

ON THE COHOMOLOGY OF p -ADIC ANALYTIC SPACES, II: THE C_{st} -CONJECTURE

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ABSTRACT. Fontaine has formulated conjectures (which are now theorems) relating étale and de Rham cohomologies of algebraic varieties over a p -adic field K . We state analogs of Fontaine’s conjectures for smooth dagger varieties over K or the completion C of its algebraic closure and we prove these conjectures for varieties which are well-behaved at infinity (which includes quasi-compact varieties, analytifications of algebraic varieties, Stein varieties or, more generally, some holomorphically convex varieties).

For varieties over K , the “pro-étale-to-de Rham” direction of these conjectures explains how to recover de Rham cohomology from pro-étale cohomology using the action of the absolute Galois group of K (the proof uses, among other tools, Fontaine’s theory of almost C -representations). For varieties over C , this Galois action is replaced by a “geometrization” of pro-étale cohomology (viewed as the C -points of a pro-étale sheaf on the category of perfectoid spaces over C). This geometrization plays a large role in the proof of the “de Rham-to-pro-étale” direction of the conjectures and makes it possible to appeal to the theory of Banach-Colmez spaces as a substitute for Fontaine’s almost C -representations.

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1. INTRODUCTION

Let \mathcal{O}_K be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p . Let \bar{K} be an algebraic closure of K , let C be its p -adic completion, and let $\bar{\mathcal{O}}_K$ denote the integral closure of \mathcal{O}_K in \bar{K} . Let $W(k)$ be the ring of Witt vectors of k with fraction field F (i.e. $W(k) = \mathcal{O}_F$) and let φ be the absolute Frobenius on $W(\bar{k})$. Set $\mathcal{G}_K = \text{Gal}(\bar{K}/K)$.

1.1. Comparison theorems for algebraic varieties. In order to put our results into perspective, let us first recall what is known for algebraic varieties. The story started with Tate conjecturing [41] the existence of a Hodge-like decomposition for the étale cohomology of smooth and proper algebraic (or even rigid analytic) varieties over K and proving the existence of such a decomposition for abelian varieties with good reduction. One upshot of Tate's results is that the p -adic periods of algebraic varieties do not live in C . Fontaine constructed [21, 22, 23] rings \mathbf{B}_{cris} , \mathbf{B}_{st} , \mathbf{B}_{dR} that should contain these periods and refined Tate's conjecture by conjecturing (first [21, 22] in the case of X with good reduction and then [24] in the case of X with semistable reduction) the existence of functorial period isomorphisms relating étale and de Rham cohomologies of smooth algebraic varieties.

Conjecture 1.1. (Fontaine) *Let X be a proper and smooth algebraic variety over K admitting a semistable model over \mathcal{O}_K . Let $i \geq 0$.*

(i) (Conjecture C_{dR}) *We have a functorial \mathcal{G}_K -equivariant isomorphism preserving filtrations*

$$H_{\text{ét}}^i(X_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}} \simeq H_{\text{dR}}^i(X) \otimes_K \mathbf{B}_{\text{dR}}$$

(ii) (Conjecture C_{st}) *We have a functorial \mathcal{G}_K -equivariant isomorphism commuting with φ and N*

$$H_{\text{ét}}^i(X_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}} \simeq H_{\text{HK}}^i(X) \otimes_F \mathbf{B}_{\text{st}},$$

compatible with the de Rham period morphism, and the natural injections $\mathbf{B}_{\text{st}} \subset \mathbf{B}_{\text{dR}}$ and $H_{\text{HK}}^i \subset H_{\text{dR}}^i$.

As we explain below (Remark 1.3) these conjectures are now theorems (even without the assumptions on existence of nice models, properness, or even smoothness).

Remark 1.2. (i) All the above cohomology groups are finite dimensional over the appropriate field (\mathbf{Q}_p for étale cohomology, K for de Rham, and F for Hyodo-Kato).

(ii) The existence of the period isomorphism implies in particular that

$$\dim_{\mathbf{Q}_p} H_{\text{ét}}^i(X_C, \mathbf{Q}_p) = \dim_K H_{\text{dR}}^i(X).$$

(iii) The Hyodo-Kato cohomology group in (ii) is an F -vector space with a semilinear Frobenius φ and a monodromy operator N satisfying $N\varphi = p\varphi N$. Moreover, there is a Hyodo-Kato isomorphism

$$\iota_{\text{HK}} : H_{\text{HK}}^i(X) \otimes_F K \xrightarrow{\sim} H_{\text{dR}}^i(X)$$

(iv) In C_{dR} , the filtration on the left hand side is the one coming from \mathbf{B}_{dR} ; the one on the right hand side is the tensor product of the Hodge filtration on $H_{\text{dR}}^i(X)$ and the filtration on \mathbf{B}_{dR} . In C_{st} , the φ and N on the left hand side are those coming from \mathbf{B}_{st} ; on the right hand side they are the tensor products of those coming from $H_{\text{HK}}^i(X)$ and \mathbf{B}_{st} .

(v) Galois properties of the rings \mathbf{B}_{st} and \mathbf{B}_{dR} make it possible to recover de Rham cohomology from étale cohomology by taking fixed points by \mathcal{G}_K : we have functorial “étale-to-de Rham” isomorphisms

$$\begin{aligned} H_{\text{dR}}^i(X) &\simeq (H_{\text{ét}}^i(X_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}})^{\mathcal{G}_K}, & \text{as filtered } K\text{-vector spaces,} \\ H_{\text{HK}}^i(X) &\simeq (H_{\text{ét}}^i(X_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}})^{\mathcal{G}_K}, & \text{as } F\text{-vector spaces with a } \varphi \text{ and a } N. \end{aligned}$$

Moreover, the Hyodo-Kato isomorphism is induced by the inclusion $\mathbf{B}_{\text{st}} \subset \mathbf{B}_{\text{dR}}$. Note that, instead of tensor products, we could have considered \mathcal{G}_K -equivariant homomorphisms: we have functorial isomorphisms

$$\begin{aligned} H_{\text{dR}}^i(X)^* &\simeq \text{Hom}_{\mathcal{G}_K}(H_{\text{ét}}^i(X_C, \mathbf{Q}_p), \mathbf{B}_{\text{dR}}), & \text{as filtered } K\text{-vector spaces,} \\ H_{\text{HK}}^i(X)^* &\simeq \text{Hom}_{\mathcal{G}_K}(H_{\text{ét}}^i(X_C, \mathbf{Q}_p), \mathbf{B}_{\text{st}}), & \text{as } F\text{-vector spaces with a } \varphi \text{ and a } N. \end{aligned}$$

(vi) It is possible to extract \mathbf{Q}_p from \mathbf{B}_{st} , using the extra structures. This induces a description of étale cohomology by de Rham cohomology (with the extra structures coming from Hyodo-Kato cohomology), i.e., it gives a “de Rham-to-étale” bicartesian diagram

$$\begin{array}{ccc} H_{\text{ét}}^i(X_C, \mathbf{Q}_p) & \longrightarrow & (H_{\text{HK}}^i(X) \otimes_F \mathbf{B}_{\text{st}})^{N=0, \varphi=1} \\ \downarrow & & \downarrow \\ F^0(H_{\text{dR}}^i(X) \otimes_K \mathbf{B}_{\text{dR}}) & \longrightarrow & H_{\text{dR}}^i(X) \otimes_K \mathbf{B}_{\text{dR}} \end{array}$$

One can refine this diagram by taking a large enough twist, which makes it possible to remove denominators¹ in t : if $r \geq i$, we have a bicartesian diagram

$$\begin{array}{ccc} H_{\text{ét}}^i(X_C, \mathbf{Q}_p(r)) & \longrightarrow & (H_{\text{HK}}^i(X) \otimes_F \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \\ \downarrow & & \downarrow \\ F^r(H_{\text{dR}}^i(X) \otimes_K \mathbf{B}_{\text{dR}}^+) & \longrightarrow & H_{\text{dR}}^i(X) \otimes_K \mathbf{B}_{\text{dR}}^+ \end{array}$$

(vii) The pair $(H_{\text{HK}}^i(X), H_{\text{dR}}^i(X))$ is a filtered (φ, N) -module in the sense of Fontaine; the fact that the above diagram is bicartesian and $H_{\text{ét}}^i(X_C, \mathbf{Q}_p)$ is finite dimensional implies, in particular, that this filtered (φ, N) -module is *weakly admissible*, a condition that can be described purely in terms of the interplay between φ , N and the filtration.

Remark 1.3. (i) As we have mentioned above, Fontaine’s conjecture is now a theorem. There have been essentially four lines of attack: the almost étale approach [17], the syntomic approach [42], the motivic approach [33], and the derived geometry approach [2]. The resulting period isomorphisms are compared in [34]. The most comprehensive results are those of Beilinson [2]: there is no assumption of properness, existence of good models or smoothness.

(ii) If we don’t assume X to have a semistable model over \mathcal{O}_K then $H_{\text{HK}}^i(X)$ has to be replaced with $H_{\text{HK}}^i(X_{\overline{K}})$, which is not an F -vector space anymore but a F^{nr} -vector space, where F^{nr} is the

¹Recall that $\mathbf{B}_{\text{dR}} = \mathbf{B}_{\text{dR}}^+[\frac{1}{t}]$, $\mathbf{B}_{\text{st}} = \mathbf{B}_{\text{st}}^+[\frac{1}{t}]$.

maximal unramified extension of F . Moreover, $H_{\text{HK}}^i(X_{\overline{K}})$ is equipped with a semi-linear action of \mathcal{G}_K commuting with φ and N and the action of \mathcal{G}_K is smooth (i.e., any element x is fixed by \mathcal{G}_L for some finite extension L of K that depends on x). In this case, the isomorphisms involving $H_{\text{HK}}^i(X)$ in (v) of Remark 1.2 involve smooth vectors for the action of \mathcal{G}_K and not only fixed vectors: i.e, we have a functorial isomorphism

$$H_{\text{HK}}^i(X_{\overline{K}})^* \simeq \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(H_{\text{ét}}^i(X_C, \mathbf{Q}_p), \mathbf{B}_{\text{st}}) \quad \text{of } (\varphi, N, \mathcal{G}_K)\text{-modules over } F^{\text{nr}}.$$

1.2. Analytic varieties. Interest in analytic varieties is more recent despite the fact that Tate [41] already formulated his conjecture for rigid analytic spaces. Scholze [38] established a version of Tate’s original conjecture for smooth and proper analytic spaces over K or C and proved the C_{dR} -conjecture for smooth and proper analytic spaces over K . We proved the C_{st} -conjecture for proper analytic spaces with a semistable model [13] (see [29] for a simplified construction) and “de Rham-to-pro-étale” versions of the C_{st} -conjecture for Stein varieties [11] over K (also with a semistable model).

In this paper, we will consider smooth dagger varieties over K or C (without any assumption on the existence of nice models):

- Any proper or partially proper rigid analytic variety has a natural dagger structure, and this includes, in particular, analytifications of algebraic varieties, Stein varieties (e.g. étale coverings of Drinfeld’s spaces in any dimension), or, more generally, holomorphically convex varieties (proper fibrations over a Stein base).

- Dagger varieties that are not necessarily partially proper include overconvergent affinoids or, more generally, quasi-compact rigid analytic varieties with an overconvergent structure.

As we have seen in the case of algebraic varieties, Fontaine’s conjectures C_{dR} and C_{st} can be split in two: a “de Rham-to-étale” direction, and a “étale-to-de Rham” direction. We are going to state analogous conjectures for analytic varieties but with étale cohomology replaced with pro-étale cohomology (i.e., we are dealing with “rational” p -adic Hodge theory and not “integral” p -adic Hodge theory).

1.2.1. *The “de Rham-to-étale” C_{dR} and C_{st} conjectures for dagger varieties.*

Conjecture 1.4. (de Rham-to-étale $C_{\text{dR}} + C_{\text{st}}$) *Let X be a smooth dagger variety over C . Let $i \leq r$. We have a functorial bicartesian diagram:*

$$\begin{array}{ccc} H_{\text{proét}}^i(X, \mathbf{Q}_p(r)) & \longrightarrow & (H_{\text{HK}}^i(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \\ \downarrow & & \downarrow \\ H^i(F^r \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+)) & \longrightarrow & H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+) \end{array}$$

Remark 1.5. (i) The \mathbf{B}_{dR}^+ -cohomology group $H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+)$ is a torsion-free \mathbf{B}_{dR}^+ -module from which one recovers the usual de Rham cohomology by moding out by t . If X is defined over K then $H_{\text{dR}}^i(X_C/\mathbf{B}_{\text{dR}}^+) \simeq H_{\text{dR}}^i(X) \widehat{\otimes}_K \mathbf{B}_{\text{dR}}^+$.

(ii) If X is defined over K , all spaces are endowed with a natural topology and an action of \mathcal{G}_K and all maps are supposed to be \mathcal{G}_K -equivariant and continuous.

(iii) As we have shown in [15], if X is defined over C , then all spaces have a natural topology and are C -points of VS’s (pro-étale sheaves on the category Perf_C of perfectoid spaces over C) and all maps are supposed to be evaluations of morphisms of VS’s and continuous.

(iv) The Hyodo-Kato cohomology group $H_{\text{HK}}^i(X)$ is a F^{nr} -module with a Frobenius φ , a monodromy operator N , a (pro-)smooth action of \mathcal{G}_K , and a Hyodo-Kato isomorphism $\iota_{\text{HK}} : H_{\text{HK}}^i(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}^+ \xrightarrow{\sim} H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+)$. The definitions of $H_{\text{HK}}^i(X)$ and ι_{HK} are adapted from the ones of Beilinson in the algebraic setting and use the alterations of Hartl and Temkin to produce good local models.

(v) In the case of proper analytic varieties, all cohomology groups in the diagram are finite dimensional (as was the case in the algebraic setting) and the kernels of the horizontal arrows are 0. This is not the case for a general analytic variety and all spaces have to be seen in the derived category of locally convex topological vector spaces over \mathbf{Q}_p ; in particular, the tensor products are (derived) completed tensor products. Even if $H_{\text{dR}}^i(X)$, for X over K , is finite dimensional, $H^i(\text{Fil}^r(\mathbf{B}_{\text{dR}}^+ \widehat{\otimes}_K \Gamma_{\text{dR}}(X)))$ may surject onto $\Omega^i(X_C)^{d=0}$ and hence can be huge (and then so is $H_{\text{proét}}^i(X_C, \mathbf{Q}_p(r))$).

1.2.2. *The “étale-to-de Rham” C_{dR} and C_{st} conjectures for dagger varieties.* In the other direction, we have the following conjectures

Conjecture 1.6. (pro-étale-to-de Rham) *Let X be a smooth dagger variety defined over K . We have functorial isomorphisms:*

$$\begin{aligned} (C_{\text{dR}}) \quad & H_{\text{dR}}^i(X)^* \simeq \text{Hom}_{\mathcal{G}_K}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p), \mathbf{B}_{\text{dR}}), \quad \text{as filtered } K\text{-vector spaces,} \\ (C_{\text{st}}) \quad & H_{\text{HK}}^i(X)^* \simeq \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p), \mathbf{B}_{\text{st}}), \quad \text{as } (\varphi, N, \mathcal{G}_K)\text{-modules over } F^{\text{nr}}. \end{aligned}$$

Remark 1.7. As we mentioned above, even if $H_{\text{dR}}^i(X)$ is finite dimensional, $H_{\text{proét}}^i(X_C, \mathbf{Q}_p)$ is, in general, huge. Hence it is a little bit of a miracle that one could recover $H_{\text{dR}}^i(X)$ from it.

The previous conjecture uses Galois action to recover de Rham and Hyodo-Kato cohomologies from pro-étale cohomology. If X is defined over C , there is no Galois action anymore but one can use the VS structure alluded to above to recover part of the structure. This leads to the following conjecture:

Conjecture 1.8. *Let X be a smooth dagger variety defined over C . We have functorial isomorphisms²:*

$$\begin{aligned} (C_{\text{dR}}) \quad & \text{Hom}_{\mathbf{B}_{\text{dR}}^+}(H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+), \mathbf{B}_{\text{dR}}) \simeq \text{Hom}_{\text{VS}}(\mathbb{H}_{\text{proét}}^i(X, \mathbf{Q}_p), \mathbb{B}_{\text{dR}}), \quad \text{as } \mathbf{B}_{\text{dR}}\text{-modules,} \\ (C_{\text{st}}) \quad & \text{Hom}_{F^{\text{nr}}}(H_{\text{HK}}^i(X), \mathbf{B}_{\text{st}}) \simeq \text{Hom}_{\text{VS}}(\mathbb{H}_{\text{proét}}^i(X, \mathbf{Q}_p), \mathbb{B}_{\text{st}}), \quad \text{as } \mathbf{B}_{\text{st}}\text{-modules.} \end{aligned}$$

Remark 1.9. (i) It is not possible to recover filtration on the \mathbf{B}_{dR}^+ -cohomology just by looking at the cohomology level because the $t^k \mathbb{B}_{\text{dR}}^+$'s are all isomorphic as VS's whereas the $t^k \mathbf{B}_{\text{dR}}^+$'s are all distinct as \mathcal{G}_K -modules.

(ii) In the same way, φ and N disappear since $M = M^{N=0, \varphi=1} \otimes_{\mathbf{B}_{\text{dR}}^{\varphi=1}} \mathbf{B}_{\text{st}}$ if $M = M_0 \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}$, where M_0 is a finite rank (φ, N) -module over F^{nr} .

(iii) One way to understand points (i) and (ii) of the remark is the following. Conjecture 1.4 represents $H_{\text{proét}}^i$ as the H^0 of a quasi-coherent sheaf on the Fargues-Fontaine curve that is obtained as the "modification" at ∞ of the sheaf associated to a φ -module. The H^0 determines the sheaf (because there is no H^1), but not the "modification" that gave rise to it. Maybe a more sophisticated formulation on the level of derived categories would allow to do that (see Remark 8.10) ?

1.3. The main results.

1.3.1. *Quasi-compact varieties.* In the case of quasi-compact varieties the $H_{\text{HK}}^i(X)$'s and the $H^i(X/\mathbf{B}_{\text{dR}}^+)$'s have finite rank, and Conjectures 1.4, 1.6 and 1.8 are proved in large parts in this paper. More precisely, we prove the following results:

- *Smooth proper varieties.*

If X is a proper variety, then Conjecture 1.4 is true. In this case, the horizontal rows in the diagram are injective and all cohomology groups are finite dimensional as in the case of algebraic varieties; once all the constructions are generalized from semistable models to analytic varieties, the proof is basically identical to that of [13], where we treated the case of varieties with semistable models over the integers.

²In all of the paper Hom_{VS} means morphisms of VS's which are continuous on C -points.

If X is defined over K , then Conjecture 1.6 is true and if X is defined over C , Conjecture 1.8 is true.

- *Smooth overconvergent affinoids.*

If X is in an overconvergent affinoid, then Conjecture 1.4 is true. In this case, the horizontal rows in the diagram are surjective, and their kernels are $(\Omega^{i-1}(X)/\text{Ker } d)(r-i)$. In particular, we have an exact sequence

$$0 \rightarrow \Omega^{r-1}(X)/\text{Ker } d \rightarrow H_{\text{proét}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^r(X) \otimes \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \rightarrow 0$$

If X is defined over K , then Conjecture 1.6 is true (the Hodge filtration is concentrated in maximal degree in that case) and if X is defined over C , Conjecture 1.8 is true.

- *General smooth quasi-compact dagger varieties.*

If X is a quasi-compact dagger variety, then Conjecture 1.4 is true. The VS's in the diagram have some finiteness properties: they are extensions of finite Dimensional Vector Spaces (also known as Banach-Colmez spaces, and referred to as BC's in the rest of the text) by torsion \mathbb{B}_{dR}^+ -Modules. In particular, such objects \mathbb{W} (referred to as qBC's, the "q" standing for "quasi") have a height $\text{ht}(\mathbb{W}) \in \mathbf{N}$ (if \mathbb{W} is the qBC attached to a finite dimensional \mathbf{Q}_p -vector space W , then $\text{ht}(\mathbb{W}) = \dim_{\mathbf{Q}_p} W$). We have the following result echoing (ii) of Remark 1.2:

Theorem 1.10. *If X is a quasi-compact smooth dagger variety, we have*

$$\text{ht}(\mathbb{H}^i(X_C, \mathbf{Q}_p)) = \dim_C(H_{\text{dR}}^i(X_C)).$$

The fact that the diagram in Conjecture 1.4 is bicartesian implies in particular that the associated sequence is exact. The exactitude on the right can be rephrased in a way echoing (vii) of Remark 1.2:

Corollary 1.11. *If X is a quasi-compact smooth dagger variety, $(H_{\text{HK}}^i(X), H^i(X/\mathbf{B}_{\text{dR}}^+))$ is an acyclic³ filtered (φ, N) -module.*

Finally, if X is defined over K , Conjecture 1.6 is true, and if X is defined over C , Conjecture 1.8 is true.

1.3.2. *Smooth partially proper rigid analytic varieties.* A partially proper rigid analytic variety can be written as an increasing union of quasi-compact dagger varieties which makes it possible to obtain results in this case from the quasi-compact case. There are nevertheless annoying problems with $\mathbf{R}^1 \lim$'s, and for our proofs to work out, we need to impose a condition "at infinity" which we call "de Rham temperedness" and which states that the naive Hodge filtration on de Rham cohomology "does not jump at infinity" (see Section 7.3 for a precise definition and Section 7.3.2 for equivalent conditions).

Remark 1.12. (i) Proper rigid analytic varieties and analytifications of algebraic varieties are de Rham tempered. We prove that some holomorphically convex varieties are also de Rham tempered (products of proper and Stein, or smooth and proper fibrations over smooth Stein curves).

(ii) We do not know any example of a dagger variety which is not de Rham tempered and it is tempting to believe that there are none.

Our main result is then:

Theorem 1.13. *If X is a smooth partially proper variety over C which is de Rham tempered, then Conjectures 1.4 and 1.8 hold, and if X is defined over K , then Conjecture 1.6 holds.*

³This means that the associated vector bundle on the Fargues-Fontaine curve has cohomology only in degree 0. It is purely a condition on the interplay between φ , N and the filtration. Weakly admissible filtered (φ, N) -modules are acyclic.

Remark 1.14. If X is smooth and Stein over C , it can be written as a strict increasing union of overconvergent affinoids. This makes it possible to use the results for affinoids instead of general quasi-compact varieties. We get that the horizontal rows in the diagram in Conjecture 1.4 are surjective and their kernels are $(\Omega^{i-1}(X)/\text{Ker } d)(r-i)$. In particular, we have an exact sequence

$$0 \rightarrow \Omega^{r-1}(X)/\text{Ker } d \rightarrow H_{\text{proét}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^r(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \rightarrow 0$$

generalizing the exact sequence of [11, Th. 1.8] which was proven under the assumption that X has a semistable model.

1.4. Proofs. As we indicated above, the case of partially proper varieties is derived from that of quasi-compact dagger varieties, so we will concentrate on the latter case. The main ingredients in the proofs are the results from [15] and the parallel theories of BC's [8, 9, 31] and Fontaine's almost C -representations [26, 27].

Let X be a quasi-compact smooth dagger variety over C . From [15] we use the existence of the basic comparison isomorphism with syntomic cohomology:

$$(1.15) \quad H_{\text{syn}}^i(X, \mathbf{Q}_p(r)) \xrightarrow{\sim} H_{\text{proét}}^i(X, \mathbf{Q}_p(r)), \quad i \leq r,$$

and the exact sequence

$$(1.16) \quad \cdots \rightarrow \text{DR}_r^{i-1}(X) \rightarrow H_{\text{syn}}^i(X, \mathbf{Q}_p(r)) \longrightarrow \text{HK}_r^i(X) \xrightarrow{\iota_{\text{HK}}} \text{DR}_r^i(X) \rightarrow \cdots,$$

where we have set:

$$\text{HK}_r^i(X) := (H_{\text{HK}}^i(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}, \quad \text{DR}_r^i(X) := H^i(\text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+)/F^r).$$

Very important for what follows is that this exact sequence and the isomorphism (1.15) can be promoted to the category of VS's (see [15, Th. 1.3, Th. 1.7]).

1.4.1. On the proof of Conjecture 1.4. Proving Conjecture 1.4 amounts to splitting the long exact sequence (1.16) into short exact sequences, which can be done directly for dagger affinoids (Theorem 5.14), going back to the definition of syntomic cohomology. The problem is to go from affinoids to varieties covered by a finite number of affinoids. This we do by induction on the number of affinoids needed, using the theory of BC's. We already used BC's in [13] to treat the proper case (with a semistable model), but there we were helped by Scholze's theorem [38, Th. 1.1] that the $H_{\text{proét}}^i(X, \mathbf{Q}_p(r))$'s are, in that case, finite dimensional \mathbf{Q}_p -vector spaces; this made it possible to use the basic theory of BC's as developed in [8, 9]. In the present paper, we need to use the more powerful point of view advocated in Le Bras' thesis [31]. The key result that comes out of the theory of BC's is the following proposition (Proposition 6.11):

Proposition 1.17. *The following properties are equivalent:*

- (a) *The diagram in Conjecture 1.4 is bicartesian.*
- (b) *$(H_{\text{HK}}^i(X), F^0 H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}))$ is acyclic, for $i = r-1$ and $i = r$.*
- (c) *The kernel and cokernel of $\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^r(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}$ have height 0.*
- (d) *$\text{ht}(\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r))) = \dim_C H_{\text{dR}}^r(X)$.*

The equivalence of (d), which is a condition only about r , and (b), which involves r and $r-1$, makes it possible to do an induction on r . Among the ingredients that go into the proof of this proposition are an interpretation (Proposition 3.24) of $\text{ht}(\mathbb{W})$ for a qBC \mathbb{W} as the rank of $\text{Hom}_{\text{VS}}(\mathbb{W}, \mathbb{B}_{\text{dR}})$ and a dichotomy (Proposition 4.21) that tells us what happens if the sequence associated to the diagram is not exact on the right. It is in this dichotomy that we use in an essential way that the degrees of involved cohomology groups are $\leq r$: this implies that the slopes of Frobenius on Hyodo-Kato cohomology are $\leq r$.

1.4.2. *On the proof of Conjectures 1.6 and 1.8.* Conjectures 1.6 and 1.8 follow from Conjecture 1.4 and results about almost C representations or BC's of the following type:

$$\mathrm{Hom}_{\mathcal{G}_K}(\mathbf{B}_{\mathrm{dR}}^+/t^k, \mathbf{B}_{\mathrm{dR}}) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathrm{VS}}(\mathbb{B}_{\mathrm{dR}}^+/t^k, \mathbb{B}_{\mathrm{dR}}) = 0$$

(Proposition 2.14 for the first statement and Corollary 3.16 for the second; this type of results allow us to get rid of the de Rham terms in the sequence of Hom's that is deduced from the exact sequence associated to the bicartesian diagram of Conjecture 1.4.)

$$\begin{aligned} \mathrm{Hom}_{\mathcal{G}_K}((M \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r}, \mathbf{B}_{\mathrm{dR}}) &= M_K^* \\ \mathrm{Hom}_{\mathrm{VS}}((M \otimes_{F^{\mathrm{nr}}} \mathbb{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r}, \mathbb{B}_{\mathrm{dR}}) &= M^* \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{dR}} \end{aligned}$$

if M is a (φ, N) -module of slopes $\leq r$ (Theorem 4.8 and corollaries for the first statement and Proposition 4.23 and 4.25 for the second).

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1.4.3. *Notation and conventions.* Let \mathcal{O}_K be a complete discrete valuation ring with fraction field K of characteristic 0 and perfect residue field k of characteristic p . Let \overline{K} be an algebraic closure of K and let $\mathcal{O}_{\overline{K}}$ denote the integral closure of \mathcal{O}_K in \overline{K} . Let $C = \widehat{\overline{K}}$ be the p -adic completion of \overline{K} . Let $W(k)$ be the ring of Witt vectors of k with fraction field F (i.e., $W(k) = \mathcal{O}_F$); let $e = e_K$ be the ramification index of K over F . Set $\mathcal{G}_K = \mathrm{Gal}(\overline{K}/K)$ and let φ be the absolute Frobenius on $W(\overline{k})$. We will denote by $\mathbf{A}_{\mathrm{cr}}, \mathbf{B}_{\mathrm{cr}}, \mathbf{B}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}$ the crystalline, semistable, and de Rham period rings of Fontaine.

We will denote by $\mathcal{O}_K, \mathcal{O}_K^\times,$ and $\mathcal{O}_K^0,$ depending on the context, the scheme $\mathrm{Spec}(\mathcal{O}_K)$ or the formal scheme $\mathrm{Spf}(\mathcal{O}_K)$ with the trivial, the canonical (i.e., associated to the closed point), and the induced by $\mathbf{N} \rightarrow \mathcal{O}_K, 1 \mapsto 0,$ log-structure, respectively. Unless otherwise stated all formal schemes are p -adic, locally of finite type, and equidimensional. For a (p -adic formal) scheme X over $\mathcal{O}_K,$ let X_0 denote the special fiber of $X;$ let X_n denote its reduction modulo $p^n.$

All rigid analytic spaces considered will be over K or $C.$ We assume that they are separated, taut, and countable at infinity.

Our cohomology groups will be equipped with a canonical topology. To talk about it in a systematic way, we will work rationally in the category of locally convex K -vector spaces and integrally in the category of pro-discrete \mathcal{O}_K -modules. For details the reader may consult [11, Sec. 2.1, 2.3]. To summarize quickly:

- (1) C_K is the category of convex K -vector spaces; it is a quasi-abelian category. We will denote the left-bounded derived ∞ -category of C_K by $\mathcal{D}(C_K).$ A morphism of complexes that is a quasi-isomorphism in $\mathcal{D}(C_K),$ i.e., its cone is strictly exact, will be called a *strict quasi-isomorphism.* The associated cohomology objects are denoted by $\tilde{H}^n(E) \in LH(C_K);$ they are called *classical* if the natural map $\tilde{H}^n(E) \rightarrow H^n(E)$ is an isomorphism.
- (2) Objects in the category PD_K of pro-discrete \mathcal{O}_K -modules are topological \mathcal{O}_K -modules that are countable inverse limits, as topological \mathcal{O}_K -modules, of discrete \mathcal{O}_K -modules $M^i,$ $i \in \mathbf{N}.$ It is a quasi-abelian category. Inside PD_K we distinguish the category PC_K of pseudocompact \mathcal{O}_K -modules, i.e., pro-discrete modules $M \simeq \lim_i M_i$ such that each M_i is of finite length (we note that if K is a finite extension of \mathbf{Q}_p this is equivalent to M being profinite). It is an abelian category.

- (3) There is a tensor product functor from the category of pro-discrete \mathcal{O}_K -modules to convex K -vector spaces:

$$(-) \otimes K : PD_K \rightarrow C_K, \quad M \mapsto M \otimes_{\mathcal{O}_K} K.$$

Since C_K admits filtered inductive limits, the functor $(-) \otimes K$ extends to a functor $(-) \otimes K : \text{Ind}(PD_K) \rightarrow C_K$. The functor $(-) \otimes K$ is right exact but not, in general, left exact. We will consider its (compatible) left derived functors

$$(-) \otimes^L K : \mathcal{D}^-(PD_K) \rightarrow \text{Pro}(\mathcal{D}^-(C_K)), \quad (-) \otimes^L K : \mathcal{D}^-(\text{Ind}(PD_K)) \rightarrow \text{Pro}(\mathcal{D}^-(C_K)).$$

If E is a complex of torsion free and p -adically complete (i.e., $E \simeq \lim_n E/p^n$) modules from PD_K then the natural map

$$E \otimes^L K \rightarrow E \otimes K$$

is a strict quasi-isomorphism [11, Prop. 2.6].

Unless otherwise stated, we work in the derived (stable) ∞ -category $\mathcal{D}(A)$ of left-bounded complexes of a quasi-abelian category A (the latter will be clear from the context). Many of our constructions will involve (pre)sheaves of objects from $\mathcal{D}(A)$. We will use a shorthand for certain homotopy limits: if $f : C \rightarrow C'$ is a map in the derived ∞ -category of a quasi-abelian category, we set

$$[C \xrightarrow{f} \gg C'] := \text{holim}(C \rightarrow C' \leftarrow 0).$$

For an operator F acting on C , we will use the brackets $[C]^F$ to denote the derived eigenspaces and the brackets $(C)^F$ or simply C^F to denote the non-derived ones.

Finally, we will use freely the notation and results from [14].

2. REVIEW OF ALMOST C -REPRESENTATIONS

We will briefly review Fontaine's theory of almost C -representations [26] (see also [27]) and some of its consequences. The theory has a satisfactory shape only when $[K : \mathbf{Q}_p] < \infty$, but some parts work for K arbitrary. Fortunately, the almost C -representations that we are going to deal with have special features and we are only going to use the results in §2.3 for which we provide alternative proofs (working for general K).

2.1. Notation. A *banach* is a Banach space over \mathbf{Q}_p (up to an equivalence of norms) and a *banach representation* of \mathcal{G}_K is a banach with a continuous and linear action of \mathcal{G}_K . Denote by $\mathcal{B}(\mathcal{G}_K)$ the category of banach representations of \mathcal{G}_K . It has a natural exact category structure: a *short exact sequence* in $\mathcal{B}(\mathcal{G}_K)$ is a sequence

$$0 \rightarrow B_1 \xrightarrow{f} B_2 \xrightarrow{g} B_3 \rightarrow 0,$$

where g is a strict epimorphism and f is a kernel of g .

A \mathbf{Q}_p -*representation* of \mathcal{G}_K is a finite dimensional \mathbf{Q}_p -vector space with a continuous and linear action of \mathcal{G}_K . Similarly, a C -*representation* of \mathcal{G}_K is a finite dimensional C -vector space with a continuous and semilinear action of \mathcal{G}_K . We will denote by $\text{Rep}_{\mathbf{Q}_p}(\mathcal{G}_K)$, resp. $\text{Rep}_C(\mathcal{G}_K)$, the category of \mathbf{Q}_p -representations, resp. C -representations. More generally, one can define the category $\text{Rep}_{\mathbf{B}_{\text{dR}}^+}(\mathcal{G}_K)$ of finite length \mathbf{B}_{dR}^+ -representations.

Fontaine proved the following surprising theorem:

Theorem 2.1. (Fontaine, [26, Th. A, Th. A']) *If $[K : \mathbf{Q}_p] < \infty$, the forgetful functors*

$$\text{Rep}_C(\mathcal{G}_K) \rightarrow \mathcal{B}(\mathcal{G}_K), \quad \text{Rep}_{\mathbf{B}_{\text{dR}}^+}(\mathcal{G}_K) \rightarrow \mathcal{B}(\mathcal{G}_K)$$

are fully faithful.

In other words, if W_1, W_2 are two C -representations of \mathcal{G}_K , all \mathbf{Q}_p -linear continuous \mathcal{G}_K -equivariant maps, of W_1 to W_2 are necessarily C -linear. Similarly, if W_1, W_2 are two \mathbf{B}_{dR}^+ -representations of \mathcal{G}_K , all \mathbf{Q}_p -linear continuous \mathcal{G}_K -equivariant maps, of W_1 to W_2 are necessarily \mathbf{B}_{dR}^+ -linear.

Remark 2.2. The proof uses Sen's theory [39] and gives a stronger result: one has the same statements for E -linear maps between E -representations for which the Hodge-Tate-Sen weights are algebraic over E (a condition that is automatic for $E = \mathbf{Q}_p$, if $[K : \mathbf{Q}_p] < \infty$, whence the theorem). In particular, we have the following fundamental result, valid for arbitrary K :

$$\text{Hom}_{\mathcal{G}_K}(C, C) \simeq K$$

2.2. Almost C -representations.

2.2.1. *The general theory.* Two banach representations W_1 and W_2 are *almost isomorphic* if there exist two finite dimensional \mathbf{Q}_p -vector spaces $V_i \subset W_i$, $i = 1, 2$, stable under \mathcal{G}_K such that $W_1/V_1 \simeq W_2/V_2$. An *almost C -representation* is a banach representation which is almost isomorphic to C^d for some $d \in \mathbf{N}$. Denote by $\mathcal{C}(\mathcal{G}_K)$ the category of almost C -representations.

Remark 2.3. The above definition makes sense for arbitrary K , but the theory has a satisfactory shape only for $[K : \mathbf{Q}_p] < \infty$; maybe the point of view developed in [27] would lead to satisfactory theory for arbitrary K ?

From now on, assume that $[K : \mathbf{Q}_p] < \infty$. Then $\mathcal{C}(\mathcal{G}_K)$ is an abelian subcategory of the exact category $\mathcal{B}(\mathcal{G}_K)$. If $W/V_2 \cong C^d/V_1$, one sets $\dim W = d$ and $\text{ht}(W) = \dim_{\mathbf{Q}_p} V_2 - \dim_{\mathbf{Q}_p} V_1$. This is independent of choices and yields additive functions [26, Th. B] – a nontrivial fact whose proof uses the theory of BC's⁴ [8], [9]. We have $d(C) = 1$, $\text{ht}(C) = 0$ and $d(V) = 0$, $\text{ht}(V) = \dim_{\mathbf{Q}_p}(V)$ if V is a \mathbf{Q}_p -representation. The category $\mathcal{C}(\mathcal{G}_K)$ contains all \mathbf{B}_{dR}^+ -representations and if W is a \mathbf{B}_{dR}^+ -representation of length d then it is almost isomorphic to C^d [26, Th. C]; we have $d(W) = d$ and $\text{ht}(W) = 0$. In particular, the \mathbf{B}_{dR}^+ -representations C and $C(1)$ are almost isomorphic.

