On the periods of motives with complex multiplication and a conjecture of Gross-Deligne

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Abstract

We prove that the existence of an automorphism of finite order on a $\overline{\mathbf{Q}}$-variety X implies the existence of algebraic linear relations between the logarithm of certain periods of X and the logarithm of special values of the Γ-function. This implies that a slight variation of results by Anderson, Colmez and Gross on the periods of CM abelian varieties is valid for a larger class of CM motives. In particular, we prove a weak form of the period conjecture of Gross-Deligne [11, p. 205][1]. Our proof relies on the arithmetic fixed point formula (equivariant arithmetic Riemann-Roch theorem) proved by K. Köhler and the second author in [13] and the vanishing of the equivariant analytic torsion for the Dolbeault complex.

1 Introduction

Let $\mathcal{M}$ be a (homological Grothendieck) motive defined over Q_0, where Q_0 is an algebraic extension of $\mathbf{Q}$ embedded in $\mathbf{C}$. We shall use the properties of the category of motives over a field which are listed at the beginning of [5]. The complex singular cohomology $H(\mathcal{M},\mathbf{C})$ of the manifold of complex points of $\mathcal{M}$ is then endowed with two natural Q_0-structures. The first one is induced by the standard Betti $\mathbf{Q}$-structure $H(\mathcal{M},\mathbf{Q})$ via the identifications $H(\mathcal{M},Q_0)=H(\mathcal{M},\mathbf{Q})\otimes_{\mathbf{Q}}Q_0$ and $H(\mathcal{M},\mathbf{C})=H(\mathcal{M},Q_0)\otimes_{Q_0}\mathbf{C}$ and will be referred to as the Betti (or singular) Q_0-structure on $H(\mathcal{M},\mathbf{C})$. The second one arises from the comparison isomorphism between $H(\mathcal{M},\mathbf{C})$ and the de Rham-Dolbeault cohomology of $\mathcal{M}$ (tensorized with $\mathbf{C}$ over Q_0) and will be referred to as the de Rham Q_0-structure. If $\mathcal{M}$ is a smooth variety $f:Y\to\operatorname{Spec}Q_0$, its de Rham-Dolbeault cohomology is by definition the Q_0-vector space $H_{\mathrm{Dlb}}(Y):=\oplus_{k\geqslant 0}\oplus_{p+q=k}R^qf_*(\Lambda^p(\Omega(f)))$ which is endowed with a natural $\mathbf{N}$ as well as $\mathbf{N}\times\mathbf{N}$ grading.

Let Q be a finite (algebraic) extension of $\mathbf{Q}$ and suppose that all the embeddings of Q into $\mathbf{C}$ factor through Q_0. Furthermore let us suppose that $\mathcal{M}$ is endowed with a Q-motive structure (over Q_0). A Q-motive is also called a motive with coefficients

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[1]This should not be confused with the conjecture by Deligne relating periods and values of L-functions.

in Q (see [5, Par. 2]). The Q-motive structure of $\mathcal{M}$ induces an inner direct sum of complex vector spaces

$$H(\mathcal{M},\mathbf{C})=\bigoplus_{\sigma\in\mathrm{Hom}(Q,\mathbf{C})} H(\mathcal{M},\mathbf{C})_\sigma$$

which respects both Q_0-structures. The notation $H(\mathcal{M},\mathbf{C})_\sigma$ refers to the complex vector subspace of $H(\mathcal{M},\mathbf{C})$ where Q acts via $\sigma\in\mathrm{Hom}(Q,\mathbf{C})$. The determinant $\det_{\mathbf{C}}(H(\mathcal{M},\mathbf{C})_\sigma)$ thus has two Q_0-structures. Let v_{sing} (resp. v_{dR}) be a non-vanishing element of $\det_{\mathbf{C}}(H(\mathcal{M},\mathbf{C})_\sigma)$ defined over Q_0 for the singular (resp. for the de Rham) Q_0-structure. We write $P_\sigma(\mathcal{M})$ for the (uniquely defined and independant of the choices made) image in $\mathbf{C}^\times/Q_0^\times$ of the complex number λ such that $v_{\mathrm{dR}}=\lambda\cdot v_{\mathrm{sing}}$.
Let χ be an odd simple Artin character of Q and let us suppose at this point that $\mathcal{M}$ is homogenous of weight k. Consider the following conjecture:

Conjecture $\mathbf{A}(\boldsymbol{\mathcal{M}},\boldsymbol{\chi})$. *The equality of complex numbers*

$$\sum_{\sigma\in\mathrm{Hom}(Q,\mathbf{C})}\log|P_\sigma(\mathcal{M})|^2\chi(\sigma)$$
$$=-\frac{L'(\chi,0)}{L(\chi,0)}\sum_{\sigma\in\mathrm{Hom}(Q,\mathbf{C})}\sum_{p+q=k}p\cdot\mathrm{rk}(H^{p,q}(\mathcal{M},\mathbf{C})_\sigma)\chi(\sigma)$$

is verified, up to addition of a term of the form $\sum_{\sigma\in\mathrm{Hom}(Q,\mathbf{C})}\log|\alpha_\sigma|\chi(\sigma)$*, where* $\alpha_\sigma\in Q_0^\times$.

This conjecture is a slight strengthening of the case $n=1$, $Y=\mathrm{Spec}\,Q_0$ of the conjecture [15, Conj. 3.1]. Notice that this conjecture has both a "motivic" and an "arithmetic" content. More precisely, *if the Hodge conjecture holds* and $Q_0=\overline{\mathbf{Q}}$, this conjecture can be reduced to the case where $\mathcal{M}$ is a submotive of an abelian variety with complex multiplication by Q. Indeed, assuming the Hodge conjecture, one can show by examining its associated Hodge structures that some exterior power of $\mathcal{M}$ (taken over Q) is isomorphic to a motive over $\overline{\mathbf{Q}}$ lying in the tannakian category generated by abelian varieties with maximal complex multiplication by Q. In this latter case, the conjecture A is contained in a conjecture of Colmez [4]. Performing this reduction to CM abelian varieties or circumventing it is the "motivic" aspect of the conjecture.
However, even in the case of CM abelian varieties, the conjecture seems far from proof: in the knowledge of the authors, only the case of Dirichlet characters has been tackled with up to now; obtaining a proof of the conjecture A *for non-abelian Artin characters* (i.e. for abelian varieties with complex multiplication by a field whose Galois group over $\mathbf{Q}$ is non-abelian) is the "arithmetic" aspect alluded above. In this text we shall be concerned with both aspects, but our original contribution concerns the "motivic" aspect, more precisely in finding a way to circumvent the Hodge conjecture.
We now state a weaker form of the conjecture A. Let χ be a simple odd Artin character of Q as before, and N be a subring of $\overline{\mathbf{Q}}$. Let $\mathcal{M}_0$ be a motive over Q_0 (not necessarily homogenous) and suppose that $\mathcal{M}_0$ is endowed with a Q-motive structure (over Q_0). Let $\mathcal{M}_0^k$ ($k\geqslant 0$) be the motive corresponding to the k-th cohomology group of $\mathcal{M}_0$.

Conjecture B($\boldsymbol{\mathcal{M}_0}$, N, χ). *The equality of complex numbers*

$$\begin{aligned}&\sum_{k\geqslant 0}(-1)^k\sum_{\sigma\in\mathrm{Hom}(Q,\mathbf{C})}\log|P_\sigma(\mathcal{M}_0^k)|^2\chi(\sigma)\\ =\ &-\sum_{k\geqslant 0}(-1)^k\frac{L'(\chi,0)}{L(\chi,0)}\sum_{\sigma\in\mathrm{Hom}(Q,\mathbf{C})}\sum_{p+q=k}p\cdot\mathrm{rk}(H^{p,q}(\mathcal{M},\mathbf{C})_\sigma)\chi(\sigma)\end{aligned}$$

is verified, up to addition of a term of the form $\sum_{\sigma\in\mathrm{Hom}(Q,\mathbf{C})}\sum_i(b_{i,\sigma}\log|\alpha_{i,\sigma}|)\chi(\sigma)$, *where* $\alpha_{i,\sigma}\in Q_0^\times$, $b_{i,\sigma}\in N$ *and* i *runs over a finite set of indices.*

Note that the conjecture A (resp. B) only depends on the vector space $H(\mathcal{M},\mathbf{C})$ (resp. $H(\mathcal{M}_0,\mathbf{C})$), together with its Hodge structure (over $\mathbf{Q}$), its de Rham Q_0-structure and its additional Q-structure. If V is a $\mathbf{Q}$-vector space together with the just described structures on $V\otimes_{\mathbf{Q}}\mathbf{C}$ (all of them satisfying the obvious compatibility relations), we shall accordingly write $\mathrm{A}(V,\chi)$ (resp. $\mathrm{B}(V,N,\chi)$) for the corresponding statement, even if V possibly does not arise from a motive.
In this article we shall prove the conjecture B (and to a lesser extent, part of the conjecture A) for a large class of motives, which includes abelian varieties with complex multiplication by an abelian extension of $\mathbf{Q}$, without assuming the Hodge conjecture (or any other conjecture about motives) to reduce the problem to abelian varieties. Let us add that even in the case of abelian varieties, our method of proof is completly different from the existing ones.
A consequence of our results is that on any $\overline{\mathbf{Q}}$-variety X, the existence of a finite group action implies the existence of non-trivial algebraic linear relations between the logarithm of the periods of the eigendifferentials of X (for the action of the group) and the logarithm of special values of the Γ-function (recall that they are related to the logarithmic derivatives of Dirichlet L-functions at 0 via the Hurwitz formula). More precisely, our results are the following:

Let X be a smooth and projective variety together with an automorphism $g:X\to X$ of order n, everything being defined over a number field Q_0. Let us denote by $\mu_n(\mathbf{C})$ (resp. $\mu_n(\mathbf{C})^\times$) the group of n-th roots of unity (resp. the set of primitive n-th roots of unity) in $\mathbf{C}$. Suppose that Q_0 is chosen large enough so that it contains $\mathbf{Q}(\mu_n)$; and let $P_n(T)\in\mathbf{Q}[T]$ be the polynomial

$$P_n(T)=\sum_{\zeta\in\mu_n(\mathbf{C})^\times}\prod_{\xi\in\mu_n(\mathbf{C})\setminus\{\zeta\}}\frac{T-\xi}{\zeta-\xi}.$$

The submotive $\mathcal{X}(g)=\mathcal{X}(X,g)$ cut out in X by the projector $P_n(g)$ is endowed by construction with a natural $Q:=\mathbf{Q}(\mu_n)$-motive structure.

Theorem 1. *For all the odd primitive Dirichlet characters χ of $\mathbf{Q}(\mu_n)$, the conjecture* $\mathrm{B}(\mathcal{X}(g),\mathbf{Q}(\mu_n),\chi)$ *holds.*

Let now Q be a finite abelian extension of $\mathbf{Q}$ with conductor f_Q and let $\mathcal{M}_0$ be the motive associated to an abelian variety defined over Q_0 with (not necessarily maximal) complex multiplication by $\mathcal{O}_Q$. We suppose that the action of $\mathcal{O}_Q$ is defined over Q_0 and that $\mathbf{Q}(\mu_{f_Q})\subseteq Q_0$.

