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# Algebraische Zahlentheorie

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## Abstracts

### Duality for $p$-adic pro-étale cohomology of analytic varieties

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(joint work with Pierre Colmez, Sally Gilles)

Let $p$ be a prime. Let $K$ be a finite extension of $\mathbf{Q}_p$. Let $\overline{K}$ be an algebraic closure of $K$ and let $C = \widehat{\overline{K}}$ be its $p$-adic completion; let $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$. Or analytic varieties are separated.

#### 1. Arithmetic duality

We have just finished writing a proof of the following result.

**Theorem 1.** (Poincaré duality for curves) *Let $X$ be a smooth, geometrically irreducible, dagger variety of dimension 1 over $K$. Then:*

(1) *There exists a natural trace map of solid $\mathbf{Q}_p$-vector spaces*

$$\mathrm{Tr}_X : H^4_{\text{proét},c}(X, \mathbf{Q}_p(2)) \xrightarrow{\sim} \mathbf{Q}_p.$$

(2) *Let $i, j \in \mathbf{Z}$. The pairing*

$$H^i_{\text{proét}}(X, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^{4-i}_{\text{proét},c}(X, \mathbf{Q}_p(2-j)) \xrightarrow{\cup} H^4_{\text{proét},c}(X, \mathbf{Q}_p(2)) \xrightarrow{\mathrm{Tr}_X} \mathbf{Q}_p[-4]$$

*is perfect, i.e., it induces isomorphisms*

$$\gamma_{X,i}: \quad H^i_{\text{proét}}(X,\mathbf{Q}_p(j)) \xrightarrow{\sim} H^{4-i}_{\text{proét},c}(X,\mathbf{Q}_p(2-j))^*,$$
$$\gamma^c_{X,i}: \quad H^i_{\text{proét},c}(X,\mathbf{Q}_p(j)) \xrightarrow{\sim} H^{4-i}_{\text{proét}}(X,\mathbf{Q}_p(2-j))^*,$$

*where* $(-)^* := \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(-,\mathbf{Q}_p)$.

*Remark* 2. (i) If $X$ is a Stein variety over $K$ with an exhaustive covering by affinoids $\{U_n\}_{n\in\mathbf{N}}$, $U_n \Subset U_{n+1}$, then the compactly supported pro-étale cohomology is defined as

$$\mathrm{R}\Gamma_{\text{proét},c}(X,\mathbf{Q}_p(j)) := \mathrm{fib}(\mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(j)) \to \mathrm{R}\Gamma_{\text{proét}}(\partial X,\mathbf{Q}_p(j))),$$
$$\mathrm{R}\Gamma_{\text{proét}}(\partial X,\mathbf{Q}_p(j)) := \mathrm{colim}_n \mathrm{R}\Gamma_{\text{proét}}(X\setminus U_n,\mathbf{Q}_p(j)).$$

If $X$ is of dimension 1 then $\mathrm{R}\Gamma_{\text{proét},c}(X,\mathbf{Q}_p)$ is the same as Huber's étale cohomology with compact support. We think that this is true in any dimension.

(ii) Let $X$ be a smooth dagger variety over $K$, of dimension 1. Then if $X$ is proper the pro-étale cohomology groups are finite; if $X$ is Stein, $H^i_{\text{proét}}(X,\mathbf{Q}_p(j))$ is nuclear Fréchet and $H^i_{\text{proét},c}(X,\mathbf{Q}_p(j))$ is of compact type; if $X$ is a dagger affinoid then it is the opposite.

(iii) If $X$ is Stein, we actually prove a derived duality in $\mathscr{D}(\mathbf{Q}_{p,\square})$, i.e., the cup product pairing gives a natural quasi-isomorphism

$$\gamma_X : \mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(j)) \xrightarrow{\sim} \mathbb{D}(\mathrm{R}\Gamma_{\text{proét},c}(X,\mathbf{Q}_p(2-j))[4]),$$

where $\mathbb{D}(-,\mathbf{Q}_p) := \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_p}(-,\mathbf{Q}_p)$. Having that, we get the quasi-isomorphism $\gamma_{X,i}$ above by taking cohomology and using the fact that $\underline{\mathrm{Ext}}^s(H^i_{\text{proét},c}(X,\mathbf{Q}_p(j)),\mathbf{Q}_p) = 0$, $s\geq 1$, because $H^i_{\text{proét},c}(X,\mathbf{Q}_p(j))$ is of compact type (hence a colimit of Smith spaces, which are projective solid objects). The quasi-isomorphism $\gamma^c_{X,i}$ follows because $H^i_{\text{proét},c}(X,\mathbf{Q}_p(j))$ is reflexive (in the classical world, in fact).

