $\chi_2 = \epsilon_{\mathrm{GL}_{\alpha_2}} \epsilon_{L_2} R_{L_2}^{\mathrm{GL}_{\alpha_2}}(\theta_2 R_{\tilde{\varphi}_2})$ for a certain $\varphi_2 \in (W_{L_2})^F$. From Lemma 5.1.4,we have an equality

$$R_L^G(\chi_1 \boxtimes \chi_2) = R_{\mathrm{GL}_{\alpha_1} \times L_2}^G(((\gamma \circ \det) \times \theta_2)(R_{\lambda_1} \boxtimes R_{\tilde{\varphi}})).$$

Let L' be the connected centralizer of

$$(\gamma \circ \det) \times \theta_2 : \mathrm{GL}_{\alpha_1}(\mathbb{F}_q) \times L_2^F \to \mathbb{C}^*.$$

By Remark 5.5.5, the character $R_L^G(\chi_1 \boxtimes \chi_2)$ belongs to the vector space spanned by irreducible characters with semisimple part $(L', (\gamma \circ \det) \times \theta_2)$.
The multiplicity $\langle (\gamma \circ \det) R_{\lambda_0}, R_L^G((\gamma \circ \det) R_{\lambda_1} \boxtimes \chi_2)\rangle$ is therefore equal to 0 if L' is different from $\mathrm{GL}_{\alpha_0}(\mathbb{F}_q)$.
To compute the right hand side of eq.(9.2.6) we can thus restrict to the case when χ_2 is given by $(\gamma \circ \det) R_{\lambda^2}$ for $\lambda^2 \in \mathcal{P}_{\alpha_2}$. From Remark 5.1.5, the right hand side of eq. (9.2.6) is thus

equal to:

$$\frac{1}{q^{(\alpha,\alpha)}}\sum_{\lambda^2\in\mathcal{P}_{\alpha_2}}\langle(\gamma\circ\det)R_{\lambda^0},(\gamma\circ\det)R_L^G(R_{\lambda^1}\boxtimes R_{\lambda^2})\rangle\frac{(-1)^{\alpha_2}}{q^{\frac{\alpha_2(\alpha_2-1)}{2}-n(\lambda^2)}H_{\lambda^2}(q)}. \tag{9.2.8}$$

From eq.(5.2.9) , the sum in eq.(9.2.8) is equal to

$$\frac{1}{q^{(\alpha,\alpha)}}\sum_{\lambda^2\in\mathcal{P}_{\alpha_2}}\langle\chi_{\lambda^0},\mathrm{Ind}_{S_{\alpha_1}\times S_{\alpha_2}}^{S_{\alpha_0}}(\chi_{\lambda^1}\boxtimes\chi_{\lambda^2})\rangle\frac{(-1)^{\alpha_2}}{q^{\frac{\alpha_2(\alpha_2-1)}{2}-n(\lambda^2)}H_{\lambda^2}(q)}=\frac{1}{q^{(\alpha,\alpha)}}\sum_{\lambda^2\in\mathcal{P}_{\alpha_2}}c_{\lambda^1,\lambda^2}^{\lambda^0}\frac{(-1)^{\alpha_2}}{q^{\frac{\alpha_2(\alpha_2-1)}{2}-n(\lambda_2)}H_{\lambda^2}(q)} \tag{9.2.9}$$

For any couple of partitions (λ,μ) denote by $C_{\lambda,\mu}(t)\in\mathbb{Q}(q)$ the function defined as

$$C_{\lambda,\mu}(t)=\begin{cases}0 \text{ if } |\lambda|<|\mu| \\ \dfrac{1}{t^{|\lambda||\mu|-|\lambda|^2-|\mu|^2}}\displaystyle\sum_{\nu\in\mathcal{P}_{|\lambda|-|\mu|}}c_{\mu,\nu}^{\lambda}\frac{(-1)^{|\lambda|-|\mu|}}{t^{\frac{|\nu|(|\nu|-1)}{2}-n(\nu)}H_\nu(t)}.\end{cases} .$$

The reasoning above shows that for any $\chi\in\mathrm{GL}_\alpha(\mathbb{F}_q)^\vee$ of type $(1,\boldsymbol{\lambda})$, there is an equality

$$\langle m_\alpha,\chi\rangle=C_{\lambda^0,\lambda^1}(q).$$

Let now $\delta\in\mathbb{N}^I$ and consider $\chi=\chi_0\boxtimes\chi_1\in\mathrm{GL}_\delta(\mathbb{F}_q)^\vee$ of type $\omega\in\mathbb{T}_\delta$, where

$$\omega=(d_1,\boldsymbol{\lambda}_1)\cdots(d_r,\boldsymbol{\lambda}_r),$$

where for $j=1,\dots,r$ we have $\boldsymbol{\lambda}_j=(\lambda_j^0,\lambda_j^1)\in\mathcal{P}^2$ and we denote by $\beta_j=|\boldsymbol{\lambda}_j|$. Consider the Levi subgroups $L_0=\prod_{j=1}^r(\mathrm{GL}_{(\beta_j)_0})_{d_j}$ and $L_1=\prod_{j=1}^r(\mathrm{GL}_{(\beta_j)_1})_{d_j}$.
There exist reduced characters $\theta^0:L_0^F\to\mathbb{C}^*$ and $\theta^1:L_1^F\to\mathbb{C}^*$ such that $\theta^0:L_0^F\to\mathbb{C}^*$ and $\theta^1:L_1^F\to\mathbb{C}^*$ are reduced and

$$\chi_0=R_{L_0}^G(\theta^0R_{\lambda_1^0}\boxtimes\cdots\boxtimes R_{\lambda_r^0})$$

and

$$\chi_1=R_{L_1}^{\mathrm{GL}_{\delta_1}}(\theta^1R_{\lambda_1^1}\boxtimes\cdots\boxtimes R_{\lambda_r^1})$$

and θ^0,θ^1 are associated to the same r-tuple $(\theta_1,\dots,\theta_r)\in\mathrm{Hom}(\mathbb{F}_{q^{d_1}}^*,\mathbb{C}^*)\times\cdots\times\mathrm{Hom}(\mathbb{F}_{q^{d_r}}^*,\mathbb{C}^*)$, via the correspondence of §5.4.1. We denote by $\overline{\boldsymbol{\lambda}_0},\overline{\boldsymbol{\lambda}_1}\in\mathcal{P}^r$ the multipartitions $\overline{\boldsymbol{\lambda}_0}=(\lambda_1^0,\dots,\lambda_r^0),\overline{\boldsymbol{\lambda}_1}=(\lambda_1^1,\dots,\lambda_r^1)$.
To verify the Dual log compatibility of the family $\{m_\alpha\}_{\alpha\in\mathbb{N}^I}$, it is enough to check that it holds:

$$\langle m_\delta,\chi\rangle H_\omega^\vee(q)=\prod_{j=1}^r C_{\lambda_j^0,\lambda_j^1}(q^{d_j})H_{(1,\boldsymbol{\lambda}_j)}^\vee(q^{d_j}) \tag{9.2.10}$$

Notice that, if $\delta \notin (\mathbb{N}^I)^*$, there must exist β_j such that $\beta_j \notin (\mathbb{N}^I)^*$. Eq.(9.2.10) therefore holds as both sides are equal to 0.

Assume then that $\delta \in (\mathbb{N}^I)^*$. From eq.(9.2.6), there is an equality

$$\langle m_\delta, \chi\rangle = \frac{1}{q^{(\delta,\delta)}} \sum_{\chi_2 \in \mathrm{GL}_{\delta_2}^\vee} \langle \chi_0, R_M^G(\chi_1 \boxtimes \chi_2)\rangle \frac{\chi_2(1)}{|\mathrm{GL}_{\delta_2}(q)|} \tag{9.2.11}$$

where $M = \mathrm{GL}_{\delta_1} \times \mathrm{GL}_{\delta_2} \subseteq G$. Let $\chi_2 = \epsilon_{\mathrm{GL}_{\delta_2}} \epsilon_{L_2} R_{L_2}^G(\theta^2 R_{\tilde{\varphi}_2})$, with $L_2 \subseteq \mathrm{GL}_{\delta_2}$ a Levi subgroup and $\theta^2 : L_2^F \to \mathbb{C}^*$ a reduced character. From Lemma 5.1.4, there is an equality

$$R_M^G(\chi_1 \boxtimes \chi_2) = R_{L_1 \times L_2}^G((\theta^1 \times \theta^2)(R_{\overline{\boldsymbol{\lambda}_1}} \boxtimes R_{\tilde{\varphi}_2})).$$

Let L' be the connected centralizer of $\theta^1 \times \theta^2 : L_1 \times L_2 \to \mathbb{C}^*$. As remarked above, the multiplicity $\langle \chi_0, R_M^G(\chi_1 \boxtimes \chi_2)\rangle = 0$ if the semisimple part $(L', \theta^1 \times \theta^2)$ is not $\mathrm{GL}_{\alpha_0}(\mathbb{F}_q)$-conjugated to (L_0, θ^0). From Remark 5.4.1, we deduce that, for $(L', \theta^1 \times \theta^2)$ to be $\mathrm{GL}_{\alpha_0}(\mathbb{F}_q)$-conjugated to (L^0, θ^0), we must have $|\lambda_j^0| \geq |\lambda_j^1|$, i.e $\beta_j \in (\mathbb{N}^I)^*$, for each $j = 1, \ldots, r$.

If there exists $j \in \{1, \ldots, r\}$ such that $\beta_j \notin (\mathbb{N}^I)^*$, eq.(9.2.10) is thus verified as both sides are equal to 0.

If $\beta_1, \ldots, \beta_r \in (\mathbb{N}^I)^*$, from Remark 5.4.1, we deduce that there exist a unique couple (L_2, θ^2), up to $\mathrm{GL}_{\delta_2}(\mathbb{F}_q)$-conjugacy, such that $(L', \theta^1 \times \theta^2)$ is G^F-conjugated to (L^0, θ^0). In particular, we can take L_2 to be

$$L_2 = \prod_{j=1}^r (\mathrm{GL}_{(\beta_j)_2})_{d_j}$$

and $\theta^2 : L_2^F \to \mathbb{C}^*$ the reduced character associated to the r-tuple $(\theta_1, \ldots, \theta_r)$. We have therefore

$$\langle m_\delta, \chi\rangle = \sum_{\substack{\overline{\boldsymbol{\lambda}_2} = (\lambda_1^2, \ldots, \lambda_r^2) \in \\ \mathcal{P}_{(\beta_1)_2} \times \cdots \times \mathcal{P}_{(\beta_r)_2}}} \frac{\langle R_{L_0}^G(\theta^0 R_{\overline{\boldsymbol{\lambda}_0}}), R_{L_1 \times L_2}^G((\theta^1 \times \theta^2) R_{\overline{\boldsymbol{\lambda}_1}} \boxtimes R_{\overline{\boldsymbol{\lambda}_2}})\rangle}{q^{(\delta,\delta)}} \frac{(-1)^{(\beta_1)_2 + \cdots (\beta_r)_2}}{q^{\frac{\delta_2(\delta_2-1)}{2} - \sum_{j=1}^r d_j n(\lambda_j^2)} \prod_{j=1}^r H_{\lambda_j^2}(q^{d_j})} \tag{9.2.12}$$

By Lemma 5.1.4 and Lemma 5.5.1, we have

$$\langle R_{L_0}^G(\theta^0 R_{\overline{\boldsymbol{\lambda}_0}}), R_{L_1 \times L_2}^G((\theta^1 \times \theta^2) R_{\overline{\boldsymbol{\lambda}_1}} \boxtimes R_{\overline{\boldsymbol{\lambda}_2}})\rangle = \prod_{j=1}^r \langle R_{\lambda_j^0}, R_{(\mathrm{GL}_{(\beta_j)_1})_{d_j} \times (\mathrm{GL}_{(\beta_j)_2})_{d_j}}^{(\mathrm{GL}_{(\beta_j)_0})_{d_j}} (R_{\lambda_j^1} \boxtimes R_{\lambda_j^2})\rangle_{(\mathrm{GL}_{(\beta_j)_0})_{d_j}(\mathbb{F}_q)}.$$

