# ON THE COHOMOLOGY OF $p$-ADIC ANALYTIC SPACES, I: THE BASIC COMPARISON THEOREM.

PIERRE COLMEZ AND WIESŁAWA NIZIOŁ

Abstract. The purpose of this paper is to prove a basic $p$-adic comparison theorem for rigid analytic and dagger varieties over the algebraic closure $C$ of a $p$-adic field: $p$-adic pro-étale cohomology, in a stable range, can be expressed as a filtered Frobenius eigenspace of de Rham cohomology (over $\mathbf{B}^+_{\mathrm{dR}}$). The key computation is the passage from absolute crystalline cohomology to Hyodo-Kato cohomology and the construction of the related Hyodo-Kato isomorphism. We also "geometrize" our comparison theorem by turning $p$-adic pro-étale and syntomic cohomologies into sheaves on the category $\mathrm{Perf}_C$ of perfectoid spaces over $C$ (this geometrization will be crucial in our proof of the $C_{\mathrm{st}}$-conjecture for holomorphically convex varieties in the sequel to this paper).

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## 1. Introduction

Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field $K$ of characteristic 0 and with perfect residue field $k$ of characteristic $p$. Let $\overline{K}$ be an algebraic closure of $K$, let $C$ be its $p$-adic completion, and let $\mathscr{O}_{\overline{K}}$ denote the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Let $W(k)$ be the ring of Witt vectors of $k$ with fraction field $F$ (i.e, $W(k)=\mathscr{O}_F$) and let $\varphi$ be the absolute Frobenius on $W(\overline{k})$. Set $\mathscr{G}_K=\operatorname{Gal}(\overline{K}/K)$.

In a joint work with Gabriel Dospinescu [13], [14] we have computed the $p$-adic (pro-)étale cohomology of certain $p$-adic symmetric spaces. A key ingredient of these computations was a one-way (de Rham-to-étale) comparison theorem for rigid analytic Stein varieties over $K$ with a semistable formal model over $\mathscr{O}_K$. This theorem had two parts: first, it related (pro-)étale cohomology to rigid analytic syntomic cohomology and, then, it expressed rigid analytic syntomic cohomology as a filtered Frobenius eigenspace associated to de Rham cohomology (tensored with $\mathbf{B}^+_{\mathrm{dR}}$). From these two parts it is the second one that had much harder proof.

The current paper is the second one in a series extending such comparison theorems to smooth rigid analytic varieties over $K$ or $C$ (without any assumption on the existence of a nice integral model). While in the first paper [16] we have focused on the arithmetic case, here we focus on the geometric case. Moreover, in comparison with [13] and [16], we significantly simplify the passage from rigid analytic syntomic cohomology to a filtered Frobenius eigenspace associated to $\mathbf{B}^+_{\mathrm{dR}}$-cohomology[1]. This requires a foundational work on Hyodo-Kato cohomology and Hyodo-Kato morphism, which occupies a good portion of this paper.

In [17], the third paper in the series, we will use the results of this paper to prove the $C_{\mathrm{st}}$-conjecture for smooth holomorphically convex varieties over $C$ (hopefully, this conjecture should hold for general smooth partially proper varieties). This includes a description of the $\mathbf{B}^+_{\mathrm{dR}}$-cohomology (with its extra-structures, namely Frobenius and monodromy) in terms of the $p$-adic pro-étale cohomology and, conversely, a description of the $p$-adic pro-étale cohomology in terms of differential forms (the $\mathbf{B}^+_{\mathrm{dR}}$-cohomology and the de Rham complex). To this end, we need to "geometrize" the comparison isomorphisms proved in this paper, i.e., view them as $C$-points of isomorphisms between Vector Spaces.

### 1.1. **Main results.**

1.1.1. *The basic comparison theorem for rigid analytic varieties.* We start the survey of our main results with the following comparison theorem:

**Theorem 1.1.** (Basic comparison theorem) *Let $X$ be a smooth rigid analytic variety over $C$. Let $r\geq 0$. There is a natural strict quasi-isomorphism*[2] (period isomorphism)*:*

$$\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))\simeq\tau_{\leq r}\big[[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r\big],\tag{1.2}$$

*where the brackets* $[...]$ *denote the homotopy fiber.*

[1] If the variety is defined over $K$, its $\mathbf{B}^+_{\mathrm{dR}}$-cohomology is just de Rham cohomology tensored with $\mathbf{B}^+_{\mathrm{dR}}$.

[2] All cohomology complexes live in the bounded below derived $\infty$-category of locally convex topological vector spaces over $\mathbf{Q}_p$. Quasi-isomorphisms in this category we call *strict quasi-isomorphisms*.

Most of the paper is devoted to the definition of the objects appearing in (1.2) as well as the period morphism itself. This can be summed up in the following theorem-construction from which Theorem 1.1 follows immediately. As before in [14], [16], there are two steps: passage from pro-étale cohomology to syntomic cohomology (easier) and a passage from syntomic cohomology to Frobenius eigenspaces of de Rham cohomology over $\mathbf{B}^+_{\mathrm{dR}}$ (more difficult).

**Theorem 1.3.** *To any smooth rigid analytic variety $X$ over $C$ there are naturally associated:*

(1) *A (rigid analytic) syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))$, $r\in\mathbf{N}$, with a natural period morphism*

$$\alpha_r:\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)),\tag{1.4}$$

*which is a strict quasi-isomorphism after truncation $\tau_{\leq r}$.*

(2) *A Hyodo-Kato cohomology $\mathrm{R}\Gamma_{\mathrm{HK}}(X)$. This is a dg $F^{\mathrm{nr}}$-algebra equipped with a Frobenius $\varphi$ and a monodromy operator $N$. We have natural Hyodo-Kato strict quasi-isomorphisms*

$$\iota_{\mathrm{HK}}:\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}C\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X),\quad \iota_{\mathrm{HK}}:\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}).$$

(3) *A distinguished triangle*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\longrightarrow[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r\tag{1.5}$$

*that can be lifted to the derived category of Vector Spaces.*

1.1.2. *Dagger varieties.* Set

$$\mathrm{HK}^i_r(X):=H^i[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r},\quad \mathrm{DR}^i_r(X):=H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r).$$

The distinguished triangle (1.5) yields a long exact sequence of cohomology groups

$$\cdots\to\mathrm{DR}^{i-1}_r(X)\to H^i_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\longrightarrow\mathrm{HK}^i_r(X)\xrightarrow{\iota_{\mathrm{HK}}}\mathrm{DR}^i_r(X)\to\cdots,\tag{1.6}$$

which, together with the period isomorphism

$$H^i_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\xrightarrow{\sim}H^i_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)),\quad i\leq r,$$

obtained from (1.4), is a starting point for our work on generalizations of the $C_{\mathrm{st}}$-conjecture to rigid analytic varieties (see the sequel to this paper [17]). This sequence is, however, difficult to use since, locally, the rigid analytic de Rham cohomology and Hyodo-Kato cohomology are, in general, very ugly: infinite dimensional and not Hausdorff. But we are mainly interested in partially proper rigid analytic varieties and these varieties have a canonical overconvergent (or dagger) structure[3]. Moreover, a dagger affinoid has de Rham cohomology that is a finite rank vector space with its natural Hausdorff topology.

Hence we are led to study dagger varieties. We prove an analog of Theorem 1.3 for smooth dagger varieties. The dagger version of (1.6) is the long exact sequence:

$$\cdots\to\mathrm{DR}^{i-1}_r(X)\to H^i_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\longrightarrow(H^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}}\mathrm{DR}^i_r(X)\to\cdots$$

But now, if $X$ is a dagger affinoid, both cohomologies $H^i_{\mathrm{HK}}(X)$ and $H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ are (free) of finite rank. If $X$ is a dagger variety the overconvergent constructions are compatible with the rigid analytic constructions for $\widehat{X}$, the completion of $X$. If $X$ is partially proper the two sets of constructions are strictly quasi-isomorphic.

[3]Recall that a dagger variety is a rigid analytic variety equipped with an overconvergent structure sheaf. See [23] for the basic definitions and properties.

1.1.3. *Geometrization.* We show in [17] that the above long exact sequence (1.6), in a stable range, splits into short exact sequences if $X$ is proper or, more generally, dagger quasi-compact, or if $X$ is Stein or, more generally, holomorphically convex (proper over a Stein base). In order to do so, we need to put some extra-structure on the terms of the exact sequence. In [15], we treated the proper case (with a semi-stable model) by using the fact that the terms in the exact sequence outside of the $H^i_{\rm syn}(X,\mathbf{Q}_p(r))$'s were naturally $C$-points of Banach-Colmez spaces (called BC's in what follows). That this is also the case of the $H^i_{\rm syn}(X,\mathbf{Q}_p(r))$'s, for $i\leq r$, follows from the comparison with pro-étale cohomology and Scholze's theorem [38] which states that these cohomology groups are in fact finite dimensional over $\mathbf{Q}_p$ and independent of the field $C$: hence they are the $C$ points of quite trivial BC's. Then the basic theory of BC's [11, 12] could be used to show that the long exact sequence splits in a stable range. (Actually, putting a BC structure on syntomic cohomology can be done directly [36], but to prove the splitting of (1.6), one still needs Scholze's finiteness theorem, if one is to stick to the methods of [15]).

In our present situation, the $H^i_{\rm pro\acute{e}t}(X,\mathbf{Q}_p(r))$'s are very much not finite dimensional over $\mathbf{Q}_p$ and depend on the field $C$. Hence they are not obviously $C$-points of anything sensible. But one can turn them into $C$ points of sheaves on ${\rm Perf}_C$, and this is a category of geometric objects (the category of Vector Spaces, VS's for short) that contains naturally the category of BC's as was advocated in Le Bras' thesis [30].

One turns the $p$-adic pro-étale cohomology into a sheaf on ${\rm Perf}_C$ by taking the sheaf associated to the presheaf $S\mapsto {\rm R}\Gamma_{\rm pro\acute{e}t}(X_S,\mathbf{Q}_p(r))$, for perfectoid algebras $S$ over $C$. Likewise, one geometrizes syntomic cohomology by geometrizing the period rings; for example, $\mathbf{B}_{\rm cr}$ becomes the functor $S\mapsto \mathbf{B}_{\rm cris}(S)$. We extend the proof of Theorem 1.3 to this geometrized context to obtain:

**Theorem 1.7.** *The quasi-isomorphisms from Theorem 1.1 and (1) of Theorem 1.3 are the evaluations on* ${\rm Spa}(C,\mathscr{O}_C)$ *of quasi-isomorphisms of Vector Spaces.*

This promotes the exact sequence (1.6) to a sequence of VS's which can be analyzed using the geometric point of view on BC's developed in [30] (this analysis is quite involved and is postponed to [17]).

1.2. **Proof of Theorems 1.1 and 1.3.** We will now sketch how Theorem 1.1 and Theorem 1.3 are proved.

(i) *Rigid-analytic varieties.* Recall that [16, Sec. 2], using the rigid analytic étale local alterations of Hartl and Temkin [28], [40], one can equip the étale topology of $X$ with a (Beilinson) base[4] consisting of semistable formal schemes (always assumed to be of finite type) over $\mathscr{O}_C$. This allows us to define sheaves by specifying them on such integral models and then sheafifying for the $\eta$-étale topology[5]. For example, in (1) the syntomic cohomology ${\rm R}\Gamma_{\rm syn}(X,\mathbf{Q}_p(r))$ of a rigid analytic variety $X$ is defined by $\eta$-étale descent from the crystalline syntomic cohomology of Fontaine-Messing. Recall that the latter is defined as the homotopy fiber ($\mathscr{X}$ is a semistable formal scheme over $\mathscr{O}_C$ equipped with its canonical log-structure)

$${\rm R}\Gamma_{\rm syn}(\mathscr{X},\mathbf{Q}_p(r)):=[F^r{\rm R}\Gamma_{\rm cr}(\mathscr{X})\xrightarrow{\varphi-p^r}{\rm R}\Gamma_{\rm cr}(\mathscr{X})],$$

where the (logarithmic) crystalline cohomology is absolute (i.e., over $\mathbf{Z}_p$). By definition, it fits into the distinguished triangle

$$(1.8)\qquad {\rm R}\Gamma_{\rm syn}(X,\mathbf{Q}_p(1))\to[{\rm R}\Gamma_{\rm cr}(X)]^{\varphi=p^r}\to{\rm R}\Gamma_{\rm cr}(X)/F^r,$$

which looks different than the triangle (1.5) that we want in (3). However, we easily find[6] that ${\rm R}\Gamma_{\rm cr}(X)/F^r\simeq {\rm R}\Gamma_{\rm dR}(X/\mathbf{B}^+_{\rm dR})/F^r$. Here ${\rm R}\Gamma_{\rm dR}(X/\mathbf{B}^+_{\rm dR})$ is the $\mathbf{B}^+_{\rm dR}$-cohomology as defined by Bhatt-Morrow-Scholze in

[4]This should be distinguished from a Verdier base; in a Beilinson base the condition on fullness of the base morphisms is dropped. See [16, 2.1].

[5]Here $\eta$-étale means topology induced from the étale topology of the rigid analytic generic fiber.

[6]The easiest way to see it is by interpreting, locally, both sides as derived de Rham cohomology.

[7], which we have redefined in the paper as $\eta$-étale descent of Hodge-completed rational absolute crystalline cohomology of semistable schemes. But the construction of an isomorphism between the middle terms in (1.8) and (1.5) requires a refined version of the Hyodo-Kato morphism.

The period map in (1), is defined by $\eta$-étale descent of Fontaine-Messing period map

$$\alpha_r : \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{ét}}(\mathscr{X}_C, \mathbf{Q}_p(r)),$$

for a semistable formal scheme $\mathscr{X}$ over $\mathscr{O}_C$. The fact that it is a strict quasi-isomorphism in a stable range follows from the computations of $p$-adic nearby cycles via syntomic complexes done by Tsuji in [41]. However, to lift it to the derived category of Vector Spaces we use its reinterpretation via $(\varphi, \Gamma)$-modules by Colmez-Nizioł and Gilles in [15], [22]. This new interpretation of the period morphism is then lifted from $C$ to perfectoid spaces over $C$ to prove Theorem 1.7.

The construction of the Hyodo-Kato morphism in (2) is quite involved; in fact, a detailed study of Hyodo-Kato cohomology and its relation to $\mathbf{B}^+_{\mathrm{dR}}$- and de Rham cohomologies occupies a large portion of this paper. The original Hyodo-Kato morphism [29] works for semistable (formal) schemes. It can not be transferred to rigid analytic varieties because, a priori, it is dependent on the choice of the uniformizer of the base field (which varies for local semistable models). Moreover, a key map in the construction[7] is defined as an element of the classsical derived category. A more careful data keeping allowed Beilinson [3] to make the Hyodo-Kato morphism independent of choices in the case of proper schemes. We adapt here his technique to formal schemes and along the way lift the morphism to derived $\infty$-category. As a byproduct we get the identification

$$[\mathrm{R}\Gamma_{\mathrm{cr}}(X)]^{\varphi=p^r} \simeq [\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}$$

and an identification of (1.8) with (1.5), as wanted.

(ii) *Dagger varieties.* The pro-étale cohomology in (1) is defined in the most naive way: if $X$ is a smooth dagger affinoid with a presentation $\{X_h\}_{h\in\mathbf{N}}$ by a pro-affinoid rigid analytic variety, we set

$$\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) := \mathrm{hocolim}_h\, \mathrm{R}\Gamma_{\text{proét}}(X_h, \mathbf{Q}_p(r));$$

then, we globalize. From this description it is clear that we have a natural map

$$\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{proét}}(\widehat{X}, \mathbf{Q}_p(r)),$$

where $\widehat{X}$ is the completion of $X$ (a rigid analytic variety). It is easy to see that in the case $X$ is partially proper, this morphism is a strict quasi-isomorphism (see [16, Prop. 3.17]).

The other overconvergent cohomologies (Hyodo-Kato, de Rham, $\mathbf{B}^+_{\mathrm{dR}}$-, syntomic) and morphisms between them can be defined in an analogous way without difficulties. In some cases though, they do however already have independent definitions: Hyodo-Kato and de Rham cohomologies were defined by Grosse-Klönne in [25] and we define syntomic cohomology as the homotopy fiber giving the following distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \longrightarrow [\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r. \tag{1.9}$$

In these cases, we prove that the two sets of definitions yield strictly quasi-isomorphic objects. As an illustration of the power of the new definitions of overconvergent cohomologies, let us look at the simple proof of the following fact, whose arithmetic analog was the main technical result of [16]:

**Proposition 1.10.** *Let $r \geq 0$. Let $X$ be a smooth dagger variety over $K$. There is a natural morphism*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X}, \mathbf{Q}_p(r)).$$

*It is a strict quasi-isomorphism if $X$ is partially proper.*

[7]For experts: the section of the projection $T \mapsto 0$.

This proposition is proved by representing, using distinguished triangles (1.5) and (1.9), both sides of the morphism by means of the rigid analytic and the overconvergent Hyodo-Kato cohomology, respectively, then passing through the rigid analytic and the overconvergent Hyodo-Kato quasi-isomorphisms (that are compatible by construction) to the de Rham cohomology, where the result is known.

*Remark* 1.11. The approach we have taken here to deal with dagger varieties is very different from the one in [14] or [16] (these two approaches also differing between themselves). That is, we do not use Grosse-Klönne's overconvergent Hyodo-Kato cohomology nor the related Hyodo-Kato morphism (which is difficult to work with and is also very different from the rigid analytic version making checking the overconvergent-rigid analytic compatibility a bit of a nightmare). Instead, we induce all the overconvergent cohomologies from their rigid analytic analogs; hence, by definition, the two constructions are compatible. This was only possible because we have constructed a functorial, $\infty$-category version of the Hyodo-Kato morphism.

*Structure of the paper.* Sections 2 and 4 are devoted to a definition of a functorial, $\infty$-categorical Hyodo-Kato quasi-isomorphism. In Section 3 we present our definition of $\mathbf{B}_{\mathrm{dR}}^+$-cohomology. Section 5 puts the above things together and introduces overconvergent geometric syntomic cohomology. In Section 6 we define comparison morphisms and in Section 7 we put a geometric structure on them.

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1.2.1. *Notation and conventions.* Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field $K$ of characteristic 0 and with perfect residue field $k$ of characteristic $p$. Let $\overline{K}$ be an algebraic closure of $K$ and let $\mathscr{O}_{\overline{K}}$ denote the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Let $C=\widehat{\overline{K}}$ be the $p$-adic completion of $\overline{K}$. Let $W(k)$ be the ring of Witt vectors of $k$ with fraction field $F$ (i.e., $W(k)=\mathscr{O}_F$); let $e=e_K$ be the ramification index of $K$ over $F$. Set $\mathscr{G}_K=\mathrm{Gal}(\overline{K}/K)$ and let $\varphi$ be the absolute Frobenius on $W(\overline{k})$. We will denote by $\mathbf{A}_{\mathrm{cr}},\mathbf{B}_{\mathrm{cr}},\mathbf{B}_{\mathrm{st}},\mathbf{B}_{\mathrm{dR}}$ the crystalline, semistable, and de Rham period rings of Fontaine.

We will denote by $\mathscr{O}_K$, $\mathscr{O}_K^\times$, and $\mathscr{O}_K^0$, depending on the context, the scheme $\mathrm{Spec}(\mathscr{O}_K)$ or the formal scheme $\mathrm{Spf}(\mathscr{O}_K)$ with the trivial, the canonical (i.e., associated to the closed point), and the induced by $\mathbf{N}\to\mathscr{O}_K, 1\mapsto 0$, log-structure, respectively. Unless otherwise stated all formal schemes are $p$-adic, locally of finite type, and equidimensional. For a ($p$-adic formal) scheme $X$ over $\mathscr{O}_K$, let $X_0$ denote the special fiber of $X$; let $X_n$ denote its reduction modulo $p^n$.

All rigid analytic spaces considered will be over $K$ or $C$. We assume that they are separated, taut, and countable at infinity. If $L=K,C$, we let $\mathrm{Sm}_L$ (resp. $\mathrm{Sm}_L^\dagger$) be the category of smooth rigid analytic (resp. dagger) varieties over $L$, and we denote by $\mathrm{Perf}_C$ the category of perfectoid spaces over $C$.

Our cohomology groups will be equipped with a canonical topology. To talk about it in a systematic way, we will work rationally in the category of locally convex $K$-vector spaces and integrally in the category of pro-discrete $\mathscr{O}_K$-modules. For details the reader may consult [14, Sec. 2.1, 2.3]. To summarize quickly:

(1) $C_K$ is the category of convex $K$-vector spaces; it is a quasi-abelian category. We will denote the left-bounded derived $\infty$-category of $C_K$ by $\mathscr{D}(C_K)$. A morphism of complexes that is a quasi-isomorphism in $\mathscr{D}(C_K)$, i.e., its cone is strictly exact, will be called a *strict quasi-isomorphism.*

The associated cohomology objects are denoted by[8] $\widetilde{H}^n(E) \in LH(C_K)$; they are called *classical* if the natural map $\widetilde{H}^n(E) \to H^n(E)$ is an isomorphism[9].

(2) Objects in the category $PD_K$ of pro-discrete $\mathscr{O}_K$-modules are topological $\mathscr{O}_K$-modules that are countable inverse limits, as topological $\mathscr{O}_K$-modules, of discrete $\mathscr{O}_K$-modules $M^i$, $i \in \mathbf{N}$. It is a quasi-abelian category. Inside $PD_K$ we distinguish the category $PC_K$ of pseudocompact $\mathscr{O}_K$-modules, i.e., pro-discrete modules $M \simeq \lim_i M_i$ such that each $M_i$ is of finite length (we note that if $K$ is a finite extension of $\mathbf{Q}_p$ this is equivalent to $M$ being profinite). It is an abelian category.

(3) There is a tensor product functor from the category of pro-discrete $\mathscr{O}_K$-modules to convex $K$-vector spaces:
$$(-)\otimes K : PD_K \to C_K, \quad M \mapsto M \otimes_{\mathscr{O}_K} K.$$
Since $C_K$ admits filtered inductive limits, the functor $(-)\otimes K$ extends to a functor $(-)\otimes K : \mathrm{Ind}(PD_K) \to C_K$. The functor $(-)\otimes K$ is right exact but not, in general, left exact. We will consider its (compatible) left derived functors
$$(-)\otimes^{\mathrm{L}} K : \mathscr{D}^-(PD_K) \to \mathrm{Pro}(\mathscr{D}^-(C_K)), \quad (-)\otimes^{\mathrm{L}} K : \mathscr{D}^-(\mathrm{Ind}(PD_K)) \to \mathrm{Pro}(\mathscr{D}^-(C_K)).$$
If $E$ is a complex of torsion free and $p$-adically complete (i.e., $E \simeq \lim_n E/p^n$) modules from $PD_K$ then the natural map
$$E\otimes^{\mathrm{L}} K \to E\otimes K$$
is a strict quasi-isomorphism [14, Prop. 2.6].

Unless otherwise stated, we work in the derived (stable) $\infty$-category $\mathscr{D}(A)$ of left-bounded complexes of a quasi-abelian category $A$ (the latter will be clear from the context). Many of our constructions will involve (pre)sheaves of objects from $\mathscr{D}(A)$. We will use a shorthand for certain homotopy limits: if $f : C \to C'$ is a map in the derived $\infty$-category of a quasi-abelian category, we set
$$[\, C \xrightarrow{f} C' \,] := \mathrm{holim}(C \to C' \leftarrow 0).$$
For an operator $F$ acting on $C$, we will use the brackets $[C]^F$ to denote the derived eigenspaces and the brackets $(C)^F$ or simply $C^F$ to denote the non-derived ones.

Finally, we will use freely the notation and results from [16].

## 2. Hyodo-Kato rigidity revisited

The original Hyodo-Kato morphism [29] works for semistable (formal) schemes. It can not be transferred to rigid analytic varieties because, a priori, it is dependent on the choice of the uniformizer of the base field (which varies for local semistable models). A more careful data keeping allowed Beilinson [3] to make it independent of choices in the case of proper schemes. We adapt here his technique to semistable formal schemes and add some extra functoriality by lifting the morphism to the derived $\infty$-category.

This gives us local Hyodo-Kato morphisms for rigid analytic varieties; the extra functoriality will be crucial for the globalization of these maps for rigid analytic and dagger varieties discussed in Chapter 4 (it makes it possible to glue local maps from an hypercover by semistable formal schemes).

2.1. **Preliminaries.** We gather in this section basic properties of period rings, isogenies, and $\varphi$-modules that we will need in the paper.

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[8] $LH$ stands for "left heart".

[9] In our situations this is usually equivalent to $H^n(E)$ being separated.

2.1.1. *Period rings.* We will review first the definitions of the rings of periods that we will need. We follow here Beilinson [3, 1.14, 1.19], where the reader can find more details. Beilinson's definitions are a slight modification of the classical ones; they stress the dependence on choices in a better way.

(i) *Arithmetic setting.* Let $S = S_K := \operatorname{Spf} \mathscr{O}_K^{\times}$, $S^0 := \operatorname{Spf} \mathscr{O}_F^0$. We denote the corresponding log-structures by $\mathscr{L} = \mathscr{L}_K$ and $\mathscr{L}_K^0$, respectively. Note that the second log-structure can be conveniently described by the pre-log structure $\mathscr{O}_K \setminus \{0\} \to \mathscr{O}_F, a \mapsto [\overline{a}]$, where $\overline{a} := a \mod \mathfrak{m}_K$ and $[-]$ denotes the Teichmüller lift.

Consider the algebra $\mathscr{O}_F[T]$ with the log-structure associated to $T$. We denote by $r_F^{\mathrm{PD}}$ the associated $p$-adic divided powers polynomial algebra. In a more natural in $K$ way, we can write $r_F^{\mathrm{PD}}$ as $r_K^{\mathrm{PD},0}$ – the $p$-adic completion of $\mathscr{O}_F < t_a >$, the divided powers polynomial algebra generated by $t_a, a \in (\mathfrak{m}_K/\mathfrak{m}_K^2)\setminus\{0\}$, with $t_{a'} = [a'/a]t_a$. We denote by $r_K^{\mathrm{PD}}$ the $p$-adic completion of the subalgebra of the PD algebra $\mathscr{O}_F < t_a >$ generated by $t_a$ and $t_a^{ne}/n!, n \geq 1$. The log-structure is given by the $t_a$'s, Frobenius action by $t_a \mapsto t_a^p$, and monodromy by the derivation sending $t_a \mapsto t_a$. Set $E = E_K := \operatorname{Spf} r_K^{\mathrm{PD}}$, $E^0 = E_K^0 := \operatorname{Spf} r_K^{\mathrm{PD},0}$. We have canonical exact embeddings $i_0 : S^0 \hookrightarrow E, i_0^*(t_a) = [a] \in \mathscr{L}_K^0$, $i_0^0 : S^0 \hookrightarrow E^0, i_0^{0,*}(t_a) = [a] \in \mathscr{L}_K^0$.

We have an exact closed embedding $S_1^0 \hookrightarrow S_1$. Retractions $\pi_l$ are given by maps $\pi_l^* : \mathscr{L}_1^0 \to \mathscr{L}_1, a \mapsto l_a$, with $l_{a'} = [a'/a]l_a$. Every retraction $\pi_l : S_1 \to S_1^0$ yields a $k^0$-structure on $S_1$, hence an exact closed embedding $i_l : S_1 \hookrightarrow E_1$, $i_l^*(t_a) = l_a$.

(ii) *Geometric setting.* Let $\overline{S} := \operatorname{Spf} \mathscr{O}_C^{\times}$. We denote its log-structure by $\overline{\mathscr{L}}$. We normalize the valuation on $C$ by $v(p) = 1$. Let $\overline{\mathscr{L}}^0$ be the log-structure on $\overline{S}^0 := \operatorname{Spf} W(\overline{k})$ generated by the pre-log structure $\mathscr{O}_C \setminus \{0\} \to W(\overline{k}), a \mapsto [\overline{a}]$, $\overline{a} := a \mod \mathfrak{m}_{\overline{K}}$. Then $\overline{\mathscr{L}}^0$ has a natural Frobenius action compatible with the Frobenius: $\varphi([a]) = [a^p]$. There is an exact embedding $\overline{S}_1^0 \hookrightarrow \overline{S}_1$.

We will denote by $\mathbf{A}_{\mathrm{cr}}^{\times}$ the period ring $\mathbf{A}_{\mathrm{cr}}$ equipped with the unique log-structure $\mathscr{L}_{\mathrm{cr}}$ extending the one on $\mathscr{O}_{C,1}^{\times}$. Let $J_{\mathrm{cr}}$ be the PD-idea, $\mathbf{A}_{\mathrm{cr}}/J_{\mathrm{cr}} \simeq \mathscr{O}_{C,1}$. Set $\mathscr{E}_{\mathrm{cr}} := \operatorname{Spf} \mathbf{A}_{\mathrm{cr}}^{\times}$. The exact embedding $\operatorname{Spec} \mathscr{O}_{C,1}^{\times} \hookrightarrow \mathscr{E}_{\mathrm{cr},n}$ given by the Fontaine map $\theta : \mathbf{A}_{\mathrm{cr}} \to \mathscr{O}_C$ is a PD-thickening in the crystalline site of $\mathscr{O}_{F,1}$.

Recall the definition of the period ring $\mathbf{B}_{\mathrm{st}}^+$. Let $\log : \mathbf{A}_{\mathrm{cr}}^*/\overline{k}^* \to \mathbf{B}_{\mathrm{cr}}^+$ be the logarithm: the unique homomorphism which extends the logarithm on $(1 + J_{\mathrm{cr}})^*$, where $J_{\mathrm{cr}} = (p, \operatorname{Ker}\theta)$. Then $\mathbf{B}_{\mathrm{st}}^+$ is defined as the universal $\mathbf{B}_{\mathrm{cr}}^+$-algebra equipped with a homomorphism of monoids $\log : \mathscr{L}_{\mathrm{cr}}/\overline{k}^* \to \mathbf{B}_{\mathrm{st}}^+$ extending the above log on $\mathbf{A}_{\mathrm{cr}}^*$. Since $v : \mathscr{L}_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}^* \xrightarrow{\sim} \mathbf{Q}_{\geq 0}$, it is clear that, for any $\lambda \in \mathscr{L}_{\mathrm{cr}}/\overline{k}^*$ with $v(\lambda) \neq 0$, the element $\log(\lambda)$ freely generates $\mathbf{B}_{\mathrm{st}}^+$ over $\mathbf{B}_{\mathrm{cr}}^+$, i.e., $\mathbf{B}_{\mathrm{cr}}^+[\log(\lambda)] \xrightarrow{\sim} \mathbf{B}_{\mathrm{st}}^+$. The Frobenius action extends to $\mathbf{B}_{\mathrm{st}}^+$ via universality. The monodromy $N$ is the $\mathbf{B}_{\mathrm{cr}}^+$-derivation on $\mathbf{B}_{\mathrm{st}}^+$ such that $N(\log(\lambda)) = -v(\lambda)$. We have $N\varphi = p\varphi N$. Moreover, any $\lambda$ as above yields a retraction $s_\lambda^* : \mathbf{B}_{\mathrm{st}}^+ \to \mathbf{B}_{\mathrm{cr}}^+, s_\lambda^*(\log(\lambda)) = 0$. If $\lambda \in \mathscr{L}_\varphi := \{\lambda \in \mathscr{L}_{\mathrm{cr}} : \varphi(\lambda) = \lambda^p\}$ then $s_\lambda^*$ is compatible with Frobenius action.

Now, recall the definition of the period ring $\widehat{\mathbf{B}}_{l,\mathrm{st}}^+$. Let $r \in \mathbf{Q}_{>0}$. Denote by $\mathbb{A}_{W(\overline{k})}^{(r)}$, the log affine space, i.e., the formal scheme $\operatorname{Spf} W(\overline{k})\{t_a\}, a \in \tau_r$, with $t_{a'} = [a'/a]t_a$. Here $\tau_r := \{a \in \overline{\mathscr{L}}_1^0 : v(a) = r\}$. The log-structure is generated by the $t_a$'s. The map $i_r : \overline{S}_1^0 \to \mathbb{A}_{W(\overline{k})}^{(r)}$, $i_r^*(t_a) = a$, can be extended to a map $i_l : \overline{S}_1 \to \mathbb{A}_{W(\overline{k})}^{(r)}$ by choosing $l_a := i_l^*(t_a) \in \overline{\mathscr{L}}_1$ that lifts $a$.

We have the commutative diagram

$$\begin{tikzcd}
 & \mathbb{A}_{W(\overline{k})}^{(r)} \times_{W(\overline{k})} \mathscr{E}_{\mathrm{cr},n} \arrow[d] \\
\overline{S}_1 \arrow[ur, hook, "{(i_l,\theta)}"] \arrow[r, hook] & \mathscr{E}_{\mathrm{cr},n}
\end{tikzcd}$$

Let $i_{l,\mathrm{st}} : \overline{S}_1 \hookrightarrow \mathscr{E}_{l,\mathrm{st},n}$ be the PD-envelope of $(i_l, \theta)$ over $\mathscr{E}_{\mathrm{cr},n}$. We write $\mathscr{E}_{l,\mathrm{st},n} = \operatorname{Spec} \widehat{\mathbf{A}}_{l,\mathrm{st},n}$ and set

$$\widehat{\mathbf{A}}_{l,\mathrm{st}} := \lim_n \widehat{\mathbf{A}}_{l,\mathrm{st},n}, \quad \widehat{\mathbf{B}}_{l,\mathrm{st}}^+ := \widehat{\mathbf{A}}_{l,\mathrm{st}}[\tfrac{1}{p}], \quad \mathscr{E}_{l,st} := \operatorname{Spf} \widehat{\mathbf{A}}_{l,\mathrm{st}}.$$

We note that $\widehat{\mathbf{B}}^+_{l,\mathrm{st}}$ is a Banach space over $F$ (which makes it easier to handle topologically than $\mathbf{B}^+_{\mathrm{st}}$). Frobenius action is given by $t_a \mapsto t_a^p$ and the monodromy operator by $N_l := t_a\partial_{t_a}$. We have the exact sequence

$$0 \to \mathbf{A}_{\mathrm{cr},n} \to \widehat{\mathbf{A}}_{l,\mathrm{st},n} \xrightarrow{N_l} \widehat{\mathbf{A}}_{l,\mathrm{st},n} \to 0. \tag{2.1}$$

Every lifting of $l$ to $\lambda \in \mathscr{L}_\varphi$ yields a map $s^*_\lambda : \widehat{\mathbf{A}}_{l,\mathrm{st},n} \to \mathbf{A}_{\mathrm{cr},n}, s^*_\lambda(t_a) := \lambda_a$, which is compatible with Frobenius action, and an identification $\widehat{\mathbf{A}}_{l,\mathrm{st},n} \xrightarrow{\sim} \mathbf{A}_{\mathrm{cr},n} < t_a\lambda_a^{-1} - 1 >$.

Let $\widehat{\mathbf{A}}^{N_l-\mathrm{nilp}}_{l,\mathrm{st}}$ be the $\mathbf{A}_{\mathrm{cr}}$-subalgebra of $\widehat{\mathbf{A}}_{l,\mathrm{st}}$ formed by the elements killed by a power of $t_a\partial_{t_a}$. It is the divided powers polynomial algebra $\mathbf{A}_{\mathrm{cr}} < \log(t_a\lambda_a^{-1}) >$. There is a $\mathbf{B}^+_{\mathrm{cr}}$-linear isomorphism

$$\kappa_l : \mathbf{B}^+_{\mathrm{st}} \xrightarrow{\sim} \widehat{\mathbf{A}}^{N_l-\mathrm{nilp}}_{l,\mathrm{st},\mathbf{Q}_p}$$

which sends a generator $\log(\lambda)$ of $\mathbf{B}^+_{\mathrm{st}}$, where $\lambda$ lifts $l$, to $-\log(t_a\lambda_a^{-1}) \in \widehat{\mathbf{A}}^{N_l-\mathrm{nilp}}_{l,\mathrm{st}}$. It is compatible with the action of $\mathscr{G}_K$, Frobenius, and it identifies $N$ on $\mathbf{B}^+_{\mathrm{st}}$ with the action of $rt_a\partial_{t_a}$.

Finally we have maps to $\mathbf{B}^+_{\mathrm{dR}}$. We will normalize them for the rest of the paper at $p$. That is, we fix a lift $[\tilde{p}] \in \mathscr{L}_\varphi$ of $p$ and define the maps:

$$\iota = \iota_p : \widehat{\mathbf{B}}^+_{p,\mathrm{st}} \to \mathbf{B}^+_{\mathrm{dR}}, \quad \iota = \iota_p := \iota_p\kappa_p : \mathbf{B}^+_{\mathrm{st}} \to \mathbf{B}^+_{\mathrm{dR}}.$$

The first map is obtained by sending $t_p$ to $p$; the second map, by sending $\log([\tilde{p}])$ to $-\log(p/[\tilde{p}])$. Otherwise saying, we can set

$$\begin{aligned}
&\widehat{\mathbf{B}}^+_{\mathrm{st}} := \widehat{\mathbf{B}}^+_{p,\mathrm{st}} := \mathbf{A}_{\mathrm{cr}} < t_p[\tilde{p}]^{-1} - 1 >^{\widehat{}} [\tfrac{1}{p}], \quad \mathbf{B}^+_{\mathrm{st}} := \mathbf{B}^+_{p,\mathrm{st}} := \mathbf{B}^+_{\mathrm{cr}}[\log([\tilde{p}])],\\
&\kappa : \mathbf{B}^+_{\mathrm{st}} \to \widehat{\mathbf{B}}^+_{\mathrm{st}},\ \log([\tilde{p}]) \mapsto -\log(t_p[\tilde{p}]^{-1}), \quad \iota : \mathbf{B}^+_{\mathrm{st}} \to \mathbf{B}^+_{\mathrm{dR}},\ \log([\tilde{p}]) \mapsto -\log(p[\tilde{p}]^{-1}),\\
&\iota : \widehat{\mathbf{B}}^+_{\mathrm{st}} \to \mathbf{B}^+_{\mathrm{dR}},\ t_p[\tilde{p}]^{-1} \mapsto p[\tilde{p}]^{-1}.
\end{aligned}$$

2.1.2. *Tensoring with period rings.* (1) Let $M$ be a bounded complex of Banach spaces, which are topological $\mathbf{B}^+_{\mathrm{cr}}$-modules. We define the topological tensor product $M \otimes_{\mathbf{B}^+_{\mathrm{cr}}} C$ as the algebraic tensor product equipped with the quotient topology induced from $M$ via the map $\theta$. This product tends to be compatible with strict quasi-isomorphisms:

**Lemma 2.2.** *Let $M, M'$ be bounded complexes of Banach spaces, which are flat $\mathbf{B}^+_{\mathrm{cr}}$-modules. Let $\alpha : M \to M'$ be a strict quasi-isomorphism. Then the induced morphism*

$$\alpha \otimes \mathrm{Id} : M \otimes_{\mathbf{B}^+_{\mathrm{cr}}} C \to M' \otimes_{\mathbf{B}^+_{\mathrm{cr}}} C$$

*is a strict quasi-isomorphism as well.*

*Proof.* Let $C(\alpha)$ denote the mapping fiber of $\alpha$. It is a bounded complex of Banach spaces. We claim that the complex

$$C(\alpha \otimes \mathrm{Id}) = [M \otimes_{\mathbf{B}^+_{\mathrm{cr}}} C \xrightarrow{\alpha\otimes\mathrm{Id}} M' \otimes_{\mathbf{B}^+_{\mathrm{cr}}} C] \simeq C(\alpha) \otimes_{\mathbf{B}^+_{\mathrm{cr}}} C$$

is strictly acyclic. Indeed, since $M, M'$ are bounded and built from flat $\mathbf{B}^+_{\mathrm{cr}}$-modules, this is so algebraically. Now the terms of $C(\alpha \otimes \mathrm{Id})$ are Banach spaces as quotients of Banach spaces by closed subspaces and the Open Mapping Theorem implies that a complex of Banach spaces is strictly acyclic if and only if it is acyclic (apply the OMT to the isomorphism $\mathrm{Im}(d_i) \to \mathrm{Ker}(d_{i+1})$ which are both Banach spaces since $d_i$ and $d_{i+1}$ are continuous). □

(2) Similarly, for a bounded complex $M$ of Banach spaces, which are topological $\mathbf{B}^+_{\mathrm{cr}}$-modules, we define the topological tensor product $M \otimes_{\mathbf{B}^+_{\mathrm{cr}}} (\mathbf{B}^+_{\mathrm{cr}}/F^i)$, $i \geq 0$, as the algebraic tensor product equipped with the quotient topology induced from $M$. We have analog of the Lemma 2.2 in this setting.

