Cohomology of p-adic Stein spaces

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(joint work with Pierre Colmez, Gabriel Dospinescu)

Let p be a prime. Let $\mathscr{O}_K$ be a complete discrete valuation ring of mixed characteristic $(0,p)$ with perfect residue field k and fraction field K. Let F be the fraction field of the ring of Witt vectors $\mathscr{O}_F = W(k)$ of k. Let $\overline{K}$ be an algebraic closure of K and let $C = \widehat{\overline{K}}$ be its p-adic completion; let $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$.

0.1. **The p-adic étale cohomology of Drinfeld half-space.** We reported on some results of our research project that aims at understanding the p-adic (pro-)étale cohomology of p-adic symmetric spaces [2]. The main question of interest being: does this cohomology realize the hoped for p-adic local Langlands correspondence in analogy with the ℓ-adic situation ? When we started this project we did not know what to expect and local computations were rather discouraging: geometric p-adic étale cohomology groups of affinoids and their interiors are huge and not invariant by base change to a bigger complete algebraically closed field. However there was one computation done long ago by Drinfeld that stood out. Let us recall it.

Assume that $[K:\mathbf{Q}_p] < \infty$ and let $\mathbb{H}_K = \mathbb{P}^1_K \setminus \mathbb{P}^1(K)$ be the Drinfeld half-plane, thought of as a rigid analytic space. It admits a natural action of $G := \mathrm{GL}_2(K)$.

Fact 1. (Drinfeld) *If ℓ is a prime number* (including $\ell = p$!), *there exists a natural isomorphism of $G \times \mathscr{G}_K$-representations*

$$H^1_{\text{ét}}(\mathbb{H}_C, \mathbf{Q}_\ell(1)) \simeq (\mathrm{Sp}^{\mathrm{cont}}(\mathbf{Q}_\ell))^*,$$

where $\mathrm{Sp}^{\mathrm{cont}}(\mathbf{Q}_\ell) := \mathscr{C}(\mathbb{P}^1(K), \mathbf{Q}_\ell)/\mathbf{Q}_\ell$ is the continuous Steinberg representation of G with coefficients in $\mathbf{Q}_\ell$ equipped with a trivial action of $\mathscr{G}_K$ and $(-)^*$ denotes the weak topological dual.

The proof is very simple: it uses Kummer theory and vanishing of the Picard groups (of the standard Stein covering of $\mathbb{H}_K$) [1, 1.4]. This result was encouraging because it showed that the p-adic étale cohomology was maybe not as pathological as one could fear.

Drinfeld's result was generalized, for $\ell \neq p$, to higher dimensions by Schneider-Stuhler [4]. Let $d \geq 1$ and let $\mathbb{H}^d_K$ be the Drinfeld half-space of dimension d, i.e.,

$$\mathbb{H}^d_K := \mathbb{P}^d_K \setminus \bigcup_{H \in \mathscr{H}} H,$$

where $\mathscr{H}$ denotes the set of K-rational hyperplanes. We set $G := \mathrm{GL}_{d+1}(K)$. If $1 \leq r \leq d$, and if ℓ is a prime number, denote by $\mathrm{Sp}_r(\mathbf{Q}_\ell)$ and $\mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Q}_\ell)$ the generalized locally constant and continuous Steinberg $\mathbf{Q}_\ell$-representations of G, respectively, equipped with a trivial action of $\mathscr{G}_K$.

Theorem 2. (Schneider-Stuhler) *Let $r \geq 0$ and let $\ell \neq p$. There are natural $G \times \mathscr{G}_K$-equivariant isomorphisms*

$$H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Q}_\ell(r)) \simeq \mathrm{Sp}^{\mathrm{cont}}_r(\mathbf{Q}_\ell)^*, \quad H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbf{Q}_\ell(r)) \simeq \mathrm{Sp}_r(\mathbf{Q}_\ell)^*.$$

The computations of Schneider-Stuhler work for any cohomology theory that satisfies certain axioms, the most important being the homotopy property with respect to the open unit ball, which fails rather dramatically for the p-adic (pro-)étale cohomology since the p-adic étale cohomology of the unit ball is huge. Nevertheless, we prove the following result.