Theorem 2. *For all the odd Dirichlet characters χ of Q, the assertion* $\mathrm{B}(\mathcal{M}_0^1,\mathbf{Q}(\mu_{f_Q}),\chi)$ *holds.*

As a consequence of the existence of the Picard variety and of theorems 1 and 2, we get:

Corollary. *Let the hypothesies of Theorem 1 hold and suppose also that X is a surface. For all the odd primitive Dirichlet characters χ of $\mathbf{Q}(\mu_n)$, the conjecture* $\mathrm{B}(H^2(\mathcal{X}(X,g)),\mathbf{Q}(\mu_n),\chi)$ *holds.*

Our method of proof relies heavily on the arithmetic fixed point formula (equivariant arithmetic Riemann-Roch theorem) proved by K. Köhler and the second author in [13]. More precisely, we write down the fixed point formula as applied to the Dolbeault complex of a variety equipped with the action of a finite group. This yields a formula for some linear combinations of logarithms of periods of the variety in terms of derivatives of (partial) Lerch ζ-functions. Using the Hurwitz formula and some combinatorics, we can translate this into Theorems 1 and 2. In general the fixed point formula of [13], like the arithmetic Riemann-Roch theorem, contains an anomaly term, given by the equivariant Ray-Singer analytic torsion, which has proved to be difficult to compute explicitly. In the case of the Dolbeault complex, this anomaly term vanishes for simple symmetry reasons. It is this fact that permits us to conclude.
When $Q_0 = \overline{\mathbf{Q}}$, Q is an abelian extension of $\mathbf{Q}$ and $\mathcal{M}_0$ is an abelian variety with maximal complex multiplication by Q, the assertion $\mathrm{A}(\mathcal{M}_0^1,\chi)$ was proved by Anderson in [1], whereas the statement $\mathrm{A}(\mathcal{M}_0^1,\chi)$ had already been proven by Gross [11, Th. 3, Par. 3, p. 204] in the case where Q is an imaginary quadratic extension of $\mathbf{Q}$, $Q_0 = \overline{\mathbf{Q}}$ and $\mathcal{M}_0$ is an abelian variety with (not necessarily maximal) complex multiplication by Q. One could probably derive Theorem 2 from the results of Anderson, using the result of Deligne on absolute Hodge cycles on abelian varieties [7] (proved after the theorem of Gross and inspired by it), which can be used as a substitute of the Hodge conjecture in this context. In the case where $\mathcal{M}_0$ is an abelian variety with maximal complex multiplication and Q is an abelian extension of $\mathbf{Q}$, Colmez [4] proves a much more precise version of $\mathrm{A}(\mathcal{M}_0^1,\chi)$. He uses the Néron model of the abelian variety to normalise the periods so as to eliminate all the indeterminacy and proves an equation similar to Theorem 2 for those periods. A slightly weaker form of his result (but still much more precise than Theorem 2) can also be obtained from the arithmetic fixed point formula, when applied to the Néron models. This is carried out in [14]. Finally, let us mention that when $\mathcal{M}_0$ is the motive of a CM elliptic curve, Theorem 2 is just a weak form of the Chowla-Selberg formula [3]. For a historical introduction to those results, see [17, p. 123–125].
In the last section of the paper, we compare the conjecture A with the period conjecture of Gross-Deligne [11, Sec. 4, p. 205]. This conjecture is a translation into the language of Hodge structures of a special case of conjecture A, with Q an abelian extension of $\mathbf{Q}$. For example, we show the following: Theorem 1 implies that if S is a surface defined over $\overline{\mathbf{Q}}$ and if S is endowed with an action of an automorphism g of finite prime order p, then the natural embedding of the Hodge structure $\det_{\mathbf{Q}(\mu_p)}(H^2(\mathcal{X}(S,g),\mathbf{Q}))$ into $H(\times_{r=1}^{\dim_{\mathbf{Q}(\mu_p)} H^2(\mathcal{X}(S,g),\mathbf{Q})} S,\mathbf{Q})$ satisfies a weak form of the period conjecture. Notice that all the cases of the period conjecture which have been verified in the litterature correspond to motives cut out in varieties with a prescribed geometric structure giving rise to (coarse) moduli spaces (abelian varieties, Fermat hypersurfaces, Kuga-Sato varieties ...) whereas this result does not specify a geometric structure on S.
In the light of the application of the arithmetic fixed point formula to the conjectures A and B, it would be interesting to investigate whether this formula is related to the construction of the cycles whose existence (postulated by the Hodge conjecture) would be necessary to reduce the conjecture A to abelian varieties.

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2 Preliminaries

2.1 Invariance properties of the conjectures

Let Q_0 and Q be number fields taken as in the Introduction, and let H be a (homogenous) Hodge structure (over $\mathbf{Q}$). The $\mathbf{C}$-vector space $H_{\mathbf{C}} := H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$ comes with a natural Q_0-structure given by $H_{\mathbf{Q}} \otimes_{\mathbf{Q}} Q_0$. Suppose that $H_{\mathbf{C}}$ is endowed with another Q_0-structure. The first of these two Q_0-structures will be referred to as the Betti (or singular) one, and the second as the de Rham Q_0-structure on $H_{\mathbf{C}}$. Suppose furthermore that $H_{\mathbf{C}}$ is endowed with an additional Q-vector space structure compatible with both the Hodge structure and the (Betti and de Rham) Q_0-structures. This Q-structure induces an inner direct sum of $\mathbf{C}$-vector spaces $H_{\mathbf{C}} := \oplus_{\sigma\in\mathrm{Hom}(Q,\mathbf{C})} H_\sigma$. Let $V := \oplus_{\sigma\in\mathrm{Hom}(Q,\mathbf{C})} \det_{\mathbf{C}}(H_\sigma)$ and let $m := \dim_Q(H)$. There is an embedding $\iota : V \hookrightarrow \otimes_{k=1}^m H_{\mathbf{C}}$ given by $\iota(\oplus_\sigma v_1^\sigma \wedge\cdots\wedge v_m^\sigma) := \sum_\sigma \mathrm{Alt}(v_1^\sigma\otimes\cdots\otimes v_m^\sigma)$. Recall that Alt is the alternation map, described by the formula $\mathrm{Alt}(x_1\otimes\cdots\otimes x_m) := \frac{1}{m!}\sum_{\pi\in\mathfrak{S}_m}\mathrm{sign}(\pi)\pi(x_1\otimes\cdots\otimes x_m)$; here $\mathfrak{S}_m$ is the permutation group on m elements and π acts on $\otimes_{k=1}^m H_{\mathbf{C}}$ by permutation of the factors.

Lemma 2.1. *The space V inherits the Hodge structure as well as the Betti and de Rham Q_0-structures of $\otimes_{k=1}^m H_{\mathbf{C}}$ via the map ι.*

Proof: The bigrading of $H_{\mathbf{C}}$ is described by the weight of H and by an action $\upsilon : \mathbf{C}^\times \to \mathrm{End}_{\mathbf{C}}(H_{\mathbf{C}})$ of the complex torus $\mathbf{C}^\times$, which commutes with complex conjugation. The bigrading of $\otimes_{k=1}^m H_{\mathbf{C}}$ is described by the weight $m\cdot\mathrm{weight}(H)$ and the tensor product action $\upsilon^{\otimes m} : \mathbf{C}^\times \to \mathrm{End}_{\mathbf{C}}(\otimes_{k=1}^m H_{\mathbf{C}})$. On the other hand we can describe a bigrading on each $\det_{\mathbf{C}}(H_\sigma)$ by the weight $m\cdot\mathrm{weight}(H)$ and by the exterior product action. The map ι commutes with both actions by construction.
To prove that V inherits the Hodge $\mathbf{Q}$-structure, consider that there is an action by $\mathbf{Q}$-vector space automorphisms of $\mathrm{Aut}(\mathbf{C})$ on $\otimes_{k=1}^m H_{\mathbf{C}}$ given by $a((h_1\otimes z_1)\otimes\cdots\otimes(h_m\otimes z_m)) := (h_1\otimes a(z_1))\otimes\cdots\otimes(h_m\otimes a(z_m))$. An element t of $\otimes_{k=1}^m H_{\mathbf{C}}$ is defined over $\mathbf{Q}$ (for the Hodge $\mathbf{Q}$-structure) if and only if $a(t) = t$ for all $a\in\mathrm{Aut}(\mathbf{C})$. For each $\sigma\in\mathrm{Hom}(Q,\mathbf{C})$, let $b_1^\sigma,\dots,b_m^\sigma$ be a a basis of H_σ, which is defined over $\sigma(Q)$ and such that $a(b_i^\sigma) = b_i^{a(\sigma)}$ for all $a\in\mathrm{Aut}(\mathbf{C})$. This can be achieved by taking the conjugates under the action of $\mathrm{Aut}(\mathbf{C})$ of a given basis. Now choose a basis $c_1,\dots,c_{d_Q}$ of Q over $\mathbf{Q}$ and let $e_i := \sum_\sigma \sigma(c_i)b_1^\sigma\wedge\cdots\wedge b_m^\sigma$. By construction, the elements $\iota(e_1),\dots,\iota(e_{d_Q})$ are invariant under $\mathrm{Aut}(\mathbf{C})$ and they are linearly independent over $\mathbf{C}$, because the determinant of the transformation matrix from the basis $\{b_1^\sigma\wedge\cdots\wedge b_m^\sigma\}_\sigma$ to the basis formed by the e_i is the discriminant of the basis e_i over $\mathbf{Q}$. They thus define over V a $\mathbf{Q}$-structure $V_{\mathbf{Q}}$ which is compatible with the Hodge $\mathbf{Q}$-structure of $\otimes_{k=1}^m H_{\mathbf{C}}$. The Betti Q_0-structure on V is then just taken to be $V_{\mathbf{Q}}\otimes_{\mathbf{Q}} Q_0$.
To show that V inherits the de Rham Q_0-structure of $\otimes_{k=1}^m H_{\mathbf{C}}$, just notice that for each $\sigma\in\mathrm{Hom}(Q,\mathbf{C})$, the space H_σ as a basis $\alpha_1^\sigma,\dots,\alpha_m^\sigma$ defined over the de Rham Q_0-structure of $H_{\mathbf{C}}$. The elements $\alpha_1^\sigma\wedge\cdots\wedge\alpha_m^\sigma$ form a basis of V and $\iota(\alpha_1^\sigma\wedge\cdots\wedge\alpha_m^\sigma)$ is by construction defined over Q_0. $\square$
In view of the last lemma the complex vector space V arises from a (homogenous) Hodge structure over $\mathbf{Q}$ that we shall denote by $\det_Q(H)$. The embedding ι arises from an embedding of Hodge structures $\det_Q(H)\hookrightarrow\otimes_{k=1}^m H$ and $\det_Q(H)$ inherits a Betti and a de Rham Q_0-structure from this embedding. If $H_0 = \oplus_{w\in\mathbf{Z}}H_w$ is a direct sum of homogenous Hodge structures (graded by the weight), each of them satisfying the hypotheses of lemma 2.1, we extend the previous definition to H_0 by letting $\det_Q(H_0) := \oplus_{w\in\mathbf{Z}}\det_Q(H_w)$.