We will denote this tensor product by

$$M \widehat{\otimes}^{\mathrm{L}}_{\mathbf{B}^+_{\mathrm{cr}}} (\mathbf{B}^+_{\mathrm{cr}}/F^i), \quad i \geq 0.$$

(3) For a bounded complex $M$ of Banach spaces, which are topological $\mathbf{B}_{\mathrm{cr}}^+$-modules, we define

$$M\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}_{\mathrm{cr}}^+}\mathbf{B}_{\mathrm{dR}}^+ := \mathrm{R}\lim_i(M\widehat{\otimes}^{\mathrm{L}}_{\mathbf{B}_{\mathrm{cr}}^+}(\mathbf{B}_{\mathrm{cr}}^+/F^i)).$$

We have analog of the Lemma 2.2 in this setting as well.

2.1.3. *Isogenies.* We recall now some terminology from [6, Sec. 1.1] (see also [1, Sec. 2.3]).

Let $\mathscr{C}$ be an additive category (or $\infty$-category). A map $f : P \to Q$ is an *isogeny* if there exists $g : Q \to P$ and an integer $N > 0$ such that $gf = N\,\mathrm{Id}_P$ and $fg = N\,\mathrm{Id}_Q$. An object $X \in \mathscr{C}$ is *bounded torsion* if it is killed by some $N$, i.e., if $N\,\mathrm{Id}_X = 0$. If $\mathscr{C}$ is an additive category, we denote by $\mathscr{C} \otimes \mathbf{Q}$ the category with the same objects as $\mathscr{C}$, with a functor $\mathscr{C} \to \mathscr{C} \otimes \mathbf{Q}, X \mapsto X_{\mathbf{Q}}$, and with $\mathrm{Hom}(X_{\mathbf{Q}}, Y_{\mathbf{Q}}) = \mathrm{Hom}(X, Y) \otimes \mathbf{Q}$. Then $\mathscr{C} \otimes \mathbf{Q}$ is the localization of $\mathscr{C}$ with respect to isogenies; for $X \in \mathscr{C}$, we have $X_{\mathbf{Q}} = 0$, i.e., $X$ is isogenous to 0, if and only if $X$ is a bounded torsion object. If $\mathscr{C}$ is abelian then $\mathscr{C} \otimes \mathbf{Q}$ is abelian as well and it equal to the quotient $\mathscr{C}_{\mathbf{Q}}$ of $\mathscr{C}$ modulo the Serre subcategory of bounded torsion objects.

Let $\mathscr{C}$ be a stable $\infty$-category equipped with a $t$-structure. If a map is an isogeny then it induces isogenies on all cohomology groups $H^n$, $n \in \mathbf{Z}$, in the heart $\mathscr{C}^{\heartsuit}$. For maps between bounded object the opposite is true as well: the map $f : P \to Q$ of bounded objects is an isogeny, if, for each $n$, the map $H^nP \to H^nQ$ is an isogeny. In particular, $X \in \mathscr{C}$ is isogeneous to 0 if each $H^n(X)$ is a bounded torsion group.

*Remark* 2.3. Consider the tensor product functor in the top row of the diagram:

$$\begin{tikzcd} \mathscr{D}(\mathrm{PD}_K) \arrow[r, "(-)\otimes^{\mathrm{L}}_K"] \arrow[d, "\mathrm{can}"'] & \mathscr{D}(C_K) \\ \mathscr{D}(\mathrm{PD}_K)_{\mathbf{Q}} \arrow[ur, dashed, "(-)_K"'] & \end{tikzcd}$$

It factors naturally through the isogeny category; we will denote so obtained functor from $\mathscr{D}(\mathrm{PD}_K)_{\mathbf{Q}}$ to $\mathscr{D}(C_K)$ by $(-)_K$.

2.1.4. *$\varphi$-modules.* A Frobenius on an $\mathscr{O}_F$-module is a $\varphi_F$-linear endomorphism. Let $R$ be an $\mathscr{O}_F$-algebra equipped with a Frobenius $\varphi_R$. For an $R$-module $M$, a Frobenius on $M$ compatible with the $R$-module structure is an $R$-linear map $\varphi_M : \varphi_R^*M \to M$. Pairs $(M, \varphi_M)$ form an abelian tensor $\mathbf{Z}_p$-category $R_\varphi$-Mod. Let $\mathscr{D}_\varphi(R)$ be its bounded derived $\infty$-category.

Consider the bounded derived $\infty$-categories $\mathscr{D}_\varphi(R), \mathscr{D}_\varphi(R)_{\mathbf{Q}}$ of bounded complexes of $\varphi$-modules over $R$. Then $\mathscr{D}_\varphi(R)_{\mathbf{Q}}$ is the quotient of $\mathscr{D}_\varphi(R)$ modulo the full subcategory of complexes with bounded torsion cohomology.

We need to discuss projective resolutions. For an $R$-module $M$, set $M_\varphi := \oplus_{n\geq 0}\varphi_R^{n*}M$ and equip it with the induced Frobenius. The functor $R$-Mod $\to R_\varphi$-Mod, $M \mapsto M_\varphi$, is left adjoint to the forgetful functor $R_\varphi$-Mod $\to R$-Mod, $(M, \varphi_M) \mapsto M$. If follows that, for a projective $R$-module $M$, the $\varphi$-module $M_\varphi$ is a projective object of $R_\varphi$-Mod.

For every $M = (M, \varphi_M) \in R_\varphi$-Mod, there is a natural short exact sequence

$$0 \to (\varphi_R^*M)_\varphi \xrightarrow{\delta} M_\varphi \xrightarrow{\varepsilon} M \to 0$$

in $R_\varphi$-Mod. The maps $\varepsilon$ and $\delta$ are induced, respectively, by adjunction from $\mathrm{Id}_M$ and the map $\varphi_R^*M \to M_\varphi$ that sends $r \otimes m$ to $\varphi_M(r \otimes m) - r \otimes m \in M \oplus \varphi_R^*M \subset M_\varphi$. Set $\widetilde{M} := \mathrm{Cone}(\delta)$, so we have a resolution $\varepsilon : \widetilde{M} \to M$. If $M$ is a projective $R$-module, this is a projective resolution in $R_\varphi$-Mod.

Let $R := r_K^{\mathrm{PD}}$. Let $P = (P, \varphi_P) \in \mathscr{D}_\varphi(R)$, $Q = (Q, \varphi_Q) \in \mathscr{D}_\varphi(W(k))$, be complexes of $p$-completely flat modules. Assume that $P$ is derived $p$-adically complete. Denote by $\widehat{P}$ the complex $\mathrm{Cone}(P \to P\widehat{\otimes}^{\mathrm{L}}_R W(k))$ viewed as an object of $\mathscr{D}_\varphi(W(k))$.

**Lemma 2.4.** *If the Frobenius on $Q_{\mathbf{Q}}$ is invertible then*

$$\mathrm{RHom}(Q, \widehat{P})_{\mathbf{Q}} = \mathrm{RHom}(Q_{\mathbf{Q}}, \widehat{P}_{\mathbf{Q}}) = 0.$$

*Proof.* We claim that we have the short exact sequence of $\varphi$-modules over $W(k)$

$$0 \to IP \to P \to P\widehat{\otimes}^{\mathrm{L}}_R W(k) \to 0, \tag{2.5}$$

where $I \subset R$ is the kernel of the projection $R \to W(k)$. Indeed, because $P_n$ is a complex of flat $R_n$-modules we have a compatible family of exact sequences

$$0 \to P_n \otimes_R I_n \to P \otimes_R R_n \to P_n \otimes_R W_n(k) \to 0$$

Passing to limit we get the short exact sequence

$$0 \to \lim_n (P_n \otimes_R I_n) \to \lim_n (P \otimes_R R_n) \to \lim_n (P_n \otimes_R W_n(k)) \to 0$$

Since $P$ is derived $p$-complete and its terms are $p$-completely flat modules, the natural morphism $P \to \lim_n (P \otimes_R R_n)$ is a quasi-isomorphism (in fact, an isomorphism) [39, Tag 091N] and the above exact sequence yields the exact sequence (2.5).

From the exact sequence (2.5) we get a distinguished triangle

$$\mathrm{RHom}(Q, IP) \to \mathrm{RHom}(Q, P) \to \mathrm{RHom}(Q, P\widehat{\otimes}^{\mathrm{L}}_R W(k)),$$

where RHom is computed in $\mathscr{D}_\varphi(W(k))$. To prove the lemma, it suffices to show that $\mathrm{RHom}(Q, IP)_{\mathbf{Q}} = 0$.

For any $\varphi$-module $M$ over $W(k)$, we can compute $\mathrm{RHom}(Q, M)$ using the above projective resolution $\widetilde{Q}$ of $Q$. We get a two-term complex $C(M)$ with $C^0(M) = \mathrm{Hom}_{W(k)}(Q, M)$, $C^1(M) = \mathrm{Hom}_{W(k)}(\varphi^*(Q), M)$ and the differential $d = d_1 - d_2$, where $d_1(a) = a\varphi_Q$, $d_2(a) = \varphi_M \varphi^*(a)$. Let $C_*(M)$ be the complex with the same terms as $C(M)$ but the differential simply $d_1$. Since we assumed that the Frobenius action on $Q_{\mathbf{Q}}$ is invertible, the complex $C_*(M)_{\mathbf{Q}}$ is acyclic.

Note that, since $Q$ is built from projective modules, the functor $C$ is exact. We will prove that $C(IP)_{\mathbf{Q}}$ is acyclic by defining a finite filtration on $IP$ such that $C(\mathrm{gr}^j IP) \simeq C_*(\mathrm{gr}^j IP)$. Let $I^{(j)}$, $j \geq 1$, be the ideal of $R$ formed by series $\sum a_i t^i$ with $a_0 = \ldots = a_{j-1} = 0$. We have $I = I^{(1)}$. Since $\varphi(I^{(j)}) \subset I^{(pj)}$, one has $C(I^{(j)}P/I^{(j+1)}P) = C_*(I^{(j)}P/I^{(j+1)}P)$. It remains to show that $C(I^{(n)}P)$ is quasi-isomorphic to $C_*(I^{(n)}P)$ for $n$ sufficiently large (then the looked for finite filtration is $I^{(j)}P$, $j \leq n$.)

By assumption, for $m$ sufficiently large, there is $\psi : Q \to \varphi^* Q$ such that $\varphi_Q \psi = p^m \mathrm{Id}_Q, \psi\varphi_Q = p^m \mathrm{Id}_{\varphi^*(Q)}$. For $n$ sufficiently large, we have $\varphi(I^{(n)}) \subset p^{m+1} I^{(pn)}$. Hence $d_2$ on $C(I^{(n)}P)$ is divisible by $p^{m+1}$. Set $f := \psi^\tau(p^{-m-1}d_2) \in \mathrm{End}(C^0(I^{(n)}P))$; then $d_2 = pd_1 f$, i.e., $d = d_1(1 - pf)$. Since $Q$ is a complex of projective modules and $IP$ is derived $p$-adically complete (as the kernel of the canonical map $P \to P\widehat{\otimes}_R W(k)$ between two derived $p$-adically complete objects), $C^0(I^{(n)}P) = \mathrm{Hom}(Q, IP) \simeq \mathrm{RHom}(Q, IP)$ is derived $p$-adically complete [39, Tag 091N]. It follows that $(1-pf)$ is a quasi-isomorphism (use derived Nakayama Lemma), so it yields $C_*(I^{(n)}P) \xrightarrow{\sim} C_*(I^{(n)}P)$, as wanted. $\square$

2.2. **Hyodo-Kato rigidity.** Now we pass to the main constructions. Till the end of this section, integrally, we will work in the category $\mathrm{PD}_K$ of pro-discrete modules only occasionally, when necessary, passing to its Ind-category.

2.2.1. *The Hyodo-Kato section.* In this section we will prove the existence of the Hyodo-Kato section in the derived $\infty$-category. We follow faithfully the arguments of Beilinson from [3, Sec. 1.14] with the following modifications:

(1) Beilinson works in the setting of proper log-smooth log-schemes hence all of his cohomology complexes are perfect; we replace them with a weaker condition of derived $p$-adically complete and $p$-completely flat,
(2) to prove that the Hyodo-Kato section (when linearized) is a quasi-isomorphism Beilinson uses finitness of Hyodo-Kato cohomology; we replace his argument with the original one due to Hyodo-Kato [29].

Since the argument of Beilinson can only be found in a preliminary version of a published paper, for the benefit of the reader and the authors, we supply all the details.

Let $f : X_1 \to S_1$ be a log-smooth map with Cartier type reduction, with $X_1$ fine. Let $f^0 : X_1^0 \to S_1^0$ be its pullback to $S_1^0$. Let $R := r_K^{\mathrm{PD}}$. Recall the definition and basic properties of the arithmetic Hyodo-Kato cohomology and the associated $r^{\mathrm{PD}}$ cohomology (in the terminology[10] from [16, 4.2]):

$$\begin{aligned}
&\mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{l,n} := \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/(S_1,E_n)), \quad i_l : S_1 \hookrightarrow E_1,\\
&\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_n := \mathrm{R}\Gamma_{\mathrm{cr}}(X_1^0/(S_1^0,S_n^0)),\\
&\mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_l := \mathrm{holim}_n\, \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{l,n}, \quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0) := \mathrm{holim}_n\, \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_n.
\end{aligned}$$

The embedding $i : X_1^0 \hookrightarrow X_1$ over $i_{l,n} : (S_1^0, S_n^0) \hookrightarrow (S_1, E_n)$ yields compatible morphisms $i_{l,n}^* : \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{l,n} \to \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_n$, $i_l^* : \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_l \to \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)$. All the above objects carry Frobenius and monodromy actions and the morphisms are compatible with these actions. Moreover,

(1) $\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)$ is a complex of $p$-completely flat[11] modules over $W(k)$, it is derived $p$-complete, and $\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_n \simeq \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)\widehat{\otimes}^{\mathrm{L}}_{W(k)} W_n(k)$;
(2) $\mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_l$ is a complex of $p$-completely flat modules over $R$, it is derived $p$-complete, and $\mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{l,n} \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_l \otimes^{\mathrm{L}}_R R_n$;
(3) we have a strict quasi-isomorphism $i_l^* : \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_l \widehat{\otimes}^{\mathrm{L}}_R W(k) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)$.

Now we present the key construction in the Hyodo-Kato theory.

**Theorem 2.6.** (1) *The Frobenius action on $\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}}$ is invertible.*
(2) *The map $i_l^* : \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{l,\mathbf{Q}} \to \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}}$ admits a unique natural $W(k)$-linear section $\iota_l$ that commutes with Frobenius. Its $R$-linear extension is a strict quasi-isomorphism:*

$$\iota_l : (R\widehat{\otimes}^{\mathrm{L}}_{W(k)} \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0))_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{l,\mathbf{Q}}.$$

*Proof.* Claim (1) is proved in [29, 2.24]. In fact, Hyodo-Kato prove more: they show that there exists a $p^d$-inverse of Frobenius, where $d = \dim X_1^0$.

For the existence part of claim (2), recall that Beilinson [3, 1.14] proved it in the case $X_1$ is proper. We will adapt his argument to our (general) local situation.

Take $P = \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_l$ in Lemma 2.4. By claim (1) the Frobenius action on $(P\widehat{\otimes}_R W(k))_{\mathbf{Q}}$ is invertible. Moreover, $P$ is derived $p$-adically complete and a complex of completely $p$-adically flat $R$-modules. Lemma 2.4 implies that the morphism $P_{\mathbf{Q}} \to (P\widehat{\otimes}^{\mathrm{L}}_R W(k))_{\mathbf{Q}}$ in $\mathscr{D}_\varphi(W(k))_{\mathbf{Q}}$ admits a unique right inverse $\iota_l$, as wanted. Consider its $R$-linearization

$$\iota_l : (R\widehat{\otimes}^{\mathrm{L}}_{W(k)} P\widehat{\otimes}^{\mathrm{L}}_R W(k))_{\mathbf{Q}} \to P_{\mathbf{Q}}.$$

[10] The notation we use here is a bit different than the one we used in [16]. This is because we have adopted here Beilinson's approach to the Hyodo-Kato morphism and with it his notation. The advantage of Beilinson's notation is that it keeps better track of the underlying data.

[11] Recall [8, 4.1] that, for a ring $A$, $M \in \mathscr{D}(A)$ is called $p$-completely flat if $M \otimes^{\mathrm{L}}_A A/p \in \mathscr{D}(A/p)$ is concentrated in degree 0.

We need to show that this is a quasi-isomorphism. But this was done by Hyodo-Kato [29] using the explicit de Rham-Witt presentation of the Hyodo-Kato complex. We are done. $\square$

2.2.2. *The Hyodo-Kato morphism.* Now, as usual, the Hyodo-Kato morphism can be obtained from the section constructed in Theorem 2.6. Let $X$ be a fine logarithmic formal scheme log-smooth over $S$. Assume that $X_1$ has Cartier type reduction over $S_1$. Let $\varpi$ be a uniformizing parameter of $\mathscr{O}_K$.

**Corollary 2.7.** *There is a natural strict quasi-isomorphism*

$$i_{\varpi}: \quad (\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)\widehat{\otimes}^{\mathrm{L}}_{W(k)}\mathscr{O}_K)_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X)_{\mathbf{Q}}.$$

*Proof.* Take $E$ with $l := \varpi \mod p\mathfrak{m}_K$. This yields an embedding $i_\pi : S \hookrightarrow E, i^*_{\varpi}(t_a) = \varpi, a := \varpi \mod \mathfrak{m}_K^2$. We start with the strict quasi-isomorphism from Theorem 2.6

$$\iota_l : (\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)\widehat{\otimes}^{\mathrm{L}}_{W(k)}R)_{\mathbf{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{l,\mathbf{Q}}.$$

Tensoring it with $\mathscr{O}_K$ (over $R$) we obtain the strict quasi-isomorphisms

$$(\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)\widehat{\otimes}^{\mathrm{L}}_{W(k)}\mathscr{O}_K)_{\mathbf{Q}} \xrightarrow{\sim} (\mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_l\widehat{\otimes}^{\mathrm{L}}_R\mathscr{O}_K)_{\mathbf{Q}} \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/\mathscr{O}_K^{\times})_{\mathbf{Q}} \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X)_{\mathbf{Q}}.$$

This is the Hyodo-Kato quasi-isomorphism $i_{\varpi}$ we wanted. $\square$

2.2.3. *Monodromy action revisited.* The purpose of this section is to prove the following:

**Proposition 2.8.** *The section*

$$\iota_l^* : \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{l,\mathbf{Q}}$$

*from Theorem 2.6 commutes with monodromy.*

Recall that the monodromy on $\mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_l$ is defined as the Gauss-Manin connection and the one on $\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)$ as its residue at $t = 0$. However, to prove Proposition 2.8 we will work with the "integration" of the monodromy action. The argument follows that of Beilinson in [3, Sec. 1.16] with the modifications mentioned earlier.

(i) *Equivariant structures.* Let $A^*$ be a cosimplicial algebra. An $A^*$*-complex* is a complex $M^*$ of cosimplicial $A^*$-modules such that, for every cosimplicial structure map $M^a \to M^b$, its $A^b$-linearization $A^b \otimes^{\mathrm{L}}_{A^a} M^a$ is a quasi-isomorphism. Denote by $\mathscr{D}(A^*)$ the derived $\infty$-category of bounded below $A^*$-complexes. We think of an element of $\mathscr{D}(A^*)$ as an $A^*$-complex with values in $\mathscr{D}(A^a)$ in degree $a$. For an endomorphism $T$ of $A^*$, $\mathscr{D}_T(A^*)$ will denote the derived $\infty$-category of bounded below $A^*$-complexes equipped with a $T$-action.

Fix an affine scheme $S$ as a base. Let $G$ be an affine group scheme acting on $X = \operatorname{Spec} A$. Let $[X/G] := EG \times_G X = \operatorname{Spec} A^*_G$ be the simplicial quotient. We have $[X/G]_m = X \times G^m$. Set $\mathscr{D}_G(A) := \mathscr{D}(A^*_G)$.

Let $\mathfrak{g}^\vee = \mathfrak{m}_e/\mathfrak{m}_e^2$ be the Lie coalgebra of $G$. Let $[X/\mathfrak{g}] := \operatorname{Spec} A^*_{\mathfrak{g}}$ be the closed subscheme of $[X/G]$ defined by the simplicial ideal generated by $\mathscr{K}^2$, where $\mathscr{K}$ is the ideal of $[X/G]_0 \subset [X/G]_1$, i.e., $\mathscr{K} = \mathfrak{m}_e \otimes A \subset \mathscr{O}(G \times X)$. We set $\mathscr{D}_{\mathfrak{g}}(A) := \mathscr{D}(A^*_{\mathfrak{g}})$, etc. There is a canonical conservative restriction functor

$$\mathrm{Lie} : \mathscr{D}_G(A) \to \mathscr{D}_{\mathfrak{g}}(A).$$

Moreover:

(1) Compatible endomorphisms $T_G$ and $T_X$ of $G$ and $X$ yield an endomorphism of $[X/G]$. We have

$$\mathscr{D}_{T,G}(A) := \mathscr{D}_T(A^*_G), \quad \mathscr{D}_{T,\mathfrak{g}}(A) := \mathscr{D}_T(A^*_{\mathfrak{g}}).$$

(2) For a group scheme $G$, we denote by $G^\natural$ its PD-completion at the unit [3, Sec. 1.2]; this is a group PD-scheme, i.e., a scheme equipped with a PD-ideal. For example, we have $\mathbb{G}_m^\natural((U,T)) = \Gamma(T, (1 + \mathscr{J}_T)^*)$. If $G$ is a group PD-scheme with PD-ideal $\mathfrak{m}_e$, then, in the above, we can also consider the Lie coalgebra in PD-sense $\mathfrak{g}^\vee := \mathfrak{m}_e/\mathfrak{m}_e^{[2]}$.

Objects of $\mathscr{D}_{T,G}(A)$, $\mathscr{D}_{T,\mathfrak{g}}(A)$ are called $G$-, resp. $\mathfrak{g}$-*equivariant $A$-complexes.* For an $A$-complex $M$, a *$G$-equivariant structure* on it is an object $M^*_G \in \mathscr{D}_G(A)$ together with a quasi-isomorphism $M^0_G \xrightarrow{\sim} M$.

(ii) *Equivariant structures on crystalline cohomology.* Let us go back to the setting of Proposition 2.8. We note that the objects $(S_1, E_n) \in (S_1/W_n(k))_{\rm cr}$ and $(S^0_1, S^0_n) \in (S^0_1/W_n(k))_{\rm cr}$ have natural $\mathbb{G}^\natural_m$-actions: $(S_1, E_n)$ is a coordinate thickening (with coordinate $t_a$), $\mathbb{G}^\natural_m$ acts on it by homotheties, and we equip $S^0_n \subset (S_1, E_n)$ with the induced action. To see the latter action explicitly, we note that, for $(U,T) \in (S^0_1/W_n)_{\rm cr}$, a map $f : (U,T) \to (S^0_1, S^0_n)$ amounts to a lifting $f([a])$ of $a \in (\mathfrak{m}_K/\mathfrak{m}^2_K) \setminus \{0\} \subset \mathscr{L}^0_1$ to $\mathscr{L}^0_T$; these liftings form a $\mathbb{G}^\natural_m((U,T))$-torsor yielding our action. This $\mathbb{G}^\natural_m$-action is compatible with the Frobenius action ($\varphi$ acts on $\mathbb{G}^\natural_m$ as $\varphi^*(t) := t^p$).

We will now show that the crystalline cohomology complexes $\mathrm{R}\Gamma_{\rm cr}(X_1/R)_{l,n}$, $\mathrm{R}\Gamma_{\rm HK}(X^0_1)_n$ are naturally equipped with $\mathbb{G}^\natural_m$-equivariant structures. Take the simplicial objects $(S_1, E_{n*})$ and $(S^0_1, S^0_{n*})$. Here, for $(U,T) \in (Z/S)_{\rm cr}$, we wrote $(U, T_*) := \check{C}((U,T)/Z)$ for the Čech nerve of the crystalline open $(U,T) \in (Z/S)_{\rm cr}$; it is a simplicial object of $(Z/S)_{\rm cr}$ with terms $(U, T_a) := (U,T)^{a+1}$ (we use the crystalline site product). It is easy to see [3, Exercise 1.7] that $(S_1, E_{n*}) = [(S_1, E_n)/\mathbb{G}^\natural_m]$ and $(S^0_1, S^0_{n*}) = [(S^0_1, S^0_n)/\mathbb{G}^\natural_m]$. Consider the objects $\mathrm{R}f_{\rm cr}(\mathscr{O}_{X_1/W_n(k)})$ and $\mathrm{R}f^0_{\rm cr}(\mathscr{O}_{X^0_1/W_n(k)})$. They are equipped with a Frobenius action. Restricting them to our simplicial objects, we get:

$$
\begin{aligned}
\mathrm{R}\Gamma_{\rm cr}(X_1/R)^*_{l,n} &:= \mathrm{R}f_{{\rm cr}\,*}(\mathscr{O}_{X_1/W_n(k)})_{(S_1,E_{n*})} \in \mathscr{D}_{\varphi,\mathbb{G}^\natural_m}(R_n),\\
\mathrm{R}\Gamma_{\rm HK}(X^0_1)^*_n &:= \mathrm{R}f^0_{{\rm cr}\,*}(\mathscr{O}_{X^0_1/W_n(k)})_{(S^0_1,S^0_{n*})} \in \mathscr{D}_{\varphi,\mathbb{G}^\natural_m}(W_n(k)).
\end{aligned}
\tag{2.9}
$$

Since $(\mathrm{R}f_{{\rm cr}\,*}(\mathscr{O}_{X_1/W_n(k)})_{(S_1,E_{n*})})^0 \simeq \mathrm{R}\Gamma_{\rm cr}(X_1/R)_{l,n}$ and $(\mathrm{R}f^0_{{\rm cr}\,*}(\mathscr{O}_{X^0_1/W_n(k)})_{(S^0_1,S^0_{n*})})^0 \simeq \mathrm{R}\Gamma_{\rm HK}(X^0_1)_n$, these are the $\mathbb{G}^\natural_m$-equivariant structures we wanted.

We are actually interested in $\mathfrak{n}$-action that comes from the above $\mathbb{G}^\natural_m$-action, where $\mathfrak{n}$ is the Lie algebra of $\mathbb{G}^\natural_m$ in PD-sense (it is a line). The objects from (2.9) form projective systems with respect to $n$. Applying $\mathrm{R}\lim_n$, we get natural $\mathfrak{n}$-structures on $\mathrm{R}\Gamma_{\rm HK}(Z^0_1)$ and $\mathrm{R}\Gamma_{\rm cr}(Z_1/R)_l$. Set $N = e^{-1}t\partial_t$, $e = [K : F]$; it is a generator of $\mathfrak{n} \otimes \mathbf{Q}$. An $\mathfrak{n}_{\mathbf{Q}}$-equivariant structure on $W(k)_{\mathbf{Q}}$-complex amounts to an endomorphism $N$. The equality $N\varphi = p\varphi N$ comes from the compatibility of the $\mathbb{G}^\natural_m$-action with Frobenius.

*Proof.* (*of Proposition 2.8*) We proceed as in the proof of Theorem 2.6 but work in the $\mathbb{G}^\natural_m$-equivariant setting. Namely, we start with the natural map $i^*_l : \mathrm{R}\Gamma_{\rm cr}(X_1/R)^*_{l,\mathbf{Q}} \to \mathrm{R}\Gamma_{\rm HK}(X^0_1)^*_{\mathbf{Q}}$, that lifts the map $i^*_l : \mathrm{R}\Gamma_{\rm cr}(X_1/R)_{l,\mathbf{Q}} \to \mathrm{R}\Gamma_{\rm HK}(X^0_1)_{\mathbf{Q}}$, and we look for its section. This will be a $\mathbb{G}^\natural_m$-equivariant lift of the section in our proposition. We set $P = \mathrm{R}\Gamma_{\rm cr}(X_1/R)^*_l$ and $Q = \mathrm{R}\Gamma_{\rm HK}(X^0_1)^*$ in Lemma 2.4 and work in the category $\mathscr{D}_{\varphi,\mathbb{G}^\natural_m}(W(k))$. We easily see that all the needed assumptions on $P, Q$ are satisfied: the only delicate point is the invertibility, up to a controlled denominator, of the Hyodo-Kato Frobenius; recall that the denominator is controlled by the dimension of the underlying scheme so here we actually need to work with truncated simplicial objects. The proof of the lemma goes through practically verbatim once we take for the projective resolution of $Q$ a cosimplicial, degree by degree, projective resolution. In the end, we get a Frobenius compatible section

$$\iota_l : \mathrm{R}\Gamma_{\rm HK}(X^0_1)^*_{\mathbf{Q}} \to \mathrm{R}\Gamma_{\rm cr}(X_1/R)^*_{l,\mathbf{Q}}.$$

The induced map $\mathrm{Lie}(\iota_l)$ yields a section between the corresponding $\mathfrak{n}_{\mathbf{Q}}$-equivariant structures. Since it lifts the original section $\iota_l$ we get the wanted compatibility of the latter with monodromy. □

2.3. **Geometric absolute crystalline cohomology and Hyodo-Kato cohomology.** We are now ready to prove the existence of geometric Hyodo-Kato quasi-isomorphisms.

2.3.1. *The comparison theorem.* Let now $\overline{f} : X_1 \to \overline{S}_1$ be a map of log-schemes with $X_1$ integral and quasi-coherent. Assume that $\overline{f}$ is the base change of a log-scheme $f : Z_1 \to S_1$, which is log-smooth and with Cartier type reduction. Choose $l$, hence $(S_1, E)$, as in Section 2.1.1. Choose a Frobenius compatible map $\theta_\lambda : (\overline{S}_1, \mathscr{E}_{\rm cr}) \to (S_1, E)$ of PD-thickenings that extends the map $\theta_1$, where $\theta$ is the canonical map $\theta : \overline{S} \to S$. This amounts to a choice of $\lambda_a := \theta_\lambda(t_a) \in \mathscr{L}_\varphi$ that lifts $l_a \in \mathscr{L}_1 \subset \overline{\mathscr{L}}_1$.

The following well-known corollary of Theorem 2.4 describes geometric absolute crystalline cohomology via Hyodo-Kato cohomology (but loosing the Galois action).

**Corollary 2.10.** (1) *There is a functorial system of compatible quasi-isomorphisms*

$$\varepsilon^R_{\lambda,n} : \mathrm{R}\Gamma_{\rm cr}(Z_1/R)_{l,n} \otimes^{\rm L}_{R_n} \mathbf{A}_{{\rm cr},n} \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm cr}(X_1)_n.$$

*They are compatible with Frobenius. Here the tensor product is taken with respect to the map* $\theta^*_{\lambda,n} : R_n \to \mathbf{A}_{{\rm cr},n}$.

(2) *There is a natural strict quasi-isomorphism*

$$\varepsilon^R_\lambda : \mathrm{R}\Gamma_{\rm cr}(Z_1/R)_l \widehat{\otimes}^{\rm L}_R \mathbf{A}_{\rm cr} \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm cr}(X_1)$$

*compatible with Frobenius.*

(3) *There is a natural strict quasi-isomorphism of* $\mathbf{B}^+_{\rm cr}$*-complexes*

$$\varepsilon^{\rm HK}_\lambda : (\mathrm{R}\Gamma_{\rm HK}(Z^0_1)\widehat{\otimes}^{\rm L}_{W(k)} \mathbf{A}_{\rm cr})_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm cr}(X_1)_{\mathbf{Q}}$$

*compatible with Frobenius.*

*Proof.* Since $Z_1$ is log-smooth, claim (1) follows by base change. Claim (2) follows from claim (1) by taking limits. Claim (3) follows from Claim (2) and Theorem 2.4. $\square$

Let $\overline{f}^0 : X^0_1 \to \overline{S}^0_1$ be the pullback of $\overline{f}$ to $\overline{S}^0_1$. We have the *completed geometric Hyodo-Kato cohomology*

$$\mathrm{R}\Gamma_{\rm HK}(X^0_1) := \mathrm{R}\Gamma_{\rm cr}(X^0_1/\overline{S}^0).$$

It is a $W(\overline{k})$-module. It compares with the arithmetic Hyodo-Kato cohomology via the base change quasi-isomorphism

$$\beta : \quad \mathrm{R}\Gamma_{\rm HK}(Z^0_1)\widehat{\otimes}^{\rm L}_{W(k_L)} W(\overline{k}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm HK}(X^0_1), \tag{2.11}$$

which is compatible with Frobenius.

**Theorem 2.12.** *There is a natural strict quasi-isomorphism*

$$\varepsilon^{\rm HK}_{\rm st} : \mathrm{R}\Gamma_{\rm HK}(X^0_1)_{\mathbf{Q}_p} \widehat{\otimes}_{\breve{F},\iota} \mathbf{B}^+_{\rm st} \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm cr}(X_1)_{\mathbf{Q}_p} \widehat{\otimes}_{\mathbf{B}^+_{\rm cr},\iota} \mathbf{B}^+_{\rm st}$$

*compatible with the tensor product, Frobenius and monodromy actions, and such that* $\varepsilon^{\rm HK}_\lambda = s^*_\lambda \varepsilon^{\rm HK}_{\rm st}\beta$.

Here, for $M = \mathrm{R}\Gamma_{\rm HK}(X^0_1), \mathrm{R}\Gamma_{\rm cr}(X_1)$, we have defined[12]

$$M_{\mathbf{Q}_p} \widehat{\otimes}_{\breve{F},\iota} \mathbf{B}^+_{\rm st} := \mathrm{L}\operatorname{colim}_r(M_{\mathbf{Q}_p} \widehat{\otimes}^{\rm R}_{\breve{F}} \mathbf{B}^{\leq r}_{\rm st}), \quad M_{\mathbf{Q}_p} \widehat{\otimes}_{\mathbf{B}^+_{\rm cr},\iota} \mathbf{B}^+_{\rm st} := \mathrm{L}\operatorname{colim}_r(M_{\mathbf{Q}_p} \widehat{\otimes}^{\rm R}_{\mathbf{B}^+_{\rm cr}} \mathbf{B}^{\leq r}_{\rm st}), \tag{2.13}$$

respectively, where $\mathbf{B}^{\leq r}_{\rm st} := \oplus^r_{i=0} \mathbf{B}^+_{\rm cr} u^i_\lambda$, $u_\lambda = \log(\lambda)$, for fixed $\lambda$.

*Proof.* The proof of the theorem runs over sections 2.3.2 (construction on the map) and 2.3.3 (compatibility with all structures). $\square$

[12] This is the only place in the paper that we use inductive tensor products.

2.3.2. *Construction of the quasi-isomophism.*
• *The index sets.* Recall that we have assumed that one can find a finite extension $L/K$ such that $\overline{f}$ is the base change of a fine log-scheme $f_L : Z_1 \to \mathrm{Spec}(\mathscr{O}_{L,1})^\times$, log-smooth and of Cartier type, by the natural map $\theta : \overline{S}_1 \to \mathrm{Spec}(\mathscr{O}_{L,1})^\times$. That is, we have a map $\theta_L : X_1 \to Z_1$ such that the square $(\overline{f}, f_L, \theta, \theta_L)$ is Cartesian. Such data $\Sigma := \{(L, f_L, \theta_L)\}$ clearly form a filtered set[13].

We have similar data $\Sigma^0 := \{(\theta^0, f^0, \theta^0_L)\}$:

(1) $\theta^0 : \overline{S}^0 \to S^{\prime,0}$ is a map of log-schemes over $S^0_K$ with $S^{\prime,0} = \mathrm{Spf}\, W(k')$, where $k' \subset \overline{k}$ is finite over $k$ and the log-structure of $S^{\prime,0}$ is generated by one element; the Frobenius on the log-scheme $S^{\prime,0}$ is induced, via the map $\theta^0$, from the Frobenius on $\overline{S}^0$;
(2) $f^0 : Z^0_1 \to S^{\prime,0}_1$ is log-smooth, fine and integral, of Cartier type;
(3) $\theta^0_X : X^0_1 \to Z^0_1$ is such that the square $(\overline{f}^0, f^0, \theta^0_1, \theta^0_X)$ is Cartesian.

Such data again form a filtered set. There is a map of filtered sets $\Sigma \to \Sigma^0$, $Z_1/S_L \mapsto Z^0_1/S^0_L$; it is cofinal.