Remark 2.4. The category $\mathcal{C}(\mathcal{G}_K)$ modulo almost isomorphisms is semi-simple with a single isomorphism class of simple objects, the class of C .

Fontaine reduced the computation of Ext-groups in the category $\mathcal{C}(\mathcal{G}_K)$ to the computation of Ext-groups in the category $\text{Rep}_{\mathbf{B}_{\text{dR}}^+}(\mathcal{G}_K)$ and the computation of Ext-groups in the category $\text{Rep}_{\mathbf{Q}_p}(\mathcal{G}_K)$ via the following fact (which relies on [26, Prop. 5.5] and [26, Prop. 5.6]):

Proposition 2.5. (Fontaine, [26, Prop. 6.4, Prop. 6.5])

(i) Let $X, Y \in \text{Rep}_{\mathbf{Q}_p}(\mathcal{G}_K)$. Then we have a canonical isomorphism

$$\text{Ext}_{\mathcal{G}_K}^i(X, Y) \xrightarrow{\sim} \text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^i(X, Y), \quad i \geq 0.$$

(ii) Let $X, Y \in \text{Rep}_{\mathbf{B}_{\text{dR}}^+}(\mathcal{G}_K)$. Then we have a canonical isomorphism

$$\text{Ext}_{\mathbf{B}_{\text{dR}}^+(\mathcal{G}_K)}^i(X, Y) \xrightarrow{\sim} \text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^i(X, Y), \quad i \geq 0.$$

He proved the following result:

Theorem 2.6. (Fontaine, [26, Th. 6.1, Prop. 6.8, Prop. 6.9]) Let $X, Y \in \mathcal{C}(\mathcal{G}_K)$.

(i) The \mathbf{Q}_p -vector spaces $\text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^i(X, Y)$ have finite rank and are trivial for $i \geq 3$.

(ii) $\sum_{i=0}^2 (-1)^i \dim_{\mathbf{Q}_p} \text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^i(X, Y) = -[K : \mathbf{Q}_p] \text{ht}(X) \text{ht}(Y)$.

(iii) There exists a natural trace map $\text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^2(X, X(1)) \rightarrow \mathbf{Q}_p$ and, for $0 \leq i \leq 2$, the map

$$\text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^i(X, Y) \times \text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^{2-i}(Y, X(1)) \rightarrow \text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^2(X, X(1)) \rightarrow \mathbf{Q}_p$$

⁴More specifically, the proof uses the result which says that, if S is an effective BC of dimension 1 and height h and if $f : S \rightarrow C$ is a morphism of BC's whose image is not finite-dimensional, then f is surjective and its kernel has dimension 0 and height h .

defines a perfect duality.

2.2.2. \mathbf{B}_{dR}^+ -representations. \mathbf{B}_{dR}^+ -representations are objects of $\mathcal{C}(\mathcal{G}_K)$ and we have a recipe for computing Ext groups between these objects. We still assume $[K : \mathbf{Q}_p] < \infty$, but the results below are valid in greater generality (see Remark 2.2).

Let χ be the cyclotomic character. Let $K_\infty \subset K(\mu_{p^\infty})$ denote the cyclotomic \mathbf{Z}_p -extension of K . Let γ be a topological generator of $\text{Gal}(K_\infty/K)$. We choose a sequence $\{\zeta_{p^n}\}_{n \geq 1}$ of primitive p^n 'th roots of unity $\zeta_{p^n}, n \geq 1$, such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. Let $t \in \mathbf{B}_{\text{dR}}^+$ be the uniformizer associated to $\{\zeta_{p^n}\}_{n \geq 1}$. We will also use its twisted form $t' := t/\pi_t$ defined in [26, Sec. 2.1]; it is a uniformizer of \mathbf{B}_{dR}^+ as well, fixed by $\text{Gal}(\overline{K}/K_\infty)$ whereas t is only fixed by $\text{Gal}(\overline{K}/K(\mu_{p^\infty}))$.

Proposition 2.7. *Let W be a \mathbf{B}_{dR}^+ -representation. The groups $\text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^i(C, W)$ are computed by the complex*

$$W_{(0)} \xrightarrow{x \mapsto (t'x, (\gamma-1)x)} W_{(1)} \oplus W_{(0)} \xrightarrow{(x,y) \mapsto (t'y - (\chi^{-1}(\gamma)\gamma-1)x)} W_{(1)},$$

where $W_{(0)}$ is the space of generalized invariants⁵ and $W_{(1)} = t'((t')^{-1}W)_{(0)}$.

Proof. By Proposition 2.5, the natural map

$$(2.8) \quad \text{Ext}_{\mathbf{B}_{\text{dR}}^+(\mathcal{G}_K)}^i(C, W) \rightarrow \text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^i(C, W)$$

from the Ext-groups in the category of \mathbf{B}_{dR}^+ -representations is an isomorphism. Our proposition is now [26, Th. 2.14]. \square

We will state two simple consequences of the above proposition.

Corollary 2.9. *If $N \geq 0$ and $1 \leq i < j$, then*

$$\text{Hom}_{\mathcal{C}(\mathcal{G}_K)}(\mathbf{B}_{\text{dR}}^+/t^i, t^{-N}\mathbf{B}_{\text{dR}}^+/t^j) = 0 \quad \text{and} \quad \text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^1(\mathbf{B}_{\text{dR}}^+/t^i, t^{-N}\mathbf{B}_{\text{dR}}^+/t^j) = 0.$$

Proof. By devissage (and twisting by χ^{-j} for $0 \leq j \leq i-1$), we can reduce to the case $i = 1$. If $W = t^{-N}\mathbf{B}_{\text{dR}}^+/t^j$, we have $W_{(0)} = K$ and $W_{(1)} = Kt'$, and our result follows from Proposition 2.7 (multiplication by t' induces an isomorphism of $W_{(0)}$ with $W_{(1)}$ and the other maps are identically zero, and it follows that the complex in the proposition is acyclic). \square

Lemma 2.10. *Let $j \in \mathbf{Z}, k \geq 1$. Then*

$$\begin{aligned} \text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^0(\mathbf{B}_{\text{dR}}^+/t^k, C(j)) &\simeq \begin{cases} 0 & \text{if } j \neq 0 \\ K & \text{if } j = 0, \end{cases} \\ \text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^1(\mathbf{B}_{\text{dR}}^+/t^k, C(j)) &\simeq \begin{cases} 0 & \text{if } j \neq 0, k \\ K & \text{if } j = 0, k, \end{cases} \\ \text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^2(\mathbf{B}_{\text{dR}}^+/t^k, C(j)) &\simeq \begin{cases} 0 & \text{if } j \neq k \\ K & \text{if } j = k. \end{cases} \end{aligned}$$

Proof. We will use Proposition 2.7 with $W = (\mathbf{B}_{\text{dR}}^+/t^k)(1-j)$ as well as the duality between $\text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^i(C, W)$ and $\text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^{2-i}(W, C(1))$ from Theorem 2.6. We have $W_{(0)} = Kt^{j-1}(1-j)$ if $1 \leq j \leq k$, $W_{(0)} = 0$ if $j \leq 0$ or $j \geq k+1$, and $W_{(1)} \simeq Kt't^{j-1}(1-j)$ if $0 \leq j \leq j-1$, $W_{(1)} = 0$ if $j \leq -1$ or $j \geq k$. In the complex from Proposition 2.7 computing the Ext-groups $\text{Ext}_{\mathcal{C}(\mathcal{G}_K)}^i(C, W)$, the only nonzero maps are the multiplications by $t' : W_{(0)} \rightarrow W_{(1)}$ which are isomorphisms unless exactly one of the two groups is trivial (i.e., $j = 0$ or $j = k$). Our result follows. \square

Remark 2.11. Every non-trivial extension of $\mathbf{B}_{\text{dR}}^+/t^k$ by $C(k)$ is isomorphic to $\mathbf{B}_{\text{dR}}^+/t^{k+1}$.

⁵The elements killed by $(g_1 - 1) \cdots (g_r - 1)$ for all $g_1, \dots, g_r \in \mathcal{G}_K$, for r big enough.

Example 2.12. *Extensions of Tate twists.* We have $\mathrm{Ext}_{\mathcal{G}(\mathcal{G}_K)}^1(C, C) \simeq K$ (the K -vector space is generated by the class of $C \otimes (\mathbf{Q}_p \oplus \mathbf{Q}_p \log t)$). Since $\mathrm{Ext}_{\mathcal{G}(\mathcal{G}_K)}^0(C, C) \simeq K$ by Remark 2.2, this implies that $\mathrm{Ext}_{\mathcal{G}(\mathcal{G}_K)}^2(C, C) = 0$ by Theorem 2.6. We also have, by duality, $\mathrm{Ext}_{\mathcal{G}(\mathcal{G}_K)}^1(C, C(1)) \simeq K$ (generated by the class of $\mathbf{B}_{\mathrm{dR}}^+/t^2$) and $\mathrm{Ext}_{\mathcal{G}(\mathcal{G}_K)}^2(C, C(1)) \simeq K$, and all the other $\mathrm{Ext}_{\mathcal{G}(\mathcal{G}_K)}^i(C, C(j))$ are trivial.

2.3. Morphisms of $\mathbf{B}_{\mathrm{dR}}^+$ -representations. In this section, K is arbitrary. We are going to derive consequences of the following two results which are valid for such K . These are the results that we will use in the rest of the paper.

Proposition 2.13. (i) $\mathrm{Hom}_{\mathcal{G}_K}(C, C) \simeq K$.

(ii) \overline{K} is dense in $\mathbf{B}_{\mathrm{dR}}^+$.

Proof. Fontaine's proof [25] of (i) works for arbitrary K (the alternative proof in [28], which uses class field theory, works only for $[K : \mathbf{Q}_p] < \infty$). For (ii), see [10]. \square

Proposition 2.14. *We have*

(i) $\mathrm{Hom}_{\mathcal{G}_K}(C, C) \simeq K$, $\mathrm{Hom}_{\mathcal{G}_K}(C, C(j)) = 0$ if $j \neq 0$.

(ii) $\mathrm{Hom}_{\mathcal{G}_K}(\mathbf{B}_{\mathrm{dR}}^+, t^{-N}\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+) \simeq K$ for $N \geq 0$, $k \geq 1$; compatibly in k .

(iii) $\mathrm{Hom}_{\mathcal{G}_K}(\mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}^+) \simeq K$, $\mathrm{Hom}_{\mathcal{G}_K}(\mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{dR}}) \simeq K$.

(iv) $\mathrm{Hom}_{\mathcal{G}_K}(t^j\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}) = 0$, $\mathrm{Hom}_{\mathcal{G}_K}(\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}^+/t^\ell\mathbf{B}_{\mathrm{dR}}^+) = 0$ if $\ell > k$.

Proof. The first claim of (i) is Fontaine's theorem. The second claim of (i) follows from the fact that $C(j)$ does not have elements on which the action of \mathcal{G}_K factors through a finite quotient and hence $\lambda \in \mathrm{Hom}_{\mathcal{G}_K}(C, C(j))$ is identically zero on \overline{K} , and thus on C by continuity and density of \overline{K} in C .

Let us prove (ii). Any map $\lambda : \mathbf{B}_{\mathrm{dR}}^+ \rightarrow t^{-N}\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+$ send \overline{K} to \overline{K} since elements of \overline{K} are the smooth vectors for the action of \mathcal{G}_K in $\mathbf{B}_{\mathrm{dR}}^+$ and $t^{-N}\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+$. By continuity, and density of \overline{K} in $\mathbf{B}_{\mathrm{dR}}^+$ one sees that $\lambda(\mathbf{B}_{\mathrm{dR}}^+) \subset \mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+$.

Now, if $k = 1$, $\lambda(\overline{K})$ is sent to 0 in $\mathbf{B}_{\mathrm{dR}}^+/t = C$ since $C(-1)$ does not have elements on which the action of \mathcal{G}_K factors through a finite quotient. But \overline{K} is dense in $t\mathbf{B}_{\mathrm{dR}}^+$, hence λ factors through $\mathbf{B}_{\mathrm{dR}}^+/t = C$. It follows from (i) that λ is the multiplication by $\kappa \in K$ (composed with $\mathbf{B}_{\mathrm{dR}}^+ \rightarrow C$). Now, if $k \geq 1$, we can compose with $\mathbf{B}_{\mathrm{dR}}^+/t^k \rightarrow C$ to deduce that there exist $\kappa \in K$ such that $\lambda - \kappa$ has values in $t\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+$. But the image of \overline{K} by $\lambda - \kappa$ is identically 0 for the same reasons as above; hence $\lambda = \kappa$ and we are done.

The first statement of (iii) follows from (ii) since

$$\mathrm{Hom}_{\mathcal{G}_K}(\mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}^+) = \lim_k \mathrm{Hom}_{\mathcal{G}_K}(\mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+)$$

For the second statement, let $\lambda \in \mathrm{Hom}_{\mathcal{G}_K}(\mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{dR}})$. By the same arguments as above, $\lambda(\overline{K}) \subset \overline{K}$; hence, by continuity, $\lambda(\mathbf{B}_{\mathrm{dR}}^+) \subset \mathbf{B}_{\mathrm{dR}}^+$ and the restriction of λ to $\mathbf{B}_{\mathrm{dR}}^+$ is the multiplication by an element κ of K . But λ_N defined by $\lambda_N(x) = t^N\lambda(t^{-N}x)$ also belongs to $\mathrm{Hom}_{\mathcal{G}_K}(\mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{dR}})$ and is the multiplication by κ on $t^N\mathbf{B}_{\mathrm{dR}}^+$. It follows that λ_N is the multiplication by κ on $\mathbf{B}_{\mathrm{dR}}^+$, and λ is the multiplication by κ on $t^{-N}\mathbf{B}_{\mathrm{dR}}^+$. Hence the second claim of (iii).

Finally, for (iv) we may assume $j = 0$ by twisting. Then the same arguments as for (ii) show that $\lambda(\mathbf{B}_{\mathrm{dR}}^+/t^k) \subset \mathbf{B}_{\mathrm{dR}}^+$ for the first claim, and that $\theta \circ \lambda$ is $\kappa\theta$ for some $\kappa \in K$ for both claims. One deduces that $\lambda = \kappa$ on \overline{K} . But we can find a sequence $(x_n)_{n \in \mathbf{N}}$ of elements of \overline{K} converging to t^k in $\mathbf{B}_{\mathrm{dR}}^+$ hence to 0 in $\mathbf{B}_{\mathrm{dR}}^+/t^k$, and continuity of λ implies that $0 = \lambda(0) = \kappa t^k$. Hence $\kappa = 0$ (since $\ell > k$ for the second claim), and $\lambda = 0$, which finishes the proof. \square

3. THE CATEGORIES \mathcal{BC} AND $q\mathcal{BC}$

In this chapter we recall the definition of BC's⁶ and introduce qBC's (categories \mathcal{BC} and $q\mathcal{BC}$). We will study properties of both categories, in particular, the canonical filtration and its relation to the Harder-Narasimhan filtration. Moreover, we will partially categorify height of BC's and introduce a notion of acyclic (φ, N) -modules that will play an important role later on in the paper.

3.1. The category \mathcal{BC} . We will discuss now basic properties of the category \mathcal{BC} of BC's and the canonical filtration of its objects.

3.1.1. Definitions and basic properties. Recall [8] that a BC⁷ \mathbb{W} is, morally, a finite dimensional C -vector space up to a finite dimensional \mathbf{Q}_p -vector space. It has a Dimension $\text{Dim } \mathbb{W} = (a, b)$, where $a = \dim \mathbb{W} \in \mathbf{N}$, the *dimension of \mathbb{W}* , is the dimension of the C -vector space and $b = \text{ht } \mathbb{W} \in \mathbf{Z}$, the *height of \mathbb{W}* , is the dimension of the \mathbf{Q}_p -vector space.

More precisely, a *Vector Space* (VS for short) \mathbb{W} is a pro-étale sheaf of \mathbf{Q}_p -vector spaces on $\text{Perf}_C: \Lambda \mapsto \mathbb{W}(\Lambda)$. Trivial examples of VS's are:

- finite dimensional \mathbf{Q}_p -vector spaces $V: \Lambda \mapsto V$ for all Λ ,
- \mathbb{V}^d , for $d \in \mathbf{N}$, with $\mathbb{V}^d(\Lambda) = \Lambda^d$, for all Λ .

More interesting examples are provided by Fontaine's rings:

- $\mathbb{B}_{\text{cris}}^+$, \mathbb{B}_{st}^+ , \mathbb{B}_{dR}^+ , \mathbb{B}_{cris} , \mathbb{B}_{st} , \mathbb{B}_{dR} are naturally VS's (and even Rings).
- If $m \geq 1$, then $\mathbb{B}_m := \mathbb{B}_{\text{dR}}^+ / t^m \mathbb{B}_{\text{dR}}^+$ is a VS (and also a Ring).
- Let $h \geq 1$ and $d \in \mathbf{Z}$. Then $\mathbb{U}_{h,d} = (\mathbb{B}_{\text{cris}}^+)^{\varphi^h = p^d}$ if $d \geq 0$, and $\mathbb{U}_{h,d} = \mathbb{B}_d / \mathbf{Q}_p^h$ if $d < 0$, are VS's.

A Vector Space \mathbb{W} is said to be finite Dimensional (a BC for short) if it "is equal to \mathbb{V}^d , for some $d \in \mathbf{N}$, up to finite dimensional \mathbf{Q}_p -vector spaces": there exists finite dimensional \mathbf{Q}_p -vector spaces V_1, V_2 and exact sequences⁸

$$0 \rightarrow V_1 \rightarrow \mathbb{Y} \rightarrow \mathbb{V}^d \rightarrow 0, \quad 0 \rightarrow V_2 \rightarrow \mathbb{Y} \rightarrow \mathbb{W} \rightarrow 0,$$

so that \mathbb{W} is obtained from \mathbb{V}^d by "adding V_1 and moding out by V_2 ". Then $\dim \mathbb{W} = d$ and $\text{ht } \mathbb{W} = \dim_{\mathbf{Q}_p} V_1 - \dim_{\mathbf{Q}_p} V_2$.

Remark 3.1. (i) We are, in general, only interested in $W = \mathbb{W}(C)$ but, without the extra structure, it would be impossible to speak of its Dimension (for example, C and $C \oplus \mathbf{Q}_p$ are isomorphic as topological \mathbf{Q}_p -vector spaces).

(ii) The functor $\mathbb{W} \mapsto \mathbb{W}(C)$ of C -points is faithful for BC's. Also, if \mathbb{W} is a BC, then $\mathbb{W}(\Lambda)$ is a \mathbf{Q}_p -Banach, for all Λ , and if $\mathbb{W}_1 \rightarrow \mathbb{W}_2$ is a morphism of BC's, then $\mathbb{W}_1(\Lambda) \rightarrow \mathbb{W}_2(\Lambda)$ is continuous and strict, for all Λ .

We quote [13, Prop. 5.16]:

Proposition 3.2. (i) *The Dimension of a BC is independent of the choices made in its definition.*

(ii) *If $f: \mathbb{W}_1 \rightarrow \mathbb{W}_2$ is a morphism of BC's, then $\text{Ker } f$, $\text{Coker } f$ and $\text{Im } f$ are BC's, and we have*

$$\text{Dim } \mathbb{W}_1 = \text{Dim } \text{Ker } f + \text{Dim } \text{Im } f \quad \text{and} \quad \text{Dim } \mathbb{W}_2 = \text{Dim } \text{Coker } f + \text{Dim } \text{Im } f.$$

(iii) *If $\dim \mathbb{W} = 0$, then $\text{ht } \mathbb{W} \geq 0$.*

(iv) *If \mathbb{W} has an increasing filtration such that the successive quotients are \mathbb{V}^1 , and if \mathbb{W}' is a sub-BC of \mathbb{W} , then $\text{ht } \mathbb{W}' \geq 0$.*

⁶Often called Banach-Colmez spaces.

⁷Called in [8] "finite dimensional Banach Space".

⁸In fact, by [31, Prop. 7.8], a sequence $0 \rightarrow \mathbb{W}_1 \rightarrow \mathbb{W}_2 \rightarrow \mathbb{W}_3 \rightarrow 0$ of BC's is exact if and only if $0 \rightarrow \mathbb{W}_1(\Lambda) \rightarrow \mathbb{W}_2(\Lambda) \rightarrow \mathbb{W}_3(\Lambda) \rightarrow 0$ is exact for all sympathetic algebras Λ . This implies that the latter sequence is actually strictly exact.

We will denote by \mathcal{BC} the category of BC's. It is an abelian category.

Example 3.3. The Spaces \mathbb{B}_m and $\mathbb{U}_{h,d}$ defined above are BC's. Their Dimensions are

$$\text{Dim } \mathbb{B}_m = (m, 0), \quad \text{Dim } \mathbb{U}_{h,d} = \begin{cases} (d, h) & \text{if } d \geq 0, \\ (-d, -h) & \text{if } d < 0. \end{cases}$$

3.1.2. *Canonical filtration.* In his thesis [35], [36], Plût introduced a filtration on objects of \mathcal{BC} and stated a number of results about this filtration. We will review them briefly here.

Remark 3.4. Most of these results can be recovered from the relation of \mathcal{BC} to vector bundles on the Fargues-Fontaine curve and the Harder-Narasimhan filtration studied in Le Bras' thesis (see Section 3.2.4).

Definition 3.5. (*Curvature*) Let $\mathbb{W} \in \mathcal{BC}$. We say that \mathbb{W} has:

- curvature > 0 , if $\text{Hom}(\mathbb{W}, \mathbb{V}^1) = 0$,
- curvature $= 0$, or is *affine*, if it is a successive extension of \mathbb{V}^1 ,
- curvature < 0 , if it injects into \mathbb{B}_{dR}^d (or, equivalently, into $(\mathbb{B}_{\text{dR}}^+)^d$),
- curvature ≤ 0 , if it injects into a \mathbb{B}_{dR}^+ -module,
- curvature ≥ 0 , if $\text{Hom}(\mathbb{W}, \mathbb{B}_{\text{dR}}^+) = 0$.

Remark 3.6. (i) If \mathbb{W} has curvature > 0 (resp. ≥ 0) and \mathbb{W}' has curvature ≤ 0 (resp. < 0), then $\text{Hom}_{\mathcal{BC}}(\mathbb{W}, \mathbb{W}') = 0$.

(ii) A sub-VS of an VS with curvature ≤ 0 (resp. < 0) has curvature ≤ 0 (resp. < 0).

(iii) A quotient of an VS with curvature ≥ 0 (resp. > 0) has curvature ≥ 0 (resp. > 0).

Proposition 3.7. (The canonical filtration) *Every $\mathbb{W} \in \mathcal{BC}$ have a unique filtration, called the canonical filtration,*

$$\mathbb{W}_{>0} \subset \mathbb{W}_{\geq 0} \subset \mathbb{W}$$

such that:

- $\mathbb{W}_{>0}$ has curvature > 0 ,
- $\mathbb{W}_{\geq 0}/\mathbb{W}_{>0}$ has curvature 0,
- $\mathbb{W}/\mathbb{W}_{\geq 0}$ has curvature < 0 .

Proof. One defines $\mathbb{W}_{>0}$ as the intersection of the kernels of all morphisms $\alpha : \mathbb{W} \rightarrow \mathbb{B}_m$, for $m \geq 1$, and $\mathbb{W}_{\geq 0}$ as the intersection of the kernels of all morphisms $\alpha : \mathbb{W} \rightarrow \mathbb{B}_{\text{dR}}$. \square

Remark 3.8. (i) $\mathbb{W}_{\leq 0} := \mathbb{W}/\mathbb{W}_{>0}$ is the largest quotient of curvature ≤ 0 of \mathbb{W} .

(ii) $\mathbb{W}^{=0} := \mathbb{W}_{\geq 0}/\mathbb{W}_{>0}$ is the largest affine sub-VS of $\mathbb{W}^{\leq 0}$.

3.2. **The category \mathcal{BC} and coherent sheaves on the Fargues-Fontaine curve.** The canonical filtration of BC's is closely related to the Harder-Narasimhan filtration defined, using the presentation of \mathcal{BC} via coherent sheaves on the Fargues-Fontaine curve, by Le Bras [31]. We will now explain this relation.

3.2.1. *The Fargues-Fontaine curve.* The (algebraic) Fargues-Fontaine curve $X = X_{\text{FF}}$ is the projective scheme attached to the graded \mathbf{Q}_p -algebra $\bigoplus_{d \geq 0} \mathbb{U}_d$, where $\mathbb{U}_d = (\mathbf{B}_{\text{cris}}^+)^{\varphi=p^d}$. The closed points of X are in bijection with the \mathbf{Q}_p -lines of \mathbb{U}_1 ; if $x \in X$, we fix a basis t_x of the corresponding line. The field C corresponds to a specific point ∞ of X , and $t = t_\infty$ is Fontaine's p -adic $2\pi i$. We denote by C_x the residue field at x ; it is an algebraically close field, complete for v_p , and $C_x^b = C^b$, but C_x is not necessarily isomorphic to C . The residue field at ∞ is C itself.

The completed local ring $\widehat{\mathcal{O}}_{X,x}$ at x is the ring $\mathbb{B}_{\text{dR}}^+(C_x)$, and t_x is a uniformizer (if $x = \infty$, then $\mathbb{B}_{\text{dR}}^+(C_x) = \mathbf{B}_{\text{dR}}^+$).

3.2.2. *Harder-Narasimhan categories.* A *Harder-Narasimhan category* is an exact category with two real valued functions rk and deg (rank and degree) – which are additive in short exact sequences – satisfying extra conditions (see [19, 5.5.1], [1]). This allows to define a slope function $\mu = \frac{\text{deg}}{\text{rk}}$ taking values in $\mathbf{R} \cup \{\pm\infty\}$ (endowed with the obvious ordering).

An object \mathcal{E} is *of slope* λ if $\mu(\mathcal{E}) = \lambda$. It is *semistable* if $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$ for all strict subobjects $\mathcal{E}' \subset \mathcal{E}$. It is *stable* if $\mu(\mathcal{E}') < \mu(\mathcal{E})$ for all strict subobjects $\mathcal{E}' \subset \mathcal{E}$. Any object \mathcal{E} has a canonical decreasing filtration (the *Harder-Narasimhan filtration*) by strict subobjects $\mathcal{E}^{\geq \lambda}$ (with $\mathcal{E}^{\geq \lambda} \subset \mathcal{E}^{\geq \mu}$ if $\lambda \geq \mu$), such that, if $\mathcal{E}^{> \lambda} = \cup_{\mu > \lambda} \mathcal{E}^{\geq \mu}$, then $\mathcal{E}^{\geq \lambda} / \mathcal{E}^{> \lambda}$ is semistable of slope λ .

We say that \mathcal{E} has slopes $\geq \lambda$ (resp. $> \lambda$) if $\mathcal{E} = \mathcal{E}^{\geq \lambda}$ (resp. $\mathcal{E} = \mathcal{E}^{> \lambda}$), and has slopes $\leq \lambda$ (resp. $< \lambda$) if $\mathcal{E}^{> \lambda} = 0$ (resp. $\mathcal{E}^{\geq \lambda} = 0$).

This has a number of consequences:

- If \mathcal{E}_1 is of slopes $\geq \lambda$ and \mathcal{E}_2 is of slopes $< \lambda$, then $\text{Hom}(\mathcal{E}_1, \mathcal{E}_2) = 0$.
- A quotient of an object of slopes $\geq \lambda$ has slopes $\geq \lambda$.
- A subobject of an object of slopes $\leq \lambda$ has slopes $\leq \lambda$.
- If $\lambda_1 \leq \lambda_2$, if \mathcal{E}_1 is of slopes $\geq \lambda_1$ and \mathcal{E}_2 of slopes $\leq \lambda_2$, then an extension of \mathcal{E}_2 by \mathcal{E}_1 has slopes in $[\lambda_1, \lambda_2]$.

3.2.3. *Vector bundles.* To a vector bundle \mathcal{E} on X , one can attach its rank $\text{rk}(\mathcal{E}) \in \mathbf{N}$, its degree $\text{deg}(\mathcal{E}) \in \mathbf{Z}$, and its slope $\mu(\mathcal{E}) = \frac{\text{deg}(\mathcal{E})}{\text{rk}(\mathcal{E})}$. These definitions can be extended to torsion coherent sheaves by additivity in short exact sequences. In particular, torsion sheaves have rank 0, degree > 0 and slope $+\infty$. Endowed with rk and deg , the category Coh_X of coherent sheaves on X is a Harder-Narasimhan category.

The following result [19] is fundamental:

Theorem 3.9. (Fargues-Fontaine)

(i) *If $\lambda = \frac{d}{h}$ (in lowest terms), there exists, up to isomorphism, a unique stable vector bundle $\mathcal{O}(\lambda)$ of slope λ ; its rank is h and its degree d .*

(ii) *Every vector bundle \mathcal{E} on X is a direct sum*

$$\mathcal{E} = \mathcal{O}(\lambda_1) \oplus \cdots \oplus \mathcal{O}(\lambda_r).$$

In particular the Harder-Narasimhan filtration splits (non canonically).

(iii) *Every coherent sheaf \mathcal{E} on X is a direct sum*

$$\mathcal{E} = \mathcal{O}(\lambda_1) \oplus \cdots \oplus \mathcal{O}(\lambda_r) \oplus \left(\bigoplus_{x \in X} \mathcal{F}_x \right),$$

where \mathcal{F}_x is a torsion coherent sheaf, supported at x and zero for almost all x .

The $\lambda_1, \dots, \lambda_r$ above are *the slopes of \mathcal{E}* (to which one has to add $+\infty$ if one of the \mathcal{F}_x is non zero).

We have, by [19, prop. 8.2.3],

$$(3.10) \quad \begin{aligned} H^0(X, \mathcal{O}(\lambda)) &= (\mathbf{B}_{\text{cris}}^+)^{\varphi^d = p^h}, & H^1(X, \mathcal{O}(\lambda)) &= 0, & \text{if } \lambda \geq 0, \\ H^0(X, \mathcal{O}(\lambda)) &= 0, & H^1(X, \mathcal{O}(\lambda)) &= \mathbf{B}_{\text{dR}}^+ / (t^h \mathbf{B}_{\text{dR}}^+ \oplus \mathbf{Q}_p^h), & \text{if } \lambda < 0. \end{aligned}$$

Note that $H^i(X, \mathcal{O}(\lambda))$, for $i = 0, 1$, is the space of C -points of a BC $\mathbb{H}^i(X, \mathcal{O}(\lambda))$, and we have

$$\begin{aligned} \text{Dim}(\mathbb{H}^0(X, \mathcal{O}(\lambda))) &= (d, h), & \text{Dim}(\mathbb{H}^1(X, \mathcal{O}(\lambda))) &= 0, & \text{if } \lambda \geq 0, \\ \text{Dim}(\mathbb{H}^0(X, \mathcal{O}(\lambda))) &= 0, & \text{Dim}(\mathbb{H}^1(X, \mathcal{O}(\lambda))) &= (d, -h), & \text{if } \lambda < 0. \end{aligned}$$

Note also that $\mathbb{H}^i(X, \mathcal{O}(\lambda))$, for $i = 0, 1$, clearly depends only on C^b if $\lambda \geq 0$; this is less clear when $\lambda < 0$ (since t depends on C) but it is still true.

3.2.4. *The category $\text{Coh}_{\bar{X}}$.* In his thesis [31], Le Bras shows that \mathcal{BC} is the smallest sub-abelian category of the category of VS's, stable by extensions, and containing \mathbf{Q}_p and \mathbb{V}^1 ; this gives an efficient alternative definition of \mathcal{BC} (in particular, it shows that a VS extension of two BC's is a BC).

We note $\text{Coh}_{\bar{X}}$ the sub-category of $D^b(\text{Coh}_X)$ of complexes \mathcal{E}_{\bullet} such that $H^i(\mathcal{E}_{\bullet}) = 0$ if $i \neq -1, 0$, $H^{-1}(\mathcal{E}_{\bullet})$ has slopes < 0 and $H^0(\mathcal{E}_{\bullet})$ has slopes ≥ 0 . Any object of $\text{Coh}_{\bar{X}}$ can be represented by a complex $\mathcal{E}_{-1} \xrightarrow{0} \mathcal{E}_0$ of coherent sheaves such that $H^0(X, \mathcal{E}_{-1}) = 0$ (i.e., \mathcal{E}_{-1} has slopes < 0) and $H^1(X, \mathcal{E}_0) = 0$ (i.e., \mathcal{E}_0 has slopes ≥ 0). Le Bras defines an exact functor

$$\text{BC} : \text{Coh}_{\bar{X}} \rightarrow \mathcal{BC}$$

(denoted by $R^0\tau_*$ in [31, 6.2]): for a complex of coherent sheaves \mathcal{F} on X , $\text{BC}(\mathcal{F})$ is the sheaf associated to the presheaf $S \mapsto H^0(X_S, \mathcal{F}_S)$. By definition of this functor⁹, one gets an exact sequence in \mathcal{BC} :

$$0 \rightarrow \mathbb{H}^1(X, \mathcal{E}_{-1}) \rightarrow \text{BC}(\mathcal{E}_{-1} \xrightarrow{0} \mathcal{E}_0) \rightarrow \mathbb{H}^0(X, \mathcal{E}_0) \rightarrow 0.$$

If $\mathcal{E}_{-1} \xrightarrow{0} \mathcal{E}_0$ et $\mathcal{F}_{-1} \xrightarrow{0} \mathcal{F}_0$ are objects of $\text{Coh}_{\bar{X}}$, then

$$\text{Hom}_{\text{BC}}(\text{BC}(\mathcal{E}_{-1} \xrightarrow{0} \mathcal{E}_0), \text{BC}(\mathcal{F}_{-1} \xrightarrow{0} \mathcal{F}_0)) = \begin{pmatrix} \text{Hom}(\mathcal{E}_{-1}, \mathcal{F}_{-1}) & \text{Ext}^1(\mathcal{E}_0, \mathcal{F}_{-1}) \\ 0 & \text{Hom}(\mathcal{E}_0, \mathcal{F}_0) \end{pmatrix}$$

The following result is the main result of [31].

Theorem 3.11. (Le Bras, [31, Th. 1.2]) *The functor BC realizes an equivalence of categories*

$$\text{Coh}_{\bar{X}} \simeq \mathcal{BC}.$$

3.2.5. *Harder-Narasimhan filtration on BC's.* One endows $\text{Coh}_{\bar{X}}$ with functions rank^- , degree deg^- and slope $\mu^- = \frac{\text{deg}^-}{\text{rk}^-}$, by setting:

$$\text{rk}^-(\mathcal{E}_{-1} \xrightarrow{0} \mathcal{E}_0) = \text{deg}(\mathcal{E}_0) - \text{deg}(\mathcal{E}_{-1}), \quad \text{deg}^-(\mathcal{E}_{-1} \xrightarrow{0} \mathcal{E}_0) = \text{rk}(\mathcal{E}_{-1}) - \text{rk}(\mathcal{E}_0),$$

which turn it into a Harder-Narasimhan category.

By transport of structure, this endows also \mathcal{BC} with functions rank^- , degree deg^- and slope μ^- ; we have

$$\text{rk}^- = \dim \quad \text{and} \quad \text{deg}^- = -\text{ht}.$$

If \mathcal{F}_x is torsion, then $\mu^-(\text{BC}(0 \rightarrow \mathcal{F}_x)) = 0$.

If $\lambda = \frac{d}{h}$ (in lowest terms), denote by \mathbb{U}_{λ} the BC defined by $\mathbb{U}_{\lambda} := \mathbb{U}_{h,d}$. Note that, if $\lambda = \frac{d}{h}$ is in lowest terms and $e \geq 1$, then $\mathbb{U}_{eh,ed} = \mathbb{U}_{\lambda}^e$. Then we have

$$\mathbb{U}_{\lambda} = \begin{cases} \mathbb{H}^0(X, \mathcal{O}(\lambda)), & \text{if } \lambda \geq 0, \\ \mathbb{H}^1(X, \mathcal{O}(\lambda)), & \text{if } \lambda < 0. \end{cases}$$

Alternatively,

$$\mathbb{U}_{\lambda} = \begin{cases} \text{BC}(0 \rightarrow \mathcal{O}(\lambda)), & \text{if } \lambda \geq 0, \\ \text{BC}(\mathcal{O}(\lambda) \rightarrow 0), & \text{if } \lambda < 0. \end{cases}$$

Then

$$\text{rk}^-(\mathbb{U}_{\lambda}) = \text{sign}(\lambda) d, \quad \text{deg}^-(\mathbb{U}_{\lambda}) = -\text{sign}(\lambda) h, \quad \mu^-(\mathbb{U}_{\lambda}) = \frac{-1}{\lambda}.$$

Remark 3.12. (i) Since $\mathbf{Q}_p = \mathbb{U}_0$, we have $\mu^-(\mathbf{Q}_p) = -\infty$.

(ii) BC's are naturally diamonds (and were amongst the first non trivial examples of diamonds) and, as such, have connected components. If \mathbb{W} is a BC, then $\mathbb{W}^{>-\infty}$ is the connected component of 0 and the quotient $\mathbb{W}^{-\infty}$ is the largest étale quotient (group of connected components, a finite dimensional \mathbf{Q}_p -vector space).

