Mémoire

présenté pour obtenir le

Diplôme d'habilitation à diriger des recherches en mathématiques de l'Université Pierre et Marie Curie

par

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Titre du mémoire:

Géométrie symplectique C^0 et sélecteurs d'action.

Soutenue le 16 novembre 2017

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Remerciements

Je tiens avant tout à remercier les membres du jury, Alberto Abbondandolo, Marie-Claude Arnaud, Patrice Le Calvez, Dusa McDuff et Claude Viterbo. La présence de chacun d'eux m'honore profondément.

Je suis infiniment reconnaissant envers Marie-Claude Arnaud, Viktor Ginzburg et Dusa McDuff, qui ont accepté sans hésiter de passer une partie de leur été à écrire un rapport sur mon mémoire, malgré leurs emplois du temps très chargés.

J'adresse un remerciement particulier à Claude Viterbo, qui m'a initié à la fois au métier de chercheur et à la topologie symplectique, pour son soutien et pour son influence bénéfique, toujours présente.

Le métier d'enseignant-chercheur est très varié et ce mémoire d'HDR, ne rend compte que de mon travail de recherche. Est-il raisonnable d'employer le mot "travail" alors qu'il n'est pour moi que plaisir de faire, écouter, et parler des mathématiques? Mais mon plaisir n'est pas due aux mathématiques seules, il tient surtout à la bonne humeur et à la sérénité qui règne dans mon cadre de vie, dont je veux remercier tous les acteurs.

La qualité de mon environement professionel doit beaucoup aux membres de la joyeuse et gourmande équipe "analyse algébrique", aux membres de l'équipe administrative performante et souriante de l'IMJ-PRG, à la communauté des symplecticiens aussi sympathique que stimulante, enfin aux étudiants motivés de tout niveau.

J'ai eu tout particulièrement du plaisir à travailler avec Lev Buhovsky, Rémi Leclercq, Frédéric Le Roux, Emmanuel Opshtein, Nicolas Roy, Sobhan Seyfaddini et Nicolas Vichery. Il est tellement plus agréable de réfléchir et d'écrire à plusieurs que je ne pourrai probablement plus jamais écrire d'article autrement qu'en collaboration!

Enfin, tout cela ne serait rien sans l'aide d'Isabelle, Gabriel et Maylis, qui font régner bonne humeur et sérénité au quotidien.

Contents

Introduction (en Français)

La géométrie symplectique C^0 naît au début des années 80 avec le théorème de Gromov-Eliashberg [27]. Celui-ci affirme: *Un difféomorphisme qui est limite uniforme de difféomorphismes symplectiques est nécessairement symplectique.* Cet énoncé est surprenant puisque la condition d'être symplectique porte sur la différentielle, et n'a donc a priori aucune raison de bien se comporter par limite uniforme. C'est une manifestation de la rigidité symplectique.

Ce théorème donne naissance à une définition et un problème. Définition: *on appellera homéomorphisme symplectique tout homéomorphisme qui est (localement) limite uniforme de difféomorphismes symplectiques.* Problème: *existe-t-il une géométrie symplectique C^0?* Autrement dit, peut-on faire de la géométrie symplectique avec des objets seulement continus et non lisses? C'est cette définition et ce problème qui sont au coeur de mes recherches et de ce mémoire.

Depuis le théorème de Gromov-Eliashberg, les indices de l'existence d'une géométrie symplectique C^0 s'accumulent lentement mais sûrement, avec par exemple l'introduction des capacités symplectiques par Ekeland et Hofer [25], l'apparition de la distance de Hofer [54], la rigidité des lagrangiennes par Laudenbach et Sikorav [78], la rigidité C^0 du crochet de Poisson par Cardin et Viterbo [18]. En 2006, Oh et Müller définissent une notion d'homéomorphisme hamiltonien qui attire l'attention [101]. Depuis lors, les nouveaux résultats s'enchaînent rapidement et on en sait maintenant beaucoup sur les homéomorphismes symplectiques. On sait par exemple qu'ils préservent les sous-variétés coisotropes et leurs feuilletages caractéristiques [65], mais qu'ils ne préservent pas nécessairement le volume symplectique des sous-variétés [16]. On sait que les homéomorphismes hamiltoniens vérifient la conjecture d'Arnold sur les surfaces [85] mais pas en dimension supérieure [15]. Le subtil mélange de rigidité et de flexibilité qui caractérise la topologie symplectique se manifeste donc aussi en géométrie symplectique C^0.

Il est difficile d'étudier la géométrie symplectique C^0 sans s'intéresser aux sélecteurs d'action (souvent aussi appelés invariants spectraux). Ces invariants sont définis [135, 121, 98, 82] à partir de la fonctionnelle d'action hamiltonienne $\mathcal{A}_H$ associée à un hamiltonien H, dont les points critiques correspondent aux orbites du flot hamiltonien de H. Ils associent à tout hamiltonien H une valeur critique de $\mathcal{A}_H$, de manière canonique et continue en H. Autrement dit, ils "sélectionnent" une valeur critique préférée. La théorie de Floer, qui est intuitivement la theorie de Morse de la fonctionnelle d'action, a une structure algébrique très riche, qui induit de nombreuses propriétés sur les sélecteurs d'action. Grâce à ces propriétés, la liste des applications est immense; cf. page 19.

La dimension 2 est souvent considérée comme triviale ou inintéressante pour le symplecticien. Elle mérite néanmoins qu'on s'y attarde. D'abord, la petite dimension facilite la visualisation; on peut donc espérer y comprendre plus profondément des invariants symplectiques qui demeurent mystérieux (tels la distance de Hofer ou les sélecteurs d'action). C'est aussi en dimension 2 que la topologie symplectique C^0 est la plus développée. En effet, un homéomorphisme symplectique d'une surface n'est rien d'autre qu'un homéomorphisme préservant l'aire orientée et ceux-ci ont été beaucoup étudiés. Enfin, comme souvent en géométrie, la petite dimension a ses spécificités. Des outils puissants ont été développés (par ex. [80])pour les besoins de la dynamique topologique, indépendamment de ceux des symplecticiens, mais ces deux types d'outils démontrent souvent des résultats similaires. Selon la question étudiée, les uns ou les autres s'avèrent meilleurs. Comprendre les liens entre ces deux théories constitue un problème très intéressant.

Ce que l'on trouve dans ce mémoire.

Ce mémoire reprend la plupart des résultats de mes recherches (la plupart en collaboration) depuis la fin de ma thèse. Plus précisément, il expose les résultats détaillés dans les articles [63, 66, 65, 67, 64, 15, 14]. Le mémoire ne traite pas séparément des différents articles mais effectue une synthèse.

Le chapitre 0 est un chapitre préliminaire qui ne contient aucun de mes résultats personnels. Sa fonction est simplement d'introduire certaines définitions et notations, en particulier concernant l'homologie de Floer. Il prépare surtout le chapitre 1.

Le chapitre 1 introduit les sélecteurs d'actions, liste leur propriétés et expose mes contributions à cette théorie. Ceux-ci peuvent être définis dans une multitude de contextes différents. Nous nous concentrerons seulement ceux reliés à mes travaux, sans recherche de généralité maximale. En revanche, j'ai essayé de mettre l'accent sur un principe général: *toute propriété algébrique de l'homologie de Floer se reflète en une propriété des sélecteurs d'action.* Ce chapitre pourra sembler austère, car technique et pauvre en applications. Cependant, mes contributions sont toutes motivées par des résultats exposés dans les chapitres suivants, qui, je l'espère, seront plus agréables à lire.

Le chapitre 2 est un survol général de la géométrie symplectique C^0, qui s'attarde sur mes propres travaux. Sa lecture ne nécessite pas forcément d'avoir lu au préalable les chapitres précédents. En effet, les sélecteurs d'action y sont bien utilisés mais apparaissent souvent seulement brièvement dans des morceaux de démonstration.

Le chapitre 3 présente mes efforts pour essayer de comprendre aussi bien que possible d'une part la distance de Hofer Lagrangienne et d'autre part les sélecteurs d'action Hamiltoniens sur les surfaces. Il ne devrait pas non plus être nécessaire d'avoir lu en détail les chapitres précédents pour lire ce chapitre, même si bien sûr, la définition de sélecteur d'action sera nécessaire.

Si je ne devais garder que les trois résultats les plus importants à mes yeux, ce serait:

- Le théorème de rigidité C^0 des sous-variétés coisotropes (théorème 2.4.2, issu de [65]),
- Le contre-exemple C^0 à la conjecture d'Arnold (théorème 2.5.1, issu de [15]),
- Une formule explicite pour les sélecteurs d'action des Hamiltoniens autonomes sur les surfaces (théorème 3.2.1, issu de [64])

Si l'on m'autorisait à en garder dix, j'ajouterais à la liste:

- L'inégalité énergie-capacité relative (théorème 1.3.12, issu de [65]),
- L'inégalité énergie-capacité duale (théorème 1.3.14, issu de [66]),
- La formule du max (théorème 1.4.1, issu de [64]),
- Les résultats de rigidité des homéomorphismes hamiltoniens (théorèmes 2.3.11 et 2.3.12, issus de [65]),
- Le théorème sur la réduction des homéomorphismes symplectiques (théorème 2.4.11, issu de [67]),
- Les efforts pour sauver la conjecture d'Arnold C^0 (théorème 2.5.6, issu de [14]),
- La borne supérieure sur la distance de Hofer Lagrangienne dans le disque (théorème 3.1.5, issu de [63]).

Enfin, le mémoire contient quelques propositions jamais publiées car trop peu significatives, mais tout de même digne d'intérêt. Ce sont les propositions 2.4.4, 2.4.16 and 3.1.10.

Ce mémoire ne traite pas des travaux effectués pendant ma thèse, ni de l'article [68], trop déconnecté du reste.

Introduction (in English)

The field of C^0 symplectic geometry is born in the early eighties with the Gromov-Eliashberg theorem [27], which states: *If a diffeomorphism is a uniform limit of symplectic diffeomorphisms, then it is symplectic.* This is a surprising statement since being symplectic is a condition on the differential, hence has a priori no reason to behave well under uniform limits a priori. This is a manifestation of symplectic rigidity.

This theorem gives rise to a definition and a problem. Definition: *We call a symplectic homeomorphism every homeomorphism which (locally) a uniform limit of symplectic diffeormophisms.* Problem: *Is there C^0-symplectic geometry?* In other words, can one do symplectic geometry with objects that are only continuous and not smooth? This definition and this problem are at the core of my research and this memoir.

Since the Gromov-Eliashberg theorem, the indications in favor of the existence of C^0-symplectic geometry accumulate slowly but surely, with for instance the introduction of symplectic capacities Ekeland and Hofer [25], the appearance of Hofer's distance [54], Lagrangian rigidity by Laudenbach and Sikorav [78], the C^0 rigidity of the Poisson bracket by Cardin and Viterbo [18]. In 2006, Oh and Müller define a notion of Hamiltonian homeomorphism which draws the attention to the subject [101]. Since then, new results come out rapidly and we now know much of the properties of symplectic homeomorphisms. For example, we know that they preserve coisotropic submanifolds and their characteristic foliation [65], but that they do not preserve the symplectic volume of submanifolds [16]. We know that Hamiltonian homeomorphisms satisfy the Arnold conjecture on surfaces [85] but not in higher dimension [15]. The subtle mixture of rigidity and flexibility which characterizes symplectic topology manifests itself as well in C^0 symplectic geometry.

It is difficult to study C^0 symplectic geometry without getting interested in action selectors (often also called spectral invariants). These invariants are defined [135, 121, 98, 82] from the Hamiltonian action functional $\mathcal{A}_H$ associated to a Hamiltonian H, whose critical points correspond to Hamiltonian orbits of H. They associate to each Hamiltonian H a critical value of $\mathcal{A}_H$, in a canonical and continuous in H way. In other words, they "select" a prefered critical value. Floer theory, which is intuitively the Morse theory of the action functional, has a very rich alegebraic structure, which in turn induces many properties of the action selectors. Thanks to these properties, the list of their applications is immense; see Page 19.

Dimension 2 is often considered as trivial or uninteresting for the symplectic geometer. Nevertheless, it deserves some study. First, small dimension facilitate pictures and visualization; we may hope to understand more deeply in dimension 2 certain symplectic invariants which remain mysterious (such as Hofer's distance or action selectors). This is also in dimension 2 that C^0-symplectic topology is the most developed. Indeed, on a surface, a symplectic homeomorphism is nothing but an (oriented) area preserving homeomorphism, and those have been extensively studied. Finally, as often in geometry, small dimension has its own specificities. Powerful tools have been developed (e.g [80]) for the needs of topological dynamics, independently of those of symplectic geometry, but these two types of tools often lead to similar results. Which one lead to the best results depends on the question studied. Understaning the links between these two theories is a very interesting and challenging problem.

What is in this dissertation.

This memoir reviews most the results that I obtained (most of them in collaboration) since the end of my PhD. More precisely, it presents the results detailes in the papers [63, 66, 65, 67, 64, 15, 14]. It is a synthesis of these articles rather than a collection of separate reviews.

Chapter 0 is a preliminary chapter. It does not contain any personal contribution. Its role is simply to introduce definitions and notations, related in particular to Floer homology. It mainly prepares Chapter 1.

Chapter 1 introduces action selectors, reviews their properties and present my contributions to this theory. Action selectors may be defined in a multitude of different settings. We will concentrate only on those settings that are related to my work, without aiming at a maximal generalization. I have tried to emphasize the following general principle: *every algebraic property of Floer homology is reflected in a property of action selectors.* This chapter may seem austere, since it is technical and lacks applications. However, my contributions are all motivated by results presented in the next chapters, which, I hope, should be more pleasant to read.

Chapter 2 is a biased survey of C^0 symplectic geometry, which focuses on my own contributions. Its reading does not necessarily require to have read the preceeding chapters. Indeed, action selectors are present but mostly only appear briefly in pieces of proofs.

Chapter 3 present my efforts to understand as deeply as possible on surfaces the Lagrangian Hofer distance and the Hamiltonian action selectors. It should not be necessary either to have read in details the preceeding chapters, even though, of course, the definition of action selector should be useful for the second part.

If I had to only keep the three results that are the most important to me, I would certainly keep:

- The C^0 rigidity of coisotropic submanifolds (Theorem 2.4.2, from [65]),
- The C^0 counter-example to the Arnold conjecture (Theorem 2.5.1, from [15]),
- An explicit formula for action selectors of autonomous Hamiltonians on surfaces (Theorem 3.2.1, from [64])

If I was allowed to keep ten of them, I would add to the list:

- The relative energy-capacity inequality (Theorem 1.3.12, from [65]),
- The dual energy-capacity inequality (Theorem 1.3.14, from [66]),
- The max formula (Theorem 1.4.1, from [64]),
- The results on the rigidity of Hamiltonian homeomorphisms (Theorems 2.3.11 and 2.3.12, from [65]),
- The theorem on the reduction of symplectic homeomorphisms (Theorems 2.4.11, from [67]),
- The efforts to rescue the C^0 Arnold conjecture (Theorem 2.5.6, from [14]),
- The upper bound for the Lagrangian Hofer distance in the disk (Theorem 3.1.5, from [63]).

This memoir also contains a few propositions never published since not significant enough, but which might still have some interest. These are Propositions 2.4.4, 2.4.16 and 3.1.10.

This dissertation does not deal with the results obtained during my PhD, nor with the article [68] which is too disconnected from the rest.

Chapter 0

Preliminaries: basic definitions and a quick introduction to Floer theories

In this chapter, we give a quick introduction to Floer theory. It does not contain any personnal contribution. Our aim is simply to set the notations and definitions needed in the next chapters. There is no intention of exhaustivity, the Floer theories introduced are only those appearing in this dissertation.

0.1 Basic definitions and notations from symplectic geometry

Symplectic manifolds. A *symplectic manifold* is a pair (M, ω) where M is a smooth manifold of dimension $2n$, and ω is a closed non-degenerate 2-form. We will denote by $[\omega]$ the De Rham cohomology class of the symplectic form.

A compatible almost complex structure on M is a bundle map $J : TM \to TM$ such that $J^2 = -\mathrm{Id}$, $\omega(\cdot, J\cdot)$ is a Riemaniann distance and $\omega(J\cdot, J\cdot) = \omega$. A fundamental fact is that the set $\mathcal{J}$ of compatible almost complex structures on M is non-empty and contractible. The first Chern class of the complex bundle (TM, J) does not depend on $J \in \mathcal{J}$, and will be denoted by $c_1(TM)$.

A symplectic manifold is called *symplectically aspherical* if

$$\langle \omega, \pi_2(M) \rangle = 0 \quad \text{and} \quad \langle c_1(TM), \pi_2(M) \rangle = 0. \tag{1}$$

This condition is very restrictive but includes important examples like tori, or more generally surfaces of positive genus and their products.

A symplectic manifold is called *monotone* if there exists $\lambda > 0$ such that $[\omega] = \lambda c_1(TM)$. This class of symplectic manifolds is much larger. It includes in particular projective spaces. A symplectic manifold is called *rational* if the group $\langle \omega, \pi_2(M) \rangle$ is discrete. In particular, monotone symplectic manifolds are rational. Rational symplectic manifolds are ubiquitous since every symplectic form can be perturbed to be made rational.

Hamiltonian formalism. A *Hamiltonian function* is a function $H : \mathbb{S}^1 \times M \to \mathbb{R}$. It will sometimes be convenient to use the notation $H_t(x)$ for $H(t, x)$. The *Hamiltonian vector field* or *symplectic gradient* associated to H is the time-dependent vector field X_{H_t} (sometimes only denoted X_H) defined by

$$\omega(\cdot, X_{H_t}) = dH_t.$$

The vector field X_H generates an isotopy ϕ_H^t (at least for small times) which is called the Hamiltonian isotopy generated by H. We also say that ϕ_H^1 is the Hamiltonian diffeomorphism generated by H. We will say that a Hamiltonian H is supported in an open subset U if each function H_t is supported in U. We will say that a Hamiltonian diffeomorphism is supported in U if it can be generated by a Hamiltonian supported in U.

A *symplectic diffeomorphism* is a diffeomorphism which preserves the symplectic form. The group of symplectic diffeomorphisms of (M,ω) will be denoted $\mathrm{Symp}(M,\omega)$. The connected component of Id in $\mathrm{Symp}(M,\omega)$ will be denoted $\mathrm{Symp}_0(M,\omega)$. The *Hamiltonian group* of (M,ω) is the set of Hamiltonian diffeormorphisms:

$$\mathrm{Ham}(M,\omega) = \{\phi_H^1 \,|\, H : \mathbb{S}^1 \times M \to \mathbb{R}\}.$$

The set $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Symp}_0(M,\omega)$, which is itself a normal subgroup of $\mathrm{Symp}(M,\omega)$. If M is not compact, we may also consider compactly supported diffeomorphisms and Hamiltonians. The corresponding groups will be denoted $\mathrm{Ham}_c(M,\omega)$, $\mathrm{Symp}_{0,c}(M,\omega)$ and $\mathrm{Symp}_c(M,\omega)$.

The fact that $\mathrm{Ham}(M,\omega)$ forms a normal subgroup of $\mathrm{Symp}_0(M,\omega)$ is one of the consequences of the following identities, which holds for all Hamiltonians H, K.

$$\begin{array}{ll}
(\phi_H^t)^{-1} = \phi_{\overline{H}}^t, & \text{where } \overline{H} = -H(t,\phi_H^t(x)) \\
\phi_H^t \circ \phi_K^t = \phi_{H\sharp K}^t, & \text{where } H\sharp K(t,x) = H(t,x) + K(t,(\phi_H^t)^{-1}(x)) \\
(\phi_H^t)^{-1} \circ \phi_K^t = \phi_{\overline{H}\sharp K}^t, & \text{where } \overline{H}\sharp K = (K-H)(t,\phi_H^t(x)) \\
\psi^{-1} \circ \phi_H^t \circ \psi = \phi_{\psi^* H}^t, & \text{where } \psi^* H = H(t,\psi(x)) \\
\phi_H^{\rho(t)} = \phi_{H^\rho}^t & \text{where } H^\rho(t,x) = \rho'(t) H(\rho(t),x), \\
& \quad \text{and } \rho : [0,1] \to [0,1] \text{ is a non decreasing map.}
\end{array}$$

The diffeomorphism $\phi_H^1 \circ \phi_K^1$ can be generated by the Hamiltonian $H\sharp K$, but it can also be obtained by considering the "concatenated" Hamiltonian,

$$H \bullet K(t,x) = \begin{cases} 2K(2t,x), & t \in [0,\frac{1}{2}], \\ 2H(2t-1,x), & t \in (\frac{1}{2},1], \end{cases}$$

which generates the isotopy

$$\phi_{H\bullet K}^t(x) = \begin{cases} \phi_K^{2t}(x), & t \in [0,\frac{1}{2}], \\ \phi_H^{2t-1} \circ \phi_K^1(x), & t \in (\frac{1}{2},1]. \end{cases}$$

To be rigorous, $H \bullet K$ is not really a Hamiltonian since it is not smooth at $t = \frac{1}{2}$. However, by replacing H and K by appropriate time-reparametrizations H^ρ and K^ρ, we may have $K_t = 0$ for t close to 1 and $H_t = 0$ for t close to 0, which imply that $H^\rho \bullet K^\rho$ is a smooth Hamiltonian. Therefore, to simplify expositions, we will often ignore this issue and use $H \bullet K$ as if it was a smooth Hamiltonian.

Hofer norm. The *Hofer norm* of a Hamiltonian H is the quantity

$$\|H\| = \int_0^1 (\max H_t - \min H_t)\, dt.$$

It is also called *Hofer length* of the isotopy $(\phi_H^t)_{t\in[0,1]}$. The *Hofer norm* of a Hamiltonian diffeomorphism ϕ is the minimal Hofer length of an isotopy connecting Id to ϕ:

$$\|\phi\| = \inf\{\|H\| \,|\, H \text{ such that } \phi_H^1 = \phi\}.$$

It is a deep fact ([54, 106, 76]) that the Hofer norm is non-degenerate: $\|\phi\| = 0$ implies $\phi = \mathrm{Id}$. It can be deduced from the energy-capacity inequality; see Chapter 1, Lemma 1.3.6. The Hofer distance between two Hamiltonian diffeomorphisms ϕ, ψ is given by

$$d(\phi,\psi) \leqslant \|\psi \circ \phi^{-1}\|.$$

A Lagrangian submanifold is a submanifold of dimension $n = \frac{1}{2}\dim M$ such that $\omega|_{TL} = 0$. One can also define a Hofer distance on the space of Lagrangian submanifolds Hamiltonian isotopic to a given one, by setting (see [22]):

$$\delta(L, L') = \inf\{\|H\| \mid H \text{ such that } \phi_H^1(L) = L'\}.$$

0.2 Hamiltonian Floer theory on symplectically aspherical symplectic manifolds

In this paragraph, we assume that (M, ω) is a closed symplectic manifold which is symplectically aspherical. Floer homology was first introduced in this setting by Floer [33]. Standard references are [6, 88, 118].

We denote by $\mathcal{L}M$ the space of contractible smooth maps $\mathbb{S}^1 \to M$. The *Hamiltonian action functional* is defined as the map

$$\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}, \quad x \mapsto \int_{\mathbb{S}^1} H(t, x(t))\, dt - \int_{\mathbb{D}^2} u^*\omega, \tag{2}$$

where u is a capping disk of x, i.e. a smooth map $u : \mathbb{D}^2 \to M$, such that $u|_{\mathbb{S}^1} = x$. The condition $\langle \omega, \pi_2(M)\rangle = 0$ implies that the term $\int_{\mathbb{D}^2} u^*\omega$ does not depend on the choice of capping u. The action functional is a central object in Hamiltonian dynamics because of the following fact.

Lemma 0.2.1. *The critical points of $\mathcal{A}_H$ are exactly the 1-periodic orbits of the Hamitonian isotopy of H.*

We call a Hamiltonian H *non-degenerate* if its time-one map ϕ_H^1 is non-degenerate, i.e. if $d\phi_H^1 - \mathrm{Id}$ is invertible. Floer homology is essentially the Morse homology of the action functional $\mathcal{A}_H$ for a non-degenerate Hamiltonian H.

More precisely, given a non-degenerate Hamiltonian H, the Floer complex of H is defined as the $\mathbb{Z}/2$-vector space[1] spanned by $\mathrm{Crit}(\mathcal{A}_H)$, the set of critical points of the action functional. This vector space admits a graduation by the so-called Conley-Zehnder index i_{CZ}, which we will not recall here, but which is well defined thanks to the condition $\langle c_1(TM), \pi_2(M)\rangle = 0$. It can be turned into a chain complex by considering certain moduli spaces of curves that we now describe.

Pick a time-periodic family of ω-compatible almost complex structures J_t and consider maps $u : \mathbb{R} \times \mathbb{S}^1 \to M$ satisfying Floer's equation

$$\boxed{\partial_s u + J_t(u)\,(\partial_t u - X_{H_t}(u)) = 0} \tag{3}$$

This equation is a reformulation of the negative gradient equation $\frac{d}{ds}u = -\nabla \mathcal{A}_H(u)$, the gradient being defined with respect to the metric induced by J.

The set of Floer trajectories between two critical points of $\mathcal{A}_H$, x_- and x_+, is defined as

$$\widehat{\mathcal{M}}(x_-, x_+; H, J) = \left\{ u : \mathbb{R} \times \mathbb{S}^1 \to M \;\middle|\; \begin{array}{l} u \text{ satisfies (3)} \\ \forall t \in \mathbb{S}^1,\ u(\pm\infty, t) = x_\pm(t) \end{array} \right\}$$

where the limits $u(\pm\infty, t)$ are uniform in t. Note that the above set admits an $\mathbb{R}$–action by reparametrization $s \mapsto s+\tau$. The moduli space of Floer trajectories between x_- and x_+, denoted by $\mathcal{M}(x_-, x_+; H, J)$, is the quotient $\widehat{\mathcal{M}}(x_-, x_+; H, J)/\mathbb{R}$.

[1] The Floer complex can be defined with arbitrary coefficient ring. It is only for the sake of simplicity that we choose $\mathbb{Z}/2$ here.

A solution u of (3) is said to be *regular* if the linearization of the operator $u \mapsto \partial_s u + J_t(u)(\partial_t u - X_{H_t}(u))$ is surjective at u. The almost complex structure J is said to be *regular* if every $u \in \widehat{\mathcal{M}}(x_-, x_+; H, J)$, for any x_-, x_+, is regular. Regularity of J implies that the above moduli spaces are all smooth finite dimensional manifolds, and if $x_- \neq x_+$, the dimension of $\widehat{\mathcal{M}}(x_-, x_+; H, J)$ is $i_{CZ}(x_-) - i_{CZ}(x_+) - 1$. A suitably generic choice of J is regular in the following sense: The set of regular J's, denoted by $\mathcal{J}_{reg}(H)$, is of second category in the set of all 1-periodic compatible almost complex structures. If $i_{CZ}(x_-) - i_{CZ}(x_+) = 1$, the moduli space is compact, hence finite. This allows us to define the Floer boundary map $\partial : CF_*(H) \to CF_{*-1}(H)$: For a generator x_- we define ∂x_- by

$$\partial x_- = \sum_{x_+} \#\mathcal{M}(x_-, x_+; H, J) \cdot x_+$$

where the sum is taken over all 1–periodic orbits x_+ such that $i_{CZ}(x_-) - i_{CZ}(x_+) = 1$ and $\#$ denotes the mod–2 cardinality of $\mathcal{M}(x_-, x_+; H, J)$. The above definition is extended to the entire chain complex by linearity.

It can be proven that $\partial^2 = 0$ and thus ∂ defines a differential on $CF_*(H)$. The Floer homology of (H, J), denoted by $HF_*(H, J)$, is the homology of the complex $(CF_*(H), \partial)$.

Although the Floer complex depends on (H, J), the Floer homology groups are independent of this auxiliary data. Indeed, there exist morphisms

$$\Psi_{H_0,J_0}^{H_1,J_1} : CF(H_0) \to CF(H_1)$$

inducing isomorphisms in homology which are called continuation morphisms. To keep the notation light we will sometimes eliminate the almost complex structures from the notations and write $\Psi_{H_0}^{H_1}$. We now describe the morphism $\Psi_{H_0}^{H_1}$. Pick $J_i \in \mathcal{J}_{reg}(H_i)$ and take a homotopy, denoted by (H_s, J_s), $s \in \mathbb{R}$, from (H_0, J_0) to (H_1, J_1) such that

$$(H_s, J_s) = \begin{cases} (H_0, J_0) & \text{if } s \leqslant 0 \\ (H_1, J_1) & \text{if } s \geqslant 1 \end{cases}.$$

Consider maps $u : \mathbb{R} \times \mathbb{S}^1 \to M$ solving an s–dependent version of Floer's equation (3):

$$\partial_s u + J_{s,t}(u)(\partial_t u - X_{H_{(s,t)}}(u)) = 0. \tag{4}$$

For 1–periodic orbits $x_0 \in \mathrm{Crit}(\mathcal{A}_{H_0}), x_1 \in \mathrm{Crit}(\mathcal{A}_{H_1})$ define the moduli space

$$\mathcal{M}(x_0, x_1; H_s, J_s) = \left\{ u : \mathbb{R} \times S^1 \to M \;\middle|\; \begin{array}{l} u \text{ satisfies (4)} \\ u(-\infty, t) = x_0(t),\ u(+\infty, t) = x_1(t) \end{array} \right\}$$

A Floer trajectory $u \in \mathcal{M}(x_0, x_1; H_s, J_s)$ is said to be *regular* if the linearization of the operator $u \mapsto \partial_s u + J_{s,t}(u)(\partial_t u - X_{H_{(s,t)}}(u))$ is onto at u. The homotopy (H_s, J_s) is said to be regular if every $u \in \mathcal{M}(x_0, x_1; H_s, J_s)$, for any x_0, x_1, is regular. Regularity of (H_s, J_s) implies that the above moduli spaces are smooth finite dimensional manifolds of dimension $i_{CZ}(x_0) - i_{CZ}(x_1)$. A suitably generic choice of (H_s, J_s) is indeed regular. When the moduli space is zero–dimensional it is compact, hence finite. Thus, we can define

$$\Psi_{H_0}^{H_1}(x_0) = \sum_{x_1} \#\mathcal{M}(x_0, x_1; H_s, J_s) \cdot x_1$$

where the sum runs over all $x_1 \in \mathrm{Crit}(\mathcal{A}_{H_1})$ such that $i_{CZ}(x_0) = i_{CZ}(x_1)$ and $\#$ denotes mod–2 cardinality. The morphism $\Psi_{H_0}^{H_1}$ is then extended by linearity to all of $CF_*(H_0)$. It can be shown that continuation morphisms are chain maps and thus, descend to homology; we will continue to denote the maps induced on homology by the same notation. The induced map on homology does not depend on the choice of the homotopy (H_s, J_s). Furthermore, at the homology level, continuation maps satisfy the following composition rule:

$$\Psi_{H_0}^{H_0} = \mathrm{Id} \quad \text{and} \quad \Psi_{H_1}^{H_2} \circ \Psi_{H_0}^{H_1} = \Psi_{H_0}^{H_2}. \tag{5}$$

Thus $\Psi_{H_0}^{H_1}$ gives an isomorphism between $HF_*(H_0, J_0)$ and $HF_*(H_1, J_1)$.