• *Construction of* $\varepsilon^{\mathrm{HK}}_{\mathrm{st}}$. Let us first construct $\varepsilon^{\mathrm{HK}}_{\mathrm{st}}$. For $\xi^0 = Z^0_1/S^{\prime,0}_1 \in \Sigma^0$, $S' = \mathrm{Spf}\, \mathscr{O}_{K'}$, let $\Psi_{\xi^0}$ be the set of triples[14] $\pi = (\pi, \pi_S, n_\pi)$, where $n_\pi \in \mathbf{N}$ and $\pi, \pi_S$ are maps such that the diagram

$$\begin{tikzcd}
X^0_1 \arrow[r, hook] \arrow[d, "\theta^0_X"] & X_1 \arrow[r] \arrow[d, "\pi"] & \overline{S}_1 \arrow[d, "\pi_S"] \\
Z^0_1 \arrow[r, "\mathrm{Fr}^{n_\pi}"] & Z^0_1 \arrow[r] & S^{\prime,0}_1
\end{tikzcd} \tag{2.14}$$

commutes. Here we denoted by Fr he absolute Frobenius. The set $\Psi_{\xi^0}$ is ordered: $\pi_1 \leq \pi_2$ means $m = n_{\pi_2}/n_{\pi_1} \in \mathbf{Z}$ and $\pi_2 = \mathrm{Fr}^m \pi_1$. We claim that the set $\Psi_{\xi^0}$ is filtered. For that it suffices to show that, for $n \geq e_{K'}$, any two triples $\pi_1 = (\pi_1, \pi_{S,1}, n)$ and $\pi_2 = (\pi_2, \pi_{S,2}, n)$ are in fact equal, that is, $\pi_1 = \pi_2$ and $\pi_{S,1} = \pi_{S,2}$. But, for $n$ as above, we have the diagram ($\pi = \pi_1, \pi_2$)

$$\begin{tikzcd}
X^0_1 \arrow[drr, hook] \arrow[ddr, "\theta^0_X"'] & & & \\
& X_1 \arrow[ul, dashed, "f"'] \arrow[r, "\mathrm{Fr}^n"'] \arrow[d, "\pi"] & X_1 \arrow[r] \arrow[d, "\pi"] & \overline{S}_1 \arrow[d, "\pi_S"] \\
& Z^0_1 \arrow[r, "\mathrm{Fr}^n"] & Z^0_1 \arrow[r] & S^{\prime,0}_1
\end{tikzcd}$$

in which the two small squares, the square with corner $X^0_1$, and the top triangle commute. This implies that $\mathrm{Fr}^n \pi = \mathrm{Fr}^n \theta^0_X f$. Since there are no nilpotents in $\mathscr{O}_{X^0_1}$, we get $\pi = \theta^0_X f$, hence $\pi_1 = \pi_2$, as wanted.

Similarly, we have the diagram

$$\begin{tikzcd}
\overline{S}_1 \arrow[dr, dashed, "f_S"] \arrow[drr, "\mathrm{Fr}^n"] \arrow[ddr, "\pi_S"'] & & \\
& \overline{S}^0_1 \arrow[r, hook] \arrow[d, "\theta^0"] & \overline{S}_1 \arrow[d, "\pi_S"] \\
& S^{\prime,0}_1 \arrow[r, "\mathrm{Fr}^n"] & S^{\prime,0}_1
\end{tikzcd} \tag{2.15}$$

[13]In [16, 4.3.1], in the case of a semistable formal scheme $\mathscr{X}$ over $\mathscr{O}_C$, we have used a different index set $\Sigma$, call it $\Sigma^{\mathrm{old}}$. It is easy to see that we obtain the same theory with both choices of the index set: if $\mathscr{X}$ is affine then the canonical map $\Sigma^{\mathrm{old}} \to \Sigma$ makes $\Sigma^{\mathrm{old}}$ cofinal in $\Sigma$.

[14]We like to call them *Frobenius-twisted descent data.*

where the square with vertex $\overline{S}_1$ and the top triangle commute. The small square commutes as well: map the commutative diagram

$$\begin{tikzcd} X_1^0 \arrow[r, hook] \arrow[d, "\theta_X^0"] & X_1 \arrow[d, "\pi"] \\ Z_1^0 \arrow[r, "\mathrm{Fr}^n"] & Z_1^0 \end{tikzcd}$$

to it using the canonical maps and use the fact that $\overline{S}_1^0$ is a field. Diagram (2.15) now implies that $\mathrm{Fr}^n\pi_S = \mathrm{Fr}^n\theta^0 f_S$. Since $S_1^{\prime,0}$ is a field we get $\pi_S = \theta^0 f_S$, hence $\pi_{S,1} = \pi_{S,2}$, as wanted.

Let now $e$ be the ramification index of $\mathscr{O}_{K'}$. Denote by $\mathbb{A}^{(1/e)}_{W(k')}$ the formal scheme $\operatorname{Spf} W(k')\{t_a\}$, where $a \in \tau_{1/e}$ is such that $[a]$ lies in the image of the embedding $\theta^{0,*} : \mathscr{L}^{\prime,0} \hookrightarrow \overline{\mathscr{L}}^0$, with $t_{a'} = [a'/a]t_a$. The log-structure is generated by $t_a$. We have an embedding $i : S^{\prime,0} \hookrightarrow \mathbb{A}^{(1/e)}_{W(k')}$, $i^*(t_a) = [a]$. Let $R^{\prime,0} := r^{\mathrm{PD},0}_{K'}$, $E^0 := \operatorname{Spf} R^{\prime,0}$. We have the PD-thickenings $i_0^0 : (S_1^{\prime,0}, E_n^0)$, $i_0^{0,*}(t_a) = a \in \mathscr{L}_1^{\prime,0}$. The map $\pi_l := i\pi_S : \overline{S}_1 \to \mathbb{A}^{(1/e)}_{W(k')}$ induces a map $i_l : \overline{S}_1 \to \mathbb{A}^{(r)}_{W(\overline{k})}$, for $r = p^{n_\pi}/e$, which corresponds (see Section 2.1.1) to a class $l_\pi \in \Lambda$ such that $v(l_\pi) = r$. The map $\pi_S$ extends canonically to a map of PD-thickenings $\pi_{\mathrm{st}} : (\overline{S}_1, \mathscr{E}_{l_\pi,\mathrm{st},n}) \to (S_1^{\prime,0}, E_n^0)$, i.e., we have the following commutative diagram

$$\begin{tikzcd} \overline{S}_1 \arrow[r, hook, "i_l"] \arrow[d, "\pi_S"] & \mathscr{E}_{l_\pi,\mathrm{st},n} \arrow[d, "\pi_E"] \\ S_1^{\prime,0} \arrow[r, hook, "i_0^0"] & E_n^0, \end{tikzcd}$$

where the map $\pi_E$ sends $t_a \mapsto t_{a^{p^{n_\pi}}}$.

We have the maps

$$\mathrm{R}\Gamma_{\mathrm{HK}}(Z_1^0)_n \xleftarrow{i^*} \mathrm{R}\Gamma_{\mathrm{cr}}(Z_1^0/(S_1^{\prime,0}, E_n^0)) \xrightarrow{(\pi^*,\pi^*_{\mathrm{st}})} \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/(\overline{S}_1, \mathscr{E}_{l_\pi,\mathrm{st},n})) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X_1)_n \otimes^{\mathrm{L}}_{\mathbf{A}_{\mathrm{cr},n}} \widehat{\mathbf{A}}_{l_\pi,\mathrm{st},n}. \tag{2.16}$$

By applying $\mathrm{R}\lim_n$ to these complexes we can remove $n$. Now, $i^*_{\mathbf{Q}}$ has a section $\iota$ (use Theorem 2.6 for $E = E^0$). Composing it with the rest of the maps from (2.16) we get a map

$$\varepsilon_{\xi^0,\pi} : \mathrm{R}\Gamma_{\mathrm{HK}}(Z_1^0)_{\mathbf{Q}} \to (\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)\widehat{\otimes}^{\mathrm{L}}_{\mathbf{A}_{\mathrm{cr}}}\widehat{\mathbf{A}}_{l_\pi,\mathrm{st}})_{\mathbf{Q}}$$

that commutes with Frobenius and monodromy.

Before proceeding let us make the following remark.

*Remark* 2.17. Let $M$ be a complex equipped with an $N$-action. Let $M^{N-\mathrm{nilp}} := [M \to M[N^{-1}]]$, where

$$M[N^{-1}] := \mathrm{L}\operatorname{colim}(M \xrightarrow{N} M \xrightarrow{N} \cdots).$$

For $M = \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)$, we have strict quasi-isomorphisms

$$M_{\mathbf{Q}_p}\widehat{\otimes}_{\breve{F},\iota}\mathbf{B}^+_{\mathrm{st}} \xleftarrow{\sim} (M_{\mathbf{Q}_p}\widehat{\otimes}_{\breve{F},\iota}\mathbf{B}^+_{\mathrm{st}})^{N-\mathrm{nilp}} \xrightarrow{\sim} (M_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\widehat{\mathbf{B}}^+_{l,\mathrm{st}})^{N-\mathrm{nilp}} \simeq (M\widehat{\otimes}^{\mathrm{L}}_{W(\overline{k})}\widehat{\mathbf{A}}_{l,\mathrm{st}})^{N-\mathrm{nilp}}_{\mathbf{Q}_p}. \tag{2.18}$$

The last quasi-isomorphism in (2.18) holds because $M$ is built from torsion-free and $p$-adically complete modules. The previous two quasi-isomorphisms are clear algebraically because we can assume that $N$ is globally nilpotent on $M$ (see [34, 0.1]); it is also clear topologically because $M_{\mathbf{Q}_p}$ is built from Banach spaces and $\widehat{\mathbf{B}}^+_{l,\mathrm{st}}$ is a Banach space (so the derived tensor product is given by the tensor product itself).

Similarly, for $M = \mathrm{R}\Gamma_{\mathrm{cr}}(X_1)$ (note that now the action of $N$ on $M$ is trivial), we have strict quasi-isomorphisms

$$M_{\mathbf{Q}_p}\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{cr}},\iota}\mathbf{B}^+_{\mathrm{st}} \xleftarrow{\sim} (M_{\mathbf{Q}_p}\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{cr}},\iota}\mathbf{B}^+_{\mathrm{st}})^{N-\mathrm{nilp}} \xrightarrow{\sim} (M\widehat{\otimes}^{\mathrm{L}}_{\mathbf{A}_{\mathrm{cr}}}\widehat{\mathbf{A}}_{l,\mathrm{st}})^{N-\mathrm{nilp}}_{\mathbf{Q}_p}.$$

Extending the map $\varepsilon_{\xi^0,\pi}$ by $\widehat{\mathbf{A}}_{l_\pi,\mathrm{st}}$-linearity and using the quasi-isomorphism (2.11) we get a map

$$\widehat{\varepsilon}_{\mathrm{st},\xi^0,\pi}: \quad (\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)\widehat{\otimes}^{\mathrm{L}}_{W(\overline{k})}\widehat{\mathbf{A}}_{l_\pi,\mathrm{st}})_{\mathbf{Q}} \to (\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)\widehat{\otimes}^{\mathrm{L}}_{\mathbf{A}_{\mathrm{cr}}}\widehat{\mathbf{A}}_{l_\pi,\mathrm{st}})_{\mathbf{Q}}$$

that is also compatible with Frobenius and monodromy. Now, we define the map

$$\varepsilon_{\mathrm{st},\xi^0,\pi} := (\widehat{\varepsilon}_{\mathrm{st},\xi^0,\pi})^{N_{l_\pi}-\mathrm{nilp}}: \quad (\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)\widehat{\otimes}^{\mathrm{L}}_{W(\overline{k})}\widehat{\mathbf{A}}_{l_\pi,\mathrm{st}})_{\mathbf{Q}}^{N_{l_\pi}-\mathrm{nilp}} \to (\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr}}}\widehat{\mathbf{A}}_{l_\pi,\mathrm{st}})_{\mathbf{Q}}^{N_{l_\pi}-\mathrm{nilp}}.$$

After passing to the category $\mathscr{D}(C_{\mathbf{Q}_p})$, we can use the quasi-isomorphisms from Remark 2.17 and get the map

$$\varepsilon_{\mathrm{st},\xi^0,\pi}: \quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}_p}\widehat{\otimes}_{\breve{F},\iota}\mathbf{B}^+_{\mathrm{st}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(X_1)_{\mathbf{Q}_p}\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{cr}},\iota}\mathbf{B}^+_{\mathrm{st}}.$$

Finally, since the Frobenius is invertible on the Hyodo-Kato cohomology, we can take the map

$$\widetilde{\varepsilon}^{\mathrm{HK}}_{\mathrm{st},\xi^0,\pi} := \varphi^{n_\pi}\varepsilon_{\mathrm{st},\xi^0,\pi}\varphi^{-n_\pi}: \quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}_p} \to \mathrm{R}\Gamma_{\mathrm{cr}}(X_1)\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{cr}},\iota}\mathbf{B}^+_{\mathrm{st}}.$$

Its $\mathbf{B}^+_{\mathrm{st}}$-linearization

$$\varepsilon^{\mathrm{HK}}_{\mathrm{st},\xi^0,\pi}: \quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}_p}\widehat{\otimes}_{\breve{F},\iota}\mathbf{B}^+_{\mathrm{st}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(X_1)_{\mathbf{Q}_p}\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{cr}},\iota}\mathbf{B}^+_{\mathrm{st}} \tag{2.19}$$

is the map we want.

• *Independence of the choice of* $\pi$ *and* $\xi^0$. Fix $\xi^0$. To show that the map $\varepsilon^{\mathrm{HK}}_{\mathrm{st},\xi^0,\pi}$ is independent of Frobenius twists, that is of the choice[15] of $\pi\in\Psi_{\xi^0}$, we note that, for $m\in\mathbf{Z}_{>0}$, there are natural compatible with Frobenius transition maps $\mu_m:\mathscr{E}_{l_\pi,\mathrm{st}}\to\mathscr{E}_{l_\pi^m,\mathrm{st}}, \mu_m^*(t_{a^m})=t_a^m$. Moreover, $\mu^{m_1}\mu^{m_2}=\mu^{m_1m_2}$ and $\mu_m^*\kappa_{l^m}=\kappa_l$. Then the transition map from $\varepsilon^{\mathrm{HK}}_{\mathrm{st},\xi^0,\pi_1}$ to $\varepsilon^{\mathrm{HK}}_{\mathrm{st},\xi^0,\pi_2}$, for $\pi_2\geq\pi_1$, is given by $(\varphi^n)^*$ acting on $\mathrm{R}\Gamma_{\mathrm{cr}}(Z_1^0/(S_1^{\prime,0},E_n^0))$ and $\mu_n^*$, for $n=n_{\pi_2}-n_{\pi_1}$. This suffices since the set $\Psi_{\xi^0}$ is filtered. We set $\varepsilon^{\mathrm{HK}}_{\mathrm{st},\xi^0}:=\varepsilon^{\mathrm{HK}}_{\mathrm{st},\xi^0,\pi}$, for any $\pi\in\Psi_{\xi^0}$.

To show that $\varepsilon^{\mathrm{HK}}_{\mathrm{st},\xi^0}$ does not depend on the choice of $\xi^0\in\Sigma^0$, we use the above maps $\mu$ to identify $\varepsilon^{\mathrm{HK}}_{\mathrm{st},\xi^0_1}$ and $\varepsilon^{\mathrm{HK}}_{\mathrm{st},\xi^0_1}$, for $\xi^0_1\leq\xi^0_2$. This suffices because the set $\Sigma^0$ is filtered. We set $\varepsilon^{\mathrm{HK}}_{\mathrm{st}}:=\varepsilon^{\mathrm{HK}}_{\mathrm{st},\xi^0}$, for any $\xi^0\in\Sigma^0$.

2.3.3. *Compatibility of the arithmetic and geometric maps* $\varepsilon^{\mathrm{HK}}$. It remains to prove that the map $\varepsilon^{\mathrm{HK}}_{\mathrm{st}}$ is a quasi-isomorphism and that the last claim of our corollary holds. For that, assume that $\xi^0$ comes from $\xi=Z_1/K'\in\Sigma$. Choose $l\in\mathscr{L}_1/k^{\prime,*}\subset\overline{\mathscr{L}}_1/\overline{k}^*$. We get a map of PD-thickenings $(\overline{S}_1,\mathscr{E}_{l,\mathrm{st},n})\to(S_1,E_n)$ that identifies the $t_a$'s. This yields the base change quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{cr}}(Z_1/R)_l\widehat{\otimes}_{R_n}\widehat{\mathbf{A}}_{l,\mathrm{st},n}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_1/(\overline{S}_1,\mathscr{E}_{l,\mathrm{st},n}))\xleftarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)\widehat{\otimes}_{W(\overline{k})}\widehat{\mathbf{A}}_{l,\mathrm{st},n}.$$

By applying $\mathrm{R}\lim_n$ we remove $n$. Composing with the $\widehat{\mathbf{A}}_{l,\mathrm{st}}$-linear extension of $\iota_l$ from Theorem 2.6 we get the strict quasi-isomorphism

$$\widehat{\varepsilon}_{\mathrm{st},l}:(\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)\widehat{\otimes}^{\mathrm{L}}_{W(\overline{k})}\widehat{\mathbf{A}}_{l,\mathrm{st}})_{\mathbf{Q}}\xrightarrow{\sim}(\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)\widehat{\otimes}^{\mathrm{L}}_{W(\overline{k})}\widehat{\mathbf{A}}_{l,\mathrm{st}})_{\mathbf{Q}}$$

Denote by

$$\varepsilon^{\mathrm{HK}}_{\mathrm{st},l}:=\kappa_l^{-1}(\widehat{\varepsilon}_{\mathrm{st},l})^{N-\mathrm{nilp}}\kappa_l: \quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}_p}\widehat{\otimes}_{\breve{F},\iota}\mathbf{B}^+_{\mathrm{st}}\to\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)_{\mathbf{Q}_p}\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{cr}},\iota}\mathbf{B}^+_{\mathrm{st}}$$

the associated map. It is a strict quasi-isomorphism. Now, choose $m$ large enough so that the action of $Fr^m$ on $Z_1$ factors as $Z_1\xrightarrow{F_m}Z_1^0\hookrightarrow Z_1$. Take $\pi:=(F_m\theta_X,F_{S,m}\theta_1,m)\in\Psi_{\xi^0}$. It is easy to see, using the uniqueness statement from Theorem 2.6, that the associated map $\varepsilon^{\mathrm{HK}}_{\mathrm{st},\xi^0,\pi}$ equals $\varepsilon^{\mathrm{HK}}_{\mathrm{st},l}$. In particular, the map $\varepsilon^{\mathrm{HK}}_{\mathrm{st}}$ is a strict quasi-isomorphism, as wanted.

The final claim of the theorem follows since $\varepsilon^{\mathrm{HK}}_{\mathrm{st},\lambda}=s_\lambda^*\varepsilon^{\mathrm{HK}}_{\mathrm{st},l}$.

[15] Up to a contractible set of choices, of course.

2.3.4. *Comparison between Hyodo-Kato and de Rham cohomologies.* Theorem 2.12 implies the following Hyodo-Kato-to-de Rham quasi-isomorphisms:

**Corollary 2.20.** *We have natural strict quasi-isomorphisms*

$$\varepsilon^{\mathrm{HK}}_{\mathrm{dR}}:\quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_1/\overline{S})_{\mathbf{Q}_p},\tag{2.21}$$

$$\varepsilon^{\mathrm{HK}}_{\mathbf{B}^+_{\mathrm{dR}}}:\quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{cr}}}\mathbf{B}^+_{\mathrm{dR}}.$$

*They are compatible via the maps $\theta:\mathbf{B}^+_{\mathrm{dR}}\to C$ and $\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)\to\mathrm{R}\Gamma_{\mathrm{cr}}(X_1/\overline{S})$.*

*Proof.* From Theorem 2.12 we have a natural strict quasi-isomorphism

$$\varepsilon^{\mathrm{HK}}_{\mathrm{st}}:\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}_p}\widehat{\otimes}_{\breve{F},\iota}\mathbf{B}^+_{\mathrm{st}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)_{\mathbf{Q}_p}\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{cr}},\iota}\mathbf{B}^+_{\mathrm{st}}.\tag{2.22}$$

Take the map $\mathbf{B}^+_{\mathrm{st}}\to\mathbf{B}^+_{\mathrm{cr}}$ given by sending $\log(\lambda_p)\mapsto 0$. It is not Galois equivariant but this will not be a problem for us. Applying it to the quasi-isomorphism (2.22), which is $\mathbf{B}^+_{\mathrm{st}}$-linear, we get a strict quasi-isomorphism

$$\varepsilon^{\mathrm{HK}}_{\mathrm{cr}}:\quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{cr}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)_{\mathbf{Q}_p}.\tag{2.23}$$

We tensor it now over $\mathbf{B}^+_{\mathrm{cr}}$ with C. By Lemma 2.2, we obtain the strict quasi-isomorphism

$$\tilde{\varepsilon}^{\mathrm{HK}}_{\mathrm{dR}}:\quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{L}}_{\mathbf{B}^+_{\mathrm{cr}}}C\tag{2.24}$$

and, composing with the strict quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{L}}_{\mathbf{B}^+_{\mathrm{cr}}}C\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_1/\overline{S})_{\mathbf{Q}_p},$$

the quasi-isomorphism $\varepsilon^{\mathrm{HK}}_{\mathrm{dR}}$ from our corollary. We note that $\varepsilon^{\mathrm{HK}}_{\mathrm{dR}}$ is compatible with the Galois action because $\sigma(\log(\lambda_p))-\log(\lambda_p)\in\operatorname{Ker}\theta$.

Proceeding as above we get the strict quasi-isomorphism

$$\tilde{\varepsilon}^{\mathrm{HK}}_{\mathrm{dR}}:\quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}(\mathbf{B}^+_{\mathrm{cr}}/F^i)\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_1)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{L}}_{\mathbf{B}^+_{\mathrm{cr}}}(\mathbf{B}^+_{\mathrm{cr}}/F^i),\quad i\geq 0.\tag{2.25}$$

Taking $\mathrm{R}\lim_i$ of both sides gives us now the second strict quasi-isomorphism of the theorem. $\square$

## 3. $\mathbf{B}^+_{\mathrm{dR}}$-COHOMOLOGY

This section is devoted to the definitions of rigid analytic and overconvergent $\mathbf{B}^+_{\mathrm{dR}}$-cohomologies $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$, for $X\in\mathrm{Sm}_C$ or $X\in\mathrm{Sm}^{\dagger}_C$, and to the study of their basic properties. These cohomologies are replacements for $\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_C\mathbf{B}^+_{\mathrm{dR}}$ which does not exist since there is no continuous ring morphism $C\to\mathbf{B}^+_{\mathrm{dR}}$ although $\overline{K}$ is naturally a subring of $\mathbf{B}^+_{\mathrm{dR}}$: if $X$ is defined over $K$, then $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\simeq\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}$. In general, we have the relation $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{dR}}}\mathbf{C}\simeq\mathrm{R}\Gamma_{\mathrm{dR}}(X)$ (see Proposition 3.11 and Proposition 3.26 for this comparison and analogous results concerning filtrations).

In the next chapter, using the Hyodo-Kato map, we will prove that, if $X\in\mathrm{Sm}_C$ is partially proper, then the rigid analytic and overconvergent $\mathbf{B}^+_{\mathrm{dR}}$-cohomologies give the same result: if $X^{\dagger}$ is the associated dagger variety, the natural map $\mathrm{R}\Gamma_{\mathrm{dR}}(X^{\dagger}/\mathbf{B}^+_{\mathrm{dR}})\to\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ is a strict quasi-isomorphism (Corollary 4.32).

Our rigid analytic $\mathbf{B}^+_{\mathrm{dR}}$-cohomology is defined by, locally, Hodge-completing absolute crystalline cohomology, but it gives the same object (Proposition 3.24) as the constructions of Bhatt-Morrow-Scholze [7] and Guo [26] via the infinitesimal site.

3.1. **CliffsNotes.** For a quick reference, we will recall now some results from [16] and add few complements.

3.1.1. *Review.* We start with a review of [16].

**Proposition 3.1.** (Colmez-Nizioł, [16, Th. 1.1])

(1) **Dagger varieties**: *To any smooth dagger variety $X$ over $L = K, C$ there are naturally associated:*
   (a) *A pro-étale cohomology* $\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r))$, $r \in \mathbf{Z}$.
   (b) *For $L = C$, a $\overline{K}$-valued rigid cohomology* $\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X)$ *and a natural strict quasi-isomorphism*[16]

$$\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X)\widehat{\otimes}^{\mathrm{R}}_{\overline{K}}C \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

   *This defines a natural $\overline{K}$-structure on the de Rham cohomology*[17].
   (c) *A Hyodo-Kato cohomology*[18] $\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)$. *This is a dg $F$-algebra if $L = K$, and a dg $F^{\mathrm{nr}}$-algebra if $L{=}C$, equipped with a Frobenius $\varphi$ and a monodromy operator $N$. For $L = C$, we have natural Hyodo-Kato strict quasi-isomorphisms*

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\overline{K} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X), \quad \iota_{\mathrm{HK}} : \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}C \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

   (d) *For $L = K$, a syntomic cohomology* $\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{syn}}(X, \mathbf{Q}_p(r))$, $r \in \mathbf{N}$, *that fits into a distinguished triangle*

$$\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \longrightarrow [\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r,$$

   *and a natural period map*

$$\alpha_r : \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)).$$

   *It is a strict quasi-isomorphism after truncation $\tau_{\leq r}$.*
   (e) (Local-global compatibility) *In the case $X$ has a semistable weak formal model the above constructions are compatible with their analogs defined using the model.*

(2) **Rigid analytic varieties**: *To any smooth rigid analytic variety $X$ over $L = K, C$ there are naturally associated:*
   (a) *For $L = C$, a $\overline{K}$-valued convergent cohomology* $\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X)$ *and a natural strict quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X)\widehat{\otimes}^{\mathrm{R}}_{\overline{K}}C \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

   *This defines a natural $\overline{K}$-structure on the de Rham cohomology.*
   (b) *A Hyodo-Kato cohomology* $\mathrm{R}\Gamma_{\mathrm{HK}}(X)$. *This is a dg $F$-algebra if $L = K$, and a dg $F^{\mathrm{nr}}$-algebra if $L{=}C$, equipped with a Frobenius $\varphi$ and a monodromy operator $N$. For $L = C$, we have natural Hyodo-Kato strict quasi-isomorphisms*

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\overline{K} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X), \quad \iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}C \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

   (c) *For $L = K, C$, a natural period map*

$$\alpha_r : \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)).$$

   *It is a strict quasi-isomorphism after truncation $\tau_{\leq r}$.*
   (d) (Local-global compatibility) *In the case $X$ has a semistable formal model the constructions in* (a), (b) *are compatible with their analogs defined using the model. This is also the case in* (c), *for $L = K$.*

---

[16] See [16, Prop. 5.20] for the definition of the tensor product.

[17] By the same procedure one can define a $F^{\mathrm{nr}}$-valued rigid cohomology $\mathrm{R}\Gamma_{\mathrm{rig},F^{\mathrm{nr}}}(X)$ and a natural strict quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{rig},F^{\mathrm{nr}}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}C \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X)$.

[18] To distinguish this overconvergent Hyodo-Kato cohomology – which was defined by Grosse-Klönne – from the Hyodo-Kato cohomology defined later in this paper we will add the subscript GK to the former. Similarly, we will distinguished the induced overconvergent syntomic cohomology.

(3) **Compatibility**: *For $L = K, C$, let $X$ be a smooth dagger variety over $L$ and let $\widehat{X}$ denote its completion. Then:*

(a) *There exists a natural map* [16, Sec. 3.2.4]

$$\iota_{\text{proét}} : \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{proét}}(\widehat{X}, \mathbf{Q}_p(r)) \quad r \in \mathbf{Z}.$$

*It is a strict quasi-isomorphism if $X$ is partially proper.*

(b) *There exists natural maps*

$$\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X) \to \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}), \quad \mathrm{R}\Gamma_{\mathrm{dR}}(X) \to \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}).$$

*The first map is compatible with Frobenius and monodromy.* If $X$ is partially proper, *the second map is a strict quasi-isomorphism; the first map is a quasi-isomorphism if $L = K$ or if $X$ comes from a dagger variety defined over a finite extension of $K$.*

(c) *For $L = K$, there is a natural map*

$$\iota^{\mathrm{GK}} : \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X}, \mathbf{Q}_p(r))$$

*and the following diagram commutes*

$$\begin{tikzcd}
\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \arrow[r, "\alpha_r"] \arrow[d, "\iota^{\mathrm{GK}}"] & \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \arrow[d, "\iota_{\text{proét}}"] \\
\mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X}, \mathbf{Q}_p(r)) \arrow[r, "\alpha_r"] & \mathrm{R}\Gamma_{\text{proét}}(\widehat{X}, \mathbf{Q}_p(r))
\end{tikzcd}$$

*Remark* 3.2. (i) Below, in Section 6.2.2, we will define the overconvergent period map in 1d over $C$ and, in Proposition 6.8, we will remove the condition $L = K$ in 3c. To do this we could not use the constructions from [14] and [16]: the first one was not functorial enough, the second one, using a "killing nilpotents" trick, just did not transfer to the geometric setting. This depressing state of affairs made us take a break of more than a year from the project before coming back to it with an approach that adapts to the analytic setting an early construction by Beilinson of the Hyodo-Kato quasi-isomorphism.

(ii) The local-global compatibility for rigid analytic geometric syntomic cohomology also holds. This will be proved in Proposition 5.3 using local-global compatibility for Hyodo-Kato and $\mathbf{B}^+_{\mathrm{dR}}$-cohomologies.

3.1.2. *Complements.* Now we pass to complementary results.

(1) *$\eta$-étale descent.* The following proposition should have been included in [16].

**Proposition 3.3.** *Let $(\mathscr{B}, F)$ be a Beilinson base*[19] *of an essentially small site $\mathscr{V}$. Then:*

(1) *The functor $F : \mathscr{B} \to \mathscr{V}$ is continuous.*

(2) *$F$ induces an equivalence of topoi*

$$\mathrm{Sh}(\mathscr{B}) \xrightarrow{\sim} \mathrm{Sh}(\mathscr{V}).$$

(3) *Let $\mathscr{D}$ be an $\infty$-category. Then $F$ induces an equivalence of $\infty$-topoi*

$$\mathrm{Sh}^{\mathrm{hyp}}(\mathscr{B}, \mathscr{D}) \xrightarrow{\sim} \mathrm{Sh}^{\mathrm{hyp}}(\mathscr{V}, \mathscr{D})$$

*of hypersheaves.*

*Proof.* Claims (1) and (2) were shown in the proof of [16, Prop. 2.2]. Claim (3) follows easily from (2) and the fact that the Jardine model structure on simplicial presheaves presents the $\infty$-topos of hypercomplete sheaves (see [31, Prop. 6.5.2.14]). $\square$

[19] Such a base was introduced by Beilinson in [2, 2.1]; it is a slightly more general notion than that of a Verdier base which is commonly used.

The example most relevant for this paper is the following: $\mathscr{V} = \mathrm{Sm}_{C,\mathrm{\acute{e}t}}$, the site of smooth rigid analytic varieties over $C$ equipped with the étale topology. $\mathscr{V}$ has a Beilinson base $(\mathscr{M}, F_\eta)$, where $\mathscr{M}$ is the category of basic semistable formal models $\mathscr{M}_C^{\mathrm{ss},b}$ or semistable formal models $\mathscr{M}_C^{\mathrm{ss}}$ and $F_\eta$ is the forgetful functor $\mathscr{X} \to \mathscr{X}_\eta$ from formal schemes to their rigid analytic generic fibers (see [16, Prop. 2.8]). We have similar constructions for the site $\mathscr{V} = \mathrm{Sm}^\dagger_{C,\mathrm{\acute{e}t}}$ of smooth dagger varieties over $C$.

If $\mathscr{F}$ is a presheaf on a Beilinson base $\mathscr{B}$, then the presheaf on $\mathscr{V}$ defined by

$$U \mapsto (\mathscr{F}^a(U) := \mathrm{L}\operatorname{colim}\mathscr{F}(V_\bullet)),$$

where the colimit is taken over hypercoverings from $\mathscr{B}$, defines a sheaf on $\mathscr{V}$. In the context of the above example of a Beilinson base we call it *$\eta$-étale descent of $\mathscr{F}$*.

(2) The following corollary removes the condition $L = K$ in 3b of Proposition 3.1 and could have been included in [16].

**Corollary 3.4.** *Let $X$ be a smooth partially proper dagger variety over $C$ and let $\widehat{X}$ denote its completion. Let $W$ be a Fréchet space over $\breve{F}$. Then the natural map*[20]

$$\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W \to \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X})\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W$$

*is a strict quasi-isomorphism.*

*Proof.* Find an admissible covering of $X$ by dagger affinoids and then look at the set of the interiors of their completions. By the definition of partially proper dagger varieties this is an admissible covering of $X$ as well. The individual varieties in the covering are partially proper and, moreover, are defined over a finite extension of $K$. The latter fact is true because the corresponding rigid affinoids are defined over a finite extension of $K$ by Elkik's theorem [18, Th. 7, Rem. 2] (the finite presentation condition there is satisfied in our case by the finiteness theorems of Grauert-Remmert-Gruson and Gruson-Raynaud [32, Th. 3.1.17, Th. 3.2.1]). Same can be said about the intersections of a finite number of them.

Now, taking the associated Čech cover and evaluating on it the morphism from the corollary we get a strict quasi-isomorphism by point 3b of Proposition 3.1. We conclude by rigid analytic descent. $\square$

(3) We will recall now a result from [16] together with a new proof (since the proof supplied in loc. cit. is a bit sketchy). This proof will serve us as a template for proofs of analogous claims.

**Proposition 3.5.** (Local-global compatibility, [16, Prop. 4.23]) *Let $\mathscr{X} \in \mathscr{M}_C^{\mathrm{ss},b}$. The natural map*

$$\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{X}_1) \to \mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{X}_C)$$

*is a strict quasi-isomorphism.*

*Proof.* It suffices to show that, for any $\eta$-étale hypercovering $\mathscr{U}_\bullet$ of $\mathscr{X}$ from $\mathscr{M}_C^{\mathrm{ss},b}$, the natural map

$$\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{X}_1) \to \mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{U}_{\bullet,1})$$

is a strict quasi-isomorphism (modulo taking a refinement of $\mathscr{U}_\bullet$). We may assume that in every degree of the hypercovering we have a finite number of formal models. Passing to cohomology ($\widetilde{H}(-)$-cohomology) and then to a truncated hypercovering we can assume that all the formal schemes mod $p$ and maps between them that are involved are defined over a common field $L$ (we will denote them with subscript $\mathscr{O}_L$), a finite extension of $K$. We may leave that way the category of semistable models but we will still be in the category of log-smooth models (with Cartier type reduction). We are reduced to showing that the map

$$\alpha : \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/S_L) \to \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{U}_{\bullet,\mathscr{O}_L,1}/S_L) \tag{3.6}$$

is a strict quasi-isomorphism.

[20] See the point (3) below for the reminder on the definition of the tensor products used.

Tensoring both sides of (3.6) with $C$ over $L$ we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/S_L)\widehat{\otimes}^{\mathrm{R}}_L C \arrow[r, "\alpha_C"] \arrow[d, "\wr"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{U}_{\bullet,\mathscr{O}_L,1}/S_L)\widehat{\otimes}^{\mathrm{R}}_L C \arrow[d, "\wr"] \\
\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C) \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{U}_{\bullet,C}).
\end{tikzcd}$$

Since the bottom map is a strict quasi-isomorphism (by étale descent) so is the top map $\alpha_C$. We claim that this, in turn, implies that the map $\alpha$ itself is a strict-quasi-isomorphism. Indeed, passing to homotopy fibers of the horizontal arrows in the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/S_L) \arrow[r, "\alpha"] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{U}_{\bullet,\mathscr{O}_L,1}/S_L) \arrow[d] \\
\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/S_L)\widehat{\otimes}^{\mathrm{R}}_L C \arrow[r, "\alpha_C"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{U}_{\bullet,\mathscr{O}_L,1}/S_L)\widehat{\otimes}^{\mathrm{R}}_L C,
\end{tikzcd}$$

we see that it suffices to prove the following claim:

if $A \in \mathscr{D}(C_L)$ is a complex such that $A\widehat{\otimes}^{\mathrm{R}}_L C$ is strictly acyclic then $A$ is strictly acyclic as well.

To show this, write $C \simeq L \oplus W$, for a Banach space $W \in C_L$, and conclude. □

(4) Let $X$ be a smooth rigid analytic variety. In [16], we have considered a number of nonstandard tensor products. For example, we have defined (see [16, 4.21]):

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\overline{K} := \mathrm{L}\operatorname{colim}((\mathrm{R}\Gamma_{\mathrm{HK}}\otimes_{F^{\mathrm{nr}}}\overline{K})(\mathscr{U}_{\bullet,1})),$$
$$\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}C := \mathrm{L}\operatorname{colim}((\mathrm{R}\Gamma_{\mathrm{HK}}\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}C)(\mathscr{U}_{\bullet,1})),$$

where the homotopy colimit is taken over affinoid $\eta$-étale hypercoverings $\mathscr{U}_\bullet$ from $\mathscr{M}^{\mathrm{ss},b}_C$. These tensor products satisfy local-global compatibility. A fact that can be proved as in the following example:

**Proposition 3.7.** (Local-global compatibility for tensor products) *Let $\mathscr{X} \in \mathscr{M}^{\mathrm{ss},b}_C$. The canonical maps*

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\widehat{\otimes}_{F^{\mathrm{nr}}}\overline{K} \to \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\overline{K},$$
$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}C \to \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}C$$

*are strict quasi-isomorphisms.*

*Proof.* For the second morphism, proceeding as in the proof of Proposition 3.5 and using its notation, we reduce to showing that the canonical map

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,1})\widehat{\otimes}^{\mathrm{R}}_{F_L}C \to \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{U}_{\bullet,\mathscr{O}_L,1})\widehat{\otimes}^{\mathrm{R}}_{F_L}C$$

is a strict quasi-isomorphism. But this is clear since, via the Hyodo-Kato morphism, this map is strictly quasi-isomorphic to the map

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C) \to \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{U}_{\bullet,C}),$$

which is a strict quasi-isomorphism.

For the first morphism, we proceed in the same way ending up with the strict quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L) \to \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{U}_{\bullet,L}).$$

Passing to homotopy colimit over finite extensions of $L$ in $\overline{K}$, we finish the argument. □

*Remark* 3.8. (1) The local-global compatibility of nonstandard tensor products (see [16, 5.16]) also holds for the dagger varieties and the Grosse-Klönne Hyodo-Kato cohomology. The proof of this fact is a simple analog of the proof of Proposition 3.7.

(2) In Proposition 3.7 we can replace $C$ with any Fréchet space $B$ over $\breve{F}$. This requires just a slight modification of the proof: pass from $\mathrm{R}\Gamma_{\mathrm{HK}}(-)\widehat{\otimes}^{\mathrm{R}}_{F_L}B$ to $(\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(-)\widehat{\otimes}^{\mathrm{R}}_{F_L}C)\widehat{\otimes}^{\mathrm{R}}_{F_L}B$, use the Hyodo-Kato morphism to pass to de Rham cohomology $\mathrm{R}\Gamma_{\mathrm{dR}}((-)_C)\widehat{\otimes}^{\mathrm{R}}_{F_L}B$, use étale descent for de Rham cohomology, and go back to $\mathrm{R}\Gamma_{\mathrm{HK}}(-)\widehat{\otimes}^{\mathrm{R}}_{F_L}B$ via $C\simeq F_L\oplus W$.