⁹More generally, $\text{BC}(\mathcal{E}_{\bullet})$ is an extension of $\mathbb{H}^0(X, H^0(\mathcal{E}_{\bullet}))$ by $\mathbb{H}^1(X, H^{-1}(\mathcal{E}_{\bullet}))$.

(iii) The Harder-Narasimhan filtration splits (non canonically), and every BC can be decomposed as

$$(3.13) \quad \mathbb{W} = \mathbb{U}_{-1/\lambda_1} \oplus \cdots \oplus \mathbb{U}_{-1/\lambda_r} \oplus \left(\oplus_x \mathbb{H}^0(X, \mathcal{F}_x) \right),$$

where the λ_i are non zero elements of $\mathbf{Q} \cup \{-\infty\}$, $\mathbb{U}_{-1/\lambda_i}$ is of slope λ_i , and \mathcal{F}_x is a torsion coherent sheaf, supported at x , zero for almost all x , and $\mathbb{H}^0(X, \mathcal{F}_x)$ is of slope 0. The λ_i are *the slopes of* \mathbb{W} (to which one has to add 0 if one of the \mathcal{F}_x is non zero).

(iv) In the exact sequence

$$0 \rightarrow \mathbb{H}^1(X, \mathcal{E}_{-1}) \rightarrow \text{BC}(\mathcal{E}_{-1} \xrightarrow{0} \mathcal{E}_0) \rightarrow \mathbb{H}^0(X, \mathcal{E}_0) \rightarrow 0,$$

the term on the left represents the subspace of slopes > 0 of $\text{BC}(\mathcal{E}_{-1} \xrightarrow{0} \mathcal{E}_0)$.

Remark 3.14. (i) The existence of the exact sequence $0 \rightarrow \mathbb{W}^{>-\infty} \rightarrow \mathbb{W} \rightarrow \mathbb{W}^{-\infty} \rightarrow 0$ makes it possible to show that a decreasing sequence of BC's is stationary: indeed, if $(\mathbb{W}_n)_{n \in \mathbf{N}}$ is such a sequence, then $\dim(\mathbb{W}_n)$ is decreasing and bounded below by 0, hence is constant for $n \geq N$. It follows that $\mathbb{W}_N/\mathbb{W}_n$ is of dimension 0 and hence is a quotient of $\mathbb{W}_N^{-\infty}$. Since $\text{ht}(\mathbb{W}_N/\mathbb{W}_n)$ is increasing and bounded by $\text{ht}(\mathbb{W}_N^{-\infty}) < \infty$, one concludes that $\mathbb{W}_N/\mathbb{W}_n$ is constant for n big enough, and that so is \mathbb{W}_n .

(ii) One can also use a presentation to reduce to the case $\mathbb{W} = \mathbb{V}^d$, and then argue by induction on d using the fact that a sub-BC of \mathbb{V}^1 is either \mathbb{V}^1 of a finite dimensional \mathbf{Q}_p -vector space. This proof applies verbatim to almost C -representations.

3.2.6. *Morphisms.* We can describe Hom and Ext^1 in the category \mathcal{BC} using the curve. For example:

(1) If D_λ is the division algebra with center \mathbf{Q}_p and invariant λ , we have

$$\text{End}_{\mathcal{BC}}(\mathbb{U}_\lambda) = \text{End}(\mathcal{O}(\lambda)) = D_\lambda.$$

(2) Recall that, if $\lambda_1 = \frac{d_1}{h_1}$, $\lambda_2 = \frac{d_2}{h_2}$, and $\lambda_1 + \lambda_2 = \frac{d}{h}$ in the minimal form, we have $\mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_2) = \mathcal{O}(\lambda_1 + \lambda_2)^{n(\lambda_1, \lambda_2)}$, where $n(\lambda_1, \lambda_2) = \frac{h_1}{h_2} h$. Also:

$$\text{Hom}(\mathcal{O}(\lambda_1), \mathcal{O}(\lambda_2)) = \text{Hom}(\mathcal{O}, \mathcal{O}(\lambda_2 - \lambda_1)), \quad \text{Hom}(\mathcal{O}, \mathcal{O}(\lambda)) = H^0(X, \mathcal{O}(\lambda)).$$

(3) There is a bijection $x \mapsto T_x$ between the closed points of X and the \mathbf{Q}_p -lines of $\mathbb{U}_1(C)$. If $x \in X$, then $\text{End}(\mathbb{U}_1/T_x) = C_x$, where C_x is the residue field of the local ring of x in X (recall that C_x is an algebraically closed field, complete for the p -adic valuation, an untild of C^b ; but C_x is not always isomorphic to C).

(4) Similarly, $\mathbb{U}_d/T_x^{\otimes d} = \text{BC}(i_{x,*}\mathbb{B}_d(C_x))$, and thus $\text{End}(\mathbb{U}_d/T_x^{\otimes d}) = \mathbb{B}_d(C_x)$.

(5) If $\lambda = \frac{d}{h} \geq 0$ then $\text{Hom}(\mathbb{U}_\lambda, \mathbb{V}^1)$ is the C -module of rank h generated by $\theta \circ \varphi^i$, for $0 \leq i \leq h-1$.

3.2.7. *Slope 0 and curvature 0.* The map $\mathcal{F} \mapsto H^0(X, \mathcal{F})$ induces an equivalence of categories from the category of torsion coherent sheaves, supported at x , to the category of finite length $\mathbb{B}_{\text{dR}}^+(C_x)$ -modules. A finite length $\mathbb{B}_{\text{dR}}^+(C_x)$ -module is a direct sum of $\mathbb{B}_m(C_x) = \mathbb{B}_{\text{dR}}^+(C_x)/t_x^m$, and the sheaf $i_{x,*}\mathbb{B}_m$ corresponding to $\mathbb{B}_m(C_x)$ (where i_x denotes the inclusion of x in X) lives in an exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{t_x^m} \mathcal{O}(m) \rightarrow i_{x,*}\mathbb{B}_m \rightarrow 0$$

Passing to the sequence of H^0 's (which is exact as $H^1(X, \mathcal{O}) = 0$), this gives an isomorphism

$$\mathbb{H}^0(X, i_{x,*}\mathbb{B}_m) = \mathbb{U}_m/\mathbf{Q}_p t_x^m.$$

One deduces from the equivalence $\text{Coh}_X^- \simeq \mathcal{BC}$ that

$$\text{End}_{\mathcal{BC}}(\mathbb{U}_m/\mathbf{Q}_p t_x^m) \simeq \text{End}_{\text{Coh}_X}(i_{x,*}\mathbb{B}_m(C_x)) \simeq \text{End}_{\mathbb{B}_{\text{dR}}^+(C_x)}(\mathbb{B}_m(C_x)) \simeq \mathbb{B}_m(C_x).$$

In particular, for $m = 1$, one obtains

$$\mathrm{End}_{\mathcal{BC}}(\mathbb{U}_1/\mathbf{Q}_p t_x) \simeq C_x.$$

Note also that, if $x \neq \infty$, then

$$\mathrm{Hom}_{\mathcal{BC}}(\mathbb{U}_1/\mathbf{Q}_p t_x, \mathbb{V}^1) \simeq \mathrm{Hom}_{\mathrm{Coh}_X}(i_{x,*}\mathbb{B}_m, i_{\infty,*}\mathbb{B}_m) = 0,$$

since the two sheaves are supported at distinct points.

In the special case $x = \infty$, which will be crucial for our results, $t_x = t$ and $\mathbb{U}_m/\mathbf{Q}_p t^m = \mathbb{B}_m$, and one can describe directly the object \mathbb{M} of \mathcal{BC} attached to a $\mathbf{B}_{\mathrm{dR}}^+$ -module of finite length M : we have

$$\mathbb{M} = M \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbb{B}_{\mathrm{dR}}^+.$$

This can be summarized by the following result.

Proposition 3.15. *The functor $M \mapsto \mathbb{M} = M \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbb{B}_{\mathrm{dR}}^+$ defines an equivalence of categories between the category of $\mathbf{B}_{\mathrm{dR}}^+$ -modules of finite length and the subcategory of \mathcal{BC} of objects of curvature 0.*

Corollary 3.16. (i) *The kernel and cokernel of a morphism of objects of curvature 0 are of curvature 0.*

(ii) *If \mathbb{W} is a torsion $\mathbf{B}_{\mathrm{dR}}^+$ -module, then $\mathrm{Hom}_{\mathrm{VS}}(\mathbb{W}, \mathbb{B}_{\mathrm{dR}}^+) = 0$ and $\mathrm{Hom}_{\mathrm{VS}}(\mathbb{W}, \mathbb{B}_{\mathrm{dR}}) = 0$.*

Proof. Point (i) is a direct consequence of Proposition 3.15. To prove point (ii) we may assume \mathbb{W} to be finitely generated (since, in any case, it is an inductive limit of finitely generated $\mathbf{B}_{\mathrm{dR}}^+$ -modules). Then, we can use Proposition 3.15 to write \mathbb{W} as $W \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbb{B}_{\mathrm{dR}}^+$ for some torsion $\mathbf{B}_{\mathrm{dR}}^+$ -module W . Now we have, using Proposition 3.15 and the fact that $\mathbb{B}_{\mathrm{dR}}^+ = \lim_k \mathbb{B}_{\mathrm{dR}}^+/t^k$,

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbb{W}, \mathbb{B}_{\mathrm{dR}}^+) = \lim_k \mathrm{Hom}_{\mathrm{VS}}(\mathbb{W}, \mathbb{B}_{\mathrm{dR}}^+/t^k) = \lim_k \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}}^+}(W, \mathbb{B}_{\mathrm{dR}}^+/t^k) = \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}}^+}(W, \mathbb{B}_{\mathrm{dR}}^+) = 0$$

This proves the result for $\mathbb{B}_{\mathrm{dR}}^+$. To prove it for \mathbb{B}_{dR} , note that W is naturally a \mathbf{Q}_p -banach and \mathbf{B}_{dR} is an inductive limit of the $t^{-N}\mathbf{B}_{\mathrm{dR}}^+$ which are \mathbf{Q}_p -Fréchet's. Hence there exists N such that W maps to $t^{-N}\mathbf{B}_{\mathrm{dR}}^+$ and then \mathbb{W} maps to $t^{-N}\mathbb{B}_{\mathrm{dR}}^+$. This makes it possible to use the case of $\mathbb{B}_{\mathrm{dR}}^+$ to finish the proof. \square

3.2.8. *Canonical and Harder-Narasimhan filtrations.* The relation between the decomposition (3.13) and the filtration of Proposition 3.7 is given by:

$$\begin{aligned} \mathbb{W}_{>0} &\simeq (\oplus_{\lambda_i > 0} \mathbb{U}_{-1/\lambda_i}) \oplus (\oplus_{x \neq \infty} \mathbb{H}^0(X, \mathcal{F}_x)), \\ \mathbb{W}_{\leq 0} &\simeq (\oplus_{\lambda_i < 0} \mathbb{U}_{-1/\lambda_i}) \oplus \mathbb{H}^0(X, \mathcal{F}_{\infty}) = \mathbb{H}^0(X, \mathcal{F}_{\infty} \oplus (\oplus_{\lambda_i < 0} \mathcal{O}(-1/\lambda_i))), \\ \mathbb{W}_{<0} &\simeq \oplus_{\lambda_i < 0} \mathbb{U}_{-1/\lambda_i}, \quad \mathbb{W}_{=0} \simeq \mathbb{H}^0(X, \mathcal{F}_{\infty}). \end{aligned}$$

From the properties of Harder-Narasimhan filtrations and the above decompositions, we can deduce the following results.

Corollary 3.17. (i) *\mathbb{W} is of curvature < 0 (resp. ≤ 0) if and only if $\mathbb{W} \simeq H^0(X, \mathcal{E})$, where \mathcal{E} is a vector bundle of slopes ≥ 0 (resp. the sum of a vector bundle of slopes ≥ 0 and a torsion sheaf supported at ∞).*

(ii) *An extension of two BC's of curvature < 0 (resp. ≤ 0) is of curvature < 0 (resp. ≤ 0).*

Corollary 3.18. The sign of the curvature determines the sign of the height:

- (a) *curvature 0 implies height 0;*
- (b) *curvature < 0 implies height > 0 ;*
- (c) *curvature > 0 implies height ≤ 0 .*

Corollary 3.19. The curvature decreases by going to a subobject and increases by taking a quotient:

- (i) A sub-BC of a BC of curvature ≤ 0 (resp. < 0) has curvature ≤ 0 (resp. < 0).
- (ii) A sub-BC of height 0 of a BC of curvature ≤ 0 has curvature 0.
- (iii) A quotient of a BC of curvature ≥ 0 (resp. > 0) has curvature ≥ 0 (resp. > 0).
- (iv) A quotient of height 0 of a BC of curvature ≥ 0 has curvature 0.

Remark 3.20. An important consequence of (ii) of Corollary 3.19 is that a sub-BC \mathbb{U} of a torsion \mathbb{B}_{dR}^+ -module \mathbb{W} satisfies $\text{ht}(\mathbb{U}) \geq 0$ and \mathbb{U} is itself a torsion \mathbb{B}_{dR}^+ -module if and only if $\text{ht}(\mathbb{U}) = 0$.

This can be proven, without the Harder-Narasimhan decomposition, by induction on the length of \mathbb{W} , using the fact that a sub-BC of \mathbb{V}^1 is either \mathbb{V}^1 or a finite dimensional \mathbf{Q}_p -vector space and the fact that an extension of \mathbb{B}_{dR}^+ -modules is itself a \mathbb{B}_{dR}^+ -module. This proof extends verbatim to almost C -representations thanks to Proposition 2.5.

3.2.9. *BC's of curvature ≤ 0 .* Plüt says that a BC of curvature ≤ 0 is *constructible*, but we will not use this terminology.

Lemma 3.21. *The following conditions are equivalent:*

- (i) \mathbb{W} is of curvature ≤ 0 .
- (ii) There is an exact sequence

$$(3.22) \quad 0 \rightarrow V \rightarrow \mathbb{W} \rightarrow \mathbb{M} \rightarrow 0,$$

where \mathbb{M} is of curvature 0 and V is finite dimensional over \mathbf{Q}_p .

Proof. Implication (i) \Rightarrow (ii) follows from the fact that, if \mathbb{W} is of curvature ≤ 0 , then

$$\mathbb{W} = \mathbb{H}^0(X, \mathcal{F}_\infty) \oplus \left(\bigoplus_{d_i/h_i \geq 0} \mathbb{U}_{d_i/h_i} \right)$$

and we have an exact sequence $0 \rightarrow \mathbf{Q}_p^{h_i} \rightarrow \mathbb{U}_{d_i/h_i} \rightarrow \mathbb{B}_{d_i} \rightarrow 0$. Then $V = \bigoplus_{d_i/h_i} \mathbf{Q}_p^{h_i}$ gives what we want.

The converse implication (ii) \Rightarrow (i) follows from the fact that \mathbb{M} is of slope 0 and V of slope $-\infty$ by assumption, so any extension has slopes in $[-\infty, 0]$. \square

3.3. **Categorification of height.** We will introduce now a partial categorification of height of BC's.

3.3.1. *The functor $\mathbb{W} \mapsto h(\mathbb{W})$.* If \mathbb{W} is a topological VS, set

$$h(\mathbb{W}) := \text{Hom}(\mathbb{W}, \mathbb{B}_{\text{dR}})$$

(Let us remind that this includes the requirement that $\mathbb{W}(C) \rightarrow \mathbf{B}_{\text{dR}}$ is continuous.) This is a \mathbf{B}_{dR} -module; hence $h(-)$ is a functor from the category \mathcal{BC} to the category of vector spaces over \mathbf{B}_{dR} . In most cases of interest, because of Lemmas 4.14 and 4.15 and Corollary 3.16, we have in fact

$$h(\mathbb{W}) = \text{Hom}^{\natural}(\mathbb{W}, \mathbb{B}_{\text{dR}}) := \text{colim}_{k \geq 0} \text{Hom}_{\text{VS}}(\mathbb{W}, t^{-k} \mathbb{B}_{\text{dR}}^+) = \text{Hom}(\mathbb{W}, \mathbb{B}_{\text{dR}}^+) \otimes_{\mathbf{B}_{\text{dR}}^+} \mathbf{B}_{\text{dR}}$$

We also set

$$\text{Ext}^{1, \natural}(\mathbb{W}, \mathbb{B}_{\text{dR}}) := \text{colim}_{k \geq 0} \text{Ext}_{\text{VS}}^1(\mathbb{W}, t^{-k} \mathbb{B}_{\text{dR}}^+).$$

This is a \mathbf{B}_{dR} -module as well.

Lemma 3.23. (i) *If \mathbb{W} is of curvature ≤ 0 , then $\text{Ext}^{1, \natural}(\mathbb{W}, \mathbb{B}_{\text{dR}}) = 0$.*

(ii) *If $0 \rightarrow \mathbb{W}_1 \rightarrow \mathbb{W} \rightarrow \mathbb{W}_2 \rightarrow 0$ is an exact sequence of BC's of curvature ≤ 0 , the sequence $0 \rightarrow h(\mathbb{W}_2) \rightarrow h(\mathbb{W}) \rightarrow h(\mathbb{W}_1) \rightarrow 0$ is exact.*

Proof. Claim (ii) follows immediately from (i).

Now, the exact sequence (3.22) shows that it suffices to prove (i) for an affine and for \mathbf{Q}_p . That $\text{Ext}_{\text{VS}}^1(\mathbf{Q}_p, \mathbb{B}_{\text{dR}}^+) = 0$ follows easily, by devissage, from the fact that the maps $\mathbb{B}_{\text{dR}}^+/t^{m+1} \rightarrow \mathbb{B}_{\text{dR}}^+/t^m$ are surjective and $\text{Ext}_{\text{VS}}^1(\mathbf{Q}_p, \mathbb{V}^1) = 0$ (see [31, Th. 4.1]). This implies that

$$\text{Ext}^{1, \natural}(\mathbf{Q}_p, \mathbb{B}_{\text{dR}}) = \text{colim}_{k \geq 0} \text{Ext}_{\text{VS}}^1(\mathbf{Q}_p, t^{-k} \mathbb{B}_{\text{dR}}^+) = 0,$$

as wanted.

To show that $\text{Ext}^{1, \natural}(\mathbb{W}, \mathbb{B}_{\text{dR}}) = 0$ for \mathbb{W} affine, it suffices, again by devissage, to show that $\text{Ext}^{1, \natural}(\mathbb{V}^1, \mathbb{B}_{\text{dR}}) = 0$. To show the latter fact we use the exact sequence

$$0 \rightarrow t^{-k} \mathbb{B}_{\text{dR}}^+ \rightarrow t^{-k-1} \mathbb{B}_{\text{dR}}^+ \rightarrow t^{-k-1} \mathbb{V}^1 \rightarrow 0$$

and the fact that $\text{Hom}_{\text{VS}}(\mathbb{V}^1, \mathbb{B}_{\text{dR}}^+) = 0$. This gives us the exact sequence

$$0 \rightarrow \text{Hom}_{\text{VS}}(\mathbb{V}^1, t^{-k-1} \mathbb{V}^1) \rightarrow \text{Ext}_{\text{VS}}^1(\mathbb{V}^1, t^{-k} \mathbb{B}_{\text{dR}}^+) \rightarrow \text{Ext}_{\text{VS}}^1(\mathbb{V}^1, t^{-k-1} \mathbb{B}_{\text{dR}}^+).$$

Since the first two terms are isomorphic to $\mathbf{B}_{\text{dR}}^+/t$ as \mathbf{B}_{dR}^+ -modules, $\text{Ext}_{\text{VS}}^1(\mathbb{V}^1, t^{-k} \mathbb{B}_{\text{dR}}^+) \rightarrow \text{Ext}_{\text{VS}}^1(\mathbb{V}^1, t^{-k-1} \mathbb{B}_{\text{dR}}^+)$ is zero. This finishes the proof. \square

Proposition 3.24. (i) *If \mathbb{W} is of curvature ≤ 0 then $\text{rk}(h(\mathbb{W})) = \text{ht}(\mathbb{W})$.*

(ii) *In general, $\text{rk}(h(\mathbb{W})) = \text{ht}(\mathbb{W}) + \text{rk}(\text{Ext}^{1, \natural}(\mathbb{W}, \mathbb{B}_{\text{dR}}))$.*

Proof. For (i), by assumption, we have the exact sequence $0 \rightarrow V \rightarrow \mathbb{W} \rightarrow \mathbb{M} \rightarrow 0$, where \mathbb{M} is affine. This yields the exact sequence

$$0 \rightarrow h(\mathbb{M}) \rightarrow h(\mathbb{W}) \rightarrow h(V) \rightarrow \text{Ext}^{1, \natural}(\mathbb{M}, \mathbb{B}_{\text{dR}}).$$

Since $h(\mathbb{M}) = 0$ and we have $\text{Ext}^{1, \natural}(\mathbb{M}, \mathbb{B}_{\text{dR}}) = 0$, by Lemma 3.23, this sequence implies that $\text{rk}(h(\mathbb{W})) = \dim_{\mathbf{Q}_p} V$. We are done because $\text{ht}(\mathbb{W}) = \dim_{\mathbf{Q}_p} V$.

(ii) follows via the exact sequence of Ext^{\natural} of a presentation: if $0 \rightarrow V \rightarrow \mathbb{W}' \rightarrow \mathbb{W} \rightarrow 0$ represents \mathbb{W} , where \mathbb{W}' is an extension of \mathbb{V}^d by a \mathbf{Q}_p -vector space V' of finite dimension, arguing as for (1) we get the following diagram with exact row:

$$\begin{array}{ccccccc} 0 & \rightarrow & h(\mathbb{W}) & \rightarrow & h(\mathbb{W}') & \rightarrow & h(V) \rightarrow \text{Ext}^{1, \natural}(\mathbb{W}, \mathbb{B}_{\text{dR}}) \rightarrow \text{Ext}^{1, \natural}(\mathbb{W}', \mathbb{B}_{\text{dR}}) \rightarrow 0 \\ & & & & \downarrow \wr & & \\ & & & & h(V') & & \end{array}$$

Since \mathbb{W}' is of curvature ≤ 0 , by Lemma 3.23, we have $\text{Ext}^{1, \natural}(\mathbb{W}', \mathbb{B}_{\text{dR}}) = 0$. It follows that

$$\text{rk}(h(\mathbb{W})) = \text{rk}(h(V')) - \text{rk}(h(V)) + \text{rk}(\text{Ext}^{1, \natural}(\mathbb{W}, \mathbb{B}_{\text{dR}})),$$

which gives us what we wanted because $\text{ht}(\mathbb{W}) = \text{rk}(h(V')) - \text{rk}(h(V))$. \square

3.4. The category $q\mathcal{BC}$. We need to enlarge the category \mathcal{BC} to allow extensions by arbitrary \mathbb{B}_m -Modules, for $m \geq 1$.

3.4.1. Definitions. A qBC (q stands for quasi) is a VS \mathbb{W} such that there exists $m \geq 1$ and a sub- \mathbb{B}_m -Module \mathbb{W}_0 of \mathbb{W} such that \mathbb{W}/\mathbb{W}_0 is a BC. We will denote by $q\mathcal{BC}$ the full subcategory of VS's consisting of qBC's.

Remark 3.25. Proposition 3.15 extends to arbitrary \mathbb{B}_m -modules: if \mathbb{W} is a \mathbb{B}_m -Module, then $\mathbb{W}(C)$ is a \mathbf{B}_m -module, and can be written as $\bigoplus_{i \in I} (\mathbf{B}_m/t^{j_i})e_i$ (with $j_i \leq m$). The natural map of \mathbb{B}_m -Modules $\mathbb{B}_m \otimes_{\mathbf{B}_m} \mathbb{W}(C) \rightarrow \mathbb{W}$ gives an isomorphism when evaluated on C -points, hence is an isomorphism (the kernel and cokernel are 0 since their C -points are 0). It follows that $\mathbb{W} \simeq \bigoplus_{i \in I} (\mathbb{B}_m/t^{j_i})e_i$, and one can deduce the result for arbitrary \mathbb{B}_m -modules from its counterpart for finite type ones.

For $\mathbb{W} \in q\mathcal{BC}$, we define:

$$\text{ht}(\mathbb{W}) := \text{ht}(\mathbb{W}/\mathbb{W}_0).$$

This does not depend on the choice of \mathbb{W}_0 : if \mathbb{W}'_0 is another choice, then $\mathbb{W}''_0 = \mathbb{W}_0 \cap \mathbb{W}'_0$ is also a possible choice, and we have exact sequences in \mathcal{BC} :

$$0 \rightarrow \mathbb{W}_0/\mathbb{W}''_0 \rightarrow \mathbb{W}/\mathbb{W}''_0 \rightarrow \mathbb{W}/\mathbb{W}_0 \rightarrow 0, \quad 0 \rightarrow \mathbb{W}'_0/\mathbb{W}''_0 \rightarrow \mathbb{W}/\mathbb{W}''_0 \rightarrow \mathbb{W}/\mathbb{W}'_0 \rightarrow 0;$$

now, since $\mathbb{W}_0/\mathbb{W}''_0$ and $\mathbb{W}'_0/\mathbb{W}''_0$ are finite length \mathbb{B}_m -Modules, their height is 0, hence

$$\text{ht}(\mathbb{W}/\mathbb{W}_0) = \text{ht}(\mathbb{W}/\mathbb{W}''_0) = \text{ht}(\mathbb{W}/\mathbb{W}'_0),$$

as wanted.

Definition 3.26. We say that \mathbb{W} has curvature ≤ 0 (resp. 0, resp. ≥ 0) if \mathbb{W}/\mathbb{W}_0 has.

This does not depend on the choice of \mathbb{W}_0 for the same reasons as above, because $\mathbb{W}'_0/\mathbb{W}''_0$ and $\mathbb{W}_0/\mathbb{W}''_0$ have curvature 0, hence $\mathbb{W}/\mathbb{W}''_0$ has curvature ≤ 0 (resp. 0, resp. ≥ 0) if and only if \mathbb{W}/\mathbb{W}'_0 has, and if and only if \mathbb{W}/\mathbb{W}_0 has.

Remark 3.27. We have analogs of Corollary 3.18 and Corollary 3.19 in the category $q\mathcal{BC}$. Indeed, in the case of the first corollary this is clear. In the case of the other one, this follows easily from the independence of height and curvature from the presentation of qBC's and the analogous results for BC's.

Lemma 3.28. *Let \mathbb{W} be a VS. Assume that \mathbb{W} has a presentation*

$$0 \rightarrow \mathbb{W}' \rightarrow \mathbb{W} \rightarrow \mathbb{V}_1 \rightarrow 0, \quad 0 \rightarrow \mathbb{V}_2 \rightarrow \mathbb{Y} \rightarrow \mathbb{W}' \rightarrow 0,$$

where \mathbb{V}_1 and \mathbb{V}_2 are BC's, and \mathbb{Y} is a \mathbb{B}_m -Module. Then \mathbb{W} is in $q\mathcal{BC}$, and $\text{ht}(\mathbb{W}) = \text{ht}(\mathbb{V}_1) - \text{ht}(\mathbb{V}_2)$.

Proof. We have to produce a sub- \mathbb{B}_m -Module \mathbb{W}_0 of \mathbb{W} such that \mathbb{W}/\mathbb{W}_0 is a BC. If we can do the same for \mathbb{W}' , then $\mathbb{W}_0 = \mathbb{W}'_0$ will work.

Hence, we can assume $\mathbb{V}_1 = 0$, in which case we have an exact sequence

$$0 \rightarrow \mathbb{V}_2 \rightarrow \mathbb{Y} \rightarrow \mathbb{W} \rightarrow 0.$$

Since \mathbb{Y} is a \mathbb{B}_m -Module, \mathbb{V}_2 has slopes ≤ 0 (hence is isomorphic to $\mathbb{H}^0(X, \mathcal{F})$, for a coherent sheaf \mathcal{F}), and the arrow $\mathbb{V}_2 \rightarrow \mathbb{Y}$ factors through $\mathbb{H}^0(X, \widehat{\mathcal{O}}_{X, \infty} \otimes \mathcal{F})$, which is a \mathbb{B}_{dR}^+ -Module of finite type. Hence the image of this arrow lands in a sub- \mathbb{B}_m -Module of finite type. This \mathbb{B}_m -Module can be included in a direct factor, still of finite type, and one takes for \mathbb{W}_0 a complementary subspace of this direct factor.

The equality of heights is clear since everything contributing lives inside a BC. \square

3.4.2. $q\text{BC}$'s and morphisms.

Lemma 3.29. ([9, Prop. 2.4]) *A sub-BC \mathbb{W} of \mathbb{V}^N , containing no \mathbb{V}^1 , satisfies $\dim(\mathbb{W}) < \text{ht}(\mathbb{W})$ or, equivalently, has slopes < -1 .*

Proof. We will use the equivalence $\text{Coh}_{\overline{X}} \simeq \mathcal{BC}$. By decomposing \mathbb{W} as a direct sum of stable BC's, it suffices to prove the statement for \mathbb{U}_λ , with $\lambda = \frac{d}{h} > 0$ (here we used the fact that \mathbb{W} does not contain \mathbb{V}^1). This amounts to showing that, if $f : \mathbb{U}_\lambda \rightarrow \mathbb{V}^N$ is an injective map then $h > d$. Passing to the category $\text{Coh}_{\overline{X}}$, we see that we need to show that if a map $f^\flat : \mathcal{O}(\lambda) \rightarrow \iota_{\infty, *} C^N$ is injective on global sections then $h > d$. But this map factors as

$$\begin{array}{ccc} \mathcal{O}(\lambda) & \xrightarrow{f^\flat} & \iota_{\infty, *} C^N \\ \downarrow & & \uparrow \\ \widehat{\mathcal{O}}_{X, \infty} \otimes \mathcal{O}(\lambda) & \rightarrow (\iota_{\infty, *} C) \otimes \mathcal{O}(\lambda) \xrightarrow{\sim} & \iota_{\infty, *} C^h \end{array}$$

Tracing this diagram from the left upper corner first vertically then horizontally we obtain a map $\bar{f} : \mathcal{O}(\lambda) \rightarrow \iota_{\infty,*}C^h$, which is injective on global sections. Now, passing back (from the category $\text{Coh}_{\bar{X}}$) to the category \mathcal{BC} we get an injective map $\tilde{f} : \mathbb{U}_{\lambda} \rightarrow \mathbb{V}^h$. Since \mathbb{U}_{λ} is of Dimension (d, h) and \mathbb{V}^h of Dimension $(h, 0)$, the existence of an injection implies $h < d$, as wanted. \square

Lemma 3.30. *Let $\pi : \mathbb{W} \rightarrow \mathbb{W}'$ be a morphism of VS's, with \mathbb{W} a \mathbb{B}_m -Module and \mathbb{W}' a BC. Then, $\text{Ker } \pi$ contains a sub- \mathbb{B}_m -Module \mathbb{W}_0 such that \mathbb{W}/\mathbb{W}_0 is a BC.*

Proof. After quotienting \mathbb{W} by a maximal sub- \mathbb{B}_m -Module of $\text{Ker } \pi$, one can assume that $\text{Ker } \pi$ contains no non-zero sub- \mathbb{B}_m -Module (the inverse image of a \mathbb{B}_m -Module of $\text{Ker } \pi/\mathbb{W}_0$ is a \mathbb{B}_m -Module of $\text{Ker } \pi$). We want to infer that then \mathbb{W} is of finite length or, equivalently, that its t -torsion sub-Module is of finite rank over \mathbb{B}_1 .

If not, this sub-Module contains an increasing sequence of sub-Modules \mathbb{X}_n , with $\mathbb{X}_n \simeq \mathbb{V}^n$. Denote by \mathbb{Y}_n the intersection of $\text{Ker } \pi$ and \mathbb{X}_n , and by \mathbb{I}_n the image of \mathbb{X}_n in \mathbb{W}' . Since $\mathbb{X}_{n+1}/\mathbb{X}_n \simeq \mathbb{V}^1$, we have an exact sequence

$$0 \rightarrow \mathbb{Y}_{n+1}/\mathbb{Y}_n \rightarrow \mathbb{V}^1 \rightarrow \mathbb{I}_{n+1}/\mathbb{I}_n \rightarrow 0.$$

It follows that one has either $\mathbb{Y}_{n+1}/\mathbb{Y}_n \simeq \mathbb{V}^1$ and $\mathbb{I}_{n+1} = \mathbb{I}_n$, or $\mathbb{Y}_{n+1}/\mathbb{Y}_n$ is of finite dimension over \mathbf{Q}_p and $\dim \mathbb{I}_{n+1} = \dim \mathbb{I}_n + 1$. Since $\dim \mathbb{W}' < \infty$, the second case can only happen for a finite number of n 's, hence $\mathbb{Y}_{n+1}/\mathbb{Y}_n \simeq \mathbb{V}^1$, if n is big enough. In particular,

$$\text{ht}(\mathbb{Y}_{n+1}) = \text{ht}(\mathbb{Y}_{n_0}) \quad \text{and} \quad \dim(\mathbb{Y}_{n+1}) = \dim(\mathbb{Y}_{n_0}) + n + 1 - n_0, \quad \text{if } n \geq n_0.$$

Now, \mathbb{Y}_n is a sub-module of $\text{Ker } \pi$, hence contains no \mathbb{V}^1 by assumption. Since \mathbb{Y}_n is a sub-BC of \mathbb{V}^n , Lemma 3.29 gives a contradiction for $n \geq \text{ht}(\mathbb{Y}_{n_0}) - \dim(\mathbb{Y}_{n_0}) + n_0$, which concludes the proof. \square

Lemma 3.31. *Let $\pi : \mathbb{W} \rightarrow \mathbb{W}'$ be a morphism of VS's, with \mathbb{W}, \mathbb{W}' $q\mathcal{BC}$'s. If \mathbb{W}'_0 is a sub- \mathbb{B}_m -Module of \mathbb{W}' such that $\mathbb{W}'/\mathbb{W}'_0$ is a BC, there exists a sub- \mathbb{B}_m -Module \mathbb{W}_0 of \mathbb{W} such that \mathbb{W}/\mathbb{W}_0 is a BC and $\pi(\mathbb{W}_0) \subset \mathbb{W}'_0$.*

Proof. By assumption, there exists a sub- \mathbb{B}_m -Module \mathbb{W}_1 of \mathbb{W} such that \mathbb{W}/\mathbb{W}_1 is a BC. Applying Lemma 3.30 to $\bar{\pi} : \mathbb{W}_1 \rightarrow \mathbb{W}'/\mathbb{W}'_0$ produces \mathbb{W}_0 such that $\mathbb{W}_1/\mathbb{W}_0$ is a BC and $\pi(\mathbb{W}_0) \subset \mathbb{W}'_0$. But then \mathbb{W}/\mathbb{W}_0 is also BC as an extension of the two BC's \mathbb{W}/\mathbb{W}_1 and $\mathbb{W}_1/\mathbb{W}_0$. \square

Proposition 3.32. *If $\pi : \mathbb{W} \rightarrow \mathbb{W}'$ is a morphism in $q\mathcal{BC}$, then $\text{Ker } \pi$, $\text{Im } \pi$, and $\text{Coker } \pi$ are in $q\mathcal{BC}$ and $\text{ht}(\mathbb{W}) = \text{ht}(\text{Ker } \pi) + \text{ht}(\text{Im } \pi)$.*

Proof. Lemma 3.31 gives us sub- \mathbb{B}_m -Modules \mathbb{W}_0 and \mathbb{W}'_0 of \mathbb{W} and \mathbb{W}' such that $\bar{\mathbb{W}} = \mathbb{W}/\mathbb{W}_0$ and $\bar{\mathbb{W}}' = \mathbb{W}'/\mathbb{W}'_0$ are BC's and $\pi(\mathbb{W}_0) \subset \mathbb{W}'_0$. Then the restriction of π to \mathbb{W}_0 is \mathbb{B}_m -linear, hence its kernel \mathbb{K}^0 and its cokernel \mathbb{C}^0 are \mathbb{B}_m -Modules. The snake Lemma give us exact sequences $0 \rightarrow \mathbb{K}^0 \rightarrow \text{Ker } \pi \rightarrow \mathbb{V}_1 \rightarrow 0$ and $0 \rightarrow \mathbb{V}_2 \rightarrow \mathbb{C}^0 \rightarrow \text{Coker } \pi \rightarrow \mathbb{V}_3 \rightarrow 0$, with $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3$ BC's. This implies that $\text{Ker } \pi$ is in $q\mathcal{BC}$ and, using Lemma 3.28, that so is $\text{Coker } \pi$.

Now $\text{Im } \pi$ is the cokernel of $\text{Ker } \pi \rightarrow \mathbb{W}$, hence is also a $q\text{BC}$. Finally the formula for the heights is easily deduced from the analogous formula for BC's. \square

Remark 3.33. (i) The above proposition implies that the category $q\mathcal{BC}$ is abelian.

(ii) It follows that a sequence $\mathbb{W}_1 \xrightarrow{f} \mathbb{W}_2 \xrightarrow{g} \mathbb{W}_3$ is exact if and only if the associated sequence $\mathbb{W}_1(C) \xrightarrow{f} \mathbb{W}_2(C) \xrightarrow{g} \mathbb{W}_3(C)$ is exact ($\mathbb{H} = (\text{Ker } g + \text{Coker } f)/(\text{Ker } g \cap \text{Coker } f)$ is a $q\text{BC}$, hence $\mathbb{H} = 0$ if and only if $\mathbb{H}(C) = 0$).