Lastly, if H is taken to be a C^2–small Morse function then the Floer homology of H coincides with its Morse homology. It follows from the above that for any regular J, we have an isomorphism $HF_*(H,J) = H_*(M, \mathbb{Z}/2)$. This isomorphism can also be established via the PSS-construction [105]. Therefore, we will call this isomorphism the PSS-isomorphism and denote it $\Phi_{H,J}$. By (5), the following diagram commutes for all (H_0, J_0), (H_1, J_1) :

$$\begin{array}{ccc} H(M,\mathbb{Z}/2) & \xrightarrow{\Phi_{H_0,J_0}} & HF(H_0,J_0) \\ & \searrow^{\Phi_{H_1,J_1}} & \downarrow \Psi_{H_0,J_0}^{H_1,J_1} \\ & & HF(H_1,J_1)\ . \end{array}$$

0.3 Hamiltonian Floer theory on more general closed symplectic manifolds

In this section we explain briefly how Floer theory is defined on general (i.e. not necessarily aspherical) symplectic manifolds. This was developped [55, 102] for semi-positive manifolds and [39, 84] on general closed symplectic manifolds[2]. We will rely on this more general framework only for Theorems 1.3.14 and 2.3.6.

The first issue when dealing with general symplectic manifolds is that neither the action functional nor the Conley Zehnder index are well defined. To overcome this difficulty, we do not work anymore on the space of contractible loops $\mathcal{L}M$ but on an appropriate cover $\widetilde{\mathcal{L}M}$. The action of a loop is defined modulo the group $\langle \omega, \pi_2(M) \rangle$ and the index modulo $\langle c_1(TM), \pi_2(M) \rangle$, it is therefore natural to consider the quotient space:

$$\widetilde{\mathcal{L}M} = \{(x,u) \,|\, x \in \mathcal{L}M, u : \mathbb{D}^2 \to M, u|_{\mathbb{S}^1} = x\} / \sim,$$

where $(x,u) \sim (x', u')$ is and only if $x = x'$, $\langle \omega, \bar{u} \# u' \rangle = 0$ and $\langle c_1(TM), \bar{u} \# u' \rangle = 0$. The equivalence class of (x,u) will be denoted $[x,u]$. The map $\widetilde{\mathcal{L}M} \to \mathcal{L}M$, $[x,u] \mapsto x$ is a covering map, whose group of deck transformations is given by $\Gamma = \pi_2(M)/(\ker[\omega] \cap \ker c_1)$.

Generalizing Equation (2), the Hamiltonian action functional is defined as:

$$\mathcal{A}_H : \widetilde{\mathcal{L}M} \to \mathbb{R}, \quad [x,u] \mapsto \int_{\mathbb{S}^1} H(t, x(t))\, dt - \int_{\mathbb{D}^2} u^*\omega, \tag{6}$$

Lemma 0.2.1 can also be generalized: The critical points of $\mathcal{A}_H$ are the elements $[x,u] \in \widetilde{\mathcal{L}M}$ such that x is a 1-periodic orbit of ϕ_H^t.

The Floer complex can be defined along the same lines as in the aspherical case (but with additional complications). For a non-degenerate Hamiltonian H, we consider a free module generated by the critical points of $\mathcal{A}_H$. But, to be able to construct the Floer differential one needs to use twisted

[2]Strictly speaking Floer homology on general symplectic manifolds beyond the semi-positive case rely on virtual cycle technics which are not yet accepted by all the experts.

coefficients: The Floer complex $CF_*(H)$ is the free-module over Λ, where Λ is the *Novikov ring*, i.e. the set of all sums

$$\sum_{A\in\Gamma} a_A\, A,$$

where every coefficients $a_A \in \mathbb{Q}$ and for all $C \in \mathbb{R}$, the set $\{A \in \Gamma \,|\, a_A \neq 0, \int_A \omega < C\}$ is finite. For a generic choice of almost complex structure J, the Floer differential is then defined by appropriately counting elements in moduli spaces of the form:

$$\widehat{\mathcal{M}}([x_-, u_-], [x_+, u_+]; H, J) = \left\{ u : \mathbb{R}\times\mathbb{S}^1 \to M \;\middle|\; \begin{array}{l} u \text{ satisfies (3)} \\ \forall t \in \mathbb{S}^1,\ u(\pm\infty, t) = x_\pm(t) \\ u_- \# u = u_+ \end{array} \right\}$$

The Floer homology groups $HF_*(H, J)$ is the homology of this complex. Continuations isomorphisms may also be defined in this generalized context, providing isomorphisms $\Psi_{H_0,J_0}^{H_1,J_1}$ between $HF_*(H_0, J_0)$ and $HF_*(H_1, J_1)$. The PSS map is now an isomorphism of Λ-modules: $\Phi_{H,J} : H_*(M) \otimes \Lambda \to HF_*(H, J)$.

0.4 Floer theory for weakly exact Lagrangian submanifolds

In this section, we review the construction of Lagrangian Floer homology in a very simple case, which was already considered in Floer's paper [33]. Later, Lagrangian Floer theory was generalized to larger settings [97, 37, 38], but we will not need this more general cases for our purposes. Let (M, ω) be a closed symplectic manifold, L_0 and L_1 two closed non-disjoint connected Lagrangian submanifolds and $p \in L_0 \cap L_1$ an intersection point. A Lagrangian submanifold L of (M, ω) is called *weakly exact*, if $\langle [\omega], \pi_2(M, L)\rangle = 0$. When the intersection $L_0 \cap L_1$ is connected, most definitions and constructions presented in the sequel will not depend on p. Therefore, we will mostly not include the point p in the notations.

We will say that a pair of Lagrangian submanifolds (L_0, L_1) is *weakly exact (with respect to p)*, if any disk in M whose boundary is on $L_0 \cup L_1$ and is "pinched" at p has vanishing symplectic area. More precisely, denote by $\mathbb{D}^2$ the unit disk in $\mathbb{C}$ centered at 0. Denote by $\mathbb{S}^1_+$ the upper half of $\mathbb{S}^1$, $\mathbb{S}^1_+ = \{z \in \mathbb{C} \,|\, |z| = 1, \mathrm{Im}(z) \geqslant 0\}$, and by $\mathbb{S}^1_-$ its lower half.

Definition 0.4.1. *The pair of Lagrangians (L_0, L_1) is weakly exact with respect to $p \in L_0 \cap L_1$ if for any map $u : (\mathbb{D}^2, \mathbb{S}^1_+, \mathbb{S}^1_-, \{-1, 1\}) \to (M, L_0, L_1, \{p\})$, we have $\int_{\mathbb{D}^2} u^*\omega = 0$.*

Notice that in this case both L_0 and L_1 are weakly exact and moreover M is symplectically aspherical.

Examples 0.4.2. Many examples of weakly exact pairs are provided by standard Lagrangian tori in standard symplectic tori. To explain this, let us first make the following observations:

- In the torus $(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, dy \wedge dx)$, denote $L = \mathbb{S}^1 \times \{0\}$ and $L' = \{0\} \times \mathbb{S}^1$. Then, (L, L), (L', L') and (L, L') are weakly exact pairs with respect to any intersection point.
- If (L_0, L_1) is weakly exact with respect to p in (M, ω) and (L_0', L_1') is weakly exact with respect to p' in (M', ω'), then $(L_0 \times L_0', L_1 \times L_1')$ is weakly exact with respect to (p, p') in $(M \times M', \omega \oplus \omega')$.

Now, a standard symplectic torus $(\mathbb{T}^{2n}, \sum dy_i \wedge dx_i)$ is a product of n copies of $(\mathbb{T}^2, dy \wedge dx)$. We call *standard Lagrangian torus* any torus obtained as a product $L_1 \times \cdots \times L_n$ where each L_i, $i = 1, \ldots, n$, is either of the form $\mathbb{S}^1 \times \{0\}$, or of the form $\{0\} \times \mathbb{S}^1$. It follows from the above two observations that: *Every pair of standard Lagrangian tori is weakly exact w.r.t any intersection point.* ◀

Let $H : [0,1]\times M \to \mathbb{R}$ be a smooth Hamiltonian function. We will say that H is *non-degenerate with respect to the pair* (L_0, L_1) if the intersection $\phi_H^1(L_0) \cap L_1$ is transverse. A Hamiltonian is generically non-degenerate with respect to (L_0, L_1).

We denote by $\Omega(L_0, L_1)$ the set of paths x from L_0 to L_1 which are in the connected component of the constant path p. Such a path admits a capping $u : [0,1] \times [0,1] \to M$ so that: For all $t \in [0,1]$, $u(0,t) = p$ and $u(1,t) = x(t)$, $[0,1] \times \{0\}$ is mapped to L_0 and $[0,1] \times \{1\}$ to L_1.

Similarly as Formula (2), we define the *action functional* as

$$\mathcal{A}_H^{L_0,L_1} : \Omega(L_0, L_1; p) \to \mathbb{R}\,, \qquad x \mapsto \int_0^1 H(t, x(t))\, dt - \int u^*\omega, \tag{7}$$

where, u is a capping of x. Note that two cappings u_1 and u_2 of $x \in \Omega(L_0, L_1)$ have the same symplectic area since $u_1 \# \bar{u}_2$ is a pinched disk as defined above. Thus the action functional is well defined.

The critical points of $\mathcal{A}_H^{L_0,L_1}$ are paths $x \in \Omega(L_0, L_1)$ which are orbits of H that is for all t, $x(t) = \phi_H^t(x(0))$. These orbits are in one-to-one correspondence with $\phi_H^1(L_0) \cap L_1$ so that their number is finite since H is non-degenerate w.r.t. (L_0, L_1) and M compact. One defines the Floer complex $CF(L_0, L_1; H)$ as the $\mathbb{Z}/2$–vector space generated by the critical points of $\mathcal{A}_H^{L_0,L_1}$. We will not need to consider any graduation on these vector space, thus we avoid any discussion on the usual index (Maslov-Viterbo index) appearing in this theory.

Floer's differential is defined as in the Hamiltonian case treated in Section 0.2. Given a 1-parameter family of compatible almost complex structures J_t. We define the set of Floer trajectories between two orbits of H, x_- and x_+, as

$$\widehat{\mathcal{M}}^{L_0,L_1}(x_-, x_+; H, J) = \left\{ u : \mathbb{R} \times [0,1] \to M \;\middle|\; \begin{array}{l} \partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0 \\ \forall t,\, u(\pm\infty, t) = x_\pm(t) \\ u(\mathbb{R} \times \{0\}) \subset L_0 \\ u(\mathbb{R} \times \{1\}) \subset L_1 \end{array} \right\}$$

where the limits $u(\pm\infty, t)$ are uniform in t. Again, there is an obvious $\mathbb{R}$–action by reparametrization $s \mapsto s + \tau$ and we define $\mathcal{M}^{L_0,L_1}(x_-, x_+; H, J)$ as the quotient $\widehat{\mathcal{M}}^{L_0,L_1}(x_-, x_+; H, J)/\mathbb{R}$.

Requiring that the pair (H, J) is *regular*, that is the linearization of the operator $\overline{\partial}_{J,H} : u \mapsto \partial_s u + J_t(u)(\partial_t u - X_{H_t}(u))$ is surjective for all $u \in \widehat{\mathcal{M}}^{L_0,L_1}(x_-, x_+; H, J)$, ensures that $\mathcal{M}^{L_0,L_1}(x_-, x_+; H, J)$ is a smooth manifold. Its 0-dimensional component $\mathcal{M}_{[0]}^{L_0,L_1}(x_-, x_+; H, J)$ is a finite set. Floer's differential is defined by linearity on $CF(L_0, L_1; p; H)$ after setting the image of a generator as

$$\partial x_- = \sum_{x_+} \#\mathcal{M}_{[0]}^{L_0,L_1}(x_-, x_+; H, J) \cdot x_+$$

where $\#\mathcal{M}$ is the mod 2 cardinality of $\mathcal{M}$ and the sum runs over all orbits x_+. As in the Hamiltonian case, it can be proven that $\partial \circ \partial = 0$ that is, ∂ is a differential.

The Floer homology of the pair (L_0, L_1) is the homology of this complex $HF(L_0, L_1; H, J) = H(CF(L_0, L_1; H), \partial_{H,J}^{L_0,L_1})$. The homology does not depend on the choice of the regular pair (H, J). Indeed, as in the Hamiltonian case, there are "continuation" morphisms

$$\Psi_{H,J}^{H',J'} : CF(L_0, L_1; H) \to CF(L_0, L_1; H')$$

inducing isomorphisms in homology. Their construction is completely analoguous to the Hamiltonian case. As above, they satisfy the composition rule

$$\Psi_{H,J}^{H,J} = \mathrm{Id} \quad \text{and} \quad \Psi_{H,J}^{H',J'} \circ \Psi_{H',J'}^{H'',J''} = \Psi_{H,J}^{H'',J''}$$

for any three regular pairs (H, J), (H', J'), and (H'', J'').

The following statement seems to be absent from the litterature: *If* (L_0, L_1) *is a weakly exact pair and if* L_0 *and* L_1 *have a connected and clean intersection*[3] *then* $HF(L_0, L_1, H, J) \simeq H(L_0 \cap L_1, \mathbb{Z}/2)$.

[3] Clean intersection means that $L_0 \cap L_1$ is a submanifold and $T_p(L_0 \cap L_1) = T_pL_0 \cap T_pL_1$ for every $p \in L_0 \cap L_1$.

It has been established in "monotone" settings by Schmäschke [120] and the proof probably applies to the weakly exact case with minor modifications. However, we will state (and need) it only in cases where it is fully established. It applies in particular in the cases where $L_0 = L_1$ is a weakly exact Lagrangian submanifold, and (L_0, L_1) is a pair of standard Lagrangian tori in a standard symplectic torus (as in Example 0.4.2).

Proposition 0.4.3. *Assume that L_0, L_1 is a pair of Lagrangian submanifolds in a split symplectic manifold $(M = M_1 \times \cdots \times M_k, \omega = \omega_1 \oplus \cdots \oplus \omega_k)$, which can be splitted in products of connected Lagrangian submanifolds $L_i = L_i^1 \times \cdots \times L_i^k$, $i = 0, 1$, so that for each $j = 1, \ldots, k$, (L_0^j, L_1^j) is a weakly exact pair in M_j satisfying: either $L_0^j = L_1^j$ or $L_0^j \cap L_1^j = \{p\}$. Then there exists isomorphisms $\Phi_{H,J}^{L_0,L_1} : H(L_0 \cap L_1, \mathbb{Z}/2) \to HF(L_0, L_1; p; H, J)$ (that will be called PSS-isomorphisms) which make the following diagrams commute*

$$\begin{array}{ccc} H(L_0 \cap L_1, \mathbb{Z}/2) & \xrightarrow{\Phi_{H,J}^{L_0,L_1}} & HF(L_0, L_1; H, J) \\ & \searrow^{\Phi_{H',J'}^{L_0,L_1}} & \downarrow \Psi_{H,J}^{H',J'} \\ & & HF(L_0, L_1; H', J')\ . \end{array}$$

Idea of the proof . The proposition follows from the following three ingredients. First, it is known (See Leclercq [82]) that for L a weakly exact Lagrangian (equivalent for (L, L) a weakly exact pair), there exist PSS-isomorphisms

$$H(L, \mathbb{Z}/2) \to HF(L, L; p; H, J).$$

Second, it is easy to check that if (L_0, L_1) is a weakly exact pair with a single intersection point, $L_0 \cap L_1 = \{p\}$, then $HF(L_0, L_1; p; H, J) \simeq \mathbb{Z}/2$ (This case was considered in [141]). Third, if (L_0, L_1) is weakly exact in (M, ω) and (L_0', L_1') is weakly exact in (M', ω'), there is a very natural Künneth-type isomorphism

$$\begin{aligned} HF(L_0, L_1; p; H, J) &\otimes HF(L_0', L_1'; p'; H', J') \\ &\simeq HF(L_0 \times L_0', L_1 \times L_1'; (p, p'); H \oplus H', J \oplus J'). \end{aligned}$$

This isomorphism is defined at the chain level by the map:

$$\begin{aligned} \operatorname{Crit} \mathcal{A}_H^{L_0,L_1} \times \operatorname{Crit} \mathcal{A}_{H'}^{L_0',L_1'} &\to \operatorname{Crit} \mathcal{A}_{H \oplus H'}^{L_0 \times L_0', L_1 \times L_1'} \\ (t \mapsto x(t), t \mapsto x'(t)) &\mapsto (t \mapsto (x(t), x'(t))). \end{aligned}$$

□

Chapter 1

Action selectors

In this chapter, we present actions selectors and our contributions to this theory.

Action selectors[1] have been introduced to symplectic topology by Viterbo [135], followed by Schwarz [121], Oh [98] and many others. They have proved to be very useful symplectic invariants. The list of their applications includes the proof of the Conley conjecture [52, 41], the proof of the Zimmer conjecture [108], the construction of symplectic capacities [135, 69], structures on Hamiltonian groups (partial orders, bi-invariant distances, quasi-morphisms...) [135, 28], the proof of the camel theorem [135], the solution to the displaced disk problem [124], variational solutions to the Hamilton-Jacobi equation [137], the proof of the higher dimensional Birkhoff theorem [1], the C^∞-closing lemma [70, 5]... Moreover, they are one of the main tools to establish rigidity in C^0-symplectic topology (See Chapter 2).

Despite this impressive list of applications, the present chapter will focus on the properties of action selectors and very little on aplications. However, all the properties described (both old and new) will be used in the subsequent chapters. The motivations for our contributions came from two sources: C^0-symplectic geometry (Chapter 2) and the study of the relations between the tools of symplectic topology and the very specific tools of topological dynamics on surfaces (Chapter 3).

Our most significant contributions to the theory are the following: the relative energy-capacity inequality (Theorem 1.3.12), the dual energy-capacity inequality (Section 1.3.3), the max formula (Section 1.4).

1.1 Presentation of action selectors

1.1.1 Toy-model: homology of the sublevel sets of a function on a closed manifold

Let f be a continuous function on a compact topological space X. The *sublevel sets* of f are the subsets

$$f^{\leqslant t} = \{x \in X \mid f(x) \leqslant t\},$$

for $t \in \mathbb{R}$. For $s \leqslant t$ the sublevel sets satisfy the inclusion $f^{\leqslant s} \subset f^{\leqslant t}$. These inclusions induce maps $\iota_s^t : H_*(f^{\leqslant s}, R) \mapsto H_*(f^{\leqslant t}, R)$ between the (singular) homology groups of the sublevel sets, with coefficients in a given ground ring R. Moreover, these maps satisfy:

- For all $t \in \mathbb{R}$, ι_t^t is the identity map $H_*(f^{\leqslant t}, R) \to H_*(f^{\leqslant t}, R)$,
- For all $r, s, t \in \mathbb{R}$, such that $r \leqslant s \leqslant t$, one has $\iota_s^t \circ \iota_r^s = \iota_r^t$.

[1]Action selectors often appear in the litterature (including my own papers) under the name of "spectral invariants" or "min-max invariants/selectors".

For t large enough, $f^{\leqslant t} = M$ thus we can give the following definition. We will write ι_s to denote ι_s^t for t large enough, hence ι_s is the map $H_*(f^{\leqslant s}, R) \to H_*(M, R)$.

Definition 1.1.1 (Homological min-max selectors)**.** *For all $\alpha \in H_*(X,R) \setminus \{0\}$ and all continuous function $f : M \to \mathbb{R}$, we define*

$$\rho(\alpha, f) = \inf_{[\sigma]=\alpha} \max_{|\sigma|} f,$$

where the infimum is over all cycles σ representing α and $|\sigma|$ stands for the support of the cycle.

The above definition can be reformulated in a more algebraic way:

$$\rho(\alpha, f) = \inf\{s \in \mathbb{R} \,|\, \alpha \in \mathrm{Im}(\iota_s)\} \tag{1.1}$$

Example 1.1.2.

1. Assume X is arcwise connected and that $\alpha = [\mathrm{pt}]$ is the class of a point in X, then $c([\mathrm{pt}], f) = \min f$.
2. Assume X is an orientable closed manifold and $\alpha = [X]$ is the fondamental class of X. Then the support of every cycle representing $[X]$, is X itself. Hence, $\rho([X], f) = \max f$.
3. If $X = \mathbb{T}^2$ is the 2-torus, the homology of X can be generated by four classes: $[pt]$, $[X]$, the class $[\alpha]$ of a meridian curve and the class $[\beta]$ of a longitude curve. Figure 1.1 represents the four corresponding values of the selector when f is the height function for an embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$.

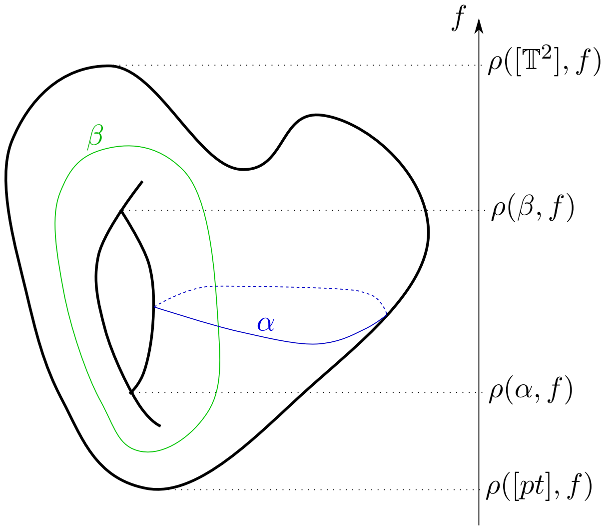


Figure 1.1: Illustration of Example 1.1.2.3

◀

Remark 1.1.3. It follows from standard Morse theory that if f is a smooth function on a smooth manifold, $\rho(\alpha, f)$ is a critical value of f. Indeed, the homology of the sublevel sets changes as we cross the value $\rho(\alpha, f)$.

Furthermore, if all the critical points with critical value $\rho(\alpha, f)$ are non-degenerate (in particular if f is Morse), then the critical value $\rho(\alpha, f)$ is attained at a critical point whose Morse index is the degree of α. ◀

The main feature of the min-max selector is its continuity:

Proposition 1.1.4. *For all homology class $\alpha \in H_*(X,R) \setminus \{0\}$, the map $\rho(\alpha,\cdot) : C^0(X,\mathbb{R}) \to \mathbb{R}$ is Lipschitzian. More precisely,*

$$\forall f, g \in C^0(M,\mathbb{R}), \quad \min(f-g) \leqslant \rho(\alpha, f) - \rho(\alpha, g) \leqslant \max(f-g).$$

This proposition is an immediate consequence of Propositions 1.1.9 and 1.1.10.

An interesting application of the homological min-max selectors is that it can be used to prove a classical lower bound on the number of critical points of a function on a manifold M: This number must be at least the *cup length* of M, a topological invariant of M defined as[2]

$$\begin{aligned}\mathrm{cl}(M) := \max\{k+1 \,:\, \exists\, a_1,\dots,a_k \in H_*(M,R),\ \forall i, \deg(a_i) \neq \dim(M) \\ \text{and } a_1 \cap \dots \cap a_k \neq 0\}.\end{aligned}$$

This is to be compared with the Morse inequalities, which provide a (better) lower bound in terms of the sum of Betti numbers of M, but under the assumption that the function is Morse (i.e. all its critical points are non-degenerate). The cuplength estimate belong to the classical Lusternik-Schnirelmann theory. A proof of this fact using selectors can be found in [137] or in [59].

1.1.2 Persistence modules, filtered complexes and selectors

It turns out that it is possible to generalize the definition of selectors to many different situations. In this section, we present an abstract framework for such a generalization. We used the terminology "persistence module" which is borrowed from topological data analysis and was introduced in symplectic topology by Polterovich and Shelukhin [110].

Definition 1.1.5 (Persistence module). *For a given ring R, a persistence module is a family of R-modules $Q^\bullet = (Q^t)_{t\in\mathbb{R}}$ endowed with maps $\iota_s^t(Q^\bullet) : Q^t \to Q^s$ for all $s \leqslant t \in \mathbb{R}$ (we will write ι_s^t when there is possible confusion), satisfying:*

- *For all $t \in \mathbb{R}$, ι_t^t is the identity map $Q^t \to Q^t$,*
- *For all $r, s, t \in \mathbb{R}$, such that $r \leqslant s \leqslant t$, one has $\iota_s^t \circ \iota_r^s = \iota_r^t$.*

We denote by Q the limit

$$Q = \varinjlim_{t\to+\infty} Q^t,$$

and $\iota_s : Q^s \to Q$ the natural map.

If $Q^t = Q^s$ and $\iota_s^t = \mathrm{Id}$ for s,t large enough, we say that the persistence module is tame. In that case, $Q = Q^t$ and $\iota_s = \iota_s^t$ for t large enough.

Many examples of persistence module arise from *filtered complexes*, i.e. chain complexes (C,∂) of R-modules endowed with a family of subcomplexes $(C^t)_{t\in\mathbb{R}}$ such that for all $s \leqslant t$, $C^s \subset C^t$.

Indeed, a persistence module is obtained from a filtered complex by taking the homology of the subcomplex $Q^t = H(C^t,\partial)$, the maps ι_s^t being induced by the inclusion maps $C^s \subset C^t$. If the filtered complex C is tame, i.e. if $C^t = C$ for t large, then the persistence module is tame as well.

Example 1.1.6. If $f : X \to \mathbb{R}$ is a continuous function on a compact topological space, the singular chain complex $C_*(X)$ admits a filtration given by the submodules of singular chains with values in $f^{\leqslant t}$. The induced (tame) persistence module is the family $Q^t = H_*(f^{\leqslant t}, R)$, the maps ι_s^t being induced by the inclusion maps $f^{\leqslant t} \to f^{\leqslant s}$. ◀

[2]Here, $\cap$ refers to the intersection product in homology. The cup length can be equivalently defined in terms of the cup product in cohomology.

To extend the definition of selectors to persistence modules, we mimic Equation (1.1):

Definition 1.1.7 (Selectors for persistence modules)**.** *Let $Q^\bullet$ be a persistence module. To every $\alpha \in Q \setminus \{0\}$, we associate:*

$$c(\alpha, Q^\bullet) = \inf\{s \in \mathbb{R} \,|\, \alpha \in \mathrm{Im}(\iota_s)\}.$$

Note that if $Q^\bullet$ is tame, then $c(\alpha, Q^\bullet)$ is finite.

The continuity of selectors can be abstracted using the notion of *interleaving* between persistence modules, which was introduced in [20] to measure distances between persistence modules.

Before we define this notion, let us introduce some more terminology. A *morphism of persistence modules* $h^\bullet$ between two persistence modules $Q^\bullet = (Q^t)_{t\in\mathbb{R}}$ and $Q'^\bullet = (Q'^t)_{t\in\mathbb{R}}$ is a family of morphisms $h^t : Q^t \to Q'^t$, $t \in \mathbb{R}$, compatible with the persistent maps $\iota_s^t(Q^\bullet)$ and $\iota_s^t(Q'^\bullet)$. In other words, for all $s \leqslant t$ we have a commutative diagram,

$$\begin{CD} Q^t @>h^t>> Q'^t \\ @A{\iota_s^t(Q^\bullet)}AA @AA{\iota_s^t(Q'^\bullet)}A \\ Q^s @>h^s>> Q'^s. \end{CD}$$

Given a persistence module $Q^\bullet$ and $\delta \in [0, +\infty)$, the *δ-shifted persistence module* is $Q^\bullet[\delta] := (Q^{t+\delta})_{t\in\mathbb{R}}$ with maps $\iota_{s+\delta}^{t+\delta}$. Morphisms of persistence modules can also be shifted by setting $h[\delta]^t := h^{t+\delta}$. There is a natural morphism of persistence modules $\sigma_{Q^\bullet}^{\delta} : Q^\bullet \to Q^\bullet[\delta]$, given by the maps $\iota_t^{t+\delta}$.

Definition 1.1.8. *We say that two persistence modules $Q^\bullet$ and $Q'^\bullet$ are (δ, ε)-interleaved if there exist two morphisms $h^\bullet : Q^\bullet \to Q'^\bullet[\delta]$ and $k^\bullet : Q'^\bullet \to Q'^\bullet[\epsilon]$ such that $k[\delta] \circ h = \sigma_{Q^\bullet}^{\delta+\varepsilon}$ and $h[\varepsilon] \circ k = \sigma_{Q'^\bullet}^{\delta+\varepsilon}$. In other words, we have the following commutative diagrams for every $s \leqslant t \in \mathbb{R}$:*

$$\begin{tikzcd} Q^t \arrow[r, "h^t"'] \arrow[rr, bend left, "\iota_t^{t+\delta+\varepsilon}(Q^\bullet)"] & Q'^{t+\delta} \arrow[r, "k^{t+\delta}"'] & Q^{t+\delta+\varepsilon} \\ Q^s \arrow[u, "\iota_s^t(Q^\bullet)"'] \arrow[r, "h^s"] \arrow[rr, bend right, "\iota_s^{s+2\delta}(Q^\bullet)"'] & Q'^{s+\delta} \arrow[u, "\iota_{s+\delta}^{t+\delta}(Q'^\bullet)"'] \arrow[r, "k^{s+\delta}"] & Q^{s+\delta+\varepsilon} \arrow[u, "\iota_{s+\delta+\varepsilon}^{t+\delta+\varepsilon}(Q^\bullet)"'] \end{tikzcd} \qquad \begin{tikzcd} Q'^t \arrow[r, "k^t"'] \arrow[rr, bend left, "\iota_t^{t+\delta+\varepsilon}(Q'^\bullet)"] & Q^{t+\varepsilon} \arrow[r, "h^{t+\varepsilon}"'] & Q'^{t+\delta+\varepsilon} \\ Q'^s \arrow[u, "\iota_s^t(Q'^\bullet)"'] \arrow[r, "k^s"] \arrow[rr, bend right, "\iota_s^{s+\delta+\varepsilon}(Q'^\bullet)"'] & Q^{s+\varepsilon} \arrow[u, "\iota_{s+\varepsilon}^{t+\varepsilon}(Q^\bullet)"'] \arrow[r, "h^{s+\varepsilon}"] & Q'^{s+\delta+\varepsilon} \arrow[u, "\iota_{s+\delta+\varepsilon}^{t+\delta+\varepsilon}(Q'^\bullet)"'] \end{tikzcd}$$

The pair (h, k) is called a (δ, ε)-interleaving.

Despite the apparent complexity of the definition, the notion of interleaving is very natural and easy to use. This is illustrated by the following two propositions, which imply Proposition 1.1.4.

Proposition 1.1.9. *Let $f, g : X \to \mathbb{R}$ be continuous functions on a compact topological space X. Then the persistence modules $(H_*(f^{\leqslant t}, R))_{t\in\mathbb{R}}$ and $(H_*(g^{\leqslant t}, R))_{t\in\mathbb{R}}$ are $(-\min(f-g), \max(f-g))$-interleaved.*

Proof . Let $\delta = -\min(f - g)$ and $\varepsilon = \max(f - g)$. Then for $s \leqslant t$, we have obvious inclusions between sublevel sets:

$$\begin{tikzcd} f^{\leqslant t} \arrow[r] \arrow[rr, bend left] & g^{\leqslant t+\delta} \arrow[r] & f^{\leqslant t+\delta+\varepsilon} \\ f^{\leqslant s} \arrow[u] \arrow[r] \arrow[rr, bend right] & g^{\leqslant s+\delta} \arrow[u] \arrow[r] & f^{\leqslant s+\delta+\varepsilon} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} g^{\leqslant t} \arrow[r] \arrow[rr, bend left] & f^{\leqslant t+\varepsilon} \arrow[r] & g^{\leqslant t+\delta+\varepsilon} \\ g^{\leqslant s} \arrow[u] \arrow[r] \arrow[rr, bend right] & f^{\leqslant s+\varepsilon} \arrow[u] \arrow[r] & g^{\leqslant s+\delta+\varepsilon} \arrow[u] \end{tikzcd}$$

These inclusions induce commutative diagrams as in Definition 1.1.8, hence a (δ, ε)-interleaving. □

Proposition 1.1.10. *Assume two persistence modules $Q^\bullet$ and $Q'^\bullet$ such that $Q = Q'$ are (δ, ε)-interleaved. Then, for all $\alpha \in Q \setminus \{0\}$,*

$$-\delta \leqslant c(\alpha, Q^\bullet) - c(\alpha, Q'^\bullet) \leqslant \varepsilon.$$

Proof . Under our assumption, by letting $t \to \infty$, the left diagram of Definition 1.1.8 gives in particular:

$$\begin{array}{ccc} & & Q \\ & \overset{\iota_s}{\nearrow} & \uparrow{\scriptstyle \iota_{s+\delta}} \\ Q^s & \xrightarrow{h^s} & Q'^{s+\delta} \end{array}$$

We deduce that if α is in the image of $\iota_s(Q^\bullet)$, then it is also in the image of $\iota_{s+\delta}(Q'^\bullet)$. This implies immediately $c(\alpha, Q'^\bullet) \leqslant c(\alpha, Q^\bullet) + \delta$. We obtain the reversed inequality similarly. □

Remark 1.1.11. The selectors of Definition 1.1.7 turn out to be part of a complete invariant of persistence modules called "barcode". Distance between barecodes can be measured with the so-called "bottleneck distance", which also admits bounds coming form interleavings. However, selectors are the only part of the barcodes that will be used in this memoir. Therefore we will not provide a definition of barcodes here. The interested reader may refer to [21, 131]. ◀

1.1.3 Filtered complexes in symplectic topology

In this section, we quickly review the filtered complexes appearing in symplectic topology. They all generalize in some way the situation of Section 1.1.1. We introduce as little as needed for our purposes.