3.2. **Geometric crystalline cohomology.** Our rigid analytic $\mathbf{B}^+_{\mathrm{dR}}$-cohomology will be defined locally as a completion of the absolute crystalline cohomology. We will start then by recalling the definition of the latter.

3.2.1. *Relative crystalline cohomology.* Let $\overline{f}:X_1\to\overline{S}_1$ be a map of log-schemes, with integral quasi-coherent source. Suppose that $\overline{f}$ is the base change of a fine log-smooth log-scheme $f_L:Z_1\to S_{L,1}$, by the natural map $\theta:\overline{S}_1\to S_{L,1}$, for a finite extension $L/K$. That is, we have a map $\theta_L:X_1\to Z_1$ such that the square $(\overline{f},f_L,\theta,\theta_L)$ is Cartesian. Such data $\Sigma_1:=\{(L,f_L,\theta_L)\}$ form a filtered set.

(a) *$C$-version.* Let $\mathscr{A}^{\mathrm{rel}}_{\mathrm{cr}}$ be the $\eta$-étale sheafification of the presheaf $\mathscr{X}\mapsto\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}/\overline{S})_{\mathbf{Q}_p}$ on $\mathscr{M}^{\mathrm{ss},b}_C$. For $X\in\mathrm{Sm}_C$, we set

$$\mathrm{R}\Gamma^{\mathrm{rel}}_{\mathrm{cr}}(X):=\mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(X,\mathscr{A}^{\mathrm{rel}}_{\mathrm{cr}}).$$

It is a dg $C$-algebra equipped with a continuous action of $\mathscr{G}_K$ if $X$ is defined over $K$. We equip it with the topology induced by $\eta$-étale descent from the topology of the $\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}/\overline{S})$'s. Since the models $\mathscr{X}$ are log-smooth over $\overline{S}$, we have natural (strict) quasi-isomorphisms

$$\mathscr{A}^{\mathrm{rel}}_{\mathrm{cr}}\simeq\mathscr{A}_{\mathrm{dR}},\quad \mathrm{R}\Gamma^{\mathrm{rel}}_{\mathrm{cr}}(X)\simeq\mathrm{R}\Gamma_{\mathrm{dR}}(X).\tag{3.9}$$

(b) *$\overline{K}$-version.* Let $\mathscr{A}_{\mathrm{cr},\overline{K}}$ be the étale sheafification of the presheaf $\mathscr{X}\mapsto\mathrm{R}\Gamma_{\mathrm{cr},\overline{K}}(\mathscr{X})$ on $\mathscr{M}^{\mathrm{ss},b}_C$, where we set

$$\mathrm{R}\Gamma_{\mathrm{cr},\overline{K}}(\mathscr{X}):=\mathrm{L}\operatorname{colim}_{\Sigma_1}\mathrm{R}\Gamma_{\mathrm{cr}}(Z_1/S_L)_{\mathbf{Q}_p}.$$

For $X\in\mathrm{Sm}_C$, we set $\mathrm{R}\Gamma_{\mathrm{cr},\overline{K}}(X):=\mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(X,\mathscr{A}_{\mathrm{cr},\overline{K}})$. It is a dg $\overline{K}$-algebra equipped with a continuous action of $\mathscr{G}_K$ if $X$ is defined over $K$ (this action is smooth if $X$ is quasi-compact). We equip it with the topology induced by $\eta$-étale descent from the topology of the $\mathrm{R}\Gamma_{\mathrm{cr}}(Z_1/S_L)$'s. There are natural continuous morphisms

$$\mathscr{A}_{\mathrm{cr},\overline{K}}\to\mathscr{A}^{\mathrm{rel}}_{\mathrm{cr}},\quad \mathrm{R}\Gamma_{\mathrm{cr},\overline{K}}(X)\to\mathrm{R}\Gamma^{\mathrm{rel}}_{\mathrm{cr}}(X).$$

**Lemma 3.10.** (1) *Let $\mathscr{X}\in\mathscr{M}^{\mathrm{ss},b}_C$. The natural map*

$$\mathrm{R}\Gamma_{\mathrm{cr},\overline{K}}(\mathscr{X})\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr},\overline{K}}(\mathscr{X}_C)$$

*is a strict quasi-isomorphism.*

(2) *For $X\in\mathrm{Sm}_K$, the natural map*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}_K\overline{K}\to\mathrm{R}\Gamma_{\mathrm{cr},\overline{K}}(X_C)$$

*is a filtered strict quasi-isomorphism.*

*Proof.* Since, for $\mathscr{Y}\in\mathscr{M}^{\mathrm{ss},b}_C$, the natural map

$$\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{Y})\to\mathrm{R}\Gamma_{\mathrm{cr},\overline{K}}(\mathscr{Y})$$

is a strict quasi-isomorphism, it induces a strict quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X)\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr},\overline{K}}(X),\quad X\in\mathrm{Sm}_C.$$

Hence our lemma follows from analogous claims for convergent $\overline{K}$-cohomology which are known (see [16, the proof of Prop. 4.23] or Section 3.1.2). $\square$

3.2.2. *Absolute crystalline cohomology.* Let $\mathscr{X} \in \mathscr{M}_C^{\rm ss}$. Recall that we have the absolute crystalline cohomology $\mathrm{R}\Gamma_{\rm cr}(\mathscr{X})_{\mathbf{Q}_p}$ equipped with the Hodge filtration $F^r\mathrm{R}\Gamma_{\rm cr}(\mathscr{X})_{\mathbf{Q}_p} := \mathrm{R}\Gamma_{\rm cr}(\mathscr{X}, \mathscr{J}^{[r]})_{\mathbf{Q}_p}$, for $r \geq 0$. Let $\mathscr{A}_{\rm cr}$ and $F^r\mathscr{A}_{\rm cr}$, $r \geq 0$, be the $\eta$-étale sheafifications of the presheaves $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\rm cr}(\mathscr{X})_{\mathbf{Q}_p}$ and $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\rm cr}(\mathscr{X}, \mathscr{J}^{[r]})_{\mathbf{Q}_p}$, respectively, on $\mathscr{M}_C^{\rm ss}$. For $X \in \mathrm{Sm}_C$, we set

$$\mathrm{R}\Gamma_{\rm cr}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\rm cr}), \quad F^r\mathrm{R}\Gamma_{\rm cr}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, F^r\mathscr{A}_{\rm cr}), \quad r \geq 0.$$

It is a dg filtered $\mathbf{B}_{\rm cr}^+$-algebra equipped with a continuous action of $\mathscr{G}_K$ if $X$ is defined over $K$. We equip it with the topology induced by $\eta$-étale descent from the topology of the $\mathrm{R}\Gamma_{\rm cr}(\mathscr{X}, \mathscr{J}^{[r]})_{\mathbf{Q}_p}$'s.

The local-global comparison requires the Hyodo-Kato quasi-isomorphism and will be proven in Lemma 5.2 below.

## 3.3. Rigid analytic $\mathbf{B}_{\rm dR}^+$-cohomology.

We will define now rigid analytic $\mathbf{B}_{\rm dR}^+$-cohomology, list its basic properties, and compare it with already existing definitions.

3.3.1. *Definition of rigid analytic $\mathbf{B}_{\rm dR}^+$-cohomology.* Let $\mathscr{X} \in \mathscr{M}_C^{\rm ss}$. To define rigid analytic $\mathbf{B}_{\rm dR}^+$-cohomology, we start with the absolute crystalline cohomology $\mathrm{R}\Gamma_{\rm cr}(\mathscr{X})_{\mathbf{Q}_p}$ and complete it with respect to the Hodge filtration $F^r\mathrm{R}\Gamma_{\rm cr}(\mathscr{X})_{\mathbf{Q}_p}$, $r \geq 0$:

$$\mathrm{R}\Gamma_{\rm cr}(\mathscr{X})\widehat{_{\mathbf{Q}_p}} := \mathrm{R}\lim_r(\mathrm{R}\Gamma_{\rm cr}(\mathscr{X})_{\mathbf{Q}_p}/F^r), \quad \mathrm{R}\Gamma_{\rm cr}(\mathscr{X}, \mathscr{J}^{[r]})\widehat{_{\mathbf{Q}_p}} := \mathrm{R}\lim_{j\geq r}(\mathrm{R}\Gamma_{\rm cr}(\mathscr{X}, \mathscr{J}^{[r]})_{\mathbf{Q}_p}/F^j).$$

This is a dg filtered $\mathbf{B}_{\rm dR}^+$-algebra. The corresponding $\eta$-étale sheafifications on $\mathscr{M}_C^{\rm ss}$ we will denote by $F^r\widehat{\mathscr{A}_{\rm cr}}$, $r \geq 0$. We have canonical maps $\kappa : F^r\mathscr{A}_{\rm cr} \to F^r\widehat{\mathscr{A}_{\rm cr}}$, $r \geq 0$. Moreover, the canonical map $\vartheta : F^r\mathrm{R}\Gamma_{\rm cr}(\mathscr{X})_{\mathbf{Q}_p} \to F^r\mathrm{R}\Gamma_{\rm cr}(\mathscr{X}/\overline{S})_{\mathbf{Q}_p}$, compatible with the map $\theta : \mathbf{B}_{\rm cr}^+ \to C$, extends to a map $\vartheta : F^r\mathrm{R}\Gamma_{\rm cr}(\mathscr{X})\widehat{_{\mathbf{Q}_p}} \to F^r\mathrm{R}\Gamma_{\rm cr}(\mathscr{X}/\overline{S})_{\mathbf{Q}_p}$, which, in turn, globalizes to a map $\vartheta : F^r\widehat{\mathscr{A}_{\rm cr}} \to F^r\mathscr{A}_{\rm cr}^{\rm rel}$.

For $X \in \mathrm{Sm}_C$, define the $\mathbf{B}_{\rm dR}^+$-cohomology:

$$\mathrm{R}\Gamma_{\rm dR}(X/\mathbf{B}_{\rm dR}^+) := \mathrm{R}\Gamma_{\text{ét}}(X, \widehat{\mathscr{A}_{\rm cr}}), \quad F^r\mathrm{R}\Gamma_{\rm dR}(X/\mathbf{B}_{\rm dR}^+) := \mathrm{R}\Gamma_{\text{ét}}(X, F^r\widehat{\mathscr{A}_{\rm cr}}), \quad r \geq 0.$$

This is a dg filtered $\mathbf{B}_{\rm dR}^+$-algebra, equipped with a continuous action of $\mathscr{G}_K$ if $X$ is defined over $K$. We equip it with the topology induced by $\eta$-étale descent from the topology of the $\mathrm{R}\Gamma_{\rm cr}(\mathscr{X}, \mathscr{J}^{[r]})\widehat{_{\mathbf{Q}_p}}$'s.

We have canonical maps

$$\begin{aligned}
\kappa : &\quad \mathrm{R}\Gamma_{\rm cr}(X) \to \mathrm{R}\Gamma_{\rm dR}(X/\mathbf{B}_{\rm dR}^+),\\
\vartheta : &\quad F^r\mathrm{R}\Gamma_{\rm dR}(X/\mathbf{B}_{\rm dR}^+) \to F^r\mathrm{R}\Gamma_{\rm dR}(X), \quad r \geq 0.
\end{aligned}$$

It is immediate from the definitions that the first map yields a filtered strict quasi-isomorphism

$$\kappa : \quad \mathrm{R}\Gamma_{\rm cr}(X)^{\widehat{}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm dR}(X/\mathbf{B}_{\rm dR}^+).$$

3.3.2. *Comparison results.* (1) We start with a comparison of $\mathbf{B}_{\rm dR}^+$- and de Rham cohomologies.

(i) *Projection from $\mathbf{B}_{\rm dR}^+$-cohomology to de Rham cohomology.*

**Proposition 3.11.** *Let $X \in \mathrm{Sm}_C$.*

(1) *We have a natural strict quasi-isomorphism*

$$\vartheta : \quad \mathrm{R}\Gamma_{\rm dR}(X/\mathbf{B}_{\rm dR}^+)\widehat{\otimes}^{\rm R}_{\mathbf{B}_{\rm dR}^+} C \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm dR}(X).$$

(2) *More generally, for $r \geq 0$, we have a natural distinguished triangle*

$$F^{r-1}\mathrm{R}\Gamma_{\rm dR}(X/\mathbf{B}_{\rm dR}^+) \xrightarrow{t} F^r\mathrm{R}\Gamma_{\rm dR}(X/\mathbf{B}_{\rm dR}^+) \xrightarrow{\vartheta} F^r\mathrm{R}\Gamma_{\rm dR}(X) \tag{3.12}$$

(3) *For $r \geq 0$, we have a natural distinguished triangle*

$$F^{r+1}\mathrm{R}\Gamma_{\rm dR}(X/\mathbf{B}_{\rm dR}^+) \to F^r\mathrm{R}\Gamma_{\rm dR}(X/\mathbf{B}_{\rm dR}^+) \xrightarrow{\beta_X} \bigoplus_{i\leq r} \mathrm{R}\Gamma(X, \Omega_X^i)(r-i)[-i] \tag{3.13}$$

*Proof.* In the first claim, the tensor product is simply defined as the homotopy fiber of the map

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)\xrightarrow{t}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+).$$

Hence it suffices to show that we have the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathbf{B}_{\mathrm{dR}}^+)\xrightarrow{t}\mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathbf{B}_{\mathrm{dR}}^+)\xrightarrow{\vartheta}\mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

Étale locally this translates into the triangle

$$\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}}\xrightarrow{t}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}}\xrightarrow{\vartheta}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}/\overline{S})_{\mathbf{Q}_p},$$

where $\mathscr{X}=\mathscr{X}_{L,C}$, for a semistable affine model $\mathscr{X}_{\mathscr{O}_L}$ over a finite extension $L$ of $K$.

This triangle fits into a commutative diagram:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}} \arrow[r, "t"] & \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}} \arrow[r, "\vartheta"] & \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}/\overline{S})_{\mathbf{Q}_p} \\
\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}_{\mathrm{dR}}^+ \arrow[u, "\hat{\iota}_{\mathrm{BK}}"', "\wr"] \arrow[r, "1\otimes t"] & \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}_{\mathrm{dR}}^+ \arrow[u, "\hat{\iota}_{\mathrm{BK}}"', "\wr"] \arrow[r, "1\otimes\theta"] & \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L C. \arrow[u, "\wr"]
\end{tikzcd}$$

Here the bottom row is a distinguished triangle. The vertical maps $\hat{\iota}_{\mathrm{BK}}:=\mathrm{R}\lim_r\iota_{\mathrm{BK},r}$, with $\iota_{\mathrm{BK},r}$ defined as the composition

$$(3.14)\qquad \iota_{\mathrm{BK},r}:\quad \begin{aligned}(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}_{\mathrm{dR}}^+)/F^r &\xrightarrow{\sim}(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L}/S_L)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_L\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{S}/S_L)_{\mathbf{Q}_p})/F^r\\ &\underset{\sim}{\xrightarrow{\cup}}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}/S_L)_{\mathbf{Q}_p}/F^r\xleftarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})_{\mathbf{Q}_p}/F^r,\end{aligned}$$

where we set

$$(3.15)\qquad F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}_{\mathrm{dR}}^+):=\mathrm{R}\lim(\mathscr{O}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L F^r\mathbf{B}_{\mathrm{dR}}^+\to\Omega^1(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L F^{r-1}\mathbf{B}_{\mathrm{dR}}^+\to\cdots).$$

It follows that the top row in our diagram is a distinguished triangle as well, as wanted.

The second claim, étale locally, reduces to showing that the triangle

$$F^{r-1}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}}\xrightarrow{t}F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}}\xrightarrow{\vartheta}F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}/\overline{S})_{\mathbf{Q}_p},$$

where $\mathscr{X}=\mathscr{X}_{L,C}$, for a semistable affine model $\mathscr{X}_{\mathscr{O}_L}$ over a finite extension $L$ of $K$, is distinguished. This triangle fits into a commutative diagram:

$$\begin{tikzcd}
F^{r-1}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}} \arrow[r, "t"] & F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}} \arrow[r, "\vartheta"] & F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}/\overline{S})_{\mathbf{Q}_p} \\
F^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}_{\mathrm{dR}}^+) \arrow[u, "\hat{\iota}_{\mathrm{BK}}"', "\wr"] \arrow[r, "1\otimes t"] & F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}_{\mathrm{dR}}^+) \arrow[u, "\hat{\iota}_{\mathrm{BK}}"', "\wr"] \arrow[r, "1\otimes\theta"] & F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L C). \arrow[u, "\wr"]
\end{tikzcd}$$

The bottom row is a distinguished triangle: use the expression (3.15) to reduce to showing that, for $r-1\geq i\geq 0$, we have a strict quasi-isomorphism

$$t:\Omega^i(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L F^{r-1-i}\mathbf{B}_{\mathrm{dR}}^+\xrightarrow{\sim}\Omega^i(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L F^{r-i}\mathbf{B}_{\mathrm{dR}}^+$$

and the triangle

$$\Omega^r(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}_{\mathrm{dR}}^+\xrightarrow{t}\Omega^r(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}_{\mathrm{dR}}^+\to\Omega^r(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L C$$

is distinguished. But the first claim is clear and the second claim was just proved in (1).

The third claim, étale locally, reduces to identifying the graded term in the distinguished triangle

$$F^{r+1}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}}\to F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}}\to\mathrm{gr}^r\,\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}/\overline{S})_{\mathbf{Q}_p},$$

where $\mathscr{X}=\mathscr{X}_{L,C}$, for a semistable affine model $\mathscr{X}_{\mathscr{O}_L}$ over a finite extension $L$ of $K$. That is, we want to show that

$$\mathrm{gr}^r\,\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}/\overline{S})_{\mathbf{Q}_p}\simeq\bigoplus_{i\leq r}\Omega^i(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L t^{r-i}C[-i].$$

The above triangle fits into a commutative diagram:

$$\begin{tikzcd}
F^{r+1}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})^{\widehat{}}_{\mathbf{Q}_p} \arrow[r] & F^{r}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})^{\widehat{}}_{\mathbf{Q}_p} \arrow[r] & \mathrm{gr}^r\,\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}/\overline{S})_{\mathbf{Q}_p} \\
F^{r+1}(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}^+_{\mathrm{dR}}) \arrow[r] \arrow[u, "\wr"', "\hat{\iota}_{\mathrm{BK}}"] & F^{r}(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}^+_{\mathrm{dR}}) \arrow[r] \arrow[u, "\wr"', "\hat{\iota}_{\mathrm{BK}}"] & \mathrm{gr}^{r}(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}^+_{\mathrm{dR}}). \arrow[u, "\wr"', "f_{\mathscr{X}}"]
\end{tikzcd}$$

Using expression (3.15) we get

$$\begin{aligned}\mathrm{gr}^{r}(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}^+_{\mathrm{dR}}) &\xrightarrow[\sim]{g_{\mathscr{X}}}\mathrm{R}\lim(\mathscr{O}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L t^rC\xrightarrow{0}\Omega^1(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L t^{r-1}C\xrightarrow{0}\cdots\xrightarrow{0}\Omega^r(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L t^0C)\\ &\simeq\bigoplus_{i\leq r}\Omega^i(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L t^{r-i}C[-i],\end{aligned}$$

as wanted. The global map $\beta_X$ is defined by globalizing the local maps $\beta_{\mathscr{X}}:=g_{\mathscr{X}}f_{\mathscr{X}}^{-1}$. $\square$

*Remark* 3.16. (1) The above proof shows that we have a distinguished triangle

$$\mathscr{A}^{\widehat{}}_{\mathrm{cr}}\xrightarrow{t}\mathscr{A}^{\widehat{}}_{\mathrm{cr}}\xrightarrow{\vartheta}\mathscr{A}^{\mathrm{rel}}_{\mathrm{cr}}.$$

(2) The maps $\hat{\iota}_{\mathrm{BK}}$ above can be defined in a more general set-up, where $\mathscr{X}_{\mathscr{O}_L}$ is assumed to be just log-syntomic over $S_L$. It is again a strict quasi-isomorphism and the proof of this claim is not much different than in the log-smooth case: The fact that the second map in the definition of $\iota_{\mathrm{BK},r}$ in (3.14) is a strict quasi-isomorphism can be seen by unwinding both sides of the cup product map: one finds a Künneth morphism for certain de Rham complexes. It is an integral quasi-isomorphism because these complexes are "flat enough" which follows from the fact that the maps $\mathscr{O}_{C,n}\to\mathscr{O}_{L,n}$ and $\mathscr{X}_{\mathscr{O}_L,n}\to\mathscr{O}_{L,n}$, for $n\geq 0$, are log-syntomic (see the proof of [41, Prop. 4.5.4] for a similar argument). The third map in (3.14) is a strict quasi-isomorphism (integrally, a quasi-isomorphism up to a constant dependent on $L$) by an argument analogous to the one given in the proof of [35, Cor. 2.4].

(ii) *Product formula.* The morphisms $\hat{\iota}_{\mathrm{BK}}$ from Remark 3.16 induce a morphism[21]

$$\iota_{\mathrm{BK}}:\quad\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}\to\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}}).$$

**Lemma 3.17.** *The morphism $\iota_{\mathrm{BK}}$ is a filtered strict quasi-isomorphism.*

*Remark* 3.18. The filtration on $\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}$ is defined by the formula

$$F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}):=\mathrm{L}\,\mathrm{colim}(F^r(\mathrm{R}\Gamma_{\mathrm{dR}}\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})(U_\bullet)),$$

where the homotopy colimit is taken over étale affinoid hypercoverings $U_\bullet$ of $X$ and, for an affinoid $U$,

$$F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(U)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}):=\mathrm{R}\lim(\mathscr{O}(U)\widehat{\otimes}^{\mathrm{R}}_K F^r\mathbf{B}^+_{\mathrm{dR}}\to\Omega^1(U)\widehat{\otimes}^{\mathrm{R}}_K F^{r-1}\mathbf{B}^+_{\mathrm{dR}}\to\cdots).$$

*Proof.* We may argue étale locally and assume that $\mathscr{X}=\mathscr{X}_{L,C}$, for a semistable affine model $\mathscr{X}_{\mathscr{O}_L}$, for a finite extension $L$ of $K$. We need to show that the map

$$\iota_{\mathrm{BK}}:\quad\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}\to(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L}/S_K)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})^{\widehat{}}\xrightarrow[\sim]{\hat{\iota}_{\mathrm{BK}}}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L}\times_{\mathscr{O}_K}\mathscr{O}_C)^{\widehat{}}_{\mathbf{Q}_p}$$

is a filtered strict quasi-isomorphism. For that, since the base change map

$$\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L}/S_K)^{\widehat{}}_{\mathbf{Q}_p}\to\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L}/S_L)^{\widehat{}}_{\mathbf{Q}_p}$$

[21] See the proof of Lemma 3.17 for details.

is a filtered quasi-isomorphism, it suffices to show that so is the canonical map

$$\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L}/S_L)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}\to\mathrm{R}\lim_r(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L}/S_L)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})/F^r. \tag{3.19}$$

But we can write (the differentials are over $L$):

$$F^j(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L}/S_L)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})=\mathrm{R}\lim(\mathscr{O}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_K F^j\mathbf{B}^+_{\mathrm{dR}}\to\Omega^1(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_K F^{j-1}\mathbf{B}^+_{\mathrm{dR}}\to\cdots)$$
$$F^j(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L}/S_L)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})/F^r=\mathrm{R}\lim(\mathscr{O}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_K(F^j\mathbf{B}^+_{\mathrm{dR}}/F^r)\to\Omega^1(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_K(F^{j-1}\mathbf{B}^+_{\mathrm{dR}}/F^{r-1})\to\cdots).$$

And now we can argue degreewise. But then, for $s\geq 0$,

$$\mathrm{R}\lim_r(\Omega^i(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_K(F^s\mathbf{B}^+_{\mathrm{dR}}/F^r))\simeq\Omega^i(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_K\mathrm{R}\lim_r(F^s\mathbf{B}^+_{\mathrm{dR}}/F^r)\simeq\Omega^i(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_K F^s\mathbf{B}^+_{\mathrm{dR}},$$

as wanted. □

(2) Now we pass to comparisons between $\mathbf{B}^+_{\mathrm{dR}}$-cohomology and crystalline cohomology.

**Lemma 3.20.** (Local-global compatibility) *Let $\mathscr{X}\in\mathscr{M}^{\mathrm{ss},b}_C$ and let $r\geq 0$. The canonical map*

$$\kappa:\quad F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})^{\widehat{}}_{\mathbf{Q}_p}\to F^r\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}^+_{\mathrm{dR}})$$

*is a strict quasi-isomorphism.*

*Proof.* Assume that $\mathscr{X}\simeq\mathscr{X}_{\mathscr{O}_L,\mathscr{O}_C}$, for a semistable model $\mathscr{X}_{\mathscr{O}_L}$ over $S_L$, $[L:K]<\infty$. From the product quasi-isomorphisms from Lemma 3.17 and its proof we get the commutative diagram

$$\begin{tikzcd} F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})^{\widehat{}}_{\mathbf{Q}_p} \arrow[r, "\kappa"] & F^r\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}^+_{\mathrm{dR}}) \\ F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}^+_{\mathrm{dR}}) \arrow[u, "\iota_{\mathrm{BK}}", "\wr"'] \arrow[r, equal] & F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}^+_{\mathrm{dR}}) \arrow[u, "\iota_{\mathrm{BK}}", "\wr"'] \end{tikzcd}$$

and the strict quasi-isomorphism we wanted. □

**Lemma 3.21.** *The canonical map*

$$\kappa\otimes 1:\quad \mathrm{R}\Gamma_{\mathrm{cr}}(X)\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{cr}}}\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}),\quad r\geq 0,$$

*is a strict quasi-isomorphism.*

Here, we set

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X)\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{cr}}}\mathbf{B}^+_{\mathrm{dR}}:=\mathrm{L}\operatorname{colim}((\mathrm{R}\Gamma_{\mathrm{cr}}\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{cr}}}\mathbf{B}^+_{\mathrm{dR}})(\mathscr{U}_\bullet)),$$

where the homotopy colimit is taken over $\eta$-étale quasi-compact hypercoverings $\mathscr{U}_\bullet$ from $\mathscr{M}^{\mathrm{ss},b}_C$.

*Proof.* It suffices to show that, for an affine $\mathscr{X}\in\mathscr{M}^{\mathrm{ss},b}_C$, the canonical map

$$\mathrm{R}\lim_r(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{cr}}}(\mathbf{B}^+_{\mathrm{cr}}/F^r))\xrightarrow{\sim}\mathrm{R}\lim_r(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})_{\mathbf{Q}_p}/F^r)$$

is a strict quasi-isomorphism. Take a log-smooth lifting $\mathscr{Y}$ of $\mathscr{X}$ over $\mathrm{Spf}(\mathbf{A}_{\mathrm{cr}})$. We have

$$\begin{aligned}(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{cr}}}(\mathbf{B}^+_{\mathrm{cr}}/F^r)&\simeq(\mathscr{O}(\mathscr{Y})_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}_{\mathrm{cr}}}(\mathbf{B}^+_{\mathrm{cr}}/F^r)\to\Omega^1_{\mathscr{Y}/\mathbf{A}_{\mathrm{cr}},\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}_{\mathrm{cr}}}(\mathbf{B}^+_{\mathrm{cr}}/F^r)\to\cdots),\\ \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})/F^r&\simeq(\mathscr{O}(\mathscr{Y})_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}_{\mathrm{cr}}}(\mathbf{B}^+_{\mathrm{cr}}/F^r)\to\Omega^1_{\mathscr{Y}/\mathbf{A}_{\mathrm{cr}},\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}_{\mathrm{cr}}}(\mathbf{B}^+_{\mathrm{cr}}/F^{r-1})\to\cdots)\end{aligned}\tag{3.22}$$

The claim in the lemma is now clear. □

3.3.3. *History.* Let $X \in \mathrm{Sm}_C$.

(i) Recall that Bhatt-Morrow-Scholze in [7, Sec. 13] introduced $\mathbf{B}_{\mathrm{dR}}^+$-cohomology of $X$, which they call[22] *crystalline cohomology of $X$ over $\mathbf{B}_{\mathrm{dR}}^+$*. We will denote it by $\mathrm{R}\Gamma_{\mathrm{dR}}^{\mathrm{BMS}}(X/\mathbf{B}_{\mathrm{dR}}^+)$. As they mention [7, Rem. 13.2], morally speaking, it is the infinitesimal cohomology of $X$ over the embedding given by the map $\theta : \mathbf{B}_{\mathrm{dR}}^+ \to C$. It is defined though in such a way that it is easy to compare it with $\mathbf{A}_{\mathrm{inf}}$-cohomology. Similarly here, we have defined $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)$ in such a way that it is easy to compare it with crystalline cohomology over $\mathbf{A}_{\mathrm{cr}}$.

(ii) The infinitesimal site definition of $\mathrm{R}\Gamma_{\mathrm{dR}}^{\mathrm{BMS}}(X/\mathbf{B}_{\mathrm{dR}}^+)$ was carried out by Guo in [26, Sec. 7.2] (see also [27]). We will denote this version of $\mathbf{B}_{\mathrm{dR}}^+$-chomology by $\mathrm{R}\Gamma_{\mathrm{dR}}^{\mathrm{Guo}}(X/\mathbf{B}_{\mathrm{dR}}^+)$ ($\mathrm{R}\Gamma_{\mathrm{dR}}^{\mathrm{Guo}}(X/\mathbf{B}_{\mathrm{dR}}^+) := \mathrm{R}\Gamma_{\mathrm{inf}}(X/\mathbf{B}_{\mathrm{dR}}^+)$). It comes equipped with a Hodge filtration (which was ignored in [7]). Moreover, Guo constructed a natural quasi-isomorphism (see [26, Cor. 1.2.9., Th. 1.2.7])

$$\mathrm{R}\Gamma_{\mathrm{dR}}^{\mathrm{Guo}}(X/\mathbf{B}_{\mathrm{dR}}^+) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}^{\mathrm{BMS}}(X/\mathbf{B}_{\mathrm{dR}}^+). \tag{3.23}$$

(iii) Our construction of $\mathbf{B}_{\mathrm{dR}}^+$-cohomology is compatible with the above constructions:

**Proposition 3.24.** *Let $X \in \mathrm{Sm}_C$.*

(1) *There is a natural quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}^{\mathrm{BMS}}(X/\mathbf{B}_{\mathrm{dR}}^+).$$

(2) *There is a natural filtered quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}^{\mathrm{Guo}}(X/\mathbf{B}_{\mathrm{dR}}^+).$$

*Proof.* Claim (1) follows from claim (2) and the quasi-isomorphism (3.23).

To prove claim (2), recall that $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)$ is defined by taking, étale locally, the Hodge completed absolute crystalline cohomology and then globalizing. More specifically, let $\mathscr{X} \in \mathscr{M}_C^{\mathrm{ss}}$. We have

$$\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}_{\mathrm{dR}}^+) &\simeq \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}} := \mathrm{R}\lim\nolimits_r(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})_{\mathbf{Q}_p}/F^r),\\
F^r\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}_{\mathrm{dR}}^+) &\simeq \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X},\mathscr{J}^{[r]})\widehat{_{\mathbf{Q}_p}} := \mathrm{R}\lim\nolimits_{j\geq r}(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X},\mathscr{J}^{[r]})_{\mathbf{Q}_p}/F^j).
\end{aligned}$$

On the other hand $\mathrm{R}\Gamma_{\mathrm{dR}}^{\mathrm{Guo}}(X/\mathbf{B}_{\mathrm{dR}}^+)$ is defined as the infinitesimal cohomology $\mathrm{R}\Gamma_{\mathrm{inf}}(X/\mathbf{B}_{\mathrm{dR}}^+)$ equipped with its natural Hodge filtration.

This means that, if $\mathscr{X}$ is affine and (exactly and) closely embedded in an affine formal log-scheme $\mathscr{Y}$, log-smooth over $\mathbf{A}_{\mathrm{cr}}$, then

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}_{\mathrm{dR}}^+)/F^r \simeq \mathrm{R}\lim((\mathscr{D}_{\mathscr{X}}(\mathscr{Y})/F^r)_{\mathbf{Q}_p} \to ((\mathscr{D}_{\mathscr{X}}(\mathscr{Y})/F^{r-1})\widehat{\otimes}_{\mathscr{O}(\mathscr{Y})}\Omega^1_{\mathscr{Y}/\mathbf{A}_{\mathrm{cr}}})_{\mathbf{Q}_p} \to \cdots),$$

where $\mathscr{D}_{\mathscr{X}}(\mathscr{Y})$ is the PD-envelope of $\mathscr{X}$ in $\mathscr{Y}$ and the tensor product is $p$-adic. On the other hand, we have

$$\mathrm{R}\Gamma_{\mathrm{inf}}(\mathscr{X}_C/\mathbf{B}_{\mathrm{dR}}^+)/F^r \simeq \mathrm{R}\lim(\mathscr{D}_{\mathscr{X}_C}(\mathscr{Y}_C)/F^r \to (\mathscr{D}_{\mathscr{X}_C}(\mathscr{Y}_C)/F^{r-1})\widehat{\otimes}_{\mathscr{O}(\mathscr{Y}_C)}\Omega^1_{\mathscr{Y}_C/\mathbf{B}_{\mathrm{dR}}^+} \to \cdots),$$

where $\mathscr{D}_{\mathscr{X}_C}(\mathscr{Y}_{\mathbf{B}})$ is the inf-envelope of $\mathscr{X}_C$ in $\mathscr{Y}_{\mathbf{B}} := \mathscr{Y}_{\mathbf{B}_{\mathrm{dR}}^+}$. Since $\mathbf{A}_{\mathrm{cr},\mathbf{Q}_p}/F^i \simeq \mathbf{B}_{\mathrm{dR}}^+/F^i$, we have a natural map

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}_{\mathrm{dR}}^+)/F^r \to \mathrm{R}\Gamma_{\mathrm{inf}}(\mathscr{X}_C/\mathbf{B}_{\mathrm{dR}}^+)/F^r.$$

This can be globalized to a map

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^r \to \mathrm{R}\Gamma_{\mathrm{dR}}^{\mathrm{Guo}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^r.$$

[22] We take here the étale version studied in [10, Sec. 6.2] and not the original analytic version. The two versions are quasi-isomorphic by [10, Sec. 6.2].

We claim that it is a filtered quasi-isomorphism. Indeed, it suffices to show this locally so we may assume that we have the data of integral models $\mathscr{X},\mathscr{Y}$ as above and, moreover, $\mathscr{Y}$ is a lifting of $\mathscr{X}$. Then

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}^+_{\mathrm{dR}})/F^r &\simeq \mathrm{R}\lim(\mathscr{O}(\mathscr{Y}_{\mathbf{B}})/F^r \to ((\mathscr{O}(\mathscr{Y})/F^{r-1})\widehat{\otimes}_{\mathscr{O}(\mathscr{Y})}\Omega^1_{\mathscr{Y}/\mathbf{A}_{\mathrm{cr}}})_{\mathbf{Q}_p} \to \cdots),\\ \mathrm{R}\Gamma^{\mathrm{Guo}}_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}^+_{\mathrm{dR}})/F^r &\simeq \mathrm{R}\lim(\mathscr{O}(\mathscr{Y}_{\mathbf{B}})/F^r \to (\mathscr{O}(\mathscr{Y}_{\mathbf{B}})/F^{r-1})\widehat{\otimes}_{\mathscr{O}(\mathscr{Y}_{\mathbf{B}})}\Omega^1_{\mathscr{Y}_{\mathbf{B}}/\mathbf{B}^+_{\mathrm{dR}}} \to \cdots).\end{aligned}$$

But we have the isomorphisms

$$\begin{aligned}((\mathscr{O}(\mathscr{Y})/F^i)\widehat{\otimes}_{\mathscr{O}(\mathscr{Y})}\Omega^j_{\mathscr{Y}/\mathbf{A}_{\mathrm{cr}}})_{\mathbf{Q}_p} &\simeq (\mathscr{O}(\mathscr{Y}_{\mathbf{B}})/F^i)\widehat{\otimes}_{\mathscr{O}(\mathscr{Y}_{\mathbf{B}})}\Omega^j_{\mathscr{Y}_{\mathbf{B},i}/\mathbf{A}_{\mathrm{cr},\mathbf{Q}_p,i}}\\ (\mathscr{O}(\mathscr{Y}_{\mathbf{B}})/F^i)\widehat{\otimes}_{\mathscr{O}(\mathscr{Y}_{\mathbf{B}})}\Omega^j_{\mathscr{Y}_{\mathbf{B}}/\mathbf{B}^+_{\mathrm{dR}}} &\simeq (\mathscr{O}(\mathscr{Y}_{\mathbf{B}})/F^i)\widehat{\otimes}_{\mathscr{O}(\mathscr{Y}_{\mathbf{B}})}\Omega^j_{\mathscr{Y}_{\mathbf{B},i}/\mathbf{B}^+_{\mathrm{dR},i}},\end{aligned}$$

where $(-)_i$ denotes moding out by $F^i$. Hence the quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}^+_{\mathrm{dR}})/F^r \xrightarrow{\sim} \mathrm{R}\Gamma^{\mathrm{Guo}}_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}^+_{\mathrm{dR}})/F^r,$$

as wanted.

Having the quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r \xrightarrow{\sim} \mathrm{R}\Gamma^{\mathrm{Guo}}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r,$$

we may take $\mathrm{R}\lim_r$ of both sides to obtain the quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \xrightarrow{\sim} \mathrm{R}\Gamma^{\mathrm{Guo}}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}).$$

This is because we have

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \xrightarrow{\sim} \mathrm{R}\lim_r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r), \quad \mathrm{R}\Gamma^{\mathrm{Guo}}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \xrightarrow{\sim} \mathrm{R}\lim_r(\mathrm{R}\Gamma^{\mathrm{G}}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r)$$

as can be easily seen by a computation similar to the one used in the proof of Lemma 3.17. Finally, to obtain the quasi-isomorphism

$$F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \xrightarrow{\sim} F^r\mathrm{R}\Gamma^{\mathrm{Guo}}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}), \quad r\geq 0,$$

we use the distinguished triangles

$$F^r\mathrm{R}\Gamma_{\mathrm{dR}} \to \mathrm{R}\Gamma_{\mathrm{dR}} \to \mathrm{R}\Gamma_{\mathrm{dR}}/F^r$$

for both cohomologies. $\square$

3.4. **Overconvergent $\mathbf{B}^+_{\mathrm{dR}}$-cohomology.** We define overconvergent $\mathbf{B}^+_{\mathrm{dR}}$-cohomology via presentations of dagger structures.

