A brief introduction to C^0-symplectic topology

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This note summarizes a talk that I gave at the "Workshop on Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry" held in the MFO Oberwolfach in July 2012. It will appear in the Oberwolfach reports. The goal of the talk was to give an idea of what is "*C^0-symplectic topology*". It was impossible to speak about all known results and point of views on this subject in 50 minutes so I decided to concentrate on three particular theorems, to motivate them and to give an idea of their proof.

We will denote by (M, ω) a symplectic manifold. A Hamiltonian is a smooth compactly supported map $H : [0,1] \times M \to \mathbb{R}$. Its symplectic gradient X_H generates a flow denoted ϕ_H^t. The poisson bracket of two smooth functions H and K is given by the formula $\{H, K\} = \omega(X_H, X_K)$. The following theorems hold on any symplectic manifold.

Theorem 1 (Gromov-Elishberg , see [6]) *Let ϕ_k be a sequence of symplectic diffeomorphisms. Suppose that it converges in the C^0-sense to some diffeomorphism ϕ. Then, ϕ is symplectic.*

Theorem 2 (Hofer [3], Lalonde-McDuff[5]) *Let H_k be a sequence of Hamiltonians. Suppose that*

1. *$\phi_{H_k}^1$ C^0-converges to some homeomorphism h,*
2. *H_k C^0-converges to 0.*

Then $h = Id$.

Theorem 3 (Cardin-Viterbo [2]) *Let F_k, G_k be sequences of Hamiltonians. Suppose that*

1. *F_k and G_k C^0-converge to some smooth functions F and G,*
2. *the Poisson bracket $\{F_k, G_k\}$ C^0-converges to 0.*

Then, $\{F, G\} = 0$.

Comments, motivations, applications

1. First note that these results are surprising! Indeed, in Theorem 1, being a symplectic diffeomorphism is a condition on the differential of the diffeomorphism. So their should be no such C^0-rigidity. Similarly, the Poisson bracket is defined only in terms of the derivatives of the functions, so in Theorem 3 the Poisson bracket should not behave well with respect to the C^0-topology.

2. Once we have these results, it is natural to wonder whether the right objects of symplectic topology are actually the smooth ones or whether they are less regular. More concretely, can one define C^0 counterparts to the classical smooth symplectic objects? For example, Theorem 1 allows to give a definition of what could be a *symplectic homeomorphism*: a homeomorphism which a C^0-limit of symplectic diffeomorphism. The question of defining a C^0-Hamiltonian dynamics is more subtle. We will discuss it later on.

3. The C^0-rigidity results can also help to understand better the smooth objects themselves. The best example of this is the recent story of the Poisson bracket. After Theorem 3 was discovered, many papers have been published to understand the phenomenon and improve this result. In the end, this has lead Buhovsky, Entov and Polterovich to define new symplectic invariants [1] and derive nice results in (smooth!) Hamiltonian dynamics.

A word on the proofs

Amazingly the three theorems above can all be deduced from the following well known result. As defined by Hofer, the energy of a Hamiltonian diffeomorphism is:

$$\|\phi\| = \inf\left\{\int_0^1 (\max H_t - \min H_t)dt \;\middle|\; \phi = \phi_H^1\right\}.$$

Theorem 4 (Hofer [3], Lalonde-McDuff[5]) *For any symplectic ball B of radius r, if a Hamiltonian diffeomorphism ϕ satisfies $\phi(B) \cap B = \emptyset$, then $\|\phi\| \geq \pi r^2$.*

Theorems 2 and 3 follows from that after some elementary differential calculus. To prove Theorem 1, a method is to define the notion of a symplectic capacity which is a way to measure the "symplectic size" of a subset of a symplectic manifold. The existence of symplectic capacities follows for example from Theorem 4. Then, one proves that a diffeomorphism is (anti-)symplectic if and only if it preserves symplectic capacities. Since the property of preserving a capacity is C^0-closed, Theorem 1 follows. This proof is nicely exposed in [6].

Attempts to define a continuous Hamiltonian dynamics

A first attempt has been proposed by Müller and Oh [7]. They define a *continuous Hamiltonian isotopy* as a path of homeomorphisms h^t with $h^0 = \mathrm{Id}$ and such that there exists a sequence of Hamiltonians H_k such that

1. $\phi^t_{H_k}$ C^0-converges to h^t,
2. H_k C^0-converges to some continuous function H.

A Hamiltonian homeomorphism is then any element of such an isotopy h^t. It follows from Theorem 2 that given a continuous H there is at most ont isotopy h^t such that the definition above is fullfilled. Therefore we can say that H "generates" h^t. Conversely, it is known (this is due independently to Viterbo and Buhovsky-Seyfaddini) that given a continuous Hamiltonian isotopy h^t there is a unique possible H up to constant. These uniqueness results show that this framework is a good generalization of what happens in the smooth case. Nevertheless, the existence problem is very hard. It is unknown which continuous functions actually generate a continuous Hamiltonian isotopy.

Another attempt (that would avoid this problem but create others) would be to work inside the completion of the Hamiltonian group for Hofer's distance. It is by definition given by $d(\phi,\psi) := \|\psi^{-1}\circ\phi\|$. The map between metric spaces $(C^\infty([0,1]\times M), \|\cdot\|_{C^0}) \to (\mathrm{Ham}(M,\omega), d)$, $H \mapsto \phi^t_H$ is Lipschitz. Thus, it extends to completions giving rise to a map $C^0([0,1]\times M) \to \overline{\mathrm{Ham}(M,\omega)}$. Hence, any continuous function has a flow in the completion. As before we can wonder whether the continuous Hamiltonian is unique up to constant. This question is answered positively on rationnal symplectic manifolds by a joint work with R. Leclercq and S. Seyfaddini [4].

Some open problems

There are many open interesting problems in this subject. Here my favorite ones:

1. Is the group of Hamiltonian diffeomorphisms C^0-closed in the group of symplectic diffeomorphisms? This is only known for surfaces, for the standard 2n-torus (Herman 83) and for a few more examples (Lalonde-McDuff-Polterovich 97).
2. Is the group of area preserving and compactly supported homeomorphisms of the 2-disk a simple group? The group of Hamiltonian homeomorphisms defined by Oh and Müller is a normal subgroup but so far no one has been able to prove that it is proper.

3. Which symplectic invariants are invariant under conjugation by a symplectic homeomorphism? For example in the case of the Calabi invariant it has been established by Gambaudo and Ghys that two Hamiltonian diffeomorphisms of the 2-disk that are conjugated by an area preserving homeomorphism have the same Calabi invariant. The analogous problem in higher dimension is open.

4. Understand "symplectically" Le Calvez's theory of area-preserving homeomorphisms of surfaces.

5. Extend Aubry-Math theory to general non-convex Hamiltonians. It is likely that one needs to consider symplectic objects (e.g., Lagrangian submanifolds) having low regularity to develop such an extension.

References

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