Polylogarithms, regulators, and Arakelov motivic complexes

A. B. Goncharov

Contents

1 Introduction 1

2 Arakelov motivic complexes 9

3 The Chow polylogarithms 20

4 The Grassmannian polylogarithms 23

5 Grassmannian polylogarithms, symmetric spaces and Borel regulators 30

6 The Chow dilogarithm and a reciprocity law 45

7 Appendix: on volumes of simplices in symmetric spaces 58

1 Introduction

Abstract. We construct an explicit regulator map from Bloch’s Higher Chow group complex to the Deligne complex of a complex algebraic variety $X$. We define the Arakelov motivic complex as the cone of this map shifted by one. Its last cohomology group is (a version of) the Arakelov Chow group defined by H. Gillet and C. Soulé.

We relate the Grassmannian $n$–logarithms (defined as in [G5]) to the geometry of symmetric space $SL_n(\mathbb{C})/SU(n)$. For $n = 2$ we recover Lobachevsky’s formula expressing the volume of an ideal geodesic simplex in the hyperbolic space via the dilogarithm. Using the relationship with symmetric spaces we construct the Borel regulator on $K_{2n-1}(\mathbb{C})$ via the Grassmannian $n$–logarithms.

We study the Chow dilogarithm and prove a reciprocity law which strengthens Suslin’s reciprocity law for Milnor’s group $K_3^M$ on curves.

Our note [G5] can serve as an introduction to this paper.
1. Beilinson’s conjectures on special values of L-functions. Let $X$ be a regular scheme. A.A. Beilinson [B1] defined the rational motivic cohomology of $X$ via Quillen’s algebraic $K$-theory of $X$ by the following formula, where $\gamma$ is the Adams filtration:

$$H^i_M(X, \mathbb{Q}(n)) = \text{Gr}^\gamma_n K_{2n-i}(X) \otimes \mathbb{Q}$$ (1)

Let $X$ be a regular complex algebraic variety. Beilinson defined the regulator $r_B : H^i_M(X, \mathbb{Q}(n)) \to H^i_{DP}(X(\mathbb{C}), \mathbb{R}(n))$ to the weight $n$ Deligne cohomology of $X(\mathbb{C})$.

Let $X$ be a regular projective variety over a number field $F$. Denote by $L(h^i(X), s)$ the $L$-function related to its $i$-dimensional cohomology. Beilinson conjectured that for any integer $n > 1 + i/2$ its special value at $s = n$ is described, up to a nonzero rational factor, by the regulator map

$$r_B : H^{i+1}_M(X, \mathbb{Q}(n)) \to H^{i+1}_{DP}(X \otimes \mathbb{R}, \mathbb{R}(n))$$

to the weight $n$ real Deligne cohomology of $X$. This conjecture is fully established only when $X = \text{Spec}(F)$ where $F$ is a number field. In this case the regulator coincides with the Borel regulator, and the relation with special values of Dedekind zeta-functions was given by the famous Borel theorem [Bo].

Although Beilinson’s conjectures are far from being proved, it is interesting to see what kind of information about the special values of $L$-functions they suggest. So we come to the problem of explicit calculation of Beilinson’s regulator. This problem is already very interesting for the Borel regulator.

2. Regulator maps on motivic complexes and Arakelov motivic cohomology. Beilinson [B2] and S. Lichtenbaum [L1] conjectured that the weight $n$ integral motivic cohomology of $X$ should appear as the cohomology of certain complexes $\Gamma(X; n)$, called the weight $n$ motivic complexes:

$$H^i_M(X, \mathbb{Z}(n)) := H^i(\Gamma(X; n))$$

These complexes are well defined as objects of the derived category.

The first motivic complexes satisfying Beilinson’s formula (1) were Bloch’s Higher Chow group complexes $\mathcal{Z}^s(X; n)$ [Bl1]. Later on A.A. Suslin and V.A. Voevodsky defined several important versions of these complexes. For another candidates for motivic complexes, called the polylogarithmic motivic complexes, see [G1-2]. They are very explicit and the smallest among possible candidates, however Beilinson’s formula (1) is far from being established for them.

The real Deligne cohomology are also cohomology of certain complexes. It was suggested in [G5] and [G7] that the regulator map should be explicitly defined on the level of complexes.
Let $X$ be a regular variety over $\mathbb{C}$. In chapter 2 we construct canonical homomorphism of complexes

Bloch’s weight $n$ Higher Chow group complex $\mathcal{Z}^\bullet(X; n)$ of $X$ \quad (2)

the weight $n$ real Deligne complex $\mathcal{C}_D^n(X(\mathbb{C}); n)$ of $X$

This construction is a version of the one given in [G5]. The complex $\mathcal{C}_D^n(X(\mathbb{C}); n)$ is the truncation $\tau_{\leq 2n}$ of the complex proposed by Deligne more then twenty years ago. The weight $n$ Arakelov motivic complex $\Gamma_A^n(X; n)$ is the cone of the map (2), shifted by $-1$:

$$\Gamma_A^n(X; n) := \text{Cone}(\mathcal{Z}^\bullet(X; n) \xrightarrow{(2)} \mathcal{C}_D^n(X(\mathbb{C}); n))[-1]$$ \quad (3)

For a regular variety $X$ over $\mathbb{R}$ the image of map (2) lies in the subcomplex

$$\mathcal{C}_D^n(X/\mathbb{R}; n) := \mathcal{C}_D^n(X(\mathbb{C}); n)^{\mathbb{F}_\infty}$$

where $\mathbb{F}_\infty$ is the De Rham involution provided by the action of complex conjugation. The weight $n$ real Arakelov motivic complex is defined as

$$\Gamma_A^n(X/\mathbb{R}; n) := \text{Cone}(\mathcal{Z}^\bullet(X; n) \xrightarrow{(2)} \mathcal{C}_D^n(X/\mathbb{R}; n))[-1]$$ \quad (4)

For a regular variety $X$ over a number field $F$ we set

$$\Gamma_A^n(X/F; n) := \text{Cone}(\mathcal{Z}^\bullet(X; n) \longrightarrow \mathcal{C}_D^n(X \otimes \mathbb{Q}_F; n))[-1]$$ \quad (5)

The Arakelov motivic cohomology are cohomology of these complexes.

Our construction works equally well for the Suslin-Voevodsky versions of the motivic complexes.

The regulator map on the polylogarithmic motivic complexes was defined in [G7] explicitly via the classical polylogarithms. The Arakelov motivic complexes constructed using regulator maps on different motivic complexes are supposed to lead to the same object of the derived category. However a precise relationship between the construction given in [G7] and the one in chapter 2 is not clear.

Higher Arakelov Chow groups. The last group of the complex $\mathcal{C}_D^n(X(\mathbb{C}); n)$ consists of closed distributions of certain type on $X(\mathbb{C})$. Replacing $\mathcal{C}$ to $\tilde{\mathcal{C}}$ in (3) we define the weight $n$ Higher Arakelov Chow group complex. Its last cohomology group is isomorphic to the Arakelov Chow group $\tilde{CH}_n^A(X(\mathbb{C}))$ as defined by Gillet and Soulé.

It would be very interesting to generalize the Arakelov–Riemann–Roch theorem proved by Gillet and Soulé to the case of Higher Arakelov Chow groups.

3. The Chow $n$–logarithm function. Let us describe the regulator map (2) in the simplest case when $X = \text{Spec}($ is a point.
Let us choose in \( \mathbb{P}^m \) a homogeneous coordinates \((z_0 : ... : z_m)\). The union of coordinate hyperplanes is a simplex \( L \). Let \( \mathbb{A}^m \) be the complement to the hyperplane \( z_1 + ... + z_m = z_0 \) in \( \mathbb{P}^m \). The abelian group \( \mathbb{Z}_m(\text{Spec}(\mathbb{C}); n) \) is freely generated by the codimension \( n \) irreducible algebraic cycles in \( \mathbb{A}^m \) intersecting properly the faces of the simplex \( L \). The intersection with codimension one faces \( L_j \) of \( L \) provide homomorphisms

\[
\partial_j : \mathbb{Z}_m(\text{Spec}(\mathbb{C}); n) \longrightarrow \mathbb{Z}_{m-1}(\text{Spec}(\mathbb{C}); n);
\partial := \sum_{j=0}^{m} (-1)^j \partial_j
\]

The weight \( n \) Higher Chow group complex over \( \text{Spec}(\mathbb{C}) \), written as a homological complex, looks as follows:

\[
... \xrightarrow{\partial} \mathbb{Z}_2(\text{Spec}(\mathbb{C}); n) \xrightarrow{\partial} \mathbb{Z}_1(\text{Spec}(\mathbb{C}); n) \xrightarrow{\partial} \mathbb{Z}_0(\text{Spec}(\mathbb{C}); n)
\]

The Deligne complex of a point reduces to the group \( \mathbb{R}(n) := (2\pi i)^n \mathbb{R} \). The regulator map (2) boils down to a construction of a homomorphism

\[
\mathbb{Z}_{2n-1}(\text{Spec}(\mathbb{C}); n) \xrightarrow{\mathcal{P}_n} \mathbb{R}(n), \text{ such that } \mathcal{P}_n \circ \partial = 0
\]

It is provided by a function \( \mathcal{P}_n \) on the space of codimension \( n \) cycles in \( \mathbb{CP}^{2n-1} \) intersecting properly faces of a simplex \( L \). This function, called the Chow \( n \)-logarithm function, was constructed in [G5]. To recall its construction, observe that a codimension \( n \) cycle in \( \mathbb{CP}^{2n-1} - L \) can be thought of as an \((n - 1)\)-dimensional variety \( X \) (a cycle) equipped with \( 2n - 1 \) rational functions \( f_1, ..., f_{2n-1} \). These functions are obtained by restriction of the coordinate functions \( x_i/x_0 \) to the cycle \( X \). We define a natural \((2n - 2)\)-form \( r_{2n-2}(f_1, ..., f_{2n-1}) \) on \( X(\mathbb{C}) \) and set

\[
\mathcal{P}_n(X; f_1, ..., f_{2n-1}) := (2\pi i)^{1-n} \int_{X(\mathbb{C})} r_{2n-2}(f_1, ..., f_{2n-1}) \tag{6}
\]

4. An example: the Chow dilogarithm. Let \( f_1, f_2, f_3 \) be three arbitrary rational functions on a complex curve \( X \). Set

\[
r_2(f_1, f_2, f_3) := \text{Alt}_3 \left( \frac{1}{6} \log |f_1| d \log |f_2| \wedge d \log |f_3| - \frac{1}{2} \log |f_1| d \arg f_2 \wedge d \arg f_3 \right)
\]

where \( \text{Alt}_3 \) is the alternation of \( f_1, f_2, f_3 \). Consider the space of quadruples \((X; f_1, f_2, f_3)\). It is a union of finite dimensional algebraic varieties. The Chow dilogarithm is a real function on its complex points defined by the formula

\[
\mathcal{P}_2(X; f_1, f_2, f_3) := \frac{1}{2\pi i} \int_{X(\mathbb{C})} r_2(f_1, f_2, f_3)
\]
The integral converges since the form $r_2(f_1, f_2, f_3)$ has only logarithmic singularities. The Chow dilogarithm provides a homomorphism

$$\Lambda^3 \mathbb{C}(X)^* \to \mathbb{R}, \quad f_1 \wedge f_2 \wedge f_3 \mapsto \mathcal{P}_2(X; f_1, f_2, f_3)$$  \hspace{1cm} (7)

Why dilogarithm appears in the name of the function $\mathcal{P}_2$? Recall the classical dilogarithm

$$\text{Li}_2(z) := -\int_0^z \frac{\log(1 - z)}{z} dz$$

It has single valued cousin, the Bloch-Wigner function:

$$L_2(z) := \text{Im} \text{Li}_2(z) + \text{arg}(1 - z) \log |z|$$

The Chow dilogarithm is defined by a two-dimensional integral over $X(\mathbb{C})$, while $L_2(z)$ is given by integral over a path on $\mathbb{CP}^1$. In chapter 6 we show that nevertheless the Chow dilogarithm can be expressed by the function $L_2(z)$. Here is how it works when $X = \mathbb{CP}^1$. For $f \in \mathbb{C}(X)$ let $v_x(f)$ be the order of zero of $f$ at $x \in X(\mathbb{C})$. Choose a point $\infty$ on $\mathbb{P}^1$. Then

$$\mathcal{P}_2(\mathbb{CP}^1; f_1, f_2, f_3) = \sum_{x_i \in \mathbb{P}^1(\mathbb{C})} v_{x_1}(f_1)v_{x_2}(f_2)v_{x_3}(f_3)L_2(r(x_1, x_2, x_3, \infty))$$  \hspace{1cm} (8)

A formula for the Chow dilogarithm on elliptic curves see in chapter 6.

The function $L_2$ satisfies Abel’s five term functional equation:

$$\sum_{i=1}^5 (-1)^i L_2(x_1, \ldots, \hat{x}_i, \ldots, x_5) = 0$$  \hspace{1cm} (9)

The Chow dilogarithm also satisfies functional equations. They appear as a reformulation of the fact that the composition

$$\mathcal{Z}_3(\text{Spec}(\mathbb{C}); 2) \xrightarrow{\partial} \mathcal{Z}_3(\text{Spec}(\mathbb{C}); 2) \xrightarrow{\mathcal{P}_2} \mathbb{R}(2)$$

is zero. Namely, let $Y$ be an algebraic surface with four rational functions $g_1, \ldots, g_4$ on it. This data provides an element of $\mathcal{Z}_4(\text{Spec}(\mathbb{C}); 2)$. To evaluate the composition on this element we do the following. Take the divisor $\operatorname{div}(g_i)$ and restrict the other functions $g_i$ on it. Then applying the Chow dilogarithm to the obtained data and taking the alternating sum over $1 \leq i \leq 4$ we get zero. In the special case when $Y = \mathbb{CP}^2$ and $\operatorname{div} g_i = l_i - l_5$, where $l_1, \ldots, l_5$ are five lines in the plane, this functional equation plus (8) is equivalent to Abel’s equation (9).

5. **The Grassmannian $n$-logarithm and symmetric space $SL_n(\mathbb{C})/SU(n)$.**

Restricting the Chow $n$-logarithm function to the subvariety of $(n - 1)$-planes in $\mathbb{CP}^{2n-1}$ in generic position to the simplex $L$ we get the Grassmannian $n$-logarithm function $\mathcal{L}_n^G$. 

5
There is a natural bijection
\[(n - 1)\text{–planes in } \mathbb{P}^{2n-1}\text{ in generic position with respect to a simplex } L \}
\subset \{(G^*_{m})^{2n-1} \}
\langle \cdots \rangle \{\text{Configurations of } 2n \text{ generic hyperplanes in } \mathbb{P}^{n-1} \}
\]
given by intersecting of an \((n - 1)\text{–plane } h \) with the codimension one faces of \( L \).

Using it we can view \( L^G_n \) as a function on the configurations of \( 2n \) hyperplanes in \( \mathbb{CP}^{n-1} \). Applying the projective duality we can consider it as a function on configurations of \( 2n \) points in \( \mathbb{CP}^{n-1} \).

In fact one can define the Grassmannian \( n \)-logarithm \( L^G_n(x_1, \ldots, x_{2n}) \) as a function on configurations of \( \text{arbitrary} \ 2n \) points in \( \mathbb{CP}^{n-1} \), see chapter 4. It is a measurable function which is real analytic on generic configurations. It satisfies the two functional equations
\[
\sum_{i=0}^{2n} (-1)^i L^G_n(x_0, \ldots, \hat{x}_i, \ldots, x_{2n}) = 0, \quad \sum_{j=0}^{2n} (-1)^j L^G_n(y_j|y_0, \ldots, \hat{y}_j, \ldots, y_{2n}) = 0
\]
(10)

In the second formula \((y_0, \ldots, y_{2n}) \) is a configuration of \( 2n + 1 \) points in \( \mathbb{CP}^{n-1} \) and \((y_j|y_0, \ldots, \hat{y}_j, \ldots, y_{2n}) \) is a configuration of \( 2n \) points in \( \mathbb{CP}^{n-1} \) obtained by projection from \( y_j \).

It follows from (8) that the Grassmannian dilogarithm is given by the Bloch-Wigner function:
\[
L^G_2(z_1, \ldots, z_4) = L_2(r(z_1, \ldots, z_4))
\]
(11)
Abel’s five term equation coincides with (10). (The two functional equations (10) are equivalent when \( n = 2 \).

Lobachevsky discovered that the dilogarithm appears in the computation of volumes of geodesic simplices in the three dimensional hyperbolic space \( \mathcal{H}_3 \). Let \( I(z_1, \ldots, z_4) \) be the ideal geodesic simplex with vertices at the points \( z_1, \ldots, z_4 \) on the absolute of \( \mathcal{H}_3 \). The absolute is naturally identified with \( \mathbb{CP}^1 \).
Lobachevsky’s formula relates its volume to the Bloch-Wigner function:

$$\text{vol}(f(z_1, ..., z_4)) = L_2(r(z_1, ..., z_4))$$

The volume function $$\text{vol}I(z_1, ..., z_4)$$ is invariant under the group $$SL_2(\mathbb{C})$$ of isometries of $$H_3$$. So it depends only on cross ratio of the points $$z_1, ..., z_4$$. It satisfies the five term equation (9). Indeed, $$\sum(-1)^iI(z_1, ..., \hat{z}_i, ..., z_5) = 0$$.

By Bloch’s theorem [Bl2] any measurable function $$f(z)$$ on $$\mathbb{C}$$ satisfying the five term equation is proportional to $$L_2(z)$$. So we get the formula up to a constant.

We generalize this picture as follows. $$\mathbb{C}P^{n-1}$$ is realized as the smallest boundary stratum of the symmetric space $$H_n := SL_n(\mathbb{C})/SU(n)$$. We define a function $$\psi_n(x_1, ..., x_{2n})$$ on configurations of $$2n$$ points of the symmetric space. The function $$\psi_n$$ is defined by an integral over $$\mathbb{C}P^{n-1}$$ similar to (6). We show that it can be naturally extended to a function $$\overline{\psi}_n$$ on configurations of $$2n$$ points in a compactification $$\overline{H}_n$$ of the symmetric space. The Grassmannian $$n$$-logarithm function turns out to be the value of the function $$\overline{\psi}_n$$ on configurations of $$2n$$ points at the smallest boundary strata, which is identified with $$\mathbb{C}P^{n-1}$$.

Now let $$n = 2$$. Then $$SL_2(\mathbb{C})/SU(2)$$ is identified with the hyperbolic space. We prove that $$\psi_2(x_1, x_2, x_3, x_4)$$ is the volume of the geodesic simplex with vertices at the points $$x_1, ..., x_4$$. Restricting to the ideal geodesic simplices and using the relation to the Grassmannian dilogarithm plus (11) we get a new proof of Lobachevsky’s formula.

6. The Grassmannian $$n$$-logarithms and the Borel regulator. For any point $$x \in \mathbb{C}P^{n-1}$$ the function

$$c_{2n-1}^n(g_1, ..., g_{2n}) := L^n_G(g_1, x, ..., g_{2n}, x)$$

(12)

is a measurable $$(2n-1)$$-cocycle of the Lie group $$GL_n(\mathbb{C})$$. Indeed, it is invariant under the diagonal action of $$GL_n(\mathbb{C})$$ and the cocycle condition is just the first functional equation for the function $$L^n_G$$. Different points $$x$$ give canonically cohomologous cocycles. However a priori it is not clear that the corresponding cohomology class is non zero.

Let $$H^{2n-1}_m(GL_n(\mathbb{C}), \mathbb{R})$$ be the space of measurable cohomology of the Lie group $$GL_n(\mathbb{C})$$. It is known that

$$H^*_m(GL_n(\mathbb{C}), \mathbb{R}) = \Lambda^*_\mathbb{R}(b_1, b_3, ..., b_{2n-1})$$

7
where $b_{2k-1} \in H_{m}^{2k-1}(GL_n(\mathbb{C}), \mathbb{R})$ are certain canonical generators called the Borel classes ([Bo1]).

**Theorem 1.1** The cohomology class of the Grassmannian cocycle (12) is a non zero rational multiple of the Borel class $b_{2n-1}$.

For normalization of the Borel classes and precise relationship between the Grassmannian polylogarithms and the Borel regulator see chapter 5, especially sections 5.4 and 5.5.

The essential role in the proof plays the fact that the Grassmannian $n$-logarithm function $\mathcal{L}^n_{GL_n}$ is a boundary value of the function $\bar{\psi}_n$. The function $\bar{\psi}_n(x_1, ..., x_{2n})$ is not continuous at certain boundary points, but always satisfies the cocycle condition. So taking any point $x \in \mathbb{H}_m$ we get a cocycle

$$c_x(g_1, ..., g_{2n-1}) := \bar{\psi}_n(g_1 x, ..., g_{2n} x)$$

of the group $GL_n(\mathbb{C})$. Its cohomology class does not depend on $x$. If $x \in \mathbb{H}_m$ the corresponding cocycle is smooth. We can differentiate it, getting a cohomology class of the Lie algebra $gl_n$, and relate it to the Borel class. On the other hand taking $x$ to be a point on the boundary stratum $\mathbb{C}^{p_n-1}$ we recover the Grassmannian cocycle (12). So we get the theorem.

Combining it with the technique developed in [G1-2] we get a simple explicit construction of the Borel regulator

$$K_{2n-1}(\mathbb{C}) \rightarrow \mathbb{R}$$

in terms of the Grassmannian $n$-logarithms. The second functional equation for $\mathcal{L}^n_{GL_n}$ plays an important role in the proof. Therefore, thanks to the Borel theorem [Bo2], this allows to express the special values of Dedekind $\zeta$-functions at $s = n$ via the Grassmannian $n$-logarithms.

The definition of the Higher Chow groups of a variety $X$ is much simpler then the definition of algebraic K-groups of $X$. The situation with the regulator maps is similar. However relating the special values of the Dedekind $\zeta$-functions to motivic cohomology of the corresponding number fields we need to work with the algebraic K-theory (or homology of $GL_n(F)$) of number fields.

**Acknowledgement.** I am grateful to Spencer Bloch for discussions of Arakelov motivic complexes. The results of this paper were discussed in my course at Brown (Spring 1998), and at the Newton Institute (Cambridge, March 1998). I am grateful to participants for their interest and useful comments.

During the work on this paper I enjoyed the hospitality and support of the MPI(Bonn) and IHES (Bures sur Yvette). I am very grateful to these institutions. This work was supported by the NSF grants DMS-9800998 and DMS-0099390.
2 Arakelov motivic complexes

1. The Higher Chow groups complex. A (non degenerate) simplex in $\mathbb{P}^m$ is an ordered collection of hyperplanes $L_0, ..., L_m$ in generic position, i.e. with empty intersection. Let us choose in $\mathbb{P}^m$ a simplex $L$ and a generic hyperplane $H$. We might think about this data as of a simplex in the $m$-dimensional affine space $\mathbb{A}^m := \mathbb{P}^m - H$. For any two nondegenerate simplices in $\mathbb{A}^m$ there is a unique affine transformation sending one simplex to the other.