3.4.1. *Definition of overconvergent* $\mathbf{B}^+_{\mathrm{dR}}$*-cohomology.* Let $X$ be a smooth dagger affinoid over $C$. Let $\mathrm{pres}(X)=\{X_h\}$ be a presentation of $X$. Define

$$F^r\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) := \mathrm{L}\operatorname{colim}_h F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X_h/\mathbf{B}^+_{\mathrm{dR}}), \quad r\geq 0.$$

For $r\geq 0$, the $\eta$-étale sheafification of $F^r\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ gives us asheaf $F^r\widehat{\mathscr{A}_{\mathrm{cr}}}$ on any smooth dagger variety $X$ over $C$. The filtered $\mathbf{B}^+_{\mathrm{dR}}$-cohomology is defined as

$$F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) := \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, F^r\widehat{\mathscr{A}_{\mathrm{cr}}}), \quad r\geq 0.$$

*Remark* 3.25. If $X$ is a smooth dagger affinoid over $C$ the above two definitions of $\mathbf{B}^+_{\mathrm{dR}}$-cohomology $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ and $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ agree. This will be shown in Corollary 4.28 below by reduction to Hyodo-Kato cohomology via the Hyodo-Kato quasi-isomorphism.

3.4.2. *Properties of overconvergent $\mathbf{B}^+_{\mathrm{dR}}$-cohomology.* We will now prove properties of overconvergent $\mathbf{B}^+_{\mathrm{dR}}$-cohomology that do not require Hyodo-Kato cohomology.

We have canonical maps

$$\begin{aligned}\vartheta: &\quad F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \to F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X), \quad X\in \mathrm{Sm}^\dagger_C, r\geq 0,\\ \iota_{\mathrm{BK}}: &\quad \mathrm{R}\Gamma_{\mathrm{dR}}(X) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}), \quad X\in \mathrm{Sm}^\dagger_K.\end{aligned}$$

induced by their rigid analytic analogs.

**Proposition 3.26.** (1) (Projection) *Let $X\in \mathrm{Sm}^\dagger_C$.*

(a) *The map $\vartheta$ defined above yields a natural strict quasi-isomorphism*

$$\vartheta: \quad \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{dR}}} C \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

(b) *More generally, for $r\geq 0$, we have a natural distinguished triangle*

$$F^{r-1}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\xrightarrow{t} F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\xrightarrow{\vartheta} F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X) \tag{3.27}$$

(c) *For $r\geq 0$, we have a natural distinguished triangle*

$$F^{r+1}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \to F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \to \bigoplus_{i\leq r}\mathrm{R}\Gamma(X,\Omega^i_X)(r-i)[-i] \tag{3.28}$$

(2) (Product formula) *Let $X\in \mathrm{Sm}^\dagger_K$. The map $\iota_{\mathrm{BK}}$ defined above yields a natural filtered quasi-isomorphism*

$$\iota_{\mathrm{BK}}: \quad \mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}} \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}}).$$

(3) ($t$-completeness) *The canonical map*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \to \mathrm{R}\lim_r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{dR}}}(\mathbf{B}^+_{\mathrm{dR}}/F^r))$$

*is a filtered strict quasi-isomorphism.*

*Remark* 3.29. In Proposition 3.26 (2), the filtration on $\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}$ is defined by the formula

$$F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}) := \mathrm{L}\operatorname{colim}(F^r(\mathrm{R}\Gamma_{\mathrm{dR}}\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})(U_\bullet)),$$

where the homotopy colimit is taken over étale dagger affinoid hypercoverings $U_\bullet$ of $X$ and, for a smooth dagger affinoid $U$,

$$F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(U)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}) := \mathrm{R}\lim(\mathscr{O}(U)\widehat{\otimes}^{\mathrm{R}}_K F^r\mathbf{B}^+_{\mathrm{dR}} \to \Omega^1(U)\widehat{\otimes}^{\mathrm{R}}_K F^{r-1}\mathbf{B}^+_{\mathrm{dR}}\to\cdots).$$

In particular, if $r=0$, $\mathrm{R}\Gamma_{\mathrm{dR}}(U)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}$ is just the usual projective tensor product.

*Proof.* To prove the first projection formula in (1) it suffices to argue locally for the dagger cohomologies. So we may assume that $X$ is a smooth dagger affinoid with the presentation $\{X_h\}$. We need to show that the projection

$$\vartheta: \quad \mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{dR}}} C \xrightarrow{\sim} \mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X)$$

is a strict quasi-isomorphism. We can write this projection more explicitly as the composition

$$\begin{aligned}\mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{dR}}} C &\simeq (\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h/\mathbf{B}^+_{\mathrm{dR}}))\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{dR}}} C \xleftarrow{\sim} \mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h/\mathbf{B}^+_{\mathrm{dR}})\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{dR}}} C\\ &\xrightarrow{\mathrm{L}\operatorname{colim}_h\vartheta} \mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\simeq \mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X).\end{aligned}$$

The second map is a strict quasi-isomorphism because the tensor product is defined as the cone of multiplication by $t$; the third map is a strict quasi-isomorphism by Proposition 3.11.

To prove the second formula in (1), we argue locally as well. We need to show that, for $r\geq 0$, we have a distinguished triangle

$$F^{r-1}\mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\xrightarrow{t} F^r\mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\xrightarrow{\vartheta} F^r\mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X), \tag{3.30}$$

where $X$ is a smooth dagger affinoid with the presentation $\{X_h\}$. But this triangle can be written as:

$$\mathrm{L}\operatorname{colim}_h F^{r-1}\mathrm{R}\Gamma_{\mathrm{dR}}(X_h/\mathbf{B}^+_{\mathrm{dR}})\xrightarrow{\mathrm{L}\operatorname{colim}_h t}\mathrm{L}\operatorname{colim}_h F^{r}\mathrm{R}\Gamma_{\mathrm{dR}}(X_h/\mathbf{B}^+_{\mathrm{dR}})\xrightarrow{\mathrm{L}\operatorname{colim}_h \vartheta}\mathrm{L}\operatorname{colim}_h F^{r}\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$$

and then it is clear that it is distinguished by Proposition 3.11.

To prove the third formula in (1), we again argue locally. We need to show that, for $r\geq 0$, we have a distinguished triangle

$$F^{r+1}\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\stackrel{\mathrm{can}}{\to}F^{r}\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\stackrel{\beta_X}{\to}\bigoplus_{i\leq r}\mathrm{R}\Gamma(X,\Omega^i_X)(r-i)[-i], \tag{3.31}$$

where $X$ is a smooth dagger affinoid with the presentation $\{X_h\}$. But we can define this triangle as:

$$\begin{tikzcd}
& \mathrm{L}\operatorname{colim}_h F^{r+1}\mathrm{R}\Gamma_{\mathrm{dR}}(X_h/\mathbf{B}^+_{\mathrm{dR}}) \arrow[dr, "\mathrm{L}\operatorname{colim}_h \mathrm{can}"] & \\
\mathrm{L}\operatorname{colim}_h \bigoplus_{i\leq r}\mathrm{R}\Gamma(X_h,\Omega^i_{X_h})(r-i)[-i] \arrow[ur, "{[1]}"] & & \mathrm{L}\operatorname{colim}_h F^{r}\mathrm{R}\Gamma_{\mathrm{dR}}(X_h/\mathbf{B}^+_{\mathrm{dR}}) \arrow[ll, "\mathrm{L}\operatorname{colim}_h \beta_{X_h}"]
\end{tikzcd}$$

and then it is clear that it is distinguished by Proposition 3.11.

In the product formula (2), the map $\iota_{\mathrm{BK}}$ is defined by globalizing maps $\iota^{\dagger}_{\mathrm{BK}}$ for dagger affinoids. To define the latter, assume that $X$ is a smooth dagger affinoid with the presentation $\{X_h\}$ and set

$$\begin{aligned}\iota^{\dagger}_{\mathrm{BK}}:\quad \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}\simeq(\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h))\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}} &\stackrel{\sim}{\leftarrow}\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}\\ &\xrightarrow{\mathrm{L}\operatorname{colim}_h\iota_{\mathrm{BK}}}\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_{h,C}/\mathbf{B}^+_{\mathrm{dR}})\simeq\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}}).\end{aligned} \tag{3.32}$$

The third map is a filtered quasi-isomorphism by Lemma 3.17. It remains to show that so is the second map, i.e., that the map

$$\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}\longrightarrow\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}, \tag{3.33}$$

is a filtered strict quasi-isomorphism. Indeed, look at the cohomology of both sides. On the right hand side, arguing as in [14, Sec. 3.2.2], we get

$$\widetilde{H}^i(\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})\simeq H^i_{\mathrm{dR}}(X)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}}.$$

For the left hand side, we compute

$$\begin{aligned}\widetilde{H}^i(\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})&\stackrel{\sim}{\to}\widetilde{H}^i(\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X^0_h)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})\simeq\operatorname{colim}_h(H^i_{\mathrm{dR}}(X^0_h)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})\\ &\stackrel{\sim}{\to}H^i_{\mathrm{dR}}(X)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}}.\end{aligned}$$

The second and the third isomorphisms above follow from:

(1) the fact that the cohomology $H^i_{\mathrm{dR}}(X^0_h)$ is a finite rank vector space over $K$ with its canonical topology (by [24, Th. 3.1]);
(2) point (1) implies the quasi-isomorphism
$$\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X^0_h)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})\simeq H^i_{\mathrm{dR}}(X^0_h)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}}$$
proved as in [14, Sec. 3.2.2];
(3) the system $\{H^i_{\mathrm{dR}}(X^0_h)\}_h$ is essentially constant and isomorphic to $H^i_{\mathrm{dR}}(X)$;
(4) point (3) implies that the system $\{H^i_{\mathrm{dR}}(X^0_h)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}}\}_h$ is essentially constant and isomorphic to $H^i_{\mathrm{dR}}(X)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}}$.

This proves that the map (3.33) is a strict quasi-isomorphism.

We shall need to argue more that it is a filtered strict quasi-isomorphism as well. We argue by induction on $r \geq 0$; the base case of $r = 0$ being proved above. For the inductive step ($r-1 \Rightarrow r$) consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{L}\operatorname{colim}_h F^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\widehat{\otimes}^{\mathrm{R}}_K \mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "\sim"] \arrow[d, "t"] & F^{r-1}(\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K \mathbf{B}^+_{\mathrm{dR}}) \arrow[d, "t"] \\
\mathrm{L}\operatorname{colim}_h F^{r}(\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\widehat{\otimes}^{\mathrm{R}}_K \mathbf{B}^+_{\mathrm{dR}}) \arrow[r] \arrow[d] & F^{r}(\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K \mathbf{B}^+_{\mathrm{dR}}) \arrow[d] \\
\mathrm{L}\operatorname{colim}_h F^{r}(\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\widehat{\otimes}^{\mathrm{R}}_K C) \arrow[r, "\sim"] & F^{r}(\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K C)
\end{tikzcd}$$

The left and the right vertical triangles are distinguished by (3.12) and (3.27), respectively. The bottom map is clearly a strict quasi-isomorphism; the top map is a strict quasi-isomorphism by the inductive assumption. It follows that so is the middle horizontal map, as wanted.

We finish the proof of the second claim of our proposition by noting that the map $\mathrm{R}\lim_h \iota_{\mathrm{BK}}$ in (3.32) is a strict quasi-isomorphism by Lemma 3.17.

For the third claim of the proposition, it suffices to argue locally for the dagger cohomologies. Hence we can assume that $X \simeq Y_C$ for a smooth dagger affinoid $Y$ defined over $K$. And then the wanted $t$-completeness follows from the second claim of the proposition and the fact that, since our tensor products are projective and the Mittag-Leffler condition is satisfied, the canonical map

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(Y)\widehat{\otimes}^{\mathrm{R}}_K \mathbf{B}^+_{\mathrm{dR}} \to \mathrm{R}\lim_r(\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(Y)\widehat{\otimes}^{\mathrm{R}}_K(\mathbf{B}^+_{\mathrm{dR}}/F^r))$$

is a filtered strict quasi-isomorphism. $\square$

## 4. Geometric Hyodo-Kato morphisms

This section is devoted to the definition of compatible rigid analytic (for $X \in \mathrm{Sm}_C$) and overconvergent (for $X \in \mathrm{Sm}^{\dagger}_C$) Hyodo-Kato cohomologies $\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)$. For a general rigid analytic variety, the Hyodo-Kato cohomology is in general quite ugly (not separated and, locally, infinite dimensional), but for dagger varieties the Hyodo-Kato cohomology has nice properties (separated and, locally, finite dimensional). On the other hand (Lemma 4.17), if $X \in \mathrm{Sm}_C$ is partially proper, then the rigid analytic and overconvergent Hyodo-Kato cohomologies give the same result: if $X^{\dagger}$ is the associated dagger variety, the natural map $\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X^{\dagger}) \to \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)$ is a strict quasi-isomorphism (Corollary 4.32).

We define $\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)$ for dagger varieties by, locally, going to the limit over a presentation in the Hyodo-Kato cohomology for rigid analytic varieties, and globalizing. This definition is much more flexible than Grosse-Klönne's [25], and we show (Lemma 4.14) that the two definitions give rise to the same cohomology.

The rigid analytic and overconvergent Hyodo-Kato cohomologies are related (Theorem 4.6 and Theorem 4.27) to the rigid analytic and overconvergent de Rham and $\mathbf{B}^+_{\mathrm{dR}}$-cohomologies by the Hyodo-Kato quasi-isomorphisms:

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}} C \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X), \quad \iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}} \mathbf{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$$

4.1. **Rigid analytic setting.** We start our definitions of Hyodo-Kato morphisms with rigid-analytic varieties.

4.1.1. *Completed Hyodo-Kato cohomology.* The completed Hyodo-Kato cohomology $\mathrm{R}\Gamma_{\mathrm{HK}}(X_1^0)$ that appeared in the proof of Theorem 2.12 has better topological properties than the classical Hyodo-Kato cohomology $\mathrm{R}\Gamma_{\mathrm{HK}}(X_1)$ (being over $p$-adically complete field $\breve{F}$ instead of $F^{\mathrm{nr}}$). Because of this we will often use it.

Let $X \in \mathrm{Sm}_C$. Let $\mathscr{A}^c_{\mathrm{HK}}$ ($c$ stands for "completion") be the $\eta$-étale sheafification of the presheaf $\mathscr{X} \to \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1^0)_{\mathbf{Q}_p}$ on $\mathscr{M}_C^{\mathrm{ss},b}$. We set

$$\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}^c_{\mathrm{HK}}).$$

It is a dg $\breve{F}$-algebra equipped with a Frobenius, monodromy action, and a continuous action of $\mathscr{G}_K$, if $X$ is defined over $K$. We equip it with the topology induced by $\eta$-étale descent from the topology of the $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1^0)_{\mathbf{Q}_p}$'s.

Unwinding the definitions, using the base change quasi-isomorphism (2.11), and globalizing we obtain that the canonical morphism

$$\beta: \quad \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\breve{F} \to \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X) \tag{4.1}$$

is a strict quasi-isomorphism. It implies:

**Lemma 4.2.** (Local-global compatibility) *For $\mathscr{X} \in \mathscr{M}_C^{\mathrm{ss}}$, the canonical morphism*

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1^0)_{\mathbf{Q}_p} \to \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(\mathscr{X}_C) \tag{4.3}$$

*is a strict quasi-isomorphism.*

*Proof.* We can pass from $K$ to $\breve{K} := K\breve{F}$ (which amounts to passing from $F$ to $\breve{F}$ for the absolutely unramified subfields) without changing the cohomologies in (4.3). And then we can simply use local-global compatibility for $(\breve{F})^{\mathrm{nr}} = \breve{F}$-cohomology (see [16, Prop. 4.23]). $\square$

4.1.2. *Geometric rigid analytic Hyodo-Kato quasi-isomorphisms.* We will now use Theorem 2.12 to define, both local and global, geometric Hyodo-Kato quasi-isomorphisms.

(i) *Local setting.* We will define two types of Hyodo-Kato morphisms: Hyodo-Kato-to-de Rham and Hyodo-Kato-to-$\mathbf{B}^+_{\mathrm{dR}}$.

Let $\mathscr{X} \in \mathscr{M}_C^{\mathrm{ss,b}}$. The Hyodo-Kato-to-de Rham morphism is defined by the composition:

$$\iota_{\mathrm{HK}}: \quad \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1^0)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C \xrightarrow[\sim]{\varepsilon^{\mathrm{HK}}_{\mathrm{dR}}} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_1/\overline{S})_{\mathbf{Q}_p} \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C). \tag{4.4}$$

It is a natural strict quasi-isomorphism.

For the Hyodo-Kato-to-$\mathbf{B}^+_{\mathrm{dR}}$ morphism we have:

**Corollary 4.5.** *Let $\mathscr{X} \in \mathscr{M}_C^{\mathrm{ss,b}}$. There exists a natural strict quasi-isomorphism*

$$\iota_{\mathrm{HK}}: \quad \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1^0)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}^+_{\mathrm{dR}}).$$

*Moreover, we have the commutative diagram*

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1^0)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}} & \xrightarrow[\sim]{\iota_{\mathrm{HK}}} & \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C/\mathbf{B}^+_{\mathrm{dR}}) \\
\downarrow{\scriptstyle 1\otimes\theta} & & \downarrow{\scriptstyle \vartheta} \\
\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1^0)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C & \xrightarrow[\sim]{\iota_{\mathrm{HK}}} & \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C).
\end{array}$$

*Proof.* To define $\iota_{\mathrm{HK}}$, we use the natural strict quasi-isomorphism

$$\varepsilon^{\mathrm{HK}}_{\mathbf{B}^+_{\mathrm{dR}}}: \quad \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1^0)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_1)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{cr}}}\mathbf{B}^+_{\mathrm{dR}}$$

from Lemma 2.20 and compose it with the strict quasi-isomorphism

$$\kappa: \quad \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_1)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{cr}}}\mathbf{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\widehat{_{\mathbf{Q}_p}}$$

from the proof of Lemma 3.21.

Commutativity of the diagram follows from Lemma 2.20. ☐

(ii) *Global setting.* We can now state the main theorem of this chapter:

**Theorem 4.6.** (Geometric Hyodo-Kato isomorphisms) *Let $X \in \mathrm{Sm}_C$. We have the natural Hyodo-Kato strict quasi-isomorphisms*

$$\iota_{\mathrm{HK}}: \quad \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X), \quad \iota_{\mathrm{HK}}: \quad \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \tag{4.7}$$

*that are compatible via the maps $\theta$ and $\vartheta$.*

*Proof.* Globalize the local strict quasi-isomorphisms from Corollary 2.20 and Corollary 4.5. ☐

(iii) *Complements.* In a similar fashion, the local strict quasi-isomorphism $\varepsilon^{\mathrm{HK}}_{\mathrm{st}}$ from Theorem 2.12 induces the natural strict quasi-isomorphism

$$\varepsilon^{\mathrm{HK}}_{\mathrm{st}}: \quad \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}_{\breve{F}}\mathbf{B}^+_{\mathrm{st}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X)\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{cr}}}\mathbf{B}^+_{\mathrm{st}}, \tag{4.8}$$

where we set

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}_{\breve{F}}\mathbf{B}^+_{\mathrm{st}} &:= \mathrm{L}\operatorname{colim}((\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}\widehat{\otimes}_{\breve{F},\iota}\mathbf{B}^+_{\mathrm{st}})(\mathscr{U}_{\bullet,1})), \\ \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{cr}}}\mathbf{B}^+_{\mathrm{st}} &:= \mathrm{L}\operatorname{colim}((\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{cr}},\iota}\mathbf{B}^+_{\mathrm{st}})(\mathscr{U}_{\bullet,1})), \end{aligned} \tag{4.9}$$

with the homotopy colimit is taken over $\eta$-étale quasi-compact hypercoverings $\mathscr{U}_\bullet$ from $\mathscr{M}^{\mathrm{ss},b}_C$. Applying the map $\mathbf{B}^+_{\mathrm{st}} \to \mathbf{B}^+_{\mathrm{cr}}$ given by sending $\log(\lambda_p) \mapsto 0$ to the morphism (4.8) we obtain the strict quasi-isomorphism

$$\varepsilon^{\mathrm{HK}}_{\mathrm{cr}}: \quad \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}_{\breve{F}}\mathbf{B}^+_{\mathrm{cr}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X). \tag{4.10}$$

4.2. **The overconvergent setting.** We are now ready to define the overconvergent geometric Hyodo-Kato morphism. We do it locally by using, via presentations, the rigid-analytic geometric Hyodo-Kato morphism constructed in the previous section and then we glue. The advantage of this approach is that, by construction, the overconvergent and the rigid analytic geometric Hyodo-Kato morphisms are compatible. This is in contrast to [14], [16], where a lot of effort was devoted to proving compatibility between the overconvergent construction due to Grosse-Klönne, and the rigid-analytic construction due to Hyodo-Kato[23].

4.2.1. *Overconvergent Hyodo-Kato cohomology via presentations of dagger structures.* In this section we introduce a definition of overconvergent Hyodo-Kato cohomology using presentations of dagger structures (see [42, Appendix], [16, Sec. 6.3]). We show that the so defined Hyodo-Kato cohomology, a priori different from the one defined by Grosse-Klönne, is, in fact, strictly quasi-isomorphic to it.

(i) *Local definition.* Let $X$ be a dagger affinoid over $L = K, C$. Let $\mathrm{pres}(X) = \{X_h\}$. Define

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X) := \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{HK}}(X_h).$$

Equip it with the induced Frobenius and monodromy. We have a natural map

$$\alpha^{\dagger}_{\mathrm{HK}}: \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X) \to \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X) \tag{4.11}$$

[23]Recently, Ertl-Yamada in [19] have introduced a particularly simple definition of overconvergent Hyodo-Kato cohomology for weak-formal semistable schemes and equally simple definition of the relevant Hyodo-Kato map. Their construction is compatible with the crystalline Hyodo-Kato analog when the scheme is proper. It is likely that their construction can be extended to the set-up needed in this paper.

defined as the composition

$$
\begin{aligned}
(4.12)\qquad \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X) = \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{HK}}(X_h) &\xrightarrow{\sim} \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{HK}}(X_h^{\mathrm{o}})\\
&\xleftarrow{\sim} \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X_h^{\mathrm{o},\dagger}) \to \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X).
\end{aligned}
$$

The third map is a strict quasi-isomorphism by Corollary 3.4: this is because the interior $X_h^{\mathrm{o}}$ is partially proper. Note that the proof of the cited corollary relies on a nontrivial comparison result between the rigid analytic and Grosse-Klönne's overconvergent Hyodo-Kato morphisms.

(ii) *Globalization.* For a general smooth dagger variety $X$ over $L$, using the natural equivalence of analytic topoi

$$(4.13)\qquad \mathrm{Sh}(\mathrm{SmAff}^{\dagger}_{L,\mathrm{\acute{e}t}}) \xrightarrow{\sim} \mathrm{Sh}(\mathrm{Sm}^{\dagger}_{L,\mathrm{\acute{e}t}})$$

we define the sheaf $\mathscr{A}^{\dagger}_{\mathrm{HK}}$ on $X_{\mathrm{\acute{e}t}}$ as the sheaf associated to the presheaf defined by $U \mapsto \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(U), U \in \mathrm{SmAff}^{\dagger}_L$, $U \to X$ an étale map. We define

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X) := \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathscr{A}^{\dagger}_{\mathrm{HK}}).$$

If $L = K$, it is a dg $F$-algebra. If $L = C$, it is a dg $F^{\mathrm{nr}}$-algebra equipped with a Frobenius, monodromy action, and a continuous action of $\mathscr{G}_K$ if $X$ is defined over $K$. We equip it with the topology induced by $\eta$-étale descent from the topology of the $\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)$'s.

Globalizing the map $\alpha^{\dagger}_{\mathrm{HK}}$ from (4.11) we obtain a natural map

$$\alpha_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X) \to \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X).$$

**Lemma 4.14.** *Let $L = K, C$.*

(1) *The above map $\alpha_{\mathrm{HK}}$ is a strict quasi-isomorphism.*
(2) (Local-global compatibility) *If $X$ is a smooth dagger affinoid the natural map*

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X) \to \mathrm{R}\Gamma_{\mathrm{HK}}(X)$$

*is a strict quasi-isomorphism.*

*Proof.* For the first claim, by étale descent, we may assume that $X$ comes from a smooth dagger affinoid. Looking at the composition (4.12) defining the map $\alpha^{\dagger}_{\mathrm{HK}}$ we see that it suffices to show that the natural map

$$(4.15)\qquad \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X_h^{\mathrm{o},\dagger}) \to \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)$$

is a strict quasi-isomorphism. But this was shown in the proof of Proposition 6.17 in [16]. We note that that proof uses the Hyodo-Kato quasi-isomorphism of Grosse-Klönne to pass to the de Rham cohomology where the analog of (4.15) is obvious.

For the second claim, consider the commutative local-global diagram

$$
\begin{tikzcd}
\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X) \arrow[r] \arrow[d, "\alpha^{\dagger}_{\mathrm{HK}}"'] & \mathrm{R}\Gamma_{\mathrm{HK}}(X) \arrow[dl, "\sim"', "\alpha_{\mathrm{HK}}"] \\
\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X) &
\end{tikzcd}
$$

The slanted arrow is a strict quasi-isomorphism by the first claim of the lemma. It suffices to show that the left vertical arrow is a strict quasi-isomorphism as well. For that, it suffices to show that the map

$$\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X_h^{\mathrm{o},\dagger}) \to \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)$$

appearing in the definition (4.12) of the map $\alpha^{\dagger}_{\mathrm{HK}}$ is a strict quasi-isomorphism but this was just shown above. $\square$

(iii) *Completed overconvergent Hyodo-Kato cohomology.* We can define the *completed overconvergent Hyodo-Kato cohomology* by a similar procedure to the one used above. It will have better topological properties than its classical version. Let $X$ be a smooth dagger affinoid over $C$. Let $\mathrm{pres}(X)=\{X_h\}$. Define

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK},\breve{F}}(X):=\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X_h).$$

For a general smooth dagger variety over $C$, we can globalize the above definition and obtain the sheaf $\mathscr{A}^{\mathrm{c}}_{\mathrm{HK}}$ and cohomology

$$\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X):=\mathrm{R}\Gamma_{\text{ét}}(X,\mathscr{A}^{\mathrm{c}}_{\mathrm{HK}}).$$

It is a dg $\breve{F}$-algebra equipped with a Frobenius, monodromy action, and a continuous action of $\mathscr{G}_K$, if $X$ is defined over $K$. We equip it with the topology induced by $\eta$-étale descent from the topology of the $\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK},\breve{F}}(X)$'s.

We have the local-global compatibility by Lemma 4.14 (replace, without loss of information, $F$ by $\breve{F}$).

(iv) *Completed overconvergent Hyodo-Kato cohomology ala Grosse-Klönne.* But we can also define the completed overconvergent Hyodo-Kato cohomology as in the rigid analytic case, by modifying the definition of the overconvergent Hyodo-Kato cohomology of Grosse-Klönne. That is, we can set $\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK},\breve{F}}(\mathscr{X}_1):=\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)$, for $\mathscr{X}\in\mathscr{M}^{\dagger,\mathrm{ss}}_C$, where $\mathscr{X}_0:=\mathscr{X}_{\overline{k}}$, and globalize. We will denote by $\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK},\breve{F}}(X)$, $X\in\mathrm{Sm}^{\dagger}_C$, the so obtained cohomology.

We easily check that we have strict quasi-isomorphisms:

$$\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK},\breve{F}}(\mathscr{X}_1)\xleftarrow{\sim}\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(\mathscr{X}_1)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\breve{F},\quad \mathscr{X}\in\mathscr{M}^{\dagger,\mathrm{ss}}, \tag{4.16}$$
$$\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK},\breve{F}}(X)\xleftarrow{\sim}\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\breve{F},\quad X\in\mathrm{Sm}^{\dagger}_C.$$

We also have local-global compatibility: pass from $F$ to $\breve{F}$ as in the proof of Lemma 4.2. This reduces the problem to the local-global compatibility for the usual Hyodo-Kato cohomology of Grosse-Klönne and this we know is true.

The two definitions of completed overconvergent Hyodo-Kato cohomology give the same objects:

**Lemma 4.17.** *Let $X\in\mathrm{Sm}^{\dagger}_C$. There exists a natural strict quasi-isomorphism*

$$\alpha_{\mathrm{HK},\breve{F}}:\quad \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\xrightarrow{\sim}\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK},\breve{F}}(X).$$

*It is compatible with Frobenius and monodromy.*

*Proof.* Pass from $F$ to $\breve{F}$ and use Lemma 4.14. □

(v) *Tensor products.* The following lemma will allow us to pass between tensor products involving the two definitions of overconvergent Hyodo-Kato cohomology.

**Lemma 4.18.** *Let $W$ be a Banach space*[24] *over $\breve{F}$.*

(1) (Local-global compatibility) *Let $X$ be a smooth dagger affinoid. The canonical map*

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W\to\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W$$

*is a strict quasi-isomorphism.*

(2) *Let $X\in\mathrm{Sm}^{\dagger}_C$. There exists a following commutative diagram*

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W \arrow[r, "\alpha_{\mathrm{HK},\breve{F}}(W)", "\sim"'] & \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W \\
\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W \arrow[u, "\wr"] \arrow[r, "\alpha_{\mathrm{HK}}(W)", "\sim"'] & \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W. \arrow[u, "\wr"]
\end{tikzcd}$$

[24] In applications, $W$ will be most often a period rings.

*Remark* 4.19. (1) The tensor product $\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W$ is defined as

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W := \mathrm{L}\operatorname{colim}_h(\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W),$$

where $\{X_h\}$ is the presentation of $X$. By globalizing, we obtain the tensor product $\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W$.

(2) **Warning**: One has to be careful with tensor products as in (1) (because we chose projective tensor products hence we lost the commutation with general inductive limits). For example, when $F^{\mathrm{nr}} = \breve{F}$, the tensor product $\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W$ is already defined. Luckily, in this case, the two definitions give the same tensor product. To see this, note that we have $\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W = (\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{HK}}(X_h))\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W$. Hence the canonical map

$$\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W \to \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W$$

induces a map $\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W \to \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W$. In the proof of Lemma 4.18 below we will show that this is a strict quasi-isomorphism.

(3) For any smooth dagger variety $X$, the tensor product $\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W$ is defined by globalizing the tensor product from (1).

*Proof.* For (1), we start with the case $W = \breve{F}$. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\breve{F} \arrow[r] \arrow[d, "\wr"] & \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\breve{F} \arrow[d, "\wr"] \\
\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK},\breve{F}}(X) \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X).
\end{tikzcd} \tag{4.20}$$

The bottom map is a strict quasi-isomorphism by Lemma 4.14 (replace $F$ by $\breve{F}$). The left vertical map is a strict quasi-isomorphism by definition and (4.1); the right vertical map is the globalization of the left vertical map hence a strict quasi-isomorphism as well. It follows that the top map is also a strict quasi-isomorphism, as wanted.

Now, for a general $W$, we take the top map in the diagram (4.20) and tensor it with $W$ over $\breve{F}$ to obtain the strict quasi-isomorphism in the top of the commutative diagram

$$\begin{tikzcd}
(\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\breve{F})\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W \arrow[r, "\sim"] & (\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\breve{F})\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W \\
\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W \arrow[r] \arrow[u] & \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W \arrow[u]
\end{tikzcd}$$

It remains to show that the left vertical map in the diagram is a strict quasi-isomorphism because then so is the right vertical map (being the globalization of the left vertical map) and then the bottom map as well, as wanted.

*Remark* 4.21. The tensor product in the top row is the usual projective tensor product. Hence the vertical maps are not identities and the statement that they are strict quasi-isomorphisms is not trivial even for $\breve{F}$.

It is clear that the left vertical map is a strict quasi-isomorphism if we drop the dagger and replace $X$ with $X_h$ for the presentation $\{X_h\}$ of $X$. It suffices thus to show that the map

$$\mathrm{L}\operatorname{colim}_h(\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\breve{F}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W) \to (\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\breve{F})\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W \tag{4.22}$$

is a strict quasi-isomorphism. Applying the Hyodo-Kato morphism we pass to the canonical map

$$\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W \to (\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{dR}}(X_h))\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}W,$$

which is a strict quasi-isomorphism by [14, 2.1.2]. Now we go back to the map (4.22) by a projection $C \to \breve{F}$.

We pass now to the second claim of the lemma. Assume first that $X$ is a smooth dagger affinoid. Then we define the map

$$\alpha_{\mathrm{HK}}(W): \quad \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W \to \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W$$

as the composition

$$\begin{aligned}\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W &= \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W \xrightarrow{\sim} \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{HK}}(X_h^{\mathrm{o}})\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W\\ &\xleftarrow{\sim} \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X_h^{\mathrm{o},\dagger})\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W \to \mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W.\end{aligned} \tag{4.23}$$

The third map is a strict quasi-isomorphism by Corollary 3.4: this is because the interior $X_h^{\mathrm{o}}$ is partially proper.

For a general $X$, we obtain the map $\alpha_{\mathrm{HK}}(W)$ by globalizing the above definition. Changing $F$ into $\breve{F}$ in the definition of $\alpha_{\mathrm{HK}}(W)$, we get the map $\alpha_{\mathrm{HK},\breve{F}}(W)$ compatible with the map $\alpha_{\mathrm{HK}}(W)$. This gives us the commutative diagram we wanted. Moreover, it is clear from the definitions that the right vertical map in the diagram is a strict quasi-isomorphism. The top map is a strict quasi-isomorphism by Lemma 4.17. The left vertical map is a strict quasi-isomorphism because we can check it locally where claim (1) reduces us to the dagger cohomology of an affinoid and there this is clear from the definitions. It follows then that the bottom map is a strict quasi-isomorphism as well, as wanted. $\square$

(vi) *Properties of overconvergent Hyodo-Kato cohomology.* Let $X$ be a smooth dagger variety over $C$. Recall that (see [16, Prop. 4.38]) the Hyodo-Kato cohomology $\widetilde{H}^*_{\mathrm{HK}}(X)$ is classical. If $X$ is quasi-compact it is a finite dimensional $F^{\mathrm{nr}}$-vector space with its natural topology. For a general $X$, it is a limit (in $C_F$) of finite dimensional $F^{\mathrm{nr}}$-vector spaces. The endomorphism $\varphi$ on $H^*_{\mathrm{HK}}(X)$ is a homeomorphism.

We will need the following computation later on:

**Proposition 4.24.** *Let $X$ be a smooth dagger variety over $C$. Let $W$ be a Banach space with an $\breve{F}$-module structure.*

1. *If $X$ is quasi-compact then the cohomology of the complex $\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W$ is classical and we have*
$$\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W) \simeq H^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}W, \quad i \geq 0.$$
2. *Take an increasing admissible covering $\{U_n\}_{n\in\mathbf{N}}$ of $X$ by quasi-compact dagger varieties $U_n$. Then we have a natural strict quasi-isomorphism*
$$\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W \xrightarrow{\sim} \mathrm{R}\lim_n(\mathrm{R}\Gamma_{\mathrm{HK}}(U_{n,C})\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W).$$
*The cohomology of $\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W$ is classical and we have, for $i \geq 0$,*
$$\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W) \simeq H^i_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W := \lim_n(H^i_{\mathrm{HK}}(U_n)\otimes_{F^{\mathrm{nr}}}W).$$
*In particular, it is a Fréchet space*[25]*.*

*Proof.* By Lemma 4.18, we may replace $\mathrm{R}\Gamma_{\mathrm{HK}}(-)$ with Grosse-Klönne's version $\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(-)$. Let $X$ be quasi-compact. Consider an étale hypercovering $\mathscr{U}_\bullet$ of $X$ built from quasi-compact models from $\mathscr{M}_C^{\dagger,\mathrm{ss},b}$. By [14, Ex. 3.16], claim (1) is true for every $\mathscr{U}_{i,C}$. Hence we have the spectral sequence

$$E_2^{i,j} = H^{\mathrm{GK},i}_{\mathrm{HK}}(\mathscr{U}_{j,C})\otimes_{F^{\mathrm{nr}}}W \Rightarrow \widetilde{H}^{i+j}(\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}W).$$

The terms of the spectral sequence are Banach spaces and the differentials in the spectral sequence are $W$-linear. Since the Hyodo-Kato cohomology groups $H^{\mathrm{GK},i}_{\mathrm{HK}}(\mathscr{U}_{j,C})$ are of finite rank, claim (1) follows.

[25] We note that $H^i_{\mathrm{HK}}(U_n)$ is a finite rank vector space over $F^{\mathrm{nr}}$ equipped with the canonical topology.

Having (1), claim (2) follows just as in the proof of [14, 3.26] (note that the system $\{H^i_{\mathrm{HK}}(U_n)\otimes_{F^{\mathrm{nr}}}W\}_{n\in\mathbf{N}}$ satisfies the Mittag-Leffler condition). $\square$

4.2.2. *Overconvergent geometric Hyodo-Kato morphism via presentations of dagger structures.* In this section we introduce a definition of overconvergent geometric Hyodo-Kato morphism using presentations of dagger structures.

(i) *Local definition.* Let $X$ be a dagger affinoid over $C$. Let $\mathrm{pres}(X)=\{X_h\}$.

Define natural Hyodo-Kato morphisms

$$\iota^{\dagger}_{\mathrm{HK}}:\quad \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK},\breve{F}}(X)\to \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X),\quad \iota^{\dagger}_{\mathrm{HK}}:\quad \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK},\breve{F}}(X)\to \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \tag{4.25}$$

as the compositions

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK},\breve{F}}(X)=\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X_h)\xrightarrow{\mathrm{L}\operatorname{colim}_h(\iota_{\mathrm{HK}})}\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)=\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X),$$
$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK},\breve{F}}(X)=\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X_h)\xrightarrow{\mathrm{L}\operatorname{colim}_h(\iota_{\mathrm{HK}})}\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h/\mathbf{B}^+_{\mathrm{dR}})=\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}).$$

They are compatible via the map $\theta:\mathbf{B}^+_{\mathrm{dR}}\to C$.

**Proposition 4.26.** *The linearizations of the Hyodo-Kato morphisms in (4.25) yield compatible natural strict Hyodo-Kato quasi-isomorphisms*

$$\iota^{\dagger}_{\mathrm{HK}}:\quad \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C\xrightarrow{\sim}\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X),\quad \iota^{\dagger}_{\mathrm{HK}}:\quad \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}).$$

*Proof.* For the first map, we need to show that the map

$$(\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X_h))\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C\xrightarrow{\mathrm{L}\operatorname{colim}_h(\iota_{\mathrm{HK}})}\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$$

is a strict quasi-isomorphism. But this map fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X_h)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C \arrow[r] \arrow[d, "\wr"', "\mathrm{L}\operatorname{colim}_h(\iota_{\mathrm{HK}})"] & (\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X_h))\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C \arrow[dl, "\mathrm{L}\operatorname{colim}_h(\iota_{\mathrm{HK}})"] \\
\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h) &
\end{tikzcd}$$

The bottom term is just the overconvergent de Rham cohomology and its cohomology is classical and a finite rank vector space over $C$ with its canonical topology. Via the vertical strict quasi-isomorphism the same is true of the upper left term. The upper right term is strictly quasi-isomorphic to $\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C$, which, by Lemma 4.18 and Proposition 4.24, also has classical cohomology that is finite rank over $C$. Hence, looking at the above diagram one cohomology degree at a time, we obtain a commutative diagram of finite rank vector spaces over $C$. These ranks are, in fact, equal: this is clear for the bottom and the upper left term; for the upper right term consider the maps:

$$\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C\simeq\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}C\xrightarrow{\iota^{\mathrm{GK}}_{\mathrm{HK}}}\mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

The first map is a strict quasi-isomorphism by Lemma 4.18. The second map is the Grosse-Klönne Hyodo-Kato morphism and it is a strict quasi-isomorphism by [16, 5.15]. Hence the rank in question is the same as that of the corresponding de Rham cohomology, as wanted.