Hamiltonian floer theory on closed symplectic manifolds

In this paragraph, we use the notations and definitions of Sections 0.2 and 0.3. The set of critical values of the action functional $\mathcal{A}_H$, often called the *action spectrum* of H, will be denoted by spec(H). It is known to be a closed subset of $\mathbb{R}$ of Lebesgue measure 0. If (M, ω) is symplectically aspherical, the action spectrum is compact. Let H be a non-degenerate Hamiltonian and J be a regular compatible almost complex structure.

Let $u : \mathbb{R} \times S^1 \to M$ denote a Floer trajectory solving either one of Equations (3), (4). The energy of u is defined as

$$E(u) := \int_{\mathbb{R}\times[0,1]} \|\partial_s u\|^2 ds dt,$$

where $\|\cdot\|$ is the norm associated to the metric $\omega(\cdot, J\cdot)$. Clearly, $E(u) \geqslant 0$.

It follows from a standard computation that if u is a Floer trajectory contributing to the boundary map, i.e. $u \in \widehat{\mathcal{M}}(x_-, x_+; H, J)$, then

$$\mathcal{A}_H(x_-) - \mathcal{A}_H(x_+) = E(u). \tag{1.2}$$

Thus the action decreases along Floer trajectories. This implies that for all $t \in \mathbb{R}$, the Λ-vector space $CF_*^t(H)$ generated by capped 1-periodic orbits $[x, u]$ of action $\leqslant t$ is a subcomplex of $CF_*(H)$. We denote by $HF_*^t(H, J)$ its homology. For $s \leqslant t$, we have inclusions $CF_*^s(H) \subset CF_*^t(H)$. We see that the family $(CF_*^t(H))_{t\in\mathbb{R}}$ has the structure of a filtered complex, hence $(HF_*^t(H, J))_{t\in\mathbb{R}}$ is a persistence module. Moreover, if the manifold is symplectically aspherical, spec(H) is compact. This implies that the filtered Floer complex and the corresponding persistence module are both tame.

It follows that Definition 1.1.7 provides selectors associated to any non zero class in $HF_*(H,J)$. Using the PSS-isomorphism $\Phi_{H,J} : H_*(M,\mathbb{Z}) \otimes \Lambda \to HF_*(H,J)$ we get a selector for any element in $H_*(M,\mathbb{Z}) \otimes \Lambda$ and any (H,J):

$$c(\alpha, H, J) = \inf\{s \in \mathbb{R} \mid \Phi_{H,J}(\alpha) \in \mathrm{Im}(\iota_s)\}. \tag{1.3}$$

For a given class α, the selectors for two distinct pairs (H_0, J_0) and (H_1, J_1) can be compared. Indeed, the continuation maps alluded at the end of Sections 0.2 and 0.3 give rise to interleavings.

Lemma 1.1.12. *The continuation map $\psi_{H_0}^{H_1}$ and its inverse $\psi_{H_1}^{H_0}$ form a (δ,ε)-interleaving for*

$$\delta = -\int_0^1 \min_{x\in M}(H_0(t,x) - H_1(t,x))\, dt, \quad \varepsilon = \int_0^1 \max_{x\in M}(H_0(t,x) - H_1(t,x))\, dt.$$

Idea of the proof . To simplify, we give the idea of the proof only in the case where (M,ω) is symplectically aspherical. We will explain why the inclusion $\Psi_{H_0}^{H_1}(CF_*^t(H_0,J_0)) \subset CF_*^{t+\delta}(H_1,J_1)$ holds true, leaving the rest of the argument to the reader. Let $\beta : \mathbb{R} \to [0,1]$ be a smooth map which equals 0 near $-\infty$ and 1 near $+\infty$. We can build $\psi_{H_0}^{H_1}$ out of a regular homotopy (H_s, J_s) chosen so that $H_{s,t}$ is arbitrarily close to $H_0 + \beta(s)(H_1 - H_0)$. To simplify again, we will assume that $H_{s,t} = H_0 + \beta(s)(H_1 - H_0)$. The map $\psi_{H_0}^{H_1}$ is constructed by counting solutions to the s-dependent Floer equation (4). Let $u \in \mathcal{M}(x_0, x_1; H_s, J_s)$ be such a solution. Then the energy of u can be computed as follows:

$$\begin{aligned}
E(u) &= \int_{-\infty}^{+\infty}\int_0^1 \|\partial_s u\|^2 \; dt\, ds \\
&= \int_{-\infty}^{+\infty}\int_0^1 \omega\left(\partial_s u, \partial_t u - X_{H_{s,t}}\right)\, dt\, ds \\
&= \int_{-\infty}^{+\infty}\int_0^1 u^*\omega\, dt\, ds - \int_{-\infty}^{+\infty}\int_0^1 dH_{s,t}\cdot \frac{\partial u}{\partial s}\, dt\, ds \\
&= \int_{\mathbb{R}\times\mathbb{S}^1} u^*\omega - \int_{-\infty}^{+\infty}\int_0^1 \tfrac{d}{ds}(H_{s,t}\circ u)\, dt\, ds + \int_{-\infty}^{+\infty}\int_0^1 \partial_s H_{s,t}\circ u\, dt\, ds \\
&= \mathcal{A}_{H_0}(x_0) - \mathcal{A}_{H_1}(x_1) + \int_{-\infty}^{+\infty}\int_0^1 \partial_s H_{s,t}\circ u\, dt\, ds.
\end{aligned}$$

We assumed that $\partial_s H_{s,t}\circ u = \beta'(s)(H_1 - H_0)$. Since $\int_{-\infty}^{+\infty}\beta'(s)\, ds = 1$, we deduce $0 \leqslant E(u) \leqslant \mathcal{A}_{H_0}(x_0) - \mathcal{A}_{H_1}(x_1) + \int_0^1 \max(H_1 - H_0)\, dt$, hence

$$\mathcal{A}_{H_1}(x_1) \leqslant \mathcal{A}_{H_0}(x_0) + \int_0^1 \max(H_1 - H_0)\, dt.$$

This shows that $\Psi_{H_0}^{H_1}(CF_*^t(H_0,J_0)) \subset CF_*^{t+\delta}(H_1,J_1)$, where $\delta = \int_0^1 \max(H_1 - H_0)\, dt = -\int_0^1 \min_{x\in M}(H_0(t,x) - H_1(t,x))\, dt$. □

Proposition 1.1.10 and the above lemma give the inequality:

$$-\int_0^1 \min_{x\in M}(H_0(t,x) - H_1(t,x))\, dt \leqslant c(\alpha, H_0, J_0) - c(\alpha, H_1, J_1) \leqslant \int_0^1 \max_{x\in M}(H_0(t,x) - H_1(t,x))\, dt \tag{1.4}$$

Proposition 1.1.13. *For every symplectic manifold, the selector $c(\alpha, H, J)$ has the following properties:*

1. Spectrality: $c(\alpha, H, J) \in \operatorname{spec}(H)$

2. *It does not depend on J, hence will be denoted $c(\alpha, H)$,*

3. Monotonicity: *if $H \leqslant K$, then $c(\alpha, H) \leqslant c(\alpha, K)$,*

4. Lipschitz continuity *(with respect to Hofer norm): $|c(\alpha, H_0) - c(\alpha, H_1)| \leqslant \|H - K\|$.*

It follows from Lipschitz continuity that the selector $c(\alpha, H)$ can be extended to all smooth functions $H : \mathbb{S}^1 \times M \to \mathbb{R}$. All the above properties extend immediately except spectrality which holds anyway:

Proposition 1.1.14 (Oh, [98])**.** *If (M, ω) is rational, then for all (not necessarily non-degenerate) smooth Hamiltonian H and all class α, $c(\alpha, H) \in \operatorname{spec}(H)$.*

The action selector associated to the fundamental class $[M] \in H_{2n}(M) \otimes \Lambda$ will play a central role in Sections 1.3, 1.4 and 3.2. Therefore, we will use the shorter notation:

$$c(H) = c([M], H). \tag{1.5}$$

Floer theory for weakly exact Lagrangians

In this paragraph, we use the definitions and notations of Section 0.4. Let (L_0, L_1) be a weakly exact pair of compact Lagrangian submanifolds in a symplectic manifold (M, ω) and let H be a Hamiltonian on M. The set of critical values of $\mathcal{A}_H^{L_0,L_1}$ is called *spectrum* and is denoted by $\operatorname{spec}(L_0, L_1; H)$.

Similarly as in the Hamiltonian case, for H non-degenerate, we denote by $CF^t(L_0, L_1; H)$ the subspace of $CF(L_0, L_1; H)$ generated by the elements $\gamma \in \operatorname{Crit}\mathcal{A}_H^{L_0,L_1}$ for which $\mathcal{A}_H^{L_0,L_1}(\gamma) \leqslant t$. An identity similar to Equation (1.2) and the positivity of energy imply that the $CF^t(L_0, L_1; H)$ is a subcomplex of $CF(L_0, L_1; H)$. Thus, we are here again in the presence of a filtered complex which induces a persistence module $HF^t(L_0, L_1; H, J)$. The spectrum can be proved to be compact, which implies that the persistence module is tame.

We can define selectors for this persistence module. This is particularly interesting when the groups $HF(L_0, L_1; H, J)$ can be computed. For example, if (L_0, L_1) satisfies the hypothesis of Proposition 0.4.3 we get for all non-zero homology class $\alpha \in H_*(L_0 \cap L_1, \mathbb{Z}/2)$ a selector

$$\ell(\alpha; L_0, L_1; H, J) = \inf\{s \in \mathbb{R} \mid \Phi_{H,J}^{L_0,L_1}(\alpha) \in \operatorname{Im}(\iota_s)\},$$

where ι_s is the natural map $HF^s(L_0, L_1; H, J) \to HF(L_0, L_1; H, J)$. A statement analogous to Lemma 1.1.12 gives the following properties.

Proposition 1.1.15. *For every pair (L_0, L_1) as in Proposition 0.4.3, and every class $\alpha \in H_*(L_0 \cap L_1, \mathbb{Z}/2) \setminus \{0\}$ the selector $\ell(\alpha; L_0, L_1; H, J)$ has the following properties:*

1. Spectrality: $\ell(\alpha; L_0, L_1; H, J) \in \operatorname{spec}(L_0, L_1; H)$

2. *It does not depend on J, hence will be denoted $\ell(\alpha; L_0, L_1; H)$,*

3. Monotonicity: *if $H \leqslant K$, then $\ell(\alpha; L_0, L_1; H) \leqslant \ell(\alpha; L_0, L_1; K)$,*

4. Lipschitz continuity: $|\ell(\alpha; L_0, L_1; H) - \ell(\alpha; L_0, L_1; H')| \leqslant \|H - H'\|$.

As in the Hamiltonian case, it follows from Lipschitz continuity that the selector $c(\alpha, H)$ can be extended to all smooth Hamiltonians $H : \mathbb{S}^1 \times M \to \mathbb{R}$. All points of Proposition 1.1.15 extend easily.

If H_t is constant on L_1 for all t, then its flow preserves L_1. If we denote by $c(t)$ this constant, the spectrum $\operatorname{spec}(L_0, L_1; H)$ is the singleton $\{\int_0^1 c(t)\,dt\}$, hence $\ell(\alpha; L_0, L_1; H) = \int_0^1 c(t)\,dt$ for all α. This simple observation, due Monzner-Vichery-Zapolsky [89], has many implications. In particular, by pushing the argument a little further, one can improve motonicity and Lipschitz continuity and get the following inequalities:

Proposition 1.1.16. *For every pair L_0, L_1 as in Proposition 0.4.3, every non-zero class $\alpha \in H_*(L_0 \cap L_1, \mathbb{Z}/2)$, and for all Hamiltonians H, K,*

$$-\int_0^1 \min_{x\in L_1}(H(t,x) - K(t,x))\,dt \leqslant \ell(\alpha, H) - \ell(\alpha, K) \leqslant \int_0^1 \max_{x\in L_1}(H(t,x) - K(t,x))\,dt$$

As in the Hamiltonian case, the fundamental class will play an important role. Let L be a weakly-exact closed Lagrangian submanifold and $[L]$ be its fundamental class. We will adopt the shorter notation:

$$\ell_L(H) = \ell([L]; L, L; H). \tag{1.6}$$

More filtered complexes

There are a lot more filtered complexes appearing in symplectic topology. Each Floer-like theory gives rise to its own set of action selectors: generating functions, symplectic homology, wrapped Lagrangian Floer theory, Rabinowitz Floer homology, embedded contact homology, microlocal sheaf theory...

Let us mention here that the first filtered complex which was used to define selectors was the one associated to generating functions of Lagrangians isotopic to the zero section in cotangent bundles. this is the original construction due to Viterbo [135]. He also used it to define two action selectors c_- and c_+ associated to every compactly supported Hamiltonian diffeomorphism of $\mathbb{R}^{2n}$. We refer to [135] or to [59] for the details of this construction.

1.1.4 Automatic properties of action selectors

Many properties of action selectors can be deduced easily directly from the their continuity and the fact that they belong to the spectrum $\mathrm{spec}(H)$ or $\mathrm{spec}_{L_0,L_1}(H)$ are everywhere discontinuous sets. Indeed, for example these properties imply that if along a family of Hamiltonians $(H^s)_{s\in[0,1]}$ the spectrum $\mathrm{spec}(H^s)$ (respectively $\mathrm{spec}_{L_0,L_1}(H^s)$) remains constant, then all selectors $c(\alpha, H^s)$ (respectively $\ell(\alpha; L_0, L_1; H)$) have to remain constant. Similarly if the spectrum is shifted ($\mathrm{spec}(H^s) = \mathrm{spec}(H^0) + f(s)$ for some continuous function $f : [0,1] \to \mathbb{R}$) or dilated ($\mathrm{spec}(H^s) = f(s)\mathrm{spec}(H^0)$ for some continuous function $f : [0,1] \to (0, +\infty)$) along the deformation, so are the selectors.

This can be used to prove the following properties.

Proposition 1.1.17.

1. *(Homotopy invariance) If two Hamiltonians H^0 and H^1 generate isotopies that are homotopic with fixed end points in* $\mathrm{Ham}(M, \omega)$*, then for all non-zero class $\alpha \in H_*(M, \mathbb{Z}) \otimes \Lambda$,*
$$c(\alpha, H^0) = c(\alpha, H^1).$$
In particular, if ρ is a non-decreasing map $[0,1] \to [0,1]$ and $H^\rho(t,x) = \rho'(t)H(\rho(t), x)$, then $c(\alpha, H) = c(\alpha, H^\rho)$.

2. *(Conjugation invariance) For all symplectic diffeomorphism ψ, all class α and all Hamiltonian H, we have $c(\psi_*^{-1}\alpha, \psi^* H) = c(\alpha, H)$. If ψ is the identity component* $\mathrm{Symp}_0(M, \omega)$*, then $c(\alpha, \psi^* H) = c(\alpha, H)$.*

3. *(Shift property) If $\sigma : \mathbb{S}^1 \to \mathbb{R}$ is a function of the time t, then*
$$c(\alpha, H + \sigma) = c(\alpha, H) + \int_0^1 \sigma(t)\,dt$$

Similar properties can be established for Lagrangian action selectors, under the same assumptions.

On symplectically aspherical manifolds another property holds, the symplectic contraction principle, which we now describe. We will call *Liouville domain* an open subset U with smooth boundary and whose closure admits a vector field ξ which is transverse to the boundary ∂U and satisfies $L_\xi \omega = -\omega$, where L is the Lie derivative. The vector field ξ is referred to as the Liouville vector field of the domain U. Note that the Liouville vector field ξ necessarily points inward along ∂U and therefore the flow $A_t : U \to U$ of ξ is defined for all positive times t. This flow "contracts" the symplectic form ω : $A_t^* \omega = e^{-t}\omega$. Recall that an open subset $U \subset M$ is called *incompressible* if the map $i_* : \pi_1(U) \to \pi_1(M)$, induced by the inclusion $i : U \to M$, is injective.

Let U denote an incompressible Liouville domain in an asymplectically aspherical manifold M and let H be a Hamiltonian supported in U. For each fixed $s \geqslant 0$ consider the Hamiltonian

$$H^s(t,x) = \begin{cases} e^{-s} H(t, A_s^{-1}(x)) & \text{if } x \in A_s(U), \\ 0 & \text{if } x \notin A_s(U). \end{cases}$$

It can be checked that the Hamiltonian flow of H^s is given by

$$\phi^t_{H^s}(x) = \begin{cases} A_s \phi^t_H A_s^{-1}(x) & \text{if } x \in A_s(U), \\ x & \text{if } x \notin A_s(U). \end{cases}$$

Since U is incompressible every contractible closed orbit in U admits a capping disk contained in U. This can be used to prove:

Proposition 1.1.18 (Symplectic contraction principle). *In the above settings,* $\operatorname{spec}(H^s) = e^{-s}\operatorname{spec}(H)$, *hence* $c(\alpha, H_s) = e^{-s} c(\alpha, H)$ *for all class* α.

1.2 From algebra to analysis

Every algebraic property of persistence modules is reflected into an analytic property of the corresponding action selectors. We have already used this general principle in the previous section: We have seen how an interleaving induces certain inequalities and applied it in the case of coninuation maps. In this section we illustrate this principle for many algebraic features of Floer homology.

C^2-small Hamiltonians

For C^2-small autonomous and Morse Hamiltonians, we have seen that Hamiltonian Floer homology correspond to Morse homology (with coefficients in the Novikov ring). This algebraic correspondance is immediately reflected on selectors in the following property.

Proposition 1.2.1 (Usher [129]). *If H is autonomous and sufficiently small for the C^2 distance, then for all homology class* $\alpha \in H(M, \mathbb{Z}/2) \setminus \{0\}$,

$$c(\alpha, H) = \rho(\alpha, H).$$

In particular, $c(H) = \max(H)$.

The function H does not need to be Morse in the previous proposition. This has a Lagrangian analogue that we now describe for the selector ℓ_L. Assume that L is a weakly exact Lagrangian submanifold. According to the Weinstein neighborhood theorem, a neighborhood of L is symplectomorphic to a neighborhood of the zero section in T^*L. Let L' be a Lagrangian which corresponds to a graph $\operatorname{graph}(df)$, for $f : L \to \mathbb{R}$, through such an idenfication.

Proposition 1.2.2 (Leclercq [82]). *With the above notations, for all Hamiltonian H such that $\phi^1_H(L) = L'$, we have*

$$\ell_L(H) - \ell_L(\bar{H}) = \max f - \min f.$$

Note that in the above statement, we cannot hope to have $\ell_L(H) = \max f$, since f is only defined up to shift by a constant.

Relating Hamiltonian and Lagrangian selectors

Hamiltonian Floer homology can be seen as a particular case of Lagrangian Floer theory, by the following trick. Given a symplectic manifold (M, ω), the diagonal $\Delta = \{(x, x) \in M \times M \,|\, x \in M\}$ is a Lagrangian submanifold of $(M \times M, (-\omega) \oplus \omega)$. If (M, ω) is symplectically aspherical, then Δ is weakly exact. Given Hamiltonians H and K on M, we define a Hamiltonian $H \oplus K$ on $M \times M$ by the formula $(H \oplus K)_t(x, y) = H_t(x) + K_t(y)$. There is a natural isomorphism

$$HF(H, J) \simeq HF(\Delta, \Delta; 0 \oplus H, -J \times J).$$

It is natural in the sense that it is compatible with the PSS-isomorphism. Morevoer, it preserves the filtration. This gives for selectors:

Proposition 1.2.3 (Leclercq [82]). *For all Hamiltonian H and all non-zero homology class $\alpha \in H(M, \mathbb{Z}/2) \simeq H(\Delta, Z/2)$,*

$$c(\alpha, H) = \ell(\alpha; \Delta, \Delta; 0 \oplus H).$$

Triangle inequality

Floer homologies also carry product structures. Given two generically chosen Hamiltonians H and H', the so-called pair-of-pants product is a map $HF(H, J) \otimes HF(H', J) \to HF(H \sharp H', J)$.

In the aspherical case, this product corresponds (through PSS-isomorphisms) to the intersection product in $H_*(M, \mathbb{Z}/2)$. On more general symplectic manifolds, the pair-of-pants product does not correspond anymore to the intersection product but to the so-called quantum product denoted $*$, which will not be defined here (See e.g. [88]). The module $H_*(M) \otimes \Lambda$ endowed with the quantum product is usually called quantum homology and is denoted $QH(M, \omega)$. If we include the filtration in the picture, we get a commutative diagram

$$\begin{array}{ccc} QH(M,\omega) \otimes QH(M,\omega) & \xrightarrow{\text{quantum}} & QH(M,\omega) \\ \text{PSS}\downarrow & & \downarrow\text{PSS} \\ HF^{s}(H,J) \otimes HF^{s'}(H',J) & \xrightarrow[\text{-pants}]{\text{pair-of}} & HF^{s+s'}(H \sharp H', J), \end{array}$$

which yields to the following inequality.

Proposition 1.2.4 (Triangle inequality, Schwarz [121], Oh [98]). *For all classes α, α' in $QH(M, \omega)$ such that $\alpha * \alpha' \neq 0$, and all Hamiltonians H, H', we have*

$$c(\alpha * \alpha', H \sharp H') \leqslant c(\alpha, H) + c(\alpha', H').$$

In particular, $c(H \sharp H') = c(H) + c(H')$.

In Lagrangian Floer theory, the pair-of-pants product becomes (in weakly-exact settings) a map

$$HF(L_0, L_1; H, J) \otimes HF(L_1, L_2; H', J) \to HF(L_0, L_2; H \sharp H', J).$$

At least in the settings of Proposition 0.4.3, it corresponds via PSS-maps to the intersection product in homology $H_*(L_0 \cap L_1, \mathbb{Z}/2) \otimes H_*(L_1 \cap L_2, \mathbb{Z}/2) \to H_*(L_0 \cap L_2, \mathbb{Z}/2)$. A triangle inequality also holds in this context, which was inspired to us by Vichery [132].

Proposition 1.2.5 (Generalized triangle inequality, H.-Leclercq-Seyfaddini [67]). *Let L_0, L_1, L_2 be such that each pair (L_0, L_1), (L_1, L_2) and (L_0, L_2) meet the requirements of Propostion 0.4.3, let H, H' be two Hamiltonians and let α, α' be non-zero homology classes in $H_*(L_0 \cap L_1, \mathbb{Z}/2)$, $H_*(L_1 \cap L_2, \mathbb{Z}/2)$ such that $\alpha \cap \alpha' \neq 0$ in $H_*(L_0 \cap L_2, \mathbb{Z}/2)$. Then,*

$$\ell(\alpha \cap \alpha'; L_0, L_2; H \sharp H') \leqslant \ell(\alpha; L_0, L_1; H) + \ell(\alpha'; L_1, L_2; H').$$

A useful particular case of the above triangle inequality is when $L_0 = L_1 = L_2 = L$ and α is the fundamental class of L. In that case we get

$$\ell_L(H\sharp H') \leqslant \ell_L(H) + \ell_L(H'). \tag{1.7}$$

Another useful case is when $L_1 = L_0 = L$, $H' = 0$, and α' is the fundamental class of L . It gives:

$$\ell(\alpha; L, L_2; H) \leqslant \ell_L(H), \tag{1.8}$$

showing that $\ell_L(H)$ is the largest Lagrangian selector involving L and H.

Splitting formula

We now consider the case of a symplectic manifold (M, ω) which is a direct product $M = M' \times M''$ with a split symplectic form $\omega = \omega' \oplus \omega''$. In that case Floer theory satisfies a Künneth-type formula, which is compatible with PSS-maps. For simplicity, we state it here only for symplectically aspherical manifolds. We have a commutative diagram of isomorphisms:

$$\begin{array}{ccc} H_*(M', \mathbb{Z}/2) \otimes H_*(M'', \mathbb{Z}/2) & \xrightarrow{\text{Künneth}} & H(M, \mathbb{Z}/2) \\ \downarrow{\scriptstyle \text{PSS}} & & \downarrow{\scriptstyle \text{PSS}} \\ HF_*^{s'}(H', J') \otimes HF_*^{s''}(H'', J'') & \longrightarrow & HF_*^{s'+s''}(H' \oplus H'', J' \oplus J''). \end{array}$$

Again, this algebraic structure is reflected on selectors as follows.

Proposition 1.2.6 (Hamiltonian splitting formula, Entov-Polterovich [29]). *Assume (M', ω') and (M'', ω'') are symplectically aspherical manifolds, H', H'' are Hamiltonians respectively on M' and M'', and α', α'' are non-zero homology classes of respectively M' and M''. Then,*

$$c(\alpha' \otimes \alpha'', H' \oplus H'') = c(\alpha', H') + c(\alpha'', H'')$$

As already mentionned in Section 0.4, a similar Künneth formula holds in the Lagrangian setting. From it we derive:

Proposition 1.2.7 (Lagrangian splitting formula, H.-Leclercq-Seyfaddini [67]). *Let (L_0', L_1') be a weakly exact pair in (M', ω') , H' a Hamiltonian on M', and α' a non-zero homology class of $L_0' \cap L_1'$. Let L_0'', L_1'', H'', α'' be similar objects on a symplectic manifold M''. Then,*

$$\ell(\alpha' \otimes \alpha''; L_0' \times L_0'', L_1' \times L_1''; H' \oplus H'') = \ell(\alpha'; L_0', L_1'; H') + \ell(\alpha''; L_0'', L_1''; H'').$$

Application: reduction inequalities

Given a smooth coisotropic submanifold C in (M, ω), with caracteristic foliation $\mathcal{F}$, the reduction $\mathcal{R}$ of C is the quotient space $C/\mathcal{F}$. Whenever this space is Hausdorff separated, it is a smooth manifolds and inherits a natural symplectic structure induced by ω. Let H be Hamiltonian whose flow preserves C, or equivalently which satisfies H_t is constant on leaves for t, then it descends to a Hamiltonian H_R on the reduction. It is natural to try to compare the action selectors of H and that of the reduced Hamiltonian H_R. For standard tori, we can prove the following.

Proposition 1.2.8 (Reduction inequality; extracted from H.-Leclercq-Seyfaddini [67]). *Let H be a Hamiltonian on a standard torus $\mathbb{T}^{2n'+2n''}$ and H_R a Hamiltonian on $T^{2n'}$, such that H coincides with $H_R \oplus 0$ on the coisotropic torus $\mathbb{T}^{2n'} \times (\mathbb{T}^n \times \{0\})$. In other words, H induces H_R on the reduction. Then,*

$$c(H_R) \leqslant c(H).$$

Such reduction inequalities were established and used in the past for action selectors constructed via generating functions (See e.g. [135, 60, 117]). But to the best of our knowledge, this has never implemented in Floer homology before. We will use this inequality for C^0-symplectic topology in Section 2.4.4. It would be an interesting problem to try to extend it to more general settings, in particular when the characteristic foliation is not a product.

With the help of Proposition 1.2.3, the proof makes use of Lagrangian action selectors. A crucial element of the proof of this proposition is the above splitting formula and which allows to compare action selectors living in different spaces. Another important ingredient if the inequality (1.8). We refer to [67], Section 4 and in particular the proofs of identities (35), (36), (37) therein for details.

More instances of the "algebra to analysis" principle

There are other algebraic properties of Floer homology that can be exploited. Since it plays no role in our work, we do not include their description here.

For example, the duality between the filtered complexes associated to H and $\bar{H}$, which is reminiscent from Poincaré duality, allows to compare the action selectors for H and $\bar{H}$ (see e.g. Usher [128]).

Another example comes from the natural action of $\pi_1(\mathrm{Ham}(M,\omega))$ introduced by Seidel [122]. When this action is trivial, one can prove that the Hamiltonian action selectors (which are a priori defined on the unversal cover of $\mathrm{Ham}(M,\omega)$ by Proposition 1.1.17) descend to $\mathrm{Ham}(M,\omega)$ (Schwarz [121], McDuff [86]). A similar study can also be performed in the Lagrangian case (Hu-Lalonde-leclercq [58]).

1.3 Energy-capacity inequalities

In this section we present the energy-capacity inequalities satisfied by the selectors introduced above. The Hamiltonian energy-capacity inequality is nowadays standard. The version we discuss is originally due to Viterbo [135] and its most general version to Usher [129].

We proved the Lagrangian version in [65], even though similar inequalities, some of them prior to our work, exist in the litterature [19, 9, 83]. Finally, the Hamiltonian and Lagrangian versions of what we call "dual energy-capacity inequality" is new and were established in [66, 65]. All these inequalities can be used to establish C^0-rigidity results; see Chapter 2.

We first introduce the Hofer-Zehnder capacity [56] and its relative version, due to Lisi and Rieser [83].

1.3.1 Hofer-Zehnder capacity and Lisi-Rieser relative capacity

We will say that a Hamiltonian H is *slow* if its Hamiltonian flow $(\phi_H^t)_t$ has no non-trivial[3] contractible periodic orbit of period at most 1. We will denote $\mathcal{H}(U)$ the set of all autonomous, slow, and non-negative smooth Hamiltonians supported in a given open subset U. The elements of $\mathcal{H}(U)$ will be called *admissible*.

Given a Lagrangian submanifold L, we will say that H is L-slow if it admits no non-trivial orbit of time-length at most 1 from L to itself, homotopic to the class of a point. We will denote $\mathcal{H}(U,L)$ the set all autonomous, L-slow and non-negative Hamiltonians which reach their extrema on L. The elements of (U,L) will be called *L-admissible*.

We are now ready to define the capacities. The definitions that we give are slight modifications of the original ones, but are equivalent to them. The Lisi-Rieser capacity can be defined more generally relatively to a coisotropic submanifold (See [83]), but we will only need the Lagrangian version here.

[3]We call an orbit trivial if it is constant. For instance, it is well-known that a C^2-small Hamiltonian has no non-trivial orbit of period $\leqslant 1$.

Definition 1.3.1. *1. (Hofer-Zehnder capacity [56]) Given an open subset U of a symplectic manifold, the Hofer-Zehnder capacity of U is defined as:*

$$c_{\mathrm{HZ}}(U) = \sup\{\max H \mid H \in \mathcal{H}(U)\}.$$

2. (Lisi-Rieser relative capacity [83]) Given an open subset U of a symplectic manifold, which intersect a Lagrangian submanifold L, the Lisi-Rieser capacity of U relatively to L is defined as:

$$c_{\mathrm{LR}}(U, L) = \sup\{\max H \mid H \in \mathcal{H}(U, L)\}.$$

The Hofer-Zehnder capacity is a capacity in the sense of Ekeland and Hofer [25], that is, it satisfies the following properties.