Let $I = (i_1, ..., i_k)$ and $L_I := L_{i_1} \cap ... \cap L_{i_k}$. Let $Z_m(X; n)$ be the free abelian group generated by irreducible codimension $n$ algebraic subvarieties in $X \times \mathbb{A}^m$ which intersect properly (i.e. with the right codimension) all faces $X \times L_I$.

Warning. We use the notation $Z_m(X; n)$ for the group denoted $Z_m(X; n)$ by Bloch. This allows us to use upper and lower indices to distinguish between the homological and cohomological notations, see below.

For a given codimension 1 face $L_i$ of a simplex $L$ in $\mathbb{A}^m$ the other faces $L_j$ cut a simplex $\hat{L}_i := \{L_i \cap L_j\}$, in $L_i$. So the intersection with codimension 1 faces $X \times L_i$ provides group homomorphisms

$$\partial_i : Z_m(X; n) \longrightarrow Z_{m-1}(X; n); \quad \partial := \sum_{i=0}^{m} (-1)^i \partial_i$$

Then $\partial^2 = 0$, so $(Z_\bullet(X; n); \partial)$ is a homological complex. Its homology are Bloch’s Higher Chow groups. By the fundamental theorem of Bloch ([Bl1-2])

$$H_i(Z_\bullet(X; n) \otimes \mathbb{Q}) = gr^i_\mathbb{A} K_i(X) \otimes \mathbb{Q}$$

Let us cook up a cohomological complex setting

$$Z^\bullet(X; n) := Z_{2n-\bullet}(X; n)$$

Its cohomology provide a definition of the integral motivic cohomology of $X$:

$$H^i_{\mathcal{A}}(X, \mathbb{Z}(n)) := H^i(Z^\bullet(X; n))$$

Bloch’s theorem guarantees Beilinson’s formula (1) for the rational motivic cohomology.

2. The Beilinson–Deligne complex. Recall that an $n$-distribution on a manifold $X$ is a continuous linear functional on the space of $(\dim X - n)$–forms with compact support. Denote by $\mathcal{D}^n_X$ the space of all real $n$–distributions on $X$. The De Rham complex of distributions $(\mathcal{D}^\bullet_X, d)$ is a resolution of the constant sheaf $\mathbb{R}$. The space $\mathcal{A}^n_X$ of all smooth $n$–forms on $X$ is a subspace of $\mathcal{D}^n_X$.

Let $X$ be a regular variety over $\mathbb{C}$. The $n$–th Beilinson-Deligne complex
\( \mathbb{R}(n)_D \) can be defined as a total complex associated with the following bicomplex:

\[
\begin{array}{ccccccc}
\mathcal{D}_X^0 & \xrightarrow{d} & \mathcal{D}_X^1 & \xrightarrow{d} & \ldots & \xrightarrow{d} & \mathcal{D}_X^n & \xrightarrow{d} & \mathcal{D}_X^{n+1} & \xrightarrow{d} & \ldots \\
\uparrow \pi_n & & \uparrow \pi_n & & & & & & \uparrow \pi_n \\
\Omega_X^n & \xrightarrow{\partial} & \Omega_X^{n+1} & \xrightarrow{\partial} & \ldots
\end{array}
\] \( \otimes \mathbb{R}(n-1) \)

Here \( \mathbb{R}(n) := (2\pi i)^n \mathbb{R} \) and

\[
\pi_n : \mathcal{D}_X^p \otimes \mathbb{C} \longrightarrow \mathcal{D}_X^p \otimes \mathbb{R}(n-1)
\]

is the projection induced by the one \( \mathbb{C} = \mathbb{R}(n-1) \oplus \mathbb{R}(n) \longrightarrow \mathbb{R}(n-1) \). Further, \( \mathcal{D}_X^n \) placed in degree 1 and \( (\Omega_X^n, \partial) \) is the De Rham complex of holomorphic forms with logarithmic singularities at infinity.

The Beilinson-Deligne complex \( \mathbb{R}(n)_D \) is quasiisomorphic to the complex

\[
\mathbb{R}(n) \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^2 \longrightarrow \ldots \longrightarrow \Omega_X^{n-1}
\]

3. The truncated Deligne complex. Let \( \mathcal{D}^{p,q}_X = \mathcal{D}^{p,q} \) be the abelian group of complex valued distributions of type \((p,q)\) on \( X(\mathbb{C}) \). Consider the following cohomological bicomplex, where \( \mathcal{D}^{n,n}_{cl} \) is the subspace of the space \( \mathcal{D}^{n,n} \) of closed currents, and \( \mathcal{D}^{0,0} \) is in degree 1:

\[
\begin{array}{ccccccc}
\mathcal{D}^{0,n-1} & \xrightarrow{\partial} & \mathcal{D}^{1,n-1} & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & \mathcal{D}^{n-1,n-1} \\
\mathcal{D}^{0,n} & \xrightarrow{\partial} & \mathcal{D}^{1,n} & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & \mathcal{D}^{n-1,n} \\
\mathcal{D}^{0,1} & \xrightarrow{\partial} & \mathcal{D}^{1,1} & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & \mathcal{D}^{n-1,1} \\
\mathcal{D}^{0,0} & \xrightarrow{\partial} & \mathcal{D}^{1,0} & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & \mathcal{D}^{n-1,0}
\end{array}
\]

Let \( \text{Tot}^\bullet \) be the total complex of this bicomplex. It is concentrated in degrees \([1, 2n]\). The complex \( C^\bullet_D(\mathcal{X}(\mathbb{C}); n) = C^\bullet_D(n) \) is a subcomplex of the complex \( \text{Tot}^\bullet \) defined as follows. Take the intersection of the part of the complex \( \text{Tot}^\bullet \) coming from the \( n \times n \) square in the diagram (and concentrated in degrees
[1, 2n − 1]) with the complex of distributions with values in \(\mathbb{R}(n − 1)\). Consider the subgroup \(D_{n, cl}(n) \subset D_{cl}(n)\) of the \(\mathbb{R}(n)\)-valued distributions of type \((n, n)\).

They form a subcomplex in \(Tot^*\) because \(\partial\theta\) sends \(\mathbb{R}(n − 1)\)-valued distributions to \(\mathbb{R}(n)\)-valued distributions. This is the complex \(\mathcal{C}^*\).

**Proposition 2.1** The complex \(\mathcal{C}^*_D(n)\) is quasiisomorphic to the truncated Beilinson-Deligne complex \(\tau_{<2n} \mathbb{R}(n)\).

**Proof.** We need the following general construction. Let \(f^* : X^* \rightarrow Y^*\) be a morphism of complexes such that the map \(f^i\) is injective for \(i \leq p\) and surjective for \(i \geq p\) (and hence is an isomorphism for \(i = p\)). Consider a complex

\[
Z^* := \text{Coker} f^p[−1] \xrightarrow{D} \text{Ker} f^{p+1}
\]

where the differential \(D : \text{Coker} f^p \rightarrow \text{Ker} f^{p+1}[1]\) is defined via the following diagram (the vertical sequences are exact):

\[
\begin{array}{ccccccc}
0 & 0 & & 0 & 0 & & 0 & 0 \\
\downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow \\
X^p & \rightarrow & X^{p-1} & \rightarrow & X^p & \rightarrow & X^{p+1} & \rightarrow & X^{p+2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y^p & \rightarrow & Y^{p-1} & \rightarrow & Y^p & \rightarrow & Y^{p+1} & \rightarrow & Y^{p+2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Coker} f^p & \rightarrow & \text{Coker} f^{p-1} & & & & & & 0 \\
\downarrow & & \downarrow & & & & & & \downarrow \\
0 & & 0 & & & & & & 0
\end{array}
\]

**Lemma 2.2** The complex \(Z^*\) is canonically quasiisomorphic to \(\text{Cone}(X^* f^* \rightarrow Y^*)\).

**Proof.** Let

\[
\tilde{\tau}_{<p} X^* := \ldots \xrightarrow{dx} X^{p-2} \xrightarrow{dx} X^{p-1} \xrightarrow{dx} \text{Im}d_X
\]

\[
\tilde{\tau}_{\geq p} Y^* := \frac{Y^p}{\text{Im}d_Y} \xrightarrow{dy} \frac{Y^{p+1}}{\text{Im}d_Y} \xrightarrow{dy} \frac{Y^{p+2}}{\text{Im}d_Y} \ldots
\]

Then there is an exact sequence of complexes \(0 \rightarrow \tilde{\tau}_{<p} X^* \rightarrow X^* \rightarrow \tilde{\tau}_{\geq p} X^* \rightarrow 0\). The conditions on the maps \(f^*\) imply that

\[
\tilde{\tau}_{<p} f^* : \tilde{\tau}_{<p} X^* \rightarrow \tilde{\tau}_{<p} Y^* \text{ is injective}
\]

\[
\tilde{\tau}_{\geq p} f^* : \tilde{\tau}_{\geq p} Y^* \rightarrow \tilde{\tau}_{\geq p} Y^* \text{ is surjective}
\]
We get maps of complexes

\[ \text{Cone}(\tilde{\tau}_{<p}X^\bullet \rightarrow f^*(\tilde{\tau}_{<p}X^\bullet)) \xrightarrow{\alpha} \text{Cone}(X^\bullet \rightarrow Y^\bullet) \xrightarrow{\beta} \text{Cone}(\tilde{\tau}_{\geq p}Y^\bullet \rightarrow \tilde{\tau}_{\geq p}Y^\bullet) \]

where \( \alpha \) is injective and \( \beta \) is surjective. The complex \( \ker(\beta)/\im(\alpha) \) looks as follows:

\[
\begin{array}{ccccccc}
0 & \rightarrow & 0 & \rightarrow & (f^p)^{-1}\im(d_Y)/\im(d_X) & \rightarrow & \ker f^{p+1} & \rightarrow & \ker f^{p+2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\coker f^{p-2} & \rightarrow & \coker f^{p-1} & \rightarrow & \im(d_Y)/f^p\im(d_X) & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

Since the map \( f^p : \im(d_Y)/\im(d_X) \rightarrow \im(d_Y)/f^p\im(d_X) \) is an isomorphism it is quasiisomorphic to \( Z^* \). The lemma is proved.

Applying the lemma to the morphism of complexes

\[ \text{Tot}(D^{\geq n,\bullet}) \xrightarrow{\pi_n} D^n_{-1,\bullet} \otimes_R R_{(n-1)} \]

we see that the complex \( R(n)_{D} \) is canonically quasiisomorphic to the following complex:

To compute the differential \( D^{n-1,n-1}_{\mathbb{R}}(n-1) \rightarrow D^{n,n}_{\mathbb{R}}(n) \) we proceed as follows. Take \( \alpha \in D^{n-1,n-1}_{\mathbb{R}}(n-1) \), so \( \alpha = (-1)^{n-1}\mathbb{R} \). Then

\[
d\alpha = \partial \alpha + \overline{\partial} \alpha = \partial \alpha + (-1)^{n-1}\overline{\partial} \alpha = 2\pi_n(\partial \alpha)
\]

Applying \( d = \partial + \overline{\partial} \) again and taking the \( (n,n) \)-component we get \( 2\overline{\partial} \partial (\alpha) \). Truncating this complex we obtain the proof of the proposition. (Note that \( d^C := (4\pi i)^{-1}(\partial - \overline{\partial}) \), so \( dd^C = (2\pi i)^{-1}\overline{\partial} \partial \).)

Now if \( X \) is a variety over \( \mathbb{R} \), then we set

\[
C^*_D(X/\mathbb{R};n) := C^*_D(X;\mathbb{R})^\infty; \quad H^i_D(X/\mathbb{R};\mathbb{R}(n)) := H^i\left(C^*_D(X/\mathbb{R};n)\right)
\]
where $F_\infty$ is the De Rham involution, i.e. the composition of the involution $F_\infty$ on $X(\mathbb{C})$ induced by the complex conjugation with the complex conjugation of coefficients.

**Theorem–Construction 2.3** a) There exists canonical homomorphism of complexes

$$\mathcal{P}^\bullet(n) : Z^\bullet(X; n) \rightarrow C^\bullet_\mathcal{F}(X; n)$$

b) If $X$ is defined over $\mathbb{R}$ then the image of the map $\mathcal{P}^\bullet(n)$ lies in the sub-complex $C^\bullet_\mathcal{F}(X/\mathbb{R}; n)$

To construct this homomorphism let us recall a remarkable differential $(m-1)$-form $r_{m-1}(L; H)$ canonically attached to the pair $(\mathbb{A}^m; L) = (\mathbb{P}^m - H, L)$ ([G5]).

4. The form $r_{m-1}(L; H)$. Let $\Omega_L$ be the canonical $m$-form in $\mathbb{P}^m - L$ with logarithmic singularities at $L$. It represents a generator of $H^m_{\text{DR}}(\mathbb{P}^m - L)$ defined over $\mathbb{Z}$. Let us give its coordinate description. Choose a homogeneous coordinates $(z_0 : ... : z_m)$ in $\mathbb{P}^m$ such that $L_i$ is given by equation $\{z_i = 0\}$.

Then

$$\Omega_L = d \log \frac{z_1}{z_0} \wedge ... \wedge d \log \frac{z_m}{z_0}$$

The form $\Omega_L$ has periods in $\mathbb{Z}(m)$. So $\pi_m(\Omega_L)$ is exact. However there is no canonical choice of a primitive $(m-1)$-form for it: the group $(\mathbb{C}^*)^m$ acting on $\mathbb{C}^m - L$ leaves the form invariant and acts non trivially on the primitives. But if we consider a simplex $L$ in the affine complex space $\mathbb{A}^m$ (or, what is the same, choose an additional hyperplane $H$ in $\mathbb{C}^m$, which should be thought of as the infinite hyperplane) then there is a canonical primitive.

To define the primitive the following construction is handy.

**The homomorphism $r_m$.** Let $X$ be a variety over $\mathbb{C}$ and $f_1, ..., f_m$ are $m$ complex-valued functions on $X(\mathbb{C})$. We attach to them the $(m-1)$-form

$$r_{m-1}(f_1, ..., f_m) :=$$

$$\text{Alt}_m \sum_{j \geq 0} c_{j,m} \log |f_1| \text{dlog} |f_2| \wedge ... \wedge \text{dlog} |f_{j+1}| \wedge \text{di arg} f_{j+2} \wedge ... \wedge \text{di arg} f_m$$

Here $c_{j,m} := \frac{1}{(2j+1)!((m-2j-1)!}$ and $\text{Alt}_m$ is the operation of alternation:

$$\text{Alt}_m F(x_1, ..., x_n) := \sum_{\sigma \in S_m} (-1)^{\sigma} F(x_{\sigma(1)}, ..., x_{\sigma(m)})$$

**Remark.** The form (68) is a part of the cocycle representing the product in real Deligne cohomology of 1-cocycles ($\log |f_i|, \text{dlog} f_i$).

So $r_{m-1}(f_1, ..., f_m)$ is an $\mathbb{R}(m-1)$-valued $(m-1)$-form and it is easy to check that

$$dr_{m-1}(f_1, ..., f_m) = \pi_m \left( \text{dlog} f_1 \wedge ... \wedge \text{dlog} f_m \right)$$
Theorem 2.4 The form $r_{m-1}(f_1, \ldots, f_m)$ defines a distribution on $X(\mathbb{C})$, and provides a group homomorphism

$$r_{m-1} : \Lambda^m \mathbb{C}(X)^* \rightarrow \mathcal{D}^{m-1}_{X(\mathbb{C})}(m-1)$$ (14)

Proof. The fact that $r_{m-1}$ is a homomorphism from $\Lambda^m \mathbb{C}(X)^*$ to differential $(m-1)$-forms is clear from the definitions.

Lemma 2.5 For any rational functions $f_1, \ldots, f_m$ on $X$ the $(m-1)$-form $r_{m-1}(f_1 \wedge \ldots \wedge f_m)$ defines a distribution on $X(\mathbb{C})$, i.e. the integral

$$\int_{X(\mathbb{C})} r_{m-1}(f_1, \ldots, f_m) \wedge \omega$$

converges for any smooth $(2\dim X - m + 1)$-form $\omega$ with compact support on $X(\mathbb{C})$.

Basic example. The integral $\int_{\mathbb{C}} \log |z| d\log |z-a| \wedge d\log |z-b|$ is divergent at infinity, where all the functions $z, z-a, z-b$ have a simple pole, since $\int_{\mathbb{C}} \log |z| \frac{dz \wedge d\overline{z}}{|z|^2}$ is divergent (both near zero and infinity). However

$$4 \cdot \int_{\mathbb{C}} \log |z| d\log |z-a| \wedge d\log |z-b| =$$

$$\int_{\mathbb{C}} \log |z| \left( d\log |z-a| \wedge d\log |z-b| + d\log |z-a| \wedge d\log (z-b) \right)$$

is convergent: the divergent parts cancelled each other. In $r_{m-1}(f_1, \ldots, f_m)$ such divergences cancelled because of multiplicativity and skew-symmetry of $r_{m-1}$.

Proof. Resolving singularities of $X$ we may assume that $X$ is nonsingular. We may suppose that the divisors of functions $f_j$ are disjoint because $r_{m-1}$ is a homomorphism. Resolving singularities we reduce lemma to the case when these divisors have normal crossing. Our statement is local, so we can assume that in local coordinates $z_1, \ldots, z_m$ one has $f_1 = z_1, \ldots, f_k = z_k$ and $\text{div} f_j$ for $j > k$ does not intersect the origin. After this the statement of lemma is obvious: each term in (68) defines a distribution near the origin. For instance the worse possible singularities has the term $\log |z_1| d\log |z_2| \wedge \ldots \wedge d\log |z_k| \wedge \omega$ where $\omega$ is smooth near the origin. It is clearly integrable with a smooth test form. The lemma is proved.

Remark. In particular if $\dim \mathbb{C} X = n$ the integral

$$\int_{X(\mathbb{C})} r_{2n}(f_1, \ldots, f_{2n+1})$$ (15)

is convergent.
Choose a coordinate system in $\mathbb{P}^m$ as above such that $H$ is given by by
\[
\{\sum_{i=1}^m z_i = z_0\}.
\]
Set
\[
r_{m-1}(L; H) := r_{m-1}(z_1/z_0, ..., z_m/z_0)
\]
Here is a more invariant definition. Choose one of the faces of the simplex $L$, say $L_0$. Consider simplex $(H, L_1, ..., L_m)$. Let $t_i$ be the rational function on $\mathbb{C}P^n$ such that $(f_i) = (L_i) - (L_0)$ normalized by $f_i(l_i) = 1$, where $l_i$ is the vertex of the simplex $(H, L_1, ..., L_m)$ opposite to the face $L_i$. Then $f_i = \frac{z_i}{z_0}$ and
\[
r_{m-1}(L; H) := r_{m-1}(f_1, ..., f_m)
\]
This form is skewsymmetric with respect to the permutation of the hyperplane faces of the simplex $L$. One has
\[
dr_{m-1}(L; H) = \pi_m(\Omega_L) \quad \text{in} \quad \mathbb{C}P^m - L
\]
So a choice of an “infinite” hyperplane $H \subset \mathbb{C}P^m$ provides the form $r_{m-1}(L; H)$.

**Example.** If $m = 1$ then
\[
A^1 = \mathbb{P}^1 - \{1\}, \quad L = \{0\} \cup \{\infty\}, \quad \Omega_L = d\log z, \quad \pi_1(d\log z) = d\log |z|,
\]
\[
r_0(\{0\} \cup \{\infty\}; \{1\}) = \log |z|
\]

A coordinate free description of the form $r_{m-1}(L; H)$. Choose a volume form $\text{vol}_m \in \text{det}V_\ast^\ast$. Set $\Delta(v_1, ..., v_m) := \langle \text{vol}_m, v_1 \wedge ... \wedge v_m \rangle \in F^\ast$.

**Lemma 2.6** For a configuration $(l_0, ..., l_m)$ of $m + 1$ vectors in generic position
\[
f_m(l_0, ..., l_m) := \sum_{i=0}^m (-1)^i \Lambda_{i \neq j} \Delta(l_0, ..., \hat{l}_j, ..., l_i, ..., l_m) \in \Lambda^m F^\ast
\]
does not depend on the choice of the volume form $\text{vol}_m$.

**Proof.** See proof of lemma 3.1 in [G3].
For a point $z \in A^m - L$ let $l_i(z)$ be the vector from $z$ to the vertex $l_i$, see the picture.
We get a canonical element
\[ f_m(l_0(z), \ldots, l_m(z)) \in \Lambda^m Q(A^m - L)^* \]

If \( F = \mathbb{C} \) applying the homomorphism \( r_m \) to this element we get a canonical \((m - 1)\)-form in \( \mathbb{C}P^m - L \). It coincides with \( r_{m-1}(L; H) \).

**Example.** If \( m = 1 \) and \((L_0, L_1, H) = (0, \infty, 1)\) then \( t = \frac{z}{z-1} \) is an affine coordinate on \( \mathbb{P}^1 \setminus \{1\} \) and
\[
  l_0(t) = \frac{z}{z-1}, \quad l_1(t) = \frac{1}{z-1}, \quad \text{so} \quad f_1(l_0(t), l_1(t)) = \frac{l_0(t)}{l_1(t)} = \frac{z}{z-1}.
\]

**Remark.** The map \( z \in A^m - L \mapsto f_m(l_0(z), \ldots, l_m(z)) \) provides an isomorphism \( F_m : CH^m(\text{Spec}(F), 0) \longrightarrow K^M_m(F) \); see [NS] where the isomorphism \( F_m \) was presented in a bit different way.

For a subvariety \( Y \subset X \) we define the \( \delta \)-distribution \( \delta_Y \) by setting
\[
  \langle \delta_Y, \omega \rangle := \int_Y \omega
\]

If \( f \) is a rational function on \( X \) we set \( \delta(f) := \delta_{\text{div}(f)} \).

Let \( Y \) be a divisor in a complex variety \( X \). Let \( g_1, \ldots, g_{n-1} \) be rational functions on \( Y \). They define an \((n - 2)\)-form \( r_{n-2}(g_1 \wedge \ldots \wedge g_{n-1}) \) on \( Y(\mathbb{C}) \), which, thanks to the lemma 2.5, defines an \( n \)-current \( r_{n-2}(g_1 \wedge \ldots \wedge g_{n-1})\delta_Y \) on \( X(\mathbb{C}) \) denoted \( \varphi(g_1 \wedge \ldots \wedge g_{n-1}) \).