For the second map in our proposition, we argue in a similar fashion. We need to show that the map

$$(\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X_h))\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\mathrm{L}\operatorname{colim}_h(\iota_{\mathrm{HK}})}\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{dR}}(X_h/\mathbf{B}^+_{\mathrm{dR}})$$

is a strict quasi-isomorphism. But this map fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X_h)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}} \arrow[r] \arrow[d, "\wr"', "\mathrm{L}\operatorname{colim}_h(\iota_{\mathrm{HK}})"] & (\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X_h))\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}} \arrow[dl, "\mathrm{L}\operatorname{colim}_h(\iota_{\mathrm{HK}})"] \\
\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{dR}}(X_h/\mathbf{B}^+_{\mathrm{dR}}) &
\end{tikzcd}$$

The vertical map is a strict quasi-isomorphism by (4.7). The horizontal map can be shown to be a strict quasi-isomorphism by an argument analogous to the one used in the proof of Proposition 3.26. It follows that so is the slanted map, as wanted. □

(ii) *Globalization.* For a general smooth dagger variety $X$ over $C$, globalizing the maps $\iota^\dagger_{\mathrm{HK}}$ from (4.25), we obtain compatible natural maps

$$\iota_{\mathrm{HK}}:\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\to\mathrm{R}\Gamma_{\mathrm{dR}}(X),\quad \iota_{\mathrm{HK}}:\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\to\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}).$$

**Theorem 4.27.** (Overconvergent Hyodo-Kato isomorphisms) *The linearizations of the above Hyodo-Kato morphisms yields compatible natural strict quasi-isomorphisms*

$$\iota_{\mathrm{HK}}:\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X),$$
$$\iota_{\mathrm{HK}}:\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}).$$

*Proof.* Looking at $\eta$-étale hypercoverings and using that our tensor products commute with products, we may assume $X$ to be a dagger affinoid and then the result is known by Proposition 4.26. □

(iii) *Application.* As an immediate application of the overconvergent Hyodo-Kato quasi-isomorphisms we get the local-global compatibility for $\mathbf{B}^+_{\mathrm{dR}}$-cohomology: .

**Corollary 4.28.** (Local-global compatibility) *Let $X$ be a smooth dagger affinoid over $C$. The canonical morphism*

$$\mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\to\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \tag{4.29}$$

*is a filtered strict quasi-isomorphism.*

*Proof.* Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma^\dagger_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}} \arrow[r, "\sim"] \arrow[d, "\wr"', "\iota^\dagger_{\mathrm{HK}}"] & \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}} \arrow[d, "\wr"', "\iota_{\mathrm{HK}}"] \\
\mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r] & \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})
\end{tikzcd}$$

The vertical arrows are strict quasi-isomorphisms by Proposition 4.26 and Theorem 4.27. The top arrow is a strict quasi-isomorphism by the local-global compatibility for completed overconvergent Hyodo-Kato cohomology. It follows that so is the bottom horizontal arrow, proving that the map (4.29) is a strict quasi-isomorphism.

To show that this map is a filtered strict quasi-isomorphism, we will argue by induction on $r\geq 0$. The inductive step uses the following commutative diagram

$$\begin{tikzcd}
F^{r-1}\mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "t"] \arrow[d, "\wr"] & F^r\mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "\vartheta"] \arrow[d] & F^r\mathrm{R}\Gamma^\dagger_{\mathrm{dR}}(X), \arrow[d, "\wr"] \\
F^{r-1}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "t"] & F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "\vartheta"] & F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X),
\end{tikzcd}$$

in which the rows are distinguished triangles by Proposition 3.26 and its proof. The first and the third vertical maps are strict quasi-isomorphism by the inductive hypothesis and by the local-global property for de Rham cohomology, respectively. It follows that the middle vertical map is a strict quasi-isomorphism as well, as wanted. □

4.2.3. *Comparison with the rigid analytic constructions.* Let $X$ be a smooth dagger variety over $L = K, C$. Let $\widehat{X}$ be its completion.

**Lemma 4.30.** (1) *There is a natural morphism*

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X) \to \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}). \tag{4.31}$$

(2) *Let $L = C$. There are compatible natural morphisms*

$$\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X) \to \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(\widehat{X}), \quad \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \to \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}/\mathbf{B}^+_{\mathrm{dR}}).$$

*They are compatible with the map (4.31).*

(3) *The morphism in (2) are compatible with the Hyodo-Kato morphisms, i.e., we have the commutative diagrams*

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X) \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}"] & \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(\widehat{X}) \arrow[d, "\iota_{\mathrm{HK}}"] \\
\mathrm{R}\Gamma_{\mathrm{dR}}(X) \arrow[r] & \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}),
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X) \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}"] & \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(\widehat{X}) \arrow[d, "\iota_{\mathrm{HK}}"] \\
\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r] & \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}/\mathbf{B}^+_{\mathrm{dR}})
\end{tikzcd}$$

*Proof.* Let $X$ be a smooth dagger affinoid over $L$ with the presentation $\{X_h\}$. Using the compatible maps $\widehat{X} \to X_h$, we define the map

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X) = \mathrm{L}\,\mathrm{colim}_h\, \mathrm{R}\Gamma_{\mathrm{HK}}(X_h) \to \mathrm{L}\,\mathrm{colim}_h\, \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}) = \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}).$$

It globalizes to give the map in (4.31).

We proceed in a similar way for the other two cohomologies. The stated compatibilities follow easily from the definitions. □

The Hyodo-Kato quasi-isomorphisms imply the following:

**Corollary 4.32.** *Let $X \in \mathrm{Sm}^{\dagger}_C$. If $X$ is partially proper, then the canonical morphisms*

$$\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X) \to \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(\widehat{X}), \quad \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \to \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}/\mathbf{B}^+_{\mathrm{dR}}) \tag{4.33}$$

*are (filtered) strict quasi-isomorphisms.*

*Proof.* For the first map, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}", "\wr"'] & \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(\widehat{X})\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}C \arrow[d, "\iota_{\mathrm{HK}}", "\wr"'] \\
\mathrm{R}\Gamma_{\mathrm{dR}}(X) \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X})
\end{tikzcd}$$

It implies that the top arrow is a strict quasi-isomorphism. Splitting off $\breve{F}$ from $C$ we obtain the claim of the corollary.

For the second map, in the unfiltered case, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}} \arrow[r, "\sim"] \arrow[d, "\iota_{\mathrm{HK}}", "\wr"'] & \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(\widehat{X})\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}} \arrow[d, "\iota_{\mathrm{HK}}", "\wr"'] \\
\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r] & \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}/\mathbf{B}^+_{\mathrm{dR}})
\end{tikzcd}$$

The top arrow is a strict quasi-isomorphism by what was just proved. It implies that the bottom arrow is a strict quasi-isomorphism, as wanted.

To treat filtrations, we proceed by induction on the filtration level $r$ (the base case of $r=0$ just proved). The inductive step ($r-1 \Rightarrow r$) uses the commutative diagram

$$\begin{tikzcd} F^{r-1}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "t"] \arrow[d, "\wr"] & F^{r}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "\vartheta"] \arrow[d] & F^{r}\mathrm{R}\Gamma_{\mathrm{dR}}(X), \arrow[d, "\wr"] \\ F^{r-1}\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "t"] & F^{r}\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r, "\vartheta"] & F^{r}\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}), \end{tikzcd}$$

in which the rows are distinguished triangles by Proposition 3.26 and Proposition 3.11. The first vertical map is a strict quasi-isomorphism by the inductive hypothesis. It follows that the middle vertical map is a strict quasi-isomorphism as well, as wanted. $\square$

## 5. Overconvergent geometric syntomic cohomology

In this section we will define overconvergent geometric syntomic cohomology and prove a comparison theorem for smooth dagger affinoids and Stein varieties over $C$.

5.1. **Local-global compatibility for rigid analytic geometric syntomic cohomology.** Recall that in [16, Sec. 4.1] the syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))$ of a rigid analytic variety $X$ is defined by $\eta$-étale descent from the crystalline syntomic cohomology of Fontaine-Messing. The latter is defined as the homotopy fiber ($\mathscr{X}$ is a semistable formal scheme over $\mathscr{O}_C$ equipped with its canonical log-structure)

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},\mathbf{Q}_p(r)) := [F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\xrightarrow{\varphi-p^r}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})],$$

where the (logarithmic) crystalline cohomology is absolute (i.e., over $\mathbf{Z}_p$). By definition, it fits into the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(1)) \to [\mathrm{R}\Gamma_{\mathrm{cr}}(X)]^{\varphi=p^r} \to \mathrm{R}\Gamma_{\mathrm{cr}}(X)/F^r \tag{5.1}$$

We were not able to prove the local-global compatibility for this syntomic cohomology in [16]: the usual technique is to pass from the second term of (5.1) to Hyodo-Kato cohomology and from the third term – to filtered de Rham cohomology; then one passes, via the Hyodo-Kato quasi-isomorphism, from Hyodo-Kato cohomology to de Rham cohomology and we do have local-global compatibility for filtered de Rham cohomology. The problem was: we did not have then the Hyodo-Kato morphism. But we have it now thanks to Theorem 4.6, so in this section we will prove the local-global compatibility for rigid analytic geometric syntomic cohomology that we will need.

We start with stating such a compatibility for absolute crystalline cohomology.

**Lemma 5.2.** (Crystalline local-global compatibility) *Let $\mathscr{X}\in\mathscr{M}^{\mathrm{ss},b}_C$. The canonical map*

$$\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})_{\mathbf{Q}_p}\to\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_C)$$

*is a strict quasi-isomorphism.*

*Proof.* We have the commutative diagram

$$\begin{tikzcd} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})_{\mathbf{Q}_p} \arrow[rr] & & \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_C) \\ \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(\mathscr{X}^0_1)_{\mathbf{Q}_p}\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{cr}} \arrow[u, "\wr", "\varepsilon^{\mathrm{HK}}_{\mathrm{cr}}"'] \arrow[rr, "\beta\otimes\mathrm{Id}"] & & \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(\mathscr{X}_C)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{cr}} \arrow[u, "\wr", "\varepsilon^{\mathrm{HK}}_{\mathrm{cr}}"'] \end{tikzcd}$$

The bottom map is a strict quasi-isomorphism by Lemma 4.2. Hence so is the top map, as wanted. $\square$

**Proposition 5.3.** (Syntomic local-global compatibility) *Let $\mathscr{X}\in\mathscr{M}^{\mathrm{ss},b}_C$. Let $r\geq 0$. The canonical map*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},\mathbf{Z}_p(r))_{\mathbf{Q}_p}\to\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_C,\mathbf{Q}_p(r))$$

*is a strict quasi-isomorphism. Here* $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},\mathbf{Z}_p(r))$ *is the syntomic cohomology of Fontaine-Messing* [21] (see also [4]).

*Proof.* Set $X := \mathscr{X}_C$. First, we define a natural strict quasi-isomorphism:

$$\begin{aligned}\iota_2:\quad & \big[[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r\big]\\ & \to [[\mathrm{R}\Gamma_{\mathrm{cr}}(X)]^{\varphi=p^r}\xrightarrow{\mathrm{can}}\mathrm{R}\Gamma_{\mathrm{cr}}(X)/F^r] = \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)).\end{aligned}$$

For that, it suffices to define the maps $\iota^1_{\mathrm{BK}}$ and $\iota^2_{\mathrm{BK}}$ in the following diagram and to show that this diagram commutes:

$$\begin{tikzcd} {[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0}} \arrow[r, "\iota_{\mathrm{HK}}\otimes\iota"] \arrow[d, "\varepsilon^{\mathrm{HK}}_{\mathrm{st}}", "\wr"'] \arrow[dd, bend right=60, "\iota^1_{\mathrm{BK}}"', "\wr"] & \mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r \arrow[dd, bend left=60, "\iota^2_{\mathrm{BK}}", "\wr"'] \\ {[\mathrm{R}\Gamma_{\mathrm{cr}}(X)\widehat{\otimes}_{\mathbf{B}^+_{\mathrm{cr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0}} \arrow[ur, "\kappa"] & \\ \mathrm{R}\Gamma_{\mathrm{cr}}(X) \arrow[u, "\wr"] \arrow[r, "\mathrm{can}"] & \mathrm{R}\Gamma_{\mathrm{cr}}(X)/F^r. \arrow[uu, "\kappa"', "\wr"] \end{tikzcd} \tag{5.4}$$

We define the maps $\iota^1_{\mathrm{BK}}$ and $\iota^2_{\mathrm{BK}}$ to make the left and the right triangles in the diagram commute. They are strict quasi-isomorphisms. The remaining pieces of the diagram commute by definition.

The morphism $\iota_2$ has a compatible local version. Now, the wanted local-global compatibility, via the strict quasi-isomorphisms $\iota_2$, follows from local-global compatibility for Hyodo-Kato cohomology and filtered $\mathbf{B}^+_{\mathrm{dR}}$-cohomology, proved in Proposition 3.1 and Lemma 3.20, respectively. □

The proof of Proposition 5.3 actually shows the following:

**Corollary 5.5.** *Let* $X\in\mathrm{Sm}_C$ *and* $r\geq 0$. *There exist a natural strict quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\simeq\big[[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r\big]. \tag{5.6}$$

We like to call the expression on the right *the Bloch-Kato syntomic cohomology* because it resembles the definition of Bloch-Kato Selmer groups in [9].

5.2. **Twisted Hyodo-Kato cohomology.** Let $X$ be a smooth dagger variety over $C$. In this section we will study the twisted Hyodo-Kato cohomology

$$\mathrm{HK}(X,r) := [\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0,\varphi=p^r},\quad r\geq 0, \tag{5.7}$$

where $\mathrm{R}\Gamma_{\mathrm{HK}}(X)$ is the geometric Hyodo-Kato cohomology defined in [16, Sec. 4.2.4] and we set

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}} := \mathrm{L}\operatorname{colim}((\mathrm{R}\Gamma_{\mathrm{HK}}\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}})(U_\bullet)),$$

where the homotopy colimit is taken over $\eta$-étale affinoid hypercoverings $U_\bullet$ from $\mathscr{M}^\dagger_C$. We wrote $[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}$ for the homotopy limit of the commutative diagram

$$\begin{tikzcd} \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}} \arrow[r, "\varphi-p^r"] \arrow[d, "N"] & \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}} \arrow[d, "N"] \\ \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}} \arrow[r, "p\varphi-p^r"] & \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}. \end{tikzcd}$$

The following proposition generalizes the computations from [14, Sec. 3.2.2] done in the case when $X$ has a semistable integral model over a finite extension of $K$.

**Proposition 5.8.** *Let* $X$ *be a smooth dagger variety over* $C$. *Let* $r\geq 0$.

(1) *If $X$ is quasi-compact then the cohomology of the complex* $\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}$ *is classical and we have*
$$\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}})\simeq H^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}},\quad i\geq 0.$$
(2) *If $X$ is quasi-compact there is a natural isomorphism*
$$\widetilde{H}^i(\mathrm{HK}(X,r))\simeq (H^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^r},\quad i\geq 0,$$
*of Banach spaces. In particular,* $\widetilde{H}^i(\mathrm{HK}(X,r))$ *is classical.*
(3) *Take an increasing admissible covering $\{U_n\}_{n\in\mathbf{N}}$ of $X$ by quasi-compact dagger varieties $U_n$. Then we have a natural strict quasi-isomorphism*
$$\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}\stackrel{\sim}{\to}\mathrm{R}\lim_n(\mathrm{R}\Gamma_{\mathrm{HK}}(U_{n,C})\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}).$$
*The cohomology of* $\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}$ *is classical and we have, for $i\geq 0$,*
$$\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}})\simeq H^i_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}:=\lim_n(H^i_{\mathrm{HK}}(U_n)\otimes_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}).$$
*In particular, it is a Fréchet space*[26].
(4) *The cohomology* $\widetilde{H}^i([\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0,\varphi=p^r})$, *$i\geq 0$, is classical and we have natural isomorphisms*
$$H^i([\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0,\varphi=p^r})\simeq (H^i_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^r},\quad i\geq 0.$$
*In particular, the space* $H^i([\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0,\varphi=p^r})$ *is Fréchet. Moreover,*
$$H^i([\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0})\simeq (H^i_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0}\simeq H^i_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{cr}},$$
*where the last isomorphism is not, in general, Galois equivariant (in the case $X$ comes from $X_K$ over $K$).*

*Proof.* Since $\widehat{\mathbf{B}}^+_{\mathrm{st}}$ is a Banach space over $\breve{F}$, claims (1) and (3) are a special case of Proposition 4.24. Claim (2) follows from (1) just as in the proof of [14, Lemma 3.20]. Finally, claim (4) follows from (3) is proved as in [14, Lemma 3.28]. □

5.2.1. *A variant of the twisted Hyodo-Kato cohomology.* There is a variant of the twisted Hyodo-Kato cohomology
$$\widetilde{\mathrm{HK}}(X,r):=[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r},\quad r\geq 0,$$
that we will often use. Here we set
$$\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}:=\mathrm{L}\,\mathrm{colim}((\mathrm{R}\Gamma_{\mathrm{HK}}\widehat{\otimes}_{F^{\mathrm{nr}},\iota}\mathbf{B}^+_{\mathrm{st}})(U_\bullet)),$$
where the homotopy colimit is taken over $\eta$-étale affinoid hypercoverings $U_\bullet$ from $\mathscr{M}^\dagger_C$. We have
$$\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}},\iota}\mathbf{B}^+_{\mathrm{st}}\simeq \mathrm{L}\,\mathrm{colim}_h(\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}},\iota}\mathbf{B}^+_{\mathrm{st}}),$$
where $\{X_h\}$ is the presentation of $X$. It is easy to check that this tensor product satisfies local-global compatibility.

**Lemma 5.9.** *Let $X\in\mathrm{Sm}^\dagger_C$. The canonical morphism*
$$\widetilde{\mathrm{HK}}(X,r)\to\mathrm{HK}(X,r),\quad r\geq 0.$$
*is a strict quasi-isomorphism.*

[26] We note that $H^i_{\mathrm{HK}}(U_n)$ is a finite rank vector space over $F^{\mathrm{nr}}$ equipped with the canonical topology.

*Proof.* It suffices to show that the canonical morphism
$$[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0}\to[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0}$$
is a strict quasi-isomorphism. For that, from the definitions of both sides, we can assume that $X$ is a dagger affinoid. Then this map can be rewritten as
$$[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}},\iota}\mathbf{B}^+_{\mathrm{st}}]^{N=0}\to[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0},$$
which, by Lemma 4.18, can be written as
$$\mathrm{L}\operatorname{colim}_h([\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\widehat{\otimes}_{F^{\mathrm{nr}},\iota}\mathbf{B}^+_{\mathrm{st}}]^{N=0})\to\mathrm{L}\operatorname{colim}_h([\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\widehat{\otimes}_{F^{\mathrm{nr}}}\widetilde{\mathbf{B}}^+_{\mathrm{st}}]^{N=0}),$$
for the presentation $\{X_h\}$ of $X$. But this map is a strict quasi-isomorphism because so is the canonical map
$$[\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\widehat{\otimes}_{F^{\mathrm{nr}},\iota}\mathbf{B}^+_{\mathrm{st}}]^{N=0}\to[\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\widehat{\otimes}_{F^{\mathrm{nr}}}\widetilde{\mathbf{B}}^+_{\mathrm{st}}]^{N=0},$$
by the same argument as the one used to show (2.18). □

5.3. **$\mathbf{B}^+_{\mathrm{dR}}$-cohomology.** Let $X$ be a smooth dagger variety over $C$. In this section we will study the filtered $\mathbf{B}^+_{\mathrm{dR}}$-cohomology $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ and its quotients
$$\mathrm{DR}(X,r):=\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r,\quad r\geq 0.$$
We note that, immediately from the distinguished triangle (3.27), we obtain

**Lemma 5.10.** *Let $X$ be a smooth dagger variety over $C$. Let $r\geq 0$. We have a distinguished triangle*
$$\mathrm{DR}(X,r-1)\xrightarrow{t}\mathrm{DR}(X,r)\xrightarrow{\vartheta}\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r$$

By Theorem 4.27, we have the strict quasi-isomorphism
$$\iota_{\mathrm{HK}}:\quad \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}).\tag{5.11}$$
It yields the following computation:

**Proposition 5.12.** *Let $X$ be a smooth dagger variety over $C$.*

(1) *If $X$ is quasi-compact then the cohomology of the complex $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ is classical and we have*
$$\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))\simeq H^i_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}},\quad i\geq 0.$$

(2) *Take an increasing admissible covering $\{U_n\}_{n\in\mathbf{N}}$ of $X$ by quasi-compact dagger varieties $U_n$. Then we have a natural strict quasi-isomorphism*
$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\xrightarrow{\sim}\mathrm{R}\lim_n\mathrm{R}\Gamma_{\mathrm{dR}}(U_n/\mathbf{B}^+_{\mathrm{dR}}).$$
*The cohomology of $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ is classical and we have, for $i\geq 0$,*
$$\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))\simeq H^i_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}\simeq\lim_n(H^i_{\mathrm{HK},\breve{F}}(U_n)\otimes_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}).$$
*In particular, it is a Fréchet space*[27].

*Proof.* Using the Hyodo-Kato morphism (5.11), we may pass to the computation of the cohomology of the complex $\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}$. Since $\mathbf{B}^+_{\mathrm{dR}}\simeq\prod_{k\geq 0}Ct^k$ in $C_{\breve{F}}$, we have
$$\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}\simeq\prod_{k\geq 0}(\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}Ct^k)$$
and we can use Lemma 4.18 to pass to $\mathrm{R}\Gamma^{\mathrm{GK}}_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}$. Then the proof of Proposition 4.24 goes through. □

[27] Recall that $H^i_{\mathrm{HK},\breve{F}}(U_n)$ is a finite rank vector space over $\breve{F}$ equipped with the canonical topology.

5.3.1. *Varieties over $K$.* Before studying filtrations on $\mathbf{B}^+_{\mathrm{dR}}$-cohomology we will look more carefully at the example of varieties defined over $K$.

Recall that (see [16, Sec. 5.1]), for a smooth dagger variety $X$ over $L$, $L=K,C$, the de Rham cohomology $\widetilde{H}^i_{\mathrm{dR}}(X)$ is classical. If $X$ is quasi-compact it is a finite dimensional $L$-vector space with its natural topology. For a general $X$, it is a limit (in $C_{\mathbf{Q}_p}$) of finite dimensional $L$-vector spaces (hence a Fréchet space).

Let $X\in \mathrm{Sm}^{\dagger}_K$. By Proposition 3.26, we have the (filtered) strict quasi-isomorphisms

$$\begin{aligned}\iota_{\mathrm{BK}}:\quad \mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}} &\xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}}),\\ \mathrm{DR}(X_C,r)&\simeq (\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})/F^r.\end{aligned}$$

(i) *Example: Stein varieties over $K$.* Assume that $X$ is Stein. We easily see that (in $\mathscr{D}(C_K)$)

$$\begin{aligned}(5.13)\quad F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}) &\simeq F^r(\Omega^\bullet(X)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})\\ &=(\mathscr{O}(X)\widehat{\otimes}_K F^r\mathbf{B}^+_{\mathrm{dR}}\to\Omega^1(X)\widehat{\otimes}_K F^{r-1}\mathbf{B}^+_{\mathrm{dR}}\to\cdots)\\ \mathrm{DR}(X_C,r)=(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})/F^r &\simeq (\Omega^\bullet(X)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r\\ &=(\mathscr{O}(X)\widehat{\otimes}_K(\mathbf{B}^+_{\mathrm{dR}}/F^r)\to\Omega^1(X)\widehat{\otimes}_K(\mathbf{B}^+_{\mathrm{dR}}/F^{r-1})\to\cdots\to\Omega^{r-1}(X)\widehat{\otimes}_K(\mathbf{B}^+_{\mathrm{dR}}/F^1)).\end{aligned}$$

In low degrees we have

$$\begin{aligned}&\mathrm{DR}(X_C,0)=0,\quad \mathrm{DR}(X_C,1)\simeq\mathscr{O}(X)\widehat{\otimes}_K C,\\ &\mathrm{DR}(X_C,2)\simeq(\mathscr{O}(X)\widehat{\otimes}_K(\mathbf{B}^+_{\mathrm{dR}}/F^2)\to\Omega^1(X)\widehat{\otimes}_K C).\end{aligned}$$

Recall that, because $X$ is Stein, the de Rham complex is built from Fréchet spaces and it has strict differentials. Arguing just as in [14, Ex. 3.30] it follows that:

(1) the complexes $F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})$ and $\mathrm{DR}(X_C,r)$ are built from Fréchet spaces;
(2) their differentials are strict;
(3) and the cohomologies $\widetilde{H}^iF^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})$ and $\widetilde{H}^i\mathrm{DR}(X_C,r)$ are classical and Fréchet.

(ii) *Example: Affinoids over $K$.* Assume now that $X$ is an affinoid. Then the computation is a bit more complicated because the spaces $\Omega^i(X)$ and $\mathbf{B}^+_{\mathrm{dR}}$ (an $LB$-space and a Fréchet space, respectively) do not work together well with tensor products. However, if we use the fact that $\mathbf{B}^+_{\mathrm{dR}}\simeq\prod_{k\geq 0}Ct^k$ in $\mathscr{D}(C_K)$, we get the strict quasi-isomorphisms

$$\Omega^i(X)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\sim}\Omega^i(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}},$$

which implies the strict quasi-isomorphisms from (5.13).

Then, arguing just as in [14, Ex. 3.30], one shows that the cohomology $\widetilde{H}^i\mathrm{DR}(X_C,r)$ is classical and that it is an $LB$-space. Also, we easily see that the differentials in the complex $F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})$ are strict; hence the cohomology $\widetilde{H}^iF^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})$ is classical.

(iii) *General varieties over $K$.* The following computation can be done in the same way as the computation in Proposition 5.12.

**Proposition 5.14.** *Let $X$ be a smooth dagger variety over $K$.*

(1) *If $X$ is quasi-compact then the cohomology of the complex $\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}$ is classical and we have*

$$\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})\simeq H^i_{\mathrm{dR}}(X)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}},\quad i\geq 0.\tag{5.15}$$

(2) *Take an increasing admissible covering $\{U_n\}_{n\in\mathbf{N}}$ of $X$ by quasi-compact dagger varieties $U_n$. Then we have a natural strict quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{R}\lim_n(\mathrm{R}\Gamma_{\mathrm{dR}}(U_n)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}).$$

*The cohomology of* $\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}$ *is classical and we have, for* $i\geq 0$,

$$\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}})\simeq H^i_{\mathrm{dR}}(X)\widehat{\otimes}^{\mathrm{R}}_K\mathbf{B}^+_{\mathrm{dR}}\simeq \lim_n(H^i_{\mathrm{dR}}(U_n)\otimes_K\mathbf{B}^+_{\mathrm{dR}}).$$

*In particular, it is a Fréchet space*[28].

5.3.2. *Stein varieties and affinoid over* $C$. If $X$ is a smooth dagger affinoid over $C$ then it is defined over a finite field extension of $K$ and its de Rham type cohomologies have properties listed in Section 5.3.1.

In the case of Stein varieties we need to argue a bit more.

**Proposition 5.16.** *Let* $X\in \mathrm{Sm}^\dagger_C$ *be Stein and* $r\geq 0$. *Then*

(1) *concerning the complex* $\mathrm{DR}(X,r)$, *we have:*
   (a) *the cohomology* $\widetilde{H}^i\mathrm{DR}(X,r)$ *is classical and Fréchet.*
   (b) *we have a strictly exact sequence*

$$0\to\Omega^i(X)/\operatorname{Im} d\to H^i\mathrm{DR}(X,r)\to H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/t^{r-i-1}\to 0$$

(2) *the cohomology* $\widetilde{H}^iF^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ *is classical and Fréchet.*

*Proof.* Concerning claim (1), cover $X$ with a Stein covering by affinoids $\{U_n\}$, $n\in\mathbf{N}$. Since every affinoid $U_n$ is defined over a finite extension of $K$, we have the strict exact sequences from [14, Ex. 3.30]

$$0\to\Omega^i(U_n)/\operatorname{Im} d\to H^i\mathrm{DR}(U_n,r)\to H^i_{\mathrm{dR}}(U_n/\mathbf{B}^+_{\mathrm{dR}})/t^{r-i-1}\to 0$$

All the terms are classical and Hausdorff. We claim that, taking their $\lim_n$, we obtain

$$\begin{aligned}(5.17)\qquad & 0\to\lim_n(\Omega^i(U_n)/\operatorname{Im} d)\to\lim_n H^i\mathrm{DR}(U_n,r)\to\lim_n(H^i_{\mathrm{dR}}(U_n/\mathbf{B}^+_{\mathrm{dR}})/t^{r-i-1})\to 0\\ & \mathrm{R}^1\lim_n H^i\mathrm{DR}(U_n,r)\simeq \mathrm{R}^1\lim_n H^i_{\mathrm{dR}}(U_n/\mathbf{B}^+_{\mathrm{dR}})/t^{r-i-1}=0\end{aligned}$$

Indeed, the sequence is strictly exact since $\mathrm{R}^1\lim_n\Omega^i(U_n)=0$. For the same reason we have the isomorphism between $\mathrm{R}^1$ lim's. Since we have Hyodo-Kato isomorphisms $H^i_{\mathrm{dR}}(U_n/\mathbf{B}^+_{\mathrm{dR}})\simeq H^i_{\mathrm{HK},\breve{F}}(U_n)\widehat{\otimes}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}$ and the Hyodo-Kato cohomology $H^i_{\mathrm{HK},\breve{F}}(U_n)$ is of finite rank, these $\mathrm{R}^1\lim_n$ vanish. From (5.17) we obtain the strictly exact sequence

$$0\to\Omega^i(X)/\operatorname{Im} d\to \widetilde{H}^i\mathrm{DR}(X,r)\to H^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/t^{r-i-1}\to 0$$

Hence, $\widetilde{H}^i\mathrm{DR}(X,r)$ is classical (as an extension of two classical objects). It is also an extension of two Fréchet spaces; which implies that it is, in particular, Hausdorff. It is also a quotient of two Fréchet spaces by construction, which implies that it is a Fréchet space itself, as wanted.

For claim (2), since we have the Hyodo-Kato strict quasi-isomorphism (from Theorem 4.27)

$$\iota_{\mathrm{HK}}:\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$$

and the cohomology

$$\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}})\simeq H^i_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}_{\breve{F}}\mathbf{B}^+_{\mathrm{dR}}$$

is classical, we get that the cohomology $\widetilde{H}^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ is also classical and Fréchet. For $i>r$, we have an isomorphism $\widetilde{H}^i(F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))\xrightarrow{\sim}\widetilde{H}^i_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$ (take an exhaustive affinoid covering and use the fact that affinoids are defined over a finite extension of $K$); hence this cohomology is also classical and Fréchet.

For $i\leq r$, we argue by induction on $r$, the base case of $r=0$ being shown above. For the inductive step ($r-1\Rightarrow r$), take the distinguished triangle (3.27) and consider the induced long exact sequence

$$0\to H^i(F^{r-1}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))\xrightarrow{t}\widetilde{H}^i(F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))\xrightarrow{\vartheta}H^i(F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X))\xrightarrow{\partial}H^{i+1}(F^{r-1}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))$$

[28] We note that $H^i_{\mathrm{dR}}(U_n)$ is a finite rank vector space over $K$ equipped with the canonical topology.

The injection on the left follows from the fact that $H^{i-1}(F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X))=0$; the terms involving $F^{r-1}$ filtration are classical by the inductive hypothesis.

• If $i<r$, then this yields an isomorphism

$$H^i(F^{r-1}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))\underset{\sim}{\xrightarrow{t}}\widetilde{H}^i(F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})),$$

showing that $\widetilde{H}^i(F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))$ is classical and Fréchet.

• For $i=r$, we get a short exact sequence

$$0\to H^i(F^{r-1}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))\xrightarrow{t}\widetilde{H}^i(F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))\xrightarrow{\vartheta}\ker\partial\to 0$$

Hence, $\widetilde{H}^i(F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}))$ is classical and a Fréchet space by the argument we have used in the case of $\widetilde{H}^i\mathrm{DR}(X,r)$ in the proof of claim (1). □

5.4. **Overconvergent geometric syntomic cohomology.** We are now ready to define overconvergent geometric syntomic cohomology and prove a comparison theorem for smooth dagger affinoids and Stein varieties.

Let $X$ be a smooth dagger variety over $C$. Take $r\geq 0$. We define the *geometric syntomic cohomology* of $X$ as the following mapping fiber (taken in $\mathscr{D}(C_{\mathbf{Q}_p})$)

$$\begin{aligned}(5.18)\qquad \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) &:= [[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r]\\ &= [\mathrm{HK}(X,r)\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\mathrm{DR}(X,r)].\end{aligned}$$

This is a generalization of the geometric syntomic cohomology introduced in [14, Sec. 3.2.2] in the case $X$ comes from a semistable model over $\mathscr{O}_K$. We will define below in Section 6.2.1 overconvergent geometric syntomic cohomology via presentations of dagger structures from rigid-analytic geometric syntomic cohomology and show in Proposition 6.6 that the two definitions give strictly quasi-isomorphic cohomologies.

The following proposition generalizes [14, Prop. 3.36].

*Remark* 5.19. We will often use an equivalent definition of overconvergent geometric syntomic cohomology:

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) := [[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r].$$

See Lemma 5.9 for why the two definitions give the same object (up to a canonical strict quasi-isomorphism).

**Proposition 5.20.** *Let $X$ be a smooth dagger affinoid or a smooth dagger Stein variety over $C$. Let $r\geq 0$. There is a natural map of strictly exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \Omega^{r-1}(X)/\operatorname{Ker} d \arrow[r, "\partial"] \arrow[d, equal] & H^r_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \arrow[r] \arrow[d, "\beta"] & (H^r_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}\otimes\theta"] & 0\\ 0 \arrow[r] & \Omega^{r-1}(X)/\operatorname{Ker} d \arrow[r, "d"] & \Omega^r(X)^{d=0} \arrow[r] & H^r_{\mathrm{dR}}(X) \arrow[r] & 0 \end{tikzcd}$$

*Moreover,* $\operatorname{Ker}(\iota_{\mathrm{HK}}\otimes\theta)\simeq(H^r_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^{r-1}}$, $H^r_{\mathrm{syn}}(X,\mathbf{Q}_p(r))$ *is LB or Fréchet, respectively, and the maps* $\beta$, $\iota_{\mathrm{HK}}\otimes\theta$ *are strict and have closed images.*

*Proof.* The diagram in the proposition arises from the commutative diagram:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(1)) \arrow[r] \arrow[d, dotted, "\tilde{\beta}"] \arrow[dd, bend right=60, "\beta"'] & {[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{\varphi=p,N=0}} \arrow[r, "\iota_{\mathrm{HK}}\otimes\iota"] \arrow[d, "\iota_{\mathrm{HK}}\otimes\iota"] & \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r \arrow[d, equal] \\
F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r] \arrow[d, "\vartheta"] & \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \arrow[r] \arrow[d, "\vartheta"] & \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r \arrow[d, "\vartheta"] \\
\Omega^{\geq r}(X) \arrow[r] & \Omega^{\bullet}(X) \arrow[r] & \Omega^{\leq r-1}(X)
\end{tikzcd}$$

The map $\tilde{\beta}$ is the map on mapping fibers induced by the commutative right square. We set $\beta := \vartheta\tilde{\beta}$. The map $\Omega^{r-1}(X)\to\Omega^r(X)$ induced from the bottom row of the above diagram is easily checked to be equal to $d$.

Applying cohomology to the above diagram we obtain a commutative diagram

$$\begin{tikzcd}
(H^{r-1}_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{\varphi=p,N=0} \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}\otimes\iota"] & \Omega^{r-1}(X)/\operatorname{Im} d \arrow[r, "\partial"] \arrow[d, equal] & \widetilde{H}^r_{\mathrm{syn}}(X,\mathbf{Q}_p(1)) \arrow[r] \arrow[d, "\beta"] & (H^r_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{\varphi=p,N=0} \arrow[d, "\iota_{\mathrm{HK}}\otimes\iota"] \\
0\to H^{r-1}_{\mathrm{dR}}(X) \arrow[r] & \Omega^{r-1}(X)/\operatorname{Im} d \arrow[r, "d"] & \Omega^r(X)^{d=0} \arrow[r] & H^r_{\mathrm{dR}}(X)
\end{tikzcd}$$

We have used here Proposition 5.8 and Proposition 5.16. We can now use the proof of Proposition 3.36 in [14] as soon as we know that, for a quasi-compact smooth dagger variety $Y$ over $C$, the slopes of Frobenius on $H^i_{\mathrm{HK}}(Y)$ are $\leq i$. But this is true when $Y=\mathscr{Y}_{K,C}$ for a semistable model over $\mathscr{O}_K$ (by the weight spectral sequence) and it follows for a general $Y$ by taking étale hypercoverings built from semistable basic models, quasi-compact in every degree. $\square$

## 6. Two comparison morphisms

In this section we define two comparison morphisms: from geometric syntomic cohomology of a smooth dagger variety to geometric syntomic cohomology of its completion and between geometric syntomic cohomology of a smooth dagger variety and its pro-étale cohomology. We also prove that the first morphism is a quasi-isomorphism for partially proper varieties (Theorem 6.2) and the second morphism is a quasi-isomorphism in a stable range (Theorem 6.9).

6.1. **From overconvergent to rigid analytic geometric syntomic cohomology.** We start with a morphism from geometric syntomic cohomology of a smooth dagger variety to geometric syntomic cohomology of its completion.

6.1.1. *Construction of the comparison morphism.* Let $X$ be a smooth dagger variety over $C$. We will construct a natural map

$$\iota:\quad \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to\mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X},\mathbf{Q}_p(r)) \tag{6.1}$$

from the syntomic cohomology of $X$ to the syntomic cohomology of its completion $\widehat{X}$.