By Proposition 5.2.2 and 5.2.9, for each $j = 1, \ldots, r$, we deduce that we have an equality

$$\langle R_{\lambda_j^0}, R_{(\mathrm{GL}_{(\beta_j)_1})_{d_j} \times (\mathrm{GL}_{(\beta_j)_2})_{d_j}}^{(\mathrm{GL}_{(\beta_j)_0})_{d_j}} (R_{\lambda_j^1} \boxtimes R_{\lambda_j^2})\rangle_{(\mathrm{GL}_{(\beta_j)_0})_{d_j}(\mathbb{F}_q)} = \langle \chi_{\lambda_j^0}, \mathrm{Ind}_{S_{(\beta_j)_1}}^{S_{(\beta_j)_0}} \times S_{(\beta_j)_2}(\chi_{\lambda_j^1} \boxtimes \chi_{\lambda_j^2})\rangle_{S_{(\beta_j)_0}} = c_{\lambda_j^1, \lambda_j^2}^{\lambda_j^0}. \tag{9.2.13}$$

From eq.(9.2.13), we deduce that we have

$$\langle m_\delta, \chi\rangle = \frac{1}{q^{(\delta,\delta)+\frac{\delta_2(\delta_2-1)}{2}}}\prod_{j=1}^{r}\left(\sum_{\lambda_j^2\in\mathcal{P}_{(\beta_j)_2}} c^{\lambda_j^0}_{\lambda_j^1,\lambda_j^2}\frac{(-1)^{(\beta_j)_2}}{q^{-d_j n(\lambda_j^2)}H_{\lambda_j^2}(q^{d_j})}\right) \tag{9.2.14}$$

From eq.(9.2.14) above, we deduce that we have:

$$\frac{\langle m_\delta, \chi\rangle}{\prod_{j=1}^r C_{\lambda_j^0,\lambda_j^1}(q^{d_j})} = \frac{\prod_{j=1}^r q^{d_j(|\lambda_j^0||\lambda_j^1|-|\lambda_j^0|^2-|\lambda_j^1|^2+\frac{|\lambda_j^2|^2-|\lambda_j^2|}{2})}}{q^{(\delta,\delta)+\frac{\delta_2(\delta_2-1)}{2}}} \tag{9.2.15}$$

From the fact that $\delta_2^2=\delta_0^2+\delta_1^2-2\delta_0\delta_1$ and, for each $j=1,\dots,r$, $|\lambda_j^2|^2=|\lambda_j^0|^2+|\lambda_j^1|^2-2|\lambda_j^0||\lambda_j^1|$ and Identity (5.8.12), we have the following equality

$$\frac{\prod_{j=1}^r q^{d_j(|\lambda_j^0||\lambda_j^1|-|\lambda_j^0|^2-|\lambda_j^1|^2+\frac{|\lambda_j^2|^2-|\lambda_j^2|}{2})}}{q^{(\delta,\delta)+\frac{\delta_2(\delta_2-1)}{2}}} = \frac{q^{\sum_{j=1}^r d_j(-\frac{|\lambda_j^0|^2}{2}-\frac{|\lambda_j^1|^2}{2})}}{q^{-\frac{\delta_0^2}{2}-\frac{\delta_1^2}{2}}} = \frac{H_\omega^\vee(q)}{\prod_{j=1}^r H^\vee_{(1,\boldsymbol{\lambda}_j)}(q^{d_j})}.$$

From the Identity right above and eq.(9.2.15), we deduce therefore equality (9.2.10).

□

We now show how Lemma 9.2.3 implies Theorem 9.2.1.

Proof of Theorem 8.2.1. We proceed by induction on the cardinality of I.

Let $|I|=1$. The quiver Q has thus 1 vertex and g loops. The argument of Example 7.2.1 shows that in this case, for each $n\in\mathbb{N}$, we have an equality $m_n=f_n^g$, where $f_n^g:\mathrm{GL}_n(\mathbb{F}_q)\to\mathbb{C}$ is the function defined as

$$f_n^g(h)=\frac{\#\{(x_1,y_1,\dots,x_g,y_g)\in\mathrm{GL}_n(\mathbb{F}_q)^{2g}\mid\prod_{i=1}^g[x_i,y_i]=h\}}{q^{(n^2(g-1))}}$$

introduced in Remark 5.8.9. It was thereby shown that $\{f_n^g\}_{n\in\mathbb{N}}$ is a dual Log compatible family.

Assume now to have shown Proposition 9.2.1 for all star-shaped quivers with m vertices and k legs and fix a star-shaped quiver $Q=(I,\Omega)$ with $|I|=m+1$. We can assume that $s_k>1$. Let $\tilde{Q}=(\tilde{I},\tilde{\Omega})$ be the subquiver of Q, with set of vertices $\tilde{I}=I-\{[k,s_k]\}$ and as set of arrows the arrows of Q between elements of $\tilde{I}$.

For a dimension vector $\alpha\in\mathbb{N}^I$, we denote by $\tilde{\alpha}$ the element of $\mathbb{N}^{\tilde{I}}$ obtained by the natural projection $\mathbb{N}^I\to\mathbb{N}^{\tilde{I}}$ and we denote by π_α the natural projection $\pi_\alpha:\mathrm{GL}_\alpha(\mathbb{F}_q)\to\mathrm{GL}_{\tilde{\alpha}}(\mathbb{F}_q)$.

For $\alpha\in\mathbb{N}^I$, let $m_{\tilde{\alpha}}$ be the function associated to the star-shaped quiver $\tilde{Q}=(\tilde{I},\tilde{\Omega})$ and $\tilde{\alpha}$ and denote by $g_\alpha:\mathrm{GL}_\alpha(\mathbb{F}_q)\to\mathbb{C}$ the class function defined by

$$g_\alpha(h) = \begin{cases} \dfrac{m_{\tilde{\alpha}}(\pi_\alpha(h))}{q^{-\alpha^2_{[k,s_k]}}} \text{ if } h_{[k,s_k]} = 1, \\ 0 \text{ otherwise} \end{cases}$$

Notice that the function g_α can be equally rewritten as:

$$g_\alpha(h) = \frac{m_{\tilde{\alpha}}(\pi_\alpha(h))}{q^{-\alpha^2_{[k,s_k]}}} \sum_{\mathcal{X} \in \mathrm{GL}_{\alpha_{[k,s_k]}}(\mathbb{F}_q)^\vee} \mathcal{X}(h_{[k,s_k]}) \frac{\mathcal{X}(1)}{|\,\mathrm{GL}_{\alpha_{[k,s_k]}}(\mathbb{F}_q)|} \tag{9.2.16}$$

Using identity (9.2.16) and the dual Log compatibility of the functions $\{m_{\tilde{\alpha}}\}$, it is not difficult to verify that the family of functions $\{g_\alpha\}_{\alpha \in \mathbb{N}^I}$ is dual Log compatible. Indeed, for each $\chi \in \mathrm{GL}_\alpha(\mathbb{F}_q)$, write $\chi = \widetilde{\chi} \boxtimes \chi_k$, with $\tilde{\chi} \in \mathrm{GL}_{\tilde{\alpha}}(\mathbb{F}_q)^\vee$ and $\chi_k \in \mathrm{GL}^\vee_{\alpha_{[k,s_k]}}(\mathbb{F}_q)$. We have

$$\langle g_\alpha, \chi \rangle = \frac{1}{|\,\mathrm{GL}_\alpha(\mathbb{F}_q)|} \sum_{h \in \mathrm{GL}_\alpha(\mathbb{F}_q)} g_\alpha(h)\chi(h) = \tag{9.2.17}$$

$$\left(\frac{1}{|\,\mathrm{GL}_{\tilde{\alpha}}(\mathbb{F}_q)|} \sum_{\tilde{h} \in \mathrm{GL}_{\tilde{\alpha}}(\mathbb{F}_q)} m_{\tilde{\alpha}}(\tilde{h})\tilde{\chi}(\tilde{h}) \right) \left(\frac{q^{\alpha^2_{[k,s_k]}}}{|\,\mathrm{GL}_{\alpha_{[k,s_k]}}(\mathbb{F}_q)|} \sum_{\substack{\mathcal{X} \in \mathrm{GL}_{\alpha_{[k,s_k]}}(\mathbb{F}_q)^\vee \\ h_k \in \mathrm{GL}_{\alpha_{[k,s_k]}}}} \mathcal{X}(h_k) \frac{\mathcal{X}(1)}{|\,\mathrm{GL}_{\alpha_{[k,s_k]}}(\mathbb{F}_q)|} \chi_k(h_k) \right) = \tag{9.2.18}$$

$$= \langle m_{\tilde{\alpha}}, \tilde{\chi} \rangle q^{\alpha^2_{[k,s_k]}} H^\vee_{\omega_{\chi_k}}(q). \tag{9.2.19}$$

and therefore, if we put $\omega_\chi = \omega$, $\omega_{\tilde{\chi}} = \tilde{\omega}$ and $\omega_{\chi_k} = \omega_k$, we have

$$\langle g_\alpha, \chi \rangle H^\vee_\omega(q) = \langle m_{\tilde{\alpha}}, \tilde{\chi} \rangle H^\vee_{\tilde{\omega}}(q) q^{\alpha^2_{[k,s_k]}} (H^\vee_{\omega_k}(q))^2. \tag{9.2.20}$$

Since the family $\{m_{\tilde{\alpha}}\}_{\widetilde{\alpha} \in \mathbb{N}^{\tilde{I}}}$ is dual Log compatible, from eq.(9.2.20), we deduce that Identity (9.2.16) is equivalent to show that, for any $n \in \mathbb{T}_n$, any $\nu \in \mathbb{T}_n$ and any $d_1, \dots, d_r$ and types $\nu_1, \dots, \nu_r$ such that $\omega = \psi_{d_1}(\nu_1) * \cdots * \psi_{d_r}(\nu_r)$, we have

$$\frac{(H^\nu(t))^2}{\prod_{j=1}^r (H^\vee_{\nu_j}(t^{d_j}))^2} = \frac{\prod_{j=1}^r t^{d_j|\nu_j|^2}}{t^n} \tag{9.2.21}$$

which is a direct consequence of eq.(5.6.2).

Let now $\overline{I} = I - \{[k, s_k - 1], [k, s_k]\}$ and, for $\alpha \in \mathbb{N}^I$, denote by $\overline{\alpha}$ the element of $\mathbb{N}^{\overline{I}}$ obtained by the natural projection $\mathbb{N}^I \to \mathbb{N}^{\overline{I}}$ and by $\overline{\pi}_\alpha : \mathrm{GL}_\alpha(\mathbb{F}_q) \to \mathrm{GL}_{\overline{\alpha}}(\mathbb{F}_q)$ the associated projection. For a couple $(\beta, \gamma) \in \mathbb{N}^2$, denote by $m^{Kr}_{(\beta,\gamma)}$ the class function associated to the Kronecker quiver and the dimension vector (β, γ) for it, which was studied in Lemma 9.2.3.

Consider then the function $k_\alpha : \mathrm{GL}_\alpha(\mathbb{F}_q) \to \mathbb{C}$ defined as

$$k_\alpha(h) = \begin{cases} \dfrac{m^{Kr}_{\alpha_{[k,s_k-1]},\alpha_{[k,s_k]}}(h_{[k,s_k-1]}, h_{[k,s_k]})}{q^{-\sum_{i\in\overline{I}}\alpha_i^2}} & \text{if } \overline{\pi}_\alpha(h) = 1 \\ 0 \text{ otherwise} \end{cases}.$$

Notice that the function k_α can be equally rewritten as

$$k_\alpha(h) = \sum_{\chi \in \mathrm{GL}^\vee_{\overline{\alpha}}(\mathbb{F}_q)} \chi(\overline{\pi}_\alpha(h)) \frac{\chi(1)}{|\,\mathrm{GL}_{\overline{\alpha}}(\mathbb{F}_q)|} \frac{m^{Kr}_{\alpha_{[k,s_k-1]},\alpha_{[k,s_k]}}(h_{[k,s_k-1]}, h_{[k,s_k]})}{q^{-\sum_{i\in\overline{I}}\alpha_i^2}}. \tag{9.2.22}$$

By identity (9.2.22) and Lemma 9.2.3,it can be verified that the family of functions $\{k_\alpha\}_{\alpha\in\mathbb{N}^I}$ is dual Log compatible in a similar way to what has been done for $\{g_\alpha\}_{\alpha\in\mathbb{N}^I}$. By Lemma 5.8.8, the family of class functions

$$\left\{ \frac{g_\alpha * k_\alpha}{q^{\sum_{i\in I}\alpha_i^2}} \right\}_{\alpha\in\mathbb{N}^I}$$

is dual Log compatible too.