Proposition 1.3.2 (Hofer-Zehnder [56]). *1. For all open subsets $U \subset V$ in a symplectic manifold M, $c_{\mathrm{HZ}}(U) \leqslant c_{\mathrm{HZ}}(V)$,*

2. For all symplectic manifolds (M, ω), (M', ω'), all open subset $U \subset M$ and all embedding $\phi : M \to M'$ such that $\phi^*\omega' = \lambda^2\omega$, $c_{\mathrm{HZ}}(\phi(U)) = \lambda^2 c_{\mathrm{HZ}}(U)$,

3. $c_{\mathrm{HZ}}(B^{2n}(1)) = c_{\mathrm{HZ}}(Z^{2n}(1)) = \pi$, where $B^{2n}(1)$ is the standard ball of radius 1 in $\mathbb{C}^{2n}$ and $Z^{2n}(1) = B^2(1) \times \mathbb{C}^{n-1}$ is the standard cylinder.

The above properties are easy to prove, except for the bound $c_{\mathrm{HZ}}(Z^{2n}(1)) \leqslant \pi$ which is a deep result. It can be proven as a consequence of the energy-capacity inequality presented below. The properties of the Lisi-Rieser capacity are analogous. Any (possibly infinite) number $c(U, L)$ satisfying this set of properties could be called a *relative capacity*.

Proposition 1.3.3 (Lisi-Rieser [83]). *1. For all open subsets $U \subset V$ in a symplectic manifold M and all Lagrangian L, $c_{\mathrm{LR}}(U, L) \leqslant c_{\mathrm{LR}}(V, L)$,*

2. For all symplectic manifolds (M, ω), (M', ω'), all open subset $U \subset M$, all Lagrangian L and all embedding $\phi : M \to M'$ such that $\phi^*\omega' = \lambda^2\omega$, $c_{\mathrm{LR}}(\phi(U), \phi(L)) = \lambda^2 c_{\mathrm{LR}}(U, L)$,

3. $c_{\mathrm{LR}}(B^{2n}(1), \mathbb{R}^n) = c(Z^{2n}(1), \mathbb{R}^n) = \frac{\pi}{2}$.

Again, the above properties are essentially straightforward except for the inequality $c(Z^{2n}(1), \mathbb{R}^n) = \frac{\pi}{2}$. It can be proven using the Lagrangian version of the Energy-capacity presented below.

REMARK 1.3.4. Action selectors of admissible and L-admissible Hamiltonians turn out to be very easy to compute. Indeed, let H be a slow autonomous Hamiltonian. It has no closed orbit of period $\leqslant 1$ except its critical points. Thus, if we denote by $\mathrm{CritVal}(H)$ its set of critical values, we have $\mathrm{spec}(sH) = s \cdot \mathrm{CritVal}(H)$ for all $s \in (0, 1]$. Since $\mathrm{CritVal}(H)$ is everywhere discontinuous (by Sard's theorem), we deduce that the continuous function $s \mapsto \frac{1}{s} c(\alpha, sH)$ has to be constant. By Proposition 1.1.15, it equals $\rho(\alpha, H)$ for s small enough. We conclude that $c(\alpha, H) = \rho(\alpha, H)$. A similar argument can be played in the Lagrangian settings using 1.2.2[4]. In particular we get:

Lemma 1.3.5. *If H is admissible, then $c(H) = \max(H)$. If H is L-admissible for L weakly exact, then $\ell_L(H) = \max(H)$.*

Other capacities can be defined as follows. The *Viterbo capacity* of an open subset U is by definition [135]:

$$c_V(U) = \sup\{c(H) \mid H \text{ any Hamiltonian supported in } U\}.$$

[4] The Lagrangian case is actually slightly more complicated. See [65], proof of Theorem 12.7 for details.

We can define a *relative Viterbo capacity* as:

$$c_V(U,L) = \sup\{\ell_L(H) \,|\, H \text{ any Hamiltonian supported in } U\}.$$

Lemma 1.3.5 implies the inequalities

$$c_{\mathrm{HZ}}(U) \leqslant c_V(U), \qquad c_{\mathrm{LR}}(U,L) \leqslant c_V(U,L),$$

for all U and L. Moreover these capacities satisfy the same properties as the Hofer-Zehnder and the Lisi-Rieser capacities listed in Propositions 1.3.2 and 1.3.3. ◀

1.3.2 The energy-capacity inequality

The energy-capacity inequality is a fundamental result. It is a very useful tool to study symplectic capacities and Hofer's distance. We will first present the Hamiltonian version of it which is the classical one and then its Lagrangian (or relative) version, which is more recent.

Hamiltonian version

Recall that a Hamiltonian diffeomorphism ϕ is said to displace a subset U if $\phi(U) \cap U = \emptyset$.

Lemma 1.3.6 (Usher [129]). *If the time-1 map of a Hamiltonian H displaces an open subset U, and if K is any Hamiltonian supported in U, then*

$$c(K) \leqslant c(H) + c(\overline{H}).$$

Remark 1.3.7. The quantity $c(H) + c(\overline{H})$ is usually called *spectral norm* or *Viterbo norm* and is denoted $\gamma(H)$. Lemma 1.3.6 implies that $\gamma(H)$ is non-negative and is a non-degenerate quantity: if $\gamma(H) = 0$, then ϕ_H^1 displaces no non-trivial open subset, hence $\phi_H^1 = \mathrm{Id}$. ◀

Together with Lemma 1.3.5, Lemma 1.3.6 implies the following inequality, sometimes refered to as "sharp energy-capacity inequality".

Theorem 1.3.8 (Usher [129]). *If the time-1 map of a Hamiltonian H displaces an open subset U, then*

$$c_{\mathrm{HZ}}(U) \leqslant \gamma(H).$$

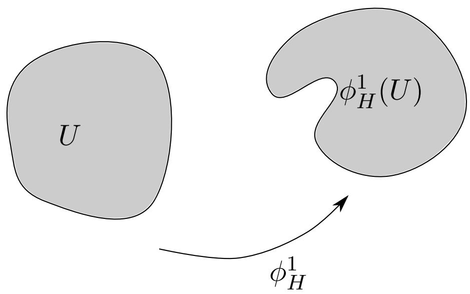


Figure 1.2: Illustration of Theorem 1.3.8

By (1.4), the classical energy-capacity inequality (originally proved by Hofer [54] in $\mathbb{R}^{2n}$) is an immediate consequence of Theorem 1.3.8. It asserts that displacing an open set requires some minimal energy, where by energy we mean Hofer norm:

Corollary 1.3.9 (Energy-capacity inequality). *If a Hamiltonian diffeomorphism ϕ displaces an open subset U, then $c_{\mathrm{HZ}}(U) \leqslant \|\phi\|$, where $\|\phi\|$ stands for the Hofer norm of ϕ.*

Note that in Theorem 1.3.8 as well as Corollary 1.3.9, the Hofer-Zehnder capacity may be replaced with the Viterbo capacity; see Remark 1.3.4.

Lagrangian version

The relative version of the energy-capacity inequality, asserts that it also requires some minimal energy to displace a Lagrangian submanifold from an open set. As for the Hamiltonian energy-capacity inequality, we can prove this result using the properties of action selectors, via the following lemma.

Lemma 1.3.10 (H.-Leclercq-Seyfaddini [65])**.** *Let L be a smooth weakly-exact closed Lagrangian submanifold and U an open subset. Assume H is a Hamiltonian such that $\phi_H^1(L) \cap U = \emptyset$ and K is a Hamiltonian supported in U. Then, $\ell_L(K) \leqslant \ell_L(H) + \ell_L(\overline{H})$.*

Remark 1.3.11. The quantity $\gamma_L(H) = \ell_L(H) + \ell_L(\overline{H})$ is usually called *Lagrangian spectral norm* or *Viterbo norm*. It is also a non-degenerate quantity (this actually follows from Lemma 1.3.10): if $\gamma_L(H) = 0$, then $\phi_H^1(L) = L$. ◀

The following theorem and corollary are then easily deduced from Lemma 1.3.10 and 1.1.16.

Theorem 1.3.12 (H.-Leclercq-Seyfaddini [65])**.** *Let L be a smooth weakly exact closed Lagrangian submanifold. Suppose that U is an open subset, with $L \cap U \neq \emptyset$. Assume H is a Hamiltonian such that $\phi_H^1(L) \cap U = \emptyset$. Then*

$$c_{\mathrm{LR}}(U; L) \leqslant \gamma_L(H).$$

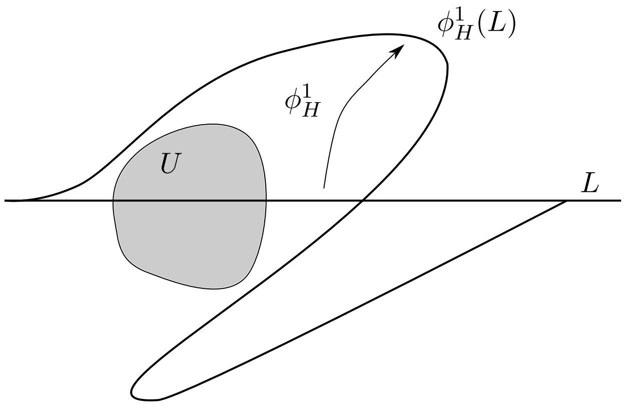


Figure 1.3: Illustration of Theorem 1.3.12

Corollary 1.3.13. *Under the hypothesis of Theorem 1.3.12,*

$$c_{\mathrm{LR}}(U; L) \leqslant \int_0^1 (\max_L H_t - \min_L H_t)\, dt.$$

Inspired by Corollary 1.3.13, Usher proved in [130] that the Lagrangian Hofer distance between two Lagrangians L, L' (See the definition at the end of Section 0.1) is given by

$$\delta(L, L') = \inf \left\{ \int_0^1 (\max_L H_t - \min_L H_t)\, dt \;\middle|\, H \text{ such that } \phi_H^1(L) = L' \right\}.$$

Therefore, the above corollary can be worded as follows: if $\phi_H^1(L) \cap U = \emptyset$, then $c_{\mathrm{LR}}(U) \leqslant \delta(L, \phi_H^1(L))$ (compare with Corollary 1.3.9).

In [65], Lemma 1.3.10 was proved for L being the zero section of a cotangent bundle. The proof is identical to the one we will now present. The method of proof is analoguous to the classical Hamiltonian case and goes back to Viterbo [135] (see also Usher's proof [129] of the analogous result for compact manifolds).

Proof of Lemma 1.3.10. Assume that $\phi_H^1(L) \cap U = \emptyset$. For $s \in [0,1]$, consider the Hamiltonian diffeomorphism $\phi_K^s \phi_H^1$, which is the end of the isotopy defined by

$$\begin{cases} \phi_H^{2t} & t \in [0, 1/2], \\ \phi_K^{(2t-1)s} \phi_H^1 & t \in [1/2, 1]. \end{cases}$$

We denote by F^s the Hamiltonian generating this isotopy. To be precise, the Hamiltonian F^s so constructed is not necessarily continuous at $t = \frac{1}{2}$, but it can be made smooth by performing appropriate time reparametrizations (as explained in Section 0.1, Page 12). Note in particular, that $F^1 = K \bullet H$ generates an isotopy homotopic with fixed end points to the isotopy generated by $K \sharp H$, hence $\ell_L(K \sharp H) = \ell_L(F^1)$.

Let us analyse the Lagrangian spectrum of F^s. A Hamiltonian orbit of F^s from L to itself is a path $t \mapsto \gamma(t)$ such that $\gamma(0) \in L$, $\gamma(1) \in L$ and for all t, $\gamma(t) = \phi_{F^s}^t(\gamma(0))$. In particular, for such an orbit, $\phi_K^s \phi_H^1(\gamma(0)) \in L$. However, since by assumption $\phi_H^1(L) \cap \operatorname{supp}(K) = \emptyset$, necessarily $\phi_H^1(\gamma(0))$ is not in the support of K. Thus, $\gamma(t) = \phi_H^{2t}(\gamma(0))$ for all $t \leqslant 1/2$ and remains constant $\gamma(t) = \phi_H^1(\gamma(0))$ for $t \geqslant 1/2$. This means that for all s, the action spectrum $\operatorname{spec}_{L,L}(F^s)$, that is the set of actions of Hamiltonian orbits from L to itself, remains constant.

Since this set is nowhere dense and ℓ_L is continuous (and takes its values in the action spectrum), the function $s \mapsto \ell_L(F^s)$ is constant so that $\ell_L(H) = \ell_L(F^1) = \ell_L(K \sharp H)$. Thus, using the triangle inequality (Proposition 1.2.5) and the identity $K = K \sharp H \sharp \bar{H}$,

$$\ell_L(K) \leqslant \ell_L(K \sharp H) + \ell_L(\overline{H}) = \ell_L(H) + \ell_L(\overline{H}).$$

□

1.3.3 The dual energy-capacity inequality

We now present a new type of inequality, whose introduction was motivated by questions in C^0-symplectic geometry (in particular Theorems 2.3.6, 2.3.11 and 2.3.12). In this new type of inequality, the conclusions obtained will be similar to that of the classical energy-capacities inequalities 1.3.8, 1.3.9, 1.3.13, 1.3.12: under some assumption, the Hofer norm, or the spectral norm of the Hamiltonian (eventually relatively to a Lagrangian) is larger than the capacity of the considered open subset. However our assumptions will be dual in some sense: the condition that ϕ_H^1 displaces the open set is replaced by the condition that H takes two large and opposite constant values on disjoint open subsets[5].

Hamiltonian version

The Hamiltonian version of the dual energy-capacity inequality can be stated as follows.

Theorem 1.3.14 (Dual energy-capacity inequality, H.-Leclercq-Seyfaddini [66]). *Let (M, ω) be a (non-negatively) monotone closed symplectic manifold. Let U_- and U_+ denote non-empty open subsets of (M, ω), and C_- and C_+ real numbers such that $C_\pm > c_{\mathrm{HZ}}(U_\pm)$. If a Hamiltonian H satisfies $H|_{U_\pm} = \pm C_\pm$, then the spectral norm $\gamma(H)$ is greater than or equal to at least one of $c_{\mathrm{HZ}}(U_-)$ and $c_{\mathrm{HZ}}(U_+)$.*

If (M, ω) is not monotone but only rational[6] and if we denote by Ω the positive generator of the group $\langle [\omega], \pi_2(M) \rangle$, then the same conclusion holds under the additional assumptions $\frac{1}{4}\Omega > C_\pm$.

[5]The use of the terminology "dual" is arguable, but these conditions can be considered as dual since "ϕ_H^1 displaces U" means that X_H is far from vanishing, and "H takes two large and opposite constant values on disjoint open subsets" means that dH is far from vanishing. Furthermore, they can be used to prove two respective statements that are more evidently dual to each other (each implication of Theorems 2.3.11).

[6]with the same reservations as for the footnote Page 15

We do not know if a similar result holds on more general symplectic manifolds. Of course the same statement holds with $\gamma(H)$ replaced with the Hofer norm of ϕ_H^1. The initial motivation for this result came from C^0-symplectic topology; see Theorem 2.3.6 or [66]. The paper [66] also contains several slight variations of Theorem 1.3.14. Unfortunately, we have not found any application of Theorem 1.3.14 beyond C^0-symplectic geometry that could not be proved with other methods.

Let us now give an idea of the proof of Theorem 1.3.14 in the simple case of a symplectically aspherical manifolds.

Idea of the proof . We first analyse what can be deduced from the fact that a Hamiltonian H vanishes on some open subset U. For any $\delta > 0$, pick an admissible Hamiltonian $f \in \mathcal{H}(U)$ such that $0 \leqslant f$, and $c_{\mathrm{HZ}}(U) - \delta \leqslant \max(f)$. By Remark 1.3.4, we have $c(f) = \max(f)$ and $c(-f) = 0$. Consider the Hamiltonian $H_s = H + sf = H\sharp sf$. Its 1-periodic orbits consist of the 1-periodic orbits of the flow of H and the critical points of f. Hence, since $H|_U = 0$ by assumption,

$$\mathrm{Spec}(H_s) = \mathrm{Spec}(H) \cup \{sf(p) \,|\, p \in \mathrm{Crit}(f)\},$$

where $\mathrm{Crit}(f)$ denotes the set of critical points of f. Similarly, define $\tilde{H}_s = \overline{H} + sf$. We have $\mathrm{Spec}(\tilde{H}_s) = \mathrm{Spec}(\overline{H}) \cup \{sf(p) \,|\, p \in \mathrm{Crit}(f)\}$. We know that $c(H_s) \in \mathrm{Spec}(H_s)$ and $c(\tilde{H}_s) \in \mathrm{Spec}(\tilde{H}_s)$. However, suppose that one of the following two situations holds:

$$c(H_s) \in \mathrm{Spec}(H) \text{ for all } s \in [0,1] \tag{1.9}$$

$$\text{or} \quad c(\tilde{H}_s) \in \mathrm{Spec}(\overline{H}) \text{ for all } s \in [0,1] \tag{1.10}$$

If (1.9) holds, then it follows from the continuity property of spectral invariants that $c(H) = c(H_1) = c(H+f)$. Using the triangle inequality we obtain $c_{\mathrm{HZ}}(U) - \delta \leqslant \max(f) = c(f) \leqslant c(\overline{H}) + c(H+f) = c(H) + c(\overline{H})$, hence $c_{\mathrm{HZ}}(U) \leqslant \gamma(H)$. We arrive at the same conclusion if (1.10) holds.

If in the contrary neither (1.9) nor (1.10) holds, then at some point in the deformation, after being constant for a while, $c(H_s)$ leaves $\mathrm{spec}(H)$ and $c(\tilde{H}_s)$ leaves $\mathrm{spec}(\overline{H})$. This means that $c(H) = s_0 f(p_0)$ and $c(\overline{H}) = s_1 f(p_1)$ for some $s_0, s_1 \in [0,1]$ and some critical points p_0, p_1 of f. Thus, $c(H) \leqslant c_{\mathrm{HZ}}(U)$ and $c(\overline{H}) \leqslant c_{\mathrm{HZ}}(U)$.

We are now ready to conclude. Assume that $H|_{U_\pm} = \pm C^\pm$ but that the conclusion of the Theorem does not hold, that is $\gamma(H) < c_{\mathrm{HZ}}(U_-)$ and $\gamma(H) < c_{\mathrm{HZ}}(U_+)$. Then, $\overline{H} + C_+$ vanishes on U_+ and the above analysis yields $c(\overline{H} + C_+) \leqslant c_{\mathrm{HZ}}(U_+)$. Similarly, $H + C_-$ vanishes on U_- and we get $c(H + C_-) \leqslant c_{\mathrm{HZ}}(U_-)$. Adding both inequalities, using the shift property of Hamiltonian selectors (Proposition 1.1.17) and the non-negativity of γ (Remark 1.3.7, we obtain:

$$C_+ + C_- \leqslant \gamma(H) + C_+ + C_- \leqslant c_{\mathrm{HZ}}(U_+) + c_{\mathrm{HZ}}(U_-),$$

hence a contradiction since by assumption $C_\pm > c_{\mathrm{HZ}}(U_\pm)$. □

Lagrangian version

The dual energy-capacity inequality admits a relative version, which we now state. It was proved in [65] for L the zero section in a cotangent bundle, but the proof applies also in the case of a weakly-exact Lagrangian submanifolds.

Theorem 1.3.15 (Relative dual energy-capacity inequality, H.-Leclercq-Seyfaddini [65])**.** *Let L be a weakly exact Lagrangian submanifold in a closed (M,ω), and U_+, U_- open subsets of T^*L such that $U_\pm \cap L \neq \emptyset$. If a Hamiltonian H satisfies $H|_{U_\pm} = \pm C_\pm$ with $C_\pm \in \mathbb{R}$ so that $C_\pm > c_{\mathrm{LR}}(U\pm; L)$, then $\gamma_L(H)$ is greater than or equal to at least one of $c_{\mathrm{HZ}}(U_-; L)$ and $c_{\mathrm{HZ}}(U_+; L)$.*

This result was also motivated by C^0-symplectic topology, in particular to prove Theorems 2.3.11 and 2.3.12 below. We will not provide a proof here, let us just mention that it is completely analoguous to the Hamiltonian case.

1.4 The Max Formula

In this part, we present a new property of action selectors, the max formula. It was proved in [64] as an important tool to compute Hamiltonian action selectors on surfaces (this work is surveyed in Chapter 3).

The max formula on closed and aspherical symplectic manifolds In this paragraph, we assume that (M, ω) is a closed symplectically aspherical symplectic manifold. Recall the definition of an incompressible Liouville domain before Proposition 1.1.18, Page 27, which in particular includes symplectic embeddings of convex subsets of $\mathbb{R}^{2n}$.

Theorem 1.4.1 (Max formula on closed aspherical manifolds, H.-Le Roux-Seyfaddini [64])**.** *Suppose that $F_1, \ldots, F_N$ are Hamiltonians whose supports are contained, respectively, in pairwise disjoint incompressible Liouville domains $U_1, \ldots, U_N$. Then,*

$$c(F_1 + \ldots + F_N) = \max\{c(F_1), \ldots, c(F_N)\}.$$

REMARK 1.4.2. Interestingly enough, the hypothesis of asphericity is not just technical: the max formula does not hold on non-aspherical manifolds. Indeed, in the first version of [64], we constructed an example of a Hamiltonian on the sphere $\mathbb{S}^2$ violating the max formula. ◀

Idea of the proof . The first step of the proof is to symplectically contract each of the F_i's, as described in Proposition 1.1.18, to obtain functions $F_{i,s}$. Proposition 1.1.18 implies that it is sufficient to prove the max formula for the $F_{i,s}$'s.

The next step is to study the Floer trajectories of (an appropriate perturbation of) $F_{1,s} + \ldots + F_{N,s}$. An application of Gromov's monotony lemma (more precisely a variant of it, Proposition 3.2 in [51]) will provide us with a positive constant $\varepsilon > 0$ such that any Floer trajectory which travels between distinct U_i and U_j has energy greater than ε. On the other hand, by picking s to be sufficiently negative we can ensure, using Proposition 1.1.18, that the spectrum of $F_{1,s} + \ldots + F_{N,s}$ is contained in $(-\frac{\epsilon}{4}, \frac{\epsilon}{4})$. By Equation (1.2) any Floer trajectory traveling between distinct U_i and U_j has energy less than $\frac{\epsilon}{2}$. As a conclusion, there exist no such Floer trajectories.

This drastically simplifies the Floer homological picture and allows us to fully describe the relations among the various Floer cycles representing the fundamental class $[M]$. □

The max formula on $\mathbb{R}^{2n}$ We also established the max formula for the Viterbo action selectors c_+ and c_- defined for compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ [135]. We already alluded to these invariants at the end of Section 1.1.3, Page 1.1.3.

We will say that N subsets $A_1, \ldots, A_N$ in $\mathbb{R}^{2n}$ are *symplectically separated* if the minimum over all indices $1 \leqslant i < j \leqslant N$ of the euclidean distance between $\psi(A_i)$ and $\psi(A_j)$ can be made arbitrary large for some symplectic diffeomorphism ψ. For example, two disjoint convex sets are always symplectically separated.

Theorem 1.4.3 (Max formula in $\mathbb{R}^{2n}$, H.-Le Roux-Seyfaddini [64])**.** *If $H_1, \ldots, H_N$ are compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ whose supports are symplectically separated, then:*

$$c_+(H_1 + \ldots + H_N) = \max(c_+(H_1), \ldots, c_+(H_N)),$$

$$c_-(H_1 + \ldots + H_N) = \min(c_-(H_1), \ldots, c_-(H_N)).$$

Idea of the proof . The proof of this theorem is by induction. For $N = 2$, the idea is that when both supports are far enough from each other (which can be achieved by a suitable symplectic diffeomorphism), then it becomes possible to build a generating function of $H_1 + H_2$ that coincides with a generating function of H_1 on some open set surrounding the support of H_1 and with a generating function of H_2 on some open set surrounding the support of H_2. Then an argument based on the Mayer-Vietoris long exact sequence, applied to the sublevels of the generating functions, allows us to compare the different action selectors. □

Discussion: possible generalizations of the max formula As mentionned above, the max formula cannot be generalized to more general closed symplectic manifolds. Indeed we found a counter-example on the sphere. However, we believe that Theorem 1.4.1 could be extended in various directions:

- The max formula probably holds for symplectically aspherical open but convex at infinity manifolds, for the Frauenfelder-Schlenk action selector [36];
- Our methods can probably be used to show that for an embedding of a ball $\psi : B \to M$ into a symplectically aspherical manifold M, ω, and for a Hamiltonian H compactly supported in $\psi(B)$, then the Floer theoretic selector $c(H)$ conincides with the Viterbo selector $c_+(H \circ \psi)$;
- On a monotone symplectic manifold, it could be the case that the max formula holds for Hamiltonians with small enough supports and/or small enough Hofer norm;
- There is certainly a similar statement for Lagrangian action selectors to be established.

1.5 New direction: action selectors and microlocal theory of sheaves

In this section, we briefly report on some joint work in progress with Nicolas Vichery.

Microlocal sheaf theory has been developped by Kashiwara and Schapira [71]. To every sheaf F (or more precisely to every object in the bounded derived category of sheaves $D^b(N)$) on a smooth manifold N is associated a *singular support* (or *microsupport*), denoted $SS(F)$. This is by construction a closed subset of T^*N which is conical, i.e. invariant by positive multiplication in the cotangent fibers. A deep theorem of Kashiwara and Schapira (Theorem 6.5.4 in [71]) asserts that $SS(F)$ is coisotropic in a generalized sense ($SS(F)$ is usually not a submanifold). This theorem shows that in principle sheaves could be useful to symplectic topology, and this principle was confirmed by the pioneering work of Tamarkin [127] and Nadler-Zaslow [95], followed by many others.

Definition 1.5.1. *We say that a subset A of $J^1N = T^*N \times \mathbb{R}$ is quantized by a sheaf F if the cone $\hat{A} = \{(x, p, t, \tau) \in T^*(N \times \mathbb{R}) \,|\, (x, \frac{p}{\tau}, t) \in A, \tau > 0\}$ is the singular support of F and if F satisfies an additional technical assumption*[7].

Example 1.5.2. Given a function $f : N \to \mathbb{R}$, the image of the 1-jet of f, $\{(x, d_xf, f(x) \,|\, x \in N\}$ is quantized by the constant sheaf on the epigraph $\{(x,t) \in N \times \mathbb{R} \,|\, f(x) \leqslant t\}$. ◀

Building on Tamarkin's paper [127], Guillermou-Kashiwara-Schapira [50] showed that if a subset can be quantized, then so are its deformations under strict contact isotopies. Guillermou proved that the Legendrian lift of every closed exact Lagrangian submanifold can be quantized [47]. These works have had already many applications: the contact non-squeezing theorem [23], a new proof of the Gromov-Eliashberg theorem [48], new proofs that the projection of a closed exact Lagrangian onto

[7] This assumption is that F should belong to the Tamarkin category, that is F is an object of $D^b(N \times \mathbb{R})$ which is invariant by convolution by the constant sheaf on $N \times [0, +\infty)$.

the zero section is a weak homotopy equivalence [47], a proof of the three cusps conjecture [49], a characterisation of augmentations of the Chekanov-Eliashberg DGA [96]...

In [132], Vichery explained how one can define action selectors for sheaves. In our joint work in progress, we use those selectors to prove cuplength estimates similar to those mentioned at the end of Section 1.1.1. In other words we apply Lusternik-Schnirelmann theory to sheaves, to obtain Arnold-type lower bounds. Before we state our result let us mention that it is possible to define a cuplength $\mathrm{cl}(F)$ for objects satisfying the technical assumption of Definition 1.5.1.

Theorem 1.5.3 (H.-Vichery, in preparation). *If $A \subset J^1N$ is quantized by F, then A admits at least $\mathrm{cl}(F)$ intersection points with the zero wall $O_N \times \mathbb{R}$, where O_N denotes the zero section in T^*N.*

This statement turns out to fit well with Principle 2.5.7. In particular, it has the following corollary.

Corollary 1.5.4. *For every smooth compact exact Lagrangian submanifold L in T^*N, and for every submanifold $V \subset N$, the number of intersection points between L with the conormal ν^*V of V is at least $\mathrm{cl}(V)$, where $\mathrm{cl}(V)$ is the classical cup length of V.*

In the case where $V = N$, the conormal ν^*V is the zero section and we recover Hofer's theorem [53].

This corollary can certainly be obtained by other methods than sheaves (Floer theory). However the approach via sheaves has several advantages that the other approaches do not have: the subset A of Theorem 1.5.3 does not need to be smooth, hence this result can probably be applied in many singular situations.

In general microlocal sheaf theory provides very promising tools for symplectic topology. It is a much more flexible theory than generating functions, and compared to Floer homology technics, it has the advantage that is does not require any kind of transversality. Applications to C^0-symplectic geometry can also be expected.

Chapter 2

C^0-symplectic geometry

C^0-symplectic geometry started when Eliashberg, building on earlier work of Gromov, proved the following celebrated theorem.

Theorem 2.0.1 (Gromov-Eliashberg [27]). *Let (M, ω) be any symplectic manifold and let ϕ_k be a sequence of symplectic diffeomorphisms. Suppose that it converges in the C^0-sense to some diffeomorphism ϕ. Then, ϕ is symplectic.*

This theorem asserts that symplectic diffeomorphisms are C^0-rigid. This is very surprising! Indeed, being a symplectic diffeomorphism is a condition on the differential of the diffeomorphism, which should not a priori behave well under C^0-limit. This is a manifestation of symplectic rigidity.

This theorem is very important in the history of symplectic topology since it is in some sense the first theorem of modern symplectic topology. It appeared before Gromov's theory of holomorphic curves. Arnold calls it the "Existence theorem of symplectic topology" [4].

The Gromov-Eliashberg theorem also gives rise to a definition. It allows to define a symplectic homeomorphism as a homeomorphism which is a C^0-limit of symplectic diffeomorphism; see Definition 2.2.1 below. More generally, it opens the field of *C^0-symplectic topology*, which may be defined as the study of the behavior of symplectic objects under C^0-limits, or more broadly, as the search of which aspects of symplectic geometry can be generalized to continuous (not smooth) settings.

Let us clarify what we mean by C^0-convergence and recall a few facts. Given two manifolds M_1, M_2, a compact subset $K \subset M_1$, a Riemannian distance d on M_2, and two maps $f, g : M_1 \to M_2$, we denote

$$d_K(f, g) = \sup_{x \in K} d(f(x), g(x)).$$

We say that a sequence of maps $f_i : M_1 \to M_2$ C^0-converges to a map $f : M_1 \to M_2$, if for every compact subset $K \subset M_1$, the sequence $d_K(f_i, f)$ converges to 0. This notion does not depend on the choice of the Riemannian metric.

An important feature of C^0-convergence is that if a sequence of homeomorphisms, f_i, C^0-converges to a homeomorphism f, then the sequence of inverses f_i^{-1} convergse to f^{-1}. However, it is possible for a sequence of homeomorphisms to converge to a map which is not a homeomorphism. To avoid this issue, some authors use as a definition of C^0-convergence that both f_i and f_i^{-1} converge. The limit of f_i is then the inverse of the limit of f_i^{-1}, and is a homeomorphism.

2.1 Why study C^0-symplectic geometry?

It is often said that symplectic topology is beautiful because it is a world divided into two countries, a rigid country and a flexible country, separated by mysterious, fuzzy and hard to reach border. C^0-

symplectic topology inherits a similar beauty, a subtle mixture of rigidity and flexibility. Moreover C^0-symplectic geometry comes with exciting open questions; for instance: Is the group of area preserving homeomorphisms of the sphere a simple group? Does $\mathbb{S}^4$ admit a C^0-symplectic structure ?

One of the goals of C^0-symplectic topology is to answer the meta-mathematical question: Is symplectic topology fundamentally about smooth objects or something about less regular objects ? To illustrate the meaning of this question, imagine that we had first defined cohomology using De Rham cohomology, then it would be natural to think that cohomology belongs to the smooth world. But of course we know that cohomology is much more general concept. Similarly, we know that the concept of curvature, which initially belonged to the smooth world, can be extended to more general metric space than just Riemannian manifolds.