Proposition 2.2. *The assertion* $\mathrm{A}(\mathcal{M},\chi)$ *(resp.* $\mathrm{B}(\mathcal{M}_0,N,\chi)$*) is equivalent to the assertion* $\mathrm{A}(\det_Q(H(\mathcal{M},\mathbf{Q})),\chi)$ *(resp.* $\mathrm{B}(\det_Q(H(\mathcal{M}_0,\mathbf{Q})),N,\chi)$*).*

Proof: We examine both sides of the equality in the assertion $\mathrm{A}(H(\mathcal{M},\mathbf{Q}),\chi)$, when $H(\mathcal{M},\mathbf{Q})$ is replaced by $\det_Q(H(\mathcal{M},\mathbf{Q}))$. From the definition of $\det_Q(H(\mathcal{M},\mathbf{Q}))$, we see that the lefthand side is unchanged. As to the righthand side, it is sufficient to show that

$$\sum_{p+q=k} p\cdot \mathrm{rk}(H_\sigma^{p,q}) = \sum_{p+q=r\cdot k} p\cdot \mathrm{rk}(\det_{\mathbf{C}}(H_\sigma)^{p,q}),$$

where $r := \mathrm{rk}(H_\sigma)$, k is the weight of $\mathcal{M}$ and $H := H(\mathcal{M},\mathbf{Q})$. To prove it, we let $v_1,\dots,v_r$ be a basis of H_σ, which is homogenous for the grading. The last equality can then be rewritten as

$$\sum_{j=1}^{r} p_{\mathcal{H}}(v_j) = p_{\mathcal{H}}(v_1\wedge\cdots\wedge v_r)$$

(where $p_{\mathcal{H}}$ stands for the Hodge p-type) which holds from the definitions. The proof of the second equivalence runs along the same lines. $\square$

Let $\mathcal{M}$, $\mathcal{M}'$ be two Q-motives (over Q_0). We recall that the *tensor product with coefficients in* Q of $\mathcal{M}$ and $\mathcal{M}'$, written $\mathcal{M}\otimes_Q\mathcal{M}'$, is the the largest direct factor of the motive $\mathcal{M}\otimes\mathcal{M}'$ on which the left and the right action of Q on $\mathcal{M}\otimes\mathcal{M}'$ coincides (see [5, Par. 2, p. 320]). If E is a finite extension of Q and χ is a character of Q, recall that $\mathrm{Ind}_Q^E(\chi)$ is the character on E (the induced character) defined by the formula $\mathrm{Ind}_Q^E(\chi)(\sigma_E) := \chi(\sigma_E|_Q)$.

Proposition 2.3. *Let E be a finite extension of Q, such that all the embeddings of E are contained in Q_0. The statement* $\mathrm{A}(\mathcal{M}\otimes_Q E, \mathrm{Ind}_Q^E(\chi))$ *(resp.* $\mathrm{B}(\mathcal{M}_0\otimes_Q E, N, \mathrm{Ind}_Q^E(\chi))$*) holds if and only if* $\mathrm{A}(\mathcal{M},\chi)$ *(resp.* $\mathrm{B}(\mathcal{M}_0,N,\chi)$*) holds.*

Proof: Let r be the dimension of E over Q. The choice of a basis $x_1,\dots,x_r$ of E as a Q-vector space induces an isomorphism of Q-motives $\mathcal{M}\otimes_Q E\simeq\oplus_{j=1}^r\mathcal{M}$ and thus an isomorphism of $\mathbf{C}$-vector spaces

$$\bigoplus_{j=1}^{r} H(\mathcal{M},\mathbf{C}) \simeq H((\mathcal{M}\otimes_Q E),\mathbf{C})$$

which respects the Hodge structure and both Q_0-structures. Under this isomorphism, we also have a decomposition

$$\bigoplus_{j=1}^{r} H(\mathcal{M},\mathbf{C})_{\sigma_Q} \simeq \bigoplus_{\sigma_E|\sigma_Q} H((\mathcal{M}\otimes_Q E),\mathbf{C})_{\sigma_E}$$

where $\sigma_Q\in\mathrm{Hom}(Q,\mathbf{C})$ and the $\sigma_E\in\mathrm{Hom}(E,\mathbf{C})$ restrict to σ_Q. This decomposition again respects the Hodge structure and both Q_0-structures. We now compute the lefthand side of the equality predicted by $\mathrm{A}(\mathcal{M}_E := \mathcal{M}\otimes_Q E, \mathrm{Ind}_Q^E(\chi))$

$$\begin{aligned}
&\sum_{\sigma_E}\log|P_{\sigma_E}(\mathcal{M}_E)|^2\,\mathrm{Ind}_Q^E(\chi)(\sigma_E)\\
&\qquad = \sum_{\sigma_Q}\chi(\sigma_Q)\sum_{\sigma_E|\sigma_Q}\log|P_{\sigma_E}(\mathcal{M}_E)|^2\\
&\qquad = \sum_{\sigma_Q}\chi(\sigma_Q)\sum_{j=1}^{r}\log|P_{\sigma_Q}(\mathcal{M})|^2 = r\cdot\sum_{\sigma_Q}\log|P_{\sigma_Q}(\mathcal{M})|^2\,\chi(\sigma_Q)
\end{aligned}$$

As for the righthand side, we compute

$$\sum_{\sigma_E}\sum_{p,q} p\cdot \mathrm{rk}(H^{p,q}(\mathcal{M}_E,\mathbf{C})_{\sigma_E})\mathrm{Ind}_Q^E(\chi)(\sigma_E)$$

$$= \sum_{\sigma_Q}\sum_{p,q} p\cdot\chi(\sigma_Q)\,\mathrm{rk}(\oplus_{\sigma_E|\sigma_Q}H^{p,q}(\mathcal{M}_E,\mathbf{C})_{\sigma_E})$$

$$= \sum_{\sigma_Q}\sum_{p,q} p\cdot\chi(\sigma_Q)\cdot r\cdot \mathrm{rk}(H^{p,q}(\mathcal{M},\mathbf{C})_{\sigma_Q})$$

dividing both sides by r, we are reduced to the conjecture $\mathrm{A}(\mathcal{M},\chi)$. The proof of the second equivalence is similar. $\square$

2.2 The arithmetic fixed point formula

For the sake of completness and in order to fix notations, we shall review in this section the arithmetic fixed point formula proved by K. Köhler and the second author in [13]. Many results will be stated without proof; we refer to [13, Sec. 4] for more details and further references to the litterature.

Let D be a regular arithmetic ring, i.e. a regular, excellent, Noetherian integral ring, together with a finite set $\mathcal{S}$ of injective ring homomorphisms of $D\hookrightarrow\mathbf{C}$, which is invariant under complex conjugation. Let μ_n be the diagonalisable group scheme over D associated to the group $\mathbf{Z}/n$. An **equivariant arithmetic variety** $f:Y\to\mathrm{Spec}\,D$ is a regular integral scheme, endowed with a μ_n-action over $\mathrm{Spec}\,D$, such that there exists a μ_n-equivariant ample line bundle on Y. We write $Y(\mathbf{C})$ for the complex manifold $\coprod_{\sigma\in\mathcal{S}}Y\otimes_{\sigma(D)}\mathbf{C}$. The group $\mu_n(\mathbf{C})$ acts on $Y(\mathbf{C})$ by holomorphic automorphisms and we shall write g for the automorphism corresponding to a fixed primitive n-th root of unity $\zeta=\zeta(g)$. The subfunctor of fixed points of the functor associated to Y is representable and we call the representing scheme the **fixed point scheme** and denote it by Y_{μ_n}. It is regular and there are natural isomorphisms of complex manifolds $Y_{\mu_n}(\mathbf{C})\simeq Y(\mathbf{C})_g$, where $Y(\mathbf{C})_g$ is the set of fixed points of Y under the action of g. We write f^{μ_n} for the map $Y_{\mu_n}\to\mathrm{Spec}\,D$ induced by f. Complex conjugation of coefficients induces an antiholomorphic automorphism of $Y(\mathbf{C})$ and $Y_{\mu_n}(\mathbf{C})$, both of which we denote by F_∞. We write $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ for $\widetilde{\mathfrak{A}}(Y(\mathbf{C})_g):=\bigoplus_{p\geqslant 0}(\mathfrak{A}^{p,p}(Y(\mathbf{C})_g)/(\mathrm{Im}\,\partial+\mathrm{Im}\,\overline{\partial}))$, where $\mathfrak{A}^{p,p}(\cdot)$ denotes the set of smooth complex differential forms ω of type (p,p) such that $F_\infty^*\omega=(-1)^p\omega$.

A hermitian equivariant sheaf (resp. vector bundle) on Y is a coherent sheaf (resp. a vector bundle) E on Y, assumed locally free on $Y(\mathbf{C})$, endowed with a μ_n-action which lifts the action of μ_n on Y and a hermitian metric h on $E_{\mathbf{C}}$, the bundle associated to E on the complex points, which is invariant under F_∞ and μ_n. We shall write (E,h) or $\overline{E}$ for an hermitian equivariant sheaf (resp. vector bundle). There is a natural $(\mathbf{Z}/n)$-grading $E|_{Y_{\mu_n}}\simeq\oplus_{k\in\mathbf{Z}/n}E_k$ on the restriction of E to Y_{μ_n}, whose terms are orthogonal, because of the invariance of the metric. We write $\overline{E}_k$ for the k-th term ($k\in\mathbf{Z}/n$), endowed with the induced metric. We shall also write $\overline{E}_{\neq 0}$ for $\oplus_{k\in(\mathbf{Z}/n)\setminus\{0\}}\overline{E}_k$.

We write $\mathrm{ch}_g(\overline{E}):=\sum_{k\in\mathbf{Z}/n}\zeta(g)^k\mathrm{ch}(\overline{E}_k)$ for the equivariant Chern character form $\mathrm{ch}_g(E_{\mathbf{C}},h)$ associated to the restriction of $(E_{\mathbf{C}},h)$ to $Y_{\mu_n}(\mathbf{C})$. Recall also that $\mathrm{Td}_g(\overline{E})$ is the differential form $\mathrm{Td}(\overline{E}_0)\Big(\sum_{i\geqslant 0}(-1)^i\mathrm{ch}_g(\Lambda^i(\overline{E}_{\neq 0}))\Big)^{-1}$. If $\mathcal{E}:0\to E'\to E\to E''\to 0$ is an exact sequence of equivariant sheaves (resp. vector bundles), we shall write $\overline{\mathcal{E}}$ for the sequence $\mathcal{E}$ together with $\mu_n(\mathbf{C})$ and F_∞-invariant hermitian metrics on $E'_{\mathbf{C}}$, $E_{\mathbf{C}}$ and $E''_{\mathbf{C}}$. To $\overline{\mathcal{E}}$ and ch_g is associated an equivariant Bott-Chern secondary class $\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}})\in\widetilde{\mathfrak{A}}(Y_{\mu_n})$, which satisfies the equation $\frac{\overline{\partial}\partial}{2\pi i}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}})=\mathrm{ch}_g(\overline{E}')+\mathrm{ch}_g(\overline{E}'')-\mathrm{ch}_g(\overline{E})$.