By direct calculation, we verify lastly that, for every $\alpha \in \mathbb{N}^I$, we have the equality

$$m_\alpha = \frac{g_\alpha * k_\alpha}{q^{\sum_{i\in I}\alpha_i^2}}.$$

Lemma 5.8.8 implies therefore the family $\{m_\alpha\}_{\alpha\in\mathbb{N}^I}$ is dual Log compatible.

9.3 Main result about non-generic character stacks

Consider a star-shaped $Q = (I, \Omega)$. For any $\sigma \in (\mathbb{C}^*)^I$ and any $\beta \in (\mathbb{N}^I)$, we will construct a spreading out of the stack $\mathcal{M}^*_{\sigma,\beta}$ in the following way.

Let $E_0 = \mathbb{Z}[x_i, x_i^{-1}]_{i\in I}$ be the ring in $|I|$ invertible variables. For any $\delta \in \mathbb{N}^I$, denote by $x^\delta \in E_0$ the element $x^\delta \coloneqq \prod_{i\in I} x_i^{\delta_i}$.

Let $\mathcal{N}_{\sigma,\beta} = (\mathbb{N}^I_{\leq\beta})^* \setminus \mathcal{H}_{\sigma,\beta}$. Consider the multiplicative set $S \subseteq E_0$ generated by the elements $x^\delta - 1$ for $\delta \in \mathcal{N}_{\sigma,\beta}$. Denote by $J \subseteq S^{-1}E_0$ the ideal generated by $(x^\delta - 1)$ for $\delta \in \mathcal{H}_{\sigma,\beta}$ and let E be the quotient

$$E \coloneqq S^{-1}E_0/J.$$

Notice that, given a field K, a map $\varphi : E \to K$ corresponds to an element $\gamma_\varphi \in (K^*)^I$ such that $\mathcal{H}_{\gamma_\varphi,\beta} = \mathcal{H}_{\sigma,\beta}$.

Let $\mathcal{A}_0$ be the polynomial E-algebra in $2\sum_{a\in\Omega} s(a)t(a)$ variables corresponding to the entries of matrices $(x_a, x_{a^*})_{a\in\Omega}$. Let $\mathcal{W} \subseteq \mathcal{A}_0$ be the multiplicative system generated by $\det(1 + x_a x_{a^*}), \det(1 + x_{a^*}x_a)$ for $a \in \Omega$ and let $\mathcal{A}'_0 \coloneqq \mathcal{W}^{-1}\mathcal{A}_0$.

Consider the ideal $\mathcal{I} \subseteq \mathcal{A}'_0$ generated by the entries of

$$\prod_{a\in\Omega}(1+x_a x_{a^*})(1+x_{a^*}x_a)^{-1} - \prod_{i\in I}(x_i I_{\alpha_i})$$

and let

$$\mathcal{A} = \mathcal{A}'_0/\mathcal{I}.$$

Let $Y = \mathrm{Spec}(A)$ and let $Y^* \subseteq Y$ be the open subset given by $y \in Y$ such that for any algebraically closed field K and any morphism $\mathrm{Spec}(K) \to Y$ with image y, the corresponding element $(x_a, x_{a^*})_{a\in\Omega} \in R(\overline{Q}, \alpha, K)$ has injective maps $(x_a)_{a\in\Omega}$.

Let now $\psi : E \to \mathbb{C}$ be the map induced by the element $\sigma \in (\mathbb{C}^*)^I$. Notice that

$$Y^* \times_{\mathrm{Spec}(E),\psi} \mathrm{Spec}(\mathbb{C}) \cong (\Phi^*_\beta)^{-1}(\sigma)$$

and therefore Y^* is a spreading out of $(\Phi^*_\beta)^{-1}(\sigma)$. Similarly, for any $\varphi : E \to \mathbb{F}_q$ corresponding to an element $\gamma_\varphi \in (\mathbb{F}_q^*)^I$ with $\mathcal{H}_{\gamma_\varphi,\beta} = \mathcal{H}_{\sigma,\beta}$, we have

$$((\Phi^*_\beta)^{-1}(\sigma))^\varphi = (\Phi^*_\beta)^{-1}(\gamma_\varphi).$$

Let $\mathrm{GL}_{\alpha,E}$ be the E-group scheme $\prod_{i\in I}\mathrm{GL}_{\alpha_i,E}$. The stack $\mathcal{Y}^* = [Y^*/\mathrm{GL}_{\alpha,E}]$ is therefore a spreading out of $\mathcal{M}^*_{\sigma,\beta}$.

By Remark 3.4.5 and the results of Theorem 5.8.4 and Theorem 9.2.1, we deduce that the stack $\mathcal{M}^*_{\sigma,\beta}$ is rational count and that we have:

$$\frac{E(\mathcal{M}^*_{\sigma,\beta}, q)}{q^{(\beta,\beta)}} = \mathrm{Coeff}_\beta\left(\mathrm{Exp}\left(\sum_{\delta\in\mathcal{H}_\sigma}\widetilde{M}_{\delta,gen}(q)y^\beta\right)\right) \tag{9.3.1}$$

where $\widetilde{M}_{\delta,gen}(t)$ are the rational functions associated to the dual Log compatible family $\{m_\delta\}_{\delta\in\mathbb{N}^I}$, as in §5.8.2. Notice that $\widetilde{M}_{\delta,gen}(t) = 0$ if $\delta \notin (\mathbb{N}^I)^*$.

In particular, if σ is such that $\mathcal{H}^*_{\sigma,\beta} = \{\beta\}$, we have that $E(\mathcal{M}^*_{\sigma,\beta}, q) = q^{-(\beta,\beta)}\widetilde{M}_{\beta,gen}(q)$. From Remark 9.1.3, we see that for any $\delta \in (\mathbb{N}^I)^*$ it holds

$$\widetilde{M}_{\delta,gen}(q) = \frac{q\mathbb{H}_\delta\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right)}{q-1}.$$

We can resume all the arguments above in the following Theorem:

Theorem 9.3.1. *For any $\beta \in (\mathbb{N}^I)^*$ and any $\sigma \in (\mathbb{C}^*)^I$, there is an equality:*

$$\frac{E(\mathcal{M}^*_{\sigma,\beta}, q)}{q^{(\beta,\beta)}} = \mathrm{Coeff}_\beta\left(\mathrm{Exp}\left(\sum_{\delta\in\mathcal{H}^*_{\sigma,\beta}}\frac{q\mathbb{H}_\delta\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right)}{q-1}y^\delta\right)\right) \tag{9.3.2}$$

9.3.1 E-series for character stacks with semisimple monodromies

From Theorem 9.3.1, we deduce the following Theorem about E-series for character stacks associated to k-tuples of semisimple orbits.

Theorem 9.3.2. *For any k-tuple $\mathcal{C}$ of semisimple orbits of* $\mathrm{GL}_n(\mathbb{C})$*, we have:*

$$\frac{E(\mathcal{M}_{\mathcal{C}},q)}{q^{(\alpha_{\mathcal{C}},\alpha_{\mathcal{C}})}} = \mathrm{Coeff}_{\alpha_{\mathcal{C}}}\left(\mathrm{Exp}\left(\sum_{\beta\in\mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}} \frac{q\mathbb{H}_\beta\left(\sqrt{q},\frac{1}{\sqrt{q}}\right)}{q-1}\right)\right). \tag{9.3.3}$$

Remark 9.3.3. Notice that Theorem 9.3.3 implies that the E-series $E(\mathcal{M}_{\mathcal{C}},q)$ does not depend on the values on the eigenvalues $\{\gamma_{j,h}\}_{\substack{j=1,\dots,k\\ h=0,\dots,s_j}}$ but only on the subset $\mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}$.

From Theorem 9.3.2, it seems natural to formulate the following generalization of Conjecture 9.1.8

Conjecture 9.3.4. *For any k-tuple of semisimple orbits $\mathcal{C}$, we have:*

$$\frac{H_c(\mathcal{M}_{\mathcal{C}},q,t)}{(qt^2)^{(\alpha_{\mathcal{C}},\alpha_{\mathcal{C}})}} = \mathrm{Coeff}_{\alpha_{\mathcal{C}}}\left(\mathrm{Exp}\left(\sum_{\beta\in\mathcal{H}^*_{\gamma_{\mathcal{C}},\alpha_{\mathcal{C}}}} \frac{(qt^2)\mathbb{H}_\beta\left(t\sqrt{q},\frac{1}{\sqrt{q}}\right)}{qt^2-1}\right)\right). \tag{9.3.4}$$

9.4 Mixed Poincaré series of character stacks for $\mathbb{P}^1_{\mathbb{C}}$ with four punctures

In this section we will verify Conjecture 9.3.4 for a certain family of non-generic character stacks.

Let $\Sigma = \mathbb{P}^1_{\mathbb{C}}$ (i.e $g=0$), $k=4$ and $n=2$. Let $D=\{x_1,\dots,x_4\}\subseteq\mathbb{P}^1_{\mathbb{C}}$. For $j=1,\dots,4$, pick $\lambda_j\in\mathbb{C}^*$ with $\lambda_j\neq\pm1$ and denote by $\mathcal{C}_j$ the conjugacy class of the diagonal matrix

$$\begin{pmatrix}\lambda_j & 0\\ 0 & \lambda_j^{-1}\end{pmatrix}.$$

Let $\mathcal{C}$ be the k-tuple $(\mathcal{C}_1,\dots,\mathcal{C}_4)$. The variety $X_{\mathcal{C}}$ is therefore

$$X_{\mathcal{C}} = \{(X_1,\dots,X_4)\in\mathcal{C}_1\times\cdots\times\mathcal{C}_4 \mid X_1X_2X_3X_4=1\}.$$

Denote by $M_{\mathcal{C}}$ the GIT quotient $M_{\mathcal{C}} := X_{\mathcal{C}}/\!/\,\mathrm{GL}_2(\mathbb{C})$. Recall that the points of $M_{\mathcal{C}}$ are in bijection with the isomorphism classes of semisimple representations of $\pi(\Sigma\setminus D)$ inside $X_{\mathcal{C}}$.

The study of the geometry of the character varieties $M_{\mathcal{C}}$ goes back to Fricke and Klein [33], who gave a description of them in terms of cubic surfaces. Denote by $a_i=\lambda_i+\lambda_i^{-1}$.
The character variety $M_{\mathcal{C}}$ is isomorphic to the cubic surface defined by the equation in 3

variables x, y, z

$$xyz+x^2+y^2+z^2-(a_1a_2+a_3a_4)x-(a_2a_3+a_1a_4)y-(a_1a_3+a_2a_4)z+a_1a_2a_3a_4+a_1^2+a_2^2+a_3^2+a_4^2-4=0. \tag{9.4.1}$$

If $\mathcal{C}$ is generic, this description identifies $M_{\mathcal{C}}$ with a smooth (affine) Del Pezzo cubic surface (see [32, Theorem 6.1.4]), i.e a smooth cubic projective surface with a triangle cut out of it. The cohomology of this kind of surfaces is well known. In particular, if $\mathcal{C}$ is generic, it holds:

$$H_c(M_{\mathcal{C}}, q, t) = q^2t^4 + 4qt^2 + t^2$$

and therefore we have

$$H_c(\mathcal{M}_{\mathcal{C}}, q, t) = \frac{q^2t^4 + 4qt^2 + t^2}{qt^2 - 1}.$$

The Identity above agrees with Hausel, Letellier, Rodriguez-Villegas Conjecture 9.1.8, as explained in [45, Paragraph 1.5].