More pragmatically, it is common in mathematics, that the study of objects with low regularity (weak topologies...) leads to progress on the smooth objects themselves. This is already true for C^0 symplectic topology. Indeed, for instance, the deep study of the C^0-rigidity of the Poisson bracket, initiated by Cardin and Viterbo, followed by Entov and Polterovich and their coauthors, lead to the discovery of the Poisson bracket invariants by Buhovsky-Entov-Polterovich, and their applications to smooth Hamiltonian dynamics (See [13] and references therein). Another example is provided by our series of paper with Leclercq and Seyfaddini [66, 65, 67]: motivated by specific questions in C^0-symplectic topology, we discovered many new properties of the action selectors of smooth maps.

From the point of view of classical mechanics (and maybe quantum mechanics?), it makes perfect sense to study the effect of C^0-perturbations of Hamiltonians, since they may serve to model the irregularities of the real world in many situations. Indeed, making a C^0-perturbation of Hamiltonian means making a perturbation which is uncontroled both in position and directions, but with small energy (imagine you want to study a system pertubed by uncontroled but small blow-ups).

C^0-symplectic topology has obtained some success in surface dynamics. Let us mention for instance the solution to the displaced disk problem by Seyfaddini [124]. To some extend the work on pseudo-rotations by Bramham [11, 10] is also related to C^0-symplectic topology.

More generally, singular symplectic objects tend to appear more and more frequently in recent research: the Entov-Polterovich theory of heavy/superheavy sets, skeletons of Weinstein manifolds, microsupports of sheaves, weak solutions to Hamilton-Jacobi equations and weak KAM theory, topological dynamics on surfaces, Ostrover's study of non-smooth convex hypersurfaces... Therefore a systematic study is needed and C^0-symplectic topology is one aspect of this general trend.

2.2 Symplectic homeomorphisms

In this section we discuss the notion of symplectic homeomorphism. Several definitions are possible, but we will adopt the following one.

Definition 2.2.1. *Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A continuous map $\theta : U \to M_2$, where $U \subset M_1$ is open, is called globally symplectic if it is the C^0-limit of a sequence of symplectic diffeomorphisms $\theta_i : U \to \theta_i(U)$.*

Let $U_1 \subset M_1$ and $U_2 \subset M_2$ be open subsets. If a homeomorphism $\theta : U_1 \to U_2$ and its inverse θ^{-1} are both globally symplectic maps, we call θ a globally symplectic homeomorphism.

We will say that a homeomorphism $\theta : M_1 \to M_2$ is a symplectic homeomorphism if every point of M_1 admits a neighborhood U such that the restriction of θ to U is a globally symplectic homeomorphism onto $\theta(U)$.

Clearly, if θ is a symplectic homeomorphism, so is θ^{-1}. By the Gromov–Eliashberg theorem (Theorem 2.0.1), a symplectic homeomorphism which is smooth is a symplectic diffeomorphism. Let us now give examples.

Example 2.2.2. For every symplectic surface (Σ, ω), it is known that every homeomorphism which preserves both the area (i.e., the measure induced by ω and the orientation is a (globally) symplectic homeomorphism in the sense of definition 2.2.1. Proofs can be found in [126] or [99]. ◀

Example 2.2.3. Let N be a smooth manifold and α a continuous 1-form which is closed in the sense of distributions[1], then the map

$$\theta : T^*M \to T^*M, \quad (q,p) \mapsto (q, p + \alpha(q))$$

is a symplectic homeomorphism.

This can be proved using approximation by convolution which allows to show that, locally, a continuous 1-form which is closed in the sense of distributions can be uniformly approximated by smooth closed 1-forms. Hence θ can locally be approximated by smooth symplectic diffeomorphisms. See Proposition 26 in [65] for details. ◀

Example 2.2.4. We will say that a bilipschitz[2] map ϕ is symplectic if $\phi^*\omega = \omega$ almost everywhere. It follows from a refinement of the Gromov-Eliashberg theorem (see Theorem 3, p. 59, in [56]) that a bilipschitz homeomorphism which is a symplectic homeomorphism is symplectic in this sense. It would be interesting to know whether the converse is true, that is, whether every symplectic bilipschitz map can be locally C^0-approximated by smooth symplectic maps. A partial result in this direction can be extracted from in [62]:

Proposition 2.2.5. *Let E be a connected component of the group of symplectic bilipschitz maps (endowed with the Lipschitz topology). If E contains a smooth symplectic map, then every element in E is a symplectic homeomorphism.*

The proof follows from the fact that a bilipschitz symplectic map which is Lipschitz-close to the identity map can be represented by a $C^{1,1}$ generating function. A smooth sequence of functions which C^1-converges to this generating function generates a sequence of smooth symplectic maps which C^0-converges to the initial bilipschitz homeomorphism. A similar argument is performed in dimension 2 in Section 3 of [62].

◀

Example 2.2.6. (Symplectic maps with smoothable isolated singularities in dimension 4) Let ϕ be a homeomorphism defined near a point p, with $\phi(p) = p$, and which is smooth in the complement of p. We will say that the singularity p of ϕ is smoothable if for every neighborhood U of p, there exists a smooth diffeomorphism ψ which coincides with ϕ on the complement of U[3].

Proposition 2.2.7. *Let ϕ be a homeomorphism defined in the neighborhood of a fixed point p in a symplectic manifold of dimension 4. Assume that ϕ is a symplectic diffeomorphism in the complement of p and that the singularity is smoothable. Then, ϕ is a symplectic homeomorphism.*

Proof . We need to show that the singularity at p is smoothable not only by diffeomorphisms but by symplectic diffeomorphism. By assumption, for all ball B centered at p and compactly included in the domain of ϕ, since the singularity is smoothable, if B is small enough, there exist a diffeomorphism ψ, that coincides with ϕ in the complement of B. Let ω denote the ambiant symplectic form. The form $\psi^*\omega$ coincides with ω outside B. By 0.3.C in Gromov's paper [45] (this is where we use dimension 4), we know that $\psi^*\omega$ is standard, there is a diffeomorphism f supported in a slightly larger ball than B, and such that $f^*\psi^*\omega = \omega$. The diffeomorphism $\psi \circ f$ is symplectic as required. □ ◀

[1] Such a 1-form α is closed in the sense of distribution if and only if its integral over every smooth contractible curve vanishes and if and only if it is locally the differential of C^1 function.

[2] A bilipschitz map is a Lipschitz homeomorphism ϕ whose inverse ϕ^{-1} is also Lipschitz.

[3] In dimension larger than 4, every such singularity is smoothable, as it follows from the h-cobordism theorem. I do not know if this is the case in dimension 4

Of course, many more examples can be obtained by taking compositions and products: If ϕ and ψ are symplectic homeomorphisms, so are $\phi \circ \psi$ and $\phi \times \psi$, whenever these maps are defined.

Discussion: alternative definitions for symplectic homeomorphisms There are several possible alternative definitions of symplectic homeomorphisms. Indeed, any C^0-formulated property which characterizes symplectic diffeomorphisms among all diffeomorphisms gives rise to a definition. Let us give a few examples.

Let C be a symplectic capacity[4]. It is known that a smooth diffeomorphism ϕ which preserves C, that is $\rho(C(U)) = C(U)$ for all (small) open subset U, must be symplectic or anti-symplectic (See [56] or [87]). Therefore we could have defined symplectic homeomorphisms as homeomorphisms which preserve C. We will call such maps *capacity-preserving* homeomorphisms.

Another possible definition is to consider *selector-preserving* maps, that is homeomorphisms ϕ such that for all Hamiltonian function H supported in small enough balls, we have

$$c(H \circ \phi) = c(H),$$

where c is the action selector introduced in Equation (1.5), in Section 1.1.3. Clearly, a selector-preserving map preserves the Viterbo capacity (See Remark 1.3.4).

Thanks to the Lipschitz continuity of action selectors, if ϕ_i are symplectic diffeomorphisms which C^0-converges to ϕ, then $H \circ \phi_i$ also C^0-converges to $H \circ \phi$, for all Hamiltonian H. Since $c(H \circ \phi_i) = c(H)$ for all i, we deduce that $c(H \circ \phi) = c(H)$. In words, globally symplectic homeomorphisms are selector-preserving maps. Of course, it follows that globally symplectic homeomorphisms also preserve the Viterbo capacity. Similarly, symplectic homeomorphisms preserve capacities of small sets. It is not known is there exists a capacity-preserving homeomorphism which is not a symplectic homeomorphism.

There are more possible definitions: for instance, Müller proved that "Shape"-preserving homeomorphisms could also be considered as symplectic homeomorphisms [90]. However, our definition of symplectic homeomorphisms (Definition 2.2.1), is the only one for which a theory has been developped. Therefore, we will limit ourselves to this definition in the sequel, except in Section 2.4.4, where selector-preserving maps will make a short appearance.

2.3 Hamiltonian homeomorphisms

2.3.1 Definitions

Now that we have defined symplectic homeomorphisms, it is natural to try to give a definition for Hamiltonian homemorphisms. Unfortunately, the situation is more intricated in this context, since there is no C^0-characterisation of Hamiltonian diffeomorphisms which is as neat as the one provided by the Gromov-Eliashberg theorem for symplectic diffeomorphisms. Still, the following conjecture has been proved in many cases.

Conjecture 2.3.1 (C^0-flux conjecture)**.** *The Hamiltonian group* $\mathrm{Ham}(M,\omega)$ *is* C^0*-closed in* $\mathrm{Symp}_0(M,\omega)$, *the identity component of* $\mathrm{Symp}(M,\omega)$.

This conjecture was initially stated by Banyaga [7]. Hermann first observed in 1983 that the conjecture is true on standard tori, more cases were obtained by Lalonde-McDuff-Polterovich [77] and then by Buhovsky [12]. Understanding the C^0-closure of $\mathrm{Ham}(M,\omega)$ in $\mathrm{Symp}(M,\omega)$ (hence in $\mathrm{Diff}(M)$ by Gromov-Eliashberg Theorem), is still a widely open problem. The analoguous problem for C^1-topology was already difficult and established in full generality by Ono [103].

[4] A symplectic capacity is a map defined on the open subsets with values in $[0, +\infty]$ of a symplectic manifolds which satisfies the conclusions of Proposition 1.3.2.

In the case of a symplectic surface (Σ, ω), the C^0-closure of $\mathrm{Ham}(\Sigma, \omega)$ is well understood. It is the set of homeomorphisms ϕ which preserve the oriented area, which acts trivially on the fundamental group of Σ and whose rotation vector $\rho(f)$ vanishes[5]. In particular, if such a homeomorphism is a diffeomorphism, it is Hamiltonian. Therefore, these homeomorphisms are often called Hamiltonian homeomorphisms in papers dealing with 2-dimensional dynamics. We will call them weakly-Hamiltonian homeomorphisms.

Definition 2.3.2 (Weakly-Hamiltonian homeomorphisms). *We say that a homeomorphism is weakly-Hamiltonian if it is a C^0-limit of Hamiltonian diffeomorphisms.*

Another (perhaps stronger) definition of Hamiltonian homeomorphism has been introduced by Oh and Müller [101]. It has the advantage that Hamiltonian homeomorphisms are generated in some sense by some continuous Hamiltonian. Therefore, it provides settings which are closer to familiar smooth situation.

Definition 2.3.3 (Hamiltonian homeomorphisms, Oh-Müller [101]). *Let $(\phi^t)_{t\in[0,1]}$ be an isotopy of homeomorphisms of M. We say that ϕ^t is a continuous Hamiltonian isotopy (or "hameotopy" for short) if there exists a compact subset $K \subset M$ and a sequence of smooth Hamiltonians $H_i : [0,1] \times M \to \mathbb{R}$ with support in K such that:*

1. *The sequence of flows $\phi^t_{H_i}$ C^0-converges to ϕ^t, uniformly in t as $i \to \infty$.*
2. *The sequence of Hamiltonians H_i converges uniformly to a continuous function $H : [0,1] \times M \to \mathbb{R}$, i.e. $\|H_i - H\|_\infty \to 0$ as $i \to \infty$, where $\|\cdot\|_\infty$ denotes the sup norm. Furthermore,*

We say that H generates ϕ^t, denote $\phi^t = \phi^t_H$, and call H a continuous Hamiltonian. A homeomorphism is called a Hamiltonian homeomorphism if it is the time–1 map of a continuous Hamiltonian flow.

The following theorem justifies in particular the wording "H generates ϕ^t", and the notation $\phi^t = \phi^t_H$. It also shows that these generalized Hamiltonians and isotopies behave similarly to the smooth ones.

Theorem 2.3.4 (Uniqueness theorems).
1. *(Oh-Müller [101]) A continuous Hamiltonian in the above sense generates a unique hameotopy.*
2. *(Viterbo [136]) A given hameotopy can be generated by a unique continuous Hamiltonian (up to addition of a function of time).*

The first point of the above theorem was observed by Müller and Oh [101]. It is a consequence of the energy-capacity inequality 1.3.9, as we now explain briefly.

Idea of the proof . Assume that H generates two distincts isotopies ϕ^t and ψ^t. Then, the zero Hamiltonian would generate $\phi^t \circ (\psi^t)^{-1} \neq \mathrm{Id}$, that is we would get a sequence H_i of Hamiltonians uniformly converging to 0 and whose isotopy $\phi^t_{H_i}$ would C^0-converge to $\phi^t \circ (\psi^t)^{-1}$. Since $\phi^t \circ (\psi^t)^{-1} \neq \mathrm{Id}$, there would be some t and some ball B such that $\phi^t_{H_i}$ displaces B. The energy-capacity inequality would then imply that H_i cannot be arbitrary close to 0. □

The second point of Theorem 2.3.4 was proved by Viterbo [136] and Buhovsky-Seyfaddini [17]. It is a dual statement to the first point, and we gave another proof of it in [66] using the dual energy-capacity inequality 1.3.14.

REMARK 2.3.5. The second point of Theorem 2.3.4 can also be stated as follows: *If a sequence of Hamiltonian isotopies $\phi^t_{H_i}$ C^0 converges to* Id *uniformly in $t \in [0,1]$, and if H_i C^0-converges to some continuous function H, then H depends only on time.*

[5]The rotation vector $\rho(\tilde{f}) \in H_1(\Sigma, \mathbb{R})$ is initially defined for a homotopy class $\tilde{f}$ of an isotopy of preserving a measure homeomorphisms (here the measure induced by the area form ω). For smooth isotopies it is Poincaré dual to the flux of the isotopy. If $\Sigma = \mathbb{T}^2$ it induces a rotation vector $\rho(f) \in H_1(\mathbb{T}^2, \mathbb{R})/\mathbb{Z}^2$ for all homeomorphism f; if Σ has genus $g > 1$, then it descends to an element $\rho(f) \in H_1(\Sigma, \mathbb{R})$

In [66], we used our dual energy-capacity inequality to prove a variant of this statement where the C^0-convergence of the isotopies is replaced by the convergence for the spectral norm γ. Interestingly enough, the result we obtain is partially local: the convergence of the Hamiltonian only needs to occur on some open set. This allowed us to prove a partially local version of the second point of Theorem 2.3.4. Our precise result is:

Theorem 2.3.6. *Let (M,ω) denote a rational[6] symplectic manifold, let U be a non-empty open subset of M, I be a non-empty open interval in $\mathbb{R}$ and H_i be a sequence of smooth Hamiltonians such that:*

(i) For any $s \in I$, $\gamma\left(\widetilde{(\phi_{H_i}^{st})_{t\in[0,1]}}\right)$ converges to zero, where by $\widetilde{(\phi_{H_i}^{st})_{t\in[0,1]}}$ we mean the element in the universal cover of $\mathrm{Ham}(M,\omega)$ *generated by the isotopy $t \mapsto (\phi_{H_i}^{st})_{t\in[0,1]}$,*

(ii) H_i converge uniformly on $I \times U$ to continuous functions H,

Then, H depends only on the time variable on $I \times U$.

Of course, the same holds for the Hofer norm instead of the spectral norm. ◀

Remark 2.3.7. One can easily check that generators of continuous Hamiltonian flows satisfy the same composition formulas as their smooth counterparts (See Section 0.1, Page 12). Namely,

1. If ϕ_H^t is a hameotopy, then $(\phi_H^t)^{-1}$ is a hameotopy generated by $\overline{H}(t,x) = -H(t,\phi_H^t(x))$,
2. If ϕ_H^t and ϕ_K^t are hameotopies, then $\phi_H^t \circ \phi_K^t$ is also a hameotopy, generated by $H\sharp K(t,x) = H(t,x) + K(t,(\phi_H^t)^{-1}(x))$,
3. If ϕ_H^t is a hameotopy and ψ is a globally symplectic homeomorphism, then $\psi^{-1} \circ \phi_H^t \circ \psi$ is a hameotopy generated by $H \circ \psi$.

◀

Following [101], let us introduce some notations for the sets of homeomorphisms we are considering. We will denote by $\mathrm{Sympeo}(M,\omega)$ the set of all compactly supported globally symplectic homeomorphisms, by $\overline{\mathrm{Ham}}(M,\omega)$ the set of weakly-Hamiltonian homeomorphisms and $\mathrm{Hameo}(M,\omega)$ the set of all Hamiltonian homeomorphisms. It is obvious from the definitions that $\mathrm{Hameo}(M,\omega) \subset \overline{Ham}(M,\omega) \subset \mathrm{Sympeo}(M,\omega)$. Since $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Symp}(M,\omega)$, taking C^0-closure, we deduce that $\overline{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Sympeo}(M,\omega)$.

The following propostion is an immediate consequence of Remark 2.3.7.

Proposition 2.3.8. *The set of Hamiltonian homeomorphisms* $\mathrm{Hameo}(M,\omega)$ *is a normal subgroup of* $\mathrm{Sympeo}_c(M,\omega)$.

The first consequence of the above proposition is that there are a lot of Hamiltonian homeomorphisms. However, at this point there is no general understanding of which homeomorphisms are Hamiltonian or not. This is a very interesting question which has attracted a lot of attention in the past 10 years in the case of the 2-disk. In this case, the space $\mathrm{Sympeo}(M,\omega)$ is the set of compactly supported area preserving homeomorphisms. The following questions are open:

Question 2.3.9. *1. Is the group of area preserving compactly supported homeomorphisms of $\mathbb{D}^2$ a simple group?*

2. Is the group $\mathrm{Hameo}(\mathbb{D}^2,\omega_{std})$ *equal to that group?*

The first question is rather old and arose from the work of Fathi [31] who proved the simpleness for compactly supported volume preserving homeomorphisms of the ball in dimension at least 3. The second question was raised by Yong-Geun Oh [100]. Of course, a negative answer to question 2 would give a negative answer to the long standing first question. The second question is actually open on every symplectic manifold.

[6] with the save reservation as in the footnote Page 15

2.3.2 Rigidity of Hamiltonian homeomorphisms

Now that the bases of the theory of Hamiltonian homeomorphisms is settled, the general problem we want to study is the following:

Problem 2.3.10. *Is the behavior of Hamiltonian homeomorphisms similar to that of Hamiltonian diffeomorphisms (i.e. does C^0-rigidity holds) or is it different (i.e. is there some flexibility)?*

It turns out that several rigidity phenomena have been established. Before we state our main results, let us recall a few facts about coisotropic submanifolds.

A submanifold C of a symplectic manifold (M, ω) is called *coisotropic* if for all $p \in C$, $(T_pC)^\omega \subset T_pC$ where $(T_pC)^\omega$ denotes the symplectic orthogonal of T_pC. For instance, hypersurfaces and Lagrangian submanifolds are coisotropic. A coisotropic submanifold carries a natural foliation $\mathcal{F}$ which integrates the distribution $(TC)^\omega$; $\mathcal{F}$ is called the *characteristic foliation* of C.

Let H be a smooth Hamiltonian on a symplectic manifold (M, ω). Recall the following two dynamical properties of a coisotropic submanifold C: Assume that C is properly embedded, then:

1. The restriction $H|_C$ is a function of time if and only if ϕ_H^t (preserves C and) flows along the characteristic foliation of C. By flowing along characteristics we mean that for any point $p \in C$ and any time $t \geqslant 0$, $\phi_H^t(p) \in \mathcal{F}(p)$, where $\mathcal{F}(p)$ stands for the characteristic leaf through p.

2. For each $p \in C$, $H|_{\mathcal{F}(p)}$ is a function of time if and only if the flow ϕ_H^t preserves C.

The next theorems show that the above two properties hold for continuous Hamiltonians, thus the answer to Problem 2.3.10 is rigidity in this context. This answers questions originally raised by Emmanuel Opshtein.

Theorem 2.3.11 (H.-Leclercq-Seyfaddini [65])**.** *Denote by C a connected coisotropic submanifold of a symplectic manifold (M, ω) which is c is properly embedded in M. Let $H \in C^0_{\mathrm{Ham}}$ with induced hameotopy ϕ_H^t. The restriction of H to C is a function of time if and only if ϕ_H^t preserves C and flows along the leaves of its characteristic foliation.*

Theorem 2.3.11 can be seen as a drastic generalization of the uniqueness theorem (Theorem 2.3.4). Indeed, note that in the particular case where $C = M$, then the characteristic foliation consists of the points of M, Theorem 2.3.11 reduces to Theorem 2.3.4 in that case.

Theorem 2.3.12 (H.-Leclercq-Seyfaddini [65])**.** *Denote by C a connected coisotropic submanifold of a symplectic manifold (M, ω) which is properly embedded in M. Let $H \in C^0_{\mathrm{Ham}}$ with induced hameotopy ϕ_H^t. The restriction of H to each leaf of the characteristic foliation of C is a function of time if and only if the flow ϕ_H preserves C.*

When C is a Lagrangian, Theorems 2.3.11 and 2.3.12 coincide and both state that: *The restriction of H to L is a function of time if and only if $\phi_H^t(L) = L$ for all t.* In an interesting manifestation of Weinstein's creed, "Everything is a Lagrangian submanifold!", the general case of Theorems 2.3.11 and 2.3.12 can be deduced from the a priori particular case of Lagrangians. Let us now give the main ideas of the proof of these results.

Idea of the proof . As explained in the previous paragraph, the idea of the proof of Theorems 2.3.11 and 2.3.12 is to first prove the case where C is Lagrangian (we will denote C by L in the case where it is Lagrangian). One of the main features of these theorems is the fact that they are local statements: the manifold M is not assumed to be closed nor controlled at infinity in any way. Thus, there is a technical (but elementary) first step which consists in reducing the proof to the case where (M, ω) is a closed symplectic manifold with L a weakly exact Lagrangian submanifold, and H a continuous Hamiltonian. In this case we want to prove:

Claim 2.3.13. *The restriction of H to L is a function of time if and only if ϕ_H^t preserves L.*

Let us prove the direct implication of this claim. This will be similar to the proof of the first point of Theorem 2.3.4, except that we will use the relative energy-capacity inequality instead of the standard energy-capacity inequality.

Proof . Suppose that $H_t|_L = c(t)$, where $c(t)$ is a function of time only. For a contradiction assume that ϕ_H^t does not preserve L, then for some s we have $\phi_H^s(L) \not\subset L$, and by a time reparametrization $t \mapsto st$ we may assume that $s = 1$, that is, $\phi_H^1(L) \not\subset L$.

Since $H \in C^0_{\mathrm{Ham}}$ there exists a sequence of smooth Hamiltonians H_i such that H_i converges uniformly to H and $\phi_{H_i}^t$ C^0-converges to ϕ_H^t for all t.

Because $\phi_H^1(L) \not\subset L$, there exists a ball B such that $B \cap L \neq \emptyset$ and $\phi_H^1(B) \cap L = \emptyset$. It follows that $\phi_{H_i}^{-1}(L) \cap B = \emptyset$ for large i. And so, by the relative energy-capacity inequality (Corollary 1.3.13)

$$\int_0^1 (\max H_i(t,\cdot)|_L - \min H_i(t,\cdot)|_L)\, dt \geqslant c_{\mathrm{LR}}(B; L_0),$$

contradicting the fact that $H|_L$ is a function of time. We conclude that ϕ_H^t preserves L_0. □

We have seen that the direct implication of Claim 2.3.13 can be deduced from the relative energy-capacity inequality. We will not provide the details here, but it turns out that the converse implication (which is somehow dual) can be similarly deduced from the dual relative energy-capacity inequality 1.3.15.

Theorem 2.3.11 follows from a local version of Claim 2.3.13 (which no longer assumes the Lagrangian to be closed) by the following trick: Locally, a coisotropic submanifold of codimension k is symplectomorphic to the subspace $\mathbb{C}^{n-k} \times \mathbb{R}^k$ in $\mathbb{C}^n = \mathbb{C}^{n-k} \times \mathbb{C}^k$, and the characteristic leaf through the point $(0,0)$ is $\{0\} \times \mathbb{R}^k$. Consider all the Lagrangian subspaces of the form $L = L' \times \mathbb{R}^k$, for L' a Lagrangian subspace of $\mathbb{C}^{n-k}$. Our coisotropic submanifold is locally the union of all these subspaces and the characteristic leaf is their intersection. Thus, Theorem 2.3.11 is obtained by applying (a local version of) Claim 2.3.13 to all these subspaces.

Theorem 2.3.12 also follows from a local version of Claim 2.3.13. Indeed, consider the "graph of the characteristic foliation", that is the set

$$\Gamma(\mathcal{F}) = \{(x, x') \in C \times C \,|\, x \text{ and } x' \text{ belong to the same leaf}\}.$$

It can be checked that $\Gamma(\mathcal{F})$ is (at least locally) a Lagrangian submanifold. By applying a local version of Claim 2.3.13 to $\Gamma(\mathcal{F})$ and to the Hamiltonian $H \oplus (-H)$ (which generate $\phi_H^t \times \phi_H^t$ on $(M \times M, \omega \oplus (-\omega)))$, one deduces Theorem 2.3.12. □

Several other rigid behaviors of Hamiltonian homeomorphisms have been established. For instance, in their original paper [101], Oh and Müller observed that the uniqueness theorems (Theorem 2.3.4) imply the following statement.

Proposition 2.3.14 (Oh-Müller, [101]). *A continuous Hamiltonian H is autonomous if and only if it generates a one parameter subgroup (i.e. $\phi_H^{t+s} = \phi_H^t \circ \phi_H^s$ for all $s, t \in \mathbb{R}$). Furthermore, in this case, H remains constant along the hameotopy is generates, (i.e. $H \circ \phi_H^t = H$).*

We proved the following proposition as an application of the variant of the uniqueness theorem (Theorem 2.3.6). It is a generalization of the previous proposition. Recall that two smooth autonomous Hamiltonians H, K are said to *commute* if their Poisson bracket $\{H, K\} = \omega(X_H, X_K)$ vanishes. In this case their flows commute and $K \circ \phi_H^t = K$.

Proposition 2.3.15 (H.-Leclercq-Seyfaddini). *Assume that (M,ω) is rational*[7]*. Let $H \in C^0_{\mathrm{Ham}}$ be an autonomous continuous Hamiltonian and $K \in C^0(M)$ be a continuous function. If H and K commute in the sense of Cardin and Viterbo*[8]*, then K is constant along the hameotopy generated by H, i.e. $K \circ \phi_H^t = K$. If $K \in C^0_{\mathrm{Ham}}$, then ϕ_H^t and ϕ_K^t commute.*

Despite all these rigidity results, some flexibility holds and Hamiltonian homeomorphisms with exotic behavior have been exhibited. Such phenomena will appear in the subsequent sections; cf. Corollary 2.4.6, Theorem 2.5.1.

2.4 Symplectic homeomorphisms and submanifolds

In this section, we continue our exploration of rigidity/flexiblity properties of C^0 symplectic geometry and turn our attention to the following general problem.

Problem 2.4.1. *Given a smooth submanifold V satisfying a property (P) in a symplectic manifold, and given a symplectic homeomorphism θ such that $\theta(V)$ is smooth, does $\theta(V)$ also satisfy (P) ?*

The property (P) can be for instance, being Lagrangian, symplectic, isotropic, coisotropic, having a certain Maslov class, a certain symplectic volume...

As we will see, the results tend to be compatible with the usual meta-mathematical principles. For instance, coisotropic submanifolds, including Lagrangian submanifolds, are known to be rigid in some sense (for instance they often have non trivial (leafwise) intersections [40]). Whereas isotropic submanifolds and symplectic submanifolds of codimension at least 4 are known to satisfy h-principle [46, 26], hence are very flexible. We will see that similar rigidity/flexibility phenomena will appear when studying Problem 2.4.1.

2.4.1 Coisotropic rigidity

Let us consider Problem 2.4.1 for (P) the property of being coisotropic. The following theorem shows that rigidity holds in this case.

Theorem 2.4.2 (H.-Leclercq-Seyfaddini[65]). *Let C be a smooth coisotropic submanifold of a symplectic manifold (M,ω). Let θ be a symplectic homeomorphism defined on M. If $\theta(C)$ is a smooth submanifold, then it is coisotropic. Furthermore, θ maps the characteristic foliation of C to that of $\theta(C)$.*

Before giving an idea of the proof of this theorem let us make a few comments. First note that it is a local statement: neither C nor M are assumed to be closed.

Theorem 2.4.2 uncovers a link between two previous rigidity results and demonstrates that they are in fact extreme cases of a single rigidity phenomenon. One extreme case, where C is a hypersurface, was established by Opshtein [104]. Clearly, in this case, the interesting part is the assertion on rigidity of characteristics, as the first assertion is trivially true. Lagrangians constitute the other extreme case. When C is Lagrangian (and connected), its characteristic foliation consists of one leaf, C itself. In this case the theorem reads: *If θ is a symplectic homeomorphism and $\theta(C)$ is smooth, then $\theta(C)$ is Lagrangian.* In [78], Laudenbach–Sikorav proved a similar result: *Let L be a closed manifold and ι_k denote a sequence of Lagrangian embeddings $L \to (M,\omega)$ which C^0–converges to an embedding ι. If $\iota(L)$ is smooth, then (under some technical assumptions) $\iota(L)$ is Lagrangian.* On one hand, their

[7] with the same reservation as Page 15

[8] H and K commute in the sense of Cardin and Viterbo if there exist smooth autonomous Hamiltonians H_i, K_i for all positive integer i, such that when i goes to infinity, H_i C^0-converges to H, K_i C^0 converges to K and the Poisson bracket $\{H_i, K_i\}$ C^0 converges to 0; see [18]

result only requires convergence of embeddings while Theorem 2.4.2 requires convergence of symplectic diffeomorphisms. On the other hand, Theorem 2.4.2 is local: It does not require the Lagrangian nor the symplectic manifold to be closed.

The above discussion raises the following question.

Question 2.4.3. *What can one say about C^0–limits of coisotropic embeddings and their characteristic foliations?*

In relation to this question, Ginzburg and Gürel constructed in [42] an example of an embedding of a hypersurface Σ which is a C^0-limit of a sequence embeddings of hypersurfaces, which are all Hamiltonian isotopic, and whose characteristic foliation does not converge to that of Σ.

Let us also remark that Theorem 2.4.2 is a coisotropic generalization of the Gromov–Eliashberg Theorem. Indeed, it implies that if the graph of a symplectic homeomorphism is smooth, then it is Lagrangian.

We now sketch the proof of Theorem 2.4.2.

Idea of the proof . The proof exploits the following dynamical characterization of coisotropic submanifolds: *A submanifold V is coisotropic if and only if for every (locally defined) autonomous Hamiltonian H which is constant on V, the flow of H preserves V.*

Let C be a smooth coisotropic submanifold, and θ be a symplectic homeomorphism. Assume $C' = \theta(C)$ is smooth. Let H be a smooth function, which is constant on C'. The crucial point is that $H \circ \theta$ is a C^0 Hamiltonian. Moreover, it is constant on C. By Theorem 2.3.11, its continuous Hamiltonian flow, which is nothing but $\theta^{-1} \circ \phi_H^t \circ \theta$, preserves C and flows along charateristics. In particular, C is preserved by this flow and it follows that C' is preserved by the flow of H. The above characterization shows that C is coisotropic.