There is the residue homomorphism
\[
  \text{Res} : \Lambda^n C(X)^* \longrightarrow \oplus_{Y \subset X \text{ (1)}} \Lambda^{n-1} C(Y)^*,
\]
given by the formula
\[
  f_1 \wedge \ldots \wedge f_n \mapsto \sum_{i=1}^{n} (-1)^{i-1} v_Y(f_i) \cdot f_{1|Y} \wedge \ldots \wedge \hat{f}_{i|Y} \wedge \ldots \wedge f_{n|Y}
\]

Here is a more rigorous definition. Let \( K \) be a field with a discrete valuation \( v \) and the residue field \( k_v \). The group of units \( U \) has a natural homomorphism
$U \rightarrow k_v^*,$ $\ u \mapsto \overline{u}$. An element $\pi \in K^*$ is prime if $\text{ord}_{\nu} \pi = 1$. There is a homomorphism $\text{res}_v : \Lambda^n K^* \rightarrow \Lambda^{n-1} k_v^*$ uniquely defined by the properties $(u_i \in U)$:

$$\text{res}_v(\pi \land u_1 \land \cdots \land u_{n-1}) = \overline{u}_1 \land \cdots \land \overline{u}_{n-1} \quad \text{and} \quad \text{res}_v(u_1 \land \cdots \land u_n) = 0$$

It does not depend on the choice of $\pi$.

We set $\text{Res} := \sum \text{res}_v$ where the sum is over all valuations of the field $\mathbb{C}(X)$ corresponding to the codimension one points of $X$.

The $(n-1)$-form $r_{n-1}(L; H)$ provides an $(n-1)$-distribution on $\mathbb{C}P^n$. Recall the simplex $\tilde{L}_i$ cut by $L$ in the hyperplane $L_i$, and $H_i = L_i \cap H$. Consider the $(n-2)$-form $r_{n-2}(\tilde{L}_i; H_i)$ on the hyperplane $L_i$ as $n$-distribution in $\mathbb{C}P^n$. We denote it as $r_{n-2}(\tilde{L}_i; H_i) \cdot \delta_{L_i}$.

More generally, a current $\alpha$ on a divisor in $X$ provides a current $\varphi(\alpha)$ on $X$.

**Lemma 2.7** a) Let $f_1, \ldots, f_n$ be rational functions on a variety $X$ over $\mathbb{C}$. Then

$$dr_{n-1}(f_1 \land \ldots \land f_n) = \pi_n \left( d\log f_1 \land \cdots \land d\log f_n \right) + 2\pi i \cdot \varphi \left( r_{n-2}\text{Res}(f_1 \land \ldots \land f_n) \right) \quad (17)$$

b) In particular one has

$$dr_{n-1}(L; H) = \pi_n(\Omega_L) + 2\pi i \cdot \sum_{i=0}^{n} (-1)^i r_{n-2}(\tilde{L}_i; H_i) \delta_{L_i} \quad (18)$$

**Proof.** The part b) follows immediately from a). Using the formula

$$d(d i \arg z) = 2\pi i \delta(z) := 2\pi i (\delta_0 - \delta_{\infty})$$

we get $d(d i \arg f) = 2\pi i \delta(f)$. The part a) follows from this.

**5. The main construction.** We have to construct a morphism of complexes

$$... \rightarrow Z^1(X; n) \rightarrow \cdots \rightarrow Z^{2n-1}(X; n) \rightarrow Z^{2n}(X; n)$$

$$\downarrow \mathcal{P}^1(n) \quad \cdots \quad \downarrow \mathcal{P}^{2n-1}(n) \quad \downarrow \mathcal{P}^{2n}(n)$$

$$0 \rightarrow D_{\mathbb{R}}^{0,0}(n-1) \rightarrow \cdots \rightarrow D_{\mathbb{R}}^{n-1,n-1}(n-1) \xrightarrow{\partial Y} D_{\mathbb{R}}^{n,n}(n)$$

Let $Y \in Z^{2n}(X; n)$ be a codimension $n$ cycle in $X$. By definition

$$\mathcal{P}^{2n}(n)(Y) := (2\pi i)^n \delta_Y$$

Let us construct homomorphisms

$$\mathcal{P}^{2n-i}(n) : Z^{2n-i}(X; n) \rightarrow D_{X(\mathbb{C})}^{2n-i-1}(n-1), \quad i > 0$$
Denote by $\pi_A$ (resp. $\pi_X$) the projection of $X \times A^i$ to $A^i$ (resp. $X$). Let $k = 2\dim Y - i + 1$. Notice that

$$2\dim X - k = 2(\dim X + i - \dim Y) - i - 1 = 2n - i - 1$$

So $(2n - i - 1)$-distributions are functionals on smooth $k$-forms with compact support on $X(\mathbb{C})$. Let $\omega$ be such a $k$-form and $Y \in Z^{2n-i}(X;n)$. Then we set

$$< P^{2n-i}(n)(Y), \omega > := (2\pi i)^{n-i} \int_{Y(\mathbb{C})} \pi_X^* \omega \wedge \pi_A^* \pi_{r_i-1}(L;H)$$  \hspace{1cm} (19)

**Theorem 2.8** The integral (19) is convergent.

**Proof.** Let $Y \subset X \times P^i$ be the closure of the cycle $Y$ and $\pi_X, \pi_A$ the corresponding projections from $Y$ to $X$ and $P^i$. Then the form $\pi_X^* \omega$ has compact support on $Y$, so, we can apply lemma 2.5. The theorem is proved.

**Remark.** We just proved that the product of distributions $\delta_Y \wedge \pi_A^* \pi_{r_i-1}(L;H)$ makes sense and

$$P^{2n-i}(n)(Y) = (2\pi i)^{n-i} \pi_X^* (\delta_Y \wedge \pi_A^* \pi_{r_i-1}(L;H))$$

Since the form $r_{i-1}(L;H)$ is $\mathbb{R}(i-1)$-valued, for $i > 0$ the distribution $P^{2n-i}(n)(Y)$ takes values in $\mathbb{R}(n-1)$. Further, $P^0_{2n}(Y)$ is obviously an $\mathbb{R}(n)$-valued distribution.

Let us show that for $i > 0$ the distribution $P^{2n-i}(n)(Y)$ lies precisely in the left bottom $(n-1) \times (n-1)$ square of the Dolbeault bicomplex. The integral (19) is non zero only if $\pi_X^* \omega \wedge \pi_A^* \pi_{r_i-1}(L;H)$ is of type

$$(\dim Y, \dim Y) = (\dim X + i - n, \dim X + i - n)$$

Since $r_{i-1}(L;H)$ is an $(i-1)$-form we see that the integral vanish if $\omega$ is a form of type $(p,q)$ where $p$ or $q$ bigger then $\dim X + 1 - n$. The statement follows.

Therefore we have constructed the maps $P^i(n)$.

**Theorem 2.9** $P^\bullet(n)$ is a homomorphism of complexes.

**Proof.** One has

$$< dP^{2n-i}(n), \omega >= - < P^{2n-i}(n), d\omega > = - \int_{Y(\mathbb{C})} \pi_X^* d\omega \wedge \pi_A^* \pi_{r_i-1}(L;H) =$$

$$\int_{Y(\mathbb{C})} \pi_X^* \omega \wedge d\pi_A^* \pi_{r_i-1}(L;H)$$  \hspace{1cm} (20)

We need the following lemma

**Lemma 2.10** $\int_{Y(\mathbb{C})} \pi_X^* \omega \wedge \pi_A^* (\Omega_L) = 0$. 

18
Proposition 2.12

The Higher Arakelov Chow groups are defined by Gillet-Soulé as follows:

- **Definition 2.11**
  
  Consider the cone of the homomorphism:
  
  \[
  \text{Cone} \left( \mathcal{Z}^*(X; n) \longrightarrow \tilde{C}^*_D(X(\mathbb{C}); n) \right)[-1]
  \]

**Theorem 2.9**

The Higher Arakelov Chow groups are denoted by \( \hat{C}^n_H(X; 0) = \hat{C}^n_H(X) \).

Proof. Let us look at the very right part of the complex \( \hat{Z}^*(X; n) \):

- \[ \ldots \longrightarrow \mathcal{Z}^{2n-1}(X; n) \longrightarrow \mathcal{Z}^{2n}(X; n) \]

\[
\downarrow \mathcal{D}_n^{(2n-1)} \quad \downarrow \mathcal{D}_n^{(2n)}
\]

\[
(D_n^{2, n-1} \oplus D_n^{1, n-2}) \mathbb{R}(n-1) \xrightarrow{(\partial, \mathcal{J})} D_n^{1, n-1} \mathbb{R}(n-1) \xrightarrow{\mathcal{J}} D_n^{n, n}(n) / A^{n, n}(n)
\]
Consider the very end of the Gersten complex on $X$:

$$
\prod_{Y \in X_{n-2}} \Lambda^2 \mathcal{C}(Y)^* \xrightarrow{\partial} \prod_{Y \in X_{n-1}} \mathcal{C}(Y)^* \longrightarrow Z_0(X; n)
$$

where $\partial$ is the tame symbol. It maps to the complex $\hat{\mathcal{Z}}^\bullet(X; n)$, i.e. to the top row of the bicomplex above, as follows. Recall that $Z_0(X; n) = Z^{2n}(X; n)$, so the very right component of our map is provided by this identification. Further, a pair $(Y; f)$ where $Y$ is an irreducible codimension $n-1$ subvariety of $X$ maps to the cycle $(y, f(y)) \subset X \times \mathbb{A}^1$. Similarly any element in $\Lambda^2 \mathcal{C}(Y)^*$ can be represented as a linear combination of elements $\sum (Y; f_i \wedge g_i)$ where $Y$ is an irreducible codimension $n-2$ subvariety of $X$ and $f_i, g_i$ are rational functions on $Y$ such that $\text{div} f_i$ and $\text{div} g_i$ share no irreducible divisors. Then we send $(Y; f_i \wedge g_i)$ to the cycle $(y, f_i(y), g_i(y)) \subset X \times \mathbb{A}^2$. It is well known that this way we get an isomorphism on the last two cohomology groups. Computing the composition of this map with the homomorphism $\mathcal{P}^\bullet(n)$ we end up precisely with the denominator in (22). The proposition is proved.

### 3 The Chow polylogarithms

Suppose $X = \text{Spec}(\mathbb{C})$. Then $\mathcal{P}_n(Y) := \mathcal{P}_n^{(1)}(Y)$ is a function on the space of all codimension $n$ cycles in $\mathbb{P}^{2n-1}$ intersecting properly faces of the simplex $L$. It is called the Chow polylogarithm function. For $i > 0$ all the distributions $\mathcal{P}_n^{(i)}(Y)$ are zero. However modifying the construction of the previous chapter we get a very interesting object, the Chow polylogarithm, even when $X$ is a point. The Chow polylogarithm function is the first component of the Chow polylogarithm. One can define the Chow polylogarithm for an arbitrary variety $X$, but we spell the details in the most interesting case when $X$ is a point.

1 Chow polylogarithms [G5]. Recall that $L$ is a simplex in $\mathbb{P}^{p+q}$, $H$ is a hyperplane in generic position to $L$, and $H_i := H \cap L_i$.

Let $\mathcal{Z}^q_p(L)$ be the variety of all codimension $q$ effective algebraic cycles in $\mathbb{P}^{p+q}$ which intersect properly, i.e. each irreducible component in the right codimension, all faces of the simplex $L$. It is a union of infinite number of finite dimensional complex algebraic varieties.

**Example.** $\mathbb{P}^{n} - L = (\mathbb{C}^*)^n$ is an irreducible component of $\mathcal{Z}^0_0(L)$ parametrizing the irreducible subvarieties, i.e. points.

The intersection of a cycle with a codimension 1 face $L_i$ of the simplex $L$ provides a map

$$
a_i : \mathcal{Z}^q_p(L) \longrightarrow \mathcal{Z}^q_{p-1}(L_i)
$$

Further, projection with the center at the vertex $l_j$ of $L$ defines a map

$$
b_j : \mathcal{Z}^q_p(L) \longrightarrow \mathcal{Z}^q_{p-1}(L_j)
$$
Theorem–Construction 3.1 For given $q \geq 0$ there is an explicitly constructed chain of $(q - p - 1)$-distributions $\omega^q_p = \omega^q_p(L; H)$ on $Z^q_p(L)$ such that

$$i) \quad d\omega^q_p(L, H) = \pi_q(\Omega_L) \tag{23}$$

$$ii) \quad d\omega^q_p(L; H) = \sum_{i=0}^{p+q} (-1)^i a^*_i \omega^q_{p-1}(L; H_i) \tag{24}$$

$$iii) \quad \sum_{j=0}^{p+q+1} (-1)^j b^*_j \omega^q_p(L; H) = 0 \tag{25}$$

The restriction of $\omega^q_p$ to the subvariety $\bar{Z}^q_p(L)$ of smooth cycles in generic position with respect to the simplex $L$ is a real-analytic differential $(q - p - 1)$-form.

For a given positive integer $q$ the collection $\{\omega^q_p\}$ is called the $q$-th Chow polylogarithm.

The varieties $Z^q_p(L)$ for $p \geq 0$ form a truncated simplicial variety $Z^q_\bullet$. The conditions i) and ii) just mean that the sequence of forms $\omega^q_p$ is a $2q$-cocycle in the bicomplex computing the Deligne cohomology $H^{2q}(Z^q_\bullet, \mathbb{R}D(q))$.

**Proof.** We define $\omega^q_p$ as the Radon transform of the distribution $r^p_{p+q-1}(L; H)$ in $\mathbb{C}P^{p+q}$ over the family of cycles $Y_\xi$ parametrized by $Z^q_p(L)$. This means the following. Consider the incidence variety:

$$\Gamma_p := \{(x, \xi) \in \mathbb{C}P^{p+q} \times Z^q_p(L)(\mathbb{C}) \text{ such that } x \in Y_\xi\}$$

where $Y_\xi$ is the cycle in $\mathbb{C}P^{p+q}$ corresponding to $\xi \in Z^q_p(L)$. We get a double bundle

$$\begin{array}{ccc}
\pi_1 & \nearrow & \pi_2 \\
\mathbb{C}P^{p+q} & \downarrow & Z^q_p(L)(\mathbb{C})
\end{array}$$

Then

$$\omega^q_p := \pi_2 \ast \text{Res}_{r^p_p} \pi_1^*(2\pi i)^{-q} r^p_{p+q}(L; H)$$

Observe that since $r^p_{p+q}(L; H)$ is a distribution on $\mathbb{C}P^{p+q}$, and hence $\pi_1^* r^p_{p+q}(L; H)$ is a distribution on $\mathbb{C}P^{p+q} \times Z^q_p(L)(\mathbb{C})$. The fact that this distribution can be restricted to $\Gamma_p$ is a version of theorem 2.4, and is proved the same way. The push forward $\pi_2\ast$ of this distribution is well defined since $\pi_2$ is a proper map.

The property i) is true by the very definition.

**Lemma 3.2** $\sum_{j=0}^{p+q+1} (-1)^j b^*_j \omega^q_0 = 0$. 

21
Proof. Let \( s(z_0, \ldots, z_n) := z_1/z_0 \wedge \ldots \wedge z_n/z_0 \). Lemma follows from the identity

\[
\sum_{j=0}^{n+1} (-1)^j s(z_0, \ldots, \hat{z_j}, \ldots, z_{n+1}) = 0
\]

So we have iii). To check ii) observe that the push forward \( \pi_2^* \) of distributions commutes with the De Rham differential. The theorem is proved.

2. Properties of the Chow polylogarithm function. The function \( P_q := \omega_q^0 \) on \( Z_q^0 \) is called the Chow \( q \)-logarithm function. It satisfies two functional equations:

\[
\sum_{i=0}^{2q} (-1)^i a_i P_q = 0 \quad \sum_{j=0}^{2q} (-1)^j b_j P_q = 0
\]

Theorem 3.3 Suppose that \( \dim X = n \) and \( f_1, \ldots, f_{2n+1} \) are rational functions on \( X \). Then the integral

\[
(2\pi i)^{1-n} \int_{X(C)} \rho_{2n}(f_1, \ldots, f_{2n+1})
\]

does not change if we multiply one of the functions \( f_i \) by a non zero constant.

Proof. Multiplying, say, \( f_1 \) by \( \lambda \) we see that the difference between the two integrals is

\[
\log |\lambda| \sum_k a_k \int_{X(C)} \text{Alt}_{2n} d\log |f_2| \wedge \ldots \wedge d\log |f_{2k-1}| \wedge d\arg f_{2k} \wedge \ldots \wedge d\arg f_{2n+1}
\]

where \( a_k \) are some rational constants (easy computable from (68)). We will prove that for each \( k \) the corresponding integral in this sum is already zero. Using the identity

\[
(d\log |f_2| + id\arg f_2) \wedge \ldots \wedge (d\log |f_{2n+1}| + id\arg f_{2n+1}) = 0
\]

we can rewrite the integral

\[
\int_{X(C)} d\arg |f_2| \wedge \ldots \wedge d\arg f_{2n+1}
\]

as a sum of similar integrals containing \( d\log |f_i| \). Our statement follows from

Proposition 3.4 Suppose that \( \dim X = n \). Then

\[
d\text{Alt}_{2n} \left( \text{log} |f_2| d\log |f_3| \wedge \ldots \wedge d\log |f_{2k-1}| \wedge d\arg f_{2k} \wedge \ldots \wedge d\arg f_{2n+1} \right) =
\]

\[
\text{Alt}_{2n} \left( d\log |f_2| \wedge \ldots \wedge d\log |f_{2k-1}| \wedge d\arg f_{2k} \wedge \ldots \wedge d\arg f_{2n+1} \right)
\]

in the sense of distributions.
Proof. Since $ddi \arg f = 2\pi i \delta(f)$, the left hand side is equal to the right hand side plus the following terms concentrated on the divisors $f_{2k+j} = 0$:

$$4\pi \cdot 2(n-k+1) \text{Alt}_{2n} \left( \delta(f_{2n+1}) \log |f_2| d \log |f_3| \wedge \ldots \wedge d \log |f_{2k-1}| \wedge d \arg f_{2k} \wedge \ldots \wedge d \arg f_{2n} \right)$$

However all these additional terms vanish thanks to the following proposition.

Proposition 3.5 Suppose that $\dim X = n$. Then for each $0 \leq j \leq n - 1$ one has

$$\text{Alt}_{2n} \left( d \log |f_1| \wedge \ldots \wedge d \log |f_{2j+1}| \wedge d \arg f_{2j+2} \wedge \ldots \wedge d \arg f_{2n} \right) = 0 \quad (28)$$

$$\text{Alt}_{2n} \left( d \log |f_1| \wedge \ldots \wedge d \log |f_{2j}| \wedge d \arg f_{2j+1} \wedge \ldots \wedge d \arg f_{2n} \right) = \frac{(2n)! \binom{n}{j}}{(2j)!} \cdot d \log |f_1| \wedge \ldots \wedge d \log |f_{2n}| \quad (29)$$

Proof. The idea is this. One has $n$ equations

$$d \log f_1 \wedge \ldots \wedge d \log f_{2n} = 0$$
$$d \log |f_1| \wedge d \log f_2 \wedge \ldots \wedge d \log f_{2n} = 0$$

$$\ldots$$

$$d \log |f_1| \wedge \ldots \wedge d \log |f_{n-1}| \wedge d \log f_n \wedge \ldots \wedge d \log f_{2n} = 0 \quad (30)$$

Taking imaginary part of each of them and alternating $f_1$, ..., $f_{2n}$ we get $n$ linear equations. Solving them we get the proposition. See the details in the Appendix in [GZ].

Therefore the Chow polylogarithm function is invariant under the natural action of the torus $(\mathbb{C}^*)^{p+q}$ on $Z_p^q(\mathbb{C})$. In particular it does not depend on the choice of the hyperplane $H$. These statements are no longer true for the forms $\omega_p^q$ for $p < q - 1$.

4 The Grassmannian polylogarithms

1. Configurations of vectors and Grassmannians: a dictionary. Let $X$ be a $G$-set. We define configurations of $m$ points of $X$ as $G$-orbits in $X^m$.

Example 1. If $X := V$ is a vector space and $G := GL(V)$ we get configurations of vectors in $V$. A configuration of vectors $(l_1, \ldots, l_m)$ is in generic position if each $k \leq \dim V$ of the vectors are linearly independent.

Example 2. If $X := \mathbb{P}(V)$ is a projective space and $G := PGL(V)$ we get configurations $(x_1, \ldots, x_m)$ of $m$ points in $\mathbb{P}(V)$. A configuration of points is in generic position if each $k \leq \dim V$ of them generate a plane of dimension $k - 1$.

Let $T_{p+q}$ be the quotient of the torus $\mathbb{G}_m^{p+q+1}$ by the diagonal subgroup $\mathbb{G}_m = (t, \ldots, t)$.\]
Lemma–Construction 4.1  i) There are canonical isomorphisms between the following sets of geometric objects:

(a) Configurations of $p+q+1$ vectors in generic position in $V_q$.
(b) Isomorphism classes of triples (a projective space $\mathbb{P}^{p+q}$ together with a simplex $L$, an “infinite” hyperplane $H$ in generic position to $L$, and a $p$-dimensional plane in generic position to $L$ (but not necessarily to $H$).
(c) Isomorphism classes of triples (a vector space $V_{p+q+1}$, a basis $(e_0, \ldots, e_{p+q})$ of $V_{p+q+1}$, and a $p+1$-dimensional subspace of $V_{p+q+1}$ in generic position with respect to the coordinate hyperplanes).

ii) The torus $T_{p+q}$ acts naturally, and without stable points, on each of the objects $a)$, $b)$, $c)$, and the isomorphisms above are compatible with this action.

Proof. i) (a) $\rightarrow$ (c). Take the kernel of the linear map from $V_{p+q+1}$ to $V_q$ sending $e_i$ to $l_i$.

(c) $\rightarrow$ (a). Take the quotient of $V_{p+q+1}/h$ along the given subspace $h$ and consider the images of the vectors $(e_0, \ldots, e_{p+q})$ there.

(c) $\rightarrow$ (b). Let $\mathbb{P}^{p+q} := P(V_{p+q+1})$. Let $A^{p+q}$ be the affine hyperplane in $V_{p+q+1}$ passing through the ends of the basis vectors $e_i$. Then $A^{p+q} \subset \mathbb{P}^{p+q}$. The coordinate hyperplanes in $V_{p+q+1}$ provide a simplex $L_{p+q} \subset \mathbb{P}^{p+q}$. The projectivization of a generic subspace $h$ in $V_{p+q+1}$ gives a $p$-plane $\tilde{h}$ in generic position with respect to this simplex. (Notice that we do not impose any condition on the mutual location of $H$ and $\tilde{h}$. For instance $\tilde{h}$ may be inside of $H$.)