Pick now $\lambda_1, \dots, \lambda_4 \in \mathbb{C}^* \setminus \{1, -1\}$ with the following property. For $\epsilon_1, \dots, \epsilon_4 \in \{1, -1\}$ such that $\lambda_1^{\epsilon_1} \cdots \lambda_4^{\epsilon_4} = 1$, then either $\epsilon_1 = \cdots = \epsilon_4 = 1$ or $\epsilon_1 = \cdots = \epsilon_4 = -1$. Notice that in this case, the associated k-tuple $\mathcal{C}$ is not generic.

In the following section, we will compute the mixed Poincaré series $H_c(\mathcal{M}_{\mathcal{C}}, q, t)$ and verify that it respects Conjecture 9.3.4.

For the character stack $\mathcal{M}_{\mathcal{C}}$, the associated quiver $Q = (I, \Omega)$ is the star-shaped quiver with one central vertex and four arrows pointing inwards. We denote the central vertex by 0 and the other vertices by $[i, 1]$ for $i = 1, \dots, 4$.
The dimension vector $\alpha_{\mathcal{C}}$ is the dimension vector for Q defined as $(\alpha_{\mathcal{C}})_0 = 2$ and $(\alpha_{\mathcal{C}})_{[i,1]} = 1$ for $i = 1, \dots, 4$. The quiver Q with the dimension vector α is depicted below.

$$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow & & \\ 1 & \longrightarrow & 2 & \longleftarrow & 1 \\ & & \uparrow & & \\ & & 1 & & \end{array}$$

The associated parameter $\gamma_{\mathcal{C}}$ is given by

$$(\gamma_{\mathcal{C}})_0 = (\lambda_1\lambda_2\lambda_3\lambda_4)^{-1} = 1$$

and, for $i = 1, \dots, 4$

$$(\gamma_{\mathcal{C}})_{[i,1]} = \lambda_i^2.$$

Denote by $\beta_1, \beta_2 \in (\mathbb{N}^I)^*$ the elements defined as $(\beta_1)_0 = 1, (\beta_1)_{[i,1]} = 1$ and $(\beta_2)_1 =$

$1, (\beta_2)_{[i,1]} = 0$ for $i = 1, \ldots, 4$. Notice that it holds $\mathcal{H}^*_{\gamma_\mathcal{C}, \alpha_\mathcal{C}} = \{\alpha, \beta_1, \beta_2\}$. There are equalities

$$\mathbb{H}_{\beta_1}\left(t\sqrt{q}, \frac{1}{\sqrt{q}}\right) = \mathbb{H}_{\beta_2}\left(t\sqrt{q}, \frac{1}{\sqrt{q}}\right) = 1.$$

Conjecture 9.3.4 predicts then the following equality

$$H_c(\mathcal{M}_\mathcal{C}, q, t) = \frac{qt^2\mathbb{H}_{\alpha_\mathcal{C}}\left(t\sqrt{q}, \frac{1}{\sqrt{q}}\right)}{qt^2-1} + \frac{q^2t^4\mathbb{H}_{\beta_1}\left(t\sqrt{q}, \frac{1}{\sqrt{q}}\right)\mathbb{H}_{\beta_2}\left(t\sqrt{q}, \frac{1}{\sqrt{q}}\right)}{(qt^2-1)^2} = \tag{9.4.2}$$

$$= \frac{q^2t^4 + 4qt^2 + t^2}{qt^2-1} + \frac{q^2t^4}{(qt^2-1)^2} = \frac{q^3t^6 + 4q^2t^4 + qt^4 - 4qt^2 - t^2}{(qt^2-1)^2}. \tag{9.4.3}$$

Denote by $\mathcal{M}'_\mathcal{C}$ the quotient stack $\mathcal{M}'_\mathcal{C} = [X_\mathcal{C}/\operatorname{PGL}_2]$. Notice that $\mathcal{M}_\mathcal{C}$ is a $\mathbb{G}_m$-gerbe over $\mathcal{M}'_\mathcal{C}$ and from Lemma 3.4.8 we have

$$H_c(\mathcal{M}_\mathcal{C}, q, t) = \frac{H_c(\mathcal{M}'_\mathcal{C}, q, t)}{qt^2-1}. \tag{9.4.4}$$

We can thus reduce ourselves to compute the cohomology of the stack $\mathcal{M}'_\mathcal{C}$.

Inside $X_\mathcal{C}$ there is the open (dense) subset which we denote by $X^s_\mathcal{C} \subseteq X_\mathcal{C}$, given by quadruple $(X_1, X_2, X_3, X_4) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_4$ corresponding to irreducible representations of $\pi_1(\Sigma \setminus D)$. Recall that $X^s_\mathcal{C}$ is smooth (see for example [31, Proposition 5.2.8]).

Denote by $\mathcal{N}^s_\mathcal{C}$ the quotient stack $[X^s_\mathcal{C}/\operatorname{PGL}_2]$. Notice that the action of PGL_2 is schematically free on $X^s_\mathcal{C}$ and therefore by Lemma 3.3.1 the stack $\mathcal{N}^s_\mathcal{C}$ is an algebraic variety.

The non irreducible representations of $X_\mathcal{C}$ all have the same semisimplification, up to isomorphism, which corresponds to the point $m \in M_\mathcal{C}$, associated to the isomorphism class of the representation

$$m = \left(\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_1^{-1}\end{pmatrix}, \begin{pmatrix}\lambda_2 & 0\\ 0 & \lambda_2^{-1}\end{pmatrix}, \begin{pmatrix}\lambda_3 & 0\\ 0 & \lambda_3^{-1}\end{pmatrix}, \begin{pmatrix}\lambda_4 & 0\\ 0 & \lambda_4^{-1}\end{pmatrix}\right). \tag{9.4.5}$$

We denote by $O \subseteq X_\mathcal{C}$ the $\operatorname{GL}_2(\mathbb{C})$-orbit associated to m. A representation $x \in X_\mathcal{C}$ which is neither irreducible nor semisimple, i.e neither inside $X^s_\mathcal{C}$ nor inside O, can be of the following two types. Either x is isomorphic to a quadruple of the form

$$m^+_{a,b,c} := \left(\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_1^{-1}\end{pmatrix}, \begin{pmatrix}\lambda_2 & a\\ 0 & \lambda_2^{-1}\end{pmatrix}, \begin{pmatrix}\lambda_3 & b\\ 0 & \lambda_3^{-1}\end{pmatrix}, \begin{pmatrix}\lambda_4 & c\\ 0 & \lambda_4^{-1}\end{pmatrix}\right) \tag{9.4.6}$$

with $(a, b, c) \in \mathbb{C}^3 - \{(0,0,0)\}$ and

$$\lambda_1\lambda_2\lambda_3 c + \lambda_1\lambda_2\mu_4 b + \lambda_1\lambda_2\lambda_3 c = 0 \tag{9.4.7}$$

or of the form

$$m^-_{a,b,c} := \left(\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_1^{-1}\end{pmatrix}, \begin{pmatrix}\lambda_2 & 0\\ a & \lambda_2^{-1}\end{pmatrix}, \begin{pmatrix}\lambda_3 & 0\\ b & \lambda_3^{-1}\end{pmatrix}, \begin{pmatrix}\lambda_4 & 0\\ c & \lambda_4^{-1}\end{pmatrix}\right)$$

with $(a,b,c) \in \mathbb{C}^3 - \{(0,0,0)\}$ and

$$\lambda_4\lambda_1^{-1}\lambda_3 a + \lambda_1^{-1}\lambda_2^{-1}\lambda_4 b + \lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1} c = 0. \tag{9.4.8}$$

We denote by $Z^+_{\mathcal{C}} \subseteq X_{\mathcal{C}}$ and by $Z^-_{\mathcal{C}} \subseteq X_{\mathcal{C}}$ the locally closed subsets of representations isomorphic to elements of the form $m^+_{(a,b,c)}$ or $m^-_{(a,b,c)}$ for some $(a,b,c) \in \mathbb{C}^3 - \{(0,0,0)\}$ respecting the conditions of eq.(9.4.7), eq.(9.4.8) respectively.

9.4.1 Cohomology of the character variety in the non-generic case

As recalled before, the variety $M_{\mathcal{C}}$ is a cubic surface defined by the equation of formula (9.4.1). Denote by $\overline{M}_{\mathcal{C}} \subseteq \mathbb{P}^3_{\mathbb{C}}$ the associated projective cubic surface. Notice that $\overline{M}_{\mathcal{C}}$ is obtained by adding to $M_{\mathcal{C}}$ the triangle at infinity $xyz = 0$, which we will denote by $U \subseteq \overline{M}_{\mathcal{C}}$.
Unlike the case where $\mathcal{C}$ is generic, for our choice of quadruples the surface $\overline{M}_{\mathcal{C}}$ is singular, with m being is its only singular point. We have moreover an isomorphism

$$\mathcal{N}^s_{\mathcal{C}} \cong M_{\mathcal{C}} - \{m\}.$$

It is a well known result (see for example [60]) that for such a singular cubic surface $\overline{M}_{\mathcal{C}}$, there exists a resolution of singularities

$$f : \widetilde{M_{\mathcal{C}}} \to \overline{M}_{\mathcal{C}}$$

such that $f^{-1}(m) \cong \mathbb{P}^1_{\mathbb{C}}$ and f is an isomorphism over $\overline{M}_{\mathcal{C}} - \{m\}$, i.e

$$f^{-1}(\overline{M}_{\mathcal{C}} - \{m\}) \cong \overline{M}_{\mathcal{C}} - \{m\}.$$

Moreover, it is known that $\widetilde{M_{\mathcal{C}}}$ is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at 6 points. There is thus an equality

$$H_c(\widetilde{M_{\mathcal{C}}}, q, t) = q^2t^4 + 7qt^2 + 1.$$

Using the long exact sequence in compactly supported cohomology for the open-closed decomposition $\widetilde{M_{\mathcal{C}}} = f^{-1}(\overline{M}_{\mathcal{C}} - \{m\}) \bigsqcup f^{-1}(m)$, we find that

$$H_c(\overline{M}_{\mathcal{C}} - \{m\}, q, t) = H_c(f^{-1}(\overline{M}_{\mathcal{C}} - \{m\}), q, t) = q^2t^4 + 6qt^2$$

and so that $H_c(\overline{M}_{\mathcal{C}}, q, t) = q^2t^4 + 6qt^2 + 1$.
It is not difficult to check that the compactly supported Poincaré polynomial of U is $H_c(U, q, t) = 3qt^2 + t + 1$. Applying the long exact sequence in compactly-supported cohomology for the open-closed decomposition $\overline{M}_{\mathcal{C}} = M_{\mathcal{C}} \bigsqcup U$ we find finally that

$$H_c(M_{\mathcal{C}}, q, t) = q^2t^4 + 3qt^2 + t^2. \tag{9.4.9}$$

From eq.(9.4.9), using the long exact sequence for the open-closed decomposition $M_{\mathcal{C}} = (M_{\mathcal{C}} - \{m\}) \bigsqcup \{m\}$ we deduce that it holds

$$H_c(\mathcal{N}^s_{\mathcal{C}}, q, t) = H_c(M_{\mathcal{C}} - \{m\}, q, t) = q^2t^4 + 3qt^2 + t^2 + t. \tag{9.4.10}$$

9.4.2 Cohomology of the character stack in the non-generic case

We introduce the following notations. Let $Y_{\mathcal{C}} = X_{\mathcal{C}} - O$ and $\mathcal{N}_{\mathcal{C}} = [Y_{\mathcal{C}} / \operatorname{PGL}_2]$. Notice that the action of PGL_2 on $Y_{\mathcal{C}}$ is set-theoretically free so that $\mathcal{N}_{\mathcal{C}}$ is at least an algebraic space. Let $Y^+_{\mathcal{C}} = Y_{\mathcal{C}} \setminus Z^-_{\mathcal{C}}$ and $Y^-_{\mathcal{C}} = Y_{\mathcal{C}} \setminus Z^+_{\mathcal{C}}$ and denote their quotients by $\mathcal{N}^+_{\mathcal{C}} = [Y^+_{\mathcal{C}} / \operatorname{PGL}_2]$ and $\mathcal{N}^-_{\mathcal{C}} = [Y^-_{\mathcal{C}} / \operatorname{PGL}_2]$ respectively.