3.4.3. *Exactness of the functor $h(-)$.*

Lemma 3.34. *Let $0 \rightarrow \mathbb{W}_1 \rightarrow \mathbb{W}_2 \rightarrow \mathbb{W}_3 \rightarrow 0$ be an exact sequence of $q\text{BC}$'s. Then there exist sub- \mathbb{B}_m -Modules \mathbb{W}'_i of \mathbb{W}_i , such $\mathbb{W}_i/\mathbb{W}'_i$ are BC's, pour $i = 1, 2, 3$, and such that we have an exact sequence $0 \rightarrow \mathbb{W}'_1 \rightarrow \mathbb{W}'_2 \rightarrow \mathbb{W}'_3 \rightarrow 0$.*

Proof. Start with a \mathbb{B}_m -Module $\mathbb{W}'_3 \subset \mathbb{W}_3$ such that $\mathbb{W}_3/\mathbb{W}'_3$ is a BC. Lemma 3.31 provides $\mathbb{W}'_2 \subset \mathbb{W}_2$ with image \mathbb{W}'_3 included in \mathbb{W}'_2 . Then \mathbb{W}'_3 is a \mathbb{B}_m -Module since $\mathbb{W}'_2 \rightarrow \mathbb{W}'_3$ is a morphism of \mathbb{B}_m -Modules. And $\mathbb{W}_3/\mathbb{W}'_3$ is a quotient of $\mathbb{W}_2/\mathbb{W}'_2$, hence is BC. Finally, if \mathbb{W}'_1 is the kernel of $\mathbb{W}'_2 \rightarrow \mathbb{W}'_3$, the wanted sequence is exact, \mathbb{W}'_1 is a \mathbb{B}_m -Module as kernel of a morphism of \mathbb{B}_m -Modules, and $\mathbb{W}_1/\mathbb{W}'_1$ is a subobject of $\mathbb{W}_2/\mathbb{W}'_2$, hence is a BC. \square

Corollary 3.35. *If $0 \rightarrow \mathbb{W}_1 \rightarrow \mathbb{W}_2 \rightarrow \mathbb{W}_3 \rightarrow 0$ is an exact sequence of qBC 's, of curvatures ≤ 0 , then the sequence $0 \rightarrow h(\mathbb{W}_3) \rightarrow h(\mathbb{W}_2) \rightarrow h(\mathbb{W}_1) \rightarrow 0$ is exact.*

Proof. Choose \mathbb{W}'_i , for $i = 1, 2, 3$, fulfilling the conclusions of Lemma 3.34; set $\mathbb{W}''_i = \mathbb{W}_i/\mathbb{W}'_i$. Then $0 \rightarrow \mathbb{W}''_1 \rightarrow \mathbb{W}''_2 \rightarrow \mathbb{W}''_3 \rightarrow 0$ is an exact sequence of BC's of curvature ≤ 0 , hence the sequence $0 \rightarrow h(\mathbb{W}''_3) \rightarrow h(\mathbb{W}''_2) \rightarrow h(\mathbb{W}''_1) \rightarrow 0$ is exact by Lemma 3.23. One concludes remarking that $h(\mathbb{W}'_i) = 0$ (since \mathbb{W}'_i is a \mathbb{B}_m -Module), hence $h(\mathbb{W}''_i) \rightarrow h(\mathbb{W}_i)$ is an isomorphism. \square

4. FILTERED (φ, N) -MODULES

In this chapter we study filtered $(\varphi, N, \mathcal{G}_K)$ -modules over K or C and their relations to the categories of almost C -representations and BC's. In particular, we introduce the notion of acyclic $(\varphi, N, \mathcal{G}_K)$ -modules as a generalization of weakly-admissible $(\varphi, N, \mathcal{G}_K)$ -modules. While the $(\varphi, N, \mathcal{G}_K)$ -modules $(H_{\text{HK}}^i(X_C), H_{\text{dR}}^i(X))$ coming from algebraic geometry tend to be weakly admissible those coming from overconvergent geometry tend to be only acyclic (as shown later in this paper).

The results of this chapter will be crucial for the proofs of our results towards the C_{st} -conjecture. In particular, Theorem 4.8 and its corollaries (resp. Propositions 4.25 and 4.29) will be used to study the pro-étale-to-de Rham part of the C_{st} -conjecture for varieties over K (resp. over C). The dichotomy of Proposition 4.21 will play a big role in the proof of the de Rham-to-pro-étale part of the C_{st} -conjecture.

4.1. Filtered $(\varphi, N, \mathcal{G}_K)$ -modules over K .

4.1.1. Filtered $(\varphi, N, \mathcal{G}_K)$ -modules.

- *$(\varphi, N, \mathcal{G}_K)$ -modules.* A (φ, N) -module over F or F^{nr} is a finite dimensional F -module or F^{nr} -module M endowed with a Frobenius $\varphi : M \rightarrow M$, semilinear with respect to the absolute Frobenius on F or F^{nr} , and a linear map $N : M \rightarrow M$ satisfying $N\varphi = p\varphi N$.

More generally, a $(\varphi, N, \mathcal{G}_K)$ -module over F^{nr} is a (φ, N) -module over F^{nr} endowed with a smooth¹⁰ action of \mathcal{G}_K which commutes with φ and N .

If M is a $(\varphi, N, \mathcal{G}_K)$ -module over F or F^{nr} , we define its dual M^* as $\text{Hom}_F(M, F)$ endowed with actions of φ , N and \mathcal{G}_K given by

$$\langle \varphi(\mu), v \rangle = \varphi(\langle \mu, \varphi^{-1}(v) \rangle), \quad \langle N(\mu), v \rangle = -\langle \mu, N(v) \rangle, \quad \langle \sigma(\mu), v \rangle = \sigma(\langle \mu, \sigma^{-1}(v) \rangle), \quad \text{if } \sigma \in \mathcal{G}_K.$$

- *Filtered modules.* A filtered module (M, Fil^\bullet) over K is a K -module M together with a descending filtration Fil^\bullet on $M_K = K \otimes_F M$ by sub- K -modules $\text{Fil}^i M_K$, with $\text{Fil}^i M_K = M_K$ if $i \ll 0$ and $\text{Fil}^i M_K = 0$ if $i \gg 0$.

If (M, Fil^\bullet) is a filtered module over K , we define the dual filtered module $(M^*, \text{Fil}_\perp^\bullet)$ by endowing the K -dual $M^* = \text{Hom}_K(M, K)$ of M , with the filtration

$$\text{Fil}_\perp^i M^* = (\text{Fil}^{1-i} M)^\perp.$$

If the filtration is obvious from the context, we don't indicate it in the notation; for example, the de Rham cohomology of a variety X over K is a filtered module over K if we endow it with the Hodge filtration, and will just be denoted by $H_{\text{dR}}^\bullet(X)$, the Hodge filtration being taken for granted.

- *Filtered $(\varphi, N, \mathcal{G}_K)$ -modules.* A filtered (φ, N) -module (M, Fil^\bullet) over K is a (φ, N) -module M over F with a structure of filtered module over K on $M_K = M \otimes_F K$.

¹⁰This means that the stabilizers of elements of M are open in \mathcal{G}_K .

A filtered $(\varphi, N, \mathcal{G}_K)$ -module (M, Fil^\bullet) over K is a $(\varphi, N, \mathcal{G}_K)$ -module M over F^{nr} with a structure of filtered module over K on $M_K = (M \otimes_{F^{\text{nr}}} \overline{K})^{\mathcal{G}_K}$.

If (M, Fil^\bullet) is a filtered $(\varphi, N, \mathcal{G}_K)$ -module over K , we define its dual $(M^*, \text{Fil}_\perp^\bullet)$ as the dual $(\varphi, N, \mathcal{G}_K)$ -module M^* with the module $M_K^* = (M_K)^*$ endowed with the filtration Fil_\perp^\bullet .

As before, if the filtration is obvious from the context, we will use sometimes just M to denote a filtered (φ, N) -module (M, Fil^\bullet) over K or sometimes (M, M_K) as in the case of de Rham cohomology: if X is a smooth quasi-compact dagger variety over K then $(H_{\text{HK}}^i(X_C), H_{\text{dR}}^i(X))$ is a filtered $(\varphi, N, \mathcal{G}_K)$ -module over K thanks to the Hyodo-Kato isomorphism.

4.1.2. Acyclicity and admissibility. If M is a filtered $(\varphi, N, \mathcal{G}_K)$ -module over K , the rank $\text{rk}(M)$ of M is the dimension of M over F^{nr} . If M has rank 1, one defines the *degree* $\text{deg}(M)$ of M by the formula

$$\text{deg}(M) := t_N(M) - t_H(M),$$

where $t_N(M)$ et $t_H(M)$ are defined by choosing a basis e of M over F^{nr} :

- there exists $\lambda \in (F^{\text{nr}})^*$ such that $\varphi(e) = \lambda e$, and we set $t_N(M) = v_p(\lambda)$;
- there exists $i \in \mathbf{Z}$, unique, such that $e \in M_K^i - M_K^{i+1}$, and we set $t_H(M) = i$.

If M has rank $r \geq 2$, then $\det M = \wedge^r M$ is of rank 1, and one defines the *degree* of M by:

$$\begin{aligned} \text{deg}(M) &:= \text{deg}(\det M) = t_N(M) - t_H(M), \\ t_N(M) &:= t_N(\det(M)), \quad t_H(M) := t_H(\det M) = \sum_{i \in \mathbf{Z}} i \dim_K M_K^i / M_K^{i+1}. \end{aligned}$$

Endowed with the rank and degree functions, the category of filtered $(\varphi, N, \mathcal{G}_K)$ -modules over K is a Harder-Narasimhan \otimes -category.

Definition 4.1. A filtered $(\varphi, N, \mathcal{G}_K)$ -module over K is said to be *weakly admissible* if it is semi-stable of slope 0 (a reformulation [18] of the original notion [20]). It is said to be *acyclic* if its Harder-Narasimhan slopes are ≥ 0 .

Remark 4.2. A weakly admissible filtered $(\varphi, N, \mathcal{G}_K)$ -module is acyclic; conversely an acyclic filtered $(\varphi, N, \mathcal{G}_K)$ -module is weakly admissible if and only if it is of degree 0.

Lemma 4.3. *The following conditions are equivalent for a filtered $(\varphi, N, \mathcal{G}_K)$ -module (M, Fil^\bullet) over K :*

- (a) (M, Fil^\bullet) is acyclic.
- (b) There exists a filtration Fil_1^\bullet on M_K such that $\text{Fil}_1^i M_K \subset \text{Fil}^i M_K$ for all i , and $(M, \text{Fil}_1^\bullet)$ is weakly admissible.

Proof. To prove (a) \Rightarrow (b), it is enough to show that one can find a filtration such that $\text{Fil}_1^i M_K \subset \text{Fil}^i M_K$ for all i , there exist i with $\text{Fil}_1^i M_K \neq \text{Fil}^i M_K$, and $(M, \text{Fil}_1^\bullet)$ is acyclic. Indeed the degree of $(M, \text{Fil}_1^\bullet)$ is then strictly smaller than that of (M, Fil^\bullet) . Hence by repeating the process one ends up with a filtration such that $\text{Fil}_n^i M_K \subset \text{Fil}^i M_K$ for all i , $(M, \text{Fil}_n^\bullet)$ is acyclic and of degree 0, hence it is of Harder-Narasimhan slope 0, i.e., it is weakly admissible.

To construct such a Fil_1^\bullet , let M^1 be the largest subobject of Harder-Narasimhan slope 0, and let M^2 be the quotient M/M_1 . Then M^2 has Harder-Narasimhan slope > 0 , and if we pick up any filtration Fil_1^\bullet on M_K^2 , such that $\text{Fil}_1^i M_K^2 = \text{Fil}^i M_K^2$ for all i except i_0 , for which $\text{Fil}_1^{i_0} M_K^2 / \text{Fil}_1^{i_0+1} M_K^2$ is of dimension 1, then $(M^2, \text{Fil}_1^\bullet)$ has Harder-Narasimhan slope ≥ 0 since the degree of any subobject has decreased by at most 1 and hence is ≥ 0 . Then defining $\text{Fil}_1^i M_K$ as the inverse image of $\text{Fil}_1^i M_K^2$ in $\text{Fil}^i M_K$ gives a filtration with the desired properties.

The converse implication is obvious: the Harder-Narasimhan slope of (M, Fil^\bullet) is greater or equal to that of $(M, \text{Fil}_1^\bullet)$. \square

Lemma 4.4. *Let (M, Fil^\bullet) be an acyclic filtered $(\varphi, N, \mathcal{G}_K)$ -module over K , with φ -slopes in $[0, r]$, and $\text{Fil}^0 M_K = M_K$, $\text{Fil}^{r+1} M_K = 0$. Then there exists a filtration Fil_1^\bullet on M_K such that $(M, \text{Fil}_1^\bullet)$ is weakly admissible and $\text{Fil}_1^i M_K \subset \text{Fil}^i M_K$ for all i , $\text{Fil}_1^0 M_K = M_K$, $\text{Fil}_1^{r+1} M_K = 0$.*

Proof. In the proof of Lemma 4.3, the φ -slopes of M^2 are in $[0, r]$, and since $\deg(M^2) > 0$, this implies that $\text{Fil}^1 M_K \neq 0$. Let i_0 be the largest integer with $\text{Fil}^{i_0} M_K \neq 0$; define Fil_1^\bullet by $\text{Fil}_1^{i_0} M_K$ of codimension 1 in $\text{Fil}^{i_0} M_K$, and $\text{Fil}_1^i M_K = \text{Fil}^i M_K$ is $i \neq 0$. Then, as in the proof of Lemma 4.3, $(M, \text{Fil}_1^\bullet)$ is acyclic, $\text{Fil}_1^0 M_K = M_K$, $\text{Fil}_1^{r+1} M_K = 0$, and $\deg(M, \text{Fil}_1^\bullet) < \deg(M, \text{Fil}^\bullet)$. Iterating the process gives the wanted filtration. \square

4.1.3. *The complex attached to a filtered $(\varphi, N, \mathcal{G}_K)$ -module.* If (M, Fil^\bullet) is a filtered $(\varphi, N, \mathcal{G}_K)$ -module over K , we set

$$X_{\text{st}}(M, \text{Fil}^\bullet) := (M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}})^{N=0, \varphi=1}, \quad X_{\text{dR}}(M, \text{Fil}^\bullet) := (M_K \otimes_K \mathbf{B}_{\text{dR}}) / \text{Fil}^0(M_K \otimes_K \mathbf{B}_{\text{dR}}).$$

Then $X_{\text{st}}(M, \text{Fil}^\bullet)$ and $X_{\text{dR}}(M, \text{Fil}^\bullet)$ are inductive limits of the $X_{\text{st}}^{(r)}(M, \text{Fil}^\bullet)$ and $X_{\text{dR}}^{(r)}(M, \text{Fil}^\bullet)$ (defined by replacing \mathbf{B}_{st} and \mathbf{B}_{dR} by $t^{-r} \mathbf{B}_{\text{st}}^+$ and $t^{-r} \mathbf{B}_{\text{dR}}^+$), which are objects of $\mathcal{C}(\mathcal{G}_K)$.

The complex $X_{\text{st}}(M, \text{Fil}^\bullet) \rightarrow X_{\text{dR}}(M, \text{Fil}^\bullet)$ is called [12, §5.3] the “fundamental complex associated to M ”. Its H^0 is denoted by $V_{\text{st}}(M, \text{Fil}^\bullet)$. Hence, we have an exact sequence

$$0 \rightarrow V_{\text{st}}(M, \text{Fil}^\bullet) \rightarrow X_{\text{st}}(M, \text{Fil}^\bullet) \rightarrow X_{\text{dR}}(M, \text{Fil}^\bullet)$$

Remark 4.5. (i) The cohomology of the fundamental complex is equal to that of the complex $X_{\text{st}}^{(r)}(M, \text{Fil}^\bullet) \rightarrow X_{\text{dR}}^{(r)}(M, \text{Fil}^\bullet)$, for r big enough. It follows that its cohomology groups are objects of $\mathcal{C}(\mathcal{G}_K)$.

(ii) The pair $(X_{\text{st}}(M, \text{Fil}^\bullet), X_{\text{dR}}(M, \text{Fil}^\bullet))$ is a B -pair in the sense of Berger [3, 4]; attached to it is a \mathcal{G}_K -equivariant vector bundle $\mathcal{E}(M, \text{Fil}^\bullet)$ on the Fargues-Fontaine curve [19, §10.1] and the fundamental complex computes the cohomology (not the \mathcal{G}_K -equivariant) of this vector bundle. Since the HN-slope of $\mathcal{E}(M, \text{Fil}^\bullet)$ is that of (M, Fil^\bullet) , the vector bundle $\mathcal{E}(M, \text{Fil}^\bullet)$ has vanishing H^1 if and only if (M, Fil^\bullet) is acyclic (see Theorem 3.9 and formulas (3.10)). It follows that, if (M, Fil^\bullet) is acyclic, we have an exact sequence

$$0 \rightarrow V_{\text{st}}(M, \text{Fil}^\bullet) \rightarrow X_{\text{st}}(M, \text{Fil}^\bullet) \rightarrow X_{\text{dR}}(M, \text{Fil}^\bullet) \rightarrow 0$$

(iii) If (M, Fil^\bullet) is acyclic, then $V_{\text{st}}(M, \text{Fil}^\bullet)$ is a finite dimensional \mathbf{Q}_p -vector space if and only if (M, Fil^\bullet) is of slope 0 (i.e., is weakly admissible). This implies it is *admissible*: the natural maps give \mathcal{G}_K -equivariant isomorphisms

$$\begin{aligned} V_{\text{st}}(M, \text{Fil}^\bullet) \otimes \mathbf{B}_{\text{st}} &\xrightarrow{\sim} M \otimes_F \mathbf{B}_{\text{st}} && \text{of } \mathbf{B}_{\text{st}}\text{-modules commuting with } \varphi \text{ and } N, \\ V_{\text{st}}(M, \text{Fil}^\bullet) \otimes \mathbf{B}_{\text{dR}} &\xrightarrow{\sim} M_K \otimes_K \mathbf{B}_{\text{dR}} && \text{of filtered } \mathbf{B}_{\text{dR}}\text{-modules,} \end{aligned}$$

and $V := V_{\text{st}}(M, \text{Fil}^\bullet)$ is a potentially semi-stable representation of \mathcal{G}_K , with $D_{\text{dR}}(V) = M_K$ and $D_{\text{pst}}(V) = M$.

(iv) If (M, Fil^\bullet) is acyclic, but not admissible, the natural map

$$V_{\text{st}}(M, \text{Fil}^\bullet) \otimes \mathbf{B}_{\text{dR}}^+ \rightarrow \text{Fil}^0(M_K \otimes_K \mathbf{B}_{\text{dR}})$$

is not an isomorphism (the kernel is huge), but it is surjective. Indeed, one can pick a filtration Fil_1^\bullet on M_K , such that $\text{Fil}_1^i M_K \subset \text{Fil}^i M_K$ for all i , and $(M, \text{Fil}_1^\bullet)$ is admissible. Then we have an exact sequence

$$(4.6) \quad 0 \rightarrow V_{\text{st}}(M, \text{Fil}_1^\bullet) \rightarrow V_{\text{st}}(M, \text{Fil}^\bullet) \rightarrow \text{Fil}^0(M_K \otimes_K \mathbf{B}_{\text{dR}}) / \text{Fil}_1^0(M_K \otimes_K \mathbf{B}_{\text{dR}}) \rightarrow 0$$

and $\mathbf{B}_{\text{dR}}^+ \cdot V_{\text{st}}(M, \text{Fil}^\bullet)$ contains $\mathbf{B}_{\text{dR}}^+ \cdot V_{\text{st}}(M, \text{Fil}_1^\bullet)$ which, by admissibility of $(M, \text{Fil}_1^\bullet)$, is equal to $\text{Fil}_1^0(M_K \otimes_K \mathbf{B}_{\text{dR}})$ and the above exact sequence gives the desired surjectivity.

4.2. Filtered $(\varphi, N, \mathcal{G}_K)$ -modules and almost C -representations.

4.2.1. *The functors D_{dR}^* and D_{st}^* .* The definitions of the classical contravariant functors D_{dR}^* , D_{st}^* and D_{pst}^* for finite dimensional \mathbf{Q}_p -representations of \mathcal{G}_K extend to objects of $\mathcal{C}(\mathcal{G}_K)$ contrarily to those of the (more commonly used) covariant functors D_{dR} , D_{st} and D_{pst} . If $W \in \mathcal{C}(\mathcal{G}_K)$, set

$$D_{\text{st}}^*(W) := \text{Hom}_{\mathcal{G}_K}(W, \mathbf{B}_{\text{st}}), \quad D_{\text{dR}}^*(W) := \text{Hom}_{\mathcal{G}_K}(W, \mathbf{B}_{\text{dR}}).$$

Then $D_{\text{st}}^*(W)$ is a (φ, N) -module over K (with $\langle \varphi(\mu), v \rangle = \varphi(\langle \mu, v \rangle)$ and $\langle N(\mu), v \rangle = N(\langle \mu, v \rangle)$) and $D_{\text{dR}}^*(W)$ is a filtered module over K (with $\text{Fil}^i D_{\text{dR}}^*(W) = \{\mu, \mu(W) \subset t^i \mathbf{B}_{\text{dR}}^+\}$). We also define $D_{\text{pst}}^*(W)$ as:

$$D_{\text{pst}}^*(W) := \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(W, \mathbf{B}_{\text{st}}) := \varinjlim_{[L:K] < \infty} \text{Hom}_{\mathcal{G}_L}(W, \mathbf{B}_{\text{st}})$$

This is a $(\varphi, N, \mathcal{G}_K)$ -module over F^{nr} .

Remark 4.7. The natural maps

$$D_{\text{st}}^*(W) \otimes_F K \rightarrow D_{\text{dR}}^*(W), \quad (D_{\text{pst}}^*(W) \otimes_{F^{\text{nr}}} \overline{K})^{\mathcal{G}_K} \rightarrow D_{\text{dR}}^*(W)$$

induced by the injections $\mathbf{B}_{\text{st}} \otimes_F K \hookrightarrow \mathbf{B}_{\text{dR}}$ and $\mathbf{B}_{\text{st}} \otimes_{F^{\text{nr}}} \overline{K} \hookrightarrow \mathbf{B}_{\text{dR}}$ are injective.

The following result is an extension to acyclic filtered (φ, N) -modules of a classical result for admissible filtered (φ, N) -modules.

Theorem 4.8. *If (M, Fil^\bullet) is an acyclic filtered (φ, N) -module over K , then*

$$\begin{aligned} D_{\text{st}}^*(V_{\text{st}}(M, \text{Fil}^\bullet)) &\simeq M^* && \text{as } (\varphi, N)\text{-modules over } F, \\ D_{\text{dR}}^*(V_{\text{st}}(M, \text{Fil}^\bullet)) &\simeq (M_K^*, \text{Fil}_\perp^\bullet) && \text{as filtered } K\text{-modules.} \end{aligned}$$

Proof. We start with noticing that the natural pairing injects M^* into $\text{Hom}_{\mathcal{G}_K}(X_{\text{st}}(M), \mathbf{B}_{\text{st}})$ (as a (φ, N) -module) and M_K^* into $\text{Hom}_{\mathcal{G}_K}(X_{\text{st}}(M), \mathbf{B}_{\text{dR}})$. Moreover, the natural map from $K \otimes_F \text{Hom}_{\mathcal{G}_K}(X_{\text{st}}(M), \mathbf{B}_{\text{st}})$ to $\text{Hom}_{\mathcal{G}_K}(X_{\text{st}}(M), \mathbf{B}_{\text{dR}})$ is injective (see Remark 4.7), and one can deduce that $D_{\text{st}}^* \rightarrow M^*$ is an isomorphism from the same result for D_{dR}^* .

Hence, we just have to prove the result for D_{dR}^* . To do so, pick a filtration Fil_1^\bullet as in (iv) of Remark 4.5: $\text{Fil}_1^i M_K \subset \text{Fil}^i M_K$ for all i and $(M, \text{Fil}_1^\bullet)$ is admissible. Let $V := V_{\text{st}}(M, \text{Fil}_1^\bullet)$, $V_1 := V_{\text{st}}(M, \text{Fil}_1^\bullet)$, and $W = \text{Fil}^0(M_K \otimes_K \mathbf{B}_{\text{dR}}) / \text{Fil}_1^0(M_K \otimes_K \mathbf{B}_{\text{dR}})$, so that the exact sequence (4.6) becomes $0 \rightarrow V_1 \rightarrow V \rightarrow W \rightarrow 0$.

Now, W is a direct sum of factors of the form $t^{k_1} \mathbf{B}_{\text{dR}}^+ / t^{k_2} \mathbf{B}_{\text{dR}}^+$. Hence $D_{\text{dR}}^*(W) = 0$ thanks to Proposition 2.14, and we get injections $M_K^* \hookrightarrow D_{\text{dR}}^*(V) \hookrightarrow D_{\text{dR}}^*(V_1)$. But $\dim_K D_{\text{dR}}^*(V_1) \leq \dim_{\mathbf{Q}_p} V_1 = \dim_K M_K$ (the first inequality is true for any finite dimensional \mathbf{Q}_p -representation of \mathcal{G}_K and the second equality is true because V_1 is de Rham). It follows that these injections are in fact isomorphisms, which proves what we want except for the equality of the filtrations on M_K^* and $D_{\text{dR}}^*(V)$.

So let $\mu \in M_K^*$. One can extend μ to a \mathbf{B}_{dR} -linear map $\mathbf{B}_{\text{dR}} \otimes_K M_K \rightarrow \mathbf{B}_{\text{dR}}$. Thanks to (iv) of Remark 4.5, one sees that $\mu(V) \subset t^i \mathbf{B}_{\text{dR}}^+$ is equivalent to $\mu(\text{Fil}^0(M_K \otimes_K \mathbf{B}_{\text{dR}})) \subset t^i \mathbf{B}_{\text{dR}}^+$. But $\text{Fil}^0(M_K \otimes_K \mathbf{B}_{\text{dR}}) = \sum_n \text{Fil}^n M_K \otimes_K t^{-n} \mathbf{B}_{\text{dR}}^+$, and $\mu(M_K) \subset K$, hence $\mu(\text{Fil}^0(M_K \otimes_K \mathbf{B}_{\text{dR}})) \subset t^i \mathbf{B}_{\text{dR}}^+$ if and only if $\mu(\text{Fil}^n M_K) = 0$ for $n \geq 1 - i$. By definition this translates into $\mu \in \text{Fil}_\perp^i M_K^*$, as wanted. \square

Corollary 4.9. *If (M, Fil^\bullet) is an acyclic filtered $(\varphi, N, \mathcal{G}_K)$ -module over K , then*

$$\begin{aligned} D_{\text{pst}}^*(V_{\text{st}}(M, \text{Fil}^\bullet)) &\simeq M^* && \text{as } (\varphi, N, \mathcal{G}_K)\text{-modules over } F^{\text{nr}}, \\ D_{\text{dR}}^*(V_{\text{st}}(M, \text{Fil}^\bullet)) &\simeq (M_K^*, \text{Fil}_\perp^\bullet) && \text{as filtered } K\text{-modules.} \end{aligned}$$

Proof. As in the proof of the above theorem, we have an injection of the right hand sides into the left hand sides, and to check that these are isomorphisms, we just have to bound the dimensions of the left hand sides. This can be achieved by passing to a finite extension L of K such that the

action of \mathcal{G}_L on M is unramified, and applying the proposition to a (φ, N) -module $M(L)$ over F_L such that $M = F^{\text{nr}} \otimes_{F_L} M(L)$. \square

Corollary 4.10. *Let (M, Fil^\bullet) be an acyclic filtered $(\varphi, N, \mathcal{G}_K)$ -module over K , with φ -slopes in $[0, r]$, and $\text{Fil}^0 M_K = M_K$, $\text{Fil}^{r+1} M_K = 0$. Set*

$$V_{\text{st}}^r(M, \text{Fil}^\bullet) := \text{Ker}((M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \rightarrow (M_K \otimes_K \mathbf{B}_{\text{dR}}^+)/\text{Fil}^r)$$

Then¹¹

$$\begin{aligned} D_{\text{pst}}^*(V_{\text{st}}^r(M, \text{Fil}^\bullet)) &\simeq M^*\{r\} && \text{as } (\varphi, N, \mathcal{G}_K)\text{-modules over } F^{\text{nr}}, \\ D_{\text{dR}}^*(V_{\text{st}}^r(M, \text{Fil}^\bullet)) &\simeq (M_K^*, \text{Fil}_\perp^\bullet\{r\}) && \text{as filtered } K\text{-modules.} \end{aligned}$$

Proof. The conditions imply that $V_{\text{st}}^r(M, \text{Fil}^\bullet) = t^r V_{\text{st}}(M, \text{Fil}^\bullet)$. Hence

$$\text{Hom}_{\mathcal{G}_K}(V_{\text{st}}^r(M, \text{Fil}^\bullet), \mathbf{B}_?) = t^r \text{Hom}_{\mathcal{G}_K}(V_{\text{st}}(M, \text{Fil}^\bullet), \mathbf{B}_?)$$

and the result follows. \square

Example 4.11. If M in Corollary 4.10 satisfies $\text{Fil}^r M_K = M_K$, then $(M_K \otimes_K \mathbf{B}_{\text{dR}}^+)/\text{Fil}^r = 0$. Hence

$$V_{\text{st}}^r(M, \text{Fil}^\bullet) = X_{\text{st}}^r(M) := (M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}$$

The corollary then becomes:

$$\begin{aligned} \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(X_{\text{st}}^r(M), \mathbf{B}_{\text{st}}) &\simeq M^*, \quad \text{as } (\varphi, N, \mathcal{G}_K)\text{-modules over } F^{\text{nr}}, \\ \text{Hom}_{\mathcal{G}_K}(X_{\text{st}}^r(M), t^j \mathbf{B}_{\text{dR}}^+) &\simeq \begin{cases} M_K & \text{if } j \leq 0, \\ 0 & \text{if } j \geq 1. \end{cases} \end{aligned}$$

Lemma 4.12. *Under the hypothesis of Corollary 4.10, we have $\text{Hom}_{\mathcal{G}_K}(V_{\text{st}}^r(M, \text{Fil}^\bullet), C(j)) = 0$, for $j \geq r + 1$.*

Proof. We can write $V := V_{\text{st}}^r(M, \text{Fil}^\bullet)$ in the form $0 \rightarrow V_1 \rightarrow V \rightarrow W \rightarrow 0$, as in the proof of Theorem 4.8, with V_1 of dimension $\dim(M)$, de Rham with Hodge-Tate weights in $[0, r]$, and W a finite type \mathbf{B}_{dR}^+ -module, sum of $t^a \mathbf{B}_{\text{dR}}^+/t^b \mathbf{B}_{\text{dR}}^+$'s, with $0 \leq a \leq b \leq r + 1$.

We have an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{E}(\mathcal{G}_K)}(W, C(j)) \rightarrow \text{Hom}_{\mathcal{E}(\mathcal{G}_K)}(V, C(j)) \rightarrow \text{Hom}_{\mathcal{E}(\mathcal{G}_K)}(V_1, C(j))$$

Since $j \geq r + 1$, by Lemma 2.10, we have $\text{Hom}_{\mathcal{E}(\mathcal{G}_K)}(t^a \mathbf{B}_{\text{dR}}^+/t^b \mathbf{B}_{\text{dR}}^+, C(j)) = 0$ if $b \leq r + 1$; hence $\text{Hom}_{\mathcal{E}(\mathcal{G}_K)}(W, C(j)) = 0$. Since the Hodge-Tate weights of V_1 are $\leq r$, we also have $\text{Hom}_{\mathcal{E}(\mathcal{G}_K)}(V_1, C(j)) = 0$.

This concludes the proof. \square

4.2.2. *Projective limits of filtered $(\varphi, N, \mathcal{G}_K)$ -modules.* Let

$$(M, \text{Fil}^\bullet) = \lim_n (M_n, \text{Fil}_n^\bullet),$$

where the $(M_n, \text{Fil}_n^\bullet)$'s are finite dimensional filtered $(\varphi, N, \mathcal{G}_K)$ -modules over K (hence $M_K = \lim_n M_{n,K}$ is endowed with the projective limit filtration). We define the dual $(M^*, \text{Fil}_\perp^\bullet)$ of (M, Fil^\bullet) as the inductive limit $\text{colim}_n (M_n^*, \text{Fil}_{n,\perp}^\bullet)$ of the duals of the $(M_n, \text{Fil}_n^\bullet)$'s.

Proposition 4.13. *Suppose the $(M_n, \text{Fil}_n^\bullet)$'s are acyclic, with φ -slopes in $[0, r]$, and $\text{Fil}^0 M_{n,K} = M_{n,K}$, $\text{Fil}^{r+1} M_{n,K} = 0$. Set*

$$V_{\text{st}}^r(M, \text{Fil}^\bullet) := \text{Ker}((M \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \rightarrow (M_K \widehat{\otimes}_K \mathbf{B}_{\text{dR}}^+)/\text{Fil}^r)$$

¹¹The notations $M\{r\}$ and $\text{Fil}_\perp^\bullet\{r\}$ mean that the action of φ is multiplied by p^r and that the filtration is shifted by r : we have $\text{Fil}_\perp^i\{r\} = \text{Fil}_\perp^{i-r}$.

(i) We have a strictly exact sequence

$$0 \rightarrow V_{\text{st}}^r(M, \text{Fil}^\bullet) \rightarrow (M \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \rightarrow (M_K \widehat{\otimes}_K \mathbf{B}_{\text{dR}}^+) / \text{Fil}^r \rightarrow 0.$$

(ii) We have strict isomorphisms

$$\begin{aligned} D_{\text{pst}}^*(V_{\text{st}}^r(M, \text{Fil}^\bullet)) &\simeq M^*\{r\} && \text{as } (\varphi, N, \mathcal{G}_K)\text{-modules over } F^{\text{nr}}, \\ D_{\text{dR}}^*(V_{\text{st}}^r(M, \text{Fil}^\bullet)) &\simeq (M_K^*, \text{Fil}_\perp^*\{r\}) && \text{as filtered } K\text{-modules.} \end{aligned}$$

Proof. Let $V_n^r := V_{\text{st}}^r(M'_n, \text{Fil}_n^\bullet)$ with $M'_n := \text{Im}(M \rightarrow M_n)$, so that $V^r := V_{\text{st}}^r(M, \text{Fil}^\bullet) \simeq \lim_n V_n^r$, and $M \simeq \lim_n M'_n$ with surjective transition maps. Set $M'_{n,K} := (M'_n \otimes_{F^{\text{nr}}} \overline{K})^{\mathcal{G}_K}$.

Since M_n is of finite rank over F^{nr} (hence F^{nr} -linear maps from M'_n extend to M_n , etc), and the jumps of the filtrations are uniformly bounded, we have strict isomorphisms

$$\text{colim}_n M_n^* \xrightarrow{\sim} \text{colim}_n (M'_n)^*, \quad \text{colim}_n M'_{n,K} \xrightarrow{\sim} \text{colim}_n (M'_{n,K})^*.$$

Hence we can assume, from the start, that the transition maps are surjective and that the morphisms of filtered modules are strict.

To prove (i), one just needs to check surjectivity on the right, but this follows from the exact sequences for the M_n 's and the fact that the $V_{\text{st}}(M_n, \text{Fil}_n^\bullet)$'s are C -points of BC's, hence their $\mathbf{R}^1 \text{lim}$'s vanish (see Remark 3.14).

To prove (ii), granting Corollary 4.10, we are reduced to showing that any continuous $\lambda : V_{\text{st}}^r(M, \text{Fil}^\bullet) \rightarrow \mathbf{B}_{\text{dR}}$ commuting with \mathcal{G}_K factors through V_n^r , for n big enough. Now, it follows from Lemma 4.15 below) that $\lambda(V^r) \subset t^{-N} \mathbf{B}_{\text{dR}}^+$, for some $N \in \mathbf{N}$. But V^r is the projective limit of the V_n^r which are Banach spaces, and the transition maps are surjective which makes it possible to choose continuous (not necessarily Galois equivariant !) sections $V_{n-1}^r \rightarrow V_n^r$, and write V^r as a product $\prod_n V'_n$, with $V'_n = \text{Ker}(V_n^r \rightarrow V_{n-1}^r)$ (and $V_n^r = V_{n-1}^r \oplus V'_n$). Since $t^{-N} \mathbf{B}_{\text{dR}}^+ / t^k \mathbf{B}_{\text{dR}}^+$ is a Banach space, it follows, by Lemma 4.14 below, that $\lambda_k : V^r \rightarrow t^{-N} \mathbf{B}_{\text{dR}}^+ / t^k \mathbf{B}_{\text{dR}}^+$ factors through a finite product of V'_n and, hence, through some $V_{n_k}^r$. Let Y_n^r be the kernel of $V^r \rightarrow V_n^r$. Then λ_k induces a \mathcal{G}_L -equivariant map $Y_{n_k} / Y_{n_{k+1}} \rightarrow C(k)$. But $Y_{n_k} / Y_{n_{k+1}} \cong V_{\text{st}}^r(\text{Ker}(M_{n_{k+1}} \rightarrow M_{n_k}), \text{Fil}^\bullet)$, and such a map is identically 0, by Lemma 4.12, if $k \geq r + 1$. Hence $\lambda(Y_{n_{r+1}}) = 0$; i.e., λ factors through $V_{n_{r+1}}^r$.