Definition 2.4. *The arithmetic equivariant Grothendieck group* $\widehat{K}_0^{\mu_n}{}'(Y)$ *(resp.* $\widehat{K}_0^{\mu_n}(Y)$*) of* Y *is the free abelian group generated by the elements of* $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ *and by the equivariant isometry classes of hermitian equivariant sheaves (resp. vector bundles), together with the relations*

(i) *for every exact sequence* $\overline{\mathcal{E}}$ *as above,* $\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}}) = \overline{E}' - \overline{E} + \overline{E}''$*;*

(ii) *if* $\eta \in \widetilde{\mathfrak{A}}(Y_{\mu_n})$ *is the sum in* $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ *of two elements* η' *and* η''*, then* $\eta = \eta' + \eta''$ *in* $\widehat{K}_0^{\mu_n}{}'(Y)$ *(resp.* $\widehat{K}_0^{\mu_n}(Y)$*).*

We shall now define a product on $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$). Let $\overline{V}, \overline{V}'$ be hermitian equivariant sheaves (resp. vector bundles) and let η, η' be elements of $\widetilde{\mathfrak{A}}(Y_{\mu_n})$. We define a product $\cdot$ on the generators of $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$) by the rules $\overline{V} \cdot \overline{V}' := \overline{V} \otimes \overline{V}'$, $\overline{V} \cdot \eta = \eta \cdot \overline{V} := \mathrm{ch}_g(\overline{V}) \wedge \eta$ and $\eta \cdot \eta' := \frac{\overline{\partial}\partial}{2\pi i} \eta \wedge \eta'$ and we extend it by linearity. This product is compatible with the relations defining $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$) and defines a commutative ring structure on $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$).
Suppose now that f is proper. Fix a F_∞-invariant Kähler metric on $Y(\mathbf{C})$, with Kähler form ω_Y and suppose that $\mu_n(\mathbf{C})$ acts by isometries with respect to this Kähler metric. Let $\overline{E} := (E, h)$ be an equivariant hermitian sheaf on Y. We write $T_g(\overline{E})$ for the equivariant analytic torsion $T_g(E_{\mathbf{C}}, h) \in \mathbf{C}$ of $(E_{\mathbf{C}}, h)$ over $Y(\mathbf{C})$; see [12, Sec. 2] or subsection 2.3 for the definition. Let $f : Y \to \operatorname{Spec} D$ be the structure morphism. We let $R^i f_* \overline{E}$ be the i-th direct image sheaf of E endowed with its natural equivariant structure and L^2-metric (see [9, Par. 1.1, p. 22] for the definition of the latter). We also write $\overline{H^i(Y, E)}$ for $R^i f_* \overline{E}$ and $R^{\cdot} f_* \overline{E}$ for the linear combination $\sum_{i \geqslant 0} (-1)^i R^i f_* \overline{E}$. Let $\eta \in \widetilde{\mathfrak{A}}(Y_{\mu_n})$ and consider the rule which associates the element $R^{\cdot} f_* \overline{E} - T_g(\overline{E})$ of $\widehat{K}_0^{\mu_n}{}'(D)$ to $\overline{E}$ and the element $\int_{Y(\mathbf{C})_g} \mathrm{Td}_g(\overline{TY}) \eta \in \widehat{K}_0^{\mu_n}{}'(D)$ to η.

Proposition 2.5. *The above rule descends to a well defined group homomorphism* $f_* : \widehat{K}_0^{\mu_n}{}'(Y) \to \widehat{K}_0^{\mu_n}{}'(D)$.

One can show that $\widehat{K}_0^{\mu_n}(D)$ is isomorphic to $\widehat{K}_0^{\mu_n}{}'(D)$ via the natural map so that by composition the last proposition yields a map $\widehat{K}_0^{\mu_n}(Y) \to \widehat{K}_0^{\mu_n}(D)$, which we shall also call f_*.
Finally, to formulate the fixed point theorem, we define the homomorphism $\rho : \widehat{K}_0^{\mu_n}(Y) \to \widehat{K}_0^{\mu_n}(Y_{\mu_n})$, which is obtained by restricting all the involved objects from Y to Y_{μ_n}. If $\overline{E}$ is a hermitian vector bundle on Y, we write $\lambda_{-1}(\overline{E}) := \sum_{k=0}^{\mathrm{rk}(E)} (-1)^k \Lambda^k(\overline{E}) \in \widehat{K}_0^{\mu_n}(Y)$, where $\Lambda^k(\overline{E})$ is the k-th exterior power of $\overline{E}$, endowed with its natural hermitian and equivariant structure. If $\overline{E}$ is the orthogonal direct sum of two hermitian equivariant vector bundles $\overline{E}'$ and $\overline{E}''$, then $\lambda_{-1}(\overline{E}) = \lambda_{-1}(\overline{E}') \cdot \lambda_{-1}(\overline{E}'')$. Let $R(\mu_n)$ be the Grothendieck group of finitely generated projective μ_n-comodules over D. There are natural isomorphisms $R(\mu_n) \simeq K_0(D)[\mathbf{Z}/n] \simeq K_0(D)[T]/(1 - T^n)$. Let $\overline{I}$ be the μ_n-comodule whose term of degree 1 is D endowed with the trivial metric and whose other terms are 0. We make $\widehat{K}_0^{\mu_n}(D)$ an $R(\mu_n)$-algebra under the ring morphism which sends T to $\overline{I}$. In the next theorem, which is the arithmetic fixed point formula, let $\mathcal{R}$ be any $R(\mu_n)$-algebra such that the elements $1 - T^k$ $(k = 1, \ldots, n-1)$ are invertible in $\mathcal{R}$. The algebra which is minimal with respect to this property is the ring $R(\mu_n)_{\{1-T^k\}_{k=1,\ldots,n-1}}$, the localization of $R(\mu_n)$ at the multiplicative subset generated by the elements $\{1 - T^k\}_{k=1,\ldots,n-1}$.
Let now $\theta \in \mathbf{R}$. For all $s \in \mathbf{C}$ such that $\Re(s) > 1$ we define Lerch's partial ζ-functions $\zeta(\theta, s) := \sum_{n \geqslant 1} \frac{\cos(n\theta)}{n^s}$ and $\eta(\theta, s) := \sum_{n \geqslant 1} \frac{\sin(n\theta)}{n^s}$, and using analytic

continuation, we extend them to meromorphic functions of s over $\mathbf{C}$. Let $R(\theta,t)$ be the formal power series

$$\sum_{n\geqslant 1,\ n\ \mathbf{odd}}(2\zeta'(\theta,-n)+\sum_{j=1}^{n}\frac{\zeta(\theta,-n)}{j})\frac{t^n}{n!}+i\sum_{n\geqslant 0,\ n\ \mathbf{even}}(2\eta'(\theta,-n)+\sum_{j=1}^{n}\frac{\eta(\theta,-n)}{j})\frac{t^n}{n!}.$$

We shall need $R(\theta,(\cdot))$ which is the unique additive characteristic class on holomorphic vector bundles such that $R(\theta,L)=R(\theta,c_1(L))$ for each line bundle L. Let V be a μ_n-equivariant vector bundle on Y; we define

$$R_g(V):=\sum_{k=1}^{\mathrm{rk}(V)}R(\arg(\zeta(g)^k),V_k).$$

In the next theorem, we consider that the values of R_g lie in $\widetilde{\mathfrak{A}}(Y_{\mu_n})$.

Theorem 2.6. *Let $\overline{N}_{Y/Y_{\mu_n}}$ be the normal bundle of Y_{μ_n} in Y, endowed with its quotient equivariant structure and quotient metric structure (which is F_∞-invariant).*

(i) *The element $\Lambda:=\lambda_{-1}(\overline{N}^{\vee}_{Y/Y_{\mu_n}})$ has an inverse in $\widehat{K}_0^{\mu_n}(Y_{\mu_n})\otimes_{R(\mu_n)}\mathcal{R}$;*

(ii) *let $\Lambda_R:=\Lambda\cdot(1+R_g(N_{Y/Y_{\mu_n}}))$; the diagram*[2]

$$\begin{array}{ccc}\widehat{K}_0^{\mu_n}(Y) & \xrightarrow{\Lambda_R^{-1}\cdot\rho} & \widehat{K}_0^{\mu_n}(Y_{\mu_n})\otimes_{R(\mu_n)}\mathcal{R}\\ \downarrow f_* & & \downarrow f_*^{\mu_n}\\ \widehat{K}_0^{\mu_n}(D) & \xrightarrow{\mathrm{Id}\otimes 1} & \widehat{K}_0^{\mu_n}(D)\otimes_{R(\mu_n)}\mathcal{R}\end{array}$$

commutes.

The proof of this theorem is the object of [13], it combines the deformation to the normal cone technique with deep results of Bismut on the behaviour of equivariant analytic torsion under immersions [2].

2.3 The equivariant analytic torsion and the L^2-metric of the Dolbeault complex

In this subsection, we shall prove the vanishing of the equivariant analytic torsion for the Dolbeault complex. Before doing so, we shall review some results on the polarisation induced by an ample line bundle on the singular cohomology of a complex manifold.

So let M be a complex projective manifold of dimension d and L be an ample line bundle on M. Let us denote by $\omega\in H^2(M,\mathbf{C})$ the first Chern class of L and for $k\leqslant d$, let us note $P^k(M,\mathbf{C})\subseteq H^k(M,\mathbf{C})$ the primitive cohomology associated to ω; this is a Hodge substructure of $H^k(M,\mathbf{C})$. Recall that for any $k\geqslant 0$, the primitive decomposition theorem establishes an isomorphism

$$H^k(M,\mathbf{C})\simeq\oplus_{r\geqslant\max(k-d,0)}\,\omega^r\wedge P^{k-2r}(M,\mathbf{C}).$$

Define the cohomological star operator $*:H^k(M,\mathbf{C})\to H^{2d-k}(M,\mathbf{C})$ by the rule $*\,\omega^r\wedge\phi:=i^{p-q}(-1)^{(p+q)(p+q+1)/2}\frac{r!}{(d-p-q-r)!}\omega^{d-p-q-r}\wedge\phi)$ if ϕ is a primitive element

[2] Note that a misprint found its way into the statement [13, Th. 4.4] (= Th. 2.6). In [13, Th. 4.4] the term $\Lambda\cdot(1-R_g(N_{Y/Y_{\mu_n}}))$ has to be replaced by $\Lambda\cdot(1+R_g(N_{Y/Y_{\mu_n}}))$ in the expression for Λ_R.

of Hodge pure type (p,q) and extend it by additivity. We can now define a hermitian metric on $H^k(M,\mathbf{C})$ by the formula

$$(\nu,\eta)_L := \int_M \nu\wedge *\overline{\eta}$$

for any $\nu,\eta\in H^k(M,\mathbf{C})$. This metric is sometimes called the Hodge metric. The next lemma follows from the definition of the L^2-metric, Hodge's theorem on the representability of cohomology classes by harmonic forms and the Hodge-Kähler identities.

Lemma 2.7. *Endow M with a Kähler metric whose Kähler form represents the cohomology class of ω and equip the bundles $\Lambda^p(\Omega_M)$ with the corresponding metrics. Endow $\oplus_{p+q=k}H^q(\Lambda^p(\Omega_M))$ with the L^2-metric and $H^k(M,\mathbf{C})$ with the Hodge metric. The Hodge-de Rham isomorphism $H^k(M,\mathbf{C})\simeq\oplus_{p+q=k}H^q(\Lambda^p(\Omega_M))$ is an isometry.*

Suppose now that M is a complex compact Kähler manifold endowed with a unitary automorphism g, and let $\overline{E}$ be a hermitian holomorphic vector bundle on M which is equipped with a unitary lifting of the action of g. Let $\Box_q^E$ be the differential operator $(\overline{\partial}+\overline{\partial}^*)^2$ acting on the C^∞-sections of the bundle $\Lambda^q T^{*(0,1)}M\otimes E$. This space of sections is equipped with the L^2-metric and the operator $\Box_q^E$ is symmetric for that metric; we let $\mathrm{Sp}(\Box_q^E)\subseteq\mathbf{R}$ be the set of eigenvalues of $\Box_q^E$ (which is discrete and bounded from below) and we let $\mathrm{Eig}_q^E(\lambda)$ be the eigenspace associated to an eigenvalue λ (which is finite-dimensional). Define

$$Z(\overline{E},g,s) := \sum_{q\geqslant 1}(-1)^{q+1}q\sum_{\lambda\in\mathrm{Sp}(\Box_q^E)\setminus\{0\}}\mathrm{Tr}(g^*|_{\mathrm{Eig}_q^E(\lambda)})\lambda^{-s}$$

for $\Re(s)$ sufficiently large. The function $Z(\overline{E},g,s)$ has a meromorphic continuation to the whole plane, which is holomorphic around 0 (see [12]). By definition, the equivariant analytic torsion of $\overline{E}$ is given by $T_g(\overline{E}) := Z'(\overline{E},g,0)$.
The non-equivariant analog of the following lemma (whose proof is similar) can be found in [16].