Notice that there is an isomorphism $[O/\operatorname{PGL}_2] \cong B\mathbb{G}_m$ and an open-closed decomposition

$$\mathcal{M}'_{\mathcal{C}} = \mathcal{N}_{\mathcal{C}} \bigsqcup [O/\operatorname{PGL}_2].$$

Applying the long-exact sequence for compactly supported cohomology to the decomposition above and knowing that $H^*_c(B\mathbb{G}_m)$ is concentrated in strictly negative even degrees, we obtain

$$H_c(\mathcal{M}'_{\mathcal{C}}, q, t) = H_c(\mathcal{N}_{\mathcal{C}}, q, t) + H_c(B\mathbb{G}_m, q, t) = H_c(\mathcal{N}_{\mathcal{C}}, q, t) + \frac{1}{qt^2 - 1}. \tag{9.4.11}$$

We have then reduced ourselves to compute the cohomology of $\mathcal{N}_{\mathcal{C}}$. We will apply Lemma 3.3.4 in the following way in the case where $X = X_{\mathcal{C}}$, $G = \operatorname{PGL}_2$ and $H \subseteq \operatorname{PGL}_2$ is the maximal torus of diagonal matrices. In the following, we identify $H \cong \mathbb{G}_m$, via the map $\mathbb{G}_m \to \operatorname{PGL}_2$, which sends $z \in \mathbb{C}^*$ to the class of $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$.

Recall that there is an isomorphism $\mathcal{C}_1 \cong G/H$. Via this latter isomorphism, the projection on the first factor induces a G-equivariant morphism

$$p : X_{\mathcal{C}} \to G/H \cong \mathcal{C}_1$$

$$(X_1, X_2, X_3, X_4) \to X_1.$$

Notice that

$$(X_{\mathcal{C}})_H = \left\{ X_2 \in \mathcal{C}_2, X_3 \in \mathcal{C}_3, X_4 \in \mathcal{C}_4 \mid X_2X_3X_4 = \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_1 \end{pmatrix} \right\}.$$

Denote by $(M_{\mathcal{C}})_H \coloneqq (X_{\mathcal{C}})_H /\!/ H$. Lemma 3.3.4 implies that there is an isomorphism

$$(M_{\mathcal{C}})_H \cong M_{\mathcal{C}}.$$

We employ similar notations for $(\mathcal{N}_{\mathcal{C}})_H, (\mathcal{N}^+_{\mathcal{C}})_H, (\mathcal{N}^-_{\mathcal{C}})_H, (\mathcal{N}^s_{\mathcal{C}})_H$. Reapplying Lemma 3.3.4, we see that there is an isomorphism $(\mathcal{N}_{\mathcal{C}}^{s})_H \cong \mathcal{N}_{\mathcal{C}}^{s}$. In particular,

$$H_c((\mathcal{N}_{\mathcal{C}}^{S})_H, q, t) = q^2t^4 + 3qt^2 + t^2 + t. \tag{9.4.12}$$

Consider now the character $\theta^+ : H = \mathbb{G}_m \to \mathbb{G}_m$ given by $\theta^+(z) = z$. The character θ induces a linearization of the H-action on the affine variety $(X_{\mathcal{C}})_H$ (see for example [54, Section 2]). Using Mumford's criterion (see [54, Proposition 2.5]), we check that the semistable points $(X_{\mathcal{C}})_H^{ss,\theta^+}$ are given by

$$(X_{\mathcal{C}})_H^{ss,\theta^+} = (Y_{\mathcal{C}}^+)_H.$$

We have indeed four type of points inside $(X_{\mathcal{C}})_H$:

- Notice that $O \cap (X_{\mathcal{C}})_H$ is the singleton $\{m\}$, corresponding to the quadruple (9.4.5). The point m, being a $\mathbb{G}_m$ fixed point, is unstable. Indeed, considering the 1-parameter subgroup $\lambda : \mathbb{G}_m \to \mathbb{G}_m$ given by $\lambda(z) = z^{-1}$, we have $\langle \theta^+, \lambda \rangle = -1 < 0$ while it exists $\lim_{t\to 0} \lambda(t) \cdot m = m$.

- The points of $(X_{\mathcal{C}}^{s})_H$ are stable. Each $x \in (X_{\mathcal{C}})_H$ corresponds to an irreducible representation. For a 1-parameter subgroup $\lambda : \mathbb{G}_m \to \mathbb{G}_m$, the limit $\lim_{t\to 0} \lambda(t) \cdot x$ exists if and only if λ is trivial, i.e $\langle \theta^+, \lambda \rangle = 0$.

- The points of $(Z_{\mathcal{C}}^+)_H$ are semistable. Notice that $(Z_{\mathcal{C}}^+)_H$ is given by points of the form $m^+_{(a,b,c)}$ as in eq.(9.4.6), for $(a, b, c) \in \mathbb{C}^3 - \{(0, 0, 0)\}$ which respects eq.(9.4.7).

  For $\lambda : \mathbb{G}_m \to \mathbb{G}_m$ given by $\lambda(t) = t^n$ for $n \in \mathbb{Z}$ and $t \in \mathbb{C}^*$, we have

  $$\lambda(t) \cdot m^+_{(a,b,c)} = m_{(t^n a, t^n b, t^n c)}.$$

  In particular, we see that the limit $\lim_{t\to 0} \lambda(t) \cdot m^+_{(a,b,c)}$ exists (and it is given by m) if and only if $n \geq 0$, i.e if and only if $\langle \theta^+, \lambda \rangle \geq 0$.

- By a similar reasoning to the case of $(Z_{\mathcal{C}}^+)_H$, the points of $(Z_{\mathcal{C}}^-)_H$ are unstable.

The algebraic space $(\mathcal{N}_{\mathcal{C}}^+)_H = [(Y_{\mathcal{C}}^+)_H/H]$ is therefore an algebraic variety and the canonical map

$$f^+ : (\mathcal{N}_{\mathcal{C}}^+)_H \to (M_{\mathcal{C}})_H$$

is proper. Notice moreover that $(f^+)^{-1}(m) = (Z_{\mathcal{C}}^+)_H/H$. As recalled above, $(Z_{\mathcal{C}}^+)_H$ is isomorphic to $\mathbb{C}^2 - \{(0, 0)\}$. Via this identification $\mathbb{G}_m$ acts on $\mathbb{C}^2 - \{(0, 0)\}$ by scalar multiplication on both coordinates. We have therefore:

$$(Z_{\mathcal{C}}^+)_H/H \cong (\mathbb{C}^2 - \{(0, 0)\})/\mathbb{G}_m = \mathbb{P}^1_{\mathbb{C}}.$$

Consider now the Leray spectral sequence for compactly supported cohomology

$$E_2^{p,q} : H_c^p((M_{\mathcal{C}})_H, R^q f_*^+\mathbb{Q}) \Rightarrow H_c^{p+q}((\mathcal{N}_{\mathcal{C}}^+)_H, \mathbb{Q}).$$

Notice that $R^q f_*^+\mathbb{Q} \neq 0$ if and only if $q = 0, 2$. More precisely, we have $f_*^+\mathbb{Q} = \mathbb{Q}$ and $R^2 f_*^+\mathbb{Q} = (i_m)_*\mathbb{Q}$, where i_m is the closed embedding

$$i_m : \{m\} \to (M_{\mathcal{C}})_H.$$

Recall that the differential maps of the spectral sequence go in the direction

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

As $R^q f_*^+\mathbb{Q}$ is 0 for odd q, the differential $d_2^{p,q}$ is the zero map for each p, q and therefore we have $E_3^{p,q} = E_2^{p,q}$ for each p, q. Moreover, notice that the differentials on the third page go in the direction $d_3^{p,q} : E_3^{p,q} \to E_3^{p+3,q-2}$. Notice that, if $q \neq 0, 2$, the vector space $E_3^{p,q}$ is equal to 0.
Moreover, if $q = 0$, we have $E_3^{3,-3} = \{0\}$ and so $d_3^{p,q} = 0$. Lastly, if $q = 2$, we have $E_3^{p,q} = \{0\}$ if $p \geq 1$ and if $p = 0$, we have $E_3^{3,0} = H_c^3((M_{\mathcal{C}})_H, \mathbb{Q}) = \{0\}$.
We deduce therefore that the differential maps $d_3^{p,q}$ are all zero. In a similar way, it is possible to verify that $d_r^{p,q} = 0$ if $r \geq 2$, for any p, q and so that the spectral sequence collapses at the second page.
From the description of the sheaves $R^q f_*^+\mathbb{Q}$ given above, we deduce that we have

$$H_c((\mathcal{N}_{\mathcal{C}}^+)_H, q, t) = q^2t^4 + 4qt^2 + t^2 \tag{9.4.13}$$

Remark 9.4.1. Notice that the variety $(\mathcal{N}_{\mathcal{C}}^+)_H$ is smooth, the morphism f^+ is proper and we have $(f^+)^{-1}(m) \cong \mathbb{P}_{\mathbb{C}}^1$.
In particular, $(\mathcal{N}_{\mathcal{C}}^+)_H$ is a resolution of singularity of the variety $(M_{\mathcal{C}})_H$. By the isomorphisms of Lemma 3.3.4, we see that the variety $\mathcal{N}_{\mathcal{C}}^+$ is the canonical resolution of singularity of the singular affine cubic surface $M_{\mathcal{C}}$.
We have therefore found a natural way to build the resolution of singularity of the GIT quotient $M_{\mathcal{C}}$ as a locally closed subvariety of the quotient stack $\mathcal{M}'_{\mathcal{C}}$. It would be interesting to generalize the same approach to other type of character stacks.

A similar reasoning can be applied to the opposite linearization, induced by the character $\theta^- : \mathbb{G}_m \to \mathbb{G}_m$ given by $\theta^-(z) = z^{-1}$. In this case in a similar way we can argue that the semistable points $(X_{\mathcal{C}})_H^{ss,\theta^-}$ are given by $(Y_{\mathcal{C}}^-)_H$.
For the corresponding quotient $(\mathcal{N}_{\mathcal{C}}^-)_H$ there is therefore an equality

$$H_c((\mathcal{N}_{\mathcal{C}}^-)_H, q, t) = q^2t^4 + 4qt^2 + t^2.$$

Denote now by j^+, j^- the open embeddings $j^+ : (\mathcal{N}_{\mathcal{C}}^+)_H \to (\mathcal{N}_{\mathcal{C}})_H$ and $j^+ : (\mathcal{N}_{\mathcal{C}}^-)_H \to (\mathcal{N}_{\mathcal{C}})_H$

and by j the open embedding $(\mathcal{N}_{\mathcal{C}}^s)_H \to (\mathcal{N}_{\mathcal{C}})_H$. Notice that there is a short exact sequence of sheaves on $(\mathcal{N}_{\mathcal{C}})_H$

$$0 \longrightarrow j_!\mathbb{Q} \longrightarrow j_!^+\mathbb{Q} \oplus j_!^-\mathbb{Q} \longrightarrow \mathbb{Q} \longrightarrow 0$$

and therefore an associated long exact sequence in compactly supported cohomology

$$H_c^{i-1}((\mathcal{N}_{\mathcal{C}})_H, \mathbb{Q}) \longrightarrow H_c^i(\mathcal{N}_{\mathcal{C}}^s, \mathbb{Q}) \longrightarrow H_c^i((\mathcal{N}_{\mathcal{C}}^+)_H, \mathbb{Q}) \oplus H_c^i((\mathcal{N}_{\mathcal{C}}^-)_H, \mathbb{Q}) \longrightarrow H_c^i((\mathcal{N}_{\mathcal{C}})_H, \mathbb{Q})$$

Notice that from Lemma 3.3.4, there is an isomorphism $(\mathcal{N}_{\mathcal{C}})_H \cong \mathcal{N}_{\mathcal{C}}$. From the long exact sequence above and equations(9.4.13),(9.4.12), it is therefore not difficult to see that

$$H_c(\mathcal{N}_{\mathcal{C}}, q, t) = q^2t^4 + 5qt^2 + t^2 + 1. \tag{9.4.14}$$

Plugging this result into eq.(9.4.11) and using identity (9.4.4), we verify finally identity (9.4.3).