(i) *The map* $\iota_1$. First, we note that we have a canonical natural morphism

$$\begin{aligned}
\iota_1:\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) &= \big[[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r\big]\\
&\to\big[[\mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X})\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}/\mathbf{B}^+_{\mathrm{dR}})/F^r\big]\\
&\xleftarrow{\sim}\big[[\mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X})\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}/\mathbf{B}^+_{\mathrm{dR}})/F^r\big].
\end{aligned}$$

Indeed, for that it suffices to show that the canonical map

$$[\mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X})\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0}\to[\mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X})\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0}$$

is a strict quasi-isomorphism. We may assume that $X$ has a semistable weak formal model $\mathscr{X}$ defined over $\mathscr{O}_{K'}$. Then the above map is equal to the map

$$[\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}^0_1)\widehat{\otimes}_{K',\iota}\mathbf{B}^+_{\mathrm{st}}]^{N=0} \to [\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}^1_0)\widehat{\otimes}^{\mathrm{R}}_{K'}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0}.$$

But this is a special case of the strict quasi-isomorphism in (2.18).

(ii) *The map $\iota_2$.* Next, we use the strict quasi-isomorphism

$$\begin{aligned}\iota_2 : \quad & \big[[\mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X})_F\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}/\mathbf{B}^+_{\mathrm{dR}})/F^r\big]\\ & \to [[\mathrm{R}\Gamma_{\mathrm{cr}}(\widehat{X})]^{\varphi=p^r}\xrightarrow{\mathrm{can}}\mathrm{R}\Gamma_{\mathrm{cr}}(\widehat{X})/F^r] = \mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X},\mathbf{Q}_p(r))\end{aligned}$$

from the proof of Proposition 5.3.

(iii) Finally, we define the map

$$\iota : \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X},\mathbf{Q}_p(r))$$

as $\iota := \iota_2\iota_1$.

6.1.2. *A comparison theorem.* We are now ready to prove our comparison theorem:

**Theorem 6.2.** *Let $X$ be a partially proper smooth dagger variety over $C$. The map*

$$\iota : \quad \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X},\mathbf{Q}_p(r))$$

*is a strict quasi-isomorphism.*

*Proof.* We have $\iota = \iota_2\iota_1$ by definition and as we have seen the map $\iota_2$ is a strict quasi-isomorphism. Hence it remains to show that so is the map $\iota_1$. For that, it suffices to show that the following canonical maps

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}} \to \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X})\widehat{\otimes}^{\mathrm{R}}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}},\quad \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r \to \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}/\mathbf{B}^+_{\mathrm{dR}})/F^r \tag{6.3}$$

are strict quasi-isomorphisms. For the second map this follows from Corollary 4.32. For the first map, by Lemma 4.18, it suffices to show that the canonical map

$$\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X)\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}} \to \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(\widehat{X})\widehat{\otimes}^{\mathrm{R}}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}}$$

is a strict quasi-isomorphism. But this holds because, by Corollary 4.32, the canonical map $\mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(X) \to \mathrm{R}\Gamma_{\mathrm{HK},\breve{F}}(\widehat{X})$ is a strict quasi-isomorphism. $\square$

6.2. **From overconvergent syntomic cohomology to pro-étale cohomology.** We will construct now a comparison morphism between geometric syntomic cohomology of a smooth dagger variety and its pro-étale cohomology. We will prove that it is a strict quasi-isomorphism in a stable range.

6.2.1. *Overconvergent geometric syntomic cohomology via presentations of dagger structures.* We start with showing that the overconvergent geometric syntomic cohomology defined as in [16, Sec. 6.3] using presentations of dagger structures, a priori different from the overconvergent geometric syntomic cohomology defined as in [16, Sec. 5.4], is strictly quasi-isomorphic to it. This was shown in [16, Prop. 6.17] in the arithmetic case, where the key ingredient of the proof is the comparison theorem between arithmetic overconvergent and rigid analytic syntomic cohomology of partially proper dagger spaces. We had to wait for the geometric version of the later comparison theorem (our Theorem 6.2) to state the geometric analog of [16, Prop. 6.17].

(i) *Local definition.* Let $X$ be a dagger affinoid over $C$. Let $\mathrm{pres}(X) = \{X_h\}$. Recall that we have defined the syntomic cohomology

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) := \mathrm{L}\,\mathrm{colim}_h\,\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,\mathbf{Q}_p(r)),\quad r\in\mathbf{N}.$$

We have a natural map

$$\iota^{\dagger}_{\mathrm{syn}}\colon \mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \tag{6.4}$$

defined as the composition

$$\begin{aligned}\mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) &= \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{syn}}(X_h,\mathbf{Q}_p(r)) \xrightarrow{\sim} \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{syn}}(X_h^{\mathrm{o}},\mathbf{Q}_p(r))\\ &\xleftarrow{\sim} \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{syn}}(X_h^{\mathrm{o},\dagger},\mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)).\end{aligned} \tag{6.5}$$

The third quasi-isomorphism holds by Theorem 6.2 because $X_h^{\mathrm{o}}$ is partially proper.

(ii) *Globalization.* For a general smooth dagger variety $X$ over $C$, using the natural equivalence of analytic topoi

$$\mathrm{Sh}(\mathrm{SmAff}^{\dagger}_{C,\text{ét}}) \xrightarrow{\sim} \mathrm{Sh}(\mathrm{Sm}^{\dagger}_{C,\text{ét}}),$$

we define the sheaf $\mathscr{A}^{\dagger}_{\mathrm{syn}}(r)$, $r\in\mathbf{N}$, on $X_{\text{ét}}$ as the sheaf associated to the presheaf defined by: $U\mapsto \mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(U,\mathbf{Q}_p(r))$, $U\in\mathrm{SmAff}^{\dagger}_C$, $U\to X$ an étale map. We define[29]

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) := \mathrm{R}\Gamma_{\text{ét}}(X,\mathscr{A}^{\dagger}_{\mathrm{syn}}(r)),\quad r\in\mathbf{N}.$$

Globalizing the map $\iota^{\dagger}_{\mathrm{syn}}$ from (6.4) we obtain a natural map

$$\iota^{\dagger}_{\mathrm{syn}} : \mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)).$$

(iii) *A comparison quasi-isomorphism.*

**Proposition 6.6.** *The above map $\iota^{\dagger}_{\mathrm{syn}}$ is a strict quasi-isomorphism.*

*Proof.* By étale descent, we may assume that $X$ is a smooth dagger affinoid. Looking at the composition (6.5) defining the map $\iota^{\dagger}_{\mathrm{syn}}$ we see that it suffices to show that the natural map

$$\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{syn}}(X_h^{\mathrm{o},\dagger},\mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))$$

is a strict quasi-isomorphism. Or, from the definitions of both sides, that we have strict (filtered) quasi-isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}} &\xleftarrow{\sim} \mathrm{L}\operatorname{colim}_h(\mathrm{R}\Gamma_{\mathrm{HK}}(X_h^{\mathrm{o},\dagger})\widehat{\otimes}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}),\\ \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) &\xleftarrow{\sim} \mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{dR}}(X_h^{\mathrm{o},\dagger}/\mathbf{B}^+_{\mathrm{dR}}).\end{aligned}$$

We may assume that $X$ is defined over a finite field extension $L$ of $K$, i.e., there exists $X_L$ such that $X\simeq X_{L,C}$ Then the above maps factor as

$$\begin{aligned}&\mathrm{L}\operatorname{colim}_h(\mathrm{R}\Gamma_{\mathrm{HK}}(X_h^{\mathrm{o},\dagger})\widehat{\otimes}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}) \xrightarrow{\sim} (\mathrm{L}\operatorname{colim}_h(\mathrm{R}\Gamma_{\mathrm{HK}}(X_h))\widehat{\otimes}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}) \xrightarrow{\sim} \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}},\\ &\mathrm{L}\operatorname{colim}_h(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{L,h}^{\mathrm{o},\dagger})\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}^+_{\mathrm{dR}}) \xrightarrow{\sim} (\mathrm{L}\operatorname{colim}_h \mathrm{R}\Gamma_{\mathrm{dR}}(X_{L,h}))\widehat{\otimes}^{\mathrm{R}}_L\mathbf{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}}).\end{aligned}$$

In the Hyodo-Kato case, the first map is a strict quasi-isomorphism by definition of the dagger tensor product. In the de Rham case, the first map is a strict filtered quasi-isomorphism by the computation (3.32).

□

[29] We will show below (see Proposition 6.6) that this definition of $\mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))$, for a smooth dagger affinoid $X$, gives an object naturally strictly quasi-isomorphic to the one defined above.

6.2.2. *The geometric overconvergent period map and a comparison result.* We are now ready to define and study the overconvergent period map. Let $X\in \mathrm{Sm}^{\dagger}_C$, $r\geq 0$. Define the period map

$$\alpha_r : \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)) \tag{6.7}$$

as the composition

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\xleftarrow{\sim}\mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\xrightarrow{\alpha_r^{\dagger}}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)),$$

where the first map is the map $\iota^{\dagger}_{\mathrm{syn}}$ from Proposition 6.6 and the second map is defined by globalizing the following map defined for a dagger affinoid $X$ with the presentation $\{X_h\}$:

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))=\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,\mathbf{Q}_p(r))\xrightarrow{\alpha_r}\mathrm{L}\operatorname{colim}_h\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_h,\mathbf{Q}_p(r))\simeq\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)).$$

Here $\alpha_r$ is the rigid analytic period map (see Proposition 3.1).

We have the following compatibility with the rigid analytic period map:

**Proposition 6.8.** (Dagger-rigid analytic compatibility) *Let $X\in\mathrm{Sm}^{\dagger}_C$ and $r\geq 0$.*

(1) *The following diagram*

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \arrow[r, "\alpha_r"] \arrow[d, "\iota"] & \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)) \arrow[d, "\iota_{\mathrm{pro\acute{e}t}}"] \\
\mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X},\mathbf{Q}_p(r)) \arrow[r, "\widehat{\alpha}_r"] & \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\widehat{X},\mathbf{Q}_p(r))
\end{tikzcd}$$

*commutes.*

(2) *If $X$ is partially proper then the maps $\iota$ and $\iota_{\mathrm{pro\acute{e}t}}$ in the above diagram are strict quasi-isomorphisms.*

Here, the period map $\alpha_r$ is the one defined above. We put hat above its rigid analytic analog to distinguish it from the dagger period map.

*Proof.* For the first claim, it suffices to show that this diagram naturally commutes étale locally. So we may assume that $X$ is a smooth dagger affinoid. Then checking commutativity is straightforward from the definitions.

For the second claim, note that the map $\iota$ is a strict quasi-isomorphism by Theorem 6.2 and the map $\iota_{\mathrm{pro\acute{e}t}}$ is a strict quasi-isomorphism by Proposition 3.1, point 3a. □

The following comparison result follows immediately from its rigid analytic analog (see Proposition 3.1, point 2c):

**Theorem 6.9.** *For $X\in\mathrm{Sm}^{\dagger}_C$ and $r\geq 0$, the period map*

$$\alpha_r : \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)) \tag{6.10}$$

*is a strict quasi-isomorphism after truncation $\tau_{\leq r}$.*

And it implies the following result (which will be the starting point of our proof of $C_{\mathrm{st}}$-conjecture for smooth analytic varieties in [17]):

**Corollary 6.11.** *For $X\in\mathrm{Sm}^{\dagger}_C$ and $r\geq 0$, we have the long exact sequence*

$$\cdots\to\widetilde{H}^{i-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r)\to\widetilde{H}^i_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))\to(H^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})/F^r)\to\cdots$$

Here we set

$$H^i_{\mathrm{HK}}(X)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}:=\lim_n(H^i_{\mathrm{HK}}(U_n)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}),$$

for an exhaustive covering $\{U_n\}_n$ of $X$ by quasi-compact open (note that the groups $H^i_{\mathrm{HK}}(U_n)$ are of finite rank).

*Proof.* Use Theorem 6.9 and the obvious fact that the canonical map $[H^i_{\mathrm{HK}}(U_n)\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}}]^{N=0}\to[H^i_{\mathrm{HK}}(U_n)\widehat{\otimes}_{F^{\mathrm{nr}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0}$ is an isomorphism. □

## 7. Geometrization of period morphisms

The purpose of this section is to geometrize syntomic cohomology (and the related Hyodo-Kato and de Rham cohomologies), pro-étale cohomology, and the associated period morphisms both in the rigid analytic and the overconvergent set-ups. By "geometrization" we mean putting VS-structure. The key computation is the one showing that the rigid analytic version of Fontaine-Messing period morphism is a shadow of a VS-morphism. Both sides of the period morphism, crystalline syntomic cohomology and pro-étale cohomology, have natural VS structures. However it it not immediately clear that the period map navigates well between these two VS-structures. To show that, in fact, it does so we use the presentation of the period map via $(\varphi,\Gamma)$-modules introduced in [15], [22].

7.1. **Rigid analytic varieties, cohomologies.** We start with the geometrization of rigid analytic cohomologies.

7.1.1. *Vector Spaces.* A VS (resp. a VS$^+$) is a functor from perfectoid $C$-Banach algebras (denoted $\Lambda=(\Lambda,\Lambda^+)$ in what follows) to $\mathbf{Q}_p$-modules (res $\mathbf{Z}_p$-modules). If $\mathbb{W}$ is a VS$^+$, then $\mathbf{Q}_p\otimes_{\mathbf{Z}_p}\mathbb{W}$ is a VS. VS's form an abelian category. Trivial examples of VS's are:

- finite dimensional $\mathbf{Q}_p$-vector spaces $V$, with associated functor $\Lambda\mapsto V$ for all $\Lambda$,
- $\mathbb{V}^d$, for $d\in\mathbf{N}$, with $\mathbb{V}^d(\Lambda)=\Lambda^d$, for all $\Lambda$.

More interesting examples are provided by Fontaine's rings [20, 11]:

- $\mathbb{B}^+_{\rm cr}$, $\mathbb{B}^+_{\rm st}$, $\mathbb{B}^+_{\rm dR}$, $\mathbb{B}_{\rm cr}$, $\mathbb{B}_{\rm st}$, $\mathbb{B}_{\rm dR}$ are naturally VS's (and even Rings).
- If $m\geq 1$, then $\mathbb{B}_m:=\mathbb{B}^+_{\rm dR}/t^m\mathbb{B}^+_{\rm dR}$ is a VS (and also a Ring).
- Let $h\geq 1$ and $d\in\mathbf{Z}$. Then $\mathbb{U}_{h,d}=(\mathbb{B}^+_{\rm cr})^{\varphi^h=p^d}$ if $d\geq 0$, and $\mathbb{U}_{h,d}=\mathbb{B}_d/\mathbf{Q}_{p^h}$ if $d<0$, are VS's.

Exemples of VS$^+$'s include $\mathbb{A}_{\rm inf}$, $\mathbb{A}_{\rm cr}$, or $\mathbb{A}^{[u,v]}$ if $0<u\leq v$; the last example being the functor sending $\Lambda$ to the $p$-adic completion of $\mathbb{A}_{\rm inf}(\Lambda)[\frac{p}{[\alpha]},\frac{[\beta]}{p}]$, where $\alpha,\beta\in\mathscr{O}_C^\flat$ with $v(\alpha)=\frac{1}{v}$ and $v(\beta)=\frac{1}{u}$. By [22, Prop. 3.2], we have $\mathbb{A}_{\rm cr}(\Lambda)\subset\mathbb{A}^{[u,v]}(\Lambda)$, for $u\geq\frac{1}{p-1}$. If $v\geq 1$, we have $\mathbb{A}^{[u,v]}(\Lambda)\subset\mathbb{B}^+_{\rm dR}(\Lambda)$ and this inclusion induces a filtration on $\mathbb{A}^{[u,v]}(\Lambda)$.

If $\mathbb{W}$ is one of the above Rings, we denote by $\mathbf{W}$ the ring $\mathbb{W}(C)$: for example $\mathbf{A}^{[u,v]}=\mathbb{A}^{[u,v]}(C)$ (for the other Rings $\mathbb{A}_{\rm inf}$, $\mathbb{A}_{\rm cr}$, $\mathbb{B}^+_{\rm cr}$, $\mathbb{B}^+_{\rm dR}$, etc., one gets back the rings already defined).

*Remark* 7.1. The above definition gives presheaves on ${\rm Perf}_C$. Passing to the associated sheaves gives a natural viewpoint on VS's and VS$^+$'s; this was put to use by Le Bras in his thesis [30].

7.1.2. *Pro-étale cohomology.* Let $X$ be a smooth rigid analytic variety over $C$. If $\Lambda$ is a perfectoid $C$-Banach algebra, let $X_\Lambda$ be the scalar extension $X\times_C\Lambda$. The functor $\Lambda\mapsto H^i_{\rm proét}(X_\Lambda,\mathbf{Q}_p)$ defines a VS. That is, there exists a VS $\mathbb{H}^i_{\rm proét}(X,\mathbf{Q}_p)$ such that $\mathbb{H}^i_{\rm proét}(X,\mathbf{Q}_p)(\Lambda)=H^i_{\rm proét}(X_\Lambda,\mathbf{Q}_p)$, for all perfectoid $C$-Banach algebras. In particular, $H^i_{\rm proét}(X,\mathbf{Q}_p)$ is the space of $C$-points of $\mathbb{H}^i_{\rm proét}(X,\mathbf{Q}_p)$; we have put in this way a geometric structure on $H^i_{\rm proét}(X,\mathbf{Q}_p)$.

We can do the same with:

1. the cohomology complex: the functor
$$\mathbb{R}_{\rm proét}(X_\Lambda,\mathbf{Q}_p):\quad \Lambda\mapsto {\rm R}\Gamma_{\rm proét}(X_\Lambda,\mathbf{Q}_p)$$
defines a VS with values in $\mathscr{D}(C_{\mathbf{Q}_p})$;
2. it cohomology groups $\widetilde{\mathbb{H}}_{\rm proét}(X_\Lambda,\mathbf{Q}_p)$ form a VS with values in $LH(C_{\mathbf{Q}_p})$;
3. it algebraic cohomology groups $\mathbb{H}_{\rm proét}(X_\Lambda,\mathbf{Q}_p)$ form the VS described above. We have a natural map $\widetilde{\mathbb{H}}_{\rm proét}(X_\Lambda,\mathbf{Q}_p)\to\mathbb{H}_{\rm proét}(X_\Lambda,\mathbf{Q}_p)$.

7.1.3. *Crystalline syntomic cohomology.* To geometrize (filtered) absolute crystalline cohomology, we define the functor

$$\mathbb{F}^r\mathbb{R}_{\rm cr}(X):\quad \Lambda\mapsto F^r{\rm R}\Gamma_{\rm cr}(X)\widehat{\otimes}^R_{\mathbf{B}^+_{\rm cr}}\mathbb{B}^+_{\rm cr}(\Lambda),\quad r\geq 0, \tag{7.2}$$

that lifts the absolute crystalline cohomology $\mathrm{R}\Gamma_{\mathrm{cr}}(X)$ from Section 3.2.2. The tensor product used in (7.2) needs to be defined. We do it in the following way. We set

$$F^r\mathrm{R}\Gamma_{\mathrm{cr}}(X)\widehat{\otimes}^R_{\mathbf{B}^+_{\mathrm{cr}}}\mathbb{B}^+_{\mathrm{cr}}(\Lambda):=\mathrm{R}\Gamma_{\text{ét}}(X,F^r\mathscr{A}_{\mathrm{cr},\Lambda}),$$

where $F^r\mathscr{A}_{\mathrm{cr},\Lambda}$ is the $\eta$-étale sheafification[30] on $\mathscr{M}^{\mathrm{ss}}_C$ of the presheaf $\mathscr{X}\mapsto(F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}/\mathbf{A}_{\mathrm{cr}})\widehat{\otimes}^{\mathrm{L}}_{\mathbf{A}_{\mathrm{cr}}}\mathbb{A}_{\mathrm{cr}}(\Lambda))_{\mathbf{Q}_p}$.

We proceed similarly for rigid $\mathbf{B}^+_{\mathrm{dR}}$-cohomology (from Section 3.3.1): we define the functor

$$\mathbb{F}^r\mathbb{R}_{\mathrm{dR}}(X/\mathbb{B}^+_{\mathrm{dR}}):\quad\Lambda\mapsto F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{dR}}}\mathbb{B}^+_{\mathrm{dR}}(\Lambda),\quad r\geq 0,$$

that lifts the filtered $\mathbf{B}^+_{\mathrm{dR}}$-cohomology $F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})$. Here

$$F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}^+_{\mathrm{dR}})\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}^+_{\mathrm{dR}}}\mathbb{B}^+_{\mathrm{dR}}(\Lambda):=\mathrm{R}\Gamma_{\text{ét}}(X,F^r\mathscr{A}^{\frown}_{\mathrm{cr},\Lambda}),$$

where $F^r\mathscr{A}^{\frown}_{\mathrm{cr},\Lambda}$ is the $\eta$-étale sheafification on $\mathscr{M}^{\mathrm{ss}}_C$ of the presheaf

$$\mathscr{X}\mapsto\mathrm{R}\lim_{i\geq r}((\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X},\mathscr{J}^{[r]})/F^i)\widehat{\otimes}^{\mathrm{L}}_{(\mathbf{A}_{\mathrm{cr}}/F^i)}(\mathbb{A}_{\mathrm{cr}}(\Lambda)/F^i))_{\mathbf{Q}_p}.$$

Finally, we lift crystalline syntomic cohomology by setting

$$\mathbb{R}_{\mathrm{syn}}(X,\mathbf{Q}_p(r)):\quad\Lambda\mapsto[\mathbb{F}^r\mathbb{R}_{\mathrm{cr}}(X)(\Lambda)\xrightarrow{\varphi-p^r}\mathbb{R}_{\mathrm{dR}}(X/\mathbb{B}^+_{\mathrm{dR}})(\Lambda)/F^r].$$

7.2. **Rigid analytic varieties, period morphism.** We move now to the geometrization of rigid analytic period morphisms. We will prove the following theorem.

**Theorem 7.3.** *For $X\in\mathrm{Sm}_C$ and $r\geq 0$, the period map*

$$\alpha_r:\quad\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to\mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(r))$$

*lifts to a map of VS's:*

$$\mathbb{A}_r:\quad\mathbb{R}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to\mathbb{R}_{\text{proét}}(X,\mathbf{Q}_p(r)).$$

*It is a strict quasi-isomorphism after truncation $\tau_{\leq r}$.*

7.2.1. *Local models, lifted case.* Consider a frame

$$R^+_\square:=\mathscr{O}_C\{X,\tfrac{1}{X_1\dots X_a},\tfrac{\varpi}{X_{a+1}\dots X_d}\},\quad R_\square=R^+_\square[\tfrac{1}{p}],$$

where $X=(X_1,\dots,X_d)$ and $0\neq\varpi\in\mathscr{O}_C$. Our formal scheme $\mathscr{X}=\mathrm{Spf}\,R^+$, for an algebra $R^+$, which is the $p$-adic completion of an étale algebra over $R^+_\square$. We equip $\mathrm{Spf}(R^+_\square)$ and $\mathrm{Spf}(R^+)$ with the logarithmic structure[31] induced by the special fiber.

For $m\geq 0$, define

$$R^{m,+}_\square:=\mathscr{O}_C\{X^{1/p^m},\tfrac{1}{(X_1\dots X_a)^{1/p^m}},\tfrac{\varpi^{1/p^m}}{(X_{a+1}\dots X_d)^{1/p^m}}\},\quad R^m_\square=R^{m,+}_\square[\tfrac{1}{p}],$$

and set $R^{\infty,+}_\square$ equal to the $p$-adic completion of $\mathrm{colim}_m\,R^{m,+}_\square$. Let

$$R^{m,+}:=R^{m,+}_\square\widehat{\otimes}_{\mathscr{O}_C}R^+,\quad R^{\infty,+}:=R^{\infty,+}_\square\widehat{\otimes}_{\mathscr{O}_C}R^+,\quad R^m=R^{m,+}[\tfrac{1}{p}],\quad R^\infty=R^{\infty,+}[\tfrac{1}{p}]$$

so that $R^\infty$ is a perfectoid Banach algebra. Define $\Gamma_R:=\mathrm{Gal}(R_\infty/R)$. We have $\Gamma_R\simeq\mathbf{Z}_p^d$.

Choose $\varpi^\flat\in\mathscr{O}_C^\flat$ with $\theta([\varpi^\flat])=\varpi$. We define

$$R^+_{\mathrm{inf},\square}:=\mathbf{A}_{\mathrm{inf}}\{X,\tfrac{1}{X_1\dots X_a},\tfrac{[\varpi^\flat]}{X_{a+1}\dots X_d}\}$$

and lift the map $R^+_\square\to R^+$ to an étale map $R^+_{\mathrm{inf},\square}\to R^+_{\mathrm{inf}}$. Set

$$R^+_{\mathrm{cr}}:=R^+_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}}\mathbf{A}_{\mathrm{cr}},\quad R^{[u,v]}:=R^+_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}}\mathbf{A}^{[u,v]}.$$

[30]We do not discuss local-global compatibilities. As far as we can tell this does not cause problems.

[31]Note that we do not allow horizontal divisors at $\infty$.

Endow everything with the log-structure coming from $\mathbf{A}_{\mathrm{cr}}$ and $\mathrm{Spf}(R)$. This gives us the commutative diagram (with cartesian squares)

$$\begin{tikzcd}
\mathrm{Spf}(R) \arrow[r, hook] \arrow[d] & \mathrm{Spf}(R^+_{\mathrm{cr}}) \arrow[d] \\
\mathrm{Spf}(R_\square) \arrow[r, hook] \arrow[d] & \mathrm{Spf}(R^+_{\mathrm{cr},\square}) \arrow[d] \\
\mathrm{Spf}(\mathscr{O}_C) \arrow[r, hook] & \mathrm{Spf}(\mathbf{A}_{\mathrm{cr}})
\end{tikzcd}$$

We define the filtrations $F^\bullet R^+_{\mathrm{cr}}$ on $R^+_{\mathrm{cr}}$ and $F^\bullet R^{[u,v]}$ on $R^{[u,v]}$ by inducing them from $\mathbf{A}_{\mathrm{cr}}$ and $\mathbf{A}^{[u,v]}$. We have the corresponding filtered de Rham complex

$$F^r\Omega^\bullet_{R^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}} := F^rR^+_{\mathrm{cr}} \to F^{r-1}R^+_{\mathrm{cr}}\widehat{\otimes}_{R^+_{\mathrm{cr}}}\Omega^1_{R^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}} \to F^{r-2}R^+_{\mathrm{cr}}\widehat{\otimes}_{R^+_{\mathrm{cr}}}\Omega^2_{R^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}}\cdots$$

The crystalline syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},\mathbf{Q}_p(r))$ is computed by the complex

$$\mathrm{Syn}(R^+_{\mathrm{cr}},r) := [F^r\Omega^\bullet_{R^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}} \xrightarrow{p^r-\varphi} \Omega^\bullet_{R^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}}].$$

7.2.2. *Period rings.* Let $\Lambda = (\Lambda,\Lambda^+)$ be a perfectoid affinoid over $(C,\mathscr{O}_C)$. We refer the reader for a study of the basic properties of $\mathbb{A}_{\mathrm{inf}}(\Lambda)$ to [7, Sec. 3].

For $0<u\leq v$ and $v\geq 1$, set

$$\mathbf{A}^{[u,v]}_\Lambda := \mathbb{A}^{[u,v]}(\Lambda),\quad R^{[u,v]}_\Lambda := R^+_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}}\mathbf{A}^{[u,v]}_\Lambda$$

equipped with the filtration induced from the one on $\mathbf{A}^{[u,v]}_\Lambda$. The following lemma is proved by the same argument as [22, Lemma 5.3]:

**Lemma 7.4.** *Let $\frac{v}{p}<1<v$. Multiplication by $t^r$ induces $p^{3r}$-isomorphisms*

$$\mathbf{A}^{[u,v]}_\Lambda \xrightarrow{\sim} F^r\mathbf{A}^{[u,v]}_\Lambda,\quad \mathbf{A}^{[u,v/p]}_\Lambda \xrightarrow{\sim} \mathbf{A}^{[u,v/p]}_\Lambda.$$

Let $\overline{R}^+$ be the maximal extension of $R^+$ that is étale in characteristic 0, and $\overline{R} = \overline{R}^+[\frac{1}{p}]$. Set $G_R := \mathrm{Gal}(\overline{R}/R)$. For $0\leq i\leq d$, choose $X_i^\flat = (X_i, X_i^{1/p},\cdots)$ in $\overline{R}^\flat$ and define an embedding of $R^+_{\mathrm{inf},\square}$ in $\mathbf{A}_{\overline{R}} = W(\overline{R}^\flat)$ by sending $X_i\mapsto[X_i^\flat]$. This extends to embeddings:

$$R^+_{\mathrm{inf}}\hookrightarrow\mathbf{A}_{\overline{R}},\quad R^+_{\mathrm{cr}}\hookrightarrow\mathbb{A}_{\mathrm{cr}}(\overline{R}),\quad R^{[u,v]}\hookrightarrow\mathbf{A}^{[u,v]}_{R^\infty}\subset\mathbf{A}_{R^\infty}\subset\mathbf{A}_{\overline{R}}.$$

We can also lift the semistable period rings to VS's. We set

$$\begin{aligned}
&\widehat{\mathbb{B}}^+_{\mathrm{st}} := \widehat{\mathbb{B}}^+_{p,\mathrm{st}} := (\mathbb{A}_{\mathrm{cr}} < t_p[\widetilde{p}]^{-1}-1 >)^\frown[\tfrac{1}{p}],\quad \mathbb{B}^+_{\mathrm{st}} := \mathbb{B}^+_{p,\mathrm{st}} := \mathbb{B}^+_{\mathrm{cr}}[\log([\widetilde{p}])],\\
&\kappa:\mathbb{B}^+_{\mathrm{st}}\to\widehat{\mathbb{B}}^+_{\mathrm{st}},\ \log([\widetilde{p}])\mapsto-\log(t_p[\widetilde{p}]^{-1}),\quad \iota:\mathbb{B}^+_{\mathrm{st}}\to\mathbb{B}^+_{\mathrm{dR}},\ \log([\widetilde{p}])\mapsto-\log(p[\widetilde{p}]^{-1}),\\
&\iota:\widehat{\mathbb{B}}^+_{\mathrm{st}}\to\mathbb{B}^+_{\mathrm{dR}},\ t_p[\widetilde{p}]^{-1}\mapsto p[\widetilde{p}]^{-1}.
\end{aligned}$$

7.2.3. *The local period morphism $\alpha_{r,n}$, lifted case.*

(i) *Over $C$.* Consider the following commutative diagram:

(7.5)
$$\begin{tikzcd}
 & \mathrm{Spf}(\mathbf{E}^{\mathrm{PD}}_{\overline{R}}) \arrow[dr] & \\
\mathrm{Spf}(\overline{R}) \arrow[ur, hook] \arrow[rr, hook] \arrow[d] & & \mathrm{Spf}(\mathbb{A}_{\mathrm{cr}}(\overline{R})\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr}}}R^+_{\mathrm{cr}}) \arrow[d] \\
\mathrm{Spf}(R) \arrow[rr, hook] \arrow[d] & & \mathrm{Spf}(R^+_{\mathrm{cr}}) \arrow[d] \\
\mathrm{Spf}(\mathscr{O}_C) \arrow[rr, hook] & & \mathrm{Spf}(\mathbf{A}_{\mathrm{cr}})
\end{tikzcd}$$

Here $\mathbf{E}^{\mathrm{PD}}_{\overline{R}}$ is the PD-envelope of the closed embedding $\mathrm{Spf}(\overline{R}) \hookrightarrow \mathrm{Spf}(\mathbb{A}_{\mathrm{cr}}(\overline{R})\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr}}} R^+_{\mathrm{cr}})$.

*Remark* 7.6. (a) We take partial divided powers of level $s$, i.e., $x^{[k]} = \frac{x^k}{[k/p^s]!}$, where $s=0$ for $p\neq 2$ and $s=1$ for $p=2$.

(b) We induce the filtration on $\mathbf{E}^{\mathrm{PD}}_{\overline{R}}$ from the filtration on $R^+_{\mathrm{cr}}$ and $\mathbb{A}_{\mathrm{cr}}(\overline{R})$. See [15, Sec. 2.6.1] for details.

Set $\Omega^i_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}}} := \mathbf{E}^{\mathrm{PD}}_{\overline{R}}\widehat{\otimes}_{R^+_{\mathrm{cr}}}\Omega^i_{R^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}}$. For $r\in\mathbf{N}$, we filter the de Rham complex $\Omega^{\bullet}_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}}}$ by subcomplexes

$$F^r\Omega^{\bullet}_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}}} := F^r\mathbf{E}^{\mathrm{PD}}_{\overline{R}} \to F^{r-1}\mathbf{E}^{\mathrm{PD}}_{\overline{R}}\widehat{\otimes}_{R^+_{\mathrm{cr}}}\Omega^1_{R^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}} \to F^{r-2}\mathbf{E}^{\mathrm{PD}}_{\overline{R}}\widehat{\otimes}_{R^+_{\mathrm{cr}}}\Omega^2_{R^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}} \to \cdots$$

Define the syntomic complexes

$$\mathrm{Syn}(\overline{R},r) := [F^r\Omega^{\bullet}_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}}} \xrightarrow{\varphi-p^r} \Omega^{\bullet}_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}}}].$$

For a continuous $G_R$-module $M$, let $C(G_R,M)$ denote the complex of continuous cochains of $G_R$ with values in $M$. The Fontaine-Messing period map

$$\alpha_{r,n}:\quad \mathrm{Syn}(R,r)_n \to C(G_R,\mathbf{Z}/p^n(r)'),$$

where $\mathbf{Z}/p^n(r)' := \frac{1}{p^{a(r)}}\mathbf{Z}/p^n(r)$, for $r=(p-1)a(r)+b(r), 0\leq b(r)\leq p-1$, is defined as the composition

$$\begin{tikzcd}
\mathrm{Syn}(R,r)_n = [F^r\Omega^{\bullet}_{R^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}} \xrightarrow{\varphi-p^r} \Omega^{\bullet}_{R^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}] \arrow[d] \\
C(G_R,[F^r\Omega^{\bullet}_{\mathbf{E}^{\mathrm{PD}}_{\overline{R},n}} \xrightarrow{\varphi-p^r} \Omega^{\bullet}_{\mathbf{E}^{\mathrm{PD}}_{\overline{R},n}}]) \\
C(G_R,[F^r\mathbb{A}_{\mathrm{cr}}(\overline{R})_n \xrightarrow{\varphi-p^r} \mathbb{A}_{\mathrm{cr}}(\overline{R})_n]) \arrow[u, "\wr"'] \\
C(G_R,\mathbf{Z}/p^n(r)') \arrow[u, "\wr"']
\end{tikzcd}$$

The second quasi-isomorphism above follows from the filtered Poincaré Lemma, i.e., from the $p$-quasi-isomorphism[32]

$$F^r\mathbb{A}_{\mathrm{cr}}(\overline{R})_n \xrightarrow{\sim} F^r\Omega^{\bullet}_{\mathbf{E}^{\mathrm{PD}}_{\overline{R},n}},\quad r\geq 0,$$

proved in [22, Prop. 7.3]. The third quasi-isomorphism follows from the fundamental exact sequence

$$0\to\mathbf{Z}/p^n(r)'\to F^r\mathbb{A}_{\mathrm{cr}}(\overline{R})_n \xrightarrow{\varphi-p^r} \mathbb{A}_{\mathrm{cr}}(\overline{R})_n\to 0$$

(ii) *Over a perfectoid $C$-algebra.* The Fontaine-Messing morphism lifts to any perfectoid Banach $C$-algebra $\Lambda$. To show this we will use the commutative diagram (we set $(R_\Lambda,R^+_\Lambda) := (R,R^+)\widehat{\otimes}_{(C,\mathscr{O}_C)}(\Lambda,\Lambda^+)$; by [37, Prop. 6.18] this is a perfectoid algebra, $\overline{R}^+_\Lambda$ is the completion of the maximal extension of $R_\Lambda$ étale in characteristic 0, $\overline{R}_\Lambda = \overline{R}^+_\Lambda[\frac{1}{p}]$, and $G_{R_\Lambda}=\mathrm{Aut}(\overline{R}_\Lambda/R_\Lambda)$):

(7.7)
$$\begin{tikzcd}
 & \mathrm{Spf}(\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda}) \arrow[dr] & \\
\mathrm{Spf}(\overline{R}^+_\Lambda) \arrow[ur, hook] \arrow[rr, hook] \arrow[d] & & \mathrm{Spf}(\mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda)\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr}}}R^+_{\mathrm{cr}}) \arrow[d] \\
\mathrm{Spf}(R) \arrow[rr, hook] \arrow[d] & & \mathrm{Spf}(R^+_{\mathrm{cr}}) \arrow[d] \\
\mathrm{Spf}(\mathscr{O}_C) \arrow[rr, hook] & & \mathrm{Spf}(\mathbf{A}_{\mathrm{cr}})
\end{tikzcd}$$

[32] We call a morphism $f:A\to B$ in a derived category a $N$-quasi-isomorphism if the induced morphism on cohomology has kernel and cokernel annihilated by $N$.

Here $\mathrm{Spf}(\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda})$ is the PD-envelope of the closed embedding $\mathrm{Spf}(\overline{R}^+_\Lambda)\hookrightarrow \mathrm{Spf}(\mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda)\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr}}}R^+_{\mathrm{cr}})$. The period morphism

$$\alpha_{r,n}(\Lambda):\quad \mathrm{Syn}(R,r)_n(\Lambda)\to C(G_{R_\Lambda},\mathbf{Z}/p^n(r)')$$

is defined as the composition

$$\begin{tikzcd}
\mathrm{Syn}(R,r)_n(\Lambda)=[F^r\Omega^\bullet_{R^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}\otimes_{\mathbf{A}_{\mathrm{cr},n}}\mathbb{A}_{\mathrm{cr}}(\Lambda)_n \arrow[r, "\varphi-p^r"] & \Omega^\bullet_{R^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr},n}}\mathbb{A}_{\mathrm{cr}}(\Lambda)_n] \arrow[d] \\
C(G_{R_\Lambda},[F^r\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda},n} \arrow[r, "\varphi-p^r"] & \Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda},n}]) \\
C(G_{R_\Lambda},[F^r\mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda)_n \arrow[r, "\varphi-p^r"] & \mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda)_n]) \arrow[u, "\wr"'] \\
& C(G_{R_\Lambda},\mathbf{Z}/p^n(r)') \arrow[u, "\wr"']
\end{tikzcd}$$

7.2.4. *Proof that $\alpha_{r,n}(\Lambda)$ is a quasi-isomorphism.*

**Proposition 7.8.** *The period morphism*

$$\alpha_{r,n}(\Lambda):\quad \mathrm{Syn}(R,r)_n(\Lambda)\to C(G_{R_\Lambda},\mathbf{Z}/p^n(r)')$$

*is a $p^{cr}$-quasi-isomorphism, for a universal constant $c$.*

*Proof.* It suffices to show that the morphisms

$$\begin{tikzcd}
{[F^r\Omega^\bullet_{R^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr},n}}\mathbb{A}_{\mathrm{cr}}(\Lambda)_n} \arrow[r, "\varphi-p^r"] & \Omega^\bullet_{R^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr},n}}\mathbb{A}_{\mathrm{cr}}(\Lambda)_n] \arrow[d] \\
C(G_{R_\Lambda},[F^r\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda},n} \arrow[r, "\varphi-p^r"] & \Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda},n}])
\end{tikzcd}$$

and

$$\begin{tikzcd}
C(G_{R_\Lambda},[F^r\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda},n} \arrow[r, "\varphi-p^r"] & \Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda},n}]) \\
C(G_{R_\Lambda},[F^r\mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda)_n \arrow[r, "\varphi-p^r"] & \mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda)_n]) \arrow[u]
\end{tikzcd}$$

are $p^{cr}$-quasi-isomorphisms. The second $p$-quasi-isomorphism follows from the filtered Poincaré Lemma

$$F^r\mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda)_n\xrightarrow{\sim}F^r\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda},n},$$

which can be proved by arguments analogous to the ones used in the proof of [22, Prop. 7.3].