To prove that θ preserves the characteristic foliation, one applies a similar argument, using the following dynamical description of characteristic leaves: *The leaf through a point p is locally the union of the orbits of p under the flows of all Hamiltonians that are constant on the coisotropic submanifold.* □

An immediate, but surprising, consequence of Theorem 2.4.2 is the fact that: *If the image of a coisotropic submanifold via a symplectic homeomorphism is smooth, so is the image of its characteristic foliation.* This leads to wonder, as in [16], whether two smooth coisotropic submanifolds which are symplectic homeomorphic should be symplectic diffeomorphic. The answer turns out to be negative, as shows the following proposition (which never appeared anywhere yet).

Proposition 2.4.4. *There exists two smooth hypersurfaces which are diffeomorphic and symplectically homeomorphic but not symplectically diffeomorphic.*

Proof . Let H_1, H_2 be two smooth compactly supported functions on $\mathbb{D}^2$ that are conjugated by an area preserving homeomorphism ϕ but not by any area preserving diffeomorphism:

$$H_1 = H_2 \circ \phi.$$

Such functions can be built as follows. Let H_2 be a smooth compactly supported function which admits a segment $I = [-a, a] \times \{0\}$ as a connected component of a level set, and all of whose derivatives vanish on I. Let ϕ be an area preserving homeomorphism which is smooth on the complement of I and do with I something that a diffeomorphism could not do, for instance sending I onto a broken line like $([0, a] \times \{0\}) \cup (\{0\} \times [0, a])$. Then, $H_1 = H_2 \circ \phi$ is smooth and the time-one maps of H_1, H_2 are not conjugated by a smooth area preserving diffeomorphism.

Denote by x the points in $\mathbb{D}^2$, and by (s, σ) the points in $T^*\mathbb{S}^1$, and consider for $i = 1, 2$ the functions on $T^*\mathbb{S}^1 \times \mathbb{D}^2$ defined by,

$$\hat{H}_i(s, \sigma, x) = \sigma + H_i(x).$$

Then set $\Sigma_i = \hat{H}_i^{-1}(0)$.

The characteristic leaf through a point $(s, -H_i(x), x)$ of Σ_i is the curve parametrized by

$$t \mapsto (s + t, -H_i(x), \phi^t_{H_i}).$$

We see that on a transversal $\{s = \text{constant}\}$, the holonomy (i.e. the Poincaré first return map) is $\mathrm{Id}_{\mathbb{R}} \times \phi^1_{H_i}$. Since these holonomies are not smoothly conjugated, the two hypersurfaces cannot be symplectomorphic. But on the other hand, they are (globally) symplectic homeomorphic since

$$(\mathrm{Id}_{T^*\mathbb{S}^1} \times \phi)(\Sigma_1) = \Sigma_2.$$

□

2.4.2 Flexibility of symplectic submanifolds of codimension at least 4

An important breakthrough was made by Buhovsky and Opshtein in [16], when they exhibited an exotic example of symplectic homeomorphism of $\mathbb{C}^3$ whose restriction to $\mathbb{C} \times \{0\} \times \{0\}$ is not symplectic. More precisely they proved the following theorem.

Theorem 2.4.5 (Buhovsky-Opshtein, Theorem 1 in [16]). *For $n \geqslant m + 2$, there exists a symplectic homeomophism $h : \mathbb{C}^n \to \mathbb{C}^n$ with support in an arbitrary neighborhood of $Q = (\mathbb{D}^2)^m \times \{0\}^{n-m} \subset \mathbb{C}^n$, such that $h|_Q = \frac{1}{2}\mathrm{Id}$.*

Of course such a homeomorphism can be implanted in a Darboux chart of any symplectic manifold. This example shows that there are some aspects of symplectic geometry that do not extend to the C^0-setting. Of course, this makes C^0-symplectic geometry even more interesting!

Their construction introduces a technique which they call "quantitative h-principle techniques", and which can potentially be reused for other constructions. To justify this terminology recall that the classical C^0-dense h-principle techniques [46, 26] show for example that every smooth embedding of a polydisk P of codimension at least 4 can be obtain as a C^0-limit of symplectic embeddings of polydisks P_i, $i \in \mathbb{N}$. Therefore, the polydiscs P_i and P_j are arbitrarily C^0-close for i, j large. The quantitative h-principle techniques essentially add to this the fact that for i, j large, there is a symplectic diffeomorphism which maps P_i onto P_j and which is C^0 small everywhere (not only near the polydisks).

These new technics will also play a role in Section 2.5 in the construction of a C^0 counter-example to the Arnold conjecture.

The following corollary of Theorem 2.4.5 was communicated to us by Lev Buhovsky. It shows that Hamiltonian homeomorphisms considered in Section 2.3 may also exhibit some exotic dynamical behavior.

Corollary 2.4.6. *There exists a compactly supported continuous Hamiltonian H on $\mathbb{C}^n$, with $n \geqslant 3$, which vanishes on $\mathbb{C}^m \times \{0\}^{n-m}$, and which generates a hameotopy ϕ^t_H which preserves $\mathbb{C}^m \times \{0\}^{n-m}$ but whose restriction to $\mathbb{C}^m \times \{0\}^{n-m}$ is not the identity.*

Such a continuous Hamiltonian H can be obtained as follows. Take any compactly supported smooth Hamiltonian K which vanishes on $\mathbb{C}^m \times \{0\}^{n-m}$ and whose flow preserves $\mathbb{C}^m \times \{0\}^{n-m}$. Then let H be $H = (K \circ h)\sharp\overline{K}$, where h is the symplectic homeomorphism provided by Theorem 2.4.5. Then, by Remark 2.3.7, H is a continuous Hamiltonian, it vanishes on $\mathbb{C}^m \times \{0\}^{n-m}$, and its flow

$h^{-1} \circ \phi_K^t \circ h \circ (\phi_K^t)^{-1}$ preserves $\mathbb{C}^m \times \{0\}^{n-m}$. But for K appropriately chosen, this flow is not the identity on $\mathbb{C}^m \times \{0\}^{n-m}$.

One can also wonder if a result similar to Theorem 2.4.2 holds for symplectic submanifolds. It is conjectured that the answer should be negative, and that the technics of [16] should lead to the construction of a symplectic homeomorphism which maps a symplectic submanifold onto an isotropic submanifold. Taking the inverse, it would also give a symplectic homeomorphism which maps an isotropic submanifold onto a symplectic submanifold.

2.4.3 Rigidity of symplectic submanifolds in codimension 2

It is not known whether a smooth submanifold which is image of a symplectic submanifold of codimension 2 under a symplectic homeomorphism should be symplectic everywhere.

However, a weaker result holds. Observe that, in a symplectic vector space, a codimension 2 subspace is either symplectic or coisotropic. Let ϕ be a symplectic homeomorphism which sends a codimension 2 symplectic submanifold S onto a smooth submanifold V. If V was coisotropic on a small open subset, then this would contradict Theorem 2.4.2 applied to ϕ^{-1}. Thus every open subset of V contains a point whose tangent space is symplectic. Since such points form an open subset, we get:

Proposition 2.4.7. *If a symplectic homeomorphism ϕ maps a codimension 2 symplectic submanifold S onto a smooth submanifold, then this submanifold $\phi(S)$ is symplectic on a dense open subset.*

This rigidity of codimension 2 symplectic submanifolds is confirmed by another result of Buhovsky and Ophstein. We state it only for symplectic surfaces in dimension 4, where the formulation is much simpler. See [16], Theorem 3.1, for the full statement.

Proposition 2.4.8 (Buhovsky-Opshtein, [16])**.** *If a symplectic homeomorphism ϕ maps a symplectic surface S of symplectic manifold of dimension 4 onto another smooth symplectic surface, then the restricted homeomorphism $\phi|_S$ preserves the symplectic area.*

This is proved by a clever use of the stable energy-capacity inequality.

2.4.4 More on coisotropic rigidity: reduction of symplectic homeomorphisms

Given a smooth coisotropic submanifold C in (M, ω), with caracteristic foliation $\mathcal{F}$, the reduction $\mathcal{R}$ of C is the quotient space $C/\mathcal{F}$. Whenever this space is Hausdorff separated, it is a smooth manifold and inherits a natural symplectic structure induced by ω.

Let C and C' be two smooth coisotropic submanifolds, with characteristic foliations $\mathcal{F}$ and $\mathcal{F}'$, and reductions $\mathcal{R} = C/\mathcal{F}$ and $\mathcal{R}' = C'/\mathcal{F}'$. Assume that $C' = \phi(C)$ for some symplectic homeomorphism ϕ. Since $\phi(\mathcal{F}) = \mathcal{F}'$, by Theorem 2.4.2, the homeomorphism ϕ induces a homeomorphism $\phi_R : \mathcal{R} \to \mathcal{R}'$. It is a classical fact that when ϕ is smooth, and hence symplectic, the reduced map ϕ_R is a symplectic diffeomorphism as well. It is therefore natural to ask whether the homeomorphism ϕ_R is symplectic, in any sense, when ϕ is not assumed to be smooth.

In the case where the reduced map ϕ_R is smooth (but not ϕ), this problem can be resolved rather easily using Theorem 2.3.11 (See [16] Prop 6, or [67] Prop 2 for a proof).

Proposition 2.4.9. *Let C be a coisotropic submanifold whose reduction $\mathcal{R}$ is a symplectic manifold*[9]*, and ϕ be a symplectic homeomorphism. Assume that $C' = \phi(C)$ is smooth (hence coisotropic) and admits a reduction $\mathcal{R}'$. Denote by $\phi_R : \mathcal{R} \to \mathcal{R}'$ the map induced by ϕ. Then, if ϕ_R is smooth, it is symplectic.*

[9] This is always locally true.

When ϕ_R is not assumed to be smooth, the situation becomes far more complicated. The question of whether or not ϕ_R is a symplectic homeomorphism seems to be very difficult and, at least currently, completely out of reach. Given the difficulty of this question, Opshtein formulated the following a priori easier problem:

Question 2.4.10. *Is the reduction ϕ_R of a symplectic homeomorphism ϕ preserving a coisotropic submanifold always a capacity preserving homeomorphism?*

Partial positive results have been obtained by Buhovsky and Opshtein [16]. They proved in particular that in the case where C is a hypersurface, the map ϕ_R is a "non-squeezing map" in the sense that for every open set U containing a symplectic ball of radius r, the image $\phi_R(U)$ cannot be embedded in a symplectic cylinder over a 2–disk of radius $R < r$. Since capacity preserving maps are non-squeezing it does provide positive evidence for Question 2.4.10. In the case where C is a hypersurface in a 4-dimensional symplectic manifold, this leads to an affirmative answer to Question 2.4.10. For higher codimension coisotropic submanifolds, they conjecture that the same holds and indicate as to how one might approach this conjecture.

In [67], we decided to work in the specific setting where M is the torus $\mathbb{T}^{2(k_1+k_2)}$ equipped with its standard symplectic structure and $C = \mathbb{T}^{2k_1+k_2} \times \{0\}^{k_2}$. The reduction of C is $\mathbb{T}^{2k_1}$ with the standard symplectic structure. We answer Question 2.4.10 positively in this setting by proving that, the reduced homeomorphism ϕ_R is a selector-preserving map (as defined in Page 42), hence preserves the Viterbo capacity.

Theorem 2.4.11 (H.-Leclercq-Seyfaddini, [67])**.** *Let ϕ be a symplectic homeomorphism of the torus $\mathbb{T}^{2(k_1+k_2)}$ equipped with its standard symplectic form. Assume that ϕ preserves the coisotropic submanifold $C = \mathbb{T}^{2k_1+k_2} \times \{0\}^{k_2}$. Denote by ϕ_R the induced homeomorphism on the reduced space $\mathcal{R} = \mathbb{T}^{2k_1}$. Then, for every time-dependent continuous function H_R on $\mathbb{S}^1 \times \mathcal{R}$, supported in a small enough open subset, we have:*

$$c(H_R \circ \phi_R) = c(H_R),$$

where $H_R \circ \phi_R(t, x) = H_R(t, \phi_R(x))$. As a consequence, the Viterbo capacity c_V (defined in Remark 1.3.4) of small subsets is preserved by ϕ_R.

The condition that H is supported in a small open set can be dropped if we impose ϕ to be a globally symplectic homeomorphism.

Our proof actually shows that if a selector-preserving homeomorphism preserves the characteristic foliation, then the homeomorphism induced on the reduction is also a selector-preserving map.

Theorem 2.4.11 can probably be generalized to other settings where the reduction is a closed manifold, but we have not investigated this direction yet. We do not know if a local version of Theorem 2.4.11 can be proved (somehow, this would be an optimal statement).

Idea of the proof . For simplicity, assume that ϕ is a globally symplectic homeomorphism hence is a selector-preserving homeomorphism. Let H_R be a continuous function on $\mathbb{S}^1 \times \mathcal{R}$. Let $K_R = H_R \circ \phi_R$. We want to prove that $c(K_R) = c(H_R)$. We denote respectively $H = H_R \oplus 0$ and $K = K_R \oplus 0$ the standard lifts of H_R and K_R to $\mathbb{T}^{2(k_1+k_2)}$. The splitting formula (Proposition 1.2.6) implies that $c(K_R) = c(K)$, and since ϕ is selector preserving, $c(K) = c(K \circ \phi)$. Now, by construction, H coincides with $K \circ \phi$ on C. In other words, H_R is the reduction of $K \circ \phi$. The reduction inequality (Proposition 1.2.8) also holds for continuous maps and yields: $c(H_R) \leqslant c(K \circ \phi) = c(K_R)$. Reversing the roles of H_R and K_R, we get $c(K_R) = c(H_R)$. □

2.4.5 More on cosiotropic rigidity: defining C^0-coisotropic submanifolds

In this section we use Theorem 2.4.2 to define C^0–coisotropic submanifolds and their characteristic foliations. Recall that every coisotropic submanifold of codimension k is locally symplectomorphic to $C_k = \mathbb{C}^{n-k} \times \mathbb{R}^k \subset \mathbb{C}^n$ and that the leaf of its characteristic foliation, $\mathcal{F}_k$, passing through $p = (z, x) \in C_k$ is given by $\mathcal{F}_k(p) = \{z\}^{n-k} \times \mathbb{R}^k$. We call C_k the *standard coisotropic subspace of codimension k.*

Definition 2.4.12. *A codimension–k C^0–submanifold C of a symplectic manifold (M, ω) is C^0-coisotropic if around each point $p \in C$ there exists an an open neighborhood U of p and $\theta : U \to V \subset \mathbb{R}^{2n}$ a symplectic homeomorphism, such that $\theta(C \cap U) = C_k \cap V$. We call such θ a C^0-coisotropic chart.*

A codimension-n C^0-coisotropic submanifold is called a C^0-Lagrangian.

Example 2.4.13. Graphs of symplectic homeomorphisms are C^0-Lagrangians. Graphs of differentials of C^1 functions and, more generally, graphs of C^0 1-forms, closed in the sense of distributions, are also C^0-Lagrangians. This last point follows from Example 2.2.3.

Conversely, we could ask whether every continuous 1-form whose graph is a C^0–Lagrangian is closed in the sense of distributions. An affirmative answer in a particular case appears in Viterbo [136, Corollary 22]. ◀

As a consequence of Theorem 2.4.2, we have:

Proposition 2.4.14. *Any C^0-coisotropic submanifold C admits a unique C^0-foliation $\mathcal{F}$ which is mapped to $\mathcal{F}_k$ by any C^0-coisotropic chart.*

Theorem 2.4.2 states that a smooth C^0-coisotropic submanifold is coisotropic and its natural C^0-foliation coincides with its characteristic foliation. Moreover, it is not hard to prove that Theorems 2.3.11 and 2.3.12 extend to C^0-coisotropic submanifolds.

This implies in particular that a smooth Hamiltonian is a function of time on a C^0-coisotropic submanifold if and only if its flow follows its C^0 foliation. This fact might be of some interest for the study of (smooth) Hamiltonian dynamics where non-smooth invariant Lagrangians naturally appear.

Since every smooth hypersurface is coisotropic, it is natural to wonder if every C^0-hypersurface is C^0-coisotropic in our sense. The answer turns out to be negative and Proposition 2.4.14 provides an obstruction. The next example did not appear anywhere.

Example 2.4.15. (Folding a hyperplane along a 2-plane in $\mathbb{R}^4$ with a symplectic homeomorphism) One of the simplest example of a non-smooth hypersurface is a hyperplace folded along a 2-plane in $\mathbb{R}^4$, that is a subset V constructed as follows.

Let H_1, H_2 be two distinct hyperplanes in $\mathbb{R}^4$, and let P be their intersection 2-plane. Note that like every 2-plane, P is either symplectic or Lagrangian. Let V_1 be one of the two connected components of $H_1 \setminus P$ and V_2 be one of the two connected components of $H_2 \setminus P$. Then, set

$$V = V_1 \cup P \cup V_2.$$

We may wonder whether V is a C^0-coisotropic submanifold; in other words, whether it is possible to "fold" a standard $\mathbb{R}^3$ with a symplectic homeomorphism to obtain V.

Proposition 2.4.16. *The above subset $V \subset \mathbb{R}^4$ is C^0-coisotropic in the sense of Definition 2.4.12 if and only if P is a symplectic 2-plane.*

Proof. Assume that P is Lagrangian. By applying a linear symplectic transformation, we may assume that $H_1 = \mathbb{C} \times \mathbb{R}$, $H_2 = \mathbb{R} \times \mathbb{C}$ (we identified $\mathbb{R}^4$ with $\mathbb{C}^2$), $P = \mathbb{R} \times \mathbb{R}$, $V_1 = (\mathbb{R} + i\,[0, +\infty)) \times \mathbb{R}$ and $V_2 = \mathbb{R} \times (\mathbb{R} + i\,[0, +\infty))$. If V was C^0-coisotropic, then according to Proposition 2.4.14, it would carry a C^0-foliation that would coincide with the characteristic foliations on V_1 and V_2. The characteristic foliation of V_1 is directed by the line $\{0\} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{C}$ and that of V_2 by the line $\mathbb{R} \times \{0\} \subset \mathbb{C} \times \mathbb{C}$. We see that these two foliations do not match to a foliation on V. Thus, V is not C^0-coisotropic.

Assume now that P is symplectic. Then, by applying a linear symplectic transformation, we may assume that $H_1 = \mathbb{C} \times \mathbb{R}$, $H_2 = \mathbb{C} \times i\mathbb{R}$, $P = \mathbb{C} \times \{0\}$, $V_1 = \mathbb{C} \times [0, +\infty)$ and $V_2 = \mathbb{C} \times i\,[0, +\infty)$. It is known that it is possible to find an oriented area preserving homeomorphism of $\mathbb{C}$ which maps $\mathbb{R}$ onto the broken line $[0, +\infty) \cup i\,[0, +\infty)$. Let ϕ be such a map. Then, the map $\mathrm{Id} \times \phi$ is a symplectic homeomorphism which maps $C_3 = \mathbb{C} \times \mathbb{R}$ on V.[10] □

It would be interesting to push further this example and try to characterize which polytopes can be (locally) smoothen by symplectic homeomorphisms. ◀

2.5 The Arnold conjecture and Hamiltonian homeomorphisms

2.5.1 The Arnold conjecture

Let us begin this section by stating the famous Arnold conjecture.

Conjecture (Arnold [3]). *A Hamiltonian diffeomorphism of M must have at least as many fixed points as the minimal number of critical points of a smooth function on M.*

What makes this conjecture so remarkable is the large number of fixed points that it predicts. This is often interpreted as a manifestation of symplectic rigidity. In contrast to Arnold's conjecture, the classical Lefschetz fixed-point theorem cannot predict the existence of more than one fixed point for a general diffeomorphism.

This conjecture has been very important for the development of symplectic topology. Indeed this motivated the construction of Floer homology, which establishes a variant of the Arnold conjecture on large classes of symplectic manifolds. The above version of the Arnold conjecture has been established on symplectically aspherical manifolds by Rudyak and Oprea in [116] who built on earlier works of Floer [34] and Hofer [53].

The variant of the Arnold conjecture implied by the existence of Floer homology is the following: *Every non-degenerate Hamiltonian diffeomorphism on a closed symplectic manifold M admits at least as many fixed points as the sum of the Betti numbers of M.* The relation with the Arnold conjecture is due to the fact that the obtained lower bound is the same as the lower bound for the number of critical points of a function on M provided by Morse theory. But this variant only applies to non-degenerate Hamiltonian diffeomorphisms.

As already mentioned at the end of Section 1.1.1, the classical Lusternik-Schnirelman theory provides lower bounds for the minimal number of critical points of any (not necessarily Morse) function. It shows that this minimal number is always at least the *cup length* of M, denoted $\mathrm{cl}(M)$ (See Page 21 for the definition of the cup length). Therefore, a natural interpretation of the Arnold conjecture, sometimes referred to as the *homological Arnold conjecture*, is that a Hamiltonian diffeomorphism of (M, ω) must have at least $\mathrm{cl}(M)$ fixed points. For a symplectic manifold of dimension $2n$, the cup length is at least $n+1$ (the n-th power of the cohomology class of the symplectic form does not vanish), hence we see that the Arnold conjecture predicts at least $n + 1$ fixed points for every Hamiltonian diffeomorphism. But to our knowledge, this estimate has not been established beyond symplectically aspherical manifolds and a few more cases.

[10] In that case the caracteristic foliations are respectively given by half-lines parallel to $\{0\} \times [0, +\infty)$ and $\{0\} \times i\,[0, +\infty)$. We see that they indeed match to a well defined foliation on V.

2.5.2 A C^0 counter-example to the Arnold conjecture

Now that we have a definition for Hamiltonian homeomorphisms (Definition 2.3.3), and in connection to Problem 2.3.10, it is natural to wonder whether the Arnold conjecture holds for Hamiltonian homeomorphisms or not.

In the case of closed surfaces, this problem have been studied and solved in the past. Matsumoto [85], building on an earlier paper of Franks [35], has proven that (weakly-)Hamiltonian homeomorphisms of surfaces satisfy the Arnold conjecture. Another proof were given by Le Calvez, using his theory of transverse foliations, which also allowed him to prove the Conley conjecture on periodic points of these homeomorphisms [80]. But none of the powerful tools of surface dynamics seem to generalize in an obvious manner to dimensions higher than two. In fact one can not hope to prove the Arnold conjecture in higher dimensions, as we proved with Lev Buhovsky and Sobhan Seyfaddini:

Theorem 2.5.1 (Buhovsky-H.-Seyfaddini [15]). *Let (M, ω) denote a closed and connected symplectic manifold of dimension at least 4. There exists a Hamiltonian homeomorphism f with a single fixed point.*

Furthermore, f can be chosen to satisfy either of the following additional properties.

1. *Let $\mathcal{H}$ be a normal subgroup of $\mathrm{Sympeo}(M, \omega)$ which contains $\mathrm{Ham}(M, \omega)$ as a proper subset. Then, $f \in \mathcal{H}$.*
2. *Let p denote the unique fixed point of f. Then, f is a symplectic diffeomorphism of $M \setminus \{p\}$.*

Let us make a few remarks. First, note that every Hamiltonian homeomorphism possesses at least one fixed point. This is because a Hamiltonian homeomorphism is by definition a C^0 limit of Hamiltonian diffeomorphisms. These diffeomorphisms may be assumed non-degenerate without loss of generality, hence admit at least one fixed point by Floer homology.

Second, we recall that $\mathrm{Ham}(M, \omega)$ is a normal subgroup of $\mathrm{Symp}(M, \omega)$. Hence, it is reasonable to expect that any alternative candidate for the group of Hamiltonian homeomorphisms should contain $Ham(M, \omega)$ and be a normal subgroup of $\mathrm{Sympeo}(M, \omega)$. It is indeed the case that $\mathrm{Hameo}(M, \omega) \trianglelefteq \mathrm{Sympeo}(M, \omega)$. Therefore, the first property in the above theorem states that there is no hope of proving the Arnold conjecture for any alternate definition of Hamiltonian homeomorphisms.

Some interesting questions remain unanswered. For instance, we can pose the following:

Question 2.5.2. *Does the Conley conjecture hold for Hamiltonian homeomorphisms on (say) closed symplectically aspherical manifolds ?*

In its simple form, the Conley conjecture states that a Hamiltonian diffeomorphism on an aspherical symplectic manifold has infinitely many periodic points. This conjecture was proved by Hingston [52] on tori and Ginzburg [41] in the more general setting. As mentioned earlier, the Conley conjecture has been proved for Hamiltonian homeomorphisms of surfaces by Le Calvez [80]. We have not been able to construct a counterexample to the Conley conjecture in higher dimensions.

2.5.3 A brief outline of the construction of the counter-example

The construction of the homeomorphism f, as prescribed in Theorem 2.5.1, is done in two major steps. The first step, which is the more difficult of the two, can be summarized in the following statement.

Lemma 2.5.3. *Let (M, ω) denote a closed and connected symplectic manifold of dimension at least 4. There exists $\psi \in \mathrm{Hameo}(M, \omega)$ and a continuously embedded tree $T \subset M$ such that*

1. *T is invariant under ψ, i.e. $\psi(T) = T$,*
2. *All of the fixed points of ψ are contained in T,*

3. *ψ is smooth in the complement of T.*

The proof of this lemma forms the technical heart of the proof. An important ingredient used in the construction of the invariant tree T is the Buhovsky-Opshtein quantitative h-principle technics, discussed in Section 2.4.2. The dimension at least four of M is used in a crucial way in the proof of this lemma. Roughly, the idea is to start from an arbitrary C^2-small Morse function H. Its time-1 map ϕ_H^1 admits only finitely many fixed points which correspond to the critical points of H. Given two such critical points x_1, y_1, we construct a closed curve γ_1, connecting x_1 to y_1, and a Hamiltonian homeomorphism ψ_1 which is C^0-close to Id, such that $\psi_1 \circ \phi_H^1$ has the same fixed points as ϕ_H^1 and the curve γ_1 is invariant by $\psi_1 \circ \phi_H^1$. We perform this construction sufficiently many times to construct curves $\gamma_1, \gamma_2, \ldots, \gamma_N$ whose union constitute the required tree T. The homeomorphism ψ is then obtained as a composition: $\psi = \psi_N \circ \cdots \circ \psi_1 \circ \phi_H^1$.

The second major step of our construction consists of "collapsing" the invariant tree T to a single point which will be the fixed point of our homeomorphism f. Here is a brief outline of how this is done. Fix a point $p \in M$. We construct a sequence $\phi_i \in \mathrm{Symp}(M, \omega)$ such that ϕ_i converges uniformly to a map $\phi : M \to M$ with the following two properties:

1. $\phi(T) = \{p\}$,

2. ϕ is a proper symplectic diffeomorphism from $M \setminus T$ to $M \setminus \{p\}$.

Note that the first property implies that ϕ is not a 1-1 map and hence, the sequence ϕ_i^{-1} is not convergent. Define $f : M \to M$ as follows: $f(p) = p$ and

$$\forall x \in M \setminus \{p\},\ f(x) = \phi \circ \psi \circ \phi^{-1}(x).$$

It is not difficult to see that p is the unique fixed point of f. Indeed, on $M \setminus \{p\}$, the map f is conjugate to $\psi : M \setminus T \to M \setminus T$ which is fixed point free by construction.

By picking the above sequence of symplectomorphisms ϕ_i carefully, it is possible to ensure that the sequence of conjugations $\phi_i \psi \phi_i^{-1}$ converges uniformly to f. The uniform convergence of $\phi_i \psi \phi_i^{-1}$ to f relies heavily on the invariance of the tree T and it occurs despite the fact that the sequence ϕ_i^{-1} diverges. It follows that f can be written as the uniform limit of a sequence of Hamiltonian diffeomorphisms.

It is not difficult to see that f is smooth on the complement of its unique fixed point. However, proving that f is a Hamiltonian homeomorphism and that it satisfies the first property listed in Theorem 2.5.1 requires some more work.

2.5.4 Rescuing the Arnold conjecture for homeomorphisms

Theorem 2.5.1 seems to indicate that there is no hope for developping a fixed point theory for Hamiltonian homeomorphisms. In this section we explain how one can use the action selectors (presented in Chapter 1) to obtain results that can be interpreted as variants of the Arnold conjecture for Hamiltonian homeomorphisms.

In this section we assume that (M, ω) is a closed symplectically aspherical manifold.

First note that thanks to their Lipschitz continuity with respect to Hofer's norm (Inequality (1.4)), continuous Hamiltonians admit action selectors, defined as limits:

Definition 2.5.4. *For $\alpha \in H_*(M, \mathbb{Z}/2) \setminus \{0\}$ and H a continuous Hamiltonian, we set*

$$c(\alpha, H) = \lim_{i \to +\infty} c(\alpha, H_i),$$

where H_i is any sequence of smooth Hamiltonians converging to H.

Note that it is not known whether $c(\alpha, H)$ only depends on the time-one map ϕ_H^1 (unlike the smooth case [121]) except on surfaces where this fact follows from [123].

One can observe in the construction of the counter-example (sketched in the previous section), that the action selectors of the final Hamiltonian homeomorphism are very close to that of the initial C^2-small Morse function (for which the action selectors correspond to the homological minmax selectors $\rho(\alpha, \cdot)$, by Proposition 1.2.1). In particular, despite the fact that this homeomorphism has a unique fixed point, it may have many distinct spectral invariants.

The following proposition comes from the classical Lusternik-Schnirelman theory. Proofs can be found in [137] or [59].

Proposition 2.5.5 (Lusternik-Schnirelman). *Let f be a smooth function on a closed manifold V. If the number of distinct homological minmax selectors of f is smaller than* $\mathrm{cl}(V)$*, then the set of critical points of f is homologically non-trivial.*

A subset $A \subset M$ is homologically non-trivial if for every open neighborhood U of A the map $i_* : H_j(U) \to H_j(M)$, induced by the inclusion $i : U \hookrightarrow M$, is non-trivial for some $j > 0$. Clearly, homologically non-trivial sets are infinite.

This proposition implies that our counter-example admits at least $\mathrm{cl}(M)$ distinct action selectors. It is therefore natural to wonder if this is a specific property of our counter-example or if it is general phenomenon.

In the spirit of the Arnold conjecture, Howard proved that a similar statement holds for fixed points of Hamiltonian diffeomorphism [57]. It turns out that his methods can be used for Hamiltonian homeomorphisms as well:

Theorem 2.5.6 (Buhovsky-H.-Seyfaddini, [14]). *Let (M, ω) denote a closed and symplectically aspherical manifold. Suppose that ϕ is a Hamiltonian homeomorphism generated by a continuous Hamiltonian H. If the number of distinct action selectors of H is smaller than* $\mathrm{cl}(M)$*, then the set of fixed points of ϕ is homologically non-trivial, hence it is infinite.*

This theorem shows that despite our counter-example, the (homological) Arnold conjecture holds true if we include in the count, not only fixed points but also the number of distinct action selectors.

Theorem 2.5.6 admits several variants in other settings, also proved in [14], with similar proofs. They apply to the following objects:

- On aspherical closed surfaces, a similar statement holds for weakly-Hamiltonian homeomorphism,
- C^0-Lagrangians that are Hamiltonian homeomorphic to the zero section of a cotangent bundle,
- Haussdorf limits of Legendrian submanifolds contact isotopic to the zero section in a 1-jet space.

More generally, the following principle seems to hold in general:

Principle 2.5.7. *Suppose that X is a non-smooth object for which one can define action selectors. If the number of action selectors associated to X is smaller than* $\mathrm{cl}(M)$*, then the set of fixed/intersection points of X is homologically non-trivial, hence it is infinite.*

Remark 2.5.8. This principle can also be established for micro-support of sheaves. This is the content of our work in progress with Nicolas Vichery reported in Section 1.5. ◀

2.6 Other aspects of C^0 symplectic geometry

There are many other aspects of C^0 symplectic geometry that we have not dealt with in this chapter. We mention a few of them below.

