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Anabelian Geometry and Representations of Fundamental Groups

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Abstracts

Geometric duality for p-adic pro-étale cohomology of analytic varieties

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(joint work with Pierre Colmez, Sally Gilles)

Let p be a prime. Let K be a finite extension of $\mathbf{Q}_p$. Let $\overline{K}$ be an algebraic closure of K and let $C = \widehat{\overline{K}}$ be its p-adic completion; let $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$. Or analytic varieties are separated.

We are in the process of writing down a proof of the following result.

Theorem 1. (Poincaré Duality) *Let X be a smooth, partially proper rigid analytic variety over K, connected, of dimension d. Then, for $j \in \mathbf{Z}$, there is a natural quasi-isomorphism in* TVS

$$\mathrm{R}\Gamma_{\text{proét}}(X_C, \mathbf{Q}_p(j)) \xrightarrow{\sim} \mathrm{R}\,\mathrm{Hom}_{\mathrm{TVS}}(\mathrm{R}\Gamma_{\text{proét},c}(X_C, \mathbf{Q}_p(d-j))[2d], \mathbf{Q}_p).$$

Here TVS is the category of Topological Vector Spaces, i.e., v-sheaves of enriched (in condensed sets) solid $\underline{\mathbf{Q}}_p$-vector spaces on Perf_C and pro-étale cohomology is seen in TVS; the Hom is internal.

The proof passes to syntomic cohomology (via a geometric version of a comparison theorem), represents syntomic cohomology via a complex of solid quasi-coherent sheaves on the Fargues-Fontaine curve, proves a Poincaré duality for such complex, and then projects this duality down to the TVS category. The Poincaré duality on the FF curve reduces to Hyodo-Kato duality on the whole curve and the filtered $\mathbf{B}_{\mathrm{dR}}^{+}$-duality at infinity (both of which are known). The functional analytic problems can be solved because all the infinite data "come from the base" and can be "taken out" via a projection formula. The enriched structure is necessary for the computation of Ext-groups between Banach-Colmez spaces (and enters through the enriched Yoneda Lemma). See Proposition 3.

Theorem 1 yields the computation:

Corollary 2. *There is a natural exact sequence in* TVS

$$0 \to \mathrm{Ext}^1_{\mathrm{TVS}}(H^{2d-i+1}_{\text{proét},c}(X_C, \mathbf{Q}_p(d)), \mathbf{Q}_p) \to H^i_{\text{proét}}(X_C, \mathbf{Q}_p) \to \mathrm{Hom}_{\mathrm{TVS}}(H^{2d-i}_{\text{proét},c}(X_C, \mathbf{Q}_p(d)), \mathbf{Q}_p) \to 0$$

This is proved by a reduction to the following vanishing result (a topological version of a result of Anschütz-Le Bras)

Proposition 3. *Let $\mathscr{F}_1, \mathscr{F}_2$ be Banach-Colmez spaces. Then*

$$\mathrm{Ext}^a_{\mathrm{TVS}}(\mathscr{F}_1, \mathscr{F}_2) = 0, \quad a \geq 2.$$

Which, in turn, is proved by using Mac-Lane resolutions (those are naturally enriched in this setting) and reducing, via the enriched Yoneda Lemma, to the computation of cohomologies of complexes built from cohomologies of affine spaces with values in the sheaves $\mathbf{Q}_p$ and $\mathbb{G}_a$. These complexes can be represented by complexes of Fréchet spaces and hence are exact if and only if are exact algebraically. But the algebraic complexes represent Ext-groups in VS-category, where they are known to vanish in degrees at least 2 by Anschütz-Le Bras.

Remark 4. There is an independent, ongoing project of Anschütz-Le Bras-Mann that studies 6-functor formalism for p-adic pro-étale cohomology on analytic varieties. In particular, this should include duality results of the type stated in Theorem 1.

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