(b) $\rightarrow$ (c). The triple $(\mathbb{P}^{p+q}, H, L)$ provides a unique up to an isomorphism data $(V_{p+q+1}, (e_0, \ldots, e_{p+q}))$. Namely, the partial data $(\mathbb{P}^{p+q}, H)$ provides us with $(V_{p+q+1}, \tilde{H})$ where $\tilde{H}$ is the subspace of $V_{p+q+1}$ projecting to $H$. Now the vertices $l_i$ of the simplex $L$ provide coordinate lines $\tilde{l}_i$ in $V_{p+q+1}$. Intersecting these coordinate lines with a parallel shift of the subspace $\tilde{H}$ we are getting a point on each of the coordinate lines. By definition the endpoints of the basis vectors $e_i$ are these points. Taking the subspace $\tilde{h}$ in $V_{p+q+1}$ projecting to a given plane $h$ in $\mathbb{P}^{p+q}$ we get the desired correspondence.

ii) The torus $T_{p+q}$ acts on the configurations of vectors in a) as

$$\{(l_1, \ldots, l_{p+q+1}) : (l_1, \ldots, l_{p+q+1}) \mapsto (l_1l_1, \ldots, l_{p+q+1}l_{p+q+1})$$

The torus $T_{p+q}$ is identified with $\mathbb{P}^{p+q} - L$ in b), and so acts naturally on the data in b). The action on the data c) is similar. The lemma is proved.

If we use the description c) for the Grassmannians then $b_j$ is obtained by factorization along the coordinate axe $< e_j >$.

Configurations of hyperplanes and torus quotients of Grassmannians. Let $\hat{G}^p_q$ be the Grassmannian of $p$-planes in $\mathbb{P}^{p+q}$ in generic position with respect to a given simplex $L$.

Taking the $T_{p+q}$-orbits of the objects a) and b) in the lemma we arrive to
Corollary 4.2 There is a bijective correspondence

\[ \hat{G}_p^q / T_{p+q} \not\rightarrow \{ \text{Configurations of } p + q + 1 \text{ generic hyperplanes in } \mathbb{P}^p \} \]

sending a \( p \)-plane \( h \) to the configuration \( \{ h \cap L_0, ..., h \cap L_{p+q} \} \) in \( h \).

2. The Grassmannian and bi-Grassmannian polylogarithms. Let us fix a positive integer \( q \). The operations \( a_i \) and \( b_j \) from section 2.1 transforms planes to planes. So we get the following diagram of varieties called the bi-Grassmannian \( \hat{G}(q) \):

\[
\begin{array}{ccc}
\hat{G}(q) := & \rightarrow & \hat{G}_1^{q+1} \rightarrow \hat{G}_0^{q+1} \\
\downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow \downarrow \\
\vdots & \rightarrow & \hat{G}_2^q \rightarrow \hat{G}_1^q \rightarrow \hat{G}_0^q
\end{array}
\]

Here the horizontal arrows are the arrows \( a_i \) and the vertical ones are \( b_j \).

Remark. The bi-Grassmannian \( \hat{G}(n) \) is not a (semi)bisimplicial scheme. (It is a truncated semihypersimplicial scheme. See s.2.6 in [G4]).

Let \( \psi_{\hat{G}}^q \) be the restriction of the differential form \( \omega_{\hat{G}}^q \) to \( \hat{G}_p^q \). The properties i), ii) from theorem 3.1 are exactly the defining conditions for the single–valued Grassmannian polylogarithm whose existence was conjectured in [HM], [BMS], see also [GGL].

Let us extend these forms by zero to the other rows of the bi-Grassmannian \( \hat{G}(q) \), i.e. set \( \psi_{\hat{G}}^{q+i}(q) = 0 \) if \( i > 0 \). Then the property iii) from theorem 3.1 guarantees that the forms \( \psi_{\hat{G}}^{q+i}(q) \) form a \( 2q \)-cocycle in the bicomplex computing the Deligne cohomology \( H^{2q}(\hat{G}(q)_*, \mathcal{R}(q)_D) \). It is called the bi-Grassmannian \( n \)-logarithm. ([G5]).

A sequence of multivalued analytic forms on Grassmannians satisfying conditions similar to i), ii) was defined in [HaM1], [HaM2]. Another construction of the multivalued analytic Grassmannian polylogarithms was suggested in [G5] in the more general setting of the multivalued Chow polylogarithm.

3. The Grassmannian \( n \)-logarithm function. By theorem 3.3 the Chow polylogarithm function is invariant under the action of the torus \( (\mathbb{C}^*)^{2n-1} \). So restricting it to the open Grassmannian \( \hat{G}_n^{n-1} \subset \hat{Z}_n^{n-1} \) and using the bijection

\[ \{ (n-1) \text{-planes in } \mathbb{P}^{2n-1} \text{ in generic position with respect to a simplex } L \} / (G_n^*)^{2n-1} \]

\[ \not\rightarrow \{ \text{Configurations of } 2n \text{ generic hyperplanes in } \mathbb{P}^{n-1} \} \]

we get a function on the configurations of \( 2n \) hyperplanes in \( \mathbb{C}P^{n-1} \), called the Grassmannian polylogarithm function \( L_n^G \).

25
The Grassmannian polylogarithm function has the following simple description on the language of configurations of hyperplanes. It is intersecting that in this description we can work with any configurations of \(2n\) hyperplanes, assuming nothing about their mutual location.

Let \(h_1, \ldots, h_{2n}\) be arbitrary \(2n\) hyperplanes in \(\mathbb{CP}^{n-1}\). Choose an additional hyperplane \(h_0\). Let \(f_i\) be a rational function on \(\mathbb{CP}^{n-1}\) with divisor \(h_i - h_0\). It is defined up to a scalar factor. Set

\[
L_n^G(h_1, \ldots, h_{2n}) := (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r^{2n-1} \left( \sum_{j=1}^{2n} (-1)^j f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_{2n} \right)
\]

It is skewsymmetric by the definition. Notice that

\[
\sum_{j=1}^{2n} (-1)^j f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_{2n} = \frac{f_1}{f_2} \wedge \cdots \wedge \frac{f_{2n}}{f_{2n-1}}
\]

So we can define \(L_n^G(h_1, \ldots, h_{2n})\) as follows: choose rational functions \(g_1, \ldots, g_{2n-1}\) such that \(\text{div} g_i = h_i - h_{2n}\) and put

\[
L_n^G(h_1, \ldots, h_{2n}) = (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r^{2n-1} (g_1, \ldots, g_{2n-1})
\]

Remark. The function \(L_n^G\) is defined on the set of all configurations of \(2n\) hyperplanes in \(\mathbb{CP}^{n-1}\). However it is not even continuous on this set. It is real analytic on the submanifold of generic configurations. Since we put no restrictions on the hyperplanes \(h_i\) the following theorem is stronger then theorem 3.1 in the case of linear subvarieties.

**Theorem 4.3** The function \(L_n^G\) has the following properties:

a) It does not depend on the choice of hyperplane \(h_0\).

b) For any \(2n+1\) hyperplanes in \(\mathbb{CP}^n\) one has

\[
\sum_{j=1}^{2n+1} (-1)^j L_n^G(h_j \cap h_1, \ldots, h_j \cap h_{2n+1}) = 0 \tag{31}
\]

c) For any \(2n+1\) hyperplanes in \(\mathbb{CP}^{n-1}\) one has

\[
\sum_{j=1}^{2n+1} (-1)^j L_n^G(h_1, \ldots, \hat{h}_j, \ldots, h_{2n+1}) = 0 \tag{32}
\]

**Proof.** a) Choose another hyperplane \(h'_0\). Take a rational function \(f_0\) with divisor \(h'_0 - h_0\). Set \(f'_i = \frac{f_i}{f_0}\). Then

\[
\sum_{j=1}^{2n+1} (-1)^j f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_{2n} = \sum_{j=1}^{2n+1} (-1)^j f'_1 \wedge \cdots \wedge \hat{f}'_j \wedge \cdots \wedge f'_{2n} = 0
\]
Indeed, substituting \( f'_i = \frac{f_i}{f_0} \) in this formula we find that the only possible nontrivial term \( f_0 \wedge f_1 \wedge ... \wedge \hat{f}_i \wedge ... \wedge f_{2n} \) vanish because it is symmetric in \( i,j \).

b) Let \( g_1, ..., g_{2n+1} \) be rational functions on \( \mathbb{C} \mathbb{P}^n \) with \( \text{div} g_i = h_i - h_0 \). Then

\[
\text{dr}_{2n} \left( \sum_{j=1}^{2n+1} (-1)^j g_1 \wedge ... \wedge \hat{g}_j \wedge ... \wedge g_{2n+1} \right) =
\]

\[
2\pi i \delta(f_j) df_j \wedge \text{dr}_{2n-1} \left( \sum_{j \neq 1} (-1)^j g_1 \wedge ... \wedge \hat{g}_i \wedge ... \wedge \hat{g}_j \wedge ... g_{2n+1} \right)
\]

(Notice that \( d \log g_1 \wedge ... \wedge d \log g_j \wedge ... \wedge d \log g_{2n+1} = 0 \) on \( \mathbb{C} \mathbb{P}^n \)). Integrating (33) over \( \mathbb{C} \mathbb{P}^n \) we get b).

c) is obvious: we apply \( r_{2n-1} \) to zero element. Theorem is proved.

4. \( \mathbb{P}^1 - \{0, \infty\} \) as a special stratum in the configuration space of \( 2n \) points in \( \mathbb{P}^n \). A special configuration is a configuration of \( 2n \) points

\[
(l_0, ..., l_{n-1}, m_0, ..., m_{n-1})
\]

in \( \mathbb{P}^{n-1} \) such that \( l_0, ..., l_{n-1} \) are vertices of a simplex in \( \mathbb{P}^{n-1} \) and \( m_i \) is a point on the edge \( l_i l_{i+1} \) of the simplex different from \( l_i \) and \( l_{i+1} \), as on the picture.

**Proposition 4.4** The set of special configurations of \( 2n \) points in \( \mathbb{P}^{n-1} \) is canonically identified with \( \mathbb{P}^1 \setminus \{0, \infty\} \).

**Proof.** We define the generalized cross-ratio

\[
r(l_0, ..., l_{n-1}, m_0, ..., m_{n-1}) \in F^*
\]
as follows. Consider the one-dimensional subspaces \( L_i, M_j \) in the \( n \)-dimensional vector space \( V \) projecting to the points \( l_i, m_j \) in \( \mathbb{P}^{n-1} \) respectively. The subspaces \( L_i, M_j, L_{i+1} \) generate a two dimensional subspace. Its quotient along \( M_i \) can be identified with \( L_i \) as well as with \( L_{i+1} \). So we get a canonical linear map \( \overline{M}_i : L_i \longrightarrow L_{i+1} \). The composition of these maps (the "linear monodromy")

\[
\overline{M}_0 \circ \cdots \circ \overline{M}_{n-1} : L_0 \longrightarrow L_0
\]

is a multiplication by an element of \( F^* \) called the generalized cross-ratio of the special configuration (34).

It is clearly invariant under the cyclic permutation

\[
l_0 \rightarrow l_0 \rightarrow \cdots \rightarrow l_{n-1} \rightarrow l_0; \quad m_0 \rightarrow m_1 \rightarrow \cdots \rightarrow m_{n-1} \rightarrow m_0
\]

Notice that \( r(l_0, \ldots, l_{n-1}, m_0, \ldots, m_{n-1}) = 1 \) if and only if the points \( m_0, \ldots, m_{n-1} \) belong to a hyperplane.

Let \( \tilde{m}_i \) be the point of intersection of the line \( l_i l_{i+1} \) with the hyperplane passing through all the points \( m_j \) except \( m_i \). Let \( r(x_1, \ldots, x_4) \) be the cross ratio of the four points on \( \mathbb{P}^1 \). Then

\[
r(l_0, \ldots, l_{n-1}, m_0, \ldots, m_{n-1}) = r(l_i, l_{i+1}, m_i, \tilde{m}_{i+1})
\]

The special configurations and classical polylogarithms. Consider the configuration of \( 2n \) hyperplanes in \( \mathbb{P}^{n-1} \) given by the following equations in homogeneous coordinates \( z_0 : \ldots : z_{n-1} \)

\[
z_0 = 0, \quad \ldots, \quad z_{n-1} = 0, \quad z_0 = z_1, \quad z_1 + z_2 = z_0, \\
z_2 - z_3 = 0, \quad \ldots, \quad z_{n-2} - z_{n-1} = 0, \quad z_{n-1} = a z_0
\]

It admits the following interpretation. Recall that the classical polylogarithm function \( Li_n(z) \) can be defined by an iterated integral:

\[
Li_n(a) = \int_0^a \frac{dt}{1-t} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t} = \int_{\Delta_a} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_{n-1}}{z_{n-1}}
\]
If \( a \in (0, 1] \) the simplex \( \Delta_a \) is defined by the equations
\[
\Delta_a := \{(z_1, \ldots, z_{n-1}) \in \mathbb{R}^{n-1} | \ 0 \leq 1 - z_1 \leq z_2 \leq z_3 \leq \cdots \leq z_{n-1} \leq a \}
\]
The faces of the simplex \( \Delta_a \) can be defined for arbitrary \( a \). Then the codimension one faces \( \{z_i = 0\} \) of the coordinate simplex and the codimension one faces of the simplex \( \Delta_a \) form the configuration (35).

We can reorder hyperplanes of this configuration as follows:
\[
z_0 = 0, \quad z_1 = 0, \quad z_1 = z_0, \quad z_1 + z_2 = z_0, \quad z_2 = 0,
\]
\[
z_2 = z_3, \quad z_3 = 0, \quad \cdots, \quad z_{n-2} = 0, \quad z_{n-2} = z_{n-1}, \quad z_{n-1} = az_0
\]
Applying the projective duality to this configuration of hyperplanes we get the special configuration of \( 2n \) points in \( \mathbb{P}^{n-1} \) with the generalized cross ratio \( a \).

The correspondence between the configuration (35) and the special configuration of points is illustrated in the case \( n = 3 \) on the picture below.

**Remark.** It is amusing that the special configuration of \( 2n \) points in \( \mathbb{P}^{n-1} \), which is related to the classical \( n \)-logarithm by theorem 4.5 below, is constructed using the geometry of the mixed motive corresponding to \( Li_{n-1}(a) \).

5. Restriction of the Grassmannian \( n \)-logarithm to the special stratum. The function \( Li_n(z) \) has a remarkable single-valued version ([Z1], [BD]):
\[
\mathcal{L}_n(z) := \begin{cases} 
\Re & (n \text{ : odd}) \\
\Im & (n \text{ : even}) 
\end{cases} \left( \sum_{k=0}^{n-1} \beta_k \log^k |z| \cdot Li_{n-k}(z) \right), \quad n \geq 2
\]
It is continuous on \( \mathbb{C} \mathbb{P}^1 \). Here \( \frac{2^x}{e^x-1} = \sum_{k=0}^{\infty} \beta_k x^k \), so \( \beta_k = \frac{2^k B_k}{k!} \) where \( B_k \) are Bernoulli numbers. For example \( \mathcal{L}_2(z) \) is the Bloch - Wigner function.

Let us consider the following modification of the function \( \mathcal{L}_n(z) \) proposed by A. M. Levin in [Le]:
\[
\tilde{\mathcal{L}}_n(x) := \sum_{k \text{ even}; \ 0 \leq k \leq n-2} \frac{2^k(n-2)!(2n-k-3)!}{(2n-3)!(n+1)!(n-k-2)!} L_{n-k}(x) \log^k |x|
\]

29
For example $\tilde{L}_n(x) = L_n(x)$ for $n \leq 3$, but already $\tilde{L}_4(x)$ is different from $L_4(x)$. A direct integration carried out in proposition 4.4.1 of [Le] shows that

$$-(2\pi i)^{n-1}(-1)^{(n-1)(n-2)/2} \tilde{L}_n(x) = \int_{\mathbb{C}^{2n-1}} \log |1 - z_1| \prod_{i=1}^{n-1} d \log |z_i| \wedge \prod_{i=1}^{n-2} d \log |z_i - z_{i+1}| \wedge d \log |z_{n-1} - a|$$

This combined with proposition 5.3 below implies

**Theorem 4.5** The value of the function $L^G_n$ on the special configuration (34) is equal to

$$-(-1)^{n(n-1)/2} 4^{n-1} \binom{2n-2}{n-1} \tilde{L}_n(a)$$

where $a = r(l_0, ..., l_{n-1}, m_0, ..., m_{n-1})$.

Another proof in the case $n = 2$ see in section 6.7 below.

**Conjecture 4.6** The Chow $n$-logarithm function can be expressed by the Grassmannian $n$-logarithm function.

**Remark.** Suppose that an element $\sum_k \{ f^{(k)}_1, ..., f^{(k)}_{2n+1} \} \in K^M_{2n+1}(\mathbb{C}(X))$ has zero residues at all the divisors on an $n$-dimensional variety $X$ over $\mathbb{C}$. Then it defines an element

$$\alpha \in gr^\gamma_2 K_{2n+1}(X) = Ext^{2n+1}_M(Q(0)_X, Q(2n+1)_X)$$

Its direct image to the point is an element

$$\pi_*(\alpha) \in gr^\gamma_n K_{2n+1}(Spec \mathbb{C}) = Ext^1_M(Q(0), Q(n))$$

Applying the regulators we see that the integral $\sum_k \int_{X(\mathbb{C})} r_{2n}(f^{(k)}_1, ..., f^{(k)}_{2n+1})$ coincides with the value of the Borel regulator on $\pi_*(\alpha)$ and so by results of the next chapter is expressible by the Grassmannian $n$-logarithms. Conjecture 4.6 tells us that this should be true for any element in $K^M_{2n+1}(\mathbb{C}(X))$.

## 5 Grassmannian polylogarithms, symmetric spaces and Borel regulators

1. The function $\psi_n$. Let

$$\mathbb{H}_n := \{ \text{positive definite Hermitian forms in } \mathbb{C}^n \} / \mathbb{R}^*_+ = SL_n(\mathbb{C})/SU(n)$$

$$= \{ \text{positive definite Hermitian forms in } \mathbb{C}^n \text{ with determinant } = 1 \}$$
It is a symmetric space of rank \( n - 1 \). For example \( \mathbb{H}_2 = \mathcal{H}_3 \) is the hyperbolic 3-space. Replacing positive definite by non negative definite Hermitian forms we get a compactification \( \mathbb{H}_n \) of the symmetric space \( \mathbb{H}_n \).

Let \( G_x \) be the subgroup of \( SL_n(\mathbb{C}) \) stabilizing the point \( x \in \mathbb{H}_n \). A point \( x \) defines a one dimensional vector space \( M_x := \{ \text{measures on } \mathbb{CP}^{n-1} \text{ invariant under } G_x \} \)

Namely, a point \( x \) corresponds to a hermitian metric in \( V_n \). This metric provides the Fubini-Studi metric on \( \mathbb{CP}^{n-1} = P(V_n) \). Moreover there is the Fubini-Studi Kahler form on \( \mathbb{CP}^{n-1} = P(V_n) \); its imaginary part is a symplectic form. Raising it to \( (n - 1) \)-th power we get the Fubini-Studi volume form. The elements of \( M_x \) are the multiples of the Fubini-Studi volume form.

So \( \mathbb{H}_n \) embeds to the projectivization of the space of all measures in \( \mathbb{CP}^{n-1} \). Taking its closure we get a compactification of \( \mathbb{H}_n \).

Let us choose for any point \( x \in \mathbb{H}_n \) an invariant measure \( \mu_x \in M_x \). Then \( \mu_x/\mu_y \) is a real function on \( \mathbb{CP}^{n-1} \).

Let \( x_0, \ldots, x_{2n-1} \) be points of the symmetric space \( SL_n(\mathbb{C})/SU(n) \). Consider the following function

\[
\psi_n(x_0, \ldots, x_{2n-1}) := \int_{\mathbb{CP}^{n-1}} \log \left| \frac{\mu_{x_1}}{\mu_{x_0}} \right| d \log \left| \frac{\mu_{x_2}}{\mu_{x_0}} \right| \wedge \ldots \wedge d \log \left| \frac{\mu_{x_{2n-1}}}{\mu_{x_0}} \right| \quad (36)
\]

Proposition 5.1 The integral

\[
\int_X \tau_m(\mu_0 : \ldots : \mu_{m+1})
\]

satisfies the following properties:

1) Skew symmetry with respect to the permutations of \( \mu_i \).

2) Homogeneity:

\[
\int_X \tau_m(\lambda_0 \mu_0 : \ldots : \lambda_{m+1} \mu_{m+1}) = \int_X \tau_m(\mu_0 : \ldots : \mu_{m+1})
\]

3) Additivity: for any \( m + 3 \) measures \( \mu_i \) on \( X \) one has

\[
\sum_{i=0}^{m+2} (-1)^i \int_X \tau_m(\mu_0 : \ldots : \mu_i : \ldots : \mu_{m+2}) = 0
\]

31
4) Let $g$ be a diffeomorphism of $X$. Then

$$
\int_X \tau_m(g^*\mu_0 : \ldots : g^*\mu_{m+1}) = \int_X \tau_m(\mu_0 : \ldots : \mu_{m+1})
$$

**Proof.** 1). Follows from $\log f \cdot d\log g + \log g \cdot d\log f = d(\log f \cdot \log g)$.

2) Using 1) we may assume $\lambda_i = 1$ for $i > 0$. Then

$$
\int_X (r^m(\lambda_0\mu_0 : \mu_1 : \ldots : \mu_{m+1}) - r^m(\mu_0 : \mu_1 : \ldots : \mu_{m+1})) =
- \log |\lambda| \cdot \int_X d(\log \frac{\mu_2}{\mu_0} | d\log |\frac{\mu_3}{\mu_0}| \wedge \ldots \wedge d\log |\frac{\mu_{m+2}}{\mu_0}|) = 0
$$

3) Taking into account the skewsymmetry of the integral we have to prove that

$$
\text{Alt}_{(0,\ldots,m+2)} \left\{ \log |\frac{\mu_2}{\mu_1}| d\log |\frac{\mu_3}{\mu_1}| \wedge \ldots \wedge d\log |\frac{\mu_{m+2}}{\mu_1}| \right\} = 0 \quad (38)
$$

Let us write $\frac{\mu_i}{\mu_j} = \frac{\mu_i}{\mu_0} / \frac{\mu_j}{\mu_0}$ and substitute it to (38). Then the terms in (38) where

$$
\log |\frac{\mu_2}{\mu_0}| d\log |\frac{\mu_3}{\mu_0}| \wedge \ldots \wedge d\log |\frac{\mu_{m+2}}{\mu_0}|
$$

will appear looks as follows:

$$
- \log |\frac{\mu_2}{\mu_0}| d\log |\frac{\mu_3}{\mu_0}| \wedge \ldots \wedge d\log |\frac{\mu_{m+2}}{\mu_0}| - \log |\frac{\mu_1}{\mu_2}| d\log |\frac{\mu_3}{\mu_2}| \wedge \ldots \wedge d\log |\frac{\mu_{m+2}}{\mu_2}|
$$

(The first two terms comes from $\text{Alt}_{(1,\ldots,m+2)} \tau_m(\mu_1 : \ldots : \mu_{m+2})$ and the second two from $\text{Alt}_{(0,2,\ldots,m+2)} \tau_m(\mu_0 : \mu_2 : \ldots : \mu_{m+2})$. The expression $\text{Alt}_{m+2} \tau_m(\mu_0 : \ldots : \mu_{m+2})$ provides no such terms if $i > 1$).