Lemma 2.8. *Let M be a complex compact Kähler manifold and let g be a unitary automorphism of M. The identity*

$$\sum_{p\geqslant 0}(-1)^p T_g(\Lambda^p(\overline{\Omega}_M)) = 0$$

holds.

Proof: Recall the Hodge decomposition (see [19, Chap. IV, no. 3, Cor. 2])

$$A^{p,q}(M) = \mathcal{H}^{p,q}(M)\oplus\partial(A^{p-1,q}(M))\oplus\partial^*(A^{p+1,q}(M))$$

where $\mathcal{H}^{p,q}(M)$ are the harmonic forms for the usual Kodaira-Laplace operator $\Box_q = (\partial+\partial^*)^2 = (\overline{\partial}+\overline{\partial}^*)^2$ and $A^{p,q}(M)$ is the space of C^∞-differential forms of type (p,q) on M. Let us write $A_1^{p,q}(M)$ for $\partial(A^{p-1,q}(M))$ and $A_2^{p,q}(M)$ for $\partial^*(A^{p+1,q}(M))$. The map $\partial|_{A_2^{p,q}(M))}$ is an injection and its image is $A_1^{p+1,q}(M)$. Notice also that the operator $\Box_q$ commutes with ∂ and ∂^*. Notice as well that the C^∞-sections of $\Lambda^q T^{*(0,1)}M\otimes\Lambda^p(\Omega_M)$ correspond to the space $A^{p,q}(M)$ and that $\Box_q^{\Lambda^p(\Omega)} = \Box_q|_{A^{p,q}}$. For $\lambda\in\mathbf{R}^\times$, we write $L_\lambda^{p,q} = \mathrm{Ker}(\Box_q^{\Lambda^p(\Omega)}-\lambda)$, $L_{\lambda,1}^{p,q} = L_\lambda^{p,q}\cap A_1^{p,q}$ and $L_{\lambda,2}^{p,q} = L_\lambda^{p,q}\cap A_2^{p,q}$. We compute

$$\sum_{p\geqslant 0}(-1)^p\mathrm{Tr}(g^*|_{L_\lambda^{p,q}}) = \sum_{p\geqslant 0}(-1)^p[\mathrm{Tr}(g^*|_{L_{\lambda,1}^{p,q}}) + \mathrm{Tr}\,(g^*|_{L_{\lambda,2}^{p,q}})] = 0$$

and from this, we conclude that $\sum_{p\geqslant 0}(-1)^p Z(\Lambda^p(\overline{\Omega}_M),g,s)\equiv 0$. $\Box$

2.4 An invariant of equivariant arithmetic K_0-theory

From now on, we restrict ourselves to the case $D = Q_0$, where $Q_0 \overset{\iota_0}{\hookrightarrow} \mathbf{C}$ is a number field embedded in $\mathbf{C}$, and we fix a primitive n-th root of unity $\zeta := e^{2\pi i/n}$. We use this choice to identify the set $\mu_n(\mathbf{C})^\times$ of primitive n-th roots of unity with the Galois group $G := \mathrm{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q}) = \mathrm{Hom}(\mathbf{Q}(\mu_n), \mathbf{C})$. The ring morphism $R(\mu_n) \to \mathbf{Q}(\mu_n)$ which sends the generator T on ζ makes $\mathbf{Q}(\mu_n)$ an $R(\mu_n)$-algebra and allows us to take $\mathcal{R} := \mathbf{Q}(\mu_n)$. We let $\widehat{\mathrm{CH}}(Q_0)$ be the arithmetic Chow ring of Q_0 with the set of embeddings $\mathcal{S} := \{\iota_0, \overline{\iota}_0\}$, in the sense of Gillet-Soulé (see [8]). There is a natural isomorphism $\widehat{\mathrm{CH}}(Q_0) \simeq \mathbf{Z} \oplus \mathbf{R}/\log|Q_0^\times|$ and a ring isomorphism $\widehat{K}_0(Q_0) \simeq \widehat{\mathrm{CH}}(Q_0)$ given by the arithmetic Chern character $\widehat{\mathrm{ch}}$ (see [8]), the ring $\widehat{K}_0(Q_0)$ being defined similarly to the ring $\widehat{K}_0^{\mu_1}(Q_0)$, with $\mathfrak{A}^{p,p}(\cdot)$ replaced by the space $\mathfrak{A}_{\mathbf{R}}^{p,p}(\cdot)$ of real (not complex) differential forms of type (p,p). The ring structure on $\mathbf{Z} \oplus \mathbf{R}/\log|Q_0^\times|$ is given by the formula $(r \oplus x) \cdot (r' \oplus x') := (r \cdot r', r \cdot x' + r' \cdot x)$. On generators of $\widehat{K}_0(Q_0)$, the arithmetic Chern character is defined as follows: For $\overline{V}$ a hermitian vector bundle on $\mathrm{Spec}\, Q_0$, the arithmetic Chern character $\widehat{\mathrm{ch}}(\overline{V})$ is the element $\mathrm{rk}(V) \oplus (-\log||s||^2)$, where s is a non-vanishing section of $\det(V)$ and $||\cdot||$ is the norm on $\det(V)_{\mathbf{C}}$ induced by the metric on $V_{\mathbf{C}}$. For an element $\eta \in \mathfrak{A}_{\mathbf{R}}(\mathrm{Spec}\, Q_0) \simeq \mathbf{R}$, the arithmetic Chern character $\widehat{\mathrm{ch}}(\eta)$ is the element $0 \oplus \frac{1}{2}\eta$.

Let now $\aleph_0$ be the additive subgroup of $\mathbf{C}$ generated by the elements $z \cdot \log|q_0|$ where $q_0 \in Q_0^\times$ and $z \in \mathbf{Q}(\mu_n)$. We define $\widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0) := \mathbf{Q}(\mu_n) \oplus \mathbf{C}/\aleph_0$ and we define a ring structure on $\widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$ by the rule $(z,x) \cdot (z',x') := (z \cdot z', z \cdot x' + z' \cdot x)$. Notice that there is a natural ring morphism $\psi : \widehat{\mathrm{CH}}(Q_0) \to \widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$ and that there is a natural $\mathbf{Q}(\mu_n)$-module structure on $\widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$. Define a rule which associates elements of $\widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$ to generators of $\widehat{K}_0^{\mu_n}(Q_0)$ as follows. Associate the element $\zeta^k \cdot \psi(\widehat{\mathrm{ch}}(\overline{V}))$ to a μ_n-equivariant hermitian vector bundle $\overline{V}$ of pure degree k (for the natural $(\mathbf{Z}/n)$-grading) on $\mathrm{Spec}\, Q_0$; furthermore associate the element $0 \oplus \frac{1}{2}\eta$ to $\eta \in \widetilde{\mathfrak{A}}(\mathrm{Spec}\, Q_0) \simeq \mathbf{C}$.

Lemma 2.9. *The above rule extends to a morphism of $R(\mu_n)$-modules $\widehat{\mathrm{ch}}_{\mu_n} : \widehat{K}_0^{\mu_n}(Q_0) \to \widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$.*

Proof: Let

$$\mathcal{V}: \ 0 \to V' \to V \to V'' \to 0$$

be a an exact sequence of μ_n-equivariant vector bundles (μ_n-comodules) over Q_0. We endow the members of $\mathcal{V}$ with (conjugation invariant) hermitian metrics h', h and h'' respectively, such that the pieces of the various gradings are orthogonal. The equality

$$\widetilde{\mathrm{ch}}_\zeta(\overline{\mathcal{V}}) = \sum_{k \in \mathbf{Z}/n} \zeta^k \widetilde{\mathrm{ch}}(\overline{\mathcal{V}}_k)$$

holds (see [13, Th. 3.4, Par. 3.3]). From this and the well-definedness of the arithmetic Chern character, the result follows. □

We shall write $\widehat{c}^1_{\mu_n}$ for the second component of $\widehat{\mathrm{ch}}_{\mu_n}$, i.e. the component lying in $\mathbf{C}/\aleph_0$.

Lemma 2.10. *Let $\overline{V}$ be a hermitian μ_n-equivariant vector bundle on* $\mathrm{Spec}\, Q_0$*. The equation*

$$\prod_{l \in \mathbf{Z}/n} (1-\zeta^l)^{-\mathrm{rk}(V_l)} \widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V})) = 1 \oplus (-\sum_{l \in \mathbf{Z}/n} \frac{\zeta^l}{1-\zeta^l} \widehat{c}^1_{\mu_n}(\overline{V}_l)) \qquad (1)$$

holds.

Proof: We shall make use of the canonical isomorphism

$$\det(\Lambda^k(W)) \simeq \det(W)^{\otimes \frac{(r-1)!}{(r-k)!(k-1)!}}$$

valid for any vector space W of rank r over a field and any $1 \leqslant k \leqslant r$, and constructed as follows: For any basis $b_1, \dots, b_r$ of W, the element

$$\bigwedge_{1 \leqslant i_1 < \cdots < i_k \leqslant r} (b_{i_1} \wedge \cdots \wedge b_{i_k})$$

of $\det(\Lambda^k(W))$ is sent on the element

$$\bigotimes_{j=1}^{\frac{(r-1)!}{(r-k)!(k-1)!}} (b_1 \wedge \cdots \wedge b_r)$$

of $\det(W)^{\otimes \frac{(r-1)!}{(r-k)!(k-1)!}}$. This isomorphism is by construction invariant under base change to a field extension. Furthermore, if one applies the above description to the orthonormal basis of a vector space over $\mathbf{C}$ endowed with a hermitian metric, one find that this isomorphism is also an isometry for the natural metrics on both sides. Thus, for any hermitian vector bundle $\overline{W}$ over $\operatorname{Spec} Q_0$, the isomorphism of vector bundles

$$\det(\Lambda^k(\overline{W})) \simeq \det(\overline{W})^{\otimes \frac{(r-1)!}{(r-k)!(k-1)!}}$$

is an isometry. Using the definition of the ring structure of $\widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$ and the fact that $\widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V} \oplus \overline{V}')) = \widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V})) \cdot \widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V}'))$ for any two hermitian equivariant vector bundles over Q_0, we see that as functions of $\overline{V}$, both sides of the equality in (1) are multiplicative for direct sums of hermitian vector bundles. We are thus reduced to prove the equality

$$(1-\zeta^l)^{-\mathrm{rk}(V_l)} \widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V}_l)) = 1 \oplus \frac{-\zeta^l}{1-\zeta^l} \widehat{c}^{\,1}_{\mu_n}(\overline{V}_l)$$

for all $l \in \mathbf{Z}/n$. Let $r_l := \mathrm{rk}(V_l)$ and let us write $\widehat{c}_1(\cdot)$ for $\widehat{\mathrm{ch}}^{[1]}(\cdot)$; we compute

$$\begin{aligned}
\widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V}_l)) &= \sum_{k=0}^{r_l} (-1)^k \zeta^{lk} \widehat{\mathrm{ch}}(\Lambda^k(\overline{V}_l)) \\
&= \sum_{k=0}^{r_l} (-1)^k \zeta^{lk} \frac{r_l!}{k!(r_l-k)!} \oplus (\sum_{k=1}^{r_l} (-1)^k \zeta^{lk} \frac{(r_l-1)!}{(k-1)!(r_l-k)!}) \widehat{c}_1(\overline{V}_l).
\end{aligned}$$

Using the binomial formula, we see that the last expression can be rewritten as

$$(1-\zeta^l)^{r_l} \oplus (-\zeta^l(1-\zeta^l)^{r_l-1}) \widehat{c}_1(\overline{V}_l)$$

and the result follows. $\square$

3 Proof of Theorems 1 and 2

3.1 Two lemmas

For z belonging to the unit circle $\mathcal{S}_1$, we define the Lerch's ζ-function $\zeta_L(z,s) := \sum_{k \geqslant 1} \frac{z^k}{k^s}$ for $s \in \mathbf{C}$ such that $\Re(s) > 1$, and using analytic continuation, we extend it to a meromorphic function of s over $\mathbf{C}$.