This concludes the proof. \square

Lemma 4.14. *Let $\lambda : \prod_{n \geq 0} W_n \rightarrow \mathbf{B}_{\text{dR}}$ be continuous, where the W_n 's are \mathbf{Q}_p -Banach spaces.*

- (i) *There exists N such that $\lambda(\prod_{n \geq 0} W_n) \subset t^{-N} \mathbf{B}_{\text{dR}}^+$.*
- (ii) *If $j \in \mathbf{N}$, there exists $N(j)$ such that $\lambda(\prod_{n \geq N(j)} W_n) \subset t^j \mathbf{B}_{\text{dR}}^+$.*

Proof. (i) is a consequence of the fact that $\prod_{n \geq 0} W_n$ is Fréchet and \mathbf{B}_{dR} is the inductive limit of the Fréchet's $t^{-N} \mathbf{B}_{\text{dR}}^+$.

(ii) is a consequence of the fact that $t^{-N} \mathbf{B}_{\text{dR}}^+ / t^j \mathbf{B}_{\text{dR}}^+$ is a Banach; hence $\lambda_j : \prod_{n \geq 0} W_n \rightarrow t^{-N} \mathbf{B}_{\text{dR}}^+ / t^j \mathbf{B}_{\text{dR}}^+$ factors through $\prod_{n < N(j)} W_n$. \square

Lemma 4.15. *Let $W = \varprojlim_n W_n$, where the W_n 's are \mathbf{Q}_p -Banach spaces, the transition maps are strict and the system is Mittag-Leffler.*

- (i) *If Y is a \mathbf{Q}_p -Banach space, $\text{Hom}(W, Y) = \varinjlim_n \text{Hom}(W_n, Y)$.*
- (ii) *$\text{Hom}(W, t^j \mathbf{B}_{\text{dR}}^+) = \varprojlim_k \varinjlim_n \text{Hom}(W_n, t^j \mathbf{B}_{\text{dR}}^+ / t^{j+k} \mathbf{B}_{\text{dR}}^+)$.*
- (iii) *$\text{Hom}(W, \mathbf{B}_{\text{dR}}) = \text{Hom}(W, \mathbf{B}_{\text{dR}}^+) \otimes_{\mathbf{B}_{\text{dR}}^+} \mathbf{B}_{\text{dR}}$.*

Proof. Since the system is Mittag-Leffler, one can replace W_n by the image of $W_{n+k} \rightarrow W_n$ for k big enough (this image is still a Banach space by the strictness assumption) without changing W or $\varinjlim_n \text{Hom}(W_n, Y)$. Hence we may assume $W_{n+1} \rightarrow W_n$ to be surjective, and we can choose a supplementary Banach subspace W''_n of $W'_n = \text{Ker}(W_{n+1} \rightarrow W_n)$ inside W_{n+1} : then $W''_n \rightarrow W_n$ is an isomorphism of Banach spaces, hence $W \simeq \prod_n W'_n$. This implies $\text{Hom}(W, Y) = \oplus_n \text{Hom}(W'_n, Y)$ from which (i) follows (since $W_n = \prod_{i \leq n} W'_i$).

Having written W as $\prod_n W'_n$, we can apply Lemma 4.14 to deduce (ii) and (iii) from (i), since $t^j \mathbf{B}_{\text{dR}}^+ / t^{j+k} \mathbf{B}_{\text{dR}}^+$ is a \mathbf{Q}_p -Banach space. \square

4.3. Filtered (φ, N) -modules over C .

4.3.1. *Filtered (φ, N) -modules and vector bundles.* A filtered (φ, N) -module (M, M_{dR}^+) over C is a tuple $(M, \varphi, N, M_{\text{dR}}^+)$, where:

- (1) M is a finite dimensional F^{nr} -vector space;
- (2) $\varphi : M \rightarrow M$ is a Frobenius map;
- (3) $N : M \rightarrow M$ is a F^{nr} -linear monodromy map such that $N\varphi = p\varphi N$;
- (4) $M_{\text{dR}}^+ \subset M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}$ is a sub- \mathbf{B}_{dR}^+ -lattice.

To such an (M, M_{dR}^+) one can attach a vector bundle $\mathcal{E}(M, M_{\text{dR}}^+)$ on X_{FF} , characterized by

$$H^0(X_{\text{FF}} \setminus \{\infty\}, \mathcal{E}(M, M_{\text{dR}}^+)) = (M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}})^{N=0, \varphi=1}, \quad \mathcal{E}(M, M_{\text{dR}}^+) \otimes_{\widehat{\mathcal{O}}_{X_{\text{FF}}, \infty}} = M_{\text{dR}}^+.$$

We set

$$V_{\text{st}}(M, M_{\text{dR}}^+) := H^0(X_{\text{FF}}, \mathcal{E}(M, M_{\text{dR}}^+)) = \text{Ker}((M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}})^{N=0, \varphi=1} \rightarrow (M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}) / M_{\text{dR}}^+)$$

Definition 4.16. (M, M_{dR}^+) is *weakly admissible* if $\mathcal{E}(M, M_{\text{dR}}^+)$ is semistable, of slope 0. It is *acyclic* if the slopes of $\mathcal{E}(M, M_{\text{dR}}^+)$ are ≥ 0 .

Remark 4.17. (i) If (M, Fil^\bullet) is a weakly admissible (resp. acyclic) filtered (φ, N) -module over K , the induced filtered (φ, N) -module $(M \otimes_F F^{\text{nr}}, \text{Fil}^0(M_K \otimes_K \mathbf{B}_{\text{dR}}))$ over C is weakly admissible (resp. acyclic).

(ii) It follows from the classification of vector bundles on X_{FF} that (M, M_{dR}^+) is weakly admissible if and only if it is admissible (i.e., a direct sum of trivial line bundles $\mathcal{O}_{X_{\text{FF}}}$). This translates into the following: (M, M_{dR}^+) is weakly admissible if and only if $V_{\text{st}}(M, M_{\text{dR}}^+)$ is finite dimensional over \mathbf{Q}_p and the sequence

$$0 \rightarrow V_{\text{st}}(M, M_{\text{dR}}^+) \rightarrow (M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}})^{N=0, \varphi=1} \rightarrow (M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}) / M_{\text{dR}}^+ \rightarrow 0$$

is exact. Moreover, if this is the case, the triviality of $\mathcal{E}(M, M_{\text{dR}}^+)$ implies that the natural maps

$$V_{\text{st}}(M, M_{\text{dR}}^+) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}} \rightarrow M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}, \quad V_{\text{st}}(M, M_{\text{dR}}^+) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}}^+ \rightarrow M_{\text{dR}}^+$$

are isomorphisms (for the second map this is also equivalent to the map $V_{\text{st}}(M, M_{\text{dR}}^+) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}} \rightarrow M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}$ being a filtered isomorphism).

(iii) The following conditions are equivalent:

- (M, M_{dR}^+) is acyclic,
- $H^1(X, \mathcal{E}(M, M_{\text{dR}}^+)) = 0$ (and hence $\mathcal{E}(M, M_{\text{dR}}^+)$ is acyclic),
- $(M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}})^{N=0, \varphi=1} \rightarrow (M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}) / M_{\text{dR}}^+$ is surjective.
- There exists a sub- \mathbf{B}_{dR}^+ lattice $N_{\text{dR}}^+ \subset M_{\text{dR}}^+$ such that (M, N_{dR}^+) is weakly admissible.

(The first two points are equivalent by Theorem 3.9 and formulas (3.10), the second and the third are equivalent because

$$H^1(X, \mathcal{E}(M, M_{\text{dR}}^+)) = \text{Coker}((M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}})^{N=0, \varphi=1} \rightarrow (M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}) / M_{\text{dR}}^+)$$

the first and last points are equivalent by the same arguments as in Lemma 4.3.)

Remark 4.18. Since $(M, M_{\text{dR}}^+) \mapsto \mathcal{E}(M, M_{\text{dR}}^+)$ commutes with tensor products, and since the slope of $\mathcal{E} \otimes \mathcal{E}'$ is equal to the sum of slopes of \mathcal{E} and \mathcal{E}' , the tensor product of two acyclic filtered φ -modules is again acyclic.

Remark 4.19. Assume that the φ -slopes are in $[0, r]$ and $M \otimes t^r \mathbf{B}_{\text{dR}}^+ \subset t^r M_{\text{dR}}^+ \subset M \otimes \mathbf{B}_{\text{dR}}^+$. Then the following conditions are equivalent:

- (M, M_{dR}^+) is acyclic,
- $(M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}})^{N=0, \varphi=p^r} \rightarrow (M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}) / t^r M_{\text{dR}}^+$ is surjective,

- for all $k \geq 0$, $(M \otimes_{F^{\text{nr}}} t^{-k} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \rightarrow (M \otimes_{F^{\text{nr}}} t^{-k} \mathbf{B}_{\text{dR}}^+)/t^r M_{\text{dR}}^+$ is surjective.

This follows from the fact that $\frac{(M \otimes_{F^{\text{nr}}} t^{-k} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}}{(M \otimes_{F^{\text{nr}}} t^{1-k} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}} \simeq \frac{M \otimes_{F^{\text{nr}}} t^{-k} \mathbf{B}_{\text{dR}}^+}{M \otimes_{F^{\text{nr}}} t^{1-k} \mathbf{B}_{\text{dR}}^+}$, for $k \geq 1$, as can be shown, for example, by a Dimension of BC argument. For the same reasons, for all $k \geq 0$,

$$t^r V_{\text{st}}(M, M_{\text{dR}}^+) = \text{Ker}((M \otimes_{F^{\text{nr}}} t^{-k} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \rightarrow (M \otimes_{F^{\text{nr}}} t^{-k} \mathbf{B}_{\text{dR}}^+)/t^r M_{\text{dR}}^+)$$

4.3.2. *Acyclicity and curvature.* The following results supply key arguments in the proof of our main comparison theorem (Theorem 6.14).

Lemma 4.20. *Let M be a (φ, N) -module over F^{nr} whose φ -slopes are in $[0, r]$. Then the \mathbf{Q}_p -module $(M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}$ generates the \mathbf{B}_{dR}^+ -module $M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}^+$.*

Proof. Since \mathbf{B}_{dR}^+ is a local ring, with residue field C , it suffices to show that $(M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}$ generates the C -module $M \otimes_{F^{\text{nr}}} C$.

Now, the map $x \mapsto x - uNx + \frac{u^2}{2!} N^2 x - \frac{u^3}{3!} N^3 x + \dots$, for $u \in \mathbf{B}_{\text{st}}^+$ mapping to $\log([p^b]/p)$ in \mathbf{B}_{dR}^+ , induces a φ -equivariant isomorphism $M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{cris}}^+ \rightarrow (M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0}$. Since u has image 0 in C , we are reduced to proving that $(M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{cris}}^+)^{\varphi=p^r}$ generates the C -module $M \otimes_{F^{\text{nr}}} C$.

Since $\mathbf{B}_{\text{cris}}^+$ contains $W(k_C)$, the theorem of Dieudonné-Manin allows us to reduce to the case where M is elementary with slopes $\frac{a}{h} \leq r$, i.e., it is generated by e_1, \dots, e_h with

$$\varphi(e_1) = e_2, \varphi(e_2) = e_3, \dots, \varphi(e_h) = p^a e_1.$$

The map $x \mapsto xe_1 + p^{-r} \varphi(x)e_2 + \dots + p^{-(h-1)r} \varphi^{h-1}(x)e_h$ induces an isomorphism of $U_{h, rh-a} := (\mathbf{B}_{\text{cris}}^+)^{\varphi^h = p^{rh-a}}$ with $(M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{cris}}^+)^{\varphi=p^r}$. Hence we are reduced to proving that the image of $U_{h, rh-a}$ by the map $x \mapsto (\theta(x), \theta(\varphi(x)), \dots, \theta(\varphi^{h-1}(x)))$ does not lie inside a proper sub- C -module of C^h . Assume that it does. Then there exists $\lambda_0, \dots, \lambda_{h-1} \in C$, not all zero, such that $\lambda_0 \theta(x) + \lambda_1 \theta(\varphi(x)) + \dots + \lambda_{h-1} \theta(\varphi^{h-1}(x)) = 0$ for all $x \in U_{h, rh-a}$. In particular, one can apply this to αx , for $\alpha \in \mathbf{Q}_p^h$. Setting $\mu_i(x) = \lambda_i \theta(\varphi^i(x))$, we get $\mu_0(x)\alpha + \mu_1(x)\varphi(\alpha) + \dots + \mu_{h-1}(x)\varphi^{h-1}(\alpha) = 0$, for all $\alpha \in \mathbf{Q}_p^h$. Linear independence of characters implies that $\mu_i(x) = 0$ for all i and x , and we get our contradiction. \square

Proposition 4.21. *Let M be a (φ, N) -module over F^{nr} with φ -slopes in $[0, r]$. Let M_{dR}^+ be a \mathbf{B}_{dR}^+ -lattice in $M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}$, with $M \otimes_{F^{\text{nr}}} t^r \mathbf{B}_{\text{dR}}^+ \subset t^r M_{\text{dR}}^+ \subset M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}^+$. Set $\mathbb{M}_{\text{dR}}^+ = M_{\text{dR}}^+ \otimes_{\mathbf{B}_{\text{dR}}^+} \mathbb{B}_{\text{dR}}^+$. Then one and only one of the following holds:*

- The map $(M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \rightarrow (M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}^+)/t^r M_{\text{dR}}^+$ is surjective.
- The image of the map $(M \otimes_{F^{\text{nr}}} \mathbb{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \rightarrow (M \otimes_{F^{\text{nr}}} \mathbb{B}_{\text{dR}}^+)/t^r \mathbb{M}_{\text{dR}}^+$ has height > 0 .

Proof. Since $(M \otimes_{F^{\text{nr}}} \mathbb{B}_{\text{dR}}^+)/t^r \mathbb{M}_{\text{dR}}^+$ has curvature 0, all its sub-BC's have curvature ≤ 0 . Hence, if we are not in case (b), this implies that the curvature of the image is 0 (use Corollary 3.18), and hence that the image is a \mathbb{B}_{dR}^+ -module (see Proposition 3.15). Now Lemma 4.20 implies that we are in case (a), as wanted. \square

Remark 4.22. In case (b), $\text{Coker}((M \otimes_{F^{\text{nr}}} \mathbb{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \rightarrow (M \otimes_{F^{\text{nr}}} \mathbb{B}_{\text{dR}}^+)/t^r \mathbb{M}_{\text{dR}}^+)$ has height < 0 and thus does not have curvature ≤ 0 .

4.4. **Filtered (φ, N) -modules and BC's.** The following computations supply key arguments in the proofs of our pro-étale-to-de Rham comparison theorems in Chapter 8.

4.4.1. *Finite rank (φ, N) -modules.*

Proposition 4.23. *Let M be a (φ, N) -module with φ -slopes in $[0, r]$, and let*

$$\mathbb{X}_{\text{st}}^r(M) := (M \otimes_{F^{\text{nr}}} \mathbb{B}_{\text{st}}^+)^{N=0, \varphi=p^r}.$$

Then:

$$\text{Hom}_{\text{VS}}(\mathbb{X}_{\text{st}}^r(M), \mathbb{B}_{\text{dR}}) \simeq M^* \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}, \quad \text{Hom}_{\text{VS}}(\mathbb{X}_{\text{st}}^r(M), \mathbb{B}_{\text{st}}) \simeq M^* \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}$$

Proof. $\mathbb{X}_{\text{st}}^r(M)$ is of curvature ≤ 0 , and the condition on the slopes imply ([13, Ex. 5.18]) that $\text{ht}(\mathbb{X}_{\text{st}}^r(M)) = \text{rk}(M)$. It follows from Proposition 3.24 that $\text{rk}(h(\mathbb{X}_{\text{st}}^r(M))) = \text{rk}(M)$. On the other hand, the inclusion $\mathbb{B}_{\text{st}}^+ \hookrightarrow \mathbb{B}_{\text{dR}}$ induces a natural map $M^* \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}} \rightarrow h(\mathbb{X}_{\text{st}}^r(M))$. Lemma 4.20 implies that this map is injective. Since the two modules have the same rank over the field \mathbf{B}_{dR} , this natural map is an isomorphism, which provides our first isomorphism.

For the second isomorphism, the injection of the right hand side into the left hand side is obvious (it follows, for example, from the first isomorphism). To prove the converse inclusion, it is enough, granting the first isomorphism, to show that if $\lambda \in M^* \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}$ satisfies $\lambda(X_{\text{st}}^r(M)) \subset \mathbf{B}_{\text{st}}$, then $\lambda \in M^* \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}$. For this, pick a weakly admissible filtration M_{dR}^+ on $M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}$, and let $V := V_{\text{st}}(M, M_{\text{dR}}^+)$ so that $V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}} \simeq M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}$ (see Remark 4.17). Since $t^r V \subset X_{\text{st}}^r(M)$, we have $\lambda(V) \subset \mathbf{B}_{\text{st}}$, and since V generates $M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}$, we have $\lambda(M) \subset \mathbf{B}_{\text{st}}$. This implies $\lambda \in M^* \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}$, as wanted. \square

Remark 4.24. The map $x \mapsto x - \frac{u}{1!}Nx + \frac{u^2}{2!}N^2x - \frac{u^3}{3!}N^3x + \dots$ induces isomorphisms

$$\mathbb{X}_{\text{cris}}^r(M) := (M \otimes \mathbb{B}_{\text{cris}}^+)^{\varphi=p^r} \xrightarrow{\sim} \mathbb{X}_{\text{st}}^r(M), \quad \text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{X}_{\text{st}}^r(M), \mathbb{B}_{\text{cris}}^+) \simeq \text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{X}_{\text{cris}}^r(M), \mathbb{B}_{\text{cris}}^+)$$

Since $\mathbb{B}_{\text{st}} = \mathbb{B}_{\text{cris}}^+[u, \frac{1}{t}]$, we obtain:

$$\begin{aligned} \text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{X}_{\text{st}}^r(M), \mathbb{B}_{\text{st}}) &= \text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{X}_{\text{st}}^r(M), \mathbb{B}_{\text{cris}}^+) \otimes_{\mathbf{B}_{\text{cris}}^+} \mathbf{B}_{\text{st}} \\ M^* \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{cris}}^+ &\subset \text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{X}_{\text{cris}}^r(M), \mathbb{B}_{\text{cris}}^+) \subset M^* \otimes_{F^{\text{nr}}} t^{-r} \mathbf{B}_{\text{cris}}^+ \end{aligned}$$

(For the inclusion on the right, use the fact that V in the proof of Proposition 4.23 is included in $t^{-r} X_{\text{cris}}^r(M)$ and $M \subset V \otimes \mathbf{B}_{\text{cris}}^+$.)

Proposition 4.25. *Let (M, M_{dR}^+) be an acyclic filtered (φ, N) -module over C with φ -slopes in $[0, r]$, with $t^r \mathbf{B}_{\text{dR}}^+ \otimes M \subset t^r M_{\text{dR}}^+ \subset \mathbf{B}_{\text{dR}}^+ \otimes M$. Set $\mathbb{M}_{\text{dR}}^+ := \mathbb{B}_{\text{dR}}^+ \otimes_{\mathbf{B}_{\text{dR}}^+} M_{\text{dR}}^+$ and*

$$\mathbb{V}_{\text{st}}^r(M, M_{\text{dR}}^+) := \text{Ker}((M \otimes_{F^{\text{nr}}} \mathbb{B}_{\text{st}})^{N=0, \varphi=p^r} \rightarrow (M \otimes_{F^{\text{nr}}} \mathbb{B}_{\text{dR}}^+)/t^r \mathbb{M}_{\text{dR}}^+).$$

Then

$$\text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{V}_{\text{st}}^r(M, M_{\text{dR}}^+), \mathbb{B}_{\text{dR}}) \simeq M^* \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}, \quad \text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{V}_{\text{st}}^r(M, M_{\text{dR}}^+), \mathbb{B}_{\text{st}}) \simeq M^* \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}$$

Proof. Let $\mathbb{X}_{\text{dR}}^r(M) := (M \otimes_{F^{\text{nr}}} \mathbb{B}_{\text{dR}}^+)/t^r \mathbb{M}_{\text{dR}}^+$; this is a \mathbb{B}_{dR}^+ -module killed by t^r . The hypothesis give us an exact sequence

$$0 \rightarrow \mathbb{V}_{\text{st}}^r(M, M_{\text{dR}}^+) \rightarrow \mathbb{X}_{\text{st}}^r(M) \rightarrow \mathbb{X}_{\text{dR}}^r(M) \rightarrow 0$$

The first isomorphism is then a consequence of Proposition 4.23 and vanishing of $\text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{X}_{\text{dR}}^r(M), \mathbb{B}_{\text{dR}})$ and $\text{Ext}_{\mathbf{V}_{\text{S}}}^{1, \natural}(\mathbb{X}_{\text{dR}}^r(M), \mathbb{B}_{\text{dR}})$ (Corollary 3.16 and Lemma 3.23).

For the second isomorphism pick up a weakly admissible filtration N_{dR}^+ containing M_{dR}^+ (this is possible because (M, M_{dR}^+) is acyclic), set $V := V_{\text{st}}(M, N_{\text{dR}}^+)$, and argue as in Proposition 4.23. \square

Remark 4.26. As in Remark 4.24, we have

$$\begin{aligned} \text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{V}_{\text{st}}^r(M, M_{\text{dR}}^+), \mathbb{B}_{\text{st}}) &= \text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{V}_{\text{st}}^r(M, M_{\text{dR}}^+), \mathbb{B}_{\text{cris}}^+) \otimes_{\mathbf{B}_{\text{cris}}^+} \mathbf{B}_{\text{st}} \\ M^* \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{cris}}^+ &\subset \text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{V}_{\text{st}}^r(M, M_{\text{dR}}^+), \mathbb{B}_{\text{cris}}^+) \subset M^* \otimes_{F^{\text{nr}}} t^{-r} \mathbf{B}_{\text{cris}}^+ \end{aligned}$$

Remark 4.27. We could also have argued as in the proof of Theorem 4.8 to prove Propositions 4.23 and 4.25.

Proposition 4.28. *Let (M, M_{dR}^+) be an acyclic filtered (φ, N) -module over C with φ -slopes in $[0, r]$, and $M \otimes_{F^{\text{nr}}} t^r \mathbf{B}_{\text{dR}}^+ \subset t^r M_{\text{dR}}^+ \subset M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}^+$. Let $k \geq 2r$. Then the natural map of \mathbf{B}_{dR}^+ -modules*

$$M^* \otimes_{F^{\text{nr}}} (\mathbf{B}_{\text{dR}}^+/t^k) \rightarrow \text{Hom}_{\mathbf{V}_{\text{S}}}(\mathbb{V}_{\text{st}}^r(M, M_{\text{dR}}^+), \mathbb{B}_{\text{dR}}^+/t^k)$$

induced by the inclusion $\mathbb{V}_{\text{st}}^r(M, M_{\text{dR}}^+) \subset M \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{dR}}^+$, has kernel and cokernel killed by t^{2r} .

Proof. Choose a N_{dR}^+ with $M \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{dR}}^+ \subset N_{\mathrm{dR}}^+ \subset M_{\mathrm{dR}}^+$ such that (M, N_{dR}^+) is weakly admissible. (This is possible by an adaptation of Lemma 4.4 to (φ, N) -modules over C .) Let $V_1 = \mathbb{V}_{\mathrm{st}}^r(M, N_{\mathrm{dR}}^+)$, $V = \mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+)$, and $W = V/V_1$, so that V_1 is a finite dimensional \mathbf{Q}_p -vector space, and $W = t^r M_{\mathrm{dR}}^+ / t^r N_{\mathrm{dR}}^+$ is a $\mathbb{B}_{\mathrm{dR}}^+$ -module killed by t^r .

Set $h_k(-) := \mathrm{Hom}_{\mathrm{VS}}(-, \mathbb{B}_{\mathrm{dR}}^+ / t^k)$. Since (we skipped the subscripts F^{nr})

$$M \otimes \frac{t^r \mathbb{B}_{\mathrm{dR}}^+}{t^k \mathbb{B}_{\mathrm{dR}}^+} \subset \frac{t^r N_{\mathrm{dR}}^+}{M \otimes t^k \mathbb{B}_{\mathrm{dR}}^+} \subset \frac{t^r M_{\mathrm{dR}}^+}{M \otimes t^k \mathbb{B}_{\mathrm{dR}}^+} \subset M \otimes \frac{\mathbb{B}_{\mathrm{dR}}^+}{t^k \mathbb{B}_{\mathrm{dR}}^+}$$

and $V_1 \subset \frac{t^r N_{\mathrm{dR}}^+}{M \otimes t^k \mathbb{B}_{\mathrm{dR}}^+}$, $V \subset \frac{t^r M_{\mathrm{dR}}^+}{M \otimes t^k \mathbb{B}_{\mathrm{dR}}^+}$, we have a commutative diagram

$$\begin{array}{ccccc} M^* \otimes (t^r \mathbf{B}_{\mathrm{dR}}^+ / t^k) & \longrightarrow & h_k\left(\frac{t^r M_{\mathrm{dR}}^+}{M \otimes t^k \mathbb{B}_{\mathrm{dR}}^+}\right) & \longrightarrow & h_k\left(\frac{t^r N_{\mathrm{dR}}^+}{M \otimes t^k \mathbb{B}_{\mathrm{dR}}^+}\right) \longrightarrow M^* \otimes (t^r \mathbf{B}_{\mathrm{dR}}^+ / t^k), \\ & & \downarrow & & \downarrow \\ & & h_k(V) & \longrightarrow & h_k(V_1) \end{array}$$

where the composed map $M^* \otimes (\mathbf{B}_{\mathrm{dR}}^+ / t^k) \rightarrow M^* \otimes (t^r \mathbf{B}_{\mathrm{dR}}^+ / t^k)$ is multiplication by t^r . Now, since V_1 is finite dimensional over \mathbf{Q}_p , and $N_{\mathrm{dR}}^+ = V_1 \otimes t^{-r} \mathbb{B}_{\mathrm{dR}}^+$, we have, using Proposition 3.15,

$$h_k\left(\frac{t^r N_{\mathrm{dR}}^+}{M \otimes t^k \mathbb{B}_{\mathrm{dR}}^+}\right) = \mathrm{Hom}_{\mathbb{B}_{\mathrm{dR}}^+}\left(\frac{t^r N_{\mathrm{dR}}^+}{M \otimes t^k \mathbb{B}_{\mathrm{dR}}^+}, \mathbb{B}_{\mathrm{dR}}^+ / t^k\right) = V_1^* \otimes (\mathbf{B}_{\mathrm{dR}}^+ / t^k) = h_k(V_1).$$

It follows that:

- $\mathrm{Ker}(M^* \otimes (\mathbf{B}_{\mathrm{dR}}^+ / t^k) \rightarrow h_k(V))$ is a subobject of $\mathrm{Ker}(M^* \otimes (\mathbf{B}_{\mathrm{dR}}^+ / t^k) \xrightarrow{t^r} M^* \otimes (t^r \mathbf{B}_{\mathrm{dR}}^+ / t^k))$, hence is killed by t^r .

- $\mathrm{Coker}(h_k(V) \rightarrow h_k(V_1))$ is a subquotient of $\mathrm{Coker}(M^* \otimes (\mathbf{B}_{\mathrm{dR}}^+ / t^k) \xrightarrow{t^r} M^* \otimes (t^r \mathbf{B}_{\mathrm{dR}}^+ / t^k))$, and hence is killed by t^r . The cokernel of $M^* \otimes (\mathbf{B}_{\mathrm{dR}}^+ / t^k) \rightarrow h_k(V_1)$ is also killed by t^r and, since the kernel of $h_k(V) \rightarrow h_k(V_1)$ is $h_k(W)$ which is killed by t^r , it follows that $\mathrm{Coker}(M^* \otimes (\mathbf{B}_{\mathrm{dR}}^+ / t^k) \rightarrow h_k(V))$ is killed by t^{2r} .

This concludes the proof. \square

4.4.2. *Projective limits of filtered (φ, N) -modules.* Let

$$(M, M_{\mathrm{dR}}^+) = \lim_n (M_n, M_{n, \mathrm{dR}}^+)$$

where the $(M_n, M_{n, \mathrm{dR}}^+)$'s are finite dimensional filtered (φ, N) -modules over C .

Proposition 4.29. *Suppose that the $(M_n, M_{n, \mathrm{dR}}^+)$'s are acyclic, with φ -slopes in $[0, r]$, and that $t^r M_n \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{dR}}^+ \subset t^r M_{n, \mathrm{dR}}^+ \subset M_n \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{dR}}^+$. Set $\mathbb{M}_{\mathrm{dR}}^+ = M_{\mathrm{dR}}^+ \widehat{\otimes}_{\mathbf{B}_{\mathrm{dR}}^+} \mathbb{B}_{\mathrm{dR}}^+$ and*

$$\mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+) := \mathrm{Ker}\left((M \widehat{\otimes}_{F^{\mathrm{nr}}} \mathbb{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r} \rightarrow (M \widehat{\otimes}_{F^{\mathrm{nr}}} \mathbb{B}_{\mathrm{dR}}^+) / t^r \mathbb{M}_{\mathrm{dR}}^+\right).$$

(i) *We have an exact sequence*

$$0 \rightarrow \mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+) \rightarrow (M \widehat{\otimes}_{F^{\mathrm{nr}}} \mathbb{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r} \rightarrow (M \widehat{\otimes}_{F^{\mathrm{nr}}} \mathbb{B}_{\mathrm{dR}}^+) / t^r \mathbb{M}_{\mathrm{dR}}^+ \rightarrow 0.$$

(ii) *We have isomorphisms*

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+), \mathbb{B}_{\mathrm{st}}) \simeq \mathrm{Hom}(M, \mathbf{B}_{\mathrm{st}}), \quad \text{as topological } \mathbf{B}_{\mathrm{st}}\text{-modules,}$$

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+), \mathbb{B}_{\mathrm{dR}}) \simeq \mathrm{Hom}(M, \mathbf{B}_{\mathrm{dR}}), \quad \text{as topological } \mathbf{B}_{\mathrm{dR}}\text{-modules.}$$

Proof. Point (i) follows from the analogous exact sequence for the M_n 's and vanishing of $\mathrm{R}^1 \lim_n$ of BC's (see Remark 3.14), which gives $\mathrm{R}^1 \lim_n \mathbb{V}_{\mathrm{st}}^r(M_n, M_{n, \mathrm{dR}}^+) = 0$.

Now, since $M = \lim_n M_n$ and $\mathbf{B}_{\mathrm{st}} = \mathrm{colim}_{j,k} t^{-j} (\oplus_{i \leq k} \mathbf{B}_{\mathrm{cris}}^+ u^i)$, and $t^{-j} (\oplus_{i \leq k} \mathbf{B}_{\mathrm{cris}}^+ u^i)$ is a Banach space, we have

$$\mathrm{Hom}(M, \mathbf{B}_{\mathrm{st}}) = \mathrm{colim}_{n,j,k} \mathrm{Hom}(M_n, t^{-j} (\oplus_{i \leq k} \mathbf{B}_{\mathrm{cris}}^+ u^i)) = (\mathrm{colim}_n \mathrm{Hom}(M_n, \mathbf{B}_{\mathrm{cris}}^+)) \otimes_{\mathbf{B}_{\mathrm{cris}}^+} \mathbf{B}_{\mathrm{st}}$$

For the same reasons, we have

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+), \mathbb{B}_{\mathrm{st}}) = \mathrm{Hom}_{\mathrm{VS}}(\mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+), \mathbb{B}_{\mathrm{cris}}^+) \otimes_{\mathbf{B}_{\mathrm{cris}}^+} \mathbf{B}_{\mathrm{st}}$$

Hence, to prove the first isomorphism of (ii), we are reduced to showing that

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+), \mathbb{B}_{\mathrm{cris}}^+) \simeq \mathrm{colim}_n \mathrm{Hom}(M_n, \mathbf{B}_{\mathrm{cris}}^+)$$

and, in view of Remark 4.26, we just have to show that any $\lambda : \mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+) \rightarrow \mathbb{B}_{\mathrm{cris}}^+$ factors through $\mathbb{V}_{\mathrm{st}}^r(M_n, M_{n, \mathrm{dR}}^+)$ for some n big enough.

According to Lemma 4.15 this is true for the induced morphism $\lambda_C : \mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+) \rightarrow \mathbf{B}_{\mathrm{cris}}^+$ on C -points. So let us pick n which works for λ_C and denote by \mathbb{W}_n the kernel of $\mathbb{V} \rightarrow \mathbb{V}_n$, where $\mathbb{V} := \mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+)$ and $\mathbb{V}_n := \mathbb{V}_{\mathrm{st}}^r(M_n, M_{n, \mathrm{dR}}^+)$. Since sympathetic C -algebras form a basis for pro-étale topology, it suffices to show that $\lambda_\Lambda : \mathbb{W}_n(\Lambda) \rightarrow \mathbb{B}_{\mathrm{cris}}^+(\Lambda)$ is zero for all sympathetic C -algebras Λ to conclude that λ factors through \mathbb{V}_n . But, if $\mathbb{Y} = \mathbb{V}, \mathbb{V}_n$, we have an injection $\mathbb{Y}(\Lambda) \hookrightarrow \mathcal{C}(\mathrm{Spm} \Lambda, \mathbb{Y}(C))$ for all sympathetic C -algebras Λ . It follows that we have the same result for $\mathbb{Y} = \mathbb{W}_n$ and, since $\lambda_C : \mathbb{W}_n(C) \rightarrow \mathbf{B}_{\mathrm{cris}}^+$ is identically 0, $\lambda_\Lambda : \mathbb{W}_n(\Lambda) \rightarrow \mathbb{B}_{\mathrm{cris}}^+(\Lambda)$ is identically 0 for all Λ .

This proves the first isomorphism. Let us now prove the second one. Since $\mathbf{B}_{\mathrm{dR}}^+/t^k$ is a Banach and $\mathbf{B}_{\mathrm{dR}} = \mathrm{colim}_j t^{-j}(\lim_k \mathbf{B}_{\mathrm{dR}}^+/t^k)$, we have

$$\mathrm{Hom}(M, \mathbf{B}_{\mathrm{dR}}) = \left(\lim_k \left(\mathrm{colim}_n \mathrm{Hom}(M_n, \mathbf{B}_{\mathrm{dR}}^+/t^k) \right) \right) \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbf{B}_{\mathrm{dR}}$$

For the same reasons as above, we obtain

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbb{V}_{\mathrm{st}}^r(M, M_{\mathrm{dR}}^+), \mathbb{B}_{\mathrm{dR}}) = \left(\lim_k \left(\mathrm{colim}_n \mathrm{Hom}_{\mathrm{VS}}(\mathbb{V}_{\mathrm{st}}^r(M_n, M_{n, \mathrm{dR}}^+), \mathbb{B}_{\mathrm{dR}}^+/t^k) \right) \right) \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbf{B}_{\mathrm{dR}}$$

and the second isomorphism of (ii) follows from Proposition 4.28. \square

5. COMPARISON THEOREMS: EXAMPLES AND A CONJECTURE

In this Chapter we will formulate a conjecture: the existence of the fundamental diagram for smooth dagger varieties over C . Before doing that though we will first look at examples of comparison theorems and fundamental diagrams.

5.1. Cliffs Notes. Here we make a small digression with a quick review of relevant results from [14] and [15].

Proposition 5.1. (Colmez-Nizioł, [14, Th. 1.1], [15, Th. 1.3])

(1) **Analytic varieties:** *To any smooth dagger or rigid analytic variety X over C there are naturally associated:*

- (a) *A pro-étale cohomology $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$, $r \in \mathbf{Z}$.*
- (b) *A syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r))$, $r \in \mathbf{N}$, with a natural period morphism*

$$\alpha_r : \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \rightarrow \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)),$$

which is a strict quasi-isomorphism after truncation $\tau_{\leq r}$. This morphism can be lifted to the derived category of Vector Spaces.

- (c) *A Hyodo-Kato cohomology¹² $\mathrm{R}\Gamma_{\mathrm{HK}}(X)$. This is a dg F^{nr} -algebra equipped with a Frobenius φ and a monodromy operator N . We have natural Hyodo-Kato strict quasi-isomorphisms*

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X) \widehat{\otimes}_{F^{\mathrm{nr}}}^R C \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X), \quad \iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X) \widehat{\otimes}_{F^{\mathrm{nr}}}^R \mathbf{B}_{\mathrm{dR}}^+ \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+).$$

- (d) *A distinguished triangle*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \longrightarrow [\mathrm{R}\Gamma_{\mathrm{HK}}(X) \widehat{\otimes}_{F^{\mathrm{nr}}}^R \mathbf{B}_{\mathrm{st}}^+]^{N=0, \varphi=p^r} \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^r$$

that can be lifted to the derived category of Vector Spaces.

- (e) (Local-global compatibility) *In the case X has a semistable weak formal model the above constructions are compatible with their analogs defined using the model.*

¹²We take here the Hyodo-Kato cohomology defined in [15].