4) Clear. The proposition is proved.

Recall the following general construction. Let $X$ be a $G$-set and $F$ a function on $X^n$ satisfying

$$
\sum_{i=1}^n (-1)^i F(x_1, \ldots, \hat{x_i}, \ldots, x_n) = 0
$$

Choose a point $x \in X$. Then there is an $(n-1)$-cocycle of the group $G$:

$$
f_x(g_1, \ldots, g_n) := F(g_1 x, \ldots, g_n x)
$$

**Lemma 5.2** The cohomology class of the cocycle $f_x$ does not depend on $x$. 

32
Proof. The difference \( f_y - f_x \) is the coboundary of the \((n - 2)\)-chain

\[
h_{x,y}(g_1, ..., g_{n-1}) = \sum_{i=1}^{n-1} (-1)^{i-1} f(g_1, g_2, ..., g_{i-1}x, g_{i}y, g_{i+1}y, ..., g_{n-1}y)
\]

Here is the geometric picture leading to this formula. Consider the prism \( \Delta^{(n-1)}_{g_1, ..., g_{n-1} \times \Delta^1_{x,y}} \) given by product of the \((n-1)\)-simplex with vertices \( g_1, ..., g_n \) by the 1-simplex with vertices \((x, y)\). Decomposing its side face \( \Delta^{(n-1)}_{g_1, ..., g_{n-1} \times \Delta^1_{x,y}} \) into a sum of simplices we come to the right hand side of (39). Then the terms of the formula \( f_y - f_x - \delta h_{x,y} \) correspond to the boundary faces of the prism. Cutting the prism on simplices we see that the sum of the terms corresponding to the prism boundary is zero thanks to the cocycle relation. The lemma is proved.

So for any \( x \in \mathbb{H}_n \psi_n(g_0x, ..., g_{2n-1}x) \) is a smooth \((2n - 1)\)-cocycle of \( GL_n(\mathbb{C}) \).

Remark. This cocycle is the restriction to \( GL_n(\mathbb{C}) \) of the Bott cocycle for the group of diffeomorphisms of \( \mathbb{CP}^{n-1} \).

Let \( h_0, ..., h_{2n-1} \) be any hyperplanes in \( \mathbb{CP}^{n-1} \). Recall that the Grassmannian \( n \)-logarithm is defined by

\[
L^G_n(h_0, ..., h_{2n-1}) = (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r_{2n-1}(f_1, ..., f_{2n-1})
\]

where \( f_i \) is a rational function on \( \mathbb{CP}^{n-1} \) with the divisor \((h_i) - (h_0)\).

Proposition 5.3 One has

\[
L^G_n(h_0, ..., h_{2n-1}) = \frac{(-4)^{n-1}((n-1)!)^2}{(2\pi i)^{n-1}(2n - 2)!} \int_{\mathbb{CP}^{n-1}} \log |f_1| \wedge \cdots \wedge \log |f_j|
\]

Proof. See proposition 6.2 in [ZhG].

3. The Grassmannian polylogarithm \( L^G_n \) as the boundary value of the function \( \psi_n \). We start from an explicit formula for the Fubini-Studi form. Let \( \mathbb{P}^{n-1} \) be the variety of all hyperplanes in \( \mathbb{P}^{n-1} \). Consider the incidence divisor

\[
D \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \quad D := \{(h, x) | x \in h\}
\]

where \( h \) is a hyperplane and \( x \) is a point in \( \mathbb{P}^{n-1} \).

Let \((x_0 : ... : x_{n-1})\) be homogeneous coordinates in \( \mathbb{P}^{n-1} \). Let

\[
\sigma_n(x, dx) := \sum_{i=0}^{n-1} (-1)^i x_i dx_0 \wedge ... \wedge \hat{dx}_i \wedge ... \wedge dx_{n-1} = i_E \text{vol}_x
\]

be the Leary form. Here \( \text{vol}_x = dx_0 \wedge ... \wedge dx_{n-1} \) and \( E = \sum x_i \partial x_i \).
There is canonical differential \((n-1,n-1)\)-form \(\omega_D\) on \(\hat{\mathbb{P}}^{n-1} \times \mathbb{P}^{n-1} - D\) with a polar singularity at the divisor \(D\). Namely, let \(x \in V_n\) and \(\xi \in V^*_n\). Then

\[
\omega_D := \frac{1}{(2\pi i)^{n-1}} \frac{\sigma_n(\xi, d\xi) \wedge \sigma_n(x, dx)}{<\xi, x>^n}
\]

It is \(PGL_n\)-invariant. A hermitian metric \(H\) in \(V_n\) provides an isomorphism \(H : V_n \rightarrow V^*_n\), and hence an isomorphism \(\mathbb{C}P^{n-1} \rightarrow \overline{\mathbb{C}P}^{n-1}\). The graph \(\Gamma_H\) of this map does not intersect the incidence divisor \(D\). Thus restricting the form \(\omega_D\) to \(\Gamma_H\) we get a volume form on \(\mathbb{C}P^{n-1}\):

\[
\omega_{FS}(H) := \frac{1}{(2\pi i)^{n-1}} \frac{\sigma_n(z, dz) \wedge \sigma_n(\overline{z}, d\overline{z})}{H(z, \overline{z})^n}
\]

(40)

It is clearly invariant under the group preserving the Hermitian form \(H\). Moreover, it is the Fubini-Studi volume form: a proof can be obtained by using the explicit formula for the Fubini-Studi Kahler form given in \([Ar\) complement 3].

One can realize \(\mathbb{C}P^{n-1}\) as the smallest stratum of the boundary of \(\mathbb{H}_n\). Namely, for a hyperplane \(h \in V_n\) let

\[
F_h := \{\text{nonnegative definite hermitian forms in } V_n = \mathbb{C}^n \text{ with kernel } h\} / \mathbb{R}^*_+
\]

The set of hermitian forms in \(V_n\) with the kernel \(h\) is isomorphic to \(\mathbb{R}^*_+\), so \(F_h\) defines a point on the boundary of \(\mathbb{H}_n\).

For any nonzero nonnegative definite hermitian form \(H\) one can define the corresponding Fubini-Studi form by formula (40). It is a differential form with singularities along the projectivization of kernel of \(H\). In particular if \(h\) is a hyperplane then the degenerate hermitian form \(F_h\) provides the Lebesgue measure on the affine space \(\mathbb{C}P^{n-1} - h\). Indeed, if \(h_0 = \{z_0 = 0\}\) then (40) specializes to

\[
\frac{1}{(2\pi i)^{n-1}} \frac{d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_{n-1} \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_{n-1}}{z_0}
\]

Denote by \(M_h\) the one dimensional real vector space generated by this form. For any hyperplane \(h \in \mathbb{C}P^{n-1}\) let us choose a measure \(\mu_h \in M_h\).

**Proposition 5.4** For any \(2n\) hyperplanes \(h_0, \ldots, h_{2n-1}\) in \(\mathbb{C}P^{n-1}\) the integral

\[
\psi_n(h_0, \ldots, h_{2n-1}) := \int_{\mathbb{C}P^{n-1}} \log |\mu_{h_1}| d\log |\mu_{h_2}| \wedge \ldots \wedge d\log |\mu_{h_{2n-1}}| \]

is convergent and equals to

\[
(-4)^{-n} \cdot (2\pi i)^{n-1}(2n)^{2n-1} \binom{2n-2}{n-1} \cdot L^G_n(h_0, \ldots, h_{2n-1})
\]

34
Proof. Let \( h_1, h_2 \) be hyperplanes in \( \mathbb{CP}^{n-1} \) and \( f \) be a rational function such that \( (f) = (h_1) - (h_2) \). From the explicit description of \( M_h \) given above we immediately see that
\[
\mu_{h_1}/\mu_{h_2} = \lambda \cdot |f|^{2n} \tag{42}
\]
Using this and theorem 2.4 we see that integral (41) is convergent. The second statement follows from proposition 5.3 and (42). The proposition is proved.

More generally, take any \( 2n \) Hermitian forms \( H_0, \ldots, H_{2n-1} \), possibly degenerate. For each of the forms \( H_i \) consider the corresponding measure \( \mu_{H_i} \), a multiple of the Fubini-Studi form related to \( H_i \). Using convergence of the integral (41) we can deduce that the integral
\[
\psi_n(H_0, \ldots, H_{2n-1}) := \int_{\mathbb{CP}^{n-1}} \log \left| \frac{\mu_{H_1}}{\mu_{H_0}} \right| d\log \left| \frac{\mu_{H_2}}{\mu_{H_0}} \right| \ldots \wedge d\log \left| \frac{\mu_{H_{2n-1}}}{\mu_{H_0}} \right| = \tag{43}
\]
\[-n^{2n-1} \cdot \int_{\mathbb{CP}^{n-1}} \log \left| \frac{H_1(z, \overline{z})}{H_0(z, \overline{z})} \right| d\log \left| \frac{H_2(z, \overline{z})}{H_1(z, \overline{z})} \right| \ldots \wedge d\log \left| \frac{H_{2n-1}(z, \overline{z})}{H_0(z, \overline{z})} \right|
\]
is also convergent. This enables us to extend \( \psi_n \) to the function \( \overline{\psi}_n(x_0, \ldots, x_{2n-1}) \) on the configuration space of \( 2n \) points in \( \mathbb{P}_{n-1} \). The function \( \overline{\psi}_n \) is discontinuous. For instance it is discontinuous at the point \( x_1 = \ldots = x_{2n-1} = F_h \) for a given hyperplane \( h \) in \( \mathbb{CP}^{n-1} \). It is however a smooth function on an open part of any given strata. We will keep the notation
\[
\psi_n(h_0, \ldots, h_{2n-1}) = \overline{\psi}_n(F_{h_0}, \ldots, F_{h_{2n-1}})
\]
Applying lemma 5.2 to the case when \( X \) is \( \mathbb{H}_n \) and using only the fact that the function \( \overline{\psi}_n(x_0, \ldots, x_{2n-1}) \) is well defined for any \( 2n \) points in \( \mathbb{H}_n \) and satisfies the cocycle condition for any \( 2n + 1 \) of them we get

**Corollary 5.5** Let \( x \in \mathbb{H}_n \) and \( h \) is a hyperplane in \( \mathbb{CP}^{n-1} \). Then the cohomology classes of the following cocycles coincide:
\[
\psi_n(g_0x, \ldots, g_{2n-1}x) \text{ and } \psi_n(g_0h, \ldots, g_{2n-1}h)
\]

4. A normalization of the Borel class \( b_n \). Choose a hermitian metric in \( V_n \). Let \( e \) be the corresponding point of the symmetric space \( \mathbb{H}_n \); its stabilizer is a subgroup \( SU(n) \). One has
\[
\left( \Lambda^* T^*_e \mathbb{H}_n \right)^{SU(n)} = \Lambda^*(SL_n(\mathbb{C})/SU(n))^{SL_n(\mathbb{C})}
\]
There are well known canonical ring isomorphisms (see [B2] and references there):
\[
\left( \Lambda^* T^*_e \mathbb{H}_n \right)^{SU(n)} \otimes_{\mathbb{Z}} \mathbb{C} = \Lambda^*(sl_n(\mathbb{C}))^{sl_n(\mathbb{C})} =
\]
\[
H^*(sl_n(\mathbb{C}), \mathbb{C}) \cong H_{\text{top}}^*(SU(n), \mathbb{C}) \cong H^*_m(SL_n(\mathbb{C}), \mathbb{C}) \tag{44}
\]

35
where $H^*$ is the Lie algebra cohomology, $H_{\text{top}}^*$ is the topological cohomology. The first isomorphism is obvious: $T_e \mathbb{H}_n \otimes \mathbb{R} \cong su(n)$. The map

$$\alpha_{\text{DR}} : \Lambda_n^0(sl_n) \xrightarrow{\sim} H_{\text{DR}}^*(SL_n, \mathbb{C})$$

sends an $sl_n$-invariant exterior form on $sl_n$ to the right–invariant, and hence biinvariant differential form on $SL_n(\mathbb{C})$. Let us described the map

$$\beta_{\text{DR}} : H_{\text{DR}}^*(SL_n(\mathbb{C}), \mathbb{C}) \rightarrow H_{\text{top}}^*(SL_n(\mathbb{C}), \mathbb{C})$$

Let $C$ be a biinvariant, and hence closed, differential $(2n - 1)$–form on $SL_n(\mathbb{C})$. Let us restrict it first to the Lie algebra, and then to the orthogonal complement $su(n)^\perp$ to the Lie subalgebra $su(n) \subset sl_n(\mathbb{C})$. We identify the $\mathbb{R}$–vector spaces $T_e \mathbb{H}_n$ and $su(n)^\perp$. The obtained exterior form on $T_e \mathbb{H}_n$ is restriction of an invariant differential form, denoted $\omega_C$, on the symmetric space $\mathbb{H}_n$. It is a closed differential form.

For any ordered $2n$ points $x_1, ..., x_{2n}$ in $\mathbb{H}_n$ there is a geodesic simplex $I(x_1, ..., x_{2n})$ in $\mathbb{H}_n$. It is constructed inductively as follows. Let $I(x_1, x_2)$ be the geodesic from $x_1$ to $x_2$. The geodesics from $x_3$ to the points of $I(x_1, x_2)$ form a geodesic triangle $I(x_1, x_2, x_3)$. All the geodesics from $x_4$ to the points of the geodesic triangle $I(x_1, x_2, x_3)$ form a geodesic simplex $I(x_1, x_2, x_3, x_4)$, and so on. When the rank of the symmetric space is more then $1$ (i.e. $n > 2$) the geodesic simplex $I(x_1, ..., x_k)$ depends on the ordering of the vertices $x_1, ..., x_k$.

The differential $(2n - 1)$–form $\omega_C$ on $SL_n(\mathbb{C})/SU(n)$ provides a volume of the geodesic simplex:

$$\text{vol}_C I(x_1, ..., x_{2n}) := \int_{I(x_1, ..., x_{2n})} \omega_C$$

For every $2n + 1$ points $x_1, ..., x_{2n+1}$ the boundary of the simplex $I(x_1, ..., x_{2n+1})$ is the alternated sum of the simplices $I(x_1, ..., \hat{x}_i, ..., x_{2n+1})$. Since the form $\omega_C$ is closed, the Stokes theorem yields

$$\sum_{i=1}^{2n+1} (-1)^i \int_{I(x_1, ..., \hat{x}_i, ..., x_{2n+1})} \omega_C = \int_{I(x_1, ..., x_{2n+1})} d\omega_C = 0 \quad (45)$$

This just means that for a given point $x$ the function $\text{vol}_C I(g_1 x, ..., g_{2n} x)$ is a smooth $(2n - 1)$–cocycle of the Lie group $SL_n(\mathbb{C})$. It was considered by J.Dupont [D]. By lemma 5.2 cocycles corresponding to different points $x$ are canonically cohomologous. The obtained cohomology class is the class $\beta_{\text{DR}}([C])$.

**Remark.** $\text{vol}_C I(x_0, ..., x_{2n-1})$ is independent of the ordering of its vertices. Indeed, consider $2n + 1$ points $(x_0, x_1, x_0, x_2, ..., x_{2n-1})$ and apply relation (45).

The Betti cohomology of $SL_n(\mathbb{C})$. Recall that $SU(n)$ is a retract of $SL_n(\mathbb{C})$. It is well known that

$$H_{\text{top}}^*(SU(n), \mathbb{Z}) = H_{\text{top}}^*(S^3 \times S^5 \times \ldots \times S^{2n-1}, \mathbb{Z}) = \Lambda^*(B_3, B_5, ..., B_{2n-1})$$
The restriction from $SU(n)$ to $SU(m)$ kills the classes $B_{2k-1}$ for $k > m$. If $k \leq m$ it identifies the class $B_{2k-1}$ for $SU(n)$ with the one for $SU(m)$. The class $B_{2n-1}$ for $SU(n)$ is provided by the fundamental class of the sphere $S^{2n-1} \subset \mathbb{C}^n$. Namely, it is the pull back of the fundamental class under the map $SU(n) \rightarrow S^{2n-1}$ provided by a choice of point on $S^{2n-1}$. This sphere has the orientation induced by the one of $\mathbb{C}^n$. Thus

$$Z \cdot B_n = \text{Ker} \left( H^{2n-1}_{\text{top}}(SU(n), \mathbb{Z}) \rightarrow H^{2n-1}_{\text{top}}(SU(n-1), \mathbb{Z}) \right)$$ (46)

The transgression in the Leray spectral sequence for the universal $SU(n)$-bundle $EU(n) \rightarrow BU(n)$ provides an isomorphism

$$Z \cdot B_n \rightarrow H^{2n}(BSU(n), \mathbb{Z}) \oplus_{0 < i < 2n} H^i \cdot H^{2n-i}$$

and identifies $B_n$ with the Chern class $c_n \in H^{2n}(BSU(n), \mathbb{Z})$ of the associated vector bundle.

The De Rham cohomology of $SL_n(\mathbb{C})$. Consider the differential form

$$D_n := \text{tr}(g^{-1}dg)^{2n-1} \in \Omega^{2n-1}(SL)$$ (47)

Its restriction to the subgroup $SL_m$ is zero for $m < n$. It follows that the cohomology class

$$[D_n] \in H^{2n-1}_{\text{DR}}(SL_n, \mathbb{C})$$

is a multiple of $B_n$. The Hodge considerations shows that $[D_n] \in (2\pi i)^n \mathbb{Q} \cdot B_n$.

**Lemma 5.6** The differential form $\omega_{D_n}$ is an $\mathbb{R}(n-1)$-valued form. In particular it provides a cohomology class

$$b_n := \beta_{\text{DR}}(D_n) \in H^{2n-1}_{m}(SL_n(\mathbb{C}), \mathbb{R}(n-1))$$

**Proof.** An easy calculation shows that the value of the exterior form $\omega_{D_n}|_{T_{L_n}}$ on

$$(e_{1,n} + e_{n,1}) \wedge i(e_{1,n} - e_{n,1}) \wedge \ldots \wedge (e_{n-1,n} + e_{n,n-1}) \wedge i(e_{n-1,n} - e_{n,n-1}) \wedge e_{n,n}$$

is non zero, and obviously lies $\mathbb{Q}(n-1)$.

On the other hand the values of the form $\omega_{D_n}$ lie in a one dimensional $\mathbb{R}$-vector space. Indeed, the space of $su(n)$-invariant real exterior $(2n-1)$-forms on the space of all hermitian $n \times n$ matrices, which have zero restriction to the subspace of hermitian $(n-1) \times (n-1)$ matrices is one dimensional. The exterior form $\omega_{D_n}|_{T_{L_n}}$ belongs to the complexification of this space. The lemma follows from this.

We call the cohomology class provided by this lemma the Borel class, and use it below to constuct the Borel regulator.
5. Comparison of the Grassmannian and Borel cohomology classes of $GL_n(\mathbb{C})$. Let $[C_G^G]$ be the cohomology class of the $(2n-1)$-cocycle of $GL_n(\mathbb{C})$ provided by the Grassmannian $n$–logarithm. We want to compare it with the Borel class.

Let us consider the following integral

$$\tilde{C}_n(H_1, \ldots, H_{2n-1}) := -n^{2n-1} \int_{\mathbb{P}^{2n-1}} \frac{H_1(z, \overline{z})}{(z, \overline{z})} d^2 H_2(z, \overline{z}) \wedge \ldots \wedge d^2 H_{2n-1}(z, \overline{z}) \quad (48)$$

where $H_i$ are arbitrary complex matrices and $H_i(z, \overline{z})$ as the bilinear form in $z, \overline{z}$ given by the matrix $H_i$. We claim that it is a $(2n-1)$-cocycle of the Lie algebra $gl_n(\mathbb{C})$, and it is obtained by differentiating of the group cocycle provided by the function (43). We put these facts in the following framework.

If we restrict to the case when $h_i$ are hermitian matrices, integral (48) admits the following interpretation. Let us construct a map $\tilde{M}_e: \mathbb{P}^{2n-1} \longrightarrow T^*_e \mathbb{H}_n$ which is a version of the moment map. For a point $z \in \mathbb{P}^{2n-1}$ the value of the functional $<\tilde{M}_e(z), v>$ of the functional $M(z)$ on a vector $v \in T^*_e \mathbb{H}_n$ is defined as follows. Let $e(t)$ be a path in $\mathbb{H}_n$ such that $e(0) = e$ and $\dot{e}(0) = v$. Then

$$<\tilde{M}_e(z), v> := \frac{d}{dt} \log \frac{\mu_{e(t)}(z)}{\mu_e(z)} \bigg|_{t=0}$$

Choose coordinates $z_1, \ldots, z_n$ in $V_n$ such that $(z, \overline{z}) := |z_1|^2 + \ldots + |z_n|^2$ corresponds to the point $e$. Then $T^*_e \mathbb{H}_n$ is identified with the space of hermitian matrices $H$. It follows from (40) that

$$<\tilde{M}_e(z), H> := nH(z, \overline{z}) \quad (49)$$

The map $\tilde{M}_e$ is clearly $SU(n)$-invariant. Its image is an $SU(n)$-orbit in $T^*_e \mathbb{H}_n$ isomorphic to $\mathbb{C}P^{2n-1}$.

We need the following general construction. Let $V$ be a real vector space and $M$ a compact subset of $V$ which is a closure of a $k$-dimensional submanifold. Any element $\omega \in \Lambda^k V$ can be viewed as a $k$-form $\omega$ on $V^*$. Integrating it over $M$ we get an exterior form $\int_M \omega \in \Lambda^k V^*$. If $M$ is a cone over $M'$ with the vertex at the origin then $\int_M \omega = \int_{M'} i_E \omega$ where $E$ is the Euler vector field on $V$.

Applying this construction to the cone over the orbit $M_e$ based at the origin we get a $SU(n)$-invariant element $\tilde{C}_n \in \Lambda^{2n-1} T^*_e \mathbb{H}_n$. It follows from (49) that it is given by formula (48) multiplied by $2n$.