□

10 Character stacks for non-orientable surfaces

In this chapter we study character stacks for real non-orientable surfaces, rather than Riemann surfaces. Our approach to real geometry follows the one introduced in [3], i.e a real non-orientable surface is a pair (X,σ), where X is a compact Riemann surface and $\sigma : X \to X$ an antiholomorphic involution without fixed points. The associated non-orientable surface is the quotient $X/\langle\sigma\rangle$.

In section §10.1, we review some generalities about fundamental groups of non-orientable surfaces. In section §10.2, we define character stacks for non-orientable surfaces $\mathcal{M}^{\epsilon}_{\mathcal{C}}$ associated to a k-tuple of semisimple conjugacy classes $\mathcal{C}$ and we show how they are related to involutions on character stacks for the Riemann surface X.
In section §10.3, we review the results of Letellier and Rodriguez-Villegas [65], where the authors compute the E-series $E(\mathcal{M}^{\epsilon}_{\mathcal{C}},q)$, when $\mathcal{C}$ is generic. Their formula strongly resembles Formula (9.1.6) for the E-series of character stacks for Riemann surfaces and the authors [65, Theorem 4.8] verified that a formula analogous to Conjecture 9.1.8 holds for the mixed Poincaré series $H_c(\mathcal{M}^{\epsilon}_{\mathcal{C}},q,t)$ when $X=\mathbb{P}^1_{\mathbb{C}}$ and $k=1$.
It would be natural to expect that a similar formula would be true for any X and any k.
The main result of this chapter is a counterexample to such a conjectural formula, obtained in section §10.5 by explicitly describing these spaces in the case in which X is an elliptic curve and $\mathcal{C}=\{e^{\frac{\pi i d}{n}}\}$, for $(d,n)=1$.
To completely describe these spaces we will need some general results about character stacks for non-orientable surfaces for $\mathcal{C}=\{e^{\frac{\pi i d}{n}}\}$, which we review in §10.4.

10.1 Notations and fundamental groups for real curves

Let X be a compact and connected Riemann surface of genus g. Assume to have fixed a real structure on X, i.e an antiholomorphic involution $\sigma : X \to X$. The involution σ determines a real projective curve $X_{\mathbb{R}}$, i.e a smooth projective variety of dimension 1 over $\mathrm{Spec}(\mathbb{R})$ such that

$$X_{\mathbb{R}} \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \cong X$$

and, via the isomorphism above, σ corresponds to the complex conjugation on the second factor, see for example [3].
Notice that $X_{\mathbb{R}}(\mathbb{R}) = X^{\sigma}$. From now on, we will assume that $X^{\sigma}=\varnothing$, (i.e that $X_{\mathbb{R}}$ has no real points). Notice that this implies that the action of $\langle\sigma\rangle$ on X is free.
In particular, the quotient space

$$S \coloneqq X/\langle\sigma\rangle$$

has the structure of a real non-orientable surface. We denote by $p : X \to S$ the quotient morphism. Recall that S is homeomorphic to the the connected sum of $r \coloneqq g+1$ projective planes.

Remark 10.1.1. Consider conversely a connected, compact real non-orientable surface Y, the orientation cover $p : \widetilde{Y} \to Y$ and the orientation reversing involution $\sigma : \widetilde{Y} \to \widetilde{Y}$.

Notice that $\widetilde{Y}$ is a compact, connected and orientable real surface, i.e can be equipped with the structure of a Riemann surface.With respect to this complex structure, the involution σ is antiholomorphic.

Let $k \in \mathbb{N}$ and consider a subset $D \subseteq X$ of $2k$ points $D = \{y_{1,1}, \ldots, y_{1,k}, y_{2,1}, \ldots, y_{2,k}\}$ with $\sigma(y_{1,i}) = y_{2,i}$ for each $i = 1, \ldots, k$. Fix now a point $x_0 \in X \setminus D$ and denote by

$$\Pi := \pi_1(x_0, X \setminus D)$$

the fundamental group with basepoint x_0.

Fix a path λ_σ from x_0 to $\sigma(x_0)$. Define then the morphism

$$\sigma_* : \Pi \to \Pi$$

$$\alpha \to \lambda_\sigma^{-1}\sigma(\alpha)\lambda_\sigma.$$

Notice that σ_*^2 is the conjugation for the element $\lambda_\sigma^{-1}\sigma(\lambda_\sigma^{-1})$ which, in general, is different from the identity. In particular, in general σ_*^2 is not the identity, i.e σ_* is not an involution.

Denote by $z_0 = p(x_0)$, by $x_i = p(y_{1,i})$ for each $i = 1, \ldots, k$ and by $D' = \{x_1, \ldots, x_k\}$, i.e $D' = p(D')$. We put then

$$\Pi^\epsilon := \pi_1(z_0, S \setminus D')$$

the fundamental group with basepoint z_0.

Recall that in this case, there is an explicit presentation of the fundamental group Π^ϵ given by

$$\Pi^\epsilon = \langle d_1^2 \cdots d_r^2 z_1 \cdots z_k = 1 \rangle,$$

where each z_i is a loop around x_i.

Notice that there is a short exact sequence

$$1 \longrightarrow \Pi \xrightarrow{p_*} \Pi^\epsilon \xrightarrow{\epsilon} \mathbb{Z}/(2) \longrightarrow 1 . \tag{10.1.1}$$

For each $j = 1, \ldots, r$, we have $\epsilon(d_j) = -1$.

Example 10.1.2. Consider the elliptic curve X associated to the lattice $\langle 1, i \rangle \subseteq \mathbb{C}$ i.e

$$X \cong \mathbb{C}/\langle 1, i \rangle$$

and let π be the projection $\pi : \mathbb{C} \to X$. Let $\sigma : X \to X$ be the involution without fixed points defined by

$$\sigma(z) = \bar{z} + \frac{1}{2}$$

and $p : X \to S := X/\langle\sigma\rangle$ be the associated quotient.

We fix a point $z_1 \in S$ and we let its preimage in X be $p^{-1}(z_1) = \{y_{1,1}, y_{1,2}\}$. Put $z_0 = p(0)$

and $x_0 = 0$ as base points and

$$\lambda_\sigma = \pi(\gamma(t))$$

where $\gamma(t) = \frac{1}{2}t$.

Denoting by a, b the paths $a(t) = \pi(it)$ and $b(t) = \pi(t)$, the fundamental group $\Pi = \pi_1(z_0, X \setminus \{y_{1,1}, y_{1,2}\})$ admits the presentation

$$\langle b^{-1}a^{-1}ba = x_2x_1\rangle. \tag{10.1.2}$$

where x_1, x_2 are two loops around y_1, y_2. It is not difficult to compute that

$$\sigma_*(a) = x_1a^{-1} \tag{10.1.3}$$

and

$$\sigma_*(b) = b \tag{10.1.4}$$

Moreover, the following equalities hold:

$$\lambda_\sigma^{-1}\sigma(x_1)\lambda_\sigma = ax_1^{-1}x_2^{-1}x_1a^{-1} \quad \text{and} \quad \lambda_\sigma^{-1}\sigma(x_2)\lambda_\sigma = bax_1^{-1}a^{-1}b^{-1}.$$

10.2 Character stacks for non-orientable surfaces

We fix an algebraically closed field K (which for us will be either $\mathbb{C}$ or $\overline{\mathbb{F}}_q$). We denote by G the general linear group GL_n over K and by $\theta : G \to G$ the Cartan involution $g \to ({}^tg)^{-1}$. The corresponding semidirect product will be denoted by

$$G^+ \coloneqq G \rtimes_\theta \mathbb{Z}/(2).$$

Let $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_k)$ be a k-tuple of semisimple conjugacy classes of G. We consider the variety

$$X_{\mathcal{C}}^\epsilon \coloneqq \{\rho : \Pi^\epsilon \to G^+ \mid \pi(\rho(d_j)) = -1 \text{ and } \rho(z_i) \in h(\mathcal{C}_i) \text{ for all } i, j\}$$

where $\pi : G^+ \to \mathbb{Z}/(2)$ is the natural projection and $h : G \to G^+$ the natural inclusion. Given the explicit presentation of Π^ϵ we can rewrite $X_{\mathcal{C}}^\epsilon$ as

$$X_{\mathcal{C}}^\epsilon = \{(D_1, \ldots, D_r, Z_1, \ldots, Z_k) \in G^r \times \mathcal{C}_1 \times \cdots \mathcal{C}_k \mid D_1\theta(D_1)\cdots D_r\theta(D_r)Z_1\cdots Z_k = 1\}.$$

The variety $X_{\mathcal{C}}^\epsilon$ is endowed with a G-action defined as:

$$g \cdot D_i = gD_i\,{}^tg \quad g \cdot Z_i = gZ_ig^{-1}. \tag{10.2.1}$$

The character stacks for the non-orientable surface (X, σ) are the quotient stacks

$$\mathcal{M}_{\mathcal{C}}^\epsilon = [X_{\mathcal{C}}^\epsilon/G].$$

As $X^\epsilon_{\mathcal{C}}$ is affine and G is reductive, we can also consider the GIT quotient

$$M^\epsilon_{\mathcal{C}} \coloneqq X^\epsilon_{\mathcal{C}}//G$$

and the universal map $q : \mathcal{M}^\epsilon_{\mathcal{C}} \to M^\epsilon_{\mathcal{C}}$.

The stacks $\mathcal{M}^\epsilon_{\mathcal{C}}$ admit an alternative description in terms of the so-called real σ-invariant representations (which can be found, for instance, in [81, Section 2][79, Section 3, 3.2] and [65, Remark 4.2]).
A representation $\rho \in X^\epsilon_{\mathcal{C}}$ gives by restriction a representation $\tilde{\rho} : \Pi \to G$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi & \xrightarrow{p_*} & \Pi^\epsilon & \xrightarrow{\chi} & \mathbb{Z}(2) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tilde{\rho}} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 1 & \longrightarrow & G & \xrightarrow{\iota} & G^+ & \xrightarrow{\pi} & \mathbb{Z}/(2) & \longrightarrow & 1 \end{array} \tag{10.2.2}$$

It is therefore natural to ask conversely which representations $\tilde{\rho}$ of Π can be lifted to a morphism $\rho : \Pi^\epsilon \to G^+$ which makes the diagram (10.2.2) commute. To answer to the question, it is necessary to precisely describe monodromies around the punctures, as explained in [65, Remark 4.2].

We can rewrite the standard presentation of Π as

$$\langle [a_1, b_1] \cdots [a_g, b_g] x_{1,1} \cdots x_{1,k} x_{2,1} \cdots x_{2,k} = 1 \rangle.$$

where each $x_{i,j}$ is a path around $y_{i,j}$. Let $\tilde{\mathcal{C}}$ be the $2k$-tuple $\tilde{\mathcal{C}} = (\mathcal{C}_1, \ldots, \mathcal{C}_k, \mathcal{C}_1, \ldots \mathcal{C}_k)$ and consider the associated representation variety $X_{\tilde{\mathcal{C}}}$.
For a representation $\tilde{\rho} \in X_{\tilde{\mathcal{C}}}$ we say that $\tilde{\rho}$ is σ-invariant if

$$\tilde{\rho} \cong \theta(\tilde{\rho}(\sigma_*)).$$

This is equivalent to asking for the existence of an element $h_\sigma \in G$ which verifies

$$h_\sigma \tilde{\rho} h_\sigma^{-1} = \theta(\tilde{\rho}(\sigma_*)). \tag{10.2.3}$$

Definition 10.2.1. Given a σ-invariant $\tilde{\rho} \in X_{\tilde{\mathcal{C}}}$, we say that the representation $\tilde{\rho}$ is *real* if there exists h_σ as in eq.(10.2.3) such that

$$\tilde{\rho}(\sigma(\lambda_\sigma)\lambda_\sigma) = h_\sigma^{-1}\theta(h_\sigma^{-1}). \tag{10.2.4}$$

We say that $\tilde{\rho}$ is *quaternionic* if there exists h_σ as in eq.(10.2.3) such that $\tilde{\rho}(\sigma(\lambda_\sigma)\lambda_\sigma) = -h_\sigma^{-1}\theta(h_\sigma^{-1})$.