C^0-rigidity of the Poisson bracket and symplectic function theory. This phenomenon which was discovered by Cardin and Viterbo [18], has been very much studied (See e.g. [61, 30, 13, 2]). Cardin and Viterbo proved the following: if H_i and K_i are two sequences of smooth functions which converge uniformly to smooth functions H and K, and if their Poisson bracket $\{H_i, K_i\}$ converge uniformly to 0, then $\{H, K\} = 0$. Buhovsky, Entov and Polterovich introduced new symplectic invariants based on this rigidity phenomenon, with many applications to symplectic function theory and to dynamics [13, 109].

Generalizations of the definition of Hamiltonian homeomorphisms. The definition of continuous Hamiltonian flows (hameotopies) have been generalized to give a new definition of continuous symplectic (not necessarily Hamiltonian) isotopies for which the flux homomorphism might be defined [8]. It has also been generalized to define continuous contact isotopies [93, 92, 94]. An interesting application is given in the paper [91] on the topological invariance to the Arnold-Hopf invariant (called helicity by Arnold).

Weak solutions of Hamilton-Jacobi equations Given an autonomous Hamiltonian H on a cotangent bundle T^*N, and a constant c, the Hamilton-Jacobi equation is given by $H(x, du(x)) = c$, where the unknown u is a function on N. This is a first order non-linear PDE, which is extensively studied in the case of Hamiltonians that convex in the fibers. From the point of view of symplectic geometry and Hamiltonian dynamics, u is a solution if and only if its graph (which is an exact Lagrangian submanifold) is invariant under the flow of H. The solutions studied are often only weak solutions (e.g. viscosity solution), hence provide natural example of non-smooth Lagrangians. See e.g. [137, 32]

C^0-symplectic topology and microlocal sheaf theory As already discussed in Section 1.5, microlocal sheaf theory associates in a canonical way a closed subset on T^*N to any sheaf on the base N, called the micro-support of the sheaf. This closed subset, which is in general very singular, was proved to be always coisotropic in a generalized sense [71]. Therefore sheaves provide a promising tool to study singular objects of symplectic nature. This has been already exploited by Guillermou who gave a proof of the Gromov-Eliashberg theorem, using this theory [48]. The microsupport has also been used to non-smooth analysis by Vichery [133].

2.7 New direction: C^0 symplectic structures ?

After the dynamical questions addressed in Section 2.5, one can also think about topological questions related to symplectic homeomorphisms. The most intriguing one is probably the following:

Problem 2.7.1. *What are the manifolds which admit a C^0-symplectic structure, that is, an atlas whose change of coordinates are symplectic homeomorphisms? Are there smooth manifolds which admit a C^0-symplectic structure but do not admit a smooth symplectic structure?*

For example, Hofer asked more than twenty years ago whether $\mathbb{S}^4$ admits a C^0-symplectic structure.

The most simple necessary condition for the existence of a smooth symplectic structure on a manifold of dimension $2n$ are the existence (for M compact) of a degree-2 cohomology class β such that $\beta^n \neq 0$, and the existence of an almost complex structure. Therefore we can ask the following questions:

Question 2.7.2. *Assume that a smooth compact manifold M admits a C^0-symplectic structure.*

1. *Does there exists degree-2 cohomology class β on M such that $\beta^n \neq 0$?*

2. Does M admit an almost complex structure?

In a joint work in progress with Sobhan Seyfaddini, we started to study the first of the two questions.

Proposition 2.4.8 seems to indicates that in dimension 4, symplectic homeomorphisms preserves the area of surfaces. This gives good hopes for a positive answer to this first question, in dimension 4. However, the situation is complicated by the following issue, pointed out by Lev Buhovsky. It can be proved using the technics of Buhovsky and Opshtein [16] that there exists a symplectic homeomorphism of $\mathbb{R}^4$ which sends a smooth loop γ to another smooth loop γ' having a different action (i.e. the symplectic area enclosed by γ is different from that of γ'). Note that there is no contradiction with Proposition 2.4.8. This just means that for this specific homeomorphism, there is no smooth capping disk of γ which is sent to a smooth capping disk of γ'.

However, we can probably get around this issue to prove some partial results. Note that every compact smooth symplectic manifold can be obtained by gluing a finite collection of smoothly embedded $2n$-symplices of $\mathbb{R}^{2n}$ along their faces via symplectic diffeomorphisms. Indeed this follows from the existence of a smooth triangulation[11]. We think that the following statement can be deduced from Proposition 2.4.8: *If a compact manifold M is obtained by gluing smoothly embedded 4-simplices in $\mathbb{R}^4$ along their faces via symplectic homeomorphisms, then M admits a degree-2 cohomology class β such that $\beta^2 \neq 0$.* In particular, $\mathbb{S}^4$ cannot be obtained by such a construction. This raises the (completely open) question of whether every C^0-symplectic structure can be obtained in this way.

The second question in 2.7.2 is of a different (homotopical) nature. We have not thought about it yet.

[11]Smooth triangulations always exists by Whitehead's theorem. A much more difficult (and open) problem is to determine whether there always exists a triangulation whose simplices are symplectomorphic to linear simplices.

Chapter 3

Symplectic invariants on surfaces

This chapter is made of two completely independent parts, unified by a common motivation: In dimension 2, symplectic invariants should have elementary descriptions, or at least should be easier to understand. We study two different symplectic invariants in their simplest (but already non-trivial) possible situation. The first part is based on [63] and deals with the Lagrangian Hofer distance in the standard 2-disk. It is essentially independent of the rest of the dissertation. The second part, based on our joint work [64] with Frédéric Le Roux and Sobhan Seyfaddini, studies action selectors of autonomous Hamiltonians on surfaces. The central object will be the selector $c(H)$ defined in Chapter 1, Equation (1.5). We will use some of the properties described in Chapter 1, in particular the max formulas (Theorems 1.4.1 and 1.4.3).

3.1 Lagrangian Hofer distance in the disk

3.1.1 Lagrangian Hofer distance

Let us recall the definition of the Lagrangian Hofer distance, which already appeared in Sections 0.1 and 1.3.2. Given two Hamiltonian isotopic compact Lagrangian submanifolds L and L', the Lagrangian Hofer distance is by definition

$$\delta(L, L') = \inf\{\|H\| \,|\, H \text{ such that } \phi_H^1(L) = L'\},$$

where $\|H\| = \int_0^1 (\max H_t - \min H_t)\, dt$ is the Hofer norm.

When the ambient manifold is not closed, we only consider compactly supported Hamiltonians. Accordingly, we only consider Lagrangian submanifolds which coincide near infinity.

It is rather easy to check that δ satisfies the triangle inequality (this follows from the identities on Hamiltonians Page 12). However, the non-degeneracy of δ, i.e. the fact that if $\delta(L, L') = 0$ then $L = L'$, is a deep fact due to Chekanov [22]. In the special case of weekly exact Lagrangian submanifolds, the non-degeneracy can be deduced from the relative energy-capacity inequality (Corollary 1.3.13). An important (and easily established) property is the invariance of δ: for all Hamiltonian isotopic Lagrangians L and L', and all symplectic diffeomorphisms ψ, one has

$$\delta(\psi(L), \psi(L')) = \delta(L, L').$$

On surfaces, the situation seems considerably simplified. First of all Lagrangian submanifolds are nothing but simple curves. Moreover, Hamiltonian homeomorphisms are also rather well understood. For example, it is a standard fact (which is a consequence of Moser's trick) that: *If C and C' are two sets which are both a finite union of curves in $\mathbb{R}^2$, then there is a (compactly supported) Hamiltonian diffeomorphism of $\mathbb{R}^2$ mapping C onto C' if and only if there is a compactly supported diffeomorphism*

f which maps C onto C', such that for every connected component U of $\mathbb{R}^2 \times C$, the area of $f(U)$ equals the area of U. This implies that it is possible to determine if some operation on a finite family of curves can be realized by a Hamiltonian diffeomorphism, by considering the intersection patterns of the curves and the total areas of the regions delimited by the curves (a finite number of parameters). Thus the situation is close to being combinatorial.

The minimal Hofer energy required for certain simple operations is also quite well understood. For example, assume that two pieces of curves in the plane intersects in exactly two points and hence delimit a half-disk of area A. Then for all $\varepsilon > 0$, there exists a Hamiltonian diffeomorphism of Hofer norm less than $A + \varepsilon$ which displaces one curve from the other; see Figure 3.1. Note that at least if the curves are closed, the relative energy-capacity inequality imposes that this energy is at least A.

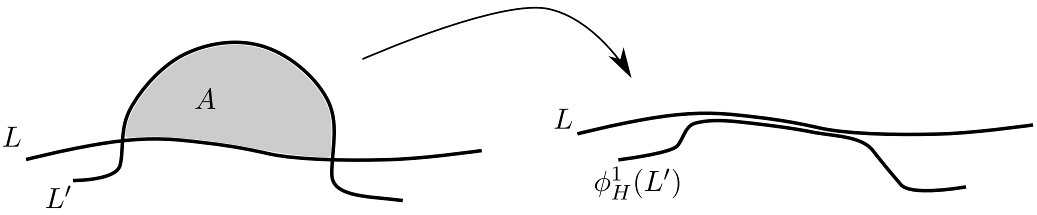


Figure 3.1: A Hamiltonian diffeomorphism ϕ^1_H with $\|H\| < A + \varepsilon$.

The remarks made in the last two paragraphs show that it is reasonnable to hope to fully describe the Lagrangian Hofer distance on the plane or more generally on surfaces. However, this is still an open problem.

An important breakthrough was made by Michael Khanevsky in the case of the standard open disk $(\mathbb{D}^2, dy \wedge dx)$. Let us call a *diameter* any curve which Hamiltonian isotopic to the *standard diameter* $L_0 = [-1, 1] \times \{0\} \subset \mathbb{D}^2$, via a Hamiltonian supported in the interior of $\mathbb{D}^2$. Equivalently, a diameter is a simple curve in $\mathbb{D}^2$ which coincides with L_0 near the boundary and cuts $\mathbb{D}^2$ in two regions of equal area.

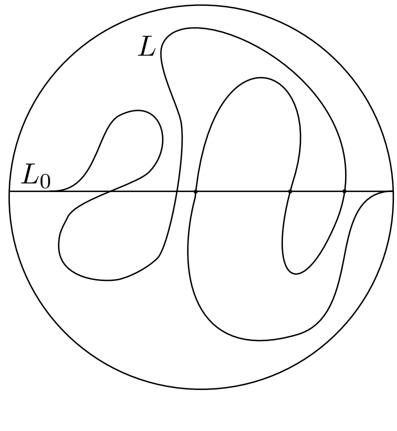


Figure 3.2: An example of diameter L.

By a clever use of the Entov-Polterovich quasi-morphism[1] on $\mathbb{S}^2$, Khanevsky proved:

Theorem 3.1.1 (Khanevsky [74]). *The space of diameters endowed with Hofer's distance is unbounded.*

This was later generalized to higher dimension balls by Seyfaddini [125].

[1]This quasi-morphism is defined on the Hamiltonian group of the sphere (and certain other symplectic manifolds) by homogenizing the Hamiltonian spectral invariant c, i.e. by taking $\lim_{k \in \mathbb{N}} \frac{1}{k} c(H^{\sharp k})$, for H mean-normalized.

Khanevsky also provided interesting upper bounds for Hofer's distance on diameters. Note that in our context, every diameter coincides with the standard diameter in the complement of a compact subset. Therefore, we will say that two diameters L and L' are transverse if we can write $L \cap L' = (-1, -1+\varepsilon] \cup \{x_i\}_{i\in\{1,\dots,N\}} \cup [1-\varepsilon', 1)$, for some $\varepsilon, \varepsilon' > 0$ and the points $x_1, \dots, x_N$ are transverse intersections (in the usual sense). We denote the number of transverse intersections N by $\sharp(L \cap L')$.

Theorem 3.1.2 (Khanevsky [74]). *For every transverse diameters L and L', we have*

$$\delta(L, L') \leqslant \pi(\tfrac{1}{8}\sharp(L \cap L') + 1).$$

This is proved by elementary methods, based on a combinatorial description of the quotient of the space of diameters by the action of the group Hamiltonian diffeomorphisms which preserve L_0 (the function "distance to L_0" descends to this space).

Khanevsky generalized this last result to other type of curves on other surfaces in [75]. We will return to it in the case of equators on the 2-sphere, in Section 3.1.3.

3.1.2 Lagrangian Hofer distance on the disk and the Maslov index

Let us start this section with the following remark.

REMARK 3.1.3. Khanevsky's upper bound given in Theorem 3.1.2 is far from being sharp in some cases. For example, pick a diameter L and for all integer $k = 1, 2, \dots$ build a diameter L_k by arranging k disjoint copies of the $\frac{1}{k}L$ as on Figure 3.3. Perturb slighly to make it transverse. Then, it can be proved that the Hofer distance between L_k and L_0 converges to 0, but clearly the number of intersection points goes to infinity. ◀

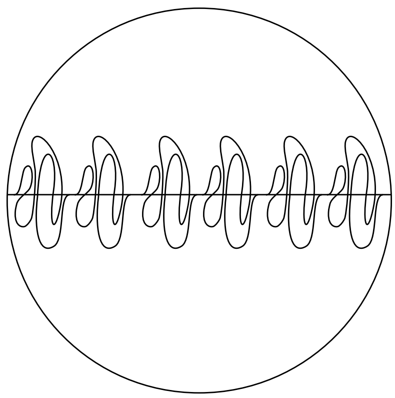


Figure 3.3: A diameter for which Khanevsky's bound is not optimal.

This motivated the search of a refinement of Khanevsky's upper bound, which could be sharper in these cases. To state our result, let us introduce the Maslov index of an intersection point between two diameters L and L' of the disk $\mathbb{D}^2$.

The Maslov index of a transverse intersection point between two given Lagrangians is an integer which is invariant under Hamiltonian transformations. Unless we choose some convention, this index is defined up to an additive constant but the index difference between two intersection points is well-defined. This index provides a graduation on Lagrangian Floer homology, but we avoided to give its abstract definition in Chapter 0. We will again avoid it here, since in the case of diameters, we can give a very simple combinatorial description of the Maslov index. The general definition can be found in [134].

First, by applying a Hamiltonian diffeomorphism, we may suppose that one of the two diameters is the standard one. Then, consider a diameter L intersecting L_0 transversely. We suppose that the points $x_1, \ldots, x_N$ are ordered by their position on L (not L_0). We choose a convention for the index of x_1, namely we set $\mu(x_1) = 0$. Then, we construct the Maslov index μ of the other intersection points inductively as follows.

We denote by D^+ the upper half-disk and D^- the lower half-disk. For $i \in \{1, \cdots, N\}$, we set $\mu(x_{i+1}) = \mu(x_i) + \delta$, where

$$\delta = \begin{cases} 1, & \text{if } L|_{[x_i, x_{i+1}]} \subset D^- \text{ and } x_i < x_{i+1} \text{ on } L_0, \\ -1, & \text{if } L|_{[x_i, x_{i+1}]} \subset D^- \text{ and } x_i > x_{i+1} \text{ on } L_0, \\ -1, & \text{if } L|_{[x_i, x_{i+1}]} \subset D^+ \text{ and } x_i < x_{i+1} \text{ on } L_0, \\ 1, & \text{if } L|_{[x_i, x_{i+1}]} \subset D^+ \text{ and } x_i > x_{i+1} \text{ on } L_0. \end{cases}$$

Intuitively, $\mu(x_i)$ measures how much T_xL "twists" in the positive direction before x reaches x_i. More precisely, it counts (with signs) the number of times where T_xL is horizontal before reaching x_i.

Remark 3.1.4. Note that by definition, the index difference between two intersection points that are consecutive on L is 1 or -1. The abstract definition of the Maslov index implies that we can invert the roles of L and L_0 and therefore that the index difference between two intersection points that are consecutive on L_0 is also either 1 or -1. ◀

To avoid any confusion, we also denote $\mu(x, L)$ for $\mu(x)$. We then set

$$\mu_{\max}(L) = \max_{i=1,\ldots,N} \mu(x_i, L) \quad \text{and} \quad \mu_{\min}(L) = \min_{i=1,\ldots,N} \mu(x_i, L).$$

We are now ready to state our result.

Theorem 3.1.5 (H. [63]). *For any diameter L transverse to L_0, with at least two transverse intersection points with L_0,*

$$\delta(L, L_0) \leqslant \pi \left(\mu_{\max}(L) - \mu_{\min}(L) - \frac{1}{2} \right).$$

The maximal index difference $\mu_{\max}(L) - \mu_{\min}(L)$ vanishes if and only if there is only one transverse intersection point. In that case, $\delta(L, L_0) \leqslant \frac{\pi}{2}$.

Remark 3.1.6. Note that Remark 3.1.4 yields the inequality $2(\mu_{\max}(L) - \mu_{\min}(L)) \leqslant \sharp(L \cap L_0) + 1$. Thus Theorem 3.1.5 implies $\delta(L, L_0) \leqslant \frac{1}{2} \cdot \sharp(L \cap L_0)$. Therefore our result implies a linear estimate as in Theorem 3.1.2 (but with $\frac{1}{2}$ as multiplicative constant, while Khanevsky's constant is $\frac{1}{8}$).

However, in the cases of considered in Remark 3.1.3, our upper bound is not optimal but much sharper since the quantity $\mu_{\max}(L_k) - \mu_{\min}(L_k)$ is independent of k. ◀

Idea of the proof . The invariance of Hofer's distance implies that we may assume that L' is the standard diameter L_0. The proof then goes by induction on $\mu_{\min}(L)$ and $\mu_{\max}(L)$. Assume that $\mu_{\min}(L) < 0$, and let x_0 be a point of minimal index.

The configuration of the curves L and L_0 near the point x_0 is as shown on the left part of Figure 3.4: There are two intersection points y, z such that y, x_0, z are in this order on L_0 and such that the arc of L between x_0 and y is in D^+ and the arc between x_0 and z is in D^-. These two arcs and L_0 delimit two half discs that we denote $r(x_0)$ and $R(x_0)$, $r(x_0)$ being the one with the smaller area. Then it is possible to construct an isotopy with Hofer length Area($r(x_0)$), which eliminate the smaller region $r(x_0)$ (in particular the point x_0 disappear), and such all other intersection point either disappear or stay fixed with fixed index; see Figure 3.4. We then show that such transformations can be performed on a certain collection of points with minimal index, and in an appropriate order, so that after these transformations, all the points with minimal index are removed, and the total Hofer energy needed for

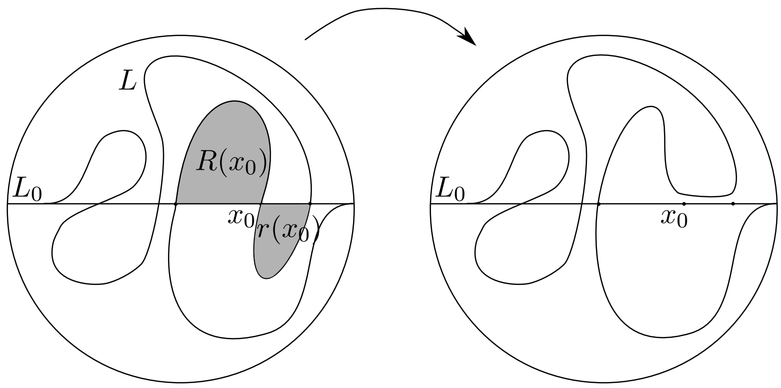


Figure 3.4: The point x_0 has minimal Maslov index.

these transformations is less than π. This leads to a new diameter L' satisfying $\mu_{\min}(L') \leqslant \mu_{\min}(L)+\pi$ satisfying $\delta(L', L) = \pi$.

We then proceed by induction until $\mu_{\min} = 0$, and work similarly for points of positive index. □

REMARK 3.1.7. An interesting consequence of the proof is that if L satisfies $\mu_{\min}(L) = 0$, then there exists a non-negative Hamiltonian function H and such that $\delta(\phi_H^1(L_0), L) \leqslant \pi$. ◀

As far as we know there is no other result of this type in the litterature. For example, nothing seems to be known in higher dimension. It is not even clear whether we can bound the Hofer distance between the standard $\mathbb{R}^n$ and a Lagrangian in $\mathbb{C}^n$ which coincides with $\mathbb{R}^n$ in the complement of the standard ball $B^{2n}(1)$, and which admits exactly one intersection point with $\mathbb{R}^n$ in the interior of $B^{2n}(1)$.

3.1.3 Open problem: Lagrangian Hofer distance on equators

It is natural to wonder if similar results as Theorems 3.1.1 and 3.1.2 holds on the sphere $\mathbb{S}^2$ for *equators*, that is simple curves Hamiltonian isotopic to a great circle L_0 called *standard equator*. Equators can also be described as simple curves which cuts the sphere in two regions (smooth open disks) of equal area.

The boundedness is an open question.

Question 3.1.8. *Is the space of equators unbounded for Hofer's distance ?*

This question is connected to other interesting questions on Hofer distance on $\mathrm{Ham}(\mathbb{S}^2)$. Khanevsky obtained the following logarithmic estimate (compare with the linear estimate of Theorem 3.1.2).

Theorem 3.1.9 (Khanevsky, [75]). *For every transverse diameters L and L', and every $\varepsilon > 0$, there exists a constant $C(\varepsilon)$,*

$$\delta(L, L') \leqslant 2\pi \log_{\frac{3}{2}-\varepsilon} \sharp(L \cap L') + C(\varepsilon).$$

Logarithms show up very rarely in symplectic topology and it is probably the first time a logarithm appears in the study of Hofer's distance. However, logarithms are common in dynamics, since they are related to entropy.

The following observation (which did not appear anywhere) does not give any answer to Question 3.1.8, but at least it gives an indication that equators far from L_0 could possibly be found by looking at images of L_0 by a the iterates of a diffeomorphism with positive topological entropy.

Proposition 3.1.10. *Let ϕ be a Hamiltonian diffeomorphism of $\mathbb{S}^2$, then we have the following inequality:*

$$\limsup_{k\to+\infty} \frac{1}{k}\delta(\phi^k(L_0), L_0) \leqslant \frac{2\pi}{\log(\frac{3}{2})}\, h_{\text{top}}(\phi),$$

where $h_{\text{top}}(\phi)$ denotes the topological entropy of ϕ. In particular, if $h_{\text{top}}(\phi) = 0$ then the growth of $\delta(\phi^k(L_0), L_0)$ is sublinear.

Proof . First note that for every great circle Λ, there is a Hamiltonian diffeomorphism of energy π which maps Λ onto L_0. Thus

$$\delta(L, L_0) \leqslant \delta(L, \lambda) + \pi.$$

Thus, by Theorem 3.1.9, we deduce

$$(\tfrac{3}{2} - \varepsilon)^{\frac{1}{2\pi}(\delta(L,L_0)-C(\varepsilon)-\pi)} \leqslant \sharp(L \cap \Lambda),$$

for every great circle Λ transverse to L. Now, the classical Crofton formula asserts that the integral of $\sharp(L \cap \Lambda)$ over all great circles[2] is $2 \cdot \text{length}(L)$, where the length is computed with respect to the standard round metric on $\mathbb{S}^2$. Thus we have $(\frac{3}{2} - \varepsilon)^{2\pi\delta(L,L_0)-C(\varepsilon)-\pi} \leqslant 2 \cdot \text{length}(L)$, hence

$$\delta(L, L_0) \leqslant 2\pi \log_{\frac{3}{2}-\varepsilon} \text{length}(L) + C'(\varepsilon), \tag{3.1}$$

where $C'(\varepsilon)$ is a real number which depends only on ε.

We can now apply Yomdin's theorem [138] and get:

$$\limsup \tfrac{1}{k} \log(\text{length}(\phi^k(L_0))) \leqslant h_{\text{top}}(\phi). \tag{3.2}$$

Inequalities (3.1) and (3.2) imply that for all ε,

$$\limsup_{k\to+\infty} \frac{1}{k}\delta(\phi^k(L_0), L_0) \leqslant \frac{2\pi}{\log(\frac{3}{2} - \varepsilon)}\, h_{\text{top}}(\phi),$$

hence the proposition. □

Remark 3.1.11. There is no hope for the reverse inequality. Indeed, if ϕ is supported in a small ball then $\limsup_{k\to+\infty} \frac{1}{k}\delta(\phi^k(L_0), L_0) = 0$ but $h_{\text{top}}(\phi)$ may be arbitrary large. ◀

3.2 Action selectors of autonomous Hamiltonians on surfaces

In this section we present the results obtained with Frédéric Le Roux and Sobhan Seyfaddini [64], which lead to an explicit formula for the Hamiltonian action selector c, introduced in Chapter 1, in Equation (1.5), for autonomous Hamiltonians on surfaces other than sphere.

[2]Great circles are parametrized by a hemisphere via the map $v \mapsto v^{\perp}$. This integral is taken with respect to the standard measure on the hemisphere.

3.2.1 Motivations and main result

This work has several motivations. One of them is the recent solution to the "displaced disks problem" of Béguin, Crovisier and Le Roux: using the action selector c, one can show that arbitrarily C^0-small area preserving homeomorphisms of a closed surface can not displace embedded disks of a given area; see [124, 24]. This problem seemed unsolvable with the usual tools of 2-dimensional dynamics. One drawback of the action selector c is the complexity of its construction which relies on the difficult machinery of Floer theory. As a consequence, despite its widespread use, c can only be computed in a handful of scenarios where the Floer theoretic picture is simple enough. Therefore one motivation was to try to interpret the invariant c in purely dynamical terms.

Another motivation in writing this article has been our hopes of better understanding the link between Hamiltonian Floer theory and Le Calvez's theory of transverse foliations for dynamical systems on surfaces [80]. In a sense, as far as surfaces are concerned, the two theories appear to be equivalent: much of what can be done via one theory can also be achieved via the other. As examples of this phenomenon, one could point to proofs of the Arnol'd conjecture and recent articles by Bramham [11, 10] and Le Calvez [81]. However, the action selector c has so far no analogue in Le Calvez's theory.

In the next section, we will introduce a very elementary invariant $\mathcal{N}(H)$, defined in purely dynamical terms, and associated to any Hamiltonian H which is either defined and compactly supported on $\mathbb{R}^2$ or defined on a closed surface of positive genus. Our main result is then:

Theorem 3.2.1 (H.-Le Roux-Seyfaddini [64])**.** *Let Σ be either $\mathbb{R}^2$ or a closed surface of positive genus. Then for all (compactly supported) Hamiltonian on Σ,*

$$c(H) = \mathcal{N}(H),$$

where c is the Viterbo selector c_+ if $\Sigma = \mathbb{R}^2$, or the Schwarz selector c otherwise.

The cases of the sphere and/or of general (non-autonomous) Hamiltonians are completely open. We will discuss the latter briefly in Section 3.2.5.

An intriguing aspect of this result is that, the obvious properties of $\mathcal{N}$ are quite different from the known properties of c. Indeed, $\mathcal{N}$ is computable in practice for autonomous Hamiltonians. Furthermore, one can easily see that it satisfies a max formula similar to the one in Theorem 1.4.1, which was not known for c before our work. On the other hand, $\mathcal{N}$ does not *a priori* seem to share the continuity properties of c.

To prove that that c and $\mathcal{N}$ coincide, our first taks was to establish the max formula for c; this is Theorems 1.4.1 and 1.4.3. Then, we established that every "formal" action selector satisfying a minimal set of axioms must coincide with $\mathcal{N}$ on autonomous Hamiltonians.

Definition 3.2.2. *A function $c : C^\infty(S^1 \times \Sigma) \to \mathbb{R}$ is a* formal *action selector if it satisfies the following four axioms:*

1. *(Spectrality) $c(H) \in \mathrm{spec}(H)$ for all $H \in C^\infty(S^1 \times \Sigma)$.*
2. *(Non-triviality) There exists a topological disk $D \subset \Sigma$ and H supported in D such that $c(H) \neq 0$.*
3. *(Continuity) c is continuous with respect to the C^∞ topology.*
4. *(Max formula) $c(H_1+\ldots+H_N) = \max\{c(H_1), \ldots, c(H_N)\}$ if the H_i's are Hamiltonians supported in pairwise disjoint disks.*

It follows from the properties listed in Chapter 1 that the Schwarz action selector c on closed surfaces of higher genus is a formal action selector. The Viterbo action selector c_+ alluded in Theorem 1.4.3, is a formal action selector on $\mathbb{R}^2$. One can also build formal action selectors on a disk by embedding the disk in a closed surface and pulling back the Schwarz selector by this embedding.

Theorem 3.2.3. *Let $c : C^\infty(S^1 \times \Sigma) \to \mathbb{R}$ denote a formal spectral invariant. Then, $c(H) = \mathcal{N}(H)$ for every $H \in C^\infty(\Sigma)$.*

We do not know if formal spectral invariants satisfy the triangle inequality, or the property that $c(H)$ is attained by an orbit of Conley–Zehnder index 2. However, it is a consequence of Theorems 3.2.1 and 3.2.3 that at the level of autonomous Hamiltonians the triangle inequality and the index property are satisfied by *formal* spectral invariants. It would be interesting to see if this can be extended to non-autonomous Hamiltonians or higher dimensional manifolds.

As we will see, for Morse Hamiltonians H, our proof gives a combinatorial description of the invariant $c(H)$ (and $\mathcal{N}(H)$) based on the Reeb graph of H; see Propositions 3.2.9 and 3.2.10.

As a byproduct of our work, we obtain a very simple description of the Entov-Polterovich partial quasi-state on closed aspherical surfaces using which we characterize heavy and super-heavy subsets of these surfaces.

3.2.2 The invariant $\mathcal{N}$

In this section we introduce the notions of unlinked sets, rotation number of a fixed point, and Hamiltonian action that lead to the definition of our invariant $\mathcal{N}$.

Unlinked sets

We consider an orientable surface Σ which may be non-compact but has empty boundary. We denote by $\mathrm{Diff}_0(\Sigma)$ the group of diffeomorphisms which are the time-one maps of a compactly supported isotopies. Given an isotopy $(\phi^t)_{t\in[0,1]}$ we denote by ϕ its time-one map ϕ^1 by ϕ.

Definition 3.2.4. *A set X of contractible fixed points of $(\phi^t)_{t\in[0,1]}$ is said* unlinked *if there exists another isotopy I whose time-one map is ϕ, which is homotopic to $(\phi^t)_{t\in[0,1]}$ in $\mathrm{Diff}_0(\Sigma)$ with fixed end-points, and that fixes every point of X.*

When $\mathrm{Diff}_0(\Sigma)$ is simply connected, the notion of unlinkedness depends only on ϕ^1. This includes the case when Σ is the disk, the plane or any closed orientable surface except the sphere and the torus ([44]). On the torus, since $\mathrm{Ham}(\mathbb{T}^2)$ is simply connected ([107], Section 7.2), it depends only on ϕ^1 if we restrict ourselves to Hamiltonian isotopies.

A basic result on unlinked sets is that *a set X of contractible fixed points of $(\phi^t)_{t\in[0,1]}$ is unlinked if and only if every finite subset of X is unlinked.* This implies in particular the existence of unlinked sets that are maximal for inclusion. See the appendix of [64] for a proof of these facts.

For finite set of points, unlinked sets admit a geometric description in terms of braids, that we now explain. Let X be a finite set of contractible fixed points. The isotopy $(\phi^t)_{t\in[0,1]}$ generates the geometric pure braid[3]

$$b_{X,(\phi^t)} : (x,t) \mapsto \phi^t(x).$$

Proposition 3.2.5. *A finite set X of contractible fixed points of $(\phi^t)_{t\in[0,1]}$ is unlinked if and only if the geometric braid $b_{X,(\phi^t)}$ represents the trivial braid.*

As a consequence, a single contractible fixed point $\{x\}$ is unlinked. For a more interesting example, let us consider a pair $\{x, y\}$ of distinct fixed points in $\Sigma = \mathbb{R}^2$. One can define the *linking number*

[3]A *geometric pure braid* (based on X) is a map $b : X \times [0,1] \to \Sigma$ such that $b(x,0) = b(x,1)$ for every x in X, and $x \mapsto b(x,t)$ is injective for every t. We say that this geometric braid *represents the trivial braid* if there exists a continuous map $B : X \times [0,1] \times [0,1] \to \Sigma$ such that $B(\cdot,\cdot,0)$ is the constant braid $(x,t) \mapsto x$, $B(\cdot,\cdot,1) = b_{X,(\phi^t)}$, and $B(\cdot,\cdot,s)$ is a geometric braid for every s.