Lemma 3.1. *Let Y be a scheme smooth over $\mathbf{C}$ and let E be a vector bundle on Y together with an automorphism $g : E \to E$ of finite order (which leaves the zero-section fixed). Let κ be the class*

$$\kappa := \mathrm{Td}(E_0) \frac{\sum_{p\geqslant 0}(-1)^p p \cdot \mathrm{ch}_g(\Lambda^p(E^\vee))}{\sum_{p\geqslant 0}(-1)^p \mathrm{ch}_g(\Lambda^p(E^\vee_{\neq 0}))}.$$

The equality

$$\kappa^{[l+\mathrm{rk}(E_0)]} = -c^{\mathrm{top}}(E_0) \sum_{z\in\mathcal{S}_1} \zeta_{\mathrm{L}}(z,-l)\, \mathrm{ch}^{[l]}(E_z^\vee)$$

holds.

Proof: According to the splitting principle, we may suppose that E is an equivariant direct sum of $r := \mathrm{rk}(E)$ line bundles F_i on which g acts by multiplication with the eigenvalue $\alpha_i^{-1} \in \mathcal{S}_1$. If we set $\gamma_i := c_1(F_i^\vee)$ for all i, we can write

$$\sum_{p\geqslant 0}(-1)^p p \cdot \mathrm{ch}_g(\wedge^p(E^\vee)) = \sum_{p\geqslant 0}(-1)^p p \sum_{1\leqslant i_1<\cdots<i_p\leqslant r} \alpha_{i_1}\ldots\alpha_{i_p} e^{\gamma_{i_1}+\cdots+\gamma_{i_p}} \tag{2}$$

If we take the formal derivative (for t) of the identity

$$\prod_{i=1}^r (1-\alpha_i e^{\gamma_i} t) = \sum_{p\geqslant 0}(-1)^p \left[\sum_{1\leqslant i_1<\cdots<i_p\leqslant r} \alpha_{i_1}\ldots\alpha_{i_p} e^{\gamma_{i_1}+\cdots+\gamma_{i_p}}\right] t^p$$

set $t=1$ and apply (2), we obtain

$$\sum_{p\geqslant 0}(-1)^p p \cdot \mathrm{ch}_g(\wedge^p(E^\vee)) = -\prod_{i=1}^r (1-\alpha_i e^{\gamma_i}) \sum_{j=1}^r \frac{\alpha_j e^{\gamma_j}}{1-\alpha_j e^{\gamma_j}}\,. \tag{3}$$

Notice now that we can write

$$\begin{aligned}\prod_{i=1}^r (1-\alpha_i e^{\gamma_i}) &= \prod_{\alpha_i\neq 1}(1-\alpha_i e^{\gamma_i}) \prod_{\alpha_i=1}(1-e^{\gamma_i}) \\ &= \sum_{p\geqslant 0}(-1)^p\, \mathrm{ch}_g(\wedge^p(E^\vee_{\neq 0}))\, \frac{c^{\mathrm{top}}(E_0)}{\mathrm{Td}(E_0)}\,,\end{aligned}$$

this together with (3) shows that

$$\kappa = -c^{\mathrm{top}}(E_0) \sum_{j=1}^r \frac{\alpha_j e^{\gamma_j}}{1-\alpha_j e^{\gamma_j}}\,. \tag{4}$$

Furthermore, notice that for any function $f(x)$ and any $k \in \mathbb{N}$, we have

$$\frac{d^k}{dt^k} f(\alpha e^t) = \left(\left[x\frac{d}{dx}\right]^k f\right)(\alpha e^t)$$

and (for $x \neq 1$)

$$\zeta_{\mathrm{L}}(x,-k) = \left[x\frac{d}{dx}\right]^k \zeta_{\mathrm{L}}(x,0)$$

and as well

$$\zeta_{\mathrm{L}}(x,0) = \frac{x}{1-x}$$

we deduce that (for $\alpha \neq 1$)

$$\frac{\alpha e^t}{1-\alpha e^t} = \sum_{p\geqslant 0} \zeta_{\mathrm{L}}(\alpha, -p)\frac{t^p}{p!}\,. \tag{5}$$

When $\alpha = 1$, we have the the classical expansion

$$\frac{e^t}{1-e^t} = -\frac{1}{t} + \sum_{p\geqslant 0} \zeta_{\mathbb{Q}}(-p)\frac{t^p}{p!}\,.$$

Using in (4) the formula just above or the formula (5) according to whether α_j is equal to 1 or not, we find that for all $l \geqslant 0$

$$\kappa^{[l+\mathrm{rk}(E_0)]} = -c^{\mathrm{top}}(E_0)\big[\sum_{z\in\mathcal{S}_1\backslash\{1\}} \zeta_{\mathrm{L}}(z,-l)\,\mathrm{ch}^{[l]}(E_z^\vee) + \zeta_{\mathbb{Q}}(-l)\,\mathrm{ch}^{[l]}(E_0^\vee)\big]$$

which, noting that $\zeta_{\mathrm{L}}(1,-l) = \zeta_{\mathbb{Q}}(-l)$, concludes the proof. $\square$

Caution. In what follows, in contradiction with classical usage and with the Introduction to this article, the notation $L(\chi, s)$ will always refer to the **non-primitive** L-function associated with a Dirichlet character χ. We shall write χ_{prim} for the primitive character associated with χ and accordingly write $L(\chi_{\mathrm{prim}}, s)$ for the associated primitive L-function.

The following lemma, proved in [14, Lemma 5.2, Sec. 5], establishes the link between Lerch ζ-functions and Dirichlet L-functions. It follows from the functional equation of Dirichlet L-functions when the character is primitive.

Lemma 3.2. *Let χ be an odd character of $G = \mathrm{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q})$. The equality*

$$\sum_{\sigma\in G} \eta(\sigma(\zeta), s)\,\overline{\chi(\sigma)} = n^{1-s}\frac{\Gamma(1-s/2)}{\Gamma((s+1)/2)}\pi^{s-1/2}L(\overline{\chi}, 1-s)$$

holds for all $s \in \mathbf{C}$.

If χ is a character of G, we shall write $\tau(\chi) := \sum_{\sigma\in G}\sigma(\zeta)\chi(\sigma)$ for the Gauss sum associated to χ. Recall that if χ is primitive (i.e. not induced from a subfield $\mathbf{Q}(\mu_m)$ with $m < n$) the following equation holds (see [18, Lemma 4.7, p. 36])

$$\sum_{\sigma\in G} \sigma(\zeta^l)\chi(\sigma) = \tau(\chi)\overline{\chi(l)} \tag{6}$$

where we used the identification $G \simeq (\mathbf{Z}/n)^\times$ via the choice $\zeta = e^{2\pi i/n}$ to give meaning to $\chi(l)$.

3.2 Computations

The notations of the sections 1 and 2, and the conventions of the subsection 2.4 are still in force. If $N^\times$ is a subgroup of $\mathbf{C}^\times$ and $z, z' \in \mathbf{C}$, we shall write $z \sim_{N^\times} z'$ if $z = \lambda\cdot z'$ with $\lambda \in N^\times$. Recall that $f : X \to \mathrm{Spec}\, Q_0$ is a smooth and projective variety acted upon by g an automorphism of order n (defined over Q_0). Suppose that Q_0 contains $\mathbf{Q}(\mu_n)$. Let L be a g-equivariant ample line bundle over X and endow $X(\mathbf{C})$ with a Kähler metric whose Kähler form represents the first Chern class of L. We will denote by $\overline{\Omega}$ the sheaf of differentials of f equipped with the induced metric.

We shall now prove Theorems 1 and 2. To do so, we first apply the arithmetic fixed point formula (Theorem 2.6) to the Dolbeault complex $\lambda_{-1}(\overline{\Omega})$.

$$
\begin{aligned}
f_*(\lambda_{-1}(\overline{\Omega})) &= T_g(\lambda_{-1}(\overline{\Omega})) \\
&\quad - \int_{X_{\mu_n}(\mathbf{C})} R_g(TX)\mathrm{Td}_g(TX)\mathrm{ch}_g(\lambda_{-1}(\Omega)) + f_*^{\mu_n}(\lambda_{-1}^{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}})\lambda_{-1}(\rho(\overline{\Omega}))) \\
&= T_g(\lambda_{-1}(\overline{\Omega})) - \int_{X(\mathbf{C})_g} R_g(TX)\mathrm{Td}(TX_g)\mathrm{ch}(\lambda_{-1}(TX_g^{\vee})) + f_*^{\mu_n}(\lambda_{-1}(\overline{\Omega}(f^{\mu_n}))) \\
&= T_g(\lambda_{-1}(\overline{\Omega})) - \int_{X(\mathbf{C})_g} R_g(TX)c^{\mathrm{top}}(TX_g) + f_*^{\mu_n}\big(\lambda_{-1}(\overline{\Omega}(f^{\mu_n}))\big)
\end{aligned}
$$

Applying $\widehat{c}^{1}_{\mu_n}(\cdot)$ to both sides of the last equality, we obtain

$$
\begin{aligned}
\widehat{c}^{1}_{\mu_n}(f_*(\lambda_{-1}(\overline{\Omega}))) &= T_g(\lambda_{-1}(\overline{\Omega})) \\
&\quad - \frac{1}{2}\int_{X(\mathbf{C})_g} R_g(TX)c^{\mathrm{top}}(TX_g) + \widehat{c}^{1}_{\mu_n}(f_*^{\mu_n}\big(\lambda_{-1}(\overline{\Omega}(f^{\mu_n}))\big))
\end{aligned} \tag{7}
$$

We deduce from lemma 2.8 that $T_g(\lambda_{-1}(\overline{\Omega})) = 0$. The following lemma shows that the third term in (7) likewise vanishes:

Lemma 3.3. *The equality* $\widehat{c}^{1}_{\mu_n}(f_*^{\mu_n}\big(\lambda_{-1}(\overline{\Omega}(f^{\mu_n}))\big)) = 0$ *holds.*

Proof: Let d_μ be the relative dimension of X_{μ_n} over $\operatorname{Spec} Q_0$. The expression $\widehat{c}^{1}_{\mu_n}(f_*^{\mu_n}\big(\lambda_{-1}(\overline{\Omega}(f^{\mu_n}))\big))$ can be subdivided into a linear combination of terms of the following kind:

$$
\widehat{c}^{1}_{\mu_n}(R^q f_*^{\mu_n}(\Lambda^p(\overline{\Omega}(f^{\mu_n})))) + \widehat{c}^{1}_{\mu_n}(R^{d_\mu - q} f_*^{\mu_n}(\Lambda^{d_\mu - p}(\overline{\Omega}(f^{\mu_n}))))
$$

By Serre duality, the spaces $R^q f_*^{\mu_n}(\Lambda^p(\Omega(f^{\mu_n})))$ and $R^{d_\mu - q} f_*^{\mu_n}(\Lambda^{d_\mu - p}(\Omega(f^{\mu_n})))$ are dual to each other, and even more, this duality is a duality of hermitian vector bundles (for the last statement, see [9]). Hence, from the definition of $\widehat{c}^{1}_{\mu_n}$, it follows that $\widehat{c}^{1}_{\mu_n}(R^q f_*^{\mu_n}(\Lambda^p(\overline{\Omega}(f^{\mu_n})))) = -\widehat{c}^{1}_{\mu_n}(R^{d_\mu - q} f_*^{\mu_n}(\Lambda^{d_\mu - p}(\overline{\Omega}(f^{\mu_n}))))$ which permits us to conclude. $\square$

We shall write $H_{\mathrm{Dlb}}(\overline{X})$ for $H_{\mathrm{Dlb}}(X)$ equipped with its natural L^2-metric. From the preceding discussion and (7), we are lead to the equality

$$
\sum_{k \geqslant 0} (-1)^k \widehat{c}^{1}_{\mu_n}(H^k_{\mathrm{Dlb}}(\overline{X})) = -\frac{1}{2}\int_{X(\mathbf{C})_g} R_g(TX)c^{\mathrm{top}}(TX_g) \tag{8}
$$

We shall now show that (8) implies Theorems 1 and 2. To do so, we will use lemma 3.2 to express derivatives of Lerch ζ-functions occuring in $R_g(TX)$ in terms of derivatives of Dirichlet L-functions, and then lemma 3.1 to give a global (cohomological) expression for the right side of (8).

Proof of Theorem 1. We compute

$$
\sum_{k \geqslant 0} (-1)^k \, \widehat{c}^{1}_{\mu_n}(H^k_{\mathrm{Dlb}}(\overline{X})) = \sum_{k \geqslant 0} (-1)^k \sum_{l \geqslant 0} \zeta^l \, \widehat{c}^{1}_{\mu_n}(H^k_{\mathrm{Dlb}}(\overline{X})_l)
$$

and

$$\sum_{\sigma\in G}\sum_{k\geqslant 0}(-1)^k\sum_{l\geqslant 0}\sigma(\zeta)^l\,\widehat{c}^1_{\mu_n}(H^k_{\mathrm{Dlb}}(\overline{X})_l)\chi(\sigma)$$
$$=\tau(\chi)\sum_{k\geqslant 0}(-1)^k\sum_{l\geqslant 0}\overline{\chi(l)}\,\widehat{c}^1_{\mu_n}(H^k_{\mathrm{Dlb}}(\overline{X})_l)$$
$$=-\tau(\chi)\int_{X(\mathbf{C})_g}L'(\overline{\chi},0)\sum_l\overline{\chi}(l)\,\mathrm{rk}(TX_l)c^{\mathrm{top}}(TX_g)$$
$$=-\tau(\chi)\int_{X(\mathbf{C})_g}\frac{L'(\overline{\chi},0)}{L(\overline{\chi},0)\tau(\chi)}\sum_{\sigma\in G}\chi(\sigma)\mathrm{Td}(TX_g)\frac{\sum_{p\geqslant 0}(-1)^p p\cdot\mathrm{ch}_{\sigma(\zeta)}(\Lambda^p(\Omega))}{\sum_{p\geqslant 0}(-1)^p\mathrm{ch}_{\sigma(\zeta)}(\Lambda^p(N^\vee))}$$
$$=-\tau(\chi)\frac{L'(\overline{\chi},0)}{L(\overline{\chi},0)}\sum_{\sigma\in G}\sum_{p,q}(-1)^{p+q}p\cdot\mathrm{rk}(H^{p,q}(X(\mathbf{C}))_\sigma)\chi(\sigma)$$

which is the result. For the first equality in the last string of equalities, we have used (6); for the second one, we have used (8) and applied (6) to the definition of Lerch's partial ζ-functions; for the third one, we used lemma 3.1; for the last equality, we have applied the holomorphic Lefschetz trace formula [10, 3.4, p. 422] to the virtual vector bundle $1-\Omega+2\cdot\Lambda^2(\Omega)-\cdots+(-1)^{\dim(X)}\dim(X)\cdot\Lambda^{\dim(X)}(\Omega)$. To conclude the proof, we still need the following lemma. Write $\mathcal{M}:=\mathcal{X}(X,g)$.

Lemma 3.4. *Let $\sigma\in\mathrm{Hom}(\mathbf{Q}(\mu_n),\mathbf{C})$. If $\omega_\sigma\in\det_{\mathbf{C}}H(\mathcal{M},\mathbf{C})_\sigma$ is non-zero and defined over Q_0 for the de Rham Q_0-structure of $H(\mathcal{M},\mathbf{C})$, then the equality $|P_\sigma(\mathcal{M})|^2\sim_{|Q_0^\times|}|\omega_\sigma|^2_{L^2}$ holds.*

Proof: Let z be a complex number such that $z\cdot\omega_\sigma$ is defined over the singular Q_0-structure of $\det_{\mathbf{C}}H(\mathcal{M},\mathbf{C})_\sigma$. Since $z\cdot\omega_\sigma$ is of pure Hodge type, the construction of the Hodge metric and lemma 2.7 shows that the number $|z\cdot\omega_\sigma|_{L^2}$ lies in $|Q_0^\times|$ and from this we can conclude. □

Proof of Theorem 2. By class-field theory and proposition 2.3, we are reduced to the case $Q=\mathbf{Q}(\mu_n)$. We may thus suppose that X is an abelian variety A with (not necessarily maximal) complex multiplication by $\mathcal{O}_{\mathbf{Q}(\mu_n)}$. Let $\overline{V}:=H^1_{\mathrm{Dlb}}(\overline{A})$. By the usual Lefschetz trace formula, the number of fixed points on $A(\mathbf{C})$ of the automorphism corresponding to $\zeta\in\mathcal{O}_{\mathbf{Q}(\mu_n)}$ is $\prod_{l\in\mathbf{Z}/n}(1-\zeta^l)^{\mathrm{rk}(V_l)}$ and therefore we get, using (8) and lemma 2.10

$$-\sum_l\frac{\zeta^l}{1-\zeta^l}\widehat{c}^1_{\mu_n}(\overline{V}_l)=-\frac{1}{2}R_g(TA|_0)=-\frac{1}{2}\sum_l 2i(\partial/\partial s)\eta(\arg(\zeta^l),0)\mathrm{rk}((TA|_0)|_l).$$

We shall write TA_l for $(TA|_0)_l$. Now notice that $\mathrm{Im}\frac{\zeta^l}{1-\zeta^l}=\eta(\arg(\zeta^l),0)$ and thus

$$2\sum_l\eta(\zeta^l,0)\widehat{c}^1_{\mu_n}(\overline{\Omega}_l)=\sum_l(\partial/\partial s)\eta(\arg(\zeta^l),0)\mathrm{rk}(TA_l).$$

Next we take the Fourier transform of both sides of the last equality for the action of $G=\mathrm{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q})$. We get:

$$-2\sum_l[\sum_{\sigma\in G}\eta(\arg(\sigma(\zeta)^l),0)\chi(\sigma)]\widehat{c}^1_{\mu_n}(\overline{\Omega}_l)=\sum_l[\sum_{\sigma\in G}(\partial/\partial s)\eta(\arg(\sigma(\zeta)^l),0)\chi(\sigma)]\mathrm{rk}(\Omega_l)$$

and changing variables

$$-2[\sum_{\sigma\in G}\eta(\arg(\sigma(\zeta)),0)\chi(\sigma)]\sum_l\overline{\chi}(l)\widehat{c}^1_{\mu_n}(\overline{\Omega}_l)$$
$$=[\sum_{\sigma\in G}(\partial/\partial s)\eta(\arg(\sigma(\zeta)),0)\chi(\sigma)]\sum_l\overline{\chi}(l)\mathrm{rk}(\Omega_l).$$

We now calculate, using lemma 3.2

$$
\begin{aligned}
&\frac{[\sum_{\sigma\in G}(\partial/\partial s)\eta(\arg(\sigma(\zeta)),0)\chi(\sigma)]}{[\sum_{\sigma\in G}\eta(\arg(\sigma(\zeta)),0)\chi(\sigma)]}\\
&\qquad = -\log(n)-\frac{1}{2}\Big(\frac{\Gamma'(1)}{\Gamma(1)}+\frac{\Gamma'(1/2)}{\Gamma(1/2)}\Big)+\log(\pi)-\frac{L'(\chi,1)}{L(\chi,1)}\\
&\qquad = -\log(n)-\frac{1}{2}\Big(\frac{\Gamma'(1)}{\Gamma(1)}+\frac{\Gamma'(1/2)}{\Gamma(1/2)}\Big)+\log(\pi)\\
&\qquad\qquad -\frac{L'(\chi_{\mathrm{prim}},1)}{L(\chi_{\mathrm{prim}},1)}-\sum_{p|n}\frac{\chi_{\mathrm{prim}}(p)}{p-\chi_{\mathrm{prim}}(p)}\log(p)\\
&\qquad = -\log(n)-\frac{1}{2}\Big(\frac{\Gamma'(1)}{\Gamma(1)}+\frac{\Gamma'(1/2)}{\Gamma(1/2)}\Big)+\log(\pi)-\log\Big(\frac{2\pi}{f_\chi}\Big)\\
&\qquad\qquad +\frac{\Gamma'(1)}{\Gamma(1)}+\frac{L'(\overline{\chi}_{\mathrm{prim}},0)}{L(\overline{\chi}_{\mathrm{prim}},0)}-\sum_{p|n}\frac{\chi_{\mathrm{prim}}(p)}{p-\chi_{\mathrm{prim}}(p)}\log(p)\\
&\qquad = \log\Big(\frac{f_\chi}{n}\Big)+\frac{L'(\overline{\chi}_{\mathrm{prim}},0)}{L(\overline{\chi}_{\mathrm{prim}},0)}-\sum_{p|n}\frac{\chi_{\mathrm{prim}}(p)}{p-\chi_{\mathrm{prim}}(p)}\log(p).
\end{aligned}
$$

Together with lemma 3.4, this allows us to conclude. We used the functional equation of primitive Dirichlet L-functions for the third equality. □