Theorem 3. *Let $r \geq 0$.*

(1) *There is a natural isomorphism of $G \times \mathscr{G}_K$-locally convex topological vector spaces (over $\mathbf{Q}_p$).*

$$H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Q}_p(r)) \simeq \mathrm{Sp}^{\mathrm{cont}}_r(\mathbf{Q}_p)^*.$$

These spaces are weak duals of Banach spaces.

(2) *There is a strictly exact sequence of $G \times \mathscr{G}_K$-Fréchet spaces*

$$0 \longrightarrow (\Omega^{r-1}(\mathbb{H}^d_K)/\ker d)\widehat{\otimes}_K C \longrightarrow H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbf{Q}_p(r)) \longrightarrow \mathrm{Sp}_r(\mathbf{Q}_p)^* \longrightarrow 0.$$

(3) *The natural map $H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Q}_p(r)) \to H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbf{Q}_p(r))$ identifies étale cohomology with the space of G-bounded vectors*[1] *in the pro-étale cohomology.*

Hence, the p-adic étale cohomology is given by the same dual of a Steinberg representation as its ℓ-adic counterpart and is invariant by scalar extension to bigger C's. However, the p-adic pro-étale cohomology is a nontrivial extension of the same dual of a Steinberg representation that describes its ℓ-adic counterpart by a huge space that depends very much on C.

Remark 4. In [1] we have generalized the above computation of Drinfeld in a different direction, namely, to the Drinfeld tower in dimension 1. We have shown that, if $K = \mathbf{Q}_p$, the p-adic local Langlands correspondence for de Rham Galois representations of dimension 2 (of Hodge-Tate weights 0 and 1 and not trianguline) can be realized inside the p-adic étale cohomology of the Drinfeld tower (see [1, Theorem 0.2] for a precise statement). The two important cohomological inputs were

(1) a p-adic comparison theorem that allows us to recover the p-adic pro-étale cohomology from the de Rham complex and the Hyodo-Kato cohomology; the latter being compared to the ℓ-adic étale cohomology computed, in turn, by non-abelian Lubin-Tate theory,
(2) the fact that the p-adic étale cohomology is equal to the space of G-bounded vectors in the p-adic pro-étale cohomology.

[1] Recall that a subset X of a locally convex vector space over $\mathbf{Q}_p$ is called *bounded* if $p^n x_n \mapsto 0$ for all sequences $\{x_n\}, n \in \mathbf{N}$, of elements of X. In the above, x is called a *G-bounded vector* if its G-orbit is a bounded set.

In contrast, here we obtain the third part of Theorem 3 only after proving the two previous parts. In fact, for a general rigid analytic variety, the natural map from p-adic étale cohomology to p-adic pro-étale cohomology is not an injection.

0.2. **A comparison theorem for p-adic pro-étale cohomology.** The proof of Theorem 3 uses the result below, which is our main theorem. It generalizes the classical p-adic comparison theorem to rigid analytic Stein spaces[2] over K with a semistable reduction. Let the field K be as at the beginning of the introduction.

Theorem 5. *Let $r \geq 0$. Let X be a semistable Stein weak formal scheme over $\mathscr{O}_K$. There exists a commutative $\mathscr{G}_K$-equivariant diagram of Fréchet spaces*[3]

$$\begin{array}{ccc} H^r_{\text{proét}}(X_C, \mathbf{Q}_p(r)) & \longrightarrow & (H^r_{\text{HK}}(X_k)\widehat{\otimes}_F \mathbf{B}^+_{\text{st}})^{N=0,\varphi=p^r} \\ \downarrow \widetilde{\beta} & & \downarrow \iota_{\text{HK}}\otimes\theta \\ \Omega^r(X_K)_{d=0}\widehat{\otimes}_K C & \xrightarrow{\text{can}} & H^r_{\text{dR}}(X_C) \end{array}$$