Another invariant $(2n-1)$-cocycle $C_n$ of the Lie algebra $gl_n$, considered by Dynkin [Dy], is given by

$$C_n(X_1, \ldots, X_{2n-1}) = \frac{1}{n!} \text{Alt}_{2n-1} \text{Tr}(X_1 X_2 \ldots X_{2n-1}) \quad (50)$$

38
Let \([C_n]\) be the cohomology class of \(GL_n(\mathbb{C})\) corresponding to the cocycle \(C_n\).

**Theorem 5.7** One has
\[
\tilde{C}_n = (-1)^{\frac{(n-1)n}{2}} \frac{(2\pi i)^{n-1} n^{2n-1} (n-1)!} {(2n-1)!} \cdot C_n
\]
and the class \([C_n^G]\) is a non zero rational multiple of \([C_n]\):
\[
[C_n^G] = (-1)^{\frac{n(n+1)}{2}} \frac{(n-1)^2 (n-1)!} {(2n-2)! (2n-1)!} [C_n]
\]

**Proof.** The second claim follows from the first using proposition 5.4.

Let us prove the first claim. The restriction of the cocycle \(\tilde{C}_n\) to the Lie subalgebra \(\mathfrak{gl}_n-1(\mathbb{C})\) equals to zero. This follows, for instance, from the Amitsur-Levitsky theorem: for any \(n\times n\) matrices \(A_1,...,A_{2n}\) one has \(\text{Alt}_{2n}(A_1,...,A_{2n}) = 0\).

On the other hand the restriction of the cocycle \(\tilde{C}_n\) to the Lie subalgebra of matrices \((a_{ij})\) where \(a_{11} = a_{22} = 0\) is zero. Indeed, in this case the form we integrate in (48) is a differential \((2n-2)\)-form in \(dz_2,...,dz_{n-1},dz_2,...,dz_{n-1}\) and thus it is zero. So thanks to (44) and (46) we conclude that the cocycle \(C_n\) is proportional to \(\tilde{C}_n\). To determine the proportionality coefficient we compute the values of the both cocycle on a special element \(E_n \in \Lambda^{2n-1} \mathfrak{gl}_n\). To write it down denote by \(e_{i,j}\) the elementary \(n \times n\) matrix whose only nonzero entry is 1 on the \((i,j)\) place. Then
\[
E_n := \bigwedge_{j=1}^{n-1} (e_{j,n} \wedge e_{n,j}) \wedge e_{n,n}
\]
A direct computation shows that
\[
\langle C_n, E_n \rangle = 1
\]
Indeed, to get a non zero trace we have to multiply \((n-1)\) blocks \(e_{n,j} e_{j,n}\), as well as \(e_{n,n}\), which can be inserted anywhere between these blocks. So there are \((n-1)!n = n!\) possibilities.

Let us compute the value of the cocycle \(\tilde{C}_n\) on \(E_n\).

**Lemma 5.8** Integral (48) equals to
\[
\frac{-n^{2n-1}}{2n-1)!} \cdot \text{Alt}_{2n-1} \int_{\mathbb{C}^{2n-1}} \frac{H_1(z,\bar{z}) dH_2(z,\bar{z}) \wedge ... \wedge dH_{2n-1}(z,\bar{z})}{(z,\bar{z})^{2n-1}}
\]

**Proof.** By proposition 5.1 integral (48) equals to
\[
\frac{-n^{2n-1}}{2n-1)!} \cdot \text{Alt}_{2n-1} \int_{\mathbb{C}^{2n-1}} \frac{H_1(z,\bar{z}) dH_2(z,\bar{z}) \wedge ... \wedge dH_{2n-1}(z,\bar{z})}{(z,\bar{z})^{2n-1}}
\]
One has
\[
\frac{dH_i(z, \overline{z})}{(z, \overline{z})^2} = \frac{(z, \overline{z})dH_i(z, \overline{z}) - H_i(z, \overline{z})d(z, \overline{z})}{(z, \overline{z})^2}
\]
Substituting
\[
-H_i(z, \overline{z})d(z, \overline{z})
\]
instead of \( \frac{dH_i(z, \overline{z})}{(z, \overline{z})^2} \) in (53)
we get zero since \( H_1 \) and \( H_6 \) appear in a symmetric way and thus disappear after the alternation. The lemma follows.

Let us calculate integral (53) in the special case
\[
H_{2n-1}(z, \overline{z}) = |z_n|^2, \quad H_{2k-1}(z, \overline{z}) = z_k \overline{z}_n, \quad H_{2k}(z, \overline{z}) = z_n \overline{z}_k
\]
so that \( H_1 \wedge \ldots \wedge H_{2n-1} = E_n \). We will restrict the integrand to the affine part \( \{ z_n = 1 \} \) and then perform the integration. Since \( dH_{2n-1}(z, \overline{z}) = 0 \) on \( \{ z_n = 1 \} \) and \( dz_k \wedge d\overline{z}_k = -2i(dx_k \wedge dy_k) \) we get
\[
-(-1)^{(n-1)n/2} (2i)^{n-1} n^{2n-1} \int_{\mathbb{C}^{n-1}} \frac{d^{n-1}x d^{n-1}y}{(1 + |z_1|^2 + \ldots + |z_{n-1}|^2)^{2n-1}} =
\]
\[
-(-1)^{(n-1)n/2} (2i)^{n-1} n^{2n-1} \cdot \text{vol}(S^{2n-3}) \int_0^{\infty} \frac{r^{2n-3} dr}{(1 + r^2)^{2n-1}} =
\]
\[
-(-1)^{(n-1)n/2} (2n-3) n^{2n-1} (n-1)! (2n-1)^{n-1}
\]
since the volume of the sphere \( S^{2n-3} \) is \( \frac{2\pi^{n-1}}{(n-2)!} \) and
\[
\int_0^{\infty} \frac{r^{2n-3} dr}{(1 + r^2)^{2n-1}} = \frac{1}{2} \int_0^{\infty} \frac{r^{n-2} dr}{(1 + r)^{2n-1}} = \frac{1}{2n-2} \left( \frac{2n-2}{n-1} \right)^{-1}
\]
(54)
To get the last equality we integrate by parts:
\[
\int_0^{\infty} \frac{r^{n-2} dr}{(1 + r)^{2n-1}} = \frac{1}{2n-2} \int_0^{\infty} r^{n-2} \left( \frac{1}{(1 + r)^{2n-2}} \right) dr =
\]
\[
\frac{n-2}{2n-2} \int_0^{\infty} \frac{r^{n-3} dr}{(1 + r)^{2n-2}} = \ldots = \frac{(n-2)!}{(2n-2)!} \int_0^{\infty} \frac{dr}{(1 + r)^{n+1}} = \frac{1}{n-1} \left( \frac{2n-2}{n-1} \right)^{-1}
\]
Theorem 5.7 is proved.

6. Construction of the Borel regulator via Grassmannian polylogarithms. Let \( G \) be a group. The diagonal map \( \Delta : G \to G \times G \) provides a homomorphism \( \Delta_* : H_n(G) \to H_n(G \times G) \). Recall that
\[
\text{Prim}H_n G := \{ x \in H_n(G) | \Delta_*(x) = x \otimes 1 + 1 \otimes x \}
\]
Set $A_Q := A \otimes \mathbb{Q}$. One has

$$K_n(F)_Q = \text{Prim}_nGL(F)_\mathbb{Q} = \text{Prim}_nGL_n(F)_\mathbb{Q}$$

where the second isomorphism is provided by Suslin’s stabilization theorem.

The Borel regulator is a map

$$r_{n}^{\text{Bo}} : K_{2n-1}(\mathbb{C})_Q \longrightarrow \mathbb{R}(n - 1)$$

provided by pairing of the class $b_n \in H^{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{R}(n - 1))$ with the subspace $K_{2n-1}(\mathbb{C})_Q \subset H_{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{Q})$.

Recall the Grassmannian complex $C_*(n)$

$$\ldots \longrightarrow C_{2n-1}(n) \stackrel{d}{\longrightarrow} C_{2n-2}(n) \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} C_0(n)$$

where $C_k(n)$ is the free abelian group generated by configurations of $k+1$ vectors $(l_0, \ldots, l_k)$ in generic position in an $n$–dimensional vector space over a field $F$, and $d$ is given by the standard formula (see s. 3.1 in [G2]). The group $C_k(n)$ is in degree $k$. Since it is a homological resolution of the trivial $GL_n(F)$–module $\mathbb{Z}$ (see lemma 3.1 in [G2]), there is canonical homomorphism

$$\varphi_{2n-1}^n : H_{2n-1}(GL_n(F)) \longrightarrow H_{2n-1}(C_*(n))$$

Thanks to lemma 5.6 the Grassmannian $n$–logarithm function provides a homomorphism

$$\mathcal{L}_n^G : C_{2n-1}(n) \longrightarrow \mathbb{R}(n - 1); \quad (l_0, \ldots, l_{2n-1}) \mapsto \mathcal{L}_n^G(l_0, \ldots, l_{2n-1}) \quad (55)$$

Thanks to the first $(2n + 1)$–term functional equation for $\mathcal{L}_n^G$, see (31), it is zero on the subgroup $dC_{2n}(n)$. So it induces a homomorphism

$$\mathcal{L}_n^G : H_{2n-1}(C_*(n)) \longrightarrow \mathbb{R}(n - 1);$$

**Lemma 5.9** The composition $\mathcal{L}_n^G \circ \varphi_{2n-1}^n$ coincides with the class $[C_n^G]$.

**Proof.** Standard, see [G4].

To construct the Borel regulator we extend, as in s. 3.10 of [G2], the class $[C_n^G]$ to a class of $GL_{2n-1}(\mathbb{C})$. Let us recall the key steps.

Let $\mathbb{Z}[S]$ be the free abelian group generated by a set $S$. Let $F$ be a field. Applying the covariant functor $\mathbb{Z} \longrightarrow \mathbb{Z}[X(F)]$ to the bi-Grassmannian $\hat{G}(n)$ (see s. 4.2), and taking the alternating sum of the obtained homomorphisms, we get a bicomplex. Using lemma 4.1 we see that it looks as follows ([G2], s. 3.7):

$$\ldots \longrightarrow C_{2n-1}(2n - 1) \downarrow \quad \ldots \longrightarrow C_{2n-1}(n + 1) \downarrow \quad \ldots \longrightarrow C_{n+1}(n) \downarrow \quad \ldots \longrightarrow C_n(n)$$

41
In particular the bottom row is the stupid truncation of the Grassmannian complex at the group $C_n(n)$. The total complex of this bicomplex is a homological complex, called the weight $n$ bi-Grassmannian complex $BC_*(n)$. In particular there is a homomorphism

$$H_{2n-1}(C_*(n)) \to H_{2n-1}(BC_*(n))$$  \hspace{1cm} (56)

In [G1-2] we proved that there are homomorphisms

$$\varphi_{2n-1}^m : H_{2n-1}(GL_m(F)) \to H_{2n-1}(BC_*(n)), \quad m \geq n$$

whose restriction to the subgroup $GL_n(F)$ coincides with the composition

$$H_{2n-1}(GL_n(F)) \xrightarrow{\varphi_{2n-1}^m} H_{2n-1}(C_*(n)) \xrightarrow{(56)} H_{2n-1}(BC_*(n)),$$

Let us extend homomorphism (55) to a homomorphism

$$\mathcal{L}_n^G : BC_{2n-1}(n) \to \mathbb{R}(n-1)$$

by setting it zero on the groups $C_{2n-1}(n+i)$ for $i > 0$. The second $(2n-1)$–term functional equation for the Grassmannian $n$–logarithm function, see (32), just means that the composition

$$C_{2n}(n+1) \to C_{2n-1}(n) \xrightarrow{\mathcal{L}_n^G} \mathbb{R}(n-1),$$

where the first map is a vertical arrow in $BC_*(n)$, is zero. Therefore we get a homomorphism

$$\mathcal{L}_n^G : H_{2n-1}(BC_*(n)) \to \mathbb{R}(n-1)$$

**Corollary 5.10** One has

$$[C_n^G] = -(-1)^{n(n+1)/2} \frac{(n-1)!^2}{(2n-2)!} \frac{b_n}{n}$$

**Proof.** Indeed,

$$\text{Alt}_{2n-1} \text{Tr}(X_1 \cdots X_{2n-1}) = \langle \text{tr}(g^{-1}dg)^{2n-1}, X_1 \wedge \cdots \wedge X_{2n-1} \rangle$$

So the claim follows from theorem 5.7 since, as it clear from comparison of formulas (50) and (47), $b_n = n![C_n]$. The corollary is proved.

**Theorem 5.11** The composition

$$K_{2n-1}(\mathbb{C}) \to \text{Prim}H_{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{Q}) \xrightarrow{\varphi_{2n-1}^{2n-1}}$$

$$H_{2n-1}(BC_*(n), \mathbb{Q}) \xrightarrow{\mathcal{L}_n^G} \mathbb{R}(n-1)$$

equals to

$$-(-1)^{n(n+1)/2} \frac{(n-1)!^2}{n(2n-2)!} \frac{b_n}{n^2}$$

$$\hspace{1cm} (58)$$

42
Proof. Recall that restriction to $GL_n$ of the map $\varphi^{2n-1}_{2n-1}$ coincides with the map $\varphi^{2n}_{2n-1}$. Therefore lemma 5.9 guarantees that restriction to $GL_n(\mathbb{C})$ of the composition of the last two arrows coincides with the map given by the class $[C^{2n}_{2n-1}]$. So corollary 5.10 implies the theorem.

7. Comparing $[D_n]$ and $B_n$. The following result is not used below.

Theorem 5.12

$[D_n] = (2\pi i)^n(2n-1)B_n$

Proof. The transgression identifies the class $B_n$ with the Chern class of the universal bundle over $BG$, where $G = GL_n(\mathbb{C})$. We will compute explicitly transgression of the $n$-th component of the Chern character of the universal vector bundle $p : E \rightarrow BG$. Let $A$ be a connection on $E$. Then the $n$-th Chern class is represented by the $2n$-form $c_n(A) := \frac{\text{tr} F_n^A}{(2\pi i)^n}$, where $F_n^A := dA + A \wedge A$ is the curvature form.

Let $q : EG \rightarrow BG$ be the principal fibration associated with $E$. Then the form $q^* c_n(A)$ is exact. If $d\omega = q^* c_n(A)$ and $F$ is a fiber of $q$ then $\omega|_F$ is closed, its cohomology class is transgressive, and goes to $[c_n(A)]$. To do the computation we choose a connection $A_0$ on $BG$ which is flat in a neighbourhood $U$ of a point $x \in BG$. It provides a trivialization of the bundle $E$ over $U$ as well as a trivialization $\varphi : EG[U] \rightarrow G \times U$.

The bundle $q^* E$ has canonical trivialization. It provides a connection $A_1$ on $q^* E$. So there are two connections, $q^* A_0$ and $A_1$ on $q^* E$. One has $A_1 = q^* A_0 + g^{-1}dg$, where $(g, u) = \varphi(x)$. Let $A(t) := tA_1 + (1-t)q^* A_0 = tg^{-1}dg + q^* A_0$

It can be thought of as a connection on the lifting of the bundle $q^* E$ to $E G \times [0, 1]$; here $t \in [0, 1]$. The curvature $F(t)$ of this connection is

$F(t) = g^{-1}dg dt + t^2 g^{-1}dg \wedge g^{-1}dg$

The push forward of the form $\text{tr} F(t)^n$ down to $EG$ is a primitive for the form $\text{tr} F^n_{A(1)} - \text{tr} F^n_{A(0)}$. It is given (in $q^{-1}U$) by

$\int_0^1 \text{tr} F(t)^n = \frac{1}{2n-1} \text{tr}(g^{-1}dg)^{2n-1}$

Theorem 5.12 is proved.

8. On the motivic nature of the Grassmannian $n$-logarithm functions. According to the results of the previous section understanding of the
Borel regulator, and hence special values of the Dedekind $\zeta$–functions, is reduced to study of properties of the Grassmannian $n$–logarithm function $L_n^G$.

Recall that a framed mixed Hodge-Tate structure has a natural $\mathbb{R}$–valued invariant, called the Lie period. Thus a variation of Hodge-Tate structures $L$ over a base $X$ provides a period function $L_n^M$ on $X(\mathbb{C})$.

**Conjecture 5.13** There exists a variation $L_n^M$ of framed mixed Tate motives over $\widetilde{G}^{n-1}$ such that:

$$\sum_{i=0}^{2n} (-1)^i a_i^* L_n^M = 0; \quad \sum_{j=0}^{2n} (-1)^j b_j^* L_n^M = 0;$$

and the Lie period $L_n^M$ of its Hodge realization satisfies

$$L_n^M - L_n^G = \sum_{i=0}^{2n-1} (-1)^i a_i^* F_n$$

where $F_n$ is a function on $\widetilde{G}_{n-2}^n(\mathbb{C})$.

b) The functional equations satisfying by $L_n^M$ essentially determine it: the space of all smooth/measurable functions satisfying these functional equations is finite dimensional.

**Remark.** The function $F_n$ is obviously not determined by (59) – add a function coming from $\widetilde{G}^{n-3}_n(\mathbb{C})$. Nevertheless we expect that there exists canonical explicit choice for $F_n$. Then formula (59) can be considered as an explicit formula for $L_n^M$ in terms of $L_n^G$’s.

Moreover we expect that there exists a canonical homotopy between the Grassmannian $n$–logarithm (understood as a cocycle in the Deligne cohomology of the bi–Grassmannian) and its “motivic” bi-Grassmannian counterpart. Observe that the motivic bi–Grassmannian $n$–logarithm should have non trivial components outside of the bottom line of the bi-Grassmannian, while the defined above (or in [G5]) Grassmannian $n$–logarithm is concentrated entirely at the bottom line.

A variation of mixed Tate motives over $\widetilde{G}^{n-1}_n$ was constructed in [HaM]. However it is not clear how to relate it to the function $L_n^G$.

**Conjecture 5.13** is known for $n = 2$ and $n = 3$.

The $n = 2$ case follows from (11), the well known motivic realization of the dilogarithm, and Bloch’s theorem characterizing the Bloch-Wigner function by Abel’s 5–term equation it satisfies.

The $n = 3$ case of conjecture 5.13 follows from the results of [G1-2], [GZ] and the motivic realization of the trilogarithm. In particular the part b) is given by theorem 1.10 in [G1].

**Examples.** 1. $n = 2$. Then $L_2^M = L_2^G$.
2. \( n = 3 \). The motivic Grassmannian trilogarithm function has been constructed in [G1-2] in terms of the classical trilogarithm function. Namely, one has

\[
\mathcal{L}_3^M(l_0, \ldots, l_5) = \frac{1}{90} \text{Alt}_6 \mathcal{L}_3 \left( \frac{\Delta(l_0, l_1, l_4) \Delta(l_1, l_2, l_4) \Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4) \Delta(l_1, l_2, l_5) \Delta(l_2, l_0, l_1)} \right)
\]

According to theorem 1.3 of [GZ] \( \mathcal{L}_3^M \) is different from \( \mathcal{L}_3^G \), and

\[
F_3(l_0, \ldots, l_4) = \frac{1}{9} \text{Alt}_5 \left( \log |\Delta(l_0, l_1, l_2)| \log |\Delta(l_1, l_2, l_3)| \Delta(l_2, l_3, l_4) \right)
\]

3. If \( n > 3 \) then \( \mathcal{L}_n^M \) is different from \( \mathcal{L}_n^G \) since it is already so for the restriction to the special configuration, see theorem 4.5.

The space of the functional equations for the function \( \mathcal{L}_3^G \) is smaller than the one for \( \mathcal{L}_3^M \), see chapter 1.5 of [G3]. A similar situation is expected for all \( n > 3 \).

The space of the functional equations for the motivic \( n \)-logarithm function \( \mathcal{L}_n^M \) should provide an explicit construction of the weight \( n \) part of the motivic Lie coalgebra of an arbitrary field \( F \), as explained in s. 4.1 in [G6], taking into account the following correction.

**Correction.** In s. 4.2 of [G6] the subgroup of the functional equations \( R_n^G \) is supposed to be defined as the subgroup of all functional equations for the function \( \mathcal{L}_n^M \), not \( \mathcal{L}_n^G \).

### 6 The Chow dilogarithm and a reciprocity law

The Chow dilogarithm provides a homomorphism \( \Lambda^3 \mathbb{C}(X)^* \to \mathbb{R} \) given by

\[
f_1 \wedge f_2 \wedge f_3 \mapsto \mathcal{P}_2(X; f_1, f_2, f_3) := \frac{1}{2\pi i} \int_{X(\mathbb{C})} r_2(f_1, f_2, f_3)
\]

In this section we show that the Chow dilogarithm can be expressed by the function \( \mathcal{L}_2(z) \). The precise versions of this claim are discussed below.

**1. The set up ([G1-2]).** For any field \( F \) we defined in [G1] the groups

\[
B_n(F) := \frac{\mathbb{Z}[F^*]}{\mathcal{R}_n(F)}, \quad n \geq 2
\]

and homomorphisms

\[
\delta_n : B_n(F) \longrightarrow B_{n-1}(F) \otimes F^*; \quad \{x\}_n \mapsto \{x\}_{n-1} \otimes x, \quad n \geq 3
\]

\[
\delta_2 : B_2(F) \longrightarrow \Lambda^2 F^*, \quad \{x\}_2 \mapsto (1 - x) \wedge x
\]

There is a complex \( \Gamma(F; n) \)

\[
B_n(F) \xrightarrow{\delta_n} B_{n-1}(F) \otimes F^* \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_n} B_2(F) \otimes \Lambda^2 F^* \xrightarrow{\delta_n} \Lambda^n F^*
\]
where $\delta_n(\{x\}_k \otimes Y) := \{x\}_{k-1} \otimes x \wedge Y$, called the weight $n$ polylogarithmic complex.

If $K$ is a field with a discrete valuation $v$ and the residue field $k_v$, then there is a homomorphism of complexes $\text{res}_v : \Gamma(K, n) \to \Gamma(k_v, n-1)[-1]$ (see §14 of §1 in [G1]). For example for $n = 3$ we have

$$
\begin{align*}
B_3(K) &\to B_2(K) \otimes K^* \quad \delta_3 \\
\downarrow \text{res}_v & \downarrow \text{res}_v \quad \Lambda^3 K^* \\
B_2(k_v) &\to \Lambda^2 k_v^* 
\end{align*}
$$

Here $\text{res}_v(\{x\}_2 \otimes y)$ is zero unless $v(x) = 0$. In the latter case it is $\text{res}_v(\{x\}_2 \otimes y) = v(y)\{\bar{x}\}_2$, where $\bar{x}$ denotes projection of $x$ to the residue field of $K$.