- (2) **Compatibility:** Let X be a smooth dagger variety over C and let \widehat{X} denote its completion. Then there exist natural compatible morphisms [14, Sec. 3.2.4]

$$\begin{aligned} \iota_{\text{proét}} &: \text{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \rightarrow \text{R}\Gamma_{\text{proét}}(\widehat{X}, \mathbf{Q}_p(r)), \quad r \in \mathbf{Z}, \\ \iota &: \text{R}\Gamma_{\text{syn}}(X, \mathbf{Q}_p(r)) \rightarrow \text{R}\Gamma_{\text{syn}}(\widehat{X}, \mathbf{Q}_p(r)), \quad r \in \mathbf{Z}, \\ &\text{R}\Gamma_{\text{dR}}(X) \rightarrow \text{R}\Gamma_{\text{dR}}(\widehat{X}), \quad \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+) \rightarrow \text{R}\Gamma_{\text{dR}}(\widehat{X}/\mathbf{B}_{\text{dR}}^+), \\ &\text{R}\Gamma_{\text{HK}}(X) \rightarrow \text{R}\Gamma_{\text{HK}}(\widehat{X}). \end{aligned}$$

They are strict quasi-isomorphisms if X is partially proper.

5.2. Proper rigid analytic varieties. We start with smooth and proper varieties.

5.2.1. Algebraic varieties. Let X_K be an algebraic variety over K and set $X = X_{K, \overline{K}}$. Recall the comparison theorem (recall that all the cohomology groups involved have finite dimension):

Theorem 5.2. *Let $r \geq 0$. There exists a natural \mathbf{B}_{st} -linear Galois equivariant period isomorphism¹³*

$$\alpha_{\text{st}} : H_{\text{ét}}^r(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}} \simeq H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}},$$

that preserves the Frobenius and the monodromy operators, and induces a filtered isomorphism

$$\alpha_{\text{dR}} : H_{\text{ét}}^r(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}} \simeq H_{\text{dR}}^r(X_K) \otimes_K \mathbf{B}_{\text{dR}}.$$

In particular, we have the natural isomorphism

$$H_{\text{ét}}^r(X, \mathbf{Q}_p) \simeq (H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}})^{\varphi=1, N=0} \cap F^0(H_{\text{dR}}^r(X_K) \otimes_K \mathbf{B}_{\text{dR}}), \quad \text{as a } \mathcal{G}_K\text{-module,}$$

as well as the natural isomorphisms

$$(5.3) \quad \begin{aligned} \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(H_{\text{ét}}^r(X, \mathbf{Q}_p), \mathbf{B}_{\text{st}}) &\simeq H_{\text{HK}}^r(X)^*, \quad \text{as a } (\varphi, N, \mathcal{G}_K)\text{-module,} \\ \text{Hom}_{\mathcal{G}_K}(H_{\text{ét}}^r(X, \mathbf{Q}_p), \mathbf{B}_{\text{dR}}) &\simeq H_{\text{dR}}^r(X_K)^*, \quad \text{as a filtered } K\text{-module.} \end{aligned}$$

5.2.2. Proper rigid analytic varieties over K . Let X_K be a proper smooth rigid analytic variety over K . Let $X = X_{K, C}$. The following result generalizes [13, Cor. 1.10], where semistable reduction was assumed. We note that all the cohomology groups involved have finite dimension: for étale cohomology this is the result of Scholze [38, Th. 1.1]; for Hyodo-Kato cohomology this follows from the Hyodo-Kato isomorphism and finiteness of de Rham cohomology.

Theorem 5.4. *Let $r \geq 0$. There exists a natural \mathbf{B}_{st} -linear and \mathcal{G}_K -equivariant period isomorphism*

$$\alpha_{\text{st}} : H_{\text{ét}}^r(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}} \simeq H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}},$$

that preserves the Frobenius and the monodromy operators, and induces a filtered isomorphism

$$\alpha_{\text{dR}} : H_{\text{ét}}^r(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}} \simeq H_{\text{dR}, \overline{K}}^r(X) \otimes_{\overline{K}} \mathbf{B}_{\text{dR}}.$$

In particular, have the following natural isomorphisms

$$(5.5) \quad \begin{aligned} \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(H_{\text{ét}}^r(X, \mathbf{Q}_p), \mathbf{B}_{\text{st}}) &\simeq H_{\text{HK}}^r(X)^*, \quad \text{as a } (\varphi, N, \mathcal{G}_K)\text{-module,} \\ \text{Hom}_{\mathcal{G}_K}(H_{\text{ét}}^r(X, \mathbf{Q}_p), \mathbf{B}_{\text{dR}}) &\simeq H_{\text{dR}}^r(X)^*, \quad \text{as a filtered } K\text{-module.} \end{aligned}$$

Proof. Take $s > r$. To define the period maps consider the following composition

$$\alpha_{\text{st}}(s) : H_{\text{ét}}^r(X, \mathbf{Q}_p(s)) \xrightarrow[\sim]{\alpha_s^{-1}} H_{\text{syn}}^r(X, s) \rightarrow (H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^s} \xrightarrow{p^{-s}} H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}},$$

where α_s is the period map from [15, Cor. 6.9]. Set

$$\alpha_{\text{st}} := t^{-s} \alpha_{\text{st}}(s) \varepsilon^s, \quad \alpha_{\text{dR}} := \iota_{\text{BK}}^{-1}(\iota_{\text{HK}} \otimes \iota) \alpha_{\text{st}},$$

¹³Note that there is no assumption on the variety. This formulation of the comparison theorem is due to Beilinson [2].

where ε is the generator of $\mathbf{Z}_p(1)$ corresponding to t .

We follow the proof of [13, Cor. 1.10], which uses BC's. We will sketch the arguments and refer the reader to [13] for details. We start with stating the isomorphism

$$(H_{\text{dR}}^r(X) \otimes_K \mathbf{B}_{\text{dR}}^+)/F^s \xrightarrow{\sim} (H^r(\text{R}\Gamma_{\text{dR}}(X) \otimes_K \mathbf{B}_{\text{dR}}^+)/F^s),$$

which follows from the degeneration of the Hodge-de Rham spectral sequence¹⁴ (proved by Scholze as a corollary of the de Rham comparison theorem [38]). Using this isomorphism, we obtain the long exact sequence (obtained from the definition of overconvergent syntomic cohomology)

$$(5.6) \quad \begin{aligned} \rightarrow (H_{\text{dR}}^{r-1}(X) \otimes_{\overline{K}} \mathbf{B}_{\text{dR}}^+)/F^s &\rightarrow H_{\text{syn}}^r(X, \mathbf{Q}_p(s)) \rightarrow (H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^s} \\ &\rightarrow (H_{\text{dR}, \overline{K}}^r(X) \otimes_{\overline{K}} \mathbf{B}_{\text{dR}}^+)/F^s \rightarrow \end{aligned}$$

We recall that the groups $H_{\text{HK}}^r(X), H_{\text{dR}}^r(X)$ have finite dimension over F^{nr} and K , respectively. The above long exact sequence yields short exact sequences

$$0 \rightarrow H_{\text{syn}}^r(X, \mathbf{Q}_p(s)) \rightarrow (H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^s} \rightarrow (H_{\text{dR}, \overline{K}}^r(X) \otimes_{\overline{K}} \mathbf{B}_{\text{dR}}^+)/F^s \rightarrow 0$$

To prove this, we observe that the map $f_r : (H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^s} \rightarrow (H_{\text{dR}, \overline{K}}^r(X) \otimes_{\overline{K}} \mathbf{B}_{\text{dR}}^+)/F^s$ is the evaluation on C of a map of BC's. But, the syntomic cohomology group $H_{\text{syn}}^r(X_C, \mathbf{Q}_p(s))$, $r \leq s$, is a finite dimensional \mathbf{Q}_p -vector space since we have the quasi-isomorphism [15, 6.10] with étale cohomology. This implies that the cokernel of f_i , viewed as a map of BC's, is of Dimension $(0, d_i)$. On the other hand, the Space $(H_{\text{dR}, \overline{K}}^r(X) \otimes_{\overline{K}} \mathbf{B}_{\text{dR}}^+)/F^s$ is a successive extension of C -vector spaces. The theory of BC's implies now that the map $(H_{\text{dR}, \overline{K}}^r(X) \otimes_{\overline{K}} \mathbf{B}_{\text{dR}}^+)/F^s \rightarrow \text{Coker } f_r$ is zero, hence $\text{Coker } f_r = 0$, as wanted.

We have the Hyodo-Kato isomorphism (see [15, Cor. 4.32])

$$\iota_{\text{HK}} : H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} C \simeq H_{\text{dR}}^r(X).$$

Taking \mathcal{G}_K -smooth vectors of both sides (note that X is quasi-compact) we get the Hyodo-Kato isomorphism

$$H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \overline{K} \simeq H_{\text{dR}}^r(X_K) \otimes_K \overline{K}.$$

Hence the pair $(H_{\text{HK}}^r(X), H_{\text{dR}}^r(X_K))$ is a $(\varphi, N, \mathcal{G}_K)$ -filtered module (in the sense of Fontaine). The above short exact sequence and a "weight" argument shows that $\mathbf{V}_{\text{st}}(H_{\text{HK}}^r(X), H_{\text{dR}}^i(X_K)) \simeq H_{\text{ét}}^r(X, \mathbf{Q}_p)$. Here $\mathbf{V}_{\text{st}}(-)$ is Fontaine's functor from filtered Frobenius modules to Galois representations. The short exact sequence and C -dimension count give also that $t_N(H_{\text{HK}}^r(X)) = t_H(H_{\text{dR}}^r(X_K))$, where $t_N(D) = v_p(\det \varphi)$ and $t_H(D) = \sum_{i \geq 0} i \dim_K(F^i D / F^{i+1} D)$. The theory of BC's now implies that the pair $(H_{\text{HK}}^r(X), H_{\text{dR}}^r(X_K))$ is weakly admissible from which our theorem follows. \square

Remark 5.7. The isomorphisms (5.3) and (5.5) are strict if we put the weak topology on the Homs.

5.2.3. *Proper rigid analytic varieties over C .* Let X be a smooth rigid analytic variety over C . Its p -adic étale cohomology is finite rank by [38, Th. 1.1]. Its Hyodo-Kato cohomology is finite rank by the Hyodo-Kato isomorphism and finiteness of de Rham cohomology. Its \mathbf{B}_{dR}^+ -cohomology is free, finite rank over \mathbf{B}_{dR}^+ , by the comparison isomorphism with Hyodo-Kato cohomology.

Theorem 5.8. *Let $r \geq 0$. There exists a natural \mathbf{B}_{st} -linear period isomorphism*

$$\alpha_{\text{st}} : H_{\text{ét}}^r(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}} \simeq H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}},$$

that preserves the Frobenius and the monodromy operators, and induces a filtered isomorphism

$$(5.9) \quad \alpha_{\text{dR}} : H_{\text{ét}}^r(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}} \simeq H_{\text{dR}}^r(X/\mathbf{B}_{\text{dR}}^+) \otimes_{\mathbf{B}_{\text{dR}}^+} \mathbf{B}_{\text{dR}}.$$

¹⁴Alternatively, one can use an argument analogous to the one we use in the proof of Theorem 5.8 below.

Here, the filtration on $H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+)$ is defined by

$$F^i H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+) := \mathrm{Im}(H^r(F^i \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)) \rightarrow H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+)).$$

Remark 5.10. A de Rham comparison isomorphism as in (5.9) was constructed earlier in [7, Th. 13.1]. It did not treat filtrations.

Proof. Take $s > r$. The period maps are define as in Theorem 5.4 but by dropping the map ι_{BK} . We note that $H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^s)$ is a BC, a successive extension of C -vector spaces of finite rank. This follows from the fact that the distinguished triangle [15, 3.28] yields the distinguished triangle

$$\bigoplus_{i \leq s} \mathrm{R}\Gamma(X, \Omega^i)(s-i)[-i] \rightarrow \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^{s+1} \rightarrow \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^s$$

and $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^1 \simeq \mathrm{R}\Gamma(X, \mathcal{O}_X)$ by [15, 3.27]. Having that, the same arguments as in the case of proper varieties over K yield short exact sequence

$$(5.11) \quad 0 \rightarrow H_{\mathrm{ét}}^r(X, \mathbf{Q}_p(s)) \rightarrow (H_{\mathrm{HK}}^r(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^s} \rightarrow H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^s) \rightarrow 0$$

Moreover, these arguments show that the canonical map

$$H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+)/F^s \rightarrow H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^s)$$

is an isomorphism. This can be proved in the following way. By (5.11), we have a surjection

$$(H_{\mathrm{HK}}^r(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^s} \twoheadrightarrow H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^s).$$

Since the above map factors through the natural map

$$(5.12) \quad H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+)/F^s \rightarrow H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^s)$$

that latter is surjective as well. But it is also injective. Indeed, we have the distinguished triangle

$$F^s \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+) \rightarrow \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+) \rightarrow \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^s$$

It yields the long exact sequence of cohomology groups

$$\rightarrow H^r(F^s \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)) \xrightarrow{f_r} H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+) \rightarrow H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^s) \rightarrow$$

Since $F^s H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+) = \mathrm{Im} f_r$, the map in (5.12) is injective. We are done.

The two isomorphisms in our theorem follow now from Remark 4.17, using the last part of Remark 4.19 (take $M = H_{\mathrm{HK}}^r(X)$, $M^+ = F^0(H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+) \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbf{B}_{\mathrm{dR}})$). \square

Remark 5.13. (i) One can restate the theorem as follows ($\mathcal{E}(-, -)$ is the associated vector bundle on the Fargues-Fontaine curve X_{FF}):

$$H_{\mathrm{ét}}^r(X, \mathbf{Q}_p) \simeq H^0(X_{\mathrm{FF}}, \mathcal{E}(H_{\mathrm{HK}}^r(X), H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+))).$$

(ii) From Theorem 5.8, we get natural isomorphisms

$$\mathrm{Hom}_{\mathrm{VS}}(H_{\mathrm{ét}}^r(X, \mathbf{Q}_p), \mathbb{B}_{\mathrm{st}}) \simeq \mathrm{Hom}_{F^{\mathrm{nr}}}(H_{\mathrm{HK}}^r(X), \mathbf{B}_{\mathrm{st}}), \quad \text{as } (\varphi, N)\text{-modules,}$$

$$\mathrm{Hom}_{\mathrm{VS}}(H_{\mathrm{ét}}^r(X, \mathbf{Q}_p), \mathbb{B}_{\mathrm{dR}}) \simeq \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}}^+}(H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+), \mathbf{B}_{\mathrm{dR}}), \quad \text{as filtered } \mathbf{B}_{\mathrm{dR}}^+\text{-modules.}$$

5.3. Dagger Stein varieties and dagger affinoids. Having the comparison result from [15, Cor. 6.9], we can now deduce a (simplified) fundamental diagram for pro-étale cohomology from the one for overconvergent syntomic cohomology.

Theorem 5.14. (Simplified fundamental diagram) *Let X be a smooth dagger Stein variety or a smooth dagger affinoid over C . Let $r \geq 0$. There is a natural map of strictly exact sequences*

$$\begin{array}{ccccc} 0 \rightarrow \Omega^{r-1}(X)/\mathrm{Ker} d & \longrightarrow & H_{\mathrm{proét}}^r(X, \mathbf{Q}_p(r)) & \longrightarrow & (H_{\mathrm{HK}}^r(X) \widehat{\otimes}_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r} \rightarrow 0 \\ & & \downarrow \tilde{\beta} & & \downarrow \iota_{\mathrm{HK}} \otimes \theta \\ 0 \rightarrow \Omega^{r-1}(X)/\mathrm{Ker} d & \xrightarrow{d} & \Omega^r(X)^{d=0} & \longrightarrow & H_{\mathrm{dR}}^r(X) \rightarrow 0 \end{array}$$

Moreover, $H_{\text{proét}}^r(X, \mathbf{Q}_p(r))$ is Fréchet or LB, respectively, the vertical maps are strict and have closed images, and $\text{Ker } \tilde{\beta} \simeq (H_{\text{HK}}^r(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^{r-1}}$.

Proof. We define $\tilde{\beta} := p^{-r} \beta \alpha_r^{-1}$, using [15, Cor. 6.9] and [15, Prop. 5.13]; the twist by p^{-r} being added to make this map compatible with symbols. The theorem follows immediately from [15, Cor. 6.9] and [15, Prop. 5.13]. \square

5.4. The fundamental diagram. We will now introduce the fundamental diagram, look at some examples, where it appears, and, finally, state a conjecture concerning it.

5.4.1. *Examples.* We start with examples.

- *Proper varieties.*

Corollary 5.15. *Let X be a smooth proper variety over C . We have the bicartesian diagram*

$$(5.16) \quad \begin{array}{ccc} H_{\text{ét}}^r(X, \mathbf{Q}_p(r)) & \xrightarrow{\alpha_{\text{st}}(r)} & (H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{\varphi=p^r, N=0} \\ \downarrow \alpha_{\text{dR}}(r) & & \downarrow \iota_{\text{HK}} \otimes \iota \\ F^r H_{\text{dR}}^r(X/\mathbf{B}_{\text{dR}}^+) & \xrightarrow{\text{can}} & H_{\text{dR}}^r(X/\mathbf{B}_{\text{dR}}^+) \end{array}$$

Recall that this means that this is a pushout and pullback diagram, or, that the sequence

$$(5.17) \quad 0 \rightarrow H_{\text{ét}}^r(X, \mathbf{Q}_p(r)) \xrightarrow{\alpha_{\text{st}}(r) \oplus \alpha_{\text{dR}}(r)} (H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \oplus F^r H_{\text{dR}}^r(X/\mathbf{B}_{\text{dR}}^+) \xrightarrow{\iota_{\text{HK}} \otimes \iota + \text{can}} H_{\text{dR}}^r(X/\mathbf{B}_{\text{dR}}^+) \rightarrow 0$$

is exact.

Proof. Follows immediately from the short exact sequence (5.11). \square

Remark 5.18. (i) The passage the other way, from diagram (5.16) to Theorem 5.8 is also possible: the exact sequence (5.17) yields the exact sequence (5.11) and we can finish as in the proof of Theorem 5.8.

(ii) We note that the natural map $H^r(F^r \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+)) \rightarrow F^r H_{\text{dR}}^r(X/\mathbf{B}_{\text{dR}}^+)$ is an isomorphism.

- *Dagger Stein varieties and dagger affinoids.*

Corollary 5.19. *Let X be a smooth dagger Stein variety or a smooth dagger affinoid over C . Let $r \geq 0$. We have the bicartesian diagram (recall that all cohomologies are classical)*

$$(5.20) \quad \begin{array}{ccc} H_{\text{ét}}^r(X, \mathbf{Q}_p(r)) & \xrightarrow{\alpha_{\text{st}}(r)} & (H_{\text{HK}}^r(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \\ \downarrow \alpha_{\text{dR}}(r) & & \downarrow \iota_{\text{HK}} \otimes \iota \\ H^r F^r \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+) & \longrightarrow & H_{\text{dR}}^r(X/\mathbf{B}_{\text{dR}}^+). \end{array}$$

Proof. Consider the following diagram of maps of distinguished triangles

$$\begin{array}{ccccc} \text{R}\Gamma_{\text{syn}}(X, \mathbf{Q}_p(r)) & \xrightarrow{f_1} & [\text{R}\Gamma_{\text{HK}}(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+]^{N=0, \varphi=p^r} & \xrightarrow{\iota_{\text{HK}} \otimes \iota} & \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+)/F^r \\ \downarrow \beta & & \downarrow \iota_{\text{HK}} \otimes \iota & & \parallel \\ F^r \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+) & \xrightarrow{f_2} & \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+) & \longrightarrow & \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+)/F^r \end{array}$$

Since $H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^r) = 0$ for $i \geq r$, the maps f_1, f_2 are surjective in degrees $\geq r$. It suffices to show that $\mathrm{Ker} f_1 \xrightarrow{\sim} \mathrm{Ker} f_2$ in degree r . We have the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Ker} f_1 & \longrightarrow & H_{\mathrm{syn}}^r(X, \mathbf{Q}_p(r)) & \xrightarrow{f_1} & (H_{\mathrm{HK}}^r(X) \widehat{\otimes}_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r} \\
& & \downarrow & & \downarrow & & \downarrow \iota_{\mathrm{HK} \otimes \iota} \\
0 & \longrightarrow & \mathrm{Ker} f_2 & \longrightarrow & H^r F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+) & \xrightarrow{f_2} & H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+) \\
& & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
0 & \longrightarrow & \mathrm{Ker} f_{2,C} & \longrightarrow & H^r F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X) & \xrightarrow{f_{2,C}} & H_{\mathrm{dR}}^r(X)
\end{array}$$

We claim that the map $\mathrm{Ker} f_2 \rightarrow \mathrm{Ker} f_{2,C}$ is an isomorphism. Indeed, we compute

$$\begin{aligned}
\mathrm{Ker} f_2 &\xleftarrow{\sim} \mathrm{Coker}(H_{\mathrm{dR}}^{r-1}(X_K) \widehat{\otimes}_K \mathbf{B}_{\mathrm{dR}}^+ \rightarrow H^{r-1}((\mathrm{R}\Gamma_{\mathrm{dR}}(X_K) \widehat{\otimes}_K \mathbf{B}_{\mathrm{dR}}^+)/F^r)) \xrightarrow{\sim} \Omega^{r-1}(X_K) \mathrm{Ker} d \widehat{\otimes}_K C, \\
\mathrm{Ker} f_{2,C} &\xleftarrow{\sim} \Omega^{r-1}(X_K) \mathrm{Ker} d \widehat{\otimes}_K C.
\end{aligned}$$

□

5.4.2. *Conjecture.* We will formulate now a conjecture describing pro-étale cohomology in terms of the de Rham complex.

Conjecture 5.21. (C_{st} -conjecture) *Let X be a smooth dagger variety over C . Let $r \geq i$. The commutative diagram*

$$\begin{array}{ccc}
\tilde{H}_{\mathrm{proét}}^i(X, \mathbf{Q}_p(r)) & \longrightarrow & (\tilde{H}_{\mathrm{HK}}^i(X) \widehat{\otimes}_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r} \\
\downarrow & & \downarrow \\
\tilde{H}^i F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+) & \longrightarrow & \tilde{H}_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+)
\end{array}$$

is bicartesian. That is, the following sequence

$$(5.22) \quad 0 \rightarrow \tilde{H}_{\mathrm{proét}}^i(X, \mathbf{Q}_p(r)) \rightarrow (\tilde{H}_{\mathrm{HK}}^i(X) \widehat{\otimes}_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r} \oplus \tilde{H}^i F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+) \rightarrow \tilde{H}_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+) \rightarrow 0$$

is exact.

Remark 5.23. (i) This conjecture is known so far in the following cases:

- X is proper (see Examples 5.4.1). In this case, the two horizontal arrows are injective.
- X is Stein or affinoid (see Examples 5.4.1). In this case, the two horizontal arrows are surjective and their kernels are $\Omega^{r-1}(X)/\mathrm{Ker} d$.

(ii) Let X be a smooth dagger variety over C . If $r \geq i$, set

$$\begin{aligned}
\tilde{H}^{r,i} &:= \tilde{H}_{\mathrm{proét}}^i(X, \mathbf{Q}_p(r)) & \tilde{X}^{r,i} &:= (\tilde{H}_{\mathrm{HK}}^i(X) \widehat{\otimes}_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r} \\
\tilde{F}^{r,i} &:= \tilde{H}^i F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+) & \tilde{B}^i &:= \tilde{H}_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+)
\end{aligned}$$

and denote by $H^{r,i}, X^{r,i}$, etc. the images of $\tilde{H}^{r,i}, \tilde{X}^{r,i}$, etc. in $C_{\mathbf{Q}_p}$. Note that the $\tilde{X}^{r,i}$'s and \tilde{B}^i 's are classical, i.e., $\tilde{X}^{r,i} \simeq X^{r,i}$ and $\tilde{B}^i \simeq B^i$.

We have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 \tilde{H}^{r,i} & \longrightarrow & \tilde{F}^{r,i} \oplus \tilde{X}^{r,i} & \longrightarrow & \tilde{B}^i \\
 \downarrow \wr & & t\downarrow \wr & & t\downarrow \\
 \tilde{H}^{r+1,i} & \longrightarrow & \tilde{F}^{r+1,i} \oplus \tilde{X}^{r+1,i} & \longrightarrow & \tilde{B}^i \\
 & & \downarrow & & \downarrow \\
 & & H_{\text{HK}}^i(X) \hat{\otimes}_{F^{\text{nr}}} C & \xrightarrow[\wr]{\sim} & H_{\text{dR}}^i(X) \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
 \end{array}$$

The vertical maps are multiplications by t (on pro-étale cohomology, this corresponds to the Tate twist); for the isomorphism $X^{r+1,i}/tX^{r,i} \simeq H_{\text{HK}}^i(X) \hat{\otimes}_{F^{\text{nr}}} C$, see Remark 4.19; the bottom isomorphism is the Hyodo-Kato map. It follows that, for fixed i , the conjecture for r is equivalent to the conjecture for $r+1$. Hence it is enough to prove it for one $r \geq i$ (for example $r=i$).

(iii) Since the \tilde{B}^i 's are actually classical, and we have a long exact sequence

$$\dots \rightarrow \tilde{X}^{r,i-1} \oplus \tilde{F}^{r,i-1} \rightarrow \tilde{B}^{i-1} \rightarrow \tilde{H}^{r,i} \rightarrow \tilde{X}^{r,i} \oplus \tilde{F}^{r,i} \rightarrow \tilde{B}^i \rightarrow \dots$$

it is enough, thanks to an induction on i , to prove surjectivity of $X^{r,i} \oplus F^{r,i} \rightarrow B^i$: this will show that $\tilde{X}^{r,i} \oplus \tilde{F}^{r,i} \rightarrow \tilde{B}^i$ is surjective (since $\tilde{B}^i \simeq B^i$) and that the long exact sequence splits into short exact sequences, as wanted.

(iv) We will prove this conjecture for quasi-compact varieties in Theorem 6.14 below.

6. DE RHAM-TO-PRO-ÉTALE COMPARISON THEOREM FOR QUASI-COMPACT VARIETIES

We will now prove Conjecture 5.21 for quasi-compact varieties. In this chapter, until Theorem 6.14, all cohomologies, unless otherwise stated, are algebraic, i.e., we ignore topological issues (see (iii) of Remark 5.23 of why this is a reasonable thing to do).

6.1. Conjectures. Let X be a smooth dagger variety over C . We will first state and discuss four conjectures, a priori unrelated, on the cohomology of X :

- Conjecture 5.21 (already stated above) describes the p -adic pro-étale cohomology of X in terms of the de Rham complex.

The remaining conjectures assume X to be quasi-compact:

- Conjecture 6.1 gives a restriction on the Hodge filtration on the de Rham cohomology in terms of the slopes of Frobenius on the Hyodo-Kato cohomology.
- Conjecture 6.4 says that, even if huge, the pro-étale cohomology groups $\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p)$ have nevertheless \mathbf{Q}_p -dimension equal to $\dim_C H_{\text{dR}}^r(X)$.
- Conjecture 6.3 says that $\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p)$ is of curvature ≤ 0 .

Next, we proceed to proving these conjectures in the case X is quasi-compact. The proof uses:

- the period quasi-isomorphism between pro-étale cohomology and syntomic cohomology from [15, 6.10],
- the canonical distinguished triangle involving syntomic cohomology from [15, 5.12],
- delicate properties of BC's (scattered through [8, 9, 31, 35] and recalled in Section 3.1 for the convenience of the reader).

These ingredients allow us to prove that the four conjectures above are, in fact, equivalent (see Proposition 6.11 as well as Lemma 6.9 and Lemma 6.10 for precise statements). We show then Conjecture 6.4 by induction on the number n of affinoids needed to cover X ; for $n=1$, Conjecture 5.21 is exactly Corollary 5.19.

6.1.1. *Acyclicity of de Rham cohomology.* For a smooth dagger variety X over C , we set

- (1) $F^i H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+) := \mathrm{Im}(H^r(F^i \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)) \rightarrow H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+));$
- (2) $H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}) := H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+) \widehat{\otimes}_{\mathbf{B}_{\mathrm{dR}}^+} \mathbf{B}_{\mathrm{dR}}$ and equip it with the induced filtration.

Conjecture 6.1. *Let X be a smooth quasi-compact dagger variety over C . For all r , the map*

$$(H_{\mathrm{HK}}^r(X) \widehat{\otimes}_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r} \rightarrow H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+)/F^r$$

is surjective, i.e., the pair $(H_{\mathrm{HK}}^r(X), F^0 H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}))$ is acyclic.

Remark 6.2. If X is proper, then the pair $(H_{\mathrm{HK}}^r(X), F^0 H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}))$ is weakly admissible hence X verifies the conjecture. If X is Stein, then $H_{\mathrm{dR}}^r(X/\mathbf{B}_{\mathrm{dR}}^+)/F^r = 0$ and X clearly satisfies the conjecture.

6.1.2. *Curvature and height of pro-étale cohomology.* Let X be a smooth quasi-compact dagger variety over C .

Conjecture 6.3. *Then, for all $r \geq 0$, $\mathbb{H}_{\mathrm{pro\acute{e}t}}^r(X, \mathbf{Q}_p)$ has curvature ≤ 0 .*

Conjecture 6.4. *For all r ,*

$$\mathrm{ht}(\mathbb{H}_{\mathrm{pro\acute{e}t}}^r(X, \mathbf{Q}_p)) = \dim_C H_{\mathrm{dR}}^r(X).$$

Remark 6.5. (i) If X is proper, Conjecture 6.4 is a theorem: we have the exact sequence of BC's (see Section 5.2.3)

$$0 \rightarrow H_{\mathrm{pro\acute{e}t}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\mathrm{HK}}^r(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r} \rightarrow H^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^r) \rightarrow 0$$

We know that $H_{\mathrm{pro\acute{e}t}}^r(X, \mathbf{Q}_p)$ has finite dimension over \mathbf{Q}_p ; hence its height is equal to its dimension over \mathbf{Q}_p . We also know that the slopes of Frobenius on $H_{\mathrm{HK}}^r(X)$ are $\leq r$, which implies that $\mathrm{ht}((H_{\mathrm{HK}}^r(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r}) = \dim_{F^{\mathrm{nr}}} H_{\mathrm{HK}}^r(X)$. Now, the above short exact sequence implies that $\mathrm{ht}(\mathbb{H}_{\mathrm{pro\acute{e}t}}^r(X, \mathbf{Q}_p)) = \dim_{F^{\mathrm{nr}}} H_{\mathrm{HK}}^r(X)$ and, since by the Hyodo-Kato isomorphism $\dim_{F^{\mathrm{nr}}} H_{\mathrm{HK}}^r(X) = \dim_C H_{\mathrm{dR}}^r(X)$, we have

$$\dim_{\mathbf{Q}_p}(H_{\mathrm{pro\acute{e}t}}^r(X, \mathbf{Q}_p)) = \dim_C H_{\mathrm{dR}}^r(X),$$

as wanted.

(ii) Does there exist non proper dagger varieties X such that the $H_{\mathrm{pro\acute{e}t}}^r(X, \mathbf{Q}_p)$'s are finite dimensional over \mathbf{Q}_p ? Already for $r = 1$, one needs $\mathcal{O}(X) = C$.

6.2. **Equivalence of conjectures.** In this section we assume X to be a smooth dagger variety over C such that $H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+)$ is (free) of finite rank over $\mathbf{B}_{\mathrm{dR}}^+$ for all i (for example, X could be quasi-compact, or the interior of a quasi-compact, or the analytification of an algebraic variety, see Remark 6.17). Conjectures 5.21, 6.1, 6.3, and 6.4 make sense in this, slightly more general, set up.

6.2.1. *The key diagram.* Fix r and, for $i \leq r$, set

$$\begin{aligned} H^{r,i} &:= H_{\mathrm{pro\acute{e}t}}^i(X, \mathbf{Q}_p(r)), & X^{r,i} &:= (H_{\mathrm{HK}}^i(X) \widehat{\otimes}_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}^+)^{N=0, \varphi=p^r}, \\ F^{r,i} &:= H^i(F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)), & \mathrm{DR}^{r,i} &:= H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^r), \\ B^i &:= H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+) \end{aligned}$$

We also denote by

$$\mathbb{H}^{r,i}, \quad \mathbb{X}^{r,i}, \quad \mathbb{F}^{r,i}, \quad \mathbb{D}\mathbb{R}^{r,i}, \quad \mathbb{B}^i$$

the associated VS's. Remark 3.33 makes it possible to navigate freely between these VS's and their C -points.

The isomorphism between pro-étale and syntomic cohomologies (see [15, 6.19]) yields a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
H^{r,i-1} & \longrightarrow & X^{r,i-1} & \longrightarrow & \text{DR}^{r,i-1} & \longrightarrow & H^{r,i} & \longrightarrow & X^{r,i} & \longrightarrow & \text{DR}^{r,i} \\
\downarrow & & \downarrow & & \parallel & & \downarrow & & \downarrow & & \parallel \\
F^{r,i-1} & \longrightarrow & B^{i-1} & \longrightarrow & \text{DR}^{r,i-1} & \longrightarrow & F^{r,i} & \longrightarrow & B^i & \longrightarrow & \text{DR}^{r,i} \\
\downarrow & & \downarrow & & \parallel & & \downarrow & & \downarrow & & \parallel \\
F^{r,i-1}/t^r & \longrightarrow & B^{i-1}/t^r F^r & \longrightarrow & \text{DR}^{r,i-1} & \longrightarrow & F^{r,i}/t^r & \longrightarrow & B^i/t^r F^r & \longrightarrow & \text{DR}^{r,i}
\end{array}$$

The bottom sequence is exact because $t^r F^{r,i} \rightarrow t^r F^r B^i$ is an isomorphism (multiplying by t^r kills the big \mathbf{B}_{dR}^+ -torsion in $F^{r,i}$, and B^i is a free finite rank \mathbf{B}_{dR}^+ -module thanks to the Hyodo-Kato isomorphism), hence we have an isomorphism:

$$(6.6) \quad \text{Ker}(F^{r,i} \rightarrow F^{r,i}/t^r) \xrightarrow{\sim} \text{Ker}(B^i \rightarrow B^i/t^r F^r).$$

All the spaces in the diagram are C -points of VS's and those in the top and the bottom rows are C -points of qBC's (this is clear for all of them except for $H^{r,i}$ (and $H^{r,i-1}$), for which one can use Lemma 3.28 and the fact that $\text{DR}^{r,i}$ is a \mathbf{B}_r -module and $X^{r,i}$ is equal to the C -points of a BC), and the above diagram lifts to a diagram of VS's (see [15, Sec. 7]).

Lemma 6.7. *For all $i \geq 0$, we have the following isomorphisms:*

$$\begin{aligned}
\text{Coker}(X^{r,i} \oplus F^{r,i} \rightarrow B^i) &\simeq \text{Coker}(X^{r,i} \rightarrow B^i/F^r), \\
\text{Ker}(H^{r,i} \rightarrow X^{r,i} \oplus F^{r,i}) &\simeq \text{Coker}(X^{r,i-1} \oplus F^{r,i-1} \rightarrow B^{i-1})
\end{aligned}$$

These isomorphisms can be lifted to the category of VS's and $\text{Coker}(X^{r,i} \rightarrow B^i/F^r)$ is the C -points of a BC.

Proof. The first isomorphism is clear. Using the snake lemma in the following commutative diagram with exact rows we prove the second isomorphism.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X^{r,i-1}/H^{r,i-1} & \longrightarrow & \text{DR}^{r,i-1} & \longrightarrow & \text{Ker}(H^{r,i} \rightarrow X^{r,i}) & \longrightarrow & 0 \\
& & \downarrow & & \parallel & & \downarrow & & \\
0 & \longrightarrow & B^{i-1}/F^r & \longrightarrow & \text{DR}^{r,i-1} & \longrightarrow & \text{Ker}(F^{r,i} \rightarrow B^i) & \longrightarrow & 0
\end{array}$$

The claim about BC's follows from Proposition 3.32 because the VS's corresponding to $X^{r,i}$ and B^i/F^r are BC's. \square

Corollary 6.8. *If $\text{Ker}(\mathbb{H}^{r,i} \rightarrow \mathbb{X}^{r,i} \oplus \mathbb{F}^{r,i}) \neq 0$, then it has height < 0 hence can not have curvature ≤ 0 .*

Proof. Use Lemma 6.7 to pass to

$$\text{Coker}(\mathbb{X}^{r,i-1} \oplus \mathbb{F}^{r,i-1} \rightarrow \mathbb{B}^{i-1}) \xrightarrow{\sim} \text{Coker}(\mathbb{X}^{r,i-1} \rightarrow \mathbb{B}^{i-1}/\mathbb{F}^r).$$

Now use Proposition 4.21 and Remark 4.22. \square

6.2.2. *Left exactness.* We will study now the exactness on the left of the sequence in Conjecture 5.21.

Lemma 6.9. *The following properties are equivalent:*

(a) *The map*

$$H_{\text{proét}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^r(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \oplus H^r(F^r \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+))$$

is injective.

(a') *The map*

$$H_{\text{proét}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^r(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \oplus H^r(F^r \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+))/t^r$$

is injective.

- (b) $(H_{\text{HK}}^{r-1}(X), F^0 H_{\text{dR}}^{r-1}(X/\mathbf{B}_{\text{dR}}))$ is acyclic.
- (c) $\text{Ker}(\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^r(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbb{B}_{\text{st}}^+)^{N=0, \varphi=p^r})$ has curvature 0.
- (c') $\text{Ker}(\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^r(X) \widehat{\otimes}_{F^{\text{nr}}} \mathbb{B}_{\text{st}}^+)^{N=0, \varphi=p^r})$ has height 0.
- (d) $\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r))$ has curvature ≤ 0 .

Proof. We note that map in (a) can be lifted to the category $q\mathcal{BC}$ and its kernel is equal to the C -points of a qBC by Proposition 3.32.