(iv) In higher dimensions we venture the following conjecture:

**Conjecture 3.** *Let $X$ be a smooth, geometrically irreducible, Stein variety of dimension $d$ over $K$. Then:*

(1) *The cohomology groups $H^i_{\text{proét}}(X,\mathbf{Q}_p(j))$ and $H^i_{\text{proét},c}(X,\mathbf{Q}_p(j))$ are nuclear Fréchet and of compact type, respectively.*
(2) *There exists a natural trace map of solid $\mathbf{Q}_p$-vector spaces*

$$\mathrm{Tr}_X : H^{2d+2}_{\text{proét},c}(X,\mathbf{Q}_p(d+1)) \xrightarrow{\sim} \mathbf{Q}_p.$$

(3) *The pairing*

$$\mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma_{\text{proét},c}(X,\mathbf{Q}_p(d+1-j)) \xrightarrow{\cup} \mathrm{R}\Gamma_{\text{proét},c}(X,\mathbf{Q}_p(d+1)) \xrightarrow{\mathrm{Tr}_X} \mathbf{Q}_p[-2d-2]$$

*is perfect, i.e., it induces (quasi-)isomorphisms*

$$\gamma_X : \mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(j)) \xrightarrow{\sim} \mathbb{D}(\mathrm{R}\Gamma_{\text{proét},c}(X,\mathbf{Q}_p(d+1-j))[2d+2]), \tag{4}$$
$$\gamma_{X,i}: \quad H^i_{\text{proét}}(X,\mathbf{Q}_p(j)) \xrightarrow{\sim} H^{2d+2-i}_{\text{proét},c}(X,\mathbf{Q}_p(d+1-j))^*,$$
$$\gamma^c_{X,i}: \quad H^i_{\text{proét},c}(X,\mathbf{Q}_p(j)) \xrightarrow{\sim} H^{2d+2-i}_{\text{proét}}(X,\mathbf{Q}_p(d+1-j))^*.$$

We note that the duality (quasi-)isomorphisms (4) hold for $X$ proper, smooth, and algebraic by Galois descent from the geometric Poincaré duality (proved by Mann and Zavyalov).

(v) The starting point of our work on arithmetic dualities was the following computation:

*Example* 5. Let $X = D$ be the open unit disc. The we compute (noncanonically)

$$H^1_{\text{proét}}(X, \mathbf{Q}_p(1)) \simeq (\mathscr{O}(D)/K) \oplus H^1(\mathscr{G}_K, \mathbf{Q}_p(1)),$$
$$H^3_{\text{proét},c}(X, \mathbf{Q}_p(1)) \simeq \mathscr{O}(\partial D)/\mathscr{O}(D) \oplus H^1(\mathscr{G}_K, \mathbf{Q}_p).$$

These groups are dual via the Galois and coherent duality:

$$H^i(\mathscr{G}_K, \mathbf{Q}_p) \simeq H^{2-i}(\mathscr{G}_K, \mathbf{Q}_p(1))^*,$$
$$H^0(D, \Omega^1_D) \simeq H^1_c(D, \mathscr{O}_D)^*.$$

We used here that $\mathscr{O}(D)/K \xrightarrow{\sim} H^0(D, \Omega^1_D)$, $\mathscr{O}(\partial D)/\mathscr{O}_D \xrightarrow{\sim} H^1_c(D, \mathscr{O}_D)$. We note that the coherent duality is a $K$-duality, which can be transformed into $\mathbf{Q}_p$-duality because $[K : \mathbf{Q}_p] < \infty$.

(vi) **Solid versus classical functional analysis**. Most of our work could be done in the set-up of classical functional analysis. We had to pass to the solid formalism because (a) we needed a well-behaved derived dual (b) we needed topological Hochschild-Serre spectral sequences.

## 2. Geometric duality

Our work on geometric dualities is still in progress. We are writing down a proof of the following result (which works in any dimension):

**Theorem 6.** (Verdier Duality) *Let $X$ be a smooth, Stein rigid analytic variety over $C$, connected, dimension $d$. Then there is a natural quasi-isomorphism*

$$\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} \mathrm{R}\,\mathrm{Hom}_{VS}(\mathrm{R}\Gamma_{\text{proét},c}(X, \mathbf{Q}_p(d+1-j))[2d], \mathbf{Q}_p(1)),$$

*where $VS$ is the category of solid Vector Spaces, i.e., $v$-sheaves of solid $\mathbf{Q}_p$-vector spaces on* $\mathrm{Perf}_C$.

The strategy is to pass to syntomic cohomology (via a geometric version of a comparison theorem), represent syntomic cohomology via a complex of solid quasi-coherent sheaves on the Fargues-Fontaine curve, prove a Poincaré duality for this complex, and then project it down to the VS category. The Poincaré duality on the curve reduces to Hyodo-Kato duality on the whole curve and $\mathbf{B}^+_{\mathrm{dR}}$-duality at infinity (both of which are known). The functional analytic problems can be solved because all the infinite data "come from the base" and can be "taken out" via a projection formula.

*Remark* 7. It is likely that Conjecture 3 will follow from the above theorem via Galois descent (as is the classical algebraic case)

References

*Reporter: Max Witzelsperger*