Let $X$ be a regular curve over an algebraically closed field $k$ and $F := k(X)^*$. Set $\text{Res} := \sum_x \text{res}_x$ where $\text{res}_x$ is the residue homomorphism for the valuation on $F$ corresponding to a point $x$ of $X$. For instance for $n = 3$ we get a morphism of complexes

$$
\begin{align*}
B_3(F) &\to B_2(F) \otimes F^* \quad \delta_3 \\
\downarrow \text{Res} & \downarrow \text{Res} \quad \Lambda^3 F^* \\
B_2(k) &\to \Lambda^2 k^* 
\end{align*}
$$

We will also need a more explicit version $B_2(F)$ of the group $B_2(F)$. Denote by $R_2(F)$ the subgroup of $\mathbb{Z}[\mathbb{P}^1(F)]$ generated by the elements

$$
\{0\}, \{\infty\} \quad \text{and} \quad \sum_{i=1}^5 (-1)^i \{r(x_1, \ldots, \bar{x}_i, \ldots, x_5)\}
$$

when $(x_1, \ldots, x_5)$ run through all 5-tuple of distinct points in $\mathbb{P}^1(F)$. Define the Bloch group $B_2(F)$ as

$$
B_2(F) := \frac{\mathbb{Z}[\mathbb{P}^1(F)]}{R_2(F)}
$$

One can show that $R_2(F) \subset \mathcal{R}_2(F)$. There is a map

$$
i : B_2(k)_\mathbb{Q} \to B_2(k)_\mathbb{Q} \quad (62)
$$

**Proposition 6.1** Let $k$ be a number field. Then (62) is an isomorphism modulo torsion.

**Proof.** The map $i$ is clearly surjective. The diagram

$$
\begin{array}{ccc}
B_2(k) & \to & B_2(k) \\
\downarrow \delta_2 & & \downarrow \delta_2 \\
\Lambda^2 k^* & = & \Lambda^2 k^*
\end{array}
$$

46
is commutative. So we need only to show that if $0 \neq x \in B_2(k) \mathbb{Q}$ and $\delta_2(x) = 0$, then $i(x) \neq 0$. This follows from the injectivity of the regulator map on $K_3^{ind}(k) \mathbb{Q}$. Indeed, by Suslin’s theorem for a field $F$ one has $K_3^{ind}(F) \mathbb{Q} = \text{Ker} \delta_2 \otimes \mathbb{Q}$. Let us identify $K_3^{ind}(\mathbb{C}) \mathbb{Q}$ with this subgroup of $B_2(\mathbb{C}) \mathbb{Q}$. The restriction of the dilogarithm map

$$B_2(k) \to B_2(k) \to (\mathbb{Z} \text{Hom}(k, \mathbb{C})) \otimes 2\pi i \mathbb{R}, \quad \{z\} \mapsto \{2\pi i \mathcal{L}_2(\sigma_i(z))\}$$

to the subgroup $\text{Ker} \delta_2 \otimes \mathbb{Q}$ gives the Borel regulator $K_3^{ind}(k) \mathbb{Q} \to \mathbb{R}^\times$ ([G1]) and thus injective by Borel’s theorem.

**Remark.** For any field $k$ the rigidity conjecture for $K_3^{ind}$ implies that the map $i$ should be an isomorphism, see [G1].

**2. The strong reciprocity law.**

**Conjecture 6.2** Let $X$ be a regular projective curve over an algebraically closed field $k$ and $F := k(X)^*$. Then there exists a canonical homomorphism of groups $h : \Lambda^3 F^* \to B_2(k)$ satisfying the following two conditions:

a) $h(k^* \wedge \Lambda^2 F^*) = 0$ and the diagram

$$\begin{array}{ccccccc}
\mathcal{B}_3(F) & \xrightarrow{\delta_2} & \mathcal{B}_2(F) \otimes F^* & \xrightarrow{\delta_2} & \Lambda^3 F^* \\
\text{Res} \downarrow & & h \downarrow & & \text{Res} \\
\mathcal{B}_2(k) & \xrightarrow{\delta_2} & \Lambda^2 k^* 
\end{array}$$

is commutative.

b) If $X$ is a curve over $\mathbb{C}$ then

$$\frac{1}{2\pi i} \int_{X(\mathbb{C})} r_2(f_1 \wedge f_2 \wedge f_3) = \mathcal{L}_2(h(f_1 \wedge f_2 \wedge f_3))$$

(64)

**Remarks.** 1. b) follows easily if we have a functorial map $h$ such that $\text{Res} = \delta_2 \circ h$, see theorem 6.10 below.

2. According to Suslin’s reciprocity law for the Milnor group $K_3(F)$ the projection of $\text{Res}(\Lambda^3 F^*) \subset \Lambda^2 k^*$ to $K_2(k)$ is zero. Since by Matsumoto’s theorem $K_2(k) = \text{Coker}(\delta_2)$, one has $\text{Res}(\Lambda^3 F^*) \subset \text{Im}(\delta_2)$. However $\text{Ker}(\delta_2)$ is nontrivial, so it is a priori unclear that we can lift naturally the map $\text{Res}$ to a map $h$. One of the reasons why we can do this is provided by (64).

We prove this conjecture in the following cases:

a). $X = \mathbb{P}^1$: we construct explicitly a reciprocity homomorphism $h : \Lambda^3 F^* \to B_2(k)$ in theorem 6.5.

b). $X$ is an elliptic curve over an algebraically closed field; we construct explicitly a reciprocity homomorphism $h : \Lambda^3 F^* \to B_2(k)$ in theorem 6.14.

c). $k = \overline{k}$, $X$ is any curve; see theorem 6.12.

In the cases a) and b) the homomorphism $h$ satisfy the following additional property. Let $F = k(X)$ and $k$ is not necessarily algebraically closed. Let $k'$ be the field of definition of the divisors $(f_1), (f_2), (f_3)$. Then $h(f_1 \wedge f_2 \wedge f_3) \in \mathcal{B}_2(k')$. 47
Conjecture 6.3 Let $X$ be a projective regular curve over an algebraically closed field $k$ and $F := k(X)$. Then the homomorphism

$$\text{Res} : \Gamma(F; n) \longrightarrow \Gamma(k; n-1)[-1]$$

is homotopic to zero.

Lemma 6.4 Assume that we have a map $h$ such that $h(k^* \wedge \Lambda^2 F^*) = 0$ and $\text{Res} = \delta_2 \circ h$. Then $h \circ \delta_3 = \text{Res}$.

Proof. The image of the group $B_2(F) \otimes F^*$ under the map $h \circ \delta_3$ belongs to the subgroup $\text{Ker}\delta_2$. Since $h(k^* \wedge \Lambda^2 F^*) = 0$ one has $h \circ \delta_3 = \text{Res}$ on $B_2(F) \otimes k^*$. Any element of $k\langle X \rangle^*$ can be connected via a curve to a constant. This together with the rigidity of $\text{Ker}\delta_2$ (built in the definition of the group $B_2(k)$) implies the result.

3. The $X = \mathbb{P}^1$ case. Recall that $v_x(f)$ is the order of zero of $f \in k(X)$ at $x$. Choose a point $\infty$ on $\mathbb{P}^1$.

Theorem 6.5 Assume that $k = \overline{k}$. Then the map $h : \Lambda^3(k(\mathbb{P}^1))^* \rightarrow B_2(k)$ given by the formula

$$h(f_1 \wedge f_2 \wedge f_3) := \sum_{x_i \in \mathbb{P}^1(k)} v_{x_i}(f_1)v_{x_2}(f_2)v_{x_3}(f_3)\{r(x_1, x_2, x_3, \infty)\}_2$$

satisfies all the conditions of conjecture 6.2 modulo 6-torsion.

Proof. Let us show that $h$ is independent of the choice of $\infty$, i.e.

$$\sum_{x_i \in \mathbb{P}^1(k)} v_{x_i}(f_1)v_{x_2}(f_2)v_{x_3}(f_3)\{r(x_1, x_2, x_3, a)\}_2 \in B_2(k)$$

does not depend on $a$. Indeed, the 5-term relation for the 5-tuple of points $(x_1, x_2, x_3, a, b)$ gives

$$\sum_{x_i \in \mathbb{P}^1(k)} v_{x_i}(f_1)v_{x_2}(f_2)v_{x_3}(f_3)\{r(x_1, x_2, x_3, a)\}_2 - \{r(x_1, x_2, x_3, b)\}_2 =$$

$$\sum_{x_i \in \mathbb{P}^1(k)} v_{x_i}(f_1)v_{x_2}(f_2)v_{x_3}(f_3)\{r(x_1, x_2, a, b)\}_2 - \{r(x_1, x_3, a, b)\}_2 + \{r(x_2, x_3, a, b)\}_2$$

Each of these 3 terms vanishes because $\sum_{x \in \mathbb{P}^1(k)} v_x(f) = 0$.

Proposition 6.6 Let $k = \overline{k}$. Then modulo 6-torsion

$$h((1 - f) \wedge f \wedge g) = \sum_{x \in \mathbb{P}^1(k)} v_x(g)\{f(x)\}_2$$

48
Lemma 6.7 Applying $\delta_2$ to both parts of (65) we get the same result modulo 6-torsion.

Proof. Choose a coordinate $t$ on $\mathbb{P}^1$ such that $f(\infty) = 1$. Then
\[
f(t) = \prod_i (a_i - t)^{\alpha_i} / \prod_k (c_k - t)^{\gamma_k}; \quad 1 - f(t) = B \prod_i (b_j - t)^{\beta_j} / \prod_k (c_k - t)^{\gamma_k},
\]
(66)
Observe that $\{f(\infty)\}_2 = 0$ modulo 6-torsion. The left hand side equals
\[
\sum_{x_i \in \mathbb{P}^1(k)} vx_1 (1 - f)vx_2 (f)\{r(x_1, x_2, 0, \infty)\}_2 = \{f(0)\}_2 - \{f(\infty)\}_2
\]
(65)

Proof. Using linearity with respect to $g$ and projective invariance of the cross–ratio we see that it is sufficient to prove the identity for $g = t$. Then it boils down to
\[
\sum_{x_i \in \mathbb{P}^1(k)} vx_1 (1 - f)vx_2 (f)\{r(x_1, x_2, 0, \infty)\}_2 = \{f(0)\}_2 - \{f(\infty)\}_2
\]

Applying $\delta_2$ to it we get
\[
\sum \beta_j \alpha_i \{b_j/a_i\}_2 = \sum \gamma_k \alpha_i \{c_k/a_i\}_2 - \sum \gamma_k \{b_j/c_k\}_2 =
\]
Applying $\delta_2$ to it we get
\[
\sum \beta_j \alpha_i \cdot \frac{a_i - b_j}{a_i} \cdot b_j \cdot a_i - \sum \gamma_k \alpha_i \cdot \frac{a_i - c_k}{a_i} \cdot c_k = \sum \gamma_k \beta_j \cdot \frac{b_j - c_k}{c_k} \cdot \frac{b_j}{c_k} =
\]
\[
- \sum i \prod_j (a_i - b_j)^{\beta_j} \cdot \frac{a_i}{a_i} + \sum j \prod_k (a_i - c_k)^{\gamma_k} \cdot b_j = \sum k \prod (a_i - c_k)^{\alpha_i} \cdot c_k \cdot \gamma_k
\]
Using (66) we see that the second line equals modulo 2–torsion to
\[
- \sum i (1 - f(a_i)) \cdot \frac{a_i}{a_i} + \sum j f(b_j) \cdot b_j - \sum k \prod (a_i - c_k)^{\alpha_i} \cdot c_k
\]
The first two terms are zero since $f(a_i) = 0$ and $f(b_j) = 1$. The third term equals to $-B \cdot \prod c_k^{\gamma_k}$ since, as it follows from (66),
\[
\prod (a_i - c_k)^{\alpha_i} / \prod (b_j - c_k)^{\beta_j} = B
\]

On the other hand to
\[
\delta_2(\{f(0)\}_2) = (1 - f(0)) \cdot f(0) = B \prod \frac{b_j^{\beta_j}}{c_k^{\gamma_k}} \cdot \frac{a_i^{\alpha_i}}{c_k^{\gamma_k}}
\]
which matches the expression we got for the left hand side. The lemma is proved.

To prove the proposition it remains to use the standard rigidity argument. Namely, we need to show that the identity is valid for some particular \( f \), which is easy, or use proposition 6.8 plus injectivity of the regulator on \( K_3(\overline{\mathbb{Q}}) \). The proposition is proved.

Now let us prove the key fact, \( \text{Res} = \delta_2 \circ h \). We need to show that for any 3 rational functions \( f_1, f_2, f_3 \) on \( \mathbb{P}^1 \)

\[
\sum_{x \in \mathbb{P}^1(\mathbb{k})} \text{res}_x (f_1 \wedge f_2 \wedge f_3) =
\delta \left( \sum_{x_i \in \mathbb{P}^1(\mathbb{k})} v_{x_1}(f_1)v_{x_2}(f_2)v_{x_3}(f_3) \{ r(x_1, x_2, x_3, \infty) \} \right) \tag{68}
\]

The both parts are obviously homomorphisms from \( \Lambda^3 F^* \) to \( \Lambda^2 k^* \) which are zero on \( k^* \wedge \Lambda^2 F^* \). (The last property for the map \( \sum \text{res}_x \) is provided by the Weil reciprocity law). We normalize the cross ratio of four points on the projective line by \( r(\infty, 0, 1, z) = z \). So it suffices to check the formula on elements

\[
\frac{z - \alpha_2}{z - \alpha_1} \wedge \frac{z - \beta_2}{z - \beta_1} \wedge \frac{z - \gamma_2}{z - \gamma_1}
\]

In this case it follows from

\[
\delta_2(\{ r(a_2, b_2, c_2, \infty) \}) = \delta_2 \left( \frac{a_2 - c_2}{b_2 - c_2} \right) = \frac{b_2 - a_2}{b_2 - c_2} \wedge \frac{a_2 - c_2}{b_2 - c_2}
\]

It remains to prove the following proposition.

**Proposition 6.8**

\[
\mathcal{P}_2(\mathbb{P}^1; f_1, f_2, f_3) = \sum_{x_i \in \mathbb{P}^1(\mathbb{C})} v_{x_1}(f_1)v_{x_2}(f_2)v_{x_3}(f_3) \mathcal{L}_2(\{ r(x_1, x_2, x_3, \infty) \})
\]

**Proof.** We immediately reduce the statement to the situation when \( f_1 = 1 - z, f_2 = z, f_3 = z - a \) which is a particular case of the following lemma

**Lemma 6.9** Let \( X \) be an arbitrary curve over \( \mathbb{C} \). Then

\[
\int_{X(\mathbb{C})} r_2((1 - f) \wedge f \wedge g) = - \sum_{x \in X(\mathbb{C})} v_x(g) \cdot \mathcal{L}_2(f(x))
\]

**Proof.** For functions \( f(z) \) and \( g(z) \) on \( X(\mathbb{C}) \) set

\[
\alpha(f, g) := \log |f|d\log |g| - \log |g|d\log |f|
\]
Consider the following 1-form on $X$ \( (C) \)
\[
\mathcal{L}_2(f) \arg g - \frac{1}{3} \alpha(1 - f, f) \log |g|
\]  
(69)
It defines a current. We claim that its derivative is equal to:
\[
2\pi \cdot \mathcal{L}_2(f) \delta(g) + r_2((1 - f) \wedge f \wedge g)
\]  
(70)
Using \( d(d \arg g) = 2\pi \cdot \delta(g) \) and
\[
\mathcal{L}_2(z) = - \log |1 - z| d \arg z + \log |z| d \arg(1 - z)
\]  
(71)
we see that the differential of the current (69) equals to
\[
2\pi \mathcal{L}_2(f) \delta(g) + \left( - \log |1 - f| d \arg f + \log |f| d \arg(1 - f) \right) \wedge d \arg g + \frac{1}{3} \left( \log |1 - f| d \log |f| - \log |f| d \log |1 - f| \right) \wedge d \log |g| - \frac{2}{3} \log |g| \cdot d \log |1 - f| \wedge d \log |f|
\]
Since \( d \log |1 - f| \wedge d \log |f| = 0 \) we have
\[
d \log |1 - f| \wedge d \log |f| = d \arg(1 - f) \wedge d \arg f
\]
Using this and writing \( r_2(f_1 \wedge f_2 \wedge f_3) \) as
\[
\frac{1}{3} (\log |f_1| d \log |f_2| \wedge d \log |f_3| + \text{cyclic permutations})
\]  
\[
-(\log |f_1| d \arg(f_2) \wedge d \arg f_3 + \text{cyclic permutations})
\]
we come to (70). Integrating we get the lemma. The theorem is proved.

4. **Expressing the Chow dilogarithms via the classical one.** Let \( \pi : Y \rightarrow S \) be a family of curves over a base \( S \) over \( \mathbb{C} \) and \( f_1, f_2, f_3 \in \mathbb{C}(Y)^* \). We get a function at the generic point of \( S \). Its value at \( s \in S \) is given by the Chow dilogarithm \( \mathcal{P}_2(Y_s; f_{s1}, f_{s2}, f_{s3}) \), where \( Y_s \) is the fiber of \( \pi \) at \( s \). Denote it by \( \mathcal{P}_2(Y \rightarrow S; f_1, f_2, f_3) \).

**Theorem 6.10 a)** Let \( \pi : Y \rightarrow S \) be a family of curves over a base \( S \) over \( \mathbb{C} \). Then there are rational functions \( \varphi_i \) on \( S \) such that
\[
\mathcal{P}_2(Y \rightarrow S; f_1, f_2, f_3) = \sum \mathcal{L}_2(\varphi_i(s))
\]
**b)** Let \( k = \mathbb{C}(S) \), \( X \) is the generic fiber of \( \pi \), and \( F = k(X) \). Suppose that there exists a map \( h : \Lambda^3 F^* \rightarrow B_2(k) \) such that \( \text{Res} = \delta_2 \circ h \). Then
\[
d \mathcal{P}_2(Y \rightarrow S; f_1, f_2, f_3) = d \mathcal{L}_2(h(f_1, f_2, f_3))
\]  
(72)
Indeed, that with the map as a 3-current on $D$. The second term in (75) is a 1-form on the divisor $p$. Choose a projection $p : X \to \mathbb{P}^1$. We may suppose without loss of generality that $p$ is a (ramified) Galois covering with the Galois group $G$. Indeed, let $p_1 : Y \to X$ be such a covering that its composition with $p$ is a Galois covering. Therefore, Lemma 6.11 proved. It remains to use lemma 6.9 and proposition 6.8. The part a) of the theorem is proved.

b) We need the following lemma.

**Lemma 6.11**

\begin{equation}
 d_4 \mathcal{P}_2(Y \to S; f_1, f_2, f_3) = (2\pi)^{-1} \cdot \text{Alt}_3 \left( e_x(f_1) \log |f_2(x)| d_4 \arg f_3(x) \right)
\end{equation}

**Proof.** Using $d \log f = 2\pi i \delta(f)$ we get an equality of 3-currents on $Y$

\begin{equation}
 dr_2(f_1, f_2, f_3) = 
\end{equation}

where the second term in (75) is a 1-form on the divisor $D := \cup_{i=1}^s \text{div}(f_i)$ considered as a 3-current on $Y$. This 1-form is the composition of the residue map

\begin{equation}
 \text{res} : \Lambda^3 \mathcal{C}(Y)^* \to \prod_{X \in Y_1} \Lambda^2 \mathcal{C}(X)^*
\end{equation}

with the map

\begin{equation}
 r_1 : \Lambda^2 \mathcal{C}(X)^* \to \mathcal{A}^1(\text{Spec}(\mathbb{C}(X))), \quad f \wedge g \mapsto -2\pi (\log |f| d \arg g - \log |g| d \arg f)
\end{equation}
The push forward of the first term in (75) vanish (since the fibers are complex curves the push down of any (3,0)-form to $S$ is zero). Integrating the second 3-current in (75) along the fibers of $Y$ we get (74). The lemma is proved.

According to lemma 6.11 and formula (71) for $dL_2$, and using $R_2 = \delta_2 \circ h$ we get the proof of the part b) of the theorem.

**Remark** The function $L_2(z)$ is continuous on $\mathbb{C}P^1$. Therefore part a) of the theorem implies that the function $P_2(Y \to S; f_1, f_2, f_3)$ can be extended to a continuous function on $S$.

**5. Conjecture 6.2 for $k = \overline{\mathbb{Q}}$.**

**Theorem 6.12** Let $X$ be a regular projective curve over $\overline{\mathbb{Q}}$ and $F := \overline{\mathbb{Q}}(X)^*$. Then there exists a homomorphism $h : \Lambda^3F^* \to B_2(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ as in conjecture 6.2 such that for any embedding $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ one has

$$
\frac{1}{2\pi i} \int_{X(\mathbb{C})} r_2(\sigma(f_1 \wedge f_2 \wedge f_3)) = L_2(\sigma(h(f_1 \wedge f_2 \wedge f_3)))
$$

(76)

**Proof.** It is similar to the proof of theorem 6.10. Choose a projection $p : X \to \mathbb{P}^1$. We may suppose that $p$ is a Galois covering with the Galois group $G$. Indeed, let $p_1 : Y \to X$ be such a covering that its composition with $p$ is a Galois covering. Setting

$$
h(f_1 \wedge f_2 \wedge f_3) := h \left( \frac{1}{\deg p_1} \cdot p_1^*(f_1 \wedge f_2 \wedge f_3) \right)
$$

we may suppose that $p$ is Galois.

Then $\sum_{g \in G} g^*\{f_1, f_2, f_3\} \in p^* kK_3^M((\mathbb{P}^1))$. So there exist $s_1^{(i)}, s_2^{(i)}, s_3^{(i)}, \in k(\mathbb{P}^1)$ and $g_j, h_j \in k(X)$ such that (73) holds. Set

$$|G| \cdot h(f_1 \wedge f_2 \wedge f_3) := \sum h(s_1^{(i)} \wedge s_2^{(i)} \wedge s_3^{(i)}) + \sum_{x \in X(\overline{\mathbb{Q}})} \{g_j(x)\}_2 \cdot v_x(h_j)
$$

**Lemma 6.13** Suppose $\sum_i (1 - f_i) \wedge f_i \wedge g_i = 0$ in $\Lambda^3\overline{\mathbb{Q}}(X)^*$. Then

$$\sum_i \sum_{x \in X(\overline{\mathbb{Q}})} v_x(g_i) \cdot \{f(x)\}_2 = 0 \text{ in the group } B_2(\overline{\mathbb{Q}}).
$$

This lemma implies that $h$ is well defined. Indeed, suppose that we have a different presentation

$$
\sum_{g \in G} g^*\{f_1, f_2, f_3\} = p^* \sum_i s_1^{(i)} \wedge s_2^{(i)} \wedge s_3^{(i)} + \sum_j (1 - \tilde{g}_j) \wedge \tilde{g}_j \wedge \tilde{h}_j
$$

We need to show that

$$
h \left( \sum i s_1^{(i)} \wedge s_2^{(i)} \wedge s_3^{(i)} - \sum s_1^{(i)} \wedge s_2^{(i)} \wedge s_3^{(i)} \right) + \quad (77)
$$
Thus for any point \( x \) of the curve \( \mathbb{P}^1 \) there is a commutative diagram

\[
\begin{array}{ccc}
B_2(F) \otimes F^* & \xrightarrow{\delta_2} & \Lambda^2 F^* \\
\text{res} \downarrow & & \downarrow \text{res} \\
B_2(k) \otimes \mathbb{Z}[X(k)] & \xrightarrow{\delta_2} & \Lambda^2 k^* \otimes \mathbb{Z}[X(k)]
\end{array}
\]

Thus for any point \( x \) of the curve \( X \) the element \( \sum_i v_x(g_i) \cdot [f_i(x)]_2 \) lies in \( \text{Ker} \delta_2 \). Therefore it defines an element \( \gamma_x \in K_3(\mathbb{Q}) \).