The horizontal maps are strictly surjective and their kernels are isomorphic to $(\Omega^{r-1}(X_K)/\ker d)\widehat{\otimes}_K C$. The maps $\widetilde{\beta}$ and $\iota_{\text{HK}} \otimes \theta$ are strict (and have closed images). Moreover,

$$\ker(\widetilde{\beta}) \simeq \ker(\iota_{\text{HK}} \otimes \theta) \simeq (H^r_{\text{HK}}(X_k)\widehat{\otimes}_F \mathbf{B}^+_{\text{st}})^{N=0,\varphi=p^{r-1}}.$$

Here $H^r_{\text{HK}}(X_k)$ is the overconvergent Hyodo-Kato cohomology of Grosse-Klönne [3], $\iota_{\text{HK}} : H^r_{\text{HK}}(X_k) \otimes_F K \xrightarrow{\sim} H^r_{\text{dR}}(X_C)$ is the Hyodo-Kato isomorphism, $\mathbf{B}^+_{\text{st}}$ is the semistable ring of periods defined by Fontaine, and $\theta : \mathbf{B}^+_{\text{st}} \to C$ is Fontaine's projection.

Example 6. In the case the Hyodo-Kato cohomology vanish we obtain a particularly simple formula. Take, for example, the rigid affine space $\mathbb{A}^d_K$. For $r \geq 1$, we have $H^r_{\text{dR}}(\mathbb{A}^d_K) = 0$ and, by the Hyodo-Kato isomorphism, also $H^r_{\text{HK}}(\mathbb{A}^d_K) = 0$. Hence the above theorem yields an isomorphism

$$H^r_{\text{proét}}(\mathbb{A}^d_C, \mathbf{Q}_p(r)) \xleftarrow{\sim} (\Omega^{r-1}(\mathbb{A}^d_K)/\ker d)\widehat{\otimes}_K C.$$

Remark 7. (i) We think of the above theorem as a one-way comparison theorem, i.e., the pro-étale cohomology $H^r_{\text{proét}}(X_C, \mathbf{Q}_p(r))$ is the pullback of the diagram

$$(H^r_{\text{HK}}(X_k)\widehat{\otimes}_F \mathbf{B}^+_{\text{st}})^{N=0,\varphi=p^r} \xrightarrow{\iota_{\text{HK}}\otimes\theta} H^r_{\text{dR}}(X_K)\widehat{\otimes}_K C \xleftarrow{\text{can}} \Omega^r(X_K)_{d=0}\widehat{\otimes}_K C$$

built from the Hyodo-Kato cohomology and a piece of the de Rham complex.

(ii) When we started doing computations of pro-étale cohomology groups (for the affine line), we could not understand why the p-adic pro-étale cohomology seemed to be so big while the Hyodo-Kato cohomology was so small (actually 0 in

[2]Recall that a rigid analytic space Y is Stein if it has an admissible affinoid covering $Y = \cup_{i\in\mathbf{N}} U_i$ such that $U_i \Subset U_{i+1}$. The key property we need is the acyclicity of cohomology of coherent sheaves.

[3]The completed tensor product is taken with respect to a Stein covering of X_K.

that case): this was against what the proper case was teaching us. If X is proper, $\Omega^{r-1}(X_K)/\ker d = 0$ and the upper line of the above diagram becomes

$$0 \to H^r_{\text{proét}}(X_C, \mathbf{Q}_p(r)) \to (H^r_{\text{HK}}(X_k)\widehat{\otimes}_F \mathbf{B}^+_{\text{st}})^{N=0,\varphi=p^r} \to (H^r_{\text{dR}}(X_K)\widehat{\otimes}\mathbf{B}^+_{\text{dR}})/\text{Fil}^r \to 0.$$

Hence the huge term on the left disappears, and an extra term on the right shows up. This seemed to indicate that there was no real hope of computing p-adic étale and pro-étale cohomologies of big spaces. It was learning about Drinfeld's result that convinced us to look further.

References

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