- We clearly have (a') \Rightarrow (a). For the opposite implication use the isomorphism (6.6).
- The equivalence of (a) and (b) is a reformulation of the isomorphisms in Lemma 6.7 (use Remark 4.19).
- We have (c) \Rightarrow (c') by Remark 3.27. And, since $\text{Ker}(\mathbb{H}^{r,r} \rightarrow \mathbb{X}^{r,r})$, which is a qBC, is a quotient of $\mathbb{D}\mathbb{R}^{r,r-1}$, which has curvature 0 (since it is a \mathbb{B}_r -module), (c') \Rightarrow (c) as well (use Remark 3.27).
- We have (a') \Rightarrow (d) since $\mathbb{H}^{r,r}$ is a submodule of $\mathbb{X}^{r,r} \oplus (\mathbb{F}^{r,r}/t^r)$, the first term of which comes from a BC of curvature ≤ 0 (since this BC is a Submodule of a \mathbb{B}_r -Module, use Remark 3.27) and the second term is an affine qBC.
- We have (d) \Rightarrow (c) since $\text{Ker}(\mathbb{H}^{r,r} \rightarrow \mathbb{X}^{r,r})$ is a sub-VS of $\mathbb{H}_{\text{proét}}^r$ and hence has curvature ≤ 0 (since so does $\mathbb{H}_{\text{proét}}^r$ by assumption), and it is also a quotient of $\mathbb{D}\mathbb{R}^{r,r-1}$ and hence has curvature ≥ 0 (since so does $\mathbb{D}\mathbb{R}^{r,r-1}$). Here we have used again Remark 3.27.
- We have (c) \Rightarrow (a') since the inclusion of the kernels implies that $\text{Ker}(H^{r,r} \rightarrow X^{r,r} \oplus (F^{r,r}/t^r))$ corresponds to a qBC of curvature ≤ 0 (by Remark 3.27), and thus it is trivial by Corollary 6.8. \square

6.2.3. *Right exactness.* We pass now to the study of the exactness on the right of the sequence in Conjecture 5.21.

Lemma 6.10. *The following properties are equivalent:*

- (a) *The map*

$$(H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \oplus H^r(F^r \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+)) \rightarrow H_{\text{dR}}^r(X/\mathbf{B}_{\text{dR}}^+)$$

is surjective.

- (a') *The map*

$$(H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \oplus H^r(F^r \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+))/t^r \rightarrow H_{\text{dR}}^r(X/\mathbf{B}_{\text{dR}}^+)/t^r F^r$$

is surjective.

- (b) $(H_{\text{HK}}^r(X), F^0 H_{\text{dR}}^r(X/\mathbf{B}_{\text{dR}}))$ is acyclic.
- (c) $\text{Coker}(\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbb{B}_{\text{st}}^+)^{N=0, \varphi=p^r})$ has curvature 0.
- (c') $\text{Coker}(\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^r(X) \otimes_{F^{\text{nr}}} \mathbb{B}_{\text{st}}^+)^{N=0, \varphi=p^r})$ has height 0.

Proof. • Clearly (a) \Rightarrow (a'). For the opposite implication use isomorphism 6.6.

- The equivalence of (a) and (b) is a reformulation of the first isomorphism from Lemma 6.7 (use Remark 4.19).
- We clearly have (c) \Rightarrow (c') and the opposite implication follows from the fact that the cokernel of $\mathbb{H}^{r,r} \rightarrow \mathbb{X}^{r,r}$ is a sub-qBC of $\mathbb{D}\mathbb{R}^{r,r}$. Since the latter has curvature 0 (because it is a \mathbb{B}_r -Module) (c) follows from Remark 3.27.
- Let I^r be the image of $\mathbb{X}^{r,r}$ in $\mathbb{D}\mathbb{R}^{r,r}$, and let J^r be the image of $\mathbb{X}^{r,r}$ in \mathbb{B}^r/F^r . Then $\text{Coker}(\mathbb{H}^{r,r} \rightarrow \mathbb{X}^{r,r}) \simeq I^r \simeq J^r$ since $\mathbb{B}^r/F^r \rightarrow \mathbb{D}\mathbb{R}^{r,r}$ is injective. It follows that $\text{Coker}(\mathbb{H}^{r,r} \rightarrow \mathbb{X}^{r,r})$ has curvature 0 if and only if J^r does. And thus if and only if $J^r = \mathbb{B}^r/F^r$ by Proposition 4.21. The equivalence of (c) and (b) follows. \square

6.2.4. *Bicartesian property.* We can now put Lemma 6.9 and Lemma 6.10 together:

Proposition 6.11. *The following properties are equivalent:*

- (a) *The diagram in Conjecture 5.21 is bicartesian.*

- (b) $(H_{\text{HK}}^i(X), F^0 H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}))$ is acyclic, for $i = r - 1$ and $i = r$.
- (c) The kernel and cokernel of $\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^r(X) \otimes \mathbb{B}_{\text{st}}^+)^{N=0, \varphi=p^r}$ have curvature 0.
- (c') The kernel and cokernel of $\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^r(X) \otimes \mathbb{B}_{\text{st}}^+)^{N=0, \varphi=p^r}$ have height 0.
- (d) $\text{ht}(\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r))) = \dim_C H_{\text{dR}}^r(X)$.

Proof. • Let us start by noting that, by Remark 5.23 (ii), in (a) we can assume $i = r$.

- Note that (a), (b), (c) and (c') are, respectively, equivalent to the conjunction of (a), (b), (c) and (c') from Lemma 6.9 and 6.10. The equivalence of (a), (b) (c) and (c') follows.
- (a) \Rightarrow (d): by assumption we have the exact sequence

$$0 \longrightarrow H^{r,r} \longrightarrow (F^{r,r}/t^r) \oplus X^{r,r} \longrightarrow B^r/t^r F^r \longrightarrow 0$$

This sequence can be lifted to the category $q\mathcal{B}\mathcal{C}$. Since $\mathbb{F}^{r,r}/t^r$ and $\mathbb{B}^r/t^r F^r$ have curvature 0, and hence height zero, and

$$\text{ht}(X^{r,r}) = \dim_{F^{\text{nr}}} H_{\text{HK}}^r(X) = \dim_C H_{\text{dR}}^r(X),$$

(d) follows. The first equality above uses the fact that the φ -slopes are $\leq r$.

- (d) \Rightarrow (b): Since $\mathbb{D}\mathbb{R}^{r,r-1}$ and $\mathbb{D}\mathbb{R}^{r,r}$ are \mathbb{B}_r -modules, the images I^{r-1} and I^r of $\mathbb{X}^{r,r-1}$ and $\mathbb{X}^{r,r}$ have height ≥ 0 . Since

$$\text{ht}(\mathbb{H}^{r,r}) = (\text{ht}(\mathbb{D}\mathbb{R}^{r,r-1}) - \text{ht}(I^{r-1})) + (\text{ht}(\mathbb{X}^{r,r}) - \text{ht}(I^r))$$

and $\text{ht}(\mathbb{D}\mathbb{R}^{r,r-1}) = 0$, the condition $\text{ht}(\mathbb{H}^{r,r}) = \text{ht}(\mathbb{X}^{r,r})$ (equivalent to (d) as we have seen) implies that $\text{ht}(I^{r-1}) = \text{ht}(I^r) = 0$. As we have seen above in proving the equivalence of (b) and (c) in Lemma 6.10, this implies (b).

This finishes the proof of our proposition. \square

Corollary 6.12. *If X is quasi-compact then Conjectures 5.21, 6.1, 6.3, and 6.4 are equivalent.*

Proof. Equivalence of (a) and (d) from Proposition 6.11 shows that, already for a fixed r , Conjectures 5.21 and 6.4 are equivalent. Moreover, taken for all r , (b) from Proposition 6.11 shows that they are equivalent to Conjecture 6.3. The equivalence of (b) and (d) from Lemma 6.9 shows that Conjectures 6.1 and 6.3 are equivalent (but one has to change r). \square

Remark 6.13. It is easy to see that Corollary 6.12 holds, more generally, if $H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+)$ is finite rank over \mathbf{B}_{dR}^+ for all i . The proof is the same as in the quasi-compact case.

6.3. Proof of the conjectures. Finally, we are ready to prove our main theorem for quasi-compact varieties.

Theorem 6.14. *Let X be a quasi-compact smooth dagger variety over C .*

- (i) For all $r \geq i$, the diagram

$$\begin{array}{ccc} \tilde{H}_{\text{proét}}^i(X, \mathbf{Q}_p(r)) & \longrightarrow & (\tilde{H}_{\text{HK}}^i(X) \hat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \\ \downarrow & & \downarrow \\ \tilde{H}^i(F^r(\text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+))) & \longrightarrow & \tilde{H}_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+) \end{array}$$

is bicartesian. Equivalently, the following sequence is exact

$$0 \rightarrow \tilde{H}_{\text{proét}}^i(X, \mathbf{Q}_p(r)) \rightarrow \tilde{H}^i(F^r(\text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+))) \oplus (\tilde{H}_{\text{HK}}^i(X) \hat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \rightarrow \tilde{H}_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+) \rightarrow 0$$

Moreover, for all r ,

- (ii) $(H_{\text{HK}}^r(X), F^0 H_{\text{dR}}^r(X/\mathbf{B}_{\text{dR}}))$ is acyclic.
- (iii) $\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r))$ has curvature ≤ 0 .
- (iv) $\text{ht}(\mathbb{H}_{\text{proét}}^r(X, \mathbf{Q}_p(r))) = \dim_C H_{\text{dR}}^r(X)$.

Proof. Let X be as in the theorem. By Remark 5.23 (iii) we can remove all the tildas in the statement of the theorem. We have seen (Corollary 6.12) that the properties (i)-(iv) for all r are equivalent (on the other hand, they are not (!) equivalent for a fixed r). It suffices thus to show (iv) for all r . This we do by induction on the number n of open affinoids necessary for covering X : the base case of $n = 1$ is Corollary 5.19.

We pass from n to $n + 1$ using Mayer-Vietoris (and the fact that an intersection of two affinoids is an affinoid, from which it follows that if U_1 is covered by n open affinoids and if U_2 is an affinoid, then $U_1 \cap U_2$ is covered by n open affinoids).

Take then U_1, U_2 as above. Let $U = U_1 \cup U_2$. If $i \geq 0$, we set

$$\begin{aligned} A_{3i} &= H_{\text{proét}}^i(U, \mathbf{Q}_p(r)), & A_{3i+1} &= H_{\text{proét}}^i(U_1 \coprod U_2, \mathbf{Q}_p(r)), \\ A_{3i+2} &= H_{\text{proét}}^i(U_1 \cap U_2, \mathbf{Q}_p(r)) \\ B_{3i} &= (H_{\text{HK}}^i(U) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}, & B_{3i+1} &= (H_{\text{HK}}^i(U_1 \coprod U_2) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}, \\ B_{3i+2} &= (H_{\text{HK}}^i(U_1 \cap U_2) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}. \end{aligned}$$

We have a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \rightarrow & A_0 & \rightarrow & A_1 & \rightarrow & \cdots & \rightarrow & A_{3r-1} & \rightarrow & A_{3r} & \rightarrow & A_{3r+1} & \rightarrow & \cdots \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & B_0 & \rightarrow & B_1 & \rightarrow & \cdots & \rightarrow & B_{3r-1} & \rightarrow & B_{3r} & \rightarrow & B_{3r+1} & \rightarrow & \cdots \end{array}$$

in which the rows are exact; in the top row we have qBC's and in the bottom row we have BC's.

We denote by K_i and C_i the kernel and cokernel of $A_i \rightarrow B_i$, respectively. It is clear from the definition that B_i has curvature ≤ 0 ; that is, it is of curvature ≤ 0 . We infer, using Proposition 3.24, that $\text{rk}(h(B_i)) = \text{ht}(B_i)$ (see Section 3.3.1 for the definition of the functor h). We know that C_i is a subobject of an affine BC and that K_i is a quotient of B_r -module. This implies in particular that $\text{ht}(A_i) \leq \text{ht}(B_i)$ with equality if and only if C_i and K_i are affine. We note that then A_i has curvature ≤ 0 because its quotient by the B_r -module K_i is a sub-BC of B_i and hence has curvature ≤ 0 (and even < 0 if is not zero).

Lemma 6.15. *If $\text{ht}(A_k) = \text{ht}(B_k)$ for all $k \leq 3r + 2$, $k \neq 3r$, and if A_{3r} has curvature ≤ 0 , then $\text{ht}(A_{3r}) = \text{ht}(B_{3r})$.*

Proof. Note that the equality $\text{ht}(A_k) = \text{ht}(B_k)$ (together with the fact that K_k is a quotient of an affine and C_k a subobject of an affine) implies that A_k and B_k have curvature ≤ 0 , that K_k and C_k are affines, and that the map $h(B_k) \rightarrow h(A_k)$ is an isomorphism. All the terms of the commutative subdiagram

$$\begin{array}{ccccccccc} A_{3r-2} & \rightarrow & A_{3r-1} & \rightarrow & A_{3r} & \rightarrow & A_{3r+1} & \rightarrow & A_{3r+2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B_{3r-2} & \rightarrow & B_{3r-1} & \rightarrow & B_{3r} & \rightarrow & B_{3r+1} & \rightarrow & B_{3r+2} \end{array}$$

have curvature ≤ 0 and so do the kernels A'_k of $A_k \rightarrow A_{k+1}$ and B'_k of $B_k \rightarrow B_{k+1}$ (as subobjects of qBC's of curvature ≤ 0). We infer, using Corollary 3.35, that the rows of the commutative diagram

$$\begin{array}{ccccccccc} h(A_{3r-2}) & \leftarrow & h(A_{3r-1}) & \leftarrow & h(A_{3r}) & \leftarrow & h(A_{3r+1}) & \leftarrow & h(A_{3r+2}) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ h(B_{3r-2}) & \leftarrow & h(B_{3r-1}) & \leftarrow & h(B_{3r}) & \leftarrow & h(B_{3r+1}) & \leftarrow & h(B_{3r+2}) \end{array}$$

are exact. If $k \neq 3r$, the fact that K_k and C_k are affine implies that $h(B_k) \rightarrow h(A_k)$ is an isomorphism; hence also $h(B_{3r}) \rightarrow h(A_{3r})$ by the 5-Lemma. Since B_{3r} has curvature ≤ 0 and so does A_{3r} by assumption, it follows from Proposition 3.24 that $\text{ht}(A_{3r}) = \text{ht}(B_{3r})$, as wanted. \square

Our theorem now follows from the following proposition. \square

Proposition 6.16. *If U_1, U_2 et $U_1 \cap U_2$ satisfy Conjectures 5.21, 6.1, and 6.4 (which are equivalent), the same holds for U .*

Proof. By assumption, U_1, U_2 and $U_1 \cap U_2$ satisfy Conjectures 5.21, 6.1, and 6.4 for all r ; we will show that the same holds for U , using induction on r . We assume thus that the result is shown up to $r - 1$ (it is trivial for $r = 0$). Hence, in particular, $\text{ht}(A_k) = \text{ht}(B_k)$ for all $k \leq 3r + 2$, $k \neq 3r$, and the problem is to show that $\text{ht}(A_{3r}) = \text{ht}(B_{3r})$.

Since the result is true for degree $r - 1$, by Lemma 6.10 we have that $(H_{\text{HK}}^{r-1}(U), F^0 H_{\text{dR}}^{r-1}(U/\mathbf{B}_{\text{dR}}))$ is acyclic. By Lemma 6.9, this implies that A_{3r} has curvature ≥ 0 . We can thus apply Lemma 6.15, and hence $\text{ht}(A_{3r}) = \text{ht}(B_{3r})$. \square

Remark 6.17. Granting Corollary 6.12 (see Remark 6.13), the same proof applies to smooth dagger varieties which can be covered by a finite number of Stein varieties U_j such that $H_{\text{dR}}^i(U_j/\mathbf{B}_{\text{dR}}^+)$ is free of finite rank over \mathbf{B}_{dR}^+ for all i, j (because the intersection of two Stein is Stein). This applies to:

- analytifications of algebraic varieties: pick a covering by affine opens, and use the analytifications of these affine opens for your Stein covering.
- the interior X of a quasi-compact rigid analytic subvariety X' of a partially proper rigid analytic variety Y : cover X' with a finite number of affinoids U_j ; choose, for each j , an affinoid $U'_j \subset Y$ strictly containing U_j , and let V_j be the intersection of X with the interior of U'_j . The V_j 's are a covering of X with the desired properties: $H_{\text{dR}}^i(V_j/\mathbf{B}_{\text{dR}}^+)$ is a free \mathbf{B}_{dR}^+ -module whose reduction modulo $\ker \theta$ is isomorphic to $H_{\text{dR}}^i(V_j)$, which is of finite rank over C by [30, Th. A].

7. DE RHAM-TO-PRO-ÉTALE COMPARISON THEOREM: MORE EXAMPLES AND A CRITERIUM

In the previous chapters we have proved the C_{st} -conjecture (Conjecture 5.21) for Stein varieties (see Examples 5.4.1) and for quasi-compact ones (Theorem 6.14). In this chapter, we give other examples, for which we can prove this conjecture, and give a sufficient criterium for its validity.

Unless otherwise stated we work in this chapter in the algebraic category.

7.1. Preliminaries. The general set-up is the following: if X is a smooth dagger variety over C , we choose a covering $\{X_n\}_{n \in \mathbf{N}}$ by increasing quasi-compact dagger subvarieties, and we want to deduce the result for X from the result for the X_n 's. To make formulas less cumbersome, let us introduce some notation.

Set

$$(7.1) \quad \begin{aligned} X^{r,i} &:= (H_{\text{HK}}^i(X) \widehat{\otimes}_{F^{\text{nr}}}^R \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}, & X_n^{r,i} &:= (H_{\text{HK}}^i(X_n) \widehat{\otimes}_{F^{\text{nr}}}^R \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}, \\ F^{r,i} &:= H^i(F^r \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+)), & F_n^{r,i} &:= H^i(F^r \text{R}\Gamma_{\text{dR}}(X_n/\mathbf{B}_{\text{dR}}^+)), \\ A^{r,i} &:= \text{Ker}(F^{r,i} \rightarrow H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+)), & A_n^{r,i} &:= \text{Ker}(F_n^{r,i} \rightarrow H_{\text{dR}}^i(X_n/\mathbf{B}_{\text{dR}}^+)), \\ \overline{F}^{r,i} &:= F^{r,i}/A^{r,i}, & \overline{F}_n^{r,i} &:= F_n^{r,i}/A_n^{r,i}. \end{aligned}$$

So that, by definition,

$$(7.2) \quad \overline{F}^{r,i} \simeq F^r H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+), \quad \overline{F}_n^{r,i} \simeq F^r H_{\text{dR}}^i(X_n/\mathbf{B}_{\text{dR}}^+).$$

In particular,

$$\overline{F}^{0,i} \simeq H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+), \quad \overline{F}_n^{0,i} \simeq H_{\text{dR}}^i(X_n/\mathbf{B}_{\text{dR}}^+).$$

We want to prove the exactness of:

$$(7.3) \quad 0 \rightarrow H^{r,i} \rightarrow X^{r,i} \oplus F_n^{r,i} \rightarrow \overline{F}^{0,i} \rightarrow 0$$

By Theorem 6.14, we have the exact sequences

$$(7.4) \quad 0 \rightarrow H_n^{r,i} \rightarrow X_n^{r,i} \oplus F_n^{r,i} \rightarrow \overline{F}_n^{0,i} \rightarrow 0$$

Hence, to prove the exactness of (7.3), we need to control various $\text{R}^1 \lim_n$.

Remark 7.5. (i) By [15, Prop. 5.8] and [15, Prop. 5.12], respectively, we have

$$R^1 \lim_n X_n^{r,i} = 0, \quad R^1 \lim_n \overline{F}_n^{r,i} = 0$$

([15, Prop. 5.12] proves the result for $\overline{F}_n^{0,i}$; the general case follows because $\overline{F}_n^{r,i}$ contains $t^r \overline{F}_n^{0,i}$, and $\overline{F}_n^{r,i}/t^r \overline{F}_n^{0,i}$ is a BC).

(ii) We have two exact sequences and isomorphisms

$$\begin{aligned} 0 \rightarrow R^1 \lim_n F_n^{r,i-1} \rightarrow F^{r,i} \rightarrow \lim_n F_n^{r,i} \rightarrow 0, \\ 0 \rightarrow R^1 \lim_n H_n^{r,i-1} \rightarrow H^{r,i} \rightarrow \lim_n H_n^{r,i} \rightarrow 0, \\ X^{r,i} \simeq \lim_n X_n^{r,i}, \quad \overline{F}^{0,i} \simeq \lim_n \overline{F}_n^{0,i}. \end{aligned}$$

(iii) Applying \lim_n to (5.11) we obtain the exact sequence

$$0 \rightarrow \lim_n H_n^{r,i} \rightarrow \lim_n X_n^{r,i} \oplus \lim_n F_n^{r,i} \rightarrow \lim_n \overline{F}_n^{0,i} \rightarrow R^1 \lim_n H_n^{r,i} \rightarrow R^1 \lim_n F_n^{r,i} \rightarrow 0$$

Hence, if we can prove that $R^1 \lim_n H_n^{r,i} = 0$, for all i , we get that $H^{r,i} \simeq \lim_n H_n^{r,i}$, that (7.3) is exact and, as a bonus, that $R^1 \lim_n F_n^{r,i} = 0$ and $F^{r,i} \simeq \lim_n F_n^{r,i}$.

Proving directly the vanishing of the $R^1 \lim_n H_n^{r,i}$'s for all i does not look too promising since pro-étale cohomology is not that easy to deal with, but one can try to prove statements about the filtered de Rham complexes that imply this vanishing. Set

$$\mathrm{DR}^{r,i} := H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^r), \quad \mathrm{DR}_n^{r,i} := H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X_n/\mathbf{B}_{\mathrm{dR}}^+)/F^r).$$

Lemma 7.6. *We have the implications*

$$(R^1 \lim_n \mathrm{DR}_n^{r,i-1} = 0) \implies (R^1 \lim_n A_n^{r,i} = 0) \implies (R^1 \lim_n H_n^{r,i} = 0).$$

Proof. We have a long exact sequence

$$\dots \overline{F}_n^{0,i-1} \rightarrow \mathrm{DR}_n^{r,i-1} \rightarrow F_n^{r,i} \rightarrow \overline{F}_n^{0,i} \rightarrow \dots$$

It follows that $A_n^{r,i}$ is a quotient of $\mathrm{DR}_n^{r,i-1}$. This proves the first implication.

For the second implication, note that $A_n^{r,i}$ is naturally a subspace of $H_n^{r,i}$; denote by $\overline{H}_n^{r,i}$ the quotient. Since $t^r \overline{F}_n^{0,i}$ is a subspace of $\overline{F}_n^{r,i}$, we have an exact sequence

$$0 \rightarrow \overline{H}_n^{r,i} \rightarrow X_n^{r,i} \oplus (\overline{F}_n^{r,i}/t^r \overline{F}_n^{0,i}) \rightarrow (\overline{F}_n^{0,i}/t^r \overline{F}_n^{0,i}) \rightarrow 0$$

in which all the terms, except $\overline{H}_n^{r,i}$ are BC's. This shows that $\overline{H}_n^{r,i}$ is a BC as well¹⁵, hence $R^1 \lim_n \overline{H}_n^{r,i} = 0$ by Remark 3.14. It follows that we have an exact sequence

$$0 \rightarrow \lim_n A_n^{r,i} \rightarrow \lim_n H_n^{r,i} \rightarrow \lim_n \overline{H}_n^{r,i} \rightarrow R^1 \lim_n A_n^{r,i} \rightarrow R^1 \lim_n H_n^{r,i} \rightarrow 0$$

which proves the second implication. \square

Remark 7.7. The above proof yields that:

$$R^1 \lim_n \overline{H}_n^{r,i} = 0$$

¹⁵To be more precise, by [15, Sec. 7.3], the analog of the exact sequence (7.24) for X_n can be lifted to the category of Vector Spaces. Moreover, since $H_{\mathrm{dR}}^i(X_n)$ is finite dimensional, all the terms in this exact sequence are BC's: this is clear for $\overline{F}_n^{0,i}/t^r \overline{F}_n^{0,i}$ and for $X_n^{r,i}$; it is true for $\overline{F}_n^{r,i}/t^r \overline{F}_n^{0,i}$ because it is a sub- $\mathbb{B}_{\mathrm{dR}}^+$ -module of $\overline{F}_n^{0,i}/t^r \overline{F}_n^{0,i}$, and, finally, it is true for $\overline{H}_n^{r,i}$ as the kernel of a morphism between BC's.

7.2. Holomorphically convex varieties. The notion of holomorphically convex analytic varieties was introduced by Berkovich in the set-up of his geometry [5, 2.6]. They have Remmert factorization property:

Theorem 7.8. (*Remmert reduction theorem*, [32, Th. 6.1]) *Let X be a smooth partially proper rigid analytic space over C . Then there exists a Stein variety over C together with a morphism $\pi : X \rightarrow S$ satisfying the following properties:*

- (i) π is proper, surjective, and with connected fibers;
- (ii) the induced morphism $\mathcal{O}_S \rightarrow \pi_* \mathcal{O}_X$ is an isomorphism.

7.2.1. Example: Proper times Stein. Let X be the dagger variety $X := Y \times_C S$, where Y and S are smooth dagger varieties over C , Y is proper and S is Stein. We cover S with a strictly increasing admissible covering $\{S_n\}_{i \in \mathbb{N}}$ by open dagger affinoids. Set $X_n := Y \times_C S_n$.

Proposition 7.9. *For $i \geq 0$, we have*

$$\mathbf{R}^1 \lim_n H_{\text{proét}}^i(X_n, \mathbf{Q}_p) = 0.$$

Proof. Since $\mathbf{R} \lim_n H_{\text{proét}}^i(X_n, \mathbf{Q}_p) = \mathbf{R} \lim_n H_{\text{proét}}^i(\widehat{X}_n, \mathbf{Q}_p)$, we can replace X_n, S_n with their completions. Since \widehat{S}_n 's are quasi-compact, we can pass to étale cohomology. We will need the following fact.

Lemma 7.10. *Let Y, T be smooth rigid analytic varieties over C . Assume that Y is proper. Let $i \geq 0$. We have the Künneth isomorphism*

$$(7.11) \quad \bigoplus_{j \geq 0} H_{\text{ét}}^j(Y, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} H_{\text{ét}}^{i-j}(T, \mathbf{Q}_p) \xrightarrow{\sim} H_{\text{ét}}^i(Y \times_C T, \mathbf{Q}_p).$$

Proof.

- *Step 1: the integral Künneth formula.* We claim that the canonical map

$$\mathbf{R}\Gamma_{\text{ét}}(Y, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p}^L \mathbf{R}\Gamma_{\text{ét}}(T, \mathbf{Z}_p) \rightarrow \mathbf{R}\Gamma_{\text{ét}}(Y \times_C T, \mathbf{Z}_p)$$

is a quasi-isomorphism. The proof of Künneth formula from [40, Tag 0F13] goes through once we have the quasi-isomorphism

$$(7.12) \quad \mathbf{R}p_{2,*} \mathbf{Z}_p \xleftarrow{\sim} \mathbf{R}\Gamma_{\text{ét}}(Y, \mathbf{Z}_p),$$

where $p_2 : Y \times_C T \rightarrow T$ is the canonical projection. This is a consequence of the proper base change theorem of Bhatt-Hansen [6, Th. 1.6]. To recall the proof from [40, Tag 0F13]: we apply $\mathbf{R}\Gamma_{\text{ét}}(T, -)$ to the quasi-isomorphism (7.12) and get a quasi-isomorphism

$$\mathbf{R}\Gamma_{\text{ét}}(Y \times_C T, \mathbf{Z}_p) \xleftarrow{\sim} \mathbf{R}\Gamma_{\text{ét}}(T, \mathbf{R}\Gamma_{\text{ét}}(Y, \mathbf{Z}_p)).$$

Now, we use the fact that the complex $\mathbf{R}\Gamma_{\text{ét}}(T, \mathbf{R}\Gamma_{\text{ét}}(Y, \mathbf{Z}_p))$ is perfect and get a quasi-isomorphism

$$\mathbf{R}\Gamma_{\text{ét}}(T, \mathbf{R}\Gamma_{\text{ét}}(Y, \mathbf{Z}_p)) \xleftarrow{\sim} \mathbf{R}\Gamma_{\text{ét}}(T, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p}^L \mathbf{R}\Gamma_{\text{ét}}(Y, \mathbf{Z}_p).$$

This finishes the proof of the integral Künneth formula.

- *Step 2: passing to \mathbf{Q}_p -coefficients.* From Step 1 and the fact that the complex $\mathbf{R}\Gamma_{\text{ét}}(Y, \mathbf{Q}_p)$ is perfect, we get the canonical quasi-isomorphism

$$\mathbf{R}\Gamma_{\text{ét}}(Y, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p}^L \mathbf{R}\Gamma_{\text{ét}}(T, \mathbf{Q}_p) \xrightarrow{\sim} \mathbf{R}\Gamma_{\text{ét}}(Y \times_C T, \mathbf{Q}_p).$$

Taking cohomology of both sides we get the isomorphism in the proposition. \square

Using the above lemma and Theorem 5.14 we get

$$\mathbf{R}^1 \lim_n H_{\text{ét}}^i(X_n, \mathbf{Q}_p) \simeq \bigoplus_{j \geq 0} H_{\text{ét}}^j(Y, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{R}^1 \lim_n H_{\text{ét}}^{i-j}(\widehat{S}_n, \mathbf{Q}_p) = 0,$$

as wanted. \square

Remark 7.13. (i) Proposition 7.9 implies the isomorphism

$$H_{\text{proét}}^i(X, \mathbf{Q}_p) \xrightarrow{\sim} \lim_n H_{\text{ét}}^i(X_n, \mathbf{Q}_p).$$

(ii) The proof of Proposition 7.9 yields also the Künneth isomorphism

$$\bigoplus_{j \geq 0} H_{\text{proét}}^j(Y, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} H_{\text{proét}}^{i-j}(S, \mathbf{Q}_p) \xrightarrow{\sim} H_{\text{proét}}^i(X, \mathbf{Q}_p).$$

This Künneth decomposition is true because we are in the very special case where the cohomology groups of one of the factors are finite dimensional. Without this assumption, such a decomposition already fails in the simplest case where both X and Y are the open unit ball in dimension 1.

Proposition 7.14. *Conjecture 5.21 holds for X .*

Proof. Let $i \leq r$. By Remark 5.23, we reduce to showing that the sequence of algebraic cohomologies

$$0 \rightarrow H_{\text{proét}}^i(X, \mathbf{Q}_p(r)) \rightarrow (H_{\text{HK}}^i(X) \widehat{\otimes}_{F^r} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r} \oplus H^i F^r \text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+) \rightarrow H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+) \rightarrow 0$$

is exact. Granting Proposition 7.9, this follows from (iii) of Remark 7.5. \square

7.2.2. *Example: Proper smooth fibration over a Stein curve.* Let X be a smooth dagger variety over C that admits a proper smooth map $\pi : X \rightarrow S$ to a smooth Stein curve over C . Cover S with a strictly increasing admissible covering $\{S_n\}_{i \in \mathbf{N}}$ by open dagger affinoids. Set $X_n := X \times_C S_n$.

Proposition 7.15. *Assume that $X, S, \{S_n\}$, and π are defined over K . Then, for $i \geq 0$, we have*

$$\text{R}^1 \lim_n H_{\text{proét}}^i(X_n, \mathbf{Q}_p) = 0.$$

Hence $H_{\text{proét}}^i(X, \mathbf{Q}_p) \xrightarrow{\sim} \lim_n H_{\text{proét}}^i(X_n, \mathbf{Q}_p)$.

Proof. Set

$$\begin{aligned} \text{DR}^{r,i} &:= \text{DR}^{r,i}(X/\mathbf{B}_{\text{dR}}^+) := H^i(\text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+)/F^r), \\ \text{DR}_n^{r,i} &:= \text{DR}^{r,i}(X_n/\mathbf{B}_{\text{dR}}^+) := H^i(\text{R}\Gamma_{\text{dR}}(X_n/\mathbf{B}_{\text{dR}}^+)/F^r). \end{aligned}$$

Similarly with C (which we omit from the notation) replacing \mathbf{B}_{dR}^+ . By Lemma 7.6, it suffices to show that, for all $r, i \geq 0$,

$$(7.16) \quad \text{DR}^{r,i} \xrightarrow{\sim} \lim_n \text{DR}_n^{r,i}.$$

• *Argument over C .* We start with showing the isomorphism¹⁶ (7.16) for $H_{\text{dR}}^i(X)$, i.e., we will prove that

$$\text{DR}^{r,i}(X) \xrightarrow{\sim} \lim_n \text{DR}^{r,i}(X_n).$$

We start with a quasi-isomorphism

$$(7.17) \quad F^r \Omega_X^\bullet \xrightarrow{\sim} [F^r \Omega_{X/S}^\bullet \xrightarrow{\nabla} F^{r-1} \Omega_{X/S}^\bullet \otimes_{\mathcal{O}_X} \pi^* \Omega_S^1],$$

where ∇ is the Gauss-Manin connection and the brackets $[\dots]$ denote mapping fiber. This yields a quasi-isomorphism

$$\Omega_X^{\leq r-1} \xrightarrow{\sim} [\Omega_{X/S}^{\leq r-1} \xrightarrow{\nabla} \Omega_{X/S}^{\leq r-2} \otimes_{\mathcal{O}_X} \pi^* \Omega_S^1].$$

Applying $\text{R}\pi_*$, we get a quasi-isomorphism

$$\text{R}\pi_* \Omega_X^{\leq r-1} \xrightarrow{\sim} [\text{R}\pi_* \Omega_{X/S}^{\leq r-1} \xrightarrow{\nabla} \text{R}\pi_* \Omega_{X/S}^{\leq r-2} \otimes_{\mathcal{O}_S} \Omega_S^1].$$

Taking cohomology we obtain the short exact sequences

$$0 \rightarrow \text{Coker} \nabla_{i-1} \rightarrow \text{DR}^{r,i}(X) \rightarrow \text{Ker} \nabla_i \rightarrow 0,$$

¹⁶Here it suffices that X, S, π are defined over C .

where $\nabla_s : \text{DR}^{r,s}(X/S) \rightarrow \text{DR}^{r-1,s}(X/S) \otimes_{\mathcal{O}(S)} \Omega^1(S)$ is the map induced by ∇ . We used here the fact that $\Omega^1(S)$ is a projective $\mathcal{O}(S)$ -module.

The above computations go through with \widehat{S}_n replacing S . We get the map of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Coker} \nabla_{i-1} & \longrightarrow & \text{DR}^{r,i}(X) & \longrightarrow & \text{Ker} \nabla_i \longrightarrow 0 \\ & & f_1 \downarrow \wr & & \downarrow f & & f_2 \downarrow \wr \\ 0 & \longrightarrow & \lim_n \text{Coker} \nabla_{i-1,n} & \longrightarrow & \lim_n \text{DR}^{r,i}(\widehat{X}_n) & \longrightarrow & \lim_n \text{Ker} \nabla_{i,n} \end{array}$$

The isomorphism f_2 follows immediately from the map of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker} \nabla_i & \longrightarrow & \text{DR}^{r,i}(X/S) & \xrightarrow{\nabla_i} & \text{DR}^{r-1,i}(X/S) \otimes_{\mathcal{O}(S)} \Omega^1(S) \\ & & \downarrow f_2 & & \downarrow \wr & & \downarrow \wr \\ 0 & \longrightarrow & \lim_n \text{Ker} \nabla_{i,n} & \longrightarrow & \lim_n \text{DR}^{r,i}(\widehat{X}_n/\widehat{S}_n) & \xrightarrow{\nabla_i} & \lim_n (\text{DR}^{r-1,i}(\widehat{X}_n/\widehat{S}_n) \otimes_{\mathcal{O}(\widehat{S}_n)} \Omega^1(\widehat{S}_n)) \end{array}$$

Here the vertical isomorphisms hold because the sheaves $R^i \pi_* \Omega_{X/S}^{\leq r-1}$ and $R^i \pi_* \Omega_{X/S}^{\leq r-2} \otimes_{\mathcal{O}_S} \Omega_S^1$ on S are coherent (because π is proper) and S is Stein. The isomorphism f_1 follows from the map of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Im} \nabla_{i-1} & \longrightarrow & \text{DR}^{r-1,i-1}(X/S) \otimes_{\mathcal{O}(S)} \Omega^1(S) & \longrightarrow & \text{Coker} \nabla_{i-1} \longrightarrow 0 \\ & & \downarrow f_3 & & \downarrow \wr & & \downarrow f_1 \\ 0 & \longrightarrow & \lim_n \text{Im} \nabla_{i-1,n} & \longrightarrow & \lim_n (\text{DR}^{r-1,i-1}(\widehat{X}_n/\widehat{S}_n) \otimes_{\mathcal{O}(\widehat{S}_n)} \Omega^1(\widehat{S}_n)) & \xrightarrow{g} & \lim_n \text{Coker} \nabla_{i-1,n} \longrightarrow 0 \end{array}$$

The map g is surjective because $R^1 \lim_n \text{Im} \nabla_{i-1,n} = 0$ since $\text{Im} \nabla_{i-1,n}$ is a quotient of $\text{DR}^{r,i-1}(\widehat{X}_n/\widehat{S}_n)$ and $R^1 \lim_n \text{DR}^{r,i-1}(\widehat{X}_n/\widehat{S}_n) = 0$ (because $R^{i-1} \pi_* \Omega_{X/S}^{\leq r-2}$ is coherent and S is Stein). It follows that the map f_1 is surjective as well. For its injectivity, it suffices to show that the map f_3 is surjective. Or that $R^1 \lim_n \text{Ker} \nabla_{i-1,n} = 0$. But this holds because $\text{Ker} \nabla_{i-1,n}$ is a finite rank C -vector space.