The first $p^{cr}$-quasi-isomorphism of the proposition is more difficult to prove. We will do it by writing the Fontaine-Messing period morphism as a sequence of morphisms inspired by the theory of $(\varphi,\Gamma)$-modules as in [15, Th. 4.16], [22, Th. 7.5] and then showing that all these morphisms are $p^{cr}$-quasi-isomorphisms after truncation $\tau_{\leq r}$. This is done by the following diagram, where we denoted by $\omega$ the map that we want to show to be a $p^{cr}$-quasi-isomorphism (after truncation $\tau_{\leq r}$, which we indicate on the diagram but will often skip in the discussion to lighten up the notation). The key ingredient to turn the complex coming from the fundamental exact sequence (i.e., $C_G(K_\varphi(F^r\mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda)))$ in the upper right corner) into something which behaves like a complex of VS's, as a functor in $\Lambda$ ($\Lambda\mapsto\overline{R}_\Lambda$ is not functorial enough), is the almost étale descent that allows to move from the second to the third line of the diagram.

We set $u=(p-1)/p, v=p-1$ if $p\geq 3$, and $u=3/4, v=3/2$ if $p=2$. The dashing of an arrow means that the arrow is very similar to the one appearing in [22, Th. 7.5].

$$\begin{tikzcd}[column sep=small]
K_{\partial,\varphi}(F^rR^+_{\mathrm{cr}}\widehat{\otimes}\mathbb{A}_{\mathrm{cr}}(\Lambda)) \arrow[r, "\omega"] \arrow[d, "\tau_{\leq r}\beta"', "\wr"] & C_G(K_{\partial,\varphi}(F^r\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda})) \arrow[d, dashed, "\wr"] & C_G(K_{\varphi}(F^r\mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda))) \arrow[l, "PL"', "\sim"] \arrow[d, dashed, "\wr"] \\
K_{\partial,\varphi}(F^rR^{[u,v]}\widehat{\otimes}\mathbb{A}^{[u,v]}(\Lambda)) \arrow[r] \arrow[dddd, dashed, bend right=20, "\wr"] \arrow[dddddd, dashed, bend right=40, "PL"', "\wr"] & C_G(K_{\partial,\varphi}(F^r\mathbf{E}^{[u,v]}_{\overline{R}_\Lambda})) & C_G(K_{\varphi}(F^r\mathbf{A}^{[u,v]}_{\overline{R}_\Lambda})) \arrow[l, "PL"', "\sim"] \\
 & C_\Gamma(K_{\partial,\varphi}(F^r\mathbf{E}^{[u,v]}_{R^\infty_\Lambda})) \arrow[u, dashed, "\mu_H", "\wr"'] & C_\Gamma(K_{\varphi}(F^r\mathbf{A}^{[u,v]}_{R^\infty_\Lambda})) \arrow[l, "PL"', "\sim"] \arrow[u, dashed, "\mu_H", "\wr"'] \\
 & C_\Gamma(K_{\partial,\varphi}(F^r\mathbf{E}^{[u,v]}_{R^\infty}\widehat{\otimes}\mathbb{A}^{[u,v]}(\Lambda))) \arrow[u, "\delta_1", "\wr"'] & C_\Gamma(K_{\varphi}(F^r\mathbf{A}^{[u,v]}_{R^\infty}\widehat{\otimes}\mathbb{A}^{[u,v]}(\Lambda))) \arrow[l, dashed, "PL"', "\sim"] \arrow[u, "\delta_2", "\wr"'] \\
 & C_\Gamma(K_{\partial,\varphi}(F^r\mathbf{E}^{[u,v]}_{R}\widehat{\otimes}\mathbb{A}^{[u,v]}(\Lambda))) \arrow[u, dashed, "\mu_\infty", "\wr"'] & C_\Gamma(K_{\varphi}(F^rR^{[u,v]}\widehat{\otimes}\mathbb{A}^{[u,v]}(\Lambda))) \arrow[l, dashed, "PL"', "\sim"] \arrow[u, dashed, "\mu_\infty", "\wr"'] \\
 & K_{\partial,\varphi,\Gamma}(F^r\mathbf{E}^{[u,v]}_{R}\widehat{\otimes}\mathbb{A}^{[u,v]}(\Lambda)) \arrow[u, dashed, "\wr"'] \arrow[d, dashed, "\mathscr{L}az", "\wr"'] & K_{\varphi,\Gamma}(F^rR^{[u,v]}\widehat{\otimes}\mathbb{A}^{[u,v]}(\Lambda)) \arrow[l, dashed, "PL"', "\sim"] \arrow[u, dashed, "\wr"'] \arrow[d, dashed, "\mathscr{L}az", "\wr"'] \\
 & K_{\partial,\varphi,\mathrm{Lie}\,\Gamma}(F^r\mathbf{E}^{[u,v]}_{R}\widehat{\otimes}\mathbb{A}^{[u,v]}(\Lambda)) & K_{\varphi,\mathrm{Lie}\,\Gamma}(F^rR^{[u,v]}\widehat{\otimes}\mathbb{A}^{[u,v]}(\Lambda)) \arrow[l, dashed, "PL"', "\sim"] \\
 & K_{\partial,\varphi,\partial_{\mathbf{A}}}(F^r\mathbf{E}^{[u,v]}_{R}\widehat{\otimes}\mathbb{A}^{[u,v]}(\Lambda)) \arrow[u, dashed, "t^\bullet", "\wr"'] & K_{\varphi,\partial_{\mathbf{A}}}(F^rR^{[u,v]}\widehat{\otimes}\mathbb{A}^{[u,v]}(\Lambda)) \arrow[l, dashed, "PL"', "\sim"] \arrow[u, dashed, "{t^\bullet,\tau_{\leq r}}", "\wr"']
\end{tikzcd}$$

Here, everything is taken modulo $p^n$. Moreover:

- tensor products with $\mathbb{A}_{\mathrm{cr}}(\Lambda)$ (resp $\mathbb{A}^{[u,v]}(\Lambda)$) are over $\mathbf{A}_{\mathrm{cr}}$ (resp. $\mathbf{A}^{[u,v]}$);
- $(R^\infty_\Lambda, R^{\infty,+}_\Lambda)=(R^\infty,R^{\infty,+})\widehat{\otimes}_{(C,\mathscr{O}_C)}(\Lambda,\Lambda^+)$; it is a perfectoid affinoid by [37, Prop. 6.18];
- the period rings $\mathbf{E}^{[u,v]}_{R^\infty},\mathbf{E}^{[u,v]}_{R}$ are as in [22, Prop. 7.2]; the period rings $\mathbf{E}^{[u,v]}_{\overline{R}_\Lambda},\mathbf{E}^{[u,v]}_{R^\infty_\Lambda}$ are defined in analogous way, using [22, Prop. 7.2] and diagram (7.7);
- $G$ and $\Gamma$ are $G_{R_\Lambda}$ and $\Gamma_R$;
- $C_G, C_\Gamma$ denote the complexes of continuous cochains on the groups $G,\Gamma$, respectively;
- $K$ denotes a complex of Koszul type:
  - — the indices indicate the operators involved in the complex:
    - ◇ $\partial$ is a shorthand for $\big(X_1\frac{\partial}{\partial X_1},\ldots,X_d\frac{\partial}{\partial X_d}\big)$,
    - ◇ $\Gamma$ is a shorthand for $(\gamma_1-1,\ldots,\gamma_d-1)$, where the $\gamma_i$'s are our chosen topological generators of $\Gamma$,
    - ◇ $\mathrm{Lie}\,\Gamma$ is a shorthand for $\big(\nabla_1,\ldots,\nabla_d\big)$, where $\nabla_i=\log\gamma_i$, so that the $\nabla_i$'s are a basis of $\mathrm{Lie}\,\Gamma$ over $\mathbf{Z}_p$,
    - ◇ $\partial_{\mathbf{A}}$ is a shorthand for $\big(X_1\frac{\partial}{\partial X_1},\ldots,X_d\frac{\partial}{\partial X_d}\big)$ viewed as operators on $R^{[u,v]}$ and $\mathbf{E}^{[u,v]}_R$.
    - ◇ $\varphi$ is a shorthand for $\varphi-p^r$.
  - — only the first term of the complex is indicated: the rest is implicit and obtained from the first term so that the maps involved make sense: $\varphi$ does not respect filtration or annulus of convergence, and $\partial$ decrease the degrees of filtration by 1.

  For example, choosing a basis of $\Omega_{R^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}}$ transforms complexes involving differentials into complexes of Koszul type: $K_{\partial,\varphi}(F^rS)$ if $S=R^+_{\mathrm{cr}}$ or $S=R^{[u,v]}$.

Let us now turn our attention to the maps between rows:

- The dashed arrows are induced by the analogous maps in the proof of [22, Th. 7.5]. They are $p^{cr}$-quasi-isomorphisms by the same argument as in loc. cit. since tensoring with $\mathbf{A}^{[u,v]}_\Lambda$ can be done outside the quasi-isomorphic complexes. We note that the tensor products $\widehat{\otimes}_{\mathbf{A}^{[u,v]}}\mathbf{A}_\Lambda^{[u,v]}$ are completed but not, a priori, derived. This does not cause problems because the $\mathbf{A}_{\mathrm{inf},n}$-module $\mathbb{A}_{\mathrm{inf}}(\Lambda)_n$ is flat for all $n$: $\mathscr{O}_C^\flat$ is a valuation ring hence the $\mathscr{O}_C^\flat$-module $\Lambda^{+,\flat}$, being torsion free, is flat.

- Going from the first row to the second row just uses the injections $R_{\mathrm{cr}}^+ \subset R^{[u,v]}$, etc.
- Going from the third row to the second row is the inflation map from $\Gamma_R$ to $G_{R_\Lambda}$, using the injection $R_\Lambda^\infty \subset \overline{R}_\Lambda$. We use almost étale descent (i.e., Faltings' almost purity theorem or its extension by Scholze or Kedlaya-Liu) to prove that it is a quasi-isomorphism.
- Going from the fifth row to the fourth row just uses the injection of $R$ into $R^\infty$.
- The maps connecting the fifth row to the sixth are the maps connecting continuous cohomology of $\Gamma_R$ to Koszul complexes.
- All the maps $\mathscr{L}az$ connecting the seventh row to the sixth are defined as in [22, Lemma 5.8].
- The maps $t^\bullet$ connecting the last row to the sixth are the maps appearing in the proof of [22, Lemma 5.7] (multiplication by suitable powers of $t$). We use here Lemma 7.4.

Let us now describe the maps between columns:

- The maps from the first column to the second one are induced by the natural injections of rings; the PL-map is a $p^{cr}$-quasi-isomorphism by [22, Lemma 7.4].
- The maps from the third column to the second are also induced by the natural injections of rings; the PL-map are $p^{cr}$-quasi-isomorphisms by [22, Prop. 7.3 ] (for the first 7 rows) and [22, Lemma 7.4] (for the last row).

Finally, the maps $\beta, \delta_1, \delta_2$ in the diagram are treated by Lemma 7.9 below. $\square$

**Lemma 7.9.** (1) *The canonical morphism*

$$\tau_{\leq r}\beta : \tau_{\leq r}K_{\partial,\varphi}(F^rR_{\mathrm{cr}}^+\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr}}}\mathbb{A}_{\mathrm{cr}}(\Lambda))_n \to \tau_{\leq r}K_{\partial,\varphi}(F^rR^{[u,v]}\widehat{\otimes}_{\mathbf{A}^{[u,v]}}\mathbf{A}_\Lambda^{[u,v]})_n$$

*is a $p^{cr}$-quasi-isomorphism.*

(2) *The canonical morphisms*

$$\delta_1 : F^r\mathbf{A}_{R^\infty}^{[u,v]}\widehat{\otimes}_{\mathbf{A}^{[u,v]}}\mathbf{A}_\Lambda^{[u,v]} \to F^r\mathbf{A}_{R_\Lambda^\infty}^{[u,v]}, \quad \delta_2 : F^r\mathbf{E}_{R^\infty}^{[u,v]}\widehat{\otimes}_{\mathbf{A}^{[u,v]}}\mathbf{A}_\Lambda^{[u,v]} \to F^r\mathbf{E}_{R_\Lambda^\infty}^{[u,v]}$$

*are isomorphisms.*

*Proof.* For the first claim, set $R_{\mathrm{cr},\Lambda}^+ := R_{\mathrm{cr}}^+\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr}}}\mathbb{A}_{\mathrm{cr}}(\Lambda)$. This ring has the same form as $R_{\mathrm{cr}}^+$ (see Section 7.2.1) but with $\mathbf{A}_{\mathrm{cr}}$ replaced by $\mathbb{A}_{\mathrm{cr}}(\Lambda)$. The above morphism can be written as

$$\tau_{\leq r}K_{\partial,\varphi}(F^rR_{\mathrm{cr},\Lambda}^+)_n \to \tau_{\leq r}K_{\partial,\varphi}(F^rR_{\mathrm{cr},\Lambda}^+\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr}}}\mathbf{A}^{[u,v]})_n.$$

Now, the proof in [22, Sec. 4.1] goes through verbatim by changing $\mathbf{A}_{\mathrm{cr}}$ to $\mathbb{A}_{\mathrm{cr}}(\Lambda)$.

For the second claim of the lemma, we start with the first morphism. By Lemma 7.4, we can replace the filtration by the one given by powers of $t$. Hence, it is enough to show that the canonical map

$$\mathbb{A}_{\mathrm{inf}}(R^\infty)\widehat{\otimes}_{\mathbf{A}_{\mathrm{inf}}}\mathbb{A}_{\mathrm{inf}}(\Lambda) \to \mathbb{A}_{\mathrm{inf}}(R_\Lambda^\infty) \tag{7.10}$$

is an isomorphism (the passage to $[u,v]$-version is obtained by taking the completed tensor product of (7.10) with $\mathbf{A}^{[u,v]}$) . Or, since both sides are $p$-adically derived complete, that so is its reduction modulo $p$:

$$R^{\infty,+,\flat}\widehat{\otimes}_{\mathscr{O}_C^\flat}\Lambda^{+,\flat} \to R_\Lambda^{\infty,+,\flat}.$$

But this can be checked modulo $p^\flat$. That is, we want the canonical map

$$(R^{\infty,+,\flat}/p^\flat) \otimes_{\mathscr{O}_C^\flat/p^\flat} (\Lambda^{+,\flat}/p^\flat) \to R_\Lambda^{\infty,+,\flat}/p^\flat$$

to be an isomorphism.

Now, this map identifies with the canonical map

$$(R^{\infty,+}/p)\otimes_{\mathscr{O}_C/p}(\Lambda^+/p)\to R^{\infty,+}_{\Lambda}/p.$$

It suffices thus to show that the canonical map

$$R^{\infty,+}\widehat{\otimes}_{\mathscr{O}_C}\Lambda^+\to R^{\infty,+}_{\Lambda}$$

is an isomorphism. But this is clear since both sides are isomorphic to the completion of the same étale extension of the tower

$$\Lambda^+\{X^{1/p^\infty},\tfrac{1}{(X_1\dots X_a)^{1/p^\infty}},\tfrac{\varpi^{1/p^\infty}}{(X_{a+1}\dots X_d)^{1/p^\infty}}\}.$$

For the second isomorphism of the second claim, by Lemma 7.4, it suffices to show that the canonical map

$$\mathbf{E}^{[u,v]}_{R^\infty}\widehat{\otimes}_{\mathbf{A}^{[u,v]}}\mathbf{A}^{[u,v]}_{\Lambda}\to\mathbf{E}^{[u,v]}_{R^\infty_\Lambda}$$

is an isomorphism. But this follows easily from the description of the structure of $\mathbf{E}^{[u,v]}_{R^\infty}$ and $\mathbf{E}^{[u,v]}_{R^\infty_\Lambda}$ from [22, Prop. 7.1] (which is a preparation for the proof of Poincaré Lemma) and the topological isomorphism (7.10). $\square$

7.2.5. *Period morphism, globalization.*

(i) *Over $C$.* Consider the following commutative diagram, a relaxed version of diagram (7.5):

(7.11)
$$\begin{tikzcd}
 & \mathrm{Spf}(\mathbf{E}^{\mathrm{PD}}_{\overline{R}}) \arrow[dd] \arrow[dr] & \\
\mathrm{Spf}(\overline{R}) \arrow[ur, hook] \arrow[rr, hook] \arrow[dd] & & \mathrm{Spf}(\mathbb{A}_{\mathrm{cr}}(\overline{R})\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr}}}R^+_{\mathrm{cr}}) \arrow[dd] \\
 & \mathscr{D}^+_{\mathrm{cr}} \arrow[dr] & \\
\mathrm{Spf}(R) \arrow[ur, hook] \arrow[rr, hook, "\iota"] \arrow[d] & & \mathrm{Spf}(R^+_{\mathrm{cr}}) \arrow[d, "\pi"] \\
\mathrm{Spf}(\mathscr{O}_C) \arrow[rr, hook] & & \mathrm{Spf}(\mathbf{A}_{\mathrm{cr}})
\end{tikzcd}$$

The map $\pi$ is log-smooth and the map $\iota$ is an exact closed immersion (and the bottom square is not necessarily cartesian). $\mathscr{D}^+_{\mathrm{cr}}$ is the PD-envelope of $\iota$. Let $r\in\mathbf{N}$. We define the filtered de Rham complex $\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}}}$ and the syntomic complex $\mathrm{Syn}(\overline{R},r)$ as above. Let

$$F^r\Omega^\bullet_{\mathscr{D}^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}}:=F^r\mathscr{D}^+_{\mathrm{cr}}\to F^{r-1}\mathscr{D}^+_{\mathrm{cr}}\otimes_{R^+_{\mathrm{cr}}}\Omega^1_{R^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}}\to F^{r-2}\mathscr{D}^+_{\mathrm{cr}}\otimes_{R^+_{\mathrm{cr}}}\Omega^2_{R^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}}\to\cdots$$

The crystalline syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},r)$ is computed by the complex

$$\mathrm{Syn}(R^+_{\mathrm{cr}},r):=[F^r\Omega^\bullet_{\mathscr{D}^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}}\xrightarrow{\varphi-p^r}\Omega^\bullet_{\mathscr{D}^+_{\mathrm{cr}}/\mathbf{A}_{\mathrm{cr}}}]$$

The Fontaine-Messing period map

$$\alpha_{r,n}:\quad\mathrm{Syn}(R,r)_n\to C(G_R,\mathbf{Z}/p^n(r)')$$

by the composition

$$\begin{tikzcd}
\mathrm{Syn}(R,r)_n \arrow[r, equal] & {[F^r\Omega^\bullet_{\mathscr{D}^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}} \arrow[r, "\varphi-p^r"] \arrow[d] & {\Omega^\bullet_{\mathscr{D}^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}]} \\
& C(G_R,[F^r\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R},n}} \arrow[r, "\varphi-p^r"] & {\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R},n}}])} \\
& C(G_R,[F^r\mathbb{A}_{\mathrm{cr}}(\overline{R})_n \arrow[r, "\varphi-p^r"] \arrow[u, "\wr"'] & {\mathbb{A}_{\mathrm{cr}}(\overline{R})_n])} \\
& C(G_R,\mathbf{Z}/p^n(r)'). \arrow[u, "\wr"'] &
\end{tikzcd}$$

The second quasi-isomorphism follows from the filtered Poincaré Lemma, i.e., from the quasi-isomorphism

$$F^r\mathbb{A}_{\mathrm{cr}}(\overline{R})_n \xrightarrow{\sim} F^r\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R},n}},\quad r\geq 0.$$

We claim that the first map in the composition is a $p^{cr}$-quasi-isomorphism, for a universal constant $c$. Indeed, since the map $\pi$ is log-smooth and the lifting $\widetilde{R}^+_{\mathrm{cr}}$ in diagram (7.5) (we put $\widetilde{\ }$ to distinguish it from $R^+_{\mathrm{cr}}$ in diagram (7.11)) is $\xi$-adically complete, for a generator $\xi$ of the kernel of the map $\theta:\mathbf{A}_{\mathrm{cr}}\to\mathscr{O}_C$, we have a map from diagram (7.11) to diagram (7.5). This map induces two compatible maps

$$F^r\Omega^\bullet_{\widetilde{R}^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}\to F^r\Omega^\bullet_{\mathscr{D}^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}},\quad F^r\Omega^\bullet_{\widetilde{\mathbf{E}}^{\mathrm{PD}}_{\overline{R},n}}\to F^r\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R},n}}. \tag{7.12}$$

These are quasi-isomorphisms since both terms in the first map compute absolute crystalline cohomology of $R$ and both terms in the second map – absolute crystalline cohomology of $\overline{R}$.

(ii) *Over a perfectoid $C$-algebra.* Let $\Lambda$ be a perfectoid affinoid over $C$. To show that the Fontaine-Messing period map, lifted to $\Lambda$, can be globalized we will use the following commutative diagram:

$$\begin{tikzcd}
& \mathrm{Spf}(\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda}) \arrow[dr] \arrow[dd] & \\
\mathrm{Spf}(\overline{R}^+_\Lambda) \arrow[ur, hook] \arrow[rr, hook] \arrow[dd] & & \mathrm{Spf}(\mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda)\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr}}}R^+_{\mathrm{cr}}) \arrow[dd] \\
& \mathscr{D}^+_{\mathrm{cr}} \arrow[dr] & \\
\mathrm{Spf}(R) \arrow[ur, hook] \arrow[rr, hook, "\iota"] \arrow[d] & & \mathrm{Spf}(R^+_{\mathrm{cr}}) \arrow[d, "\pi"] \\
\mathrm{Spf}(\mathscr{O}_C) \arrow[rr, hook] & & \mathrm{Spf}(\mathbf{A}_{\mathrm{cr}})
\end{tikzcd} \tag{7.13}$$

The Fontaine-Messing period map

$$\alpha_{r,n}(\Lambda):\quad \mathrm{Syn}(R,r)_n(\Lambda)\to C(G_{R_\Lambda},\mathbf{Z}/p^n(r)') \tag{7.14}$$

can be defined by the composition

$$\begin{array}{c}
\mathrm{Syn}(R,r)_n(\Lambda)=[F^r\Omega^\bullet_{\mathscr{D}^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr},n}}\mathbb{A}_{\mathrm{cr}}(\Lambda)_n\xrightarrow{\varphi-p^r}\Omega^\bullet_{\mathscr{D}^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr},n}}\mathbb{A}_{\mathrm{cr}}(\Lambda)_n]\\
\downarrow\\
C(G_{R_\Lambda},[F^r\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda,n}}\xrightarrow{\varphi-p^r}\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda,n}}])\\
\uparrow\wr\\
C(G_{R_\Lambda},[F^r\mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda)_n\xrightarrow{\varphi-p^r}\mathbb{A}_{\mathrm{cr}}(\overline{R}_\Lambda)_n])\\
\uparrow\wr\\
C(G_{R_\Lambda},\mathbf{Z}/p^n(r)')
\end{array}$$

We claim that the first map is a $p^{cr}$-quasi-isomorphism, for a universal constant $c$. Indeed, since the map $\pi$ is log-smooth and the lifting $\widetilde{R}^+_{\mathrm{cr}}$ in diagram (7.7) (we put $\tilde{}$ to distinguish it from $R^+_{\mathrm{cr}}$ in diagram (7.13)) is $\xi$-adically complete, we have a map from diagram (7.13) to diagram (7.7). This map induces two compatible maps

$$F^r\Omega^\bullet_{\widetilde{R}^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr},n}}\mathbb{A}_{\mathrm{cr}}(\Lambda)_n\to F^r\Omega^\bullet_{\mathscr{D}^+_{\mathrm{cr},n}/\mathbf{A}_{\mathrm{cr},n}}\widehat{\otimes}_{\mathbf{A}_{\mathrm{cr},n}}\mathbb{A}_{\mathrm{cr}}(\Lambda)_n,$$
$$F^r\Omega^\bullet_{\widetilde{\mathbf{E}}^{\mathrm{PD}}_{\overline{R}_\Lambda,n}}\to F^r\Omega^\bullet_{\mathbf{E}^{\mathrm{PD}}_{\overline{R}_\Lambda,n}}.$$

These maps are quasi-isomorphisms: the first one by the first quasi-isomorphism from (7.12) and flatness of $\mathbb{A}_{\mathrm{cr}}(\Lambda)$ over $\mathbf{A}_{\mathrm{cr}}$; the second one, via the filtered Poincaré Lemma, can be $p$-identified with the identity map

$$F^r\Omega^\bullet_{\mathbf{A}_{\overline{R}_\Lambda,n}}\to F^r\Omega^\bullet_{\mathbf{A}_{\overline{R}_\Lambda,n}}.$$

7.2.6. *Proof of Theorem 7.3.* Let now $X\in\mathrm{Sm}_C$. Using the definition (7.14) of lifted Fontaine-Messing period map we can globalize it (for example, using colimits over all the embedding data as in diagram (7.13) and then $\eta$-étale sheafification, as usual) to a period map

$$\mathbb{A}_r:\quad \mathbb{R}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to\mathbb{R}_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)),\quad r\geq 0.$$

This is a strict quasi-isomorphism after truncation $\tau_{\leq r}$, as wanted.

7.3. **Dagger varieties.** We will now geometrize cohomologies and period morphisms associated to dagger varieties.

7.3.1. *Cohomologies.* Let $X$ be a dagger affinoid over $C$, and $\{X_h\}$ be a presentation. Define the VS:

$$\mathbb{R}^\dagger_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p):=\mathrm{L}\operatorname{colim}_n\mathbb{R}_{\mathrm{pro\acute{e}t}}(X_h,\mathbf{Q}_p).$$

For a smooth dagger variety $X$ over $C$, this globalizes, via étale sheafification, to the VS $\mathbb{R}_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p)$. We set

$$\widetilde{\mathbb{H}}^i_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p),\mathbb{H}^i_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p):\quad\Lambda\mapsto\widetilde{H}^i(\mathbb{R}_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p)(\Lambda)),H^i(\mathbb{R}_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p)(\Lambda)).$$

We define similarly the Hyodo-Kato cohomology, the $\mathbb{B}^+_{\mathrm{dR}}$-cohomology, and the syntomic cohomology:

$$\mathbb{R}_{\mathrm{HK}}(X),\quad\widetilde{\mathbb{H}}^i_{\mathrm{HK}}(X);\quad\mathbb{R}_{\mathrm{dR}}(X/\mathbb{B}^+_{\mathrm{dR}}),\quad\widetilde{\mathbb{H}}^i_{\mathrm{dR}}(X/\mathbb{B}^+_{\mathrm{dR}});\quad\mathbb{R}^\dagger_{\mathrm{syn}}(X,\mathbf{Q}_p(r)),\quad\widetilde{\mathbb{H}}^{\dagger,i}_{\mathrm{syn}}(X,\mathbf{Q}_p(r));$$
$$\mathbb{R}_{\mathrm{syn}}(X,\mathbf{Q}_p(r)),\quad\widetilde{\mathbb{H}}^i_{\mathrm{syn}}(X,\mathbf{Q}_p(r)).$$

We note that the Hyodo-Kato cohomology is the constant functor equal to $\mathrm{R}\Gamma_{\mathrm{HK}}(X),\widetilde{H}^i_{\mathrm{HK}}(X)$.

7.3.2. *Period maps.* Let $X$ be a dagger affinoid over $C$, and $\{X_h\}$ be a presentation. Let $r\geq 0$. The local period morphisms

$$\mathbb{A}_r^\dagger:\quad \mathbb{R}^\dagger_{\rm syn}(X,\mathbf{Q}_p(r))\to \mathbb{R}^\dagger_{\rm pro\acute{e}t}(X,\mathbf{Q}_p(r))$$

are defined as

$$\mathbb{R}^\dagger_{\rm syn}(X,\mathbf{Q}_p(r))=\mathrm{L}\,\mathrm{colim}_h\,\mathbb{R}_{\rm syn}(X_h,\mathbf{Q}_p(r))\xrightarrow{\mathrm{L}\,\mathrm{colim}_h\,\mathbb{A}_{r,h}}\mathrm{L}\,\mathrm{colim}_h\,\mathbb{R}_{\rm pro\acute{e}t}(X_h,\mathbf{Q}_p(r))=\mathbb{R}^\dagger_{\rm pro\acute{e}t}(X,\mathbf{Q}_p(r)).$$

For a smooth dagger variety $X$, this globalizes to period morphisms

$$\mathbb{A}_r^\dagger:\quad \mathbb{R}^\dagger_{\rm syn}(X,\mathbf{Q}_p(r))\to \mathbb{R}_{\rm pro\acute{e}t}(X,\mathbf{Q}_p(r)).$$

These are strict quasi-isomorphisms after truncation $\tau_{\leq r}$ because so are the rigid analytic period morphisms $\mathbb{A}_{r,h}$ by Theorem 7.3.

Recall now that, for $X\in \mathrm{Sm}^\dagger_C$, the period morphisms

$$\alpha_r:\quad \mathrm{R}\Gamma_{\rm syn}(X,\mathbf{Q}_p(r))\to \mathrm{R}\Gamma_{\rm pro\acute{e}t}(X,\mathbf{Q}_p(r))$$

are defined as the compositions

$$\mathrm{R}\Gamma_{\rm syn}(X,\mathbf{Q}_p(r))\xleftarrow[\sim]{\iota^\dagger_{\rm syn}}\mathrm{R}\Gamma^\dagger_{\rm syn}(X,\mathbf{Q}_p(r))\xrightarrow{\alpha_r^\dagger}\mathrm{R}\Gamma_{\rm pro\acute{e}t}(X,\mathbf{Q}_p(r)).$$

These morphisms lift to VS. Indeed, it remains to show that we can lift the map $\iota^\dagger_{\rm syn}$ to a map

$$\mathbb{I}^\dagger_{\rm syn}:\quad \mathbb{R}^\dagger_{\rm syn}(X,\mathbf{Q}_p(r))\to \mathbb{R}_{\rm syn}(X,\mathbf{Q}_p(r)),$$

and that this map is a strict quasi-isomorphism. We define the map $\mathbb{I}^\dagger_{\rm syn}$ by étale sheafifying the following composition ($X$ is a smooth dagger affinoid over $C$),

$$\begin{tikzcd}
\mathbb{R}^\dagger_{\rm syn}(X,\mathbf{Q}_p(r))=\mathrm{L}\,\mathrm{colim}_h\,\mathbb{R}_{\rm syn}(X_h,\mathbf{Q}_p(r)) \arrow[r, "\sim"] & \mathrm{L}\,\mathrm{colim}_h\,\mathbb{R}_{\rm syn}(X_h^{\rm o},\mathbf{Q}_p(r)) \\
\mathbb{R}_{\rm syn}(X,\mathbf{Q}_p(r)) & \mathrm{L}\,\mathrm{colim}_h\,\mathbb{R}_{\rm syn}(X_h^{\rm o,\dagger},\mathbf{Q}_p(r)) \arrow[l, "\sim"'] \arrow[u, "\wr"', "\mathbb{I}"]
\end{tikzcd}$$

Here, the morphism $\mathbb{I}$ needs to be defined and both it and the bottom morphism need to be shown to be strict quasi-isomorphisms.

**Proposition 7.15.** (Definition of the map $\mathbb{I}$) *Let $X\in\mathrm{Sm}^\dagger_C$. We have a natural map*

$$\mathbb{I}:\quad \mathbb{R}_{\rm syn}(X,\mathbf{Q}_p(r))\to \mathbb{R}_{\rm syn}(\widehat{X},\mathbf{Q}_p(r))$$

*It is a strict quasi-isomorphism for $X$ partially proper.*

*Proof.* We will set $\mathbb{I}:=\mathbb{I}_2\mathbb{I}_1$, with the maps $\mathbb{I}_1,\mathbb{I}_2$ defined as follows.

(i) *The map* $\mathbb{I}_1$. The map $\mathbb{I}_1$ is defined as the following composition:

$$\begin{tikzcd}
\mathbb{R}_{\rm syn}(X,\mathbf{Q}_p(r)) \arrow[r, equal] & \big[[\mathbb{R}_{\rm HK}(X)\widehat{\otimes}^{\rm R}_{F^{\rm nr}}\widehat{\mathbb{B}}^+_{\rm st}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\rm HK}\otimes\iota}\mathbb{R}_{\rm dR}(X/\mathbb{B}^+_{\rm dR})/F^r\big] \arrow[d] \\
 & \big[[\mathbb{R}_{\rm HK}(\widehat{X})\widehat{\otimes}_{F^{\rm nr}}\widehat{\mathbb{B}}^+_{\rm st}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\rm HK}\otimes\iota}\mathbb{R}_{\rm dR}(\widehat{X}/\mathbb{B}^+_{\rm dR})/F^r)\big] \\
 & \big[[\mathbb{R}_{\rm HK}(\widehat{X})_F\widehat{\otimes}_{F^{\rm nr}}\mathbb{B}^+_{\rm st}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\rm HK}\otimes\iota}\mathbb{R}_{\rm dR}(\widehat{X}/\mathbb{B}^+_{\rm dR})/F^r)\big] \arrow[u, "\wr"']
\end{tikzcd}$$

It is a strict quasi-isomorphism. Indeed, for that it suffices to show that the canonical map

$$[\mathbb{R}_{\rm HK}(\widehat{X})\widehat{\otimes}_{F^{\rm nr}}\mathbb{B}^+_{\rm st}]^{N=0}\to[\mathbb{R}_{\rm HK}(\widehat{X})\widehat{\otimes}^{\rm R}_{F^{\rm nr}}\widehat{\mathbb{B}}^+_{\rm st}]^{N=0}$$

is a strict quasi-isomorphism. But this can be shown exactly as in Section 6.1.1.

(ii) *The map* $\mathbb{I}_2$. Now, we define a natural strict quasi-isomorphism $\mathbb{I}_2$ by

$$\begin{tikzcd}
\big[[\mathbb{R}_{\mathrm{HK}}(\widehat{X})_F\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbb{B}_{\mathrm{st}}^+]^{N=0,\varphi=p^r} \arrow[r, "\iota_{\mathrm{HK}}\otimes\iota"] & \mathbb{R}_{\mathrm{dR}}(\widehat{X}/\mathbb{B}_{\mathrm{dR}}^+)/F^r\big] \arrow[d] & \\
{[[\mathbb{R}_{\mathrm{cr}}(\widehat{X})]^{\varphi=p^r}} \arrow[r, "\mathrm{can}"] & \mathbb{R}_{\mathrm{cr}}(\widehat{X})/F^r] \arrow[r, equal] & \mathbb{R}_{\mathrm{syn}}(\widehat{X},\mathbf{Q}_p(r)).
\end{tikzcd}$$

For that, it suffices to define the maps $\iota_{\mathrm{BK}}^1$ and $\iota_{\mathrm{BK}}^2$ in the following diagram and to show that this diagram commutes:

$$\begin{tikzcd}
{[\mathbb{R}_{\mathrm{HK}}(\widehat{X})\widehat{\otimes}_{F^{\mathrm{nr}}}\mathbb{B}_{\mathrm{st}}^+]^{N=0}} \arrow[r, "\iota_{\mathrm{HK}}\otimes\iota"] \arrow[d, "\varepsilon_{\mathrm{st}}^{\mathrm{HK}}\otimes\mathrm{Id}", "\wr"'] \arrow[dd, bend right=60, "\iota_{\mathrm{BK}}^1"', "\wr"] & (\mathbb{R}_{\mathrm{dR}}(\widehat{X}/\mathbb{B}_{\mathrm{dR}}^+)\widehat{\otimes}^{\mathrm{R}}_{\mathbf{B}_{\mathrm{dR}}^+}\mathbb{B}_{\mathrm{dR}}^+)/F^r \arrow[dd, bend left=60, "\iota_{\mathrm{BK}}^2", "\wr"'] \\
{[\mathbb{R}_{\mathrm{cr}}(\widehat{X})\widehat{\otimes}_{\mathbf{B}_{\mathrm{cr}}^+}\mathbb{B}_{\mathrm{st}}^+]^{N=0}} \arrow[ur, "\kappa\otimes\iota"] & \\
\mathbb{R}_{\mathrm{cr}}(\widehat{X})\widehat{\otimes}_{\mathbf{B}_{\mathrm{cr}}^+}\mathbb{B}_{\mathrm{cr}}^+ \arrow[u, "\wr"] \arrow[r, "\mathrm{can}"] & (\mathbb{R}_{\mathrm{cr}}(\widehat{X})\widehat{\otimes}_{\mathbf{B}_{\mathrm{cr}}^+}\mathbb{B}_{\mathrm{cr}}^+)/F^r. \arrow[uu, "\kappa\otimes\mathrm{can}"', "\wr"]
\end{tikzcd} \tag{7.16}$$

We define the maps $\iota_{\mathrm{BK}}^1$ and $\iota_{\mathrm{BK}}^2$ to make the left and the right triangles in the diagram commute. They are strict quasi-isomorphisms. The remaining pieces of the diagram commute by definition. □

Let $X\in \mathrm{Sm}_C^\dagger$. We define the global period morphism

$$\mathbb{A}_r:\mathbb{R}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to\mathbb{R}_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))$$

as the composition $\mathbb{A}_r^\dagger(\mathbb{I}_{\mathrm{syn}}^\dagger)^{-1}$. From what we have shown above, it follows that:

**Corollary 7.17.** *The map*

$$\tau_{\leq r}\mathbb{A}_r:\tau_{\leq r}\mathbb{R}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to\tau_{\leq r}\mathbb{R}_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))$$

*is a strict quasi-isomorphism.*

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CNRS, IMJ-PRG, Sorbonne Université, 4 place Jussieu, 75005 Paris, France
*E-mail address*: `pierre.colmez@imj-prg.fr`

CNRS, IMJ-PRG, Sorbonne Université, 4 place Jussieu, 75005 Paris, France
*E-mail address*: `wieslawa.niziol@imj-prg.fr`