4 The period conjecture of Gross-Deligne

In this section, we shall indicate the consequences of Theorem 1 and Theorem 2 for the period conjecture of Gross-Deligne [11, Sec. 4, p. 205]. We first recall the latter conjecture. Let Q be a finite abelian extension of $\mathbf{Q}$ and let H be a rational and homogenous Hodge structure of dimension $[Q:\mathbf{Q}]$. Suppose that there is a morphism of rings $\iota: Q\hookrightarrow \mathrm{End}(H)$ (in other words, H has maximal complex multiplication by Q). Suppose also that H is embedded in the singular cohomology $H(X,\mathbf{Q})$ of a variety X defined over $\overline{\mathbf{Q}}$. We let f_Q be the conductor of Q and we choose an embedding of Q in $\mathbf{Q}(\mu_{f_Q})$ (this is possible by class-field theory). Choose an embedding $\varphi: Q\hookrightarrow\mathbf{C}$ and an isomorphism $\mathrm{Gal}(\mathbf{Q}(\mu_{f_Q})/\mathbf{Q})\simeq(\mathbf{Z}/f_Q)^\times$. There is a natural map $\mathrm{Gal}(\mathbf{Q}(\mu_{f_Q})/\mathbf{Q})\to\mathrm{Hom}(Q,\mathbf{C})$ given for each $\sigma\in\mathrm{Gal}(\mathbf{Q}(\mu_{f_Q})/\mathbf{Q})$ by $\varphi\circ\sigma|_Q$, and we thus obtain a map $(\mathbf{Z}/f_Q)^\times\to\mathrm{Hom}(Q,\mathbf{C})$. For each $u\in(\mathbf{Z}/f_Q)^\times$ let $\omega_u^H\in H\otimes_{\mathbf{Q}}\mathbf{C}$ be a non-vanishing element affording the embedding corresponding to u and defined over $\overline{\mathbf{Q}}$ for the de Rham $\overline{\mathbf{Q}}$-structure of $H(X,\mathbf{C})$ (such an element is well-defined up to multiplication by a non-zero algebraic number). We attach to ω_u^H a period $\mathrm{Per}(\omega_u^H):=v(\omega_u^H)$ where $v\in H^\vee$ is any (non-zero) element of the dual of the $\mathbf{Q}$-vector space H. The number $\mathrm{Per}(\omega_u^H)$ is independent of the choices of v and ω_u^H, up to multiplication by a non-zero algebraic number, and only depends on u and H. In the notation of the beginning of the Introduction, with $Q_0=\overline{\mathbf{Q}}$, we have $\mathrm{Per}(\omega_u^H)=1/P_u(H)$ (where we identify u with the corresponding embedding of Q after the equality sign). Let $(p(u),q(u))$ be the Hodge type of ω_u^H. By [6, Lemme 6.12], there exists a (non-unique) function $\epsilon^H:\mathbf{Z}/f_Q\to\mathbf{Q}$ which satisfies the equation

$$p(u)=\sum_{a\in\mathbf{Z}/f_Q}\epsilon^H(a)[u\cdot a/f_Q]$$

for all $u\in(\mathbf{Z}/f_Q)^\times$. Here $[\cdot]$ takes the fractional part. The following conjecture is formulated by Gross in [11, p. 205]; he indicates that the precise form of it was suggested to him by Deligne. This conjecture is related by Deligne to his conjecture on motives of rank 1 in [5, 8.9, p. 338].

Period conjecture. *Let $u \in (\mathbf{Z}/f_Q)^\times$. The relations*

$$|\mathrm{Per}(\omega_u^H)| \sim_{\overline{\mathbf{Q}}^\times} \prod_{a\in\mathbf{Z}/f_Q} \Gamma(1-\frac{a}{f_Q})^{\epsilon^H(a/u)} \tag{i}$$

$$\mathrm{Per}(\omega_u^H) \sim_{\overline{\mathbf{Q}}^\times} |\mathrm{Per}(\omega_u^H)| \tag{ii}$$

hold.

The relations (i) and (ii) can of course be condensed in the single relation $\mathrm{Per}(\omega_u^H) \sim_{\overline{\mathbf{Q}}^\times} \prod_{a\in\mathbf{Z}/f_Q} \Gamma(1-\frac{a}{f_Q})^{\epsilon^H(a/u)}$.
One can show that the period conjecture is independent of the choice of the function ϵ^H (see the appendix by Koblitz and Ogus to [5]).
We let Γ_{f_Q} be the algebraic extension of $\mathbf{Q}$ generated by the numbers

$$\prod_{a\in\mathbf{Z}/f_Q} \Gamma(1-\frac{a}{f_Q})^{\gamma(a/u)},$$

where γ runs on all the functions $\mathbf{Z}/f_Q \to \mathbf{Q}$ satisfying the equation

$$\sum_{a\in\mathbf{Z}/f_Q} \gamma(a)[u\cdot a/f_Q] = 0$$

for all $u \in (\mathbf{Z}/f_Q)^\times$.
Suppose now that $Q = \mathbf{Q}(\mu_p)$ for a prime number p and let $Q_0 \subseteq \overline{\mathbf{Q}}$ be a field of definition of X containing Q and Γ_p. Suppose that H inherits the de Rham Q_0-structure of $H(X,\mathbf{C})$ and that its Q-vector space structure is compatible with that Q_0-structure. Suppose furthermore that all the ω_u^H are defined over Q_0 (for the de Rham Q_0-structure) and that $\mathrm{Per}(\omega_u^H) \sim_{Q_0^\times} \frac{1}{\mathrm{Per}(\omega_{-u}^H)}$ for all u.
If χ is an Artin character, we define $\mathbf{Q}(\chi)$ as the field generated over $\mathbf{Q}$ by the values of χ. We denote by E the compositum in $\mathbf{C}$ of the field $\mathbf{Q}(\mu_p)$ and of the fields $\mathbf{Q}(\chi)$ for all the odd Artin characters χ of $\mathbf{Q}(\mu_p)$, and we let $E^+ := E \cap \mathbf{R}$.

Lemma 4.1. *If the conjecture* $\mathrm{B}(H,E,\chi)$ *holds for all the odd Artin characters χ of $Q = \mathbf{Q}(\mu_p)$, then the identity*

$$\log|\mathrm{Per}(\omega_u^H)| = \log|\prod_{a\in\mathbf{Z}/p} \Gamma(1-\frac{a}{p})^{\epsilon^H(a/u)}| + \sum_i b_{u,i}\log|a_{u,i}|$$

holds for all u. Here $b_{u,i} \in E^+$, $a_{u,i} \in Q_0$ and i runs over a finite set of indices.

Proof: We can plainly assume that $\epsilon^H(0) = 0$. We write

$$\sum_u(\log|\mathrm{Per}(\omega_u^H)| - \sum_i b_{u,i}\log|a_{u,i}|)\chi(u) = \sum_u \chi(u)\sum_a \log(\Gamma(1-a/p))\epsilon(a/u)$$
$$= \sum_a \log(\Gamma(1-a/p))\sum_u \chi(u)\epsilon(a/u) = \sum_a \log(\Gamma(1-a/p)\chi(a)\sum_u \chi(u)\epsilon(1/u).$$

Now by the definition of ϵ

$$\sum_u \chi(u)p(u) = \sum_a [a/p]\sum_u \chi(u)\epsilon(a/u) = (\sum_a \chi(a)[a/p])(\sum_u \chi(u)\epsilon(1/u))$$

and thus

$$\sum_u (\log|\mathrm{Per}(\omega_u^H)| - \sum_i b_{u,i}\log|a_{u,i}|)\chi(u) = (\sum_a \log(\Gamma(1-a/p))\chi(a))\frac{\sum_u \chi(u)p(u)}{\sum_a \chi(a)[a/p]}$$

By Hurwitz's formula, this equals

$$(\frac{L'(\chi,0)}{L(\chi,0)} + \log(p))\sum_u \chi(u)p(u)$$
$$= \frac{L'(\chi_{\mathrm{prim}},0)}{L(\chi_{\mathrm{prim}},0)}\sum_u \chi(u)p(u) + (\log(p) + \frac{\chi_{\mathrm{prim}}(p)}{1-\chi_{\mathrm{prim}}(p)}\log(p))\sum_u \chi(u)p(u)$$

which is the result. □

Suppose moreover that X is acted upon by an automorphism g (defined over Q_0) of order p, and let $d_k := \dim_Q(H^k(\mathcal{X}(g),\mathbf{Q}))$ for any $k \geqslant 0$. Arising from the Künneth isomorphism $H(\times_{j=1}^{d_k} X, \mathbf{Q}) \simeq \otimes_{j=1}^{d_k} H(X,\mathbf{Q})$ and lemma 2.1, there is a natural embedding of Hodge structures $\det_Q(H^k(\mathcal{X}(g),\mathbf{Q})) \hookrightarrow H(\times_{j=1}^{d_k} X,\mathbf{Q})$ which respects the de Rham Q_0-structures.
The next two results follow from Theorem 1 and lemma 4.1.

Corollary 4.2. *Suppose that there is at most one k such that $0 \leqslant k \leqslant \dim(X)$ and $d_k \neq 0$. The natural embedding of Hodge structures $\det_Q(H^k(\mathcal{X}(g),\mathbf{Q})) \hookrightarrow H(\times_{j=1}^{d_k} X,\mathbf{Q})$ satisfies the hypothesis of the period conjecture and the identity*

$$\log|\mathrm{Per}(\omega_u^{\det_Q(H^k(\mathcal{X}(g),\mathbf{Q}))})| = \log|\prod_{a\in\mathbf{Z}/p}\Gamma(1-\frac{a}{p})^{\epsilon^{\det_Q(H^k(\mathcal{X}(g),\mathbf{Q}))}(a/u)}| + \sum_i b_{u,i}\log|a_{u,i}|$$

holds for all u, where $b_{u,i} \in E^+$, $a_{u,i} \in Q_0$ and i runs over a finite set of indices.

Notice that in particular the last corollary applies if X is a hypersurface of a projective space which carries an automorphism extending g.

Corollary 4.3. *If X is a surface, the natural embedding of Hodge structures $\det_Q(H^2(\mathcal{X}(g),\mathbf{Q})) \hookrightarrow H(\times_{j=1}^{d_2} X,\mathbf{Q})$ satisfies the hypothesis of the period conjecture and the identity*

$$\log|\mathrm{Per}(\omega_u^{\det_Q(H^2(\mathcal{X}(g),\mathbf{Q}))})| = \log|\prod_{a\in\mathbf{Z}/p}\Gamma(1-\frac{a}{p})^{\epsilon^{\det_Q(H^2(\mathcal{X}(g),\mathbf{Q}))}(a/u)}| + \sum_i b_{u,i}\log|a_{u,i}|$$

holds for all u, where $b_{u,i} \in E^+$, $a_{u,i} \in Q_0$ and i runs over a finite set of indices.

Proof: The conjecture $\mathrm{B}(H^1(\mathcal{X}(g),\mathbf{Q}),E,\chi)$ is verified for all the odd Artin characters χ of $\mathbf{Q}(\mu_p)$ because of the existence of the Picard variety and Theorem 2. The lemma 4.1 now implies that the conjecture is verified for $H^2(\mathcal{X}(g),\mathbf{Q})$. □

Notice that when $p = 3$ in either of the last corollaries, the item (i) of the period conjecture holds for the described embedding of Hodge structures (for all $u \in (\mathbf{Z}/p)^\times$), since $E^+ = \mathbf{Q}$ in that case.
Remark. The (ii) of the period conjecture seems to be out of the reach of the techniques developped in the present paper.

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