Having that f_1 and f_2 are isomorphisms, diagram chase shows that f is an isomorphism as well. Hence we have

$$\text{DR}^{r,i}(X) \xrightarrow{\sim} \lim_n \text{DR}^{r,i}(\widehat{X}_n), \quad R^1 \lim_n \text{DR}^{r,i-1}(\widehat{X}_n) = 0.$$

The same holds with X_n replacing \widehat{X}_n because the relevant projective systems are equivalent.

• *Argument over \mathbf{B}_{dR}^+ .* Recall that we can write $F^i \mathbf{B}_{\text{dR}}^+ \simeq \prod_{j \geq i} t^j C$ as topological K -vector spaces. It follows that, since X_n is defined over K , the distinguished triangle (see [15, Lemma 5.10])

$$\text{R}\Gamma_{\text{dR}}(X_n/\mathbf{B}_{\text{dR}}^+)/F^{r-1} \xrightarrow{t} \text{R}\Gamma_{\text{dR}}(X_n/\mathbf{B}_{\text{dR}}^+)/F^r \xrightarrow{\vartheta} \text{R}\Gamma_{\text{dR}}(X_n)/F^r$$

is split compatibly with n . Hence, together with the above computations over C , we can use it in the ascending induction over r to show that $R^1 \lim_n \text{DR}^{r,i}(X_n/\mathbf{B}_{\text{dR}}^+) = 0$ assuming that we know the base case of $r = 0$. But this case is trivially true. \square

Proposition 7.18. *Assume that X, S and π are defined over K . Then Conjecture 5.21 holds for X .*

Proof. Identical to the proof of Proposition 7.14 with Proposition 7.15 replacing Proposition 7.9. \square

Remark 7.19. Proposition 7.15 and Proposition 7.18 should be true also when the varieties are defined over C : the only argument that needs to be changed is the one over \mathbf{B}_{dR}^+ in the proof of Proposition 7.15.

7.3. De Rham tempered varieties. A general smooth dagger variety X can be written as an increasing union of quasi-compact dagger varieties X_n for which we proved that Conjecture 5.21 holds. Going to the limit in the diagram for X_n involves nasty $\mathbf{R}^1 \lim_n$ problems which leads us to introduce the de Rham temperedness conditions below. We prove that Conjecture 5.21 holds for de Rham tempered varieties. Note that we don't know any example of smooth dagger varieties that are not de Rham tempered, and it is not impossible that they all are.

Definition 7.20. Let X be a smooth dagger variety over C . We say that X is *de Rham tempered* if the naive Hodge filtration on de Rham cohomology does not jump at infinity, i.e., if X can be covered by an increasing sequence $\{X_n\}_{n \in \mathbf{N}}$ of quasi-compact dagger subvarieties such that, for all i, r , the canonical map $F^r H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+) \rightarrow \lim_n F^r H_{\mathrm{dR}}^i(X_n/\mathbf{B}_{\mathrm{dR}}^+)$ is surjective.

Remark 7.21. (i) The natural map $H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+) \rightarrow \lim_n H_{\mathrm{dR}}^i(X_n/\mathbf{B}_{\mathrm{dR}}^+)$ is an isomorphism due to the finiteness of de Rham cohomology groups of the X_n . This implies that, for all $r \geq 0$, $F^r H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+) \rightarrow \lim_n F^r H_{\mathrm{dR}}^i(X_n/\mathbf{B}_{\mathrm{dR}}^+)$ is injective; hence it is an isomorphism if X is de Rham tempered.

7.3.1. *Comparison theorem for de Rham tempered dagger varieties.*

Theorem 7.22. *Conjecture 5.21 holds for de Rham tempered smooth dagger varieties.*

Proof. Let X be a smooth dagger variety over C , and let $\{X_n\}_{n \in \mathbf{N}}$ be a covering by increasing quasi-compact dagger subvarieties. We are going to prove, by induction on i , that the sequence

$$(7.23) \quad 0 \rightarrow H_{\mathrm{proét}}^i(X, \mathbf{Q}_p(r)) \rightarrow X^{r,i} \oplus F^{r,i} \rightarrow H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+) \rightarrow 0,$$

is exact. For that, we will use the fact that the analogous sequence for X_n is exact by Theorem 6.14. By Remark 5.23, this will suffice to prove Conjecture 5.21.

The induction hypothesis implies that the sequence (7.23) is exact on the left, and we are left to verify that it is exact on the right. For this, we will modify this sequence to pass to BC's. $A^{r,i}$ is a torsion $\mathbf{B}_{\mathrm{dR}}^+$ -module; it is also canonically a subgroup of $H_{\mathrm{proét}}^i(X, \mathbf{Q}_p(r))$. Let $\overline{H}^{r,i} := H_{\mathrm{proét}}^i(X, \mathbf{Q}_p(r))/A^{r,i}$. We are reduced to showing the exactness on the right of the sequence

$$0 \rightarrow \overline{H}^{r,i} \rightarrow X^{r,i} \oplus \overline{F}^{r,i} \rightarrow H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+) \rightarrow 0.$$

Note that $\overline{F}^{r,i} \hookrightarrow H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+)$. Now, the image of $\overline{F}^{r,i}$ in $H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+)$ contains $t^r H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+)$. Define $G^{r,i}$ to be the inverse image of $t^r H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+)$ in $\overline{F}^{r,i}$. We have $\overline{F}^{r,i}/G^{r,i} \hookrightarrow H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+)/t^r$ and an exact sequence

$$(7.24) \quad 0 \rightarrow \overline{H}^{r,i} \rightarrow X^{r,i} \oplus \overline{F}^{r,i}/G^{r,i} \rightarrow H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+)/t^r.$$

We want to show that this sequence is exact on the right. Define $\overline{H}_n^{r,i}$, $X_n^{r,i}$ and $G_n^{r,i}$ as the corresponding objects for X_n . The condition that X is de Rham tempered implies that the canonical map $\overline{F}^{r,i}/G^{r,i} \rightarrow \lim_n \overline{F}_n^{r,i}/G_n^{r,i}$ is an isomorphism. Since the maps $X^{r,i} \rightarrow \lim_n X_n^{r,i}$ and $H_{\mathrm{dR}}^i(X) \rightarrow \lim_n H_{\mathrm{dR}}^i(X_n)$ are isomorphisms, we also have the isomorphism $\overline{H}^{r,i} \xrightarrow{\sim} \lim_n \overline{H}_n^{r,i}$. A diagram chase shows that it suffices to prove that $\mathbf{R}^1 \lim_n \overline{H}_n^{r,i} = 0$, but this we already know by Remark 7.7. \square

7.3.2. *Equivalent conditions for de Rham temperedness.* Let X be a smooth dagger variety over C , and let $\{X_n\}_{n \in \mathbf{N}}$ be a covering by increasing quasi-compact dagger subvarieties.

Lemma 7.25. *The following conditions are equivalent:*

- X is de Rham tempered,
- $\overline{F}^{r,i} \rightarrow \lim_n \overline{F}_n^{r,i}$ is surjective,
- $\mathbf{R}^1 \lim_n A_n^{r,i} \rightarrow \mathbf{R}^1 \lim_n F_n^{r,i}$ is injective (which is also equivalent to it being an isomorphism).

Proof. The equivalence of the first two conditions is just the definition. Now, we have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 0 & \rightarrow & \lim_n A_n^{r,i} & \longrightarrow & \lim_n F_n^{r,i} & \longrightarrow & \lim_n \overline{F}_n^{r,i} & \rightarrow & \mathrm{R}^1 \lim_n A_n^{r,i} \rightarrow \mathrm{R}^1 \lim_n F_n^{r,i} \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & A^{r,i} & \longrightarrow & F^{r,i} & \longrightarrow & \overline{F}^{r,i} & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & \mathrm{R}^1 \lim_n F_n^{r,i-1} & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
 \end{array}$$

Note that the $\mathrm{R}^1 \lim_n \overline{F}_n^{r,i}$ that should appear in the first row is 0 due to the fact that the de Rham cohomology of X_n is finite dimensional. The equivalence of the second and the third conditions is obvious from the diagram. \square

Remark 7.26. The above conditions are satisfied if $\mathrm{R}^1 \lim_n A_n^{r,i} = 0$, hence also, thanks to Lemma 7.6, if $\mathrm{R}^1 \lim_n \mathrm{DR}_n^{r,i-1} = 0$.

7.3.3. *Examples of de Rham tempered dagger varieties.* The following smooth dagger varieties are de Rham tempered:

- (1) proper; more generally, quasi-compact;
- (2) analytifications of algebraic varieties;
- (3) Stein varieties;
- (4) products $Y \times S$ where Y is proper and S is Stein;
- (5) smooth proper fibrations over Stein curves.

The point (1) is clear (we can take all X_n 's equal to X).

For (2), if X is a smooth Stein dagger variety over C , then $\overline{F}^{r,i} = t^{r-i} H_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+)$. Hence X is de Rham tempered by Section 7.3.2.

For (3), use the filtered de Rham Künneth formula.

For (4), use the fact that $\mathrm{R}^1 \lim_n A_n^{r,i} = 0$ (by the proof of Proposition 7.15 and Lemma 7.6), and the equivalent conditions in Section 7.3.2.

Remark 7.27. Since, by Theorem 7.22, de Rham tempered implies that Conjecture 5.21 is true, this conjecture holds in all the above cases. However, in all these cases we have earlier used alternative methods to prove the conjecture. On the other hand we don't know any example of varieties that are not de Rham tempered. Hence, we can ask the following question, which is important in view of Theorem 7.22.

Question 7.28. Are all smooth dagger varieties de Rham tempered ?

Remark 7.29. (i) It would already be good to know whether holomorphically convex rigid analytic varieties over C are de Rham tempered.

(ii) Other interesting examples would be complements of compact subsets in proper varieties such as $\mathbf{P}^n \setminus \mathbf{P}^n(\mathbf{Q}_p)$ or, more generally, complements of higher dimensional "compact" subvarieties like the ones that appear in the definition of period domains [16].

8. GENERAL COMPARISON THEOREMS: PRO-ÉTALE-TO-DE RHAM

In this chapter we show that in the case of de Rham tempered dagger varieties we can extract the Hyodo-Kato and de Rham cohomologies from the pro-étale cohomology. If such varieties are defined over K the extraction recovers Frobenius, monodromy, and the naive Hodge filtration.

8.1. The pro-étale-to-de Rham C_{st} -conjecture for varieties over K . In this section, we study the following conjecture extracting, for analytic spaces over K , the Hyodo-Kato and de Rham cohomologies from the pro-étale cohomology. This extends to p -adic analytic spaces the C_{st} -conjecture of Fontaine. (We go back to working with locally convex topological vector spaces.)

Conjecture 8.1. *Let X be a smooth dagger variety over K . We have natural strict isomorphisms:*

$$\begin{aligned} \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p), \mathbf{B}_{\text{st}}) &\simeq H_{\text{HK}}^i(X_C)^*, \quad \text{as a } (\varphi, N, \mathcal{G}_K)\text{-module,} \\ \text{Hom}_{\mathcal{G}_K}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p), \mathbf{B}_{\text{dR}}) &\simeq H_{\text{dR}}^i(X)^*, \quad \text{as a filtered } K\text{-module.} \end{aligned}$$

Remark 8.2. Our approach uses syntomic cohomology, which gives a description of $H_{\text{proét}}^i(X_C, \mathbf{Q}_p(r))$ for $r \geq i$, rather than that of $H_{\text{proét}}^i(X_C, \mathbf{Q}_p)$. Hence we are going to consider the following equivalent form of Conjecture 8.1 (where the $\{r\}$ has the same meaning as in Corollary 4.10):

$$\begin{aligned} \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p(r)), \mathbf{B}_{\text{st}}) &\simeq H_{\text{HK}}^i(X_C)^* \{r\}, \quad \text{as a } (\varphi, N, \mathcal{G}_K)\text{-module,} \\ \text{Hom}_{\mathcal{G}_K}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p(r)), \mathbf{B}_{\text{dR}}) &\simeq H_{\text{dR}}^i(X)^* \{r\}, \quad \text{as a filtered } K\text{-module.} \end{aligned}$$

8.2. Some results towards the conjecture. We are going to prove Conjecture 8.1 (in the equivalent form from Remark 8.2) in the following cases:

Theorem 8.3. *Conjecture 8.1 holds for:*

- (a) *affinoids,*
- (b) *Stein varieties,*
- (c) *quasi-compact varieties,*
- (d) *de Rham tempered varieties.*

Of course, the last case contains all the other cases, but the arguments in the special cases are somewhat simpler.

8.2.1. Dagger affinoids. We note that it suffices to show that we have natural isomorphisms since the weak topology on the Hom-spaces is Hausdorff. Let

$$M := H_{\text{HK}}^i(X_C), \quad X_{\text{st}}^i(M) := (M \widehat{\otimes}_{F^{\text{nr}}}^R \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^i}, \quad M_K = (\overline{K} \otimes_{F^{\text{nr}}} M)^{\mathcal{G}_K} \simeq H_{\text{dR}}^i(X).$$

The last isomorphism follows from the Hyodo-Kato isomorphism [15, Th. 4.27]. Recall that we have the exact sequence (see Theorem 5.14)

$$(8.4) \quad 0 \rightarrow (\Omega_X^{i-1} / \text{Ker } d) \widehat{\otimes}_K C \rightarrow H_{\text{proét}}^i(X_C, \mathbf{Q}_p(i)) \rightarrow X_{\text{st}}^i(M) \rightarrow 0.$$

Applying $\text{Hom}_{\mathcal{G}_K}(-, \mathbf{B}_{\text{st}})$ and $\text{Hom}_{\mathcal{G}_K}(-, \mathbf{B}_{\text{dR}})$ to it, we get the following exact sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(X_{\text{st}}^i(M), \mathbf{B}_{\text{st}}) &\rightarrow \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p(i)), \mathbf{B}_{\text{st}}) \rightarrow \text{Hom}_{\mathcal{G}_K}^{\text{sm}}((\Omega_X^{i-1} / \text{Ker } d) \widehat{\otimes}_K C, \mathbf{B}_{\text{st}}), \\ 0 \rightarrow \text{Hom}_{\mathcal{G}_K}(X_{\text{st}}^i(M), \mathbf{B}_{\text{dR}}) &\rightarrow \text{Hom}_{\mathcal{G}_K}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p(i)), \mathbf{B}_{\text{dR}}) \rightarrow \text{Hom}_{\mathcal{G}_K}((\Omega_X^{i-1} / \text{Ker } d) \widehat{\otimes}_K C, \mathbf{B}_{\text{dR}}). \end{aligned}$$

Lemma 8.5. *We have*

$$\text{Hom}_{\mathcal{G}_K}^{\text{sm}}((\Omega_X^{i-1} / \text{Ker } d) \widehat{\otimes}_K C, \mathbf{B}_{\text{st}}) = 0, \quad \text{Hom}_{\mathcal{G}_K}((\Omega_X^{i-1} / \text{Ker } d) \widehat{\otimes}_K C, \mathbf{B}_{\text{dR}}) = 0.$$

Proof. Since $\text{Hom}_{\mathcal{G}_K}^{\text{sm}}((\Omega_X^{i-1} / \text{Ker } d) \widehat{\otimes}_K C, \mathbf{B}_{\text{st}}) \simeq \text{colim}_{L/K} \text{Hom}_{\mathcal{G}_L}((\Omega_X^{i-1} / \text{Ker } d) \widehat{\otimes}_K C, \mathbf{B}_{\text{st}})$ and $\mathbf{B}_{\text{st}} \hookrightarrow \mathbf{B}_{\text{dR}}$, it suffices to show that $\text{Hom}_{\mathcal{G}_L}((\Omega_X^{i-1} / \text{Ker } d) \widehat{\otimes}_K C, \mathbf{B}_{\text{dR}}) = 0$, for a finite extension L of K .

But $\text{Hom}_{\mathcal{G}_L}(C, \mathbf{B}_{\text{dR}}) = 0$ (Proposition 2.14 (iv)); hence $\text{Hom}_{\mathcal{G}_L}((\Omega_X^{i-1} / \text{Ker } d) \otimes_K C, \mathbf{B}_{\text{dR}}) = 0$. Since our maps are requested to be continuous and $(\Omega_X^{i-1} / \text{Ker } d) \otimes_K C$ is dense in $(\Omega_X^{i-1} / \text{Ker } d) \widehat{\otimes}_K C$, we get the wanted vanishing. \square

This lemma yields isomorphisms

$$\begin{aligned} \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(X_{\text{st}}^i(M), \mathbf{B}_{\text{st}}) &\xrightarrow{\sim} \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p(i)), \mathbf{B}_{\text{st}}), \\ \text{Hom}_{\mathcal{G}_K}(X_{\text{st}}^i(M), \mathbf{B}_{\text{dR}}) &\xrightarrow{\sim} \text{Hom}_{\mathcal{G}_K}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p(i)), \mathbf{B}_{\text{dR}}). \end{aligned}$$

The filtration on M_K is concentrated in degree i (i.e., $\text{Fil}^i M_K = M_K$ and $\text{Fil}^{i+1} M_K = 0$). Hence we can use Example 4.11 of Corollary 4.10 to finish the proof of Conjecture 8.1 in the case of dagger affinoids.

8.2.2. *Stein varieties.* Let now X be a smooth dagger Stein variety over K . Let

$$M := H_{\text{HK}}^i(X_C), \quad X_{\text{st}}^i(M) := (M \widehat{\otimes}_{F^{\text{nr}}}^R \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^i}.$$

Recall that we have the strictly exact sequence (see Theorem 5.14)

$$(8.6) \quad 0 \rightarrow (\Omega_X^{i-1} / \text{Ker } d) \widehat{\otimes}_K C \rightarrow H_{\text{proét}}^i(X_C, \mathbf{Q}_p(r)) \rightarrow X_{\text{st}}^i(M) \rightarrow 0.$$

Applying $\text{Hom}_{\mathcal{G}_K}(-, \mathbf{B}_{\text{st}})$ and $\text{Hom}_{\mathcal{G}_K}(-, \mathbf{B}_{\text{dR}})$ to it and using Proposition 2.14 as in Lemma 8.5, we get strict isomorphisms

$$\begin{aligned} \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(X_{\text{st}}^i(M), \mathbf{B}_{\text{st}}) &\xrightarrow{\sim} \text{Hom}_{\mathcal{G}_K}^{\text{sm}}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p(i)), \mathbf{B}_{\text{st}}), \\ \text{Hom}_{\mathcal{G}_K}(X_{\text{st}}^i(M), \mathbf{B}_{\text{dR}}) &\xrightarrow{\sim} \text{Hom}_{\mathcal{G}_K}(H_{\text{proét}}^i(X_C, \mathbf{Q}_p(i)), \mathbf{B}_{\text{dR}}). \end{aligned}$$

The Hodge filtration on $M_K = H_{\text{dR}}^i(X)$ is again concentrated in degree i . Now, choose an admissible exhaustive covering of X by dagger affinoids U_n . We have strict filtered (since all the filtrations are concentrated in degree i) isomorphisms

$$H_{\text{HK}}^i(X_C) \xrightarrow{\sim} \lim_n M_n, \quad X_{\text{st}}^i(M) \xrightarrow{\sim} \lim_n X_{\text{st}}^i(M_n), \quad H_{\text{dR}}^i(X) \xrightarrow{\sim} \lim_n H_{\text{dR}}^i(U_n) = \lim_{n,K} M_{n,K},$$

where we set

$$M_n := H_{\text{HK}}^i(U_{n,C}), \quad X_{\text{st}}^i(M_n) := (M_n \widehat{\otimes}_{F^{\text{nr}}}^R \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^i}, \quad M_{n,K} := (M_n \otimes_{F^{\text{nr}}} \overline{K})^{\mathcal{G}_K} \simeq H_{\text{dR}}^i(U_n).$$

In other words, M endowed with the Hodge filtration is the projective limit of the M_n 's. Hence we can use Proposition 4.13 to finish the proof of Conjecture 8.1 in the case of Stein varieties.

8.2.3. *Quasi-compact dagger varieties.* Let X be a quasi-compact dagger variety over K . Fix $r \geq i$. Set:

$$\begin{aligned} \widetilde{H}^{r,i} &:= \widetilde{H}_{\text{proét}}^i(X_C, \mathbf{Q}_p(r)), \quad \widetilde{F}^{r,i} := \widetilde{H}^i(F^r(\text{R}\Gamma_{\text{dR}}(X) \widehat{\otimes}_K \mathbf{B}_{\text{dR}}^+)), \\ X^{r,i} &:= (H_{\text{HK}}^i(X_C) \widehat{\otimes}_{F^{\text{nr}}} \mathbf{B}_{\text{st}}^+)^{N=0, \varphi=p^r}, \quad B^i := H_{\text{dR}}^i(X) \widehat{\otimes}_K \mathbf{B}_{\text{dR}}^+. \end{aligned}$$

Let $\widetilde{A}^{r,i}$ be the kernel of the canonical map $\widetilde{F}^{r,i} \rightarrow B^i$. Then $\widetilde{A}^{r,i}$ is also canonically a subgroup of $\widetilde{H}^{r,i}$. Let

$$\overline{H}^{r,i} := \widetilde{H}^{r,i} / \widetilde{A}^{r,i}, \quad \overline{F}^{r,i} := \widetilde{F}^{r,i} / \widetilde{A}^{r,i}.$$

Note that $\overline{F}^{r,i} / \overline{A}^{r,i}$ is a subgroup of B^i , hence it is classical.

Lemma 8.7. *We have, for all $i \leq r$,*

$$\text{Hom}_{\mathcal{G}_K}^{\text{sm}}(\widetilde{A}^{r,i}, \mathbf{B}_{\text{st}}) = 0, \quad \text{Hom}_{\mathcal{G}_K}(\widetilde{A}^{r,i}, \mathbf{B}_{\text{dR}}) = 0.$$

Proof. It suffices to show that

$$\text{Hom}_{\mathcal{G}_K}^{\text{sm}}(A^{r,i}, \mathbf{B}_{\text{st}}) = 0, \quad \text{Hom}_{\mathcal{G}_K}(A^{r,i}, \mathbf{B}_{\text{dR}}) = 0.$$

And, for that, it is enough to prove the second statement. We are going to use the fact that $A^{r,i}$ is a quotient of $\text{DR}^{r,i-1}$ (see the proof of Lemma 7.6) and prove the same statement for $\text{DR}^{r,i}$, with $i < r$.

Now $\text{DR}^{r,i}$ is the i -th hypercohomology group of the complex

$$\text{DR}^{r,\bullet} := ((\mathbf{B}_{\text{dR}}^+ / t^r) \widehat{\otimes}_K \mathcal{O} \rightarrow (\mathbf{B}_{\text{dR}}^+ / t^{r-1}) \widehat{\otimes}_K \Omega^1 \rightarrow \cdots \rightarrow (\mathbf{B}_{\text{dR}}^+ / t) \widehat{\otimes}_K \Omega^{r-1}).$$

Choose a covering of X by dagger affinoids, and denote by $Z^{r,i}$ the group of i -cocycles of the Čech double complex associated to this covering. Since $\mathrm{DR}^{r,i}$ is a quotient of $Z^{r,i}$, it is enough to prove that $\mathrm{Hom}_{\mathcal{G}_K}(Z^{r,i}, \mathbf{B}_{\mathrm{dR}}) = 0$.

Denote by $Z_K^{j,i}$ the group of i -cocycles of the Čech double complex associated to the above covering and the complex

$$\mathrm{DR}_K^{j,\bullet} := (\mathcal{O} \rightarrow \Omega^1 \rightarrow \dots \rightarrow \Omega^{j-1}).$$

We are going to prove that $\sum_{j \leq r} (t^{r-j} \mathbf{B}_{\mathrm{dR}}^+ / t^r \mathbf{B}_{\mathrm{dR}}^+) \otimes_K Z_K^{j,i} \rightarrow Z^{r,i}$ has dense image. This will allow us to conclude since $\mathrm{Hom}_{\mathcal{G}_K}((t^{r-j} \mathbf{B}_{\mathrm{dR}}^+ / t^r \mathbf{B}_{\mathrm{dR}}^+) \otimes_K Z_K^{j,i}, \mathbf{B}_{\mathrm{dR}}) = 0$ by Proposition 2.14 and our maps are assumed to be continuous.

To prove this density, choose a Banach basis over K of the K -Banach $\mathbf{B}_{\mathrm{dR}}^+ / t^r$ of the form $(t^j e_n)_{0 \leq j \leq r-1, n \in \mathbf{N}}$ (pick a family e_n of elements of $\mathbf{B}_{\mathrm{dR}}^+ / t^r$ whose images in $\mathbf{B}_{\mathrm{dR}}^+ / t = C$ form a Banach basis of C over K). Then one can use $(t^j e_n)_{0 \leq j \leq k-1, n \in \mathbf{N}}$ as a Banach basis of $\mathbf{B}_{\mathrm{dR}}^+ / t^k$, if $k \leq r$. This makes it possible to decompose $\mathrm{DR}^{r,\bullet}$ as a completed direct sum of the complexes $t^j e_n \otimes \mathrm{DR}_K^{r-j,\bullet}$'s, and then $Z^{r,i}$ is the completion of the sum of the $t^j e_n \otimes Z_K^{r-j,i}$'s. \square

It follows from Lemma 8.7 that we have isomorphisms

$$\mathrm{Hom}_{\mathcal{G}_K}^{\mathrm{sm}}(\overline{H}^{r,i}, \mathbf{B}_{\mathrm{st}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{G}_K}^{\mathrm{sm}}(\widetilde{H}^{r,i}, \mathbf{B}_{\mathrm{st}}), \quad \mathrm{Hom}_{\mathcal{G}_K}(\overline{H}^{r,i}, \mathbf{B}_{\mathrm{dR}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{G}_K}(\widetilde{H}^{r,i}, \mathbf{B}_{\mathrm{dR}})$$

Since X is quasi-compact, Conjecture 5.21 holds and we have the exact sequence (5.22):

$$0 \rightarrow \widetilde{H}^{r,i} \rightarrow X^{r,i} \oplus \widetilde{F}^{r,i} \rightarrow B^i \rightarrow 0.$$

This induces exact sequences

$$(8.8) \quad 0 \rightarrow \overline{H}^{r,i} \rightarrow X^{r,i} \oplus \overline{F}^{r,i} \rightarrow B^i \rightarrow 0, \quad 0 \rightarrow \overline{H}^{r,i} \rightarrow X^{r,i} \rightarrow B^i / \overline{F}^{r,i} \rightarrow 0,$$

Since $\overline{F}^{r,i}$ is a subgroup of B^i , all the terms in these sequences are classical. This identifies topologically $\overline{H}^{r,i}$ with $V_{\mathrm{st}}^r(H_{\mathrm{HK}}^i(X_C), H_{\mathrm{dR}}^i(X))$. Hence we can use Corollary 4.10 to finish the proof of Conjecture 8.1 in the case of quasi-compact dagger varieties.

8.2.4. De Rham tempered varieties. Let X be a de Rham tempered dagger variety over K . Write X as an increasing union of quasi-compact dagger subvarieties X_n . Define $\overline{H}^{r,i}$, $X^{r,i}$ and $\overline{F}^{r,i}$ as in the quasi-compact case, and $\overline{H}_n^{r,i}$, $X_n^{r,i}$ and $\overline{F}_n^{r,i}$ as the corresponding objects for X_n . The condition that X is de Rham tempered implies that the canonical map $\overline{F}^{r,i} \rightarrow \lim_n \overline{F}_n^{r,i}$ is an isomorphism. It follows that $(H_{\mathrm{HK}}^i(X_C), H_{\mathrm{dR}}^i(X))$ is the projective limit of the $(H_{\mathrm{HK}}^i(X_{n,C}), H_{\mathrm{dR}}^i(X_n))$'s as a filtered $(\varphi, N, \mathcal{G}_K)$ -module.

On the other hand, we have $\mathrm{Hom}_{\mathcal{G}_K}(\overline{H}^{r,i}, \mathbf{B}_?) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{G}_K}(H^{r,i}, \mathbf{B}_?)$ for the same reasons as in the quasi-compact case, and an identification $\overline{H}^{r,i} \simeq V_{\mathrm{st}}^r(H_{\mathrm{HK}}^i(X_C), H_{\mathrm{dR}}^i(X))$ as above. Hence we can use Proposition 4.13 to finish the proof of Conjecture 8.1 in the case of de Rham tempered varieties.

8.3. The étale to de Rham C_{st} -conjecture for varieties over C . The following theorem shows that one can recover de Rham cohomology (without the Hodge filtration) and Hyodo-Kato cohomology (without actions of φ and N) from pro-étale cohomology for varieties over C , despite the absence of Galois action.

Theorem 8.9. *Let X be a de Rham tempered dagger variety over C . Let $i \geq 0$.*

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbb{H}_{\mathrm{pro\acute{e}t}}^i(X, \mathbf{Q}_p), \mathbb{B}_{\mathrm{st}}) \simeq \mathrm{Hom}_{F^{\mathrm{nr}}}(\mathbb{H}_{\mathrm{HK}}^i(X), \mathbf{B}_{\mathrm{st}}), \quad \text{as a } \mathbf{B}_{\mathrm{st}}\text{-module,}$$

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbb{H}_{\mathrm{pro\acute{e}t}}^i(X, \mathbf{Q}_p), \mathbb{B}_{\mathrm{dR}}) \simeq \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}}^+}(\mathbb{H}_{\mathrm{dR}}^i(X/\mathbf{B}_{\mathrm{dR}}^+), \mathbf{B}_{\mathrm{dR}}), \quad \text{as a } \mathbf{B}_{\mathrm{dR}}\text{-module.}$$

Proof. If M is a VS, we set

$$h(M) := \text{Hom}_{\text{VS}}(M, \mathbb{B}_{\text{dR}}).$$

Fix $r \geq i$. Set:

$$\begin{aligned} \mathbb{H}^{r,i} &:= \mathbb{H}_{\text{proét}}^i(X_C, \mathbf{Q}_p(r)), & \mathbb{F}^{r,i} &:= H^i(F^r(\text{R}\Gamma_{\text{dR}}(X/\mathbf{B}_{\text{dR}}^+) \widehat{\otimes}_{\mathbf{B}_{\text{dR}}^+} \mathbb{B}_{\text{dR}}^+)), \\ \mathbb{X}^{r,i} &:= (H_{\text{HK}}^i(X_C) \widehat{\otimes}_{F^{\text{nr}} \mathbb{B}_{\text{st}}^+})^{N=0, \varphi=p^r}, & \mathbb{B}^i &:= H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+) \widehat{\otimes}_{\mathbf{B}_{\text{dR}}^+} \mathbb{B}_{\text{dR}}^+. \end{aligned}$$

Let $\mathbb{A}^{r,i}$ be the kernel of the canonical map $\mathbb{F}^{r,i} \rightarrow \mathbb{B}^i$. Then $\mathbb{A}^{r,i}$ is a torsion \mathbb{B}_{dR}^+ -module; it is also canonically a subgroup of $\mathbb{H}^{r,i}$. Let

$$\overline{\mathbb{H}}^{r,i} := \mathbb{H}^{r,i} / \mathbb{A}^{r,i}, \quad \overline{\mathbb{F}}^{r,i} := \mathbb{F}^{r,i} / \mathbb{A}^{r,i}.$$

Since $\mathbb{A}^{r,i}$ is a torsion \mathbb{B}_{dR}^+ -module, $h(\mathbb{A}^{r,i}) = 0$ by Corollary 3.16; hence we have an isomorphism

$$h(\overline{\mathbb{H}}^{r,i}) \xrightarrow{\sim} h(\mathbb{H}^{r,i}).$$

We also have natural sequences

$$\begin{aligned} 0 \rightarrow \mathbb{H}^{r,i} \rightarrow \mathbb{X}^{r,i} \oplus \mathbb{F}^{r,i} \rightarrow \mathbb{B}^i \rightarrow 0, & \quad 0 \rightarrow \mathbb{H}^{r,i} \rightarrow \mathbb{X}^{r,i} \oplus \mathbb{F}^{r,i} \rightarrow \mathbb{B}^i \rightarrow 0 \\ 0 \rightarrow \overline{\mathbb{H}}^{r,i} \rightarrow \mathbb{X}^{r,i} \rightarrow \mathbb{B}^i / \overline{\mathbb{F}}^{r,i} \rightarrow 0 \end{aligned}$$

- If X is quasi-compact, the last sequence is a sequence of BC's. It is exact (see Remark 3.33) because passing to C -points yields the sequence $0 \rightarrow \overline{H}^{r,i} \rightarrow X^{r,i} \rightarrow B^i / \overline{F}^{r,i} \rightarrow 0$ which was proven to be exact, as a consequence of the validity of Conjecture 5.21 (see (8.8)). This identifies $\overline{\mathbb{H}}^{r,i}$ with $\mathbb{V}_{\text{st}}^r(H_{\text{HK}}^i(X), t^{-r} \overline{F}^{r,i})$, which makes it possible to use Proposition 4.25 to prove Theorem 8.9 in the quasi-compact case.

- The general de Rham tempered case reduces to the quasi-compact case exactly as for varieties over K (see Section 8.2.4); one just uses Proposition 4.29 instead of Proposition 4.13 to finish the proof. \square

Remark 8.10. (i) If X is proper, then $\mathbb{H}^i(X, \mathbf{Q}_p)$ is a finite dimensional \mathbf{Q}_p -vector space with no extra structure. This shows that it is hopeless to try to recover the actions of φ and N on $H_{\text{HK}}^i(X)$ or the filtration on $H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+)$ using Theorem 8.9.

(ii) On the other hand, if X is a dagger affinoid, we know what the filtration on $H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+)$ is: we have $F^{i+k} H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+) = t^k H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+)$. Also, as we have seen¹⁷

$$(8.11) \quad (H_{\text{HK}}^i(X) \otimes \mathbb{B}_{\text{st}}^+)^{N=0, \varphi=p^i} \simeq (H_{\text{HK}}^i(X) \otimes \mathbb{B}_{\text{cris}}^+)^{\varphi=p^i} = (H_{\text{HK}}^i(X) \otimes \mathbb{B}_{\text{max}}^+)^{\varphi=p^i}$$

Using Lemma 4.20, one sees that

$$\begin{aligned} H_{\text{HK}}^i(X)^* \otimes \mathbb{B}_{\text{max}}^+ &\subset \text{Hom}_{\text{VS}}((H_{\text{HK}}^i(X) \otimes \mathbb{B}_{\text{max}}^+)^{\varphi=p^i}, \mathbb{B}_{\text{max}}^+) \\ &\subset \{\lambda \in H_{\text{HK}}^i(X)^* \otimes \mathbb{B}_{\text{max}}^+, \varphi^n(\lambda) \in H_{\text{HK}}^i(X)^* \otimes \mathbb{B}_{\text{dR}}^+, \forall n \geq 0\} \\ &= H_{\text{HK}}^i(X)^* \otimes \mathbb{B}_{\text{max}}^+ \end{aligned}$$

Since

$$\text{Hom}_{\text{VS}}(\mathbb{H}^i(X, \mathbf{Q}_p(i)), \mathbb{B}_{\text{max}}^+) \cong \text{Hom}_{\text{VS}}((H_{\text{HK}}^i(X) \otimes \mathbb{B}_{\text{max}}^+)^{\varphi=p^i}, \mathbb{B}_{\text{max}}^+)$$

by the proof of Theorem 8.9, one sees that one can recover the action of φ on $H_{\text{HK}}^i(X) \otimes \check{C}$ (by tensoring with \check{C} above $\mathbb{B}_{\text{max}}^+$ and dualizing).

(iii) If X is a general smooth dagger variety over C , point (ii) suggests that one can recover a sheafified version of the actions of φ on $H_{\text{HK}}^i(X)$ and of the filtration on $H_{\text{dR}}^i(X/\mathbf{B}_{\text{dR}}^+)$. Recovering the action of N seems out of reach by these methods.

¹⁷ $\mathbb{B}_{\text{max}} = \mathbf{A}_{\text{max}}[\frac{1}{p}]$, where \mathbf{A}_{max} is the p -adic completion of $\mathbf{A}_{\text{inf}}[\frac{[p]}{p}]$. It is a much nicer ring than $\mathbb{B}_{\text{cris}}^+$: for example, $\alpha = \frac{[\varepsilon]-1}{t} \in \mathbb{B}_{\text{cris}}^+$, $\varphi^n(\alpha) \in \mathbb{B}_{\text{dR}}^+$ for all $n \geq 0$, but $\alpha \notin \mathbb{B}_{\text{cris}}^+$, whereas if $x \in \mathbb{B}_{\text{max}}$ and $\varphi^n(x) \in \mathbb{B}_{\text{dR}}^+$ for all $n \geq 0$, then $x \in \mathbb{B}_{\text{max}}^+$. But $\varphi(\mathbb{B}_{\text{max}}^+) \subset \mathbb{B}_{\text{cris}}^+ \subset \mathbb{B}_{\text{max}}^+$, hence which explains the second equality in (8.11).

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