If the conditions of Equations (10.2.3),(10.2.4) are satisfied, the couple $(\tilde{\rho}, h_\sigma)$ can be extended to a map $\rho \in X^{\epsilon}_{\mathcal{C}}$ such that the diagram (10.2.2) commutes. Let $\tilde{\mathcal{U}}_{\mathcal{C}}$ be the variety

$$\tilde{\mathcal{U}}_{\mathcal{C}} = \{(\tilde{\rho}, h_\sigma) \in X_{\tilde{\mathcal{C}}} \times G \text{ which verify Equations 10.2.3, 10.2.4}\}. \tag{10.2.5}$$

The variety $\tilde{\mathcal{U}}_{\mathcal{C}}$ is endowed with a G-action defined by

$$g \cdot (\tilde{\rho}, h_\sigma) := (g\tilde{\rho}g^{-1}, \theta(g)hg^{-1}). \tag{10.2.6}$$

The arguments above imply the following Proposition

Proposition 10.2.2. *There is an isomorphism of quotient stacks*

$$\mathcal{M}^{\epsilon}_{\mathcal{C}} \cong [\tilde{\mathcal{U}}_{\mathcal{C}}/G].$$

Remark 10.2.3. If $\tilde{\rho}$ is an irreducible representation and h is such that there is an equality $h\tilde{\rho}h^{-1} = \theta(\tilde{\rho}(\sigma_*))$, then either $h^{-1}\theta(h^{-1}) = \tilde{\rho}(\sigma(\lambda_\sigma)\lambda_\sigma)$ or $h^{-1}\theta(h^{-1}) = -\tilde{\rho}(\sigma(\lambda_\sigma)\lambda_\sigma)$ and only one of the two is true (see [82, III.5.1.2]), i.e an irreducible σ-invariant representation is either real or quaternionic.

Remark 10.2.4. It is natural to consider the stack $\mathcal{M}_{\tilde{\mathcal{C}}}$ and the associated GIT quotient $M_{\tilde{\mathcal{C}}}$. The stacks $\mathcal{M}_{\tilde{\mathcal{C}}}$ and $M_{\tilde{\mathcal{C}}}$ admit an involution, which we denote again by σ, induced by the map

$$\sigma(\tilde{\rho}) := \theta(\tilde{\rho}(\sigma_*)).$$

We can define a morphism $f : M^{\epsilon}_{\mathcal{C}} \to M^{\sigma}_{\tilde{\mathcal{C}}}$ which maps a couple $(\tilde{\rho}, h)$ as in Equation (10.2.5) to the representation $\tilde{\rho}$.
In a slightly more involved way, it would be possible to lift the map f to a morphism of quotient stacks $F : \mathcal{M}^{\epsilon}_{\mathcal{C}} \to \mathcal{M}^{\sigma}_{\tilde{\mathcal{C}}}$. These morphisms are in general not even surjective. We will describe the image of f in certain cases in Proposition 10.4.3.

10.3 Cohomology results for generic character stacks for non-orientable surfaces

Put $K = \mathbb{C}$ and let us now explain one of the main results of [65] about the stacks $\mathcal{M}^{\epsilon}_{\mathcal{C}}$. Let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_k)$ be a k-tuple of semisimple conjugacy classes of G.
In [65, Theorem 4.6], the authors showed the following Theorem.

Theorem 10.3.1. *For any generic $\mathcal{C}$, the following equality holds:*

$$E(\mathcal{M}^{\epsilon}_{\mathcal{C}}, q) = \frac{q^{\frac{d_\mu}{2}}}{q-1}\mathbb{H}_{\boldsymbol{\mu},r}\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right) \tag{10.3.1}$$

where $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ is the multipartition given by the multiplicities of the eigenvalues of

$\mathcal{C}_1, \ldots, \mathcal{C}_k$ *respectively and*

$$d_{\boldsymbol{\mu}} = n^2(r-2+k) + 2 - \sum_{i,j} (\mu_i^j)^2.$$

This result is surprisingly similar to Theorem 9.1.6 about generic character stacks for Riemann surfaces. Notice, for instance, that for $r = 2h$ the E-polynomial of $\mathcal{M}_{\mathcal{C}}^{\epsilon}$ agrees thus with the one of $\mathcal{M}_{\mathcal{C}}$ for a Riemann surface Σ of genus h.

In [65, Theorem 4.6] it is proved that a formula analogous to Formula (9.1.6) holds in the non-orientable setting for $r = k = 1$ i.e that the following equality holds

$$H_c(\mathcal{M}_{\mathcal{C}}^{\epsilon}, q, t) = \frac{(qt^2)^{\frac{d_\mu}{2}}}{qt^2 - 1} \mathbb{H}_{\boldsymbol{\mu},1}\left(t\sqrt{q}, -\frac{1}{\sqrt{q}}\right). \tag{10.3.2}$$

It would therefore have been natural to expect that such a formula holds for all r, k i.e that

$$H_c(\mathcal{M}_{\mathcal{C}}^{\epsilon}, q, t) = \frac{(qt^2)^{\frac{d_\mu}{2}}}{qt^2 - 1} \mathbb{H}_{\boldsymbol{\mu},r}\left(t\sqrt{q}, -\frac{1}{\sqrt{q}}\right). \tag{10.3.3}$$

for a generic $\mathcal{C}$. The main result of this paper is a counterexample to Formula (10.3.3), obtained by an explicit description of these spaces in the case $r = 2$, i.e when X is an elliptic curve.

10.4 Character stacks for $k = 1$ and generic central orbit

In this section we assume that $K = \mathbb{C}$. We fix $r \geq 1$ and a Riemann surface X of genus $g \coloneqq r - 1$ with an antiholomorphic involution $\sigma : X \to X$.

Consider a point $z_1 \in S = X/\langle\sigma\rangle$ and the subset $D' \coloneqq \{z_1\} \subseteq \Sigma$ (i.e $k = 1$). Let $d, n \in \mathbb{N}$ such that d is even and $(d, n) = 1$.
Let $\mathcal{C}$ be the generic semisimple orbit of $\mathrm{GL}_n(\mathbb{C})$ given by $\mathcal{C} = \{e^{\pi i \frac{d}{n}} I_n\}$. We denote the associated character stack in this case by $\mathcal{M}_{n,d}^{\epsilon} \coloneqq \mathcal{M}_{\mathcal{C}}^{\epsilon}$ and similarly the associated GIT quotient by $M_{n,d}^{\epsilon}$.

Remark 10.4.1. As $\mathcal{C}$ is a central orbit, the character stack $\mathcal{M}_{\mathcal{C}}$ is the twisted character stack

$$\mathcal{M}_{n,d} = \left[\{A_1, B_1, \ldots, A_g, B_g \in \mathrm{GL}_n \mid [A_1, B_1] \cdots [A_g, B_g] = e^{\frac{2\pi i d}{n}}\}/\,\mathrm{GL}_n\right].$$

As d and n are coprime, the representations $\tilde{\rho} \in \mathcal{M}_{n,d}$ are irreducible, see for example [44, Lemma 2.2.6]. In this case, given an element $\rho \in X_{\mathcal{C}}^{\epsilon}$ corresponding to a couple $(\tilde{\rho}, h_\sigma)$ with $\tilde{\rho} \in X_{\tilde{\mathcal{C}}}$ we have $\mathrm{Stab}_G(\rho) = \pm 1$ (see [82, III.5.1.3]). The morphism

$$q : \mathcal{M}_{n,d}^{\epsilon} \to M_{n,d}^{\epsilon}$$

is thus a μ_2-gerbe.

Remark 10.4.2. The canonical morphism $q : \mathcal{M}_{n,d}^{\epsilon} \to M_{n,d}^{\epsilon}$, being a μ_2-gerbe, is proper. The proper base change for Artin stacks implies that for every $x \in M_{n,d}^{\epsilon}$ and for every $i \in \mathbb{Z}$ we have

$$(R^i q_* \mathbb{C})_x = H^i(B\mu_2).$$

As the rational higher cohomology of $B\mu_2$ vanishes, $R^i q_* \mathbb{C} = 0$ if $i \neq 0$ and $q_* \mathbb{C} = \mathbb{C}$. The Leray spectral sequence for cohomology with compact support implies that we have

$$H_c^p(\mathcal{M}_{n,d}^\epsilon) \cong H_c^p(M_{n,d}^\epsilon).$$

The cohomology of the quotient stack is isomorphic to that of the GIT quotient. In particular, the (compactly-supported) cohomology of $\mathcal{M}_{n,d}^\epsilon$ is 0 in negative degrees.

The main result of this paragraph is the following proposition:

Proposition 10.4.3.
(i) If r is odd there are no quaternionic representations inside $M_{n,d}^\sigma$. If r is even, $M_{n,d}^\sigma$ admits a decomposition into open-closed subvarieties

$$M_{n,d}^\sigma = M_{n,d}^{\sigma,+} \bigsqcup M_{n,d}^{\sigma,-}$$

where $M_{n,d}^{\sigma,+}, M_{n,d}^{\sigma,-}$ are given by real/quaternionic representations respectively and there is an isomorphism $M_{n,d}^{\sigma,+} \cong M_{n,d}^{\sigma,-}$.
(ii) The map $f : M_{n,d}^\epsilon \to M_{n,d}^{\sigma,+}$ introduced in Remark 10.2.4 is an isomorphism.

Remark 10.4.4. Proposition 10.4.3 and the other results of this section are probably known to the experts but we could not locate a reference in the literature. We review them here for the sake of completeness.

Before proving Proposition 10.4.3, we notice that the quaternionic and the real representations form disjoint subsets by Remark 10.2.3.
To see that there are no quaternionic representations for r odd, we will use the equivalence between quaternionic representations and quaternionic Higgs bundles. As this correspondence is crucial for the study of the varieties $M_{n,d}^\epsilon$, let us briefly review it here. For more details, see for example [11],[80],[6],[7],[9]

10.4.1 Real and quaternionic Higgs bundles

A Higgs bundle over X is a pair $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is a vector bundle over X and Φ a morphism $\Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1$.
The moduli space of (stable) Higgs bundle over $\tilde{\Sigma}$ of rank n and degree d is denoted by $M_{Dol,n,d}$ (for a definition of stability see for example [6, Section 4.1] or [11, Definition 2.3]).
It is a fundamental result (see for example [88]) that there is a homeomoprhism (called non abelian Hodge correspondence)

$$M_{Dol,n,d} \cong M_{n,d}. \tag{10.4.1}$$

We consider the involution on $M_{Dol,n,d}$, which we denote again by σ, given by

$$\sigma((\mathcal{E},\Phi)) = (\sigma^*(\overline{\mathcal{E}}), -\sigma^*(\overline{\Phi}))$$

and we say that a Higgs bundle $(\mathcal{E},\Phi)$ is σ-invariant if there exists an isomorphism

$$\alpha : (\mathcal{E},\Phi) \to \sigma((\mathcal{E},\Phi)).$$

Real Higgs bundles are pairs $((\mathcal{E},\Phi),\alpha)$ such that

$$\sigma^*(\overline{\alpha})\alpha = I_{\mathcal{E}}.$$

In a similar way, quaternionic Higgs bundles are defined by asking for the equality

$$\sigma^*(\overline{\alpha})\alpha = -I_{\mathcal{E}}.$$

In [11, Proposition 5.6],[9, Theorem 4.8] it is shown that the homemorphism (10.4.1) restricts to a homeomorphism

$$M^{\sigma}_{n,d} \cong M^{\sigma}_{Dol,n,d}.$$

In *loc.cit* it is shown moreover that this bijection sends real/quaternionic representations into real/quaternionic Higgs bundles respectively. We will denote the subsets of $M_{Dol,n,d}$ given by real/quaternionic Higgs bundles by $M^{\sigma,+}_{Dol,n,d}/M^{\sigma,-}_{Dol,n,d}$ respectively.
Notice that, as

$$\sigma : M_{Dol,n,d} \to M_{Dol,n,d}$$

is antiholomorphic, the fixed points locus $M^{\sigma}_{Dol,n,d}$ is not a complex algebraic variety anymore. It is a real analytic variety which is identified with the set of $\mathbb{R}$-points of $M_{Dol,n,d}$ with respect to the real structure induced by σ.