$\ell(x,y)$ as the degree of the circle map

$$t \mapsto \frac{\phi^t(x) - \phi^t(y)}{\|\phi^t(x) - \phi^t(y)\|}.$$

Then the pair $\{x, y\}$ is unlinked if and only if $\ell(x,y) = 0$.

Rotation number and negative unlinked sets

For simplicity we restrict ourselves to a surface Σ which is either the plane or a closed surface other than sphere. Consider a contractible fixed point x for an isotopy $(\phi^t)_{t\in[0,1]}$ as before. Since x is a contractible fixed point, there exists a "capping disk", i.e. a smooth map $u : \mathbb{D}^2 \to \Sigma$ from the unit disk $\mathbb{D}^2$ whose restriction to the unit circle is (a parametrization of) the trajectory $t \mapsto \phi^t(x)$. Since $\mathbb{D}^2$ is contractible, the pullback of $T\Sigma$ under u may be identified with the trivial bundle $\mathbb{D}^2 \times \mathbb{R}^2$. Given a unit vector v in $\mathbb{R}^2 \simeq \{x\} \times \mathbb{R}^2 \simeq T_x\Sigma$, the pullback of the path $t \mapsto (\phi^t(x), D_{\phi^t(x)}\phi^t \cdot v)$ is a path (t, v_t) in $\mathbb{D}^2 \times (\mathbb{R}^2 \setminus \{0\})$. The map

$$(t,v) \mapsto \frac{v_t}{\|v_t\|}$$

is an isotopy in the circle. We call *rotation number of* x and denote by $\rho(x)$ the rotation number of this isotopy[4].

It can be proven that the rotation number $\rho(x)$ depends only on x and ϕ^1.

In the Hamiltonian context the rotation number may be generalized to higher dimensions, and is called the mean index, see for example [119, 43].

Definition 3.2.6. *An unlinked set X is said* negative *if $\rho(x) \leqslant 0$ for every $x \in X$. We say that a negative unlinked set X is* maximal *if there is no negative unlinked set X' strictly containing X. Maximal negative unlinked set will be called "mnus" for short.*

Note that the rotation number, and hence being negatively unlinked, is invariant under conjugation in the group $\mathrm{Diff}_0(\Sigma)$. It can be proved that every negative unlinked set is contained in a maximal negative unlinked set. Furthermore, the closure of a negative unlinked set is still a negative unlinked set, and maximal negative unlinked sets are closed.

If our isotopy is Hamiltonian for a symplectic form on Σ, it is known that there always exists at least one negative contractible fixed point. In the case when Σ is not compact, every point outside the support of ϕ is a negative contractible fixed point. In the case when Σ is a compact surface, a negative contractible fixed point is provided by P. Le Calvez's proof of the Arnol'd conjecture. It also follows from Floer's proof of the Arnold conjecture.

Definition of $\mathcal{N}$

For simplicity again we restrict ourselves to a surface Σ which is either the plane $\mathbb{R}^2$, the interior of a closed disk in the plane, or a closed surface which is not the sphere, although everything works on any surface Σ for which the inclusion of $\mathrm{Ham}(\Sigma)$ into $\mathrm{Diff}_0(\Sigma)$ is trivial at the level of the fundamental groups.

Let us consider a Hamiltonian H on Σ. Remember that the notions of unlinkedness and rotation number depend only on ϕ_H^1 and not on the isotopy. We denote by $\mathrm{mnus}(\phi_H^1)$ the set of all mnus's of ϕ_H^1. We have seen that there exists at least one mnus and that furthermore, since there exists a negative contractible fixed point, every mnus is non-empty. Hence the following definition is valid.

[4] The rotation number is a real number defined as follows. We lift the isotopy to an isotopy $(F_t)_{t\in[0,1]}$ of $\mathbb{R}$, whose time-one map F_1 is a homeomorphism of the line that commutes with the translation $s \mapsto s+1$; the rotation number of the isotopy is, by definition, the translation number of F_1, $\lim_{n\to+\infty} \frac{1}{n}(F_1^n(s) - s)$ for any $s \in \mathbb{R}$ (see for example [72]).

Definition 3.2.7. *For all Hamiltonian H on Σ, we set*

$$\mathcal{N}(H) = \inf_{X \in \mathrm{mnus}(\phi_H^1)} \sup_{x \in X} \mathcal{A}_H(x).$$

If $\Sigma = \mathbb{R}^2$ then $\mathcal{A}_H(x) = \mathcal{A}_G(x)$ for every (compactly supported) Hamiltonian function G whose time-one map is ϕ. If Σ is a closed surface then the same equality holds if $\int_0^1 \int_\Sigma G d\omega\, dt = \int_0^1 \int_\Sigma H d\omega\, dt$. In particular we see that like the Viterbo and Schwarz action selector, we may also define $\mathcal{N}(\phi)$ to be $\mathcal{N}(H)$ where H is any Hamiltonian function whose time-one map is ϕ, normalized by the condition $\int_0^1 \int_\Sigma H d\omega\, dt = 0$ in the case Σ is a closed surface.

Note that $\mathcal{N}$ is invariant under conjugation by symplectic diffeomorphisms.

Remark 3.2.8: Max formula for $\mathcal{N}$. It easy to check that the max formula holds for $\mathcal{N}$: Suppose that $H_1, \ldots, H_N$ are Hamiltonians on Σ whose supports are contained in pairwise disjoint open disks $U_1, \ldots, U_N$. Then

$$\mathcal{N}(H_1 + \ldots + H_N) = \max\{\mathcal{N}(H_1), \ldots, \mathcal{N}(H_N)\}.$$

◀

Example: radial Hamiltonians

In this paragraph, we illustrate the notions introduced above on a basic but fundamental example. Let $H \in C^\infty(\mathbb{R}^2)$ be a smooth autonomous Hamiltonian on the plane, that only depends on the distance to the origin. It will be convenient to write H in the form

$$\forall x, y \in \mathbb{R},\ H(x, y) = f(\pi(x^2 + y^2)),$$

for some function $f : [0, +\infty) \to \mathbb{R}$.

Fixed points. The Hamiltonian vector field is given by

$$X_H = (-2\pi y f'(\pi(x^2 + y^2)), 2\pi x f'(\pi(x^2 + y^2)))$$

and we see that the time–1 map of the flow restricted to the circle of radius r is the rotation by $2\pi f'(\pi r^2)$. Thus, the fixed points of ϕ_H^1 are, besides the origin, the points of $\mathbb{R}^2$ whose distance to the origin r is such that $f'(\pi r^2)$ is an integer.

Rotation numbers. Let (x, y) be such a point, denote $s = \pi(x^2 + y^2)$ and $k = f'(s)$. The orbit of H makes exactly k oriented turns along the circle centered at the origin and passing through (x, y). The linearized flow of H along the orbit, i.e. the linear map $D\phi_H^t(x, y)$, acts on a vector $\vec{v}$ tangent to the circle as the rotation by angle $2\pi kt$, thus $\rho(x, y) = k$. Therefore the fixed points with non-positive rotation number correspond to values of s where f is non-increasing. Note that the rotation number of the origin is $f'(0)$.

Mnus's. Let p_1, p_2 be two distinct fixed points of ϕ_H^1. To fix ideas, assume that p_2 is no closer to the origin than p_1. Then the linking number $\ell(p_1, p_2)$ equals the rotation number $\rho(p_2)$. This immediately leads to the following complete description of the mnus's. Let X denote the set of critical points of H. For every point $p = (x, y)$ such that $f'(\pi(x^2 + y^2))$ is a negative integer, let X_p denote the union of $\{p\}$ and of the critical points of H further than p from the origin. The sets X_p are mnus's. If $f'(0) \leqslant 0$ then X is a mnus, in the opposite case $X \setminus \{0\}$ is a mnus (note that, by the intermediate value theorem, in this case this last set is not included in any of the X_p's).

Reading the Hamiltonian action on diagrams. The Hamiltonian action of these fixed points is given by

$$\mathcal{A}_H(x, y) = f(s) - sk = f(s) - sf'(s).$$

It corresponds to the intersection of the vertical axis $\{0\} \times \mathbb{R}$ with the tangent to the graph of f at the point $(s, f(s))$, see Figure 3.5. The action can also be read on the graph of minus the rotation number $-f'$. With the above notations, $\mathcal{A}_H(x, y) = -(ks + \int_s^{+\infty} f'(\sigma)d\sigma)$. This corresponds to the grey area in Figure 3.6.

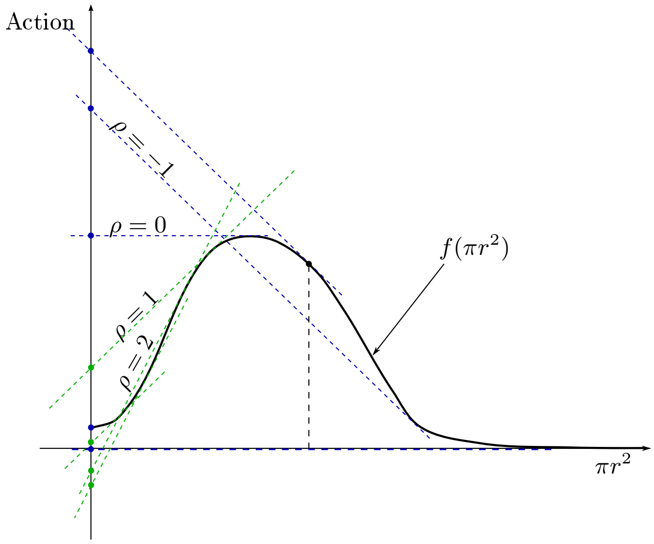


Figure 3.5: The dotted lines are the tangents to the graph of f with integer slope. Their tangency points correspond to the fixed points of ϕ_H^1. The intersections of these lines with the vertical axis (represented by thick dots) give the action. The points with non-positive rotation numbers are in blue.

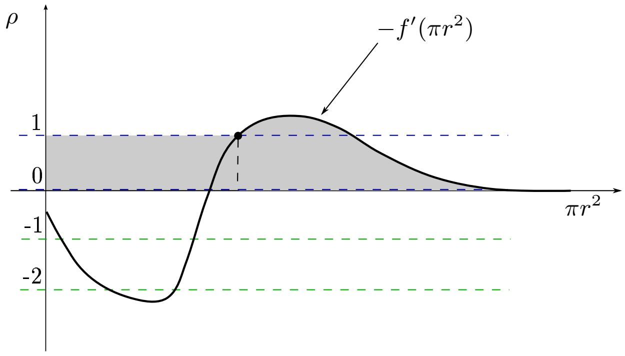


Figure 3.6: The fixed points correspond to intersections of the graph $\rho = -f'$ with the horizontal lines "$\rho =$ integer constant". The action of the thick black dot is the area of the grey region. This thick black dot corresponds to the thick black dot on Figure 3.5.

Computing $\mathcal{N}$. First assume that the function $f : [0, +\infty) \to \mathbb{R}$ is decreasing and has non-vanishing derivative on $(0, r_0)$, where $[0, r_0]$ is the support of f. Let Y be the complement in the plane of the open disk with radius r_0. The mnus's are the sets of the form $\{x\} \cup Y$ where x is any fixed point not in Y. Finally we get

$$\mathcal{N}(H) = \min_x \mathcal{A}_H(x), \tag{3.3}$$

where the minimum runs over all fixed points of ϕ_H^1 that are not in Y. With the interpretation of the action explained above, we see that it is a positive number, attained at a periodic orbit of period exactly one.

Another case when $\mathcal{N}$ is easy to compute is when f takes only non-positive values. Indeed, remember that the set of all critical points of H, taking out the origin in case $f'(0) > 0$, is a mnus. Since every critical point has a non-positive action, we see that $\mathcal{N}(H) = 0$. There does not seem to be any easy formula in the case of a general radial Hamiltonian.

3.2.3 Idea of the proof of Theorem 3.2.3

Let c be a formal action selector in the sense of Definition 3.2.2. The strategy for proving that $c = \mathcal{N}$ for autonomous Hamiltonians consists of four main steps.

Step 1: We prove that c must satisfy many more properties than just the ones of Definition 3.2.2. In particular, every *formal* action selector is monotone and Lipschitz continuous with respect to H, and satisfies the Energy-Capacity inequality: the value of c for functions supported on a disk is bounded by the area of the disk.

Step 2: We prove that $c = \mathcal{N}$ for Morse functions on the plane[5]. This is achieved by proving that $\mathcal{N}$ and c satisfy the same recursive relation; see Proposition 3.2.9.

Step 3: We prove that $c = \mathcal{N}$ for Morse functions on closed surfaces. This is done by relating the values of both $\mathcal{N}$ and c to their values on the plane; see Proposition 3.2.10.

Step 4: We complete the proof by perturbing a general Hamiltonian to a carefully chosen nearby Morse Hamiltonian; the non-triviality of this final step stems from the fact that we do not know if $\mathcal{N}(H)$ depends continuously on H.

We will now provide more explanations on Steps 2 and 3.

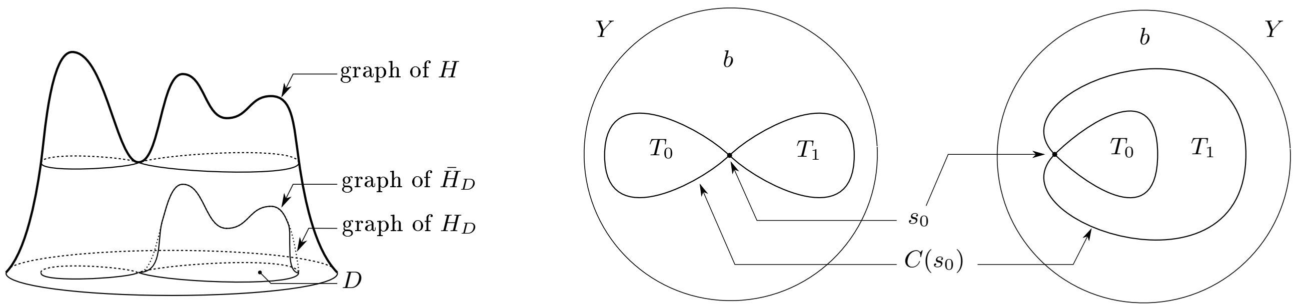
More on Step 2

Step 2 is perhaps the most difficult step in the proof. We first establish a recursive formula for $\mathcal{N}$. To state it let us introduce a few definitions.

Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a Morse function which admits at least one saddle point. For a saddle point s of H, we denote by $C(s)$ the connected component of s in the corresponding level set of H ($C(s)$ is homeomorphic to a bouquet of two circles). Let D be one of the two bounded connected components of $\mathbb{R}^2 \setminus C(s)$. We denote by $\bar{H}_D$ an appropriate smooth approximation of the function

$$H_D = \begin{cases} H - H(s) & \text{on } D \\ 0 & \text{on } \mathbb{R}^2 \setminus D. \end{cases}.$$

We call s_0 the "outer-most" saddle point of H, denote by b, T_0, T_1 the two three components of the complement of $C(s_0)$ in the interior of the support of H. By definition of s_0, b contains no critical point. There are two possible configurations that can be seen on the figure below.



We can now state the recursive formula: *Denote*

$$\mathcal{N}_b = \min\{\mathcal{A}_H(x) \,|\, x \textit{ fixed point of } \phi_H^1 \textit{ in } b\}.$$

If H has no saddle points, then

$$\mathcal{N}(H) = \begin{cases} 0 & \textit{if } H|_b < 0 \\ \mathcal{N}_b & \textit{if } H|_b > 0 \end{cases}. \tag{3.4}$$

If H has at least one saddle, then

$$\mathcal{N}(H) = \begin{cases} \max(0, H(s_0) + \max(\mathcal{N}(\bar{H}_{T_0}), \mathcal{N}(\bar{H}_{T_1}))) & \textit{if } H|_b < 0 \\ \min(\mathcal{N}_b, H(s_0) + \max(\mathcal{N}(\bar{H}_{T_0}), \mathcal{N}(\bar{H}_{T_1}))) & \textit{if } H|_b > 0 \end{cases}. \tag{3.5}$$

Step 2 follows by induction on the number of saddles once is proved the following:

[5]Here we call Morse function on the plane a function whose support is a (smooth) disk, which is Morse and admits finitely many critical points in the interior of its support

Proposition 3.2.9. *As $\mathcal{N}$, any formal action selector satisfies the identities (3.4) and (3.5).*

To get a taste for how things work, we will consider in the next paragraphs the two simplest scenarios; see Figure 3.7. We focus on a non-negative Morse function H on the plane.

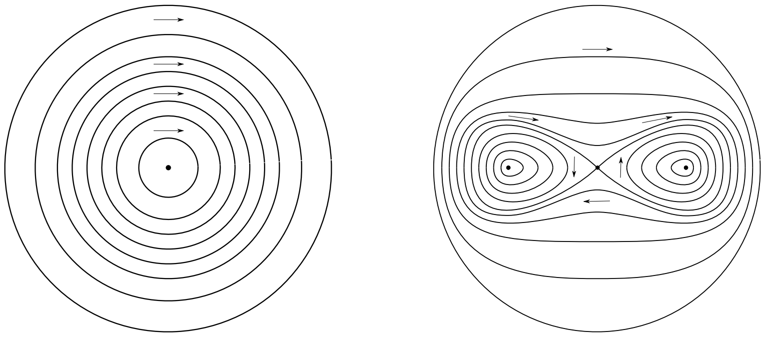


Figure 3.7: "single mountain" and "double mountain."

The first and easiest scenario is that of a function without any saddle point; the graph of such function looks like a "single mountain." Let us call trivial the fixed points lying outside the support of H. Then every two non-trivial fixed points of ϕ_H^1 are linked, and the definition of $\mathcal{N}$ implies that it coincides with the minimum value, say a, of the actions of its non-trivial fixed points. By spectrality, the value of c cannot be less than a (the value 0 can also be excluded using the other properties). On the other hand we can bound H from above by a function G which still has a as the minimal positive action, and whose other action values are larger than the area of its support. By the energy-capacity inequality $c(G)$ must be equal to a, and by monotonicity we get $c(H) \leqslant c(G) = a$, as wanted.

In the second simplest scenario, H has a single saddle point s, and is larger than $H(s)$ on the two disks T_0, T_1 bounded by the level set of s. In this case the graph of H looks like a "double mountain". Again the list of all maximal negative unlinked sets is easy to establish. Mnus's are of two kinds: in addition to the set of trivial fixed points which is contained in every mnus, the first kind consists of a single fixed point of ϕ_H^1 whose orbit surrounds the saddle point, and the second kind consists of the saddle together with one fixed point in each of the two disks T_0, T_1. Denoting by a, a_0, a_1 the minimal positive values of the action respectively outside the saddle level and inside T_0 and T_1, the definition of $\mathcal{N}$ yields

$$\mathcal{N}(H) = \min(a, \max(a_0, a_1)).$$

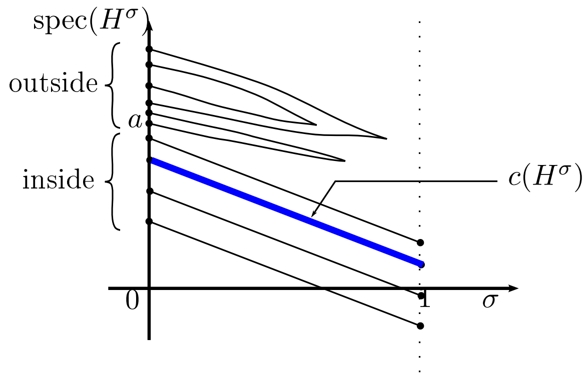
We now try to prove that $c(H)$ satisfies the same formula. Proving the upper bound $c(H) \leqslant a$ goes as in the first scenario. Then, we consider the case when the value of c is attained outside the saddle level set. Here the definition of a gives $c(H) \geqslant a$. Since $c(H) \leqslant a$, we get $c(H) = a$

If $c(H) \neq a$, we have $c(H) < a$ and c is attained in T_0 or T_1. Now let us write, $H = F + H_{T_0} + H_{T_1}$, where F equals the constant value $H(s)$ on $T_0 \cup T_1$, and H_{T_0} and H_{T_1} are supported respectively on T_0 and T_1. In this outline we will pretend that these are smooth functions; note that H_{T_0} and H_{T_1} have no saddle points and hence they are both "single mountains." By a careful analysis of the action values, we construct a deformation H^σ from $H^0 = H$ to $H^1 = H_{T_0} + H_{T_1}$ with the following properties: during the deformation,

- the part of the action spectrum corresponding to orbits in $T_0 \cup T_1$, which we will refer to the "inside" spectrum, decreases at the constant speed $v = H(s)$,

- the remainder of the spectrum, which we will refer to the "outside" spectrum, does not decrease faster than v.

Now the crucial point is that $c(H^0) < a$, whereas the "outside" spectrum for H^0 is no smaller than a. Thus in the bifurcation diagram $\sigma \mapsto \text{spec}(H^\sigma)$, the connected component of $c(H^0)$ is disjoint from the connected components of the "outside" spectrum, and this component is a single line with slope $-H(s)$.



By continuity, we get that $c(H^0) = c(H^1) + H(s)$. Then the max formula and the "single mountain" scenario give

$$c(H^1) = \max(c(H_{T_0}), c(H_{T_1})) = \max(a_0 - H(s), a_1 - H(s)).$$

We conclude that $c(H) = \max(a_0, a_1)$, as wanted.

More on Step 3

To prove the equality $c = \mathcal{N}$ for Morse functions on closed surfaces of positive genus, we first establish a formula which reduces the problem of computing $\mathcal{N}$ to computations for Hamiltonians supported in disks. We then show that this formula is also satisfied by c.

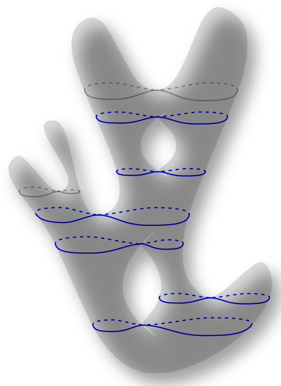


Figure 3.8: In blue, the essential saddles for the "height" function. In green, the inessential ones.

Let Σ denote a closed surface of positive genus and consider a Morse function $H : \Sigma \to \mathbb{R}$. We will use a decomposition of Σ in discs and cylinders based on the notion of "essential saddle". Let s be a saddle point of H. As above we denote by $C(s)$ the connected component of $H^{-1}(H(s))$ which contains s. We will say that the saddle s is *essential* if $C(s)$ is not contractible in Σ. Let Σ' be the open and disconnected surface obtained from Σ by removing $C(s)$ for each essential saddle s of H. Then it can be proved that every connected component S of Σ' is either a disk or an essential cylinder, and if S is a cylinder then ϕ_H^1 has no contractible fixed point in S (See Figure 3.8).

Define $\mathcal{D}$ to be the set of all the disks obtained via the above decomposition of Σ. Note that H is constant on the boundary of each of these disks. For every disk $D \in \mathcal{D}$, we denote as above by $\bar{H}_D$

be an appropriate smoothing of the fonction $H|_D - H(\partial D)$ extended by 0 outside D. It is a Morse function supported in D. It can be checked that $\mathcal{N}(H)$ admits the following formula

$$\mathcal{N}(H) = \max\{H(\partial D) + \mathcal{N}(\bar{H}_D) \mid D \in \mathcal{D}\}. \tag{3.6}$$

Inspired by this formula, we proved that c satisfies exactly the same formula:

Proposition 3.2.10.

$$c(H) = \max\{H(\partial D) + c(\bar{H}_D) \mid D \in \mathcal{D}\}.$$

In other words, $c(H)$ is determined by the values of all the $c(\bar{H}_D)$. By Step 2, $c(\bar{H}_D) = \mathcal{N}(\bar{H}_D)$. Thus, Equation (3.6) and Proposition 3.2.10 imply $c(H) = \mathcal{N}(H)$.

The proof of Proposition 3.2.10 relies crucially on the max formula verified by c.

3.2.4 Further consequences: a simple description of the Entov-Polterovich partial quasi-states

In this section, we assume that Σ is a closed surface of genus at least 1. Take c to be any *formal* action selector on Σ and define

$$\zeta(H) = \lim_{k\to\infty} \frac{1}{k} c(kH), \tag{3.7}$$

for any autonomous function H. The functional ζ was introduced by Entov and Polterovich in [28] and it is referred to as a partial symplectic quasi-state. It is well-known that the quasi-state on $\mathbb{S}^2$ admits a very simple description [28]. Proposition 3.2.10 can be used to give a simple description of ζ on aspherical surfaces.

Theorem 3.2.11. *For any Morse function H on Σ, $\zeta(H)$ is the maximum of H over all of its essential saddles. More generally, for any continuous function $H : \Sigma \to \mathbb{R}$,*

$$\zeta(H) = \inf\left\{a \mid H^{-1}(a, +\infty) \text{ is contractible in } \Sigma\right\}. \tag{3.8}$$

The quantity on the right hand side of Formula (3.8) has already appeared in the literature in a different (but related) context: it was introduced by Polterovich and Siburg in [111] to study the asymptotic behavior of Hofer's metric on open surfaces with infinite area.

A rather surprising consequence of the above result is that the functional ζ which is constructed via symplectic techniques, namely Floer theory, is in fact invariant under the action of all diffeomorphisms, i.e. $\zeta(f \circ \phi) = \zeta(f)$ for any diffeomorphism ϕ. Building on the works of Py [113, 112], Zapolsky and Rosenberg constructed genuine (and not partial) quasi-states on the torus (in [140]) and surfaces of genus higher than one (in [114]). Many other examples of quasi-states on higher genus surfaces were then provided by Zapolsky [139]. Like ζ, these quasi-states can be described by simple formulas which are different than the formula for ζ. The quasi-state on the torus is only invariant under the action of symplectomorphisms while the other ones are invariant under the action of all diffeomorphisms, like ζ.

The above theorem has some interesting corollaries. In [29], Entov and Polterovich introduced the notions of heaviness and super-heaviness. A closed subset $X \subset \Sigma$ is called *heavy* if $\zeta(H) \geqslant \inf(H|_X)$ for every function H. A closed subset X is called *superheavy* if $\zeta(H) \leqslant \sup(H|_X)$ for every function H.[6].

Proposition 3.2.12. *Let $X \subset \Sigma$ be a closed subset. Then,*

1. *X is heavy if and only if X is not included in a disk.*
2. *X is super-heavy if and only if any closed curve included in its complement is contractible in Σ.*

Partial results were also obtained by Kawasaki [73]. Since the product of two super-heavy sets is super-heavy, by Theorem 1.5 of [29], the above result can be used to construct new examples of (strongly) non-displaceable sets.

[6]Although it is not obvious from the definition, every superheavy set is necessarily heavy; see [29].

3.2.5 New direction: an action selector from quasi-positive braids?

In this very speculative section, we briefly report on our attempts (so far unsuccesful) with F. Le Roux and S. Seyfaddini to find a dynamical interpretation of the action selector c on surfaces for all Hamiltonians, not only the autonomous ones. A less optimistic but already challenging goal would be to construct a continuous action selector (not necessarily c) that would be defined in purely dynamical terms. We will limit this discussion to the case of compactly supported Hamiltonians of $\mathbb{R}^2$.

First note that there is no hope to extend the proof of $c = \mathcal{N}$ outlined in the preceeding sections since it relies crucially on the very simple structure of autonomous Hamiltonians.

The work of Le Calvez exposed in the monograph [79], provides a framework based on generating functions where unlinked sets of fixed points naturally appear. Since the Viterbo selector c_+, is defined from such generating functions, this seems to be the perfect framework to relate this selector to unlinked sets as those appearing in the definition of our invariant $\mathcal{N}$. However, a close look at [79] leads to consider other type of sets of fixed points, in particular the following:

Definition 3.2.13. *A set of fixed points X of ϕ_H^1 is weakly-positive if for every $x, y \in X$, we have $\ell(x,y) \geqslant 0$. In other words, every pair of strands of the braid b_{X,ϕ_H^t} is non-negatively linked.*

As in Section 3.2.2, we may consider maximal weakly-positive sets of fixed points, call them "mawpos" and set

$$\mathcal{N}_{\mathrm{mawpos}}(H) = \inf_{X \in \mathrm{mawpos}(\phi_1^H)} \max_{x \in X} \mathcal{A}_H(x).$$

We have checked that $\mathcal{N}_{\mathrm{mawpos}}$ satisfies the same recursive relations (3.4) and (3.5) as $\mathcal{N}$, hence $c(H) = \mathcal{N}_{\mathrm{mawpos}}(H)$ for every autonomous Hamiltonian of $\mathbb{R}^2$.

Another possible type of set of fixed points to consider could be:

Definition 3.2.14. *A set of fixed points X of ϕ_H^1 is quasi-positive if the braid b_{X,ϕ_H^t} is quasi-positive, i.e. is a positive product of conjugates of the canonical generators of the braid group.*

Calling "maqpos" the maximal quasi-positive sets of fixed points, we may define:

$$\mathcal{N}_{\mathrm{maqpos}}(H) = \inf_{X \in \mathrm{mapos}(\phi_1^H)} \max_{x \in X} \mathcal{A}_H(x).$$

Since quasi-positive sets are weakly-positive, we have $\mathcal{N}_{\mathrm{maqpos}} \leqslant \mathcal{N}_{\mathrm{mawpos}}$.

Let us now give a hint as to why quasi-positive braids appear in our problem. Given a Hamiltonian H and a loop x, the action functionnal (2) is given by $\mathcal{A}_H(x) = Q(x) + \int_{\mathbb{S}^1} H_t(x(t)\,dt$, where $Q(x)$ is the quadratic form $Q(x) = \frac{1}{2}\int_{\mathbb{S}^1} \langle -i\dot{x}(t), x(t)\rangle\,dt$. Morally, the action selector c is obtained by applying the minmax principle to a family of cycles homologuous to the negative space of Q (that we will denote by E)[7]. It follows that it is important for our problem to understand the sets of orbits corresponding to families of loops belonging to E.

Inserting the Fourier decomposition into the formula of $Q(x)$, one sees that E is the set of all loops x written in Fourier series $x(t) = \sum_{k \geqslant 0} e^{2\pi i k} x_k$ (i.e. all Fourier coefficient for $k < 0$ vanish). Thus, the elements of E are exactly the restrictions to $\mathbb{S}^1$ of holomorphic maps $\mathbb{D}^2 \to \mathbb{C}$. Now, Rudolph [115] proved that given N loops $x_1, \ldots, x_N$, such that $x_i(t) \neq x_j(t)$ for all $i \neq j$ and all t, the map $t \mapsto (x_1(t), \ldots, x_N(t))$ extends to a holomorphic map $\mathbb{D}^2 \to \mathbb{C}^N$ if and only if the geometric braid traced by the x_i's is quasi-positive.

At this point, we do not know whether the dynamically defined $\mathcal{N}_{\mathrm{mawpos}}$ or $\mathcal{N}_{\mathrm{maqpos}}$ are continuous action selectors. We think that it is possible to construct a certain continuous action selector $\tilde{c}$ (which is unfortunately not dynamically defined) which satisfies the following inequalities: $c \leqslant \tilde{c}$ and $\mathcal{N}_{\mathrm{maqpos}} \leqslant \tilde{c} \leqslant \mathcal{N}_{\mathrm{mawpos}}$. We do not know if any of these inequalities is an equality.

[7] This is how the Ekeland-Hofer selector [25] is defined. The other selectors are variants of it.

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