For any embedding \( \sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \) the value of the Borel regulator on \( \sigma(\gamma_x) \) is equal to \( \sum_{x \in X} v_x(g_i) \cdot \mathcal{L}_{\sigma}(f_i(x)) \). So by lemma 6.9 the value of the Borel regulator on \( \sum_x \sigma(\gamma_x) \) is equal to \( 2\pi \cdot \int_{S^1} r_2(\sum_i (1 - f_i) \wedge f_i \wedge g_i) \) and hence it is zero by our assumption. So Borel’s theorem implies that the element is also zero.

A similar argument using lemma 6.13 shows that the homomorphism \( h \) does not depend on the choice of the (finite) Galois extension of \( \mathbb{Q}(\mathbb{P}^1) \) containing the field \( \mathbb{Q}(X) \).

6. Explicit formulas for the reciprocity homomorphism \( h \) and the Chow dilogarithm in the case of an elliptic curve. Let \( E \) be an elliptic curve. We want to calculate the integral \( \int_{E(\mathbb{Q})} r_2(f_1 \wedge f_2 \wedge f_3) \). Let us suppose that \( E \) is realized as a plane curve. Then any rational function \( f \) on \( E \) can be written as a ratio of products of linear homogeneous functions:

\[
f = \frac{l_1 \cdots l_k}{l_{k+1} \cdots l_{2k}}
\]
So it is enough to calculate the integral \( \int_{E(C)} \frac{r_2(l_1/l_0 \wedge l_2/l_0 \wedge l_3/l_0)}{} \) where \( l_i \) are linear functions in homogeneous coordinates. We will do this in a more general setting.

**Notations.** Let \( X \) be a plane algebraic curve and \( l_i \) linear functions in homogeneous coordinates. Denote by \( L_i \) the line \( l_i = 0 \) in \( \mathbb{P}^2 \). Let \( D_i \) be the divisor \( L_i \cap X \). Set \( l_{ij} := L_i \cap L_j \). For three points \( a, b, c \) and a divisor \( D = \sum n_i(x_i) \) on a line we will use the following notation (see fig.2):

\[
\{r(a, b, c, D)\}_2 := \sum_i n_i \{r(a, b, c, x_i)\}_2
\]

**Theorem 6.14** Let \( E \) be an elliptic curve over an algebraically closed field \( k \). Then there exists a homomorphism of groups \( h : \Lambda^3 F^* \to \mathcal{B}_2(k) \) such that for any linear homogeneous functions \( l_0, ..., l_3 \) one has

\[
h(l_1/l_0 \wedge l_2/l_0 \wedge l_3/l_0) = -\sum_{i=0}^3 (-1)^i \{r(l_i, ..., l_3, D_i)\}_2
\]

and which satisfy all the properties of conjecture 6.2. In particular, if \( k = \mathbb{C} \) then

\[
\mathcal{P}_2(E; f_1 \wedge f_2 \wedge f_3) = L_2(h(f_1 \wedge f_2 \wedge f_3))
\]

**Proof.** Suppose we have four generic lines \( L_0, L_1, L_2, L_3 \) in \( \mathbb{P}^2 \). Any two of them, say \( L_0 \) and \( L_1 \), provide a canonical rational function \((l_0/l_1)\) on \( \mathbb{P}^2 \) with the divisor \((L_0) - (L_1)\) normalized by the condition that its value at the point \( l_{23} \) is equal to 1.
Lemma 6.15  a) On the line $L_3$ one has $(l_1/l_0) + (l_2/l_0) = 1$.

b) $\frac{(l_1/l_0)}{(l_2/l_0)} = -(l_1/l_2)$

Proof. Let $m$ be a point on the line $L_3$. Then

$$(l_1/l_0)(m) = r(l_{03}, l_{13}, l_{23}, m); \quad (l_2/l_0)(m) = r(l_{03}, l_{23}, l_{13}, m) \quad (81)$$

This gives a). It follows from this that if the point $m$ approaches to the point $l_03$ then $\frac{(l_1/l_0)}{(l_2/l_0)}$ tends to $-1$. This implies b).

Lemma 6.16 For any plane curve $X$ one has

$$\sum_{x \in X} \text{res}_x \left( (l_1/l_0) \land (l_2/l_0) \land (l_3/l_0) \right) = -\delta_2 \left( \sum_{i=0}^{3} (-1)^i \{r(l_0, \ldots, l_3, D_i)\}_2 \right)$$

Proof. Let us compute first the residues at the divisors $D_1, D_2, D_3$ using part a) of lemma 6.15. For example the residue at $x \in D_1$ is equal to

$$v_x((l_1/l_0)) \cdot (l_2/l_0)(x) \land (l_3/l_0)(x) = v_x((l_1/l_0)) \cdot (1 - (l_3/l_0)(x)) \land (l_3/l_0)(x) \quad (81)$$

$$v_x((l_1/l_0)) \cdot \{r(l_{13}, l_{10}, l_{12}, x)\}_2 = v_x((l_1/l_0)) \cdot \{r(l_{10}, l_{12}, l_{13}, x)\}_2$$

It remains to compute the residues on the line $L_0$. According to part b) of lemma 6.15 one has

$$(l_1/l_0) \land (l_2/l_0) \land (l_3/l_0) = -(l_0/l_1) \land (l_2/l_1) \land (l_3/l_1)$$

Using this we reduce the calculation to the previous case.

Proposition 6.17 Let $E$ be an elliptic curve over an algebraically closed field $k$. Then formula (79) provides a well defined homomorphism of groups $h : \Lambda^3 F^* \to B_2(k)$.

Proof. Let $D := \sum_{i} n_i(x_i)$ be a divisor of a rational function $f$ on $E$. To decompose it into a fraction of products of linear homogeneous functions $l_i$ we proceed as follows. Let $l_{x,y}$ (resp. $l_x$) be a linear homogeneous equation of the line in $\mathbb{P}^2$ through the points $x$ and $y$ on $E$ (resp. $x$ and $-x$). The divisor of the function $l_{x,y}/l_x$ is $(x) + (y) - (x + y) - (0)$. If $D = (x) + (y) + D_1$, we write $f = l_{x,y}/l_{x+y} \cdot f'$, so $(f') = (0) + (x + y) + D_1$. After a finite number of such steps we get a desired decomposition.
There are the following relations
\[
\frac{l_{x,y}}{l_{x+y}^2} \cdot \frac{l_{x+y, z}}{l_{y+z}^2} \cdot \frac{l_{x, y+z}}{l_{x+y+z}^2} = \text{a constant}
\]

One can prove that they generate all the relations between the functions \(l_{x,y}/l_{x+y}\). So \(h\) is well defined if it annihilates the following expression:

\[
F(x, y, z; l_0, l_2, l_3) := \left(\frac{l_{x,y}}{l_{x+y}^2} \cdot \frac{l_{x+y, z}}{l_{y+z}^2} \cdot \frac{l_{x, y+z}}{l_{x+y+z}^2}\right) \land (l_2/l_0) \land (l_3/l_0)
\]

It follows from lemma 6.16 that \(\delta_2(F(x, y, z; l_0, l_2, l_3)) = 0\). Thus according to the definition of the group \(B_2(k)\) it is enough to check that \(h(F(x', y', z'; l_0, l_2, l_3)) = 0\) for a certain triple of points \((x', y', z')\). It is easy to see that \(h(F(a, a, a; l_0, l_2, l_3)) = 0\) since then the first factor in \(F\) is a constant. The proposition is proved.

**Proposition 6.18** Let \(X\) be an algebraic curve in \(\mathbb{P}^2\) over \(\mathbb{C}\) and \(l_0, ..., l_3\) linear homogeneous functions on \(\mathbb{C}^3\). Then one has (using the notations defined above)

\[
\int_{X(\mathbb{C})} r_2(l_1/l_0 \land l_2/l_0 \land l_3/l_0) = 2\pi \cdot \sum_{i=0}^3 (-1)^i \mathcal{L}_2(r_{l_0}, ..., l_{i+1}, D_i) \quad (82)
\]

**Proof.** It follows from lemma 6.16 and theorem 6.10 that the differentials of the both parts coincide. So its difference is a constant. To show that this constant is zero we deform \(X\) to a union of lines in \(\mathbb{P}^2\). Using proposition 6.8 one sees that formula (82) is valid when \(X\) is a line in \(\mathbb{P}^2\).
Appendix: on volumes of simplices in symmetric spaces

1. Volumes of hyperbolic geodesic simplices as boundary integrals

A point \( y \) of the \( n \)-dimensional hyperbolic space \( \mathcal{H}_n \) defines a one dimensional space \( M_y \) of volume forms on the absolute \( \partial \mathcal{H}_n \). It consists of the volume forms invariant under the action of the isotropy group of \( y \). We write them as follows. Let \( x_1, ..., x_{n+1} \) be the coordinates in a vector space \( V_{n+1} \) and \( Q(x) := x_1^2 + ... + x_n^2 - x_{n+1}^2 \). Then \( \mathcal{H}_n \) can be realized as the projectivization of the cone \( Q(x) < 0 \), and its boundary is the projectivization of the cone \( Q(x) = 0 \).

Choose a point \( y \in \mathcal{H}_n \). Lifting \( y \) to a vector \( y' \in V_{n+1} \) we have the following volume form on the boundary \( \partial \mathcal{H}_n \):

\[
\delta(Q(x))\alpha_{n+1}(x, dx)\]

If \( y \) belongs to the boundary \( \partial \mathcal{H}_n \), this formula provides a space \( M_y \) of singular volume forms on the absolute; they are invariant under the isotropy group of \( y \).

Let us choose for any point \( y \) such a volume form \( \mu_y \). For two points \( x, y \) the ratio \( \mu_x/\mu_y \) is a nonzero function on the absolute.

Let \( I(y_0, ..., y_n) \) be the geodesic simplex with vertices at \( y_0, ..., y_n \) where the points \( y_i \) could be on the absolute. Denote by \( \text{vol}(I(y_0, ..., y_n)) \) the volume of this simplex with respect to the invariant volume form in \( \mathcal{H}_n \) normalized by the following condition: if we realize the hyperbolic space as the interior of the unit ball \( y_1^2 + ... + y_n^2 \leq 1 \) then the volume form restricted to the tangent space at the origin \( (0, ..., 0) \) is \( dy_1 \wedge ... \wedge dy_n \).

**Theorem 7.1** For any hyperbolic geodesic simplex \( I(y_0, ..., y_n) \) one has

\[
\frac{(n-1)n\text{vol}(S^{n-1})}{n} \cdot \text{vol}(I(y_0, ..., y_n)) = \int_{\partial \mathcal{H}_n} \log \frac{\mu_{y_1}}{\mu_{y_0}} d\log \frac{\mu_{y_2}}{\mu_{y_0}} \wedge ... \wedge d\log \frac{\mu_{y_n}}{\mu_{y_0}}
\]

(83)

Let \( \varphi(y_0, ..., y_n) \) be the function defined by the right hand side of (83). Thanks to property 2) of proposition 5.1 it does not depend on the choice of invariant volume forms \( \mu_y \).

**Proposition 7.2** The function \( \varphi(y_0, ..., y_n) \) has the following properties:

1) It is a smooth function on the vertices \( y_i \).
2) Equals to zero if three of the vertices belong to the same geodesic.
3) Additive with respect to cutting of a simplex, i.e. if \( y_0, ..., y_n \) are such points that \( y_0, y_1, y_2 \) are on the same geodesic, \( y_1 \) between \( y_0 \) and \( y_2 \), then

\[
\varphi(y_0, y_2, ..., y_{n+1}) = \varphi(y_0, y_1, y_3, ..., y_{n+1}) + \varphi(y_1, y_2, ..., y_{n+1})
\]

4) Invariant under the action of the symmetry group \( SO(n, 1) \).
To do the computation of this integral we may suppose that

Thus

Lemma 7.3 \( \varphi \) denoted when \( y \) is fixed and \( \mu _{y} \) are lifted from the line \( \mathbb{R} \). Therefore \( d \log \frac{\mu _{y_2}}{\mu _{y_1}} \wedge d \log \frac{\mu _{y_3}}{\mu _{y_1}} = 0 \). So for a degenerate simplex \( (y_0, ... , y_n) \) one has

\[
\log \frac{\mu _{y_0}}{\mu _{y_1}} d \log \frac{\mu _{y_2}}{\mu _{y_1}} \wedge ... \wedge d \log \frac{\mu _{y_n}}{\mu _{y_1}} = 0
\]

It remains to mention the skewsymmetry of the integral \((83)\). The property 2) is proved.

3) Follows from 2) and the additivity property from proposition (5.1).

4) This is clear from 4) of proposition (5.1). The proposition is proved.

The leading term of the Taylor expansion of the function \( \varphi (y_0, y_1, ..., y_n) \) when \( y_0 \) is fixed and \( y_1, ..., y_n \) are near \( y_0 \) provides an exterior \( n \)-form in \( T_{y_0}\mathcal{H} _{n} \) denoted \( \varphi _{y_0}(Y_1, ... , Y_n) \), \( Y_i \in T_{y_0}\mathcal{H} _{n} \). Let us compare it with the volume form \( \text{Vol}_{y_0}(Y_1, ... , Y_n) \) in \( T_{y_0}\mathcal{H} _{n} \) normalized as before theorem \( ?? \).

Lemma 7.3 \( \varphi _{y_0}(Y_1, ... , Y_n) = \frac{(n-1)^n}{n} \text{vol}(S^{n-1}) : \text{Vol}_{y_0}(Y_1, ... , Y_n) \).

Proof. Below we abuse notations by writing \( y \) for \( y' \). One has \( \mu _{y_1}/\mu _{y_2} = \frac{(y_2, x)^{n-1}}{(y_1, x)^{n-1}} \). So

\[
\varphi (y_0, ... , y_n) = (n-1)^n \int _{\partial \mathcal{H} _{n}} \log \frac{(y_1, x)}{(y_0, x)} d \log \frac{(y_2, x)}{(y_0, x)} \wedge ... \wedge d \log \frac{(y_n, x)}{(y_0, x)}
\]

Thus

\[
\varphi _{y_0}(Y_1, ... , Y_n) = (n-1)^n \int _{\partial \mathcal{H} _{n}} (Y_1, x) d(Y_2, x) \wedge ... \wedge d(Y_n, x)
\]

To do the computation of this integral we may suppose that \( y_0 = (0, ..., 0, 1) \), \( Y_i = \frac{\partial}{\partial x_i} \), so \( (Y_i, x) = x_i \). Then the last integral equals to

\[
(n-1)^n \int _{S^{n-1}} x_1 dx_2 \wedge ... \wedge dx_n = \frac{(n-1)^n}{n} \text{vol}(S^{n-1})
\]

where \( S^{n-1} \) is the sphere \( x_1^2 + ... + x_n^2 = 1 \). The lemma follows.

Proof of theorem 7.1. Let us suppose first that the points \( x_i \) are inside of the hyperbolic space.

59
The function \( \varphi(x_0, \ldots, x_n) \) defines an \( n \)-density \( \tilde{\varphi} \) on \( \mathcal{H}_n \). Namely, to define the integral \( \tilde{\varphi} \) over a simplex \( M \) one has to subdivide it on small simplices and take the sum of the functions \( \varphi \) corresponding to their vertices. When the simplices are getting smaller the limit exists and is by definition \( \int_M \tilde{\varphi} \). Here we used the properties 1) - 3). More precisely 1) and 2) implies that \( \varphi \) defines an additive volume form in \( \mathcal{H}_n \), and 3) (together with 1)) guarantee that this volume form is \( \sigma \)-additive.

The skewsymmetry property implies that \( \tilde{\varphi} \) is actually a differential \( n \)-form. It is invariant under the action of the group \( SO(n, 1) \). Therefore it is proportional to the standard volume form.

Now suppose that vertices \( x_i \) can be on the absolute. Then it is easy to see that the corresponding integral (37) is still convergent. Moreover, if the vertices of the geodesic simplex are in generic position then it is a continuous function of the vertices. This implies that the volume of an ideal geodesic simplex is finite (which is, of course, an elementary fact) and coincides with the corresponding integral (83).

A completely similar results are valid for the complex \( n \)-dimensional hyperbolic space \( \mathcal{H}_n^\mathbb{C} := \{ |z_1| + \ldots + |z_n|^2 < 1 \} \), \( z_i \in \mathbb{C} \), and the quaternionic hyperbolic space \( \mathcal{H}_n^\mathbb{H} := \{ |q_1| + \ldots + |q_n|^2 < 1 \} \) (\( q_i \) are quaternions). Indeed, a point \( x \) in each of these spaces defines an invariant volume form \( \mu_x \) on the boundary.

2. Calculation of volume of a three dimensional ideal geodesic simplex. If \( n = 3 \) the absolute can be identified with \( \mathbb{C}P^1 \). So

\[
f_{\infty, a}(z) = z - a
\]

and for the ideal simplex with vertices at the points \( \infty, 0, 1, a \) on the absolute we get

\[
\text{vol}(I(\infty, 0, 1, a)) = 3c_3 \int_{\mathbb{C}P^1} (\log |z| d\log |1-z| - \log |1-z| d\log |z|) \wedge d\log |z-a| =
\]

\[
3c_3 \cdot \int_{\mathbb{C}P^1} (\log |z| d\arg(1-z) - \log |1-z| d\arg(z)) \wedge d\arg(z-a)
\]

because \( d\log(z-1) \wedge d\log(z-a) = 0 \). Here \( \log f = \log |f| + i \arg(f) \). Using

\[
dL_2(z) = \log |z| d\arg(1-z) - \log |1-z| d\arg(z)
\]

we rewrite the last integral as

\[
3c_3 \cdot \int_{\mathbb{C}P^1} dL_2(z) \wedge d\arg(z-a) \tag{84}
\]

Computing the differential in the sense of distributions we get

\[
d(L_2(z) d\arg(z-a)) = 2\pi \cdot L_2(z) \delta(z-a) dxdy + dL_2(z) \wedge d\arg(z-a)
\]
So the integral of the right hand side over $\mathbb{CP}^1$ is zero, i.e. the integral (84) is equal to $-6\pi c_3 \cdot L_2(a)$. ($c_3 = -1/6\pi$).

3. Volumes of geodesic simplices in $SL_n(\mathbb{C})/SU(n)$. Recall the invariant differential $(2n-1)$–form $\omega_{D_n}$ in $\mathbb{H}_n$.

**Question.** Is it true that

$$\text{vol}_{D_n}(I(x_0, \ldots, x_{2n-1})) = \text{constant} \times \psi_n(x_0, \ldots, x_{2n-1}) \quad (85)$$

One can show following the lines of s. 7.1 that the positive answer to this question is equivalent to the following statement: if $x_0, x_1, x_2$ are on the same geodesic then $\psi_n(x_0, \ldots, x_{2n-1}) = 0$.

4. Another approach to Grassmannian polylogarithms. The following construction was suggested to the author during the Fall of 1989, independently, by M. Kontsevich and by J. Nekovar. A hyperplane $h$ in an $n$–dimensional complex vector space $V$ determines an arrow in the space of degenerate non negative definite hermitian forms in $V$ consisting of the forms with the kernel $h$. Let $h_1, \ldots, h_{2n}$ be hyperplanes in $V$. Let $C(h_1, \ldots, h_{2n})$ be projectivization of the convex hull of the arrays corresponding to these hyperplanes. It is a simplex in $\mathbb{P}_n$. The idea is to integrate the form $\omega_{D_n}$ over this simplex. If $n = 2$ this construction provides an ideal geodesic simplex in the Cayley realization of the hyperbolic space, given by the interior part of a ball in $\mathbb{R}P^3$. However the convergence of this integral for $n > 2$ has not been established yet, although it does not seem to be a very difficult problem. If the integral is convergent, we get a function on configurations of $2n$ hyperplanes in $\mathbb{CP}^{n-1}$. It would be very interesting to investigate this construction further and compare it with our construction of the Grassmannian polylogarithms.

5. A $(2n-1)$–cocycle of $GL(\mathbb{C})$. Consider the infinite dimensional vector space with a given basis $e_1, \ldots, e_m, \ldots$. The group $GL(\mathbb{C})$ is the group of automorphisms of this space moving only finite number of basis vectors.

Let us describe the restriction of the cocycle to the subgroup $GL_{n+m}(\mathbb{C})$ acting on the subspace generated by first $n+m$ vectors. Take $2n$ elements $g_1, \ldots, g_{2n}$ of this group. Consider the corresponding $2n \times (m+1)$-tuples of vectors:

$$g_1(e_n, \ldots, e_{n+m}), \ldots, g_{2n}(e_n, \ldots, e_{n+m})$$

The set of all $(m+1)$–tuples of vectors form a vector space. Let $\Delta_{2n-1}$ be the standard simplex $\sum_{j=1}^{2n} \lambda_j = 1$. Consider the set $C_n^m \subset \Delta_{2n-1}$ consisting of $(\lambda_1, \ldots, \lambda_{2n})$ that

$$\sum_{i=1}^{2n} \lambda_i \cdot g_i(e_n, \ldots, e_{n+m})$$

is an $(m+1)$–tuples of vectors not in generic position. It is a cycle of codimension $n$. The Chow polylogarithm function evaluated on it provides the desired (measurable) cocycle.
The cocycle property follows from the functional equation for the Chow polylogarithm and the following general fact. The set of those \((\lambda_1, \ldots, \lambda_{2n+1})\) that \(\sum_i \lambda_i \cdot g_i(e_n, \ldots, e_{n+m})\) is an \((m+1)\)-tuples of vectors not in generic position is also of codimension \(n\).

The construction is consistent with the restriction just by definition.

**Problem.** Show that the cohomology class of this cocycle is nontrivial and proportional to the Borel class.

### References


MacPherson R.: Gabrielov, Gelfand, Losic combinatorial formula for the first Pontryagin class Seminar Bourbaki 1976


Department of Mathematics, Brown University, Providence, RI 02912, USA.
e-mail sasha@math.brown.edu