For odd r, if a quaternionic couple $(\tilde{\rho},h)$ existed (i.e $M^{\sigma,-}_{n,d} \neq \varnothing$) there would exist a stable quaternionic Higgs bundle $(\mathcal{E},\Phi)$ on X. Its determinant $\det(\mathcal{E})$ would be a quaternionic line bundle of degree d over X.
The quaternionic condition is preserved under taking the determinant as n is odd. The existence of a quaternionic line bundle for odd r is ruled out by the topological criterion of [79, Theorem 2.4].

To prove Proposition 10.4.3, we will need the following preliminary Lemma.

Lemma 10.4.5. *Put $X_{n,d} := X_{\tilde{\mathcal{C}}}$ and let us consider the varieties $Y_{n,d}, Z_{n,d}$ defined by*

$$Y_{n,d} := X_{n,d} \times_{M_{n,d}} X_{n,d} = \{(\tilde{\rho}_1,\tilde{\rho}_2) \mid \tilde{\rho}_1 \cong \tilde{\rho}_2\}$$

and

$$Z_{n,d} := \{(\tilde{\rho}_1,\tilde{\rho}_2,h) \mid (\tilde{\rho}_1,\tilde{\rho}_2) \in Y \ , \ h \in \mathrm{GL}_n \ \mid h\tilde{\rho}_1 h^{-1} = \tilde{\rho}_2\}.$$

The projection map $\psi : Z_{n,d} \to Y_{n,d}$ is a principal $\mathbb{G}_m$-bundle for the étale topology.

Proof. The variety $Z_{n,d}$ is endowed with the $\mathbb{G}_m$ action defined as

$$t \cdot (\rho_1, \rho_2, h) = (\rho_1, \rho_2, th).$$

This action is free and transitive on the fibers of ψ, as all the representations inside $X_{n,d}$ are irreducible. Moreover $\psi(t \cdot z) = \psi(z)$ for all $z \in Z_{n,d}$. We are thus reduced to show that ψ is locally trivial for the étale topology.
As the map $\tilde{q} : X_{n,d} \to M_{n,d}$ is a principal PGL_n-bundle for the étale topology, there exists an étale open covering $\{U_i\}_{i \in I}$ of $M_{n,d}$ such that $\tilde{q}^{-1}(U_i) \cong U_i \times \mathrm{PGL}_n$ for each $i \in I$.
Put $Y_{U_i} \coloneqq Y_{n,d} \times_{M_{n,d}} U_i$ and similarly $Z_{U_i} \coloneqq Z_{n,d} \times_{M_{n,d}} U_i$. It is enough to show that the pullback map $\psi : Z_{U_i} \to Y_{U_i}$ is locally trivial in the étale topology for each $i \in I$.
Fix then $i \in I$ and put $U_i = U$. Notice that the variety Y_U admits the following isomorphism:

$$Y_U = \tilde{q}^{-1}(U) \times_U \tilde{q}^{-1}(U) \cong (U \times \mathrm{PGL}_n) \times_U (U \times \mathrm{PGL}_n) \cong U \times \mathrm{PGL}_n \times \mathrm{PGL}_n \,.$$

In a similar way, the variety Z_U is isomorphic to

$$Z_U = \psi^{-1}(Y_U) = \{(u, g, h, s) \in U \times \mathrm{PGL}_n \times \mathrm{PGL}_n \times \mathrm{GL}_n \mid gh^{-1} = [s]\}$$

so that ψ corresponds to the morphism $\psi(u, g, h, s) = (u, g, h)$. We can view Y_U as a subset of $U \times \mathrm{PGL}_n \times \mathrm{PGL}_n \times \mathrm{PGL}_n$ as

$$Y_U = \{(u, g, h, s) \in U \times \mathrm{PGL}_n \times \mathrm{PGL}_n \times \mathrm{PGL}_n \mid gh^{-1} = s\}.$$

Via these identifications, the map ψ corresponds to the restriction of the morphism

$$U \times \mathrm{PGL}_n \times \mathrm{PGL}_n \times \mathrm{GL}_n \to U \times \mathrm{PGL}_n \times \mathrm{PGL}_n \times \mathrm{PGL}_n$$

given by the identity on the first three factors and the quotient map $\mathrm{GL}_n \to \mathrm{PGL}_n$ on the last one. This is a principal $\mathbb{G}_m$-bundle because $\mathrm{GL}_n \to \mathrm{PGL}_n$ is so.

□

We now prove Proposition 10.4.3. We keep the notations of Lemma 10.4.5.

Proof of Proposition 10.4.3.
Let $\tilde{q} : X_{n,d} \to M_{n,d}$ be the quotient map. Put $X^{\sigma}_{n,d} \coloneqq \tilde{q}^{-1}(M^{\sigma}_{n,d})$ and $X^{\sigma,+}_{n,d} = \tilde{q}^{-1}(M^{\sigma,+}_{n,d})$ and similarly for quaternionic representations $X^{\sigma,-}_{n,d} = \tilde{q}^{-1}(M^{\sigma,-}_{n,d})$. The variety $X^{\sigma}_{n,d}$ is isomorphic to the closed subvariety $Y^{\sigma}_{n,d}$ of $Y_{n,d}$ given by:

$$Y^{\sigma}_{n,d} = \{(\tilde{\rho}_1, \tilde{\rho}_2) \in Y_{n,d} \mid \tilde{\rho}_2 = \theta \tilde{\rho}_1 \sigma_*\}$$

via the map $p_1|_{Y^{\sigma}_{n,d}} : Y^{\sigma}_{n,d} \to X^{\sigma}_{n,d}$, where p_1 is the projection onto the first factor of $Y_{n,d}$. Put $Y^{\sigma,+}_{n,d} = p_1^{-1}(X^{\sigma,+}_{n,d})$ and similarly $Y^{\sigma,-}_{n,d}$. From Remark 10.2.3 there is a well-defined morphism

$$p_3 : \psi^{-1}(Y^{\sigma}_{n,d}) \to \{I_n, -I_n\}$$

$$(\tilde\rho_1, \tilde\rho_2, h) \to \theta(h)h\tilde\rho(\sigma(\lambda_\sigma)\lambda_\sigma).$$

Notice that $Y^{\sigma,+}_{n,d} = \psi(p_3^{-1}(I_n))$ and $Y^{\sigma,-}_{n,d} = \psi(p_3^{-1}(-I_n))$. As ψ is open, we deduce that $X^{\sigma,+}_{n,d}$, $X^{\sigma,-}_{n,d}$ are disjoint and open and so closed too inside $X^{\sigma}_{n,d}$. The same is true then for $M^{\sigma,+}_{n,d}, M^{\sigma,-}_{n,d}$. The projection $(\tilde\rho_1, \tilde\rho_2, h) \to (\tilde\rho_1, h)$ induces an isomorphism $\psi^{-1}(Y^{\sigma,+}_{n,d}) = X^{\epsilon}_{n,d}$. By Proposition 10.4.5, the morphism

$$X^{\epsilon}_{n,d} \to X^{\sigma,+}_{n,d}$$

is thus a principal $\mathbb{G}_m$-bundle. The G-action on $X^{\epsilon}_{n,d}$ defined by the Formula (10.2.6) induces an action of the center $Z_G = \mathbb{G}_m$ which differs from the one coming from the principal $\mathbb{G}_m$-bundle structure by a square factor. The morphism $X^{\epsilon}_{n,d} \to X^{\sigma,+}_{n,d}$ induces thus a G-equivariant isomorphism

$$X^{\epsilon}_{n,d}/(\mathbb{G}_m/(\pm I_n)) \cong X^{\sigma,+}_{n,d} \tag{10.4.2}$$

We deduce the following chain of isomorphisms:

$$M^{\epsilon}_{n,d} = X^{\epsilon}_{n,d}/(\mathrm{GL}_n(\mathbb{C})/(\pm I_n)) \cong (X^{\epsilon}_{n,d}/(\mathbb{G}_m/(\pm I_n)))/(\mathrm{GL}_n(\mathbb{C})/\mathbb{G}_m) \cong M^{\sigma,+}_{n,d}.$$

□

To end the proof of Proposition 10.4.3, it actually remains to show that $M^{\sigma,+}_{n,d}, M^{\sigma,-}_{n,d}$ are isomorphic if r is even. For r even there exists a quaternionic representation $\tau \in M^{\sigma,-}_{1,0}$ of rank 1 over X (see [80, Theorem 2.4]).
Taking the tensor product by τ gives then an isomorphism $-\otimes\tau : M^{\sigma,+}_{n,d} \to M^{\sigma,-}_{n,d}$: the same proof was carried out for real and quaternionic vector bundles in [80, Theorem 1.1].

10.5 Character stacks for (real) elliptic curves

We focus now on the case $r = 2$. We consider the elliptic curve X and the antiholomorphic involution σ introduced in Example 10.1.2. We keep the notations introduced in the Example 10.1.2.
In [44, Lemma 2.2.6] it is shown that for $(n, d) = 1$ there is an isomorphism

$$M_{n,d} = \mathbb{C}^* \times \mathbb{C}^*. \tag{10.5.1}$$

To see this, notice that a representation $\tilde\rho \in M_{n,d}$ corresponds to a pair of matrices A, B such that

$$B^{-1}A^{-1}BA = e^{\frac{2\pi i d}{n}} 1_n.$$

where $\tilde\rho(a) = A$ and $\tilde\rho(b) = B$. Let $z, w \in \mathbb{C}^*$ such that $A^n = zI_n$ and $B^n = wI_n$(see [44, Theorem 2.2.17]). The isomorphism (10.5.1) is obtained by mapping $\tilde\rho$ to the couple (z, w). Via this identification, the involution σ is given by:

$$\sigma(z, w) = (z, w^{-1})$$

and so

$$M^{\sigma}_{n,d} = \mathbb{C}^* \bigsqcup \mathbb{C}^*.$$

From Equation (10.1.3) we deduce indeed that

$$\theta(\tilde{\rho}(\sigma_*(b))) = \theta(\tilde{\rho}(b)) = \theta(B)$$

and so $(\theta(\tilde{\rho}(\sigma_*(b))))^n = \theta(B)^n = w^{-1}I_n$. By Equation (10.1.4) the following equality holds:

$$\theta(\tilde{\rho}(\sigma_*(a))) = \theta(\tilde{\rho}(x_1 a^{-1})) = \theta(\tilde{\rho}(x_1))\theta(A^{-1}) = e^{-\frac{\pi d}{n}} A^t$$

and so $(\theta(\tilde{\rho}(\sigma_*(a)))^n = (A^t)^n = zI_n$. By Proposition 10.4.3, we deduce the following result :

Theorem 10.5.1. *For $r = 2$, the character variety $M^{\epsilon}_{n,d}$ is isomorphic to $\mathbb{C}^*$ as an affine variety and the character stack $\mathcal{M}^{\epsilon}_{n,d}$ is a μ_2-gerbe over $\mathbb{C}^*$.*

By Remark 10.4.2, for $r = 2$ we have the following identity:

$$H_c(\mathcal{M}^{\epsilon}_{n,d}, q, t) = qt^2 + t. \tag{10.5.2}$$

As suggested in the introduction, this does not agree with the expected formula (10.3.3). If the Formula (10.3.3) were true, the following identity would hold

$$H_c(\mathcal{M}^{\epsilon}_{n,d}, q, t) = \frac{qt^2}{qt^2 - 1}\mathbb{H}_{n,2}\left(t\sqrt{q}, -\frac{1}{\sqrt{q}}\right)$$

where $\mathbb{H}_{n,2}(z, w)$ are the functions defined in §3.8 for $\mu = ((n))$. The functions $\mathbb{H}_{n,2}(z, w)$ have been explicitly computed in [14, Theorem 1.0.2]. The result of [14] agrees with the conjectural formula (9.1.8) for the mixed Poincaré series of character varieties $\mathcal{M}_{n,d}$ for elliptic curves, i.e

$$(qt^2)\mathbb{H}_{n,2}\left(t\sqrt{q}, -\frac{1}{\sqrt{q}}\right) = (qt^2 + t)^2.$$

This implies that

$$\frac{(qt^2 + t)^2}{qt^2 - 1} = \frac{qt^2}{qt^2 - 1}\mathbb{H}_{n,2}\left(t\sqrt{q}, -\frac{1}{\sqrt{q}}\right) \neq (qt^2 + t) \tag{10.5.3}$$

giving a counterexample to the conjectural formula (10.3.3).

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