Completed cohomology of the Drinfeld tower for $\mathbf{GL_2(Q_p)}$

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(joint work with Pierre Colmez, Wiesława Nizioł and with Juan Esteban Rodriguez Camargo)

Let p be an odd prime and let $\mathrm{Gal}_{\mathbf{Q}_p}$ be the absolute Galois group of $\mathbf{Q}_p$. Let $G = \mathrm{GL}_2(\mathbf{Q}_p)$, let D be a quaternion division algebra over $\mathbf{Q}_p$ and fix a large enough finite extension L of $\mathbf{Q}_p$, serving as coefficient field for our representations. If H is a p-adic Lie group let $\mathrm{Ban}(H)$ be the category of admissible unitary L-Banach space representations of H. If Γ is a topological group let $\mathrm{Ban}(H)^\Gamma$ be the category of Γ-equivariant objects of $\mathrm{Ban}(H)$. Finally, let $\mathrm{Irr}(\mathrm{Gal}_{\mathbf{Q}_p})$ be the set of isomorphism classes of continuous 2-dimensional absolutely irreducible L-representations of $\mathrm{Gal}_{\mathbf{Q}_p}$.

The classical Jacquet-Langlands correspondence relates certain irreducible smooth representations of G over L (the discrete series) and the irreducible smooth representations of $D^\times$ over L. We are interested here in the Banach, locally analytic and modulo p versions of such a correspondence. Recall that the p-adic local Langlands correspondence establishes a bijection $V \mapsto \Pi(V), \Pi \mapsto V(\Pi)$ between $\mathrm{Irr}(\mathrm{Gal}_{\mathbf{Q}_p})$ and the set $\widehat{G}_{\mathrm{ss}}$ of isomorphism classes of absolutely irreducible objects of $\mathrm{Ban}(G)$ which are supersingular (i.e. not isomorphic to a subquotient of a unitary parabolic induction). There were several hints that this correspondence can be realised geometrically in the p-adic étale cohomology of various spaces, as we will briefly recall.

Let $X_\infty = X(p^\infty)$ be the inverse limit (in the category of $\mathbf{Q}$-schemes) of the modular curves X_n of level $\Gamma(p^n)$ as $n \to \infty$. Then $H^1_{\mathrm{et}}(X_\infty \otimes \overline{\mathbf{Q}}, L)$ is the completed cohomology of the tower of modular curves $(X_n)_{n\geq 1}$ and it belongs to $\mathrm{Ban}(G)^{\mathrm{Gal}_{\mathbf{Q}}}$. Emerton proved in [2] that under mild hypothesis on the absolutely irreducible odd representation $\rho : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(L)$, unramified outside of p one has an isomorphism in $\mathrm{Ban}(G)$

$$\mathrm{Hom}_{\mathrm{Gal}_{\mathbf{Q}}}(\rho, H^1_{\mathrm{et}}(X_\infty \otimes \overline{\mathbf{Q}}, L)) \simeq \Pi(\rho|_{\mathrm{Gal}_{\mathbf{Q}_p}}).$$

The scheme $X_\infty \otimes_{\mathbf{Q}} \mathbf{C}_p$ has an "analytification" X_∞^{an}, which is a perfectoid space over $\mathbf{C}_p$ (as a proétale sheaf on perfectoid $\mathbf{C}_p$-algebras it is the inverse limit of the sheaves represented by the analytifications of $X_n \otimes_{\mathbf{Q}} \mathbf{C}_p$). Scholze constructed a G-equivariant Hodge-Tate period map $\pi_{\mathrm{HT}} : X_\infty^{\mathrm{an}} \to \mathbf{P}^1$ to the adic projective line over $\mathbf{C}_p$. Let $\Omega = \mathbf{P}^1 \setminus \mathbf{P}^1(\mathbf{Q}_p)$ be Drinfeld's upper half-plane, an open rigid subvariety of $\mathbf{P}^1$ stable under the action of G. The inverse image $\pi_{\mathrm{HT}}^{-1}(\Omega)$ is the disjoint union of finitely many copies of a perfectoid space LT_∞ with a continuous G-action, the Lubin-Tate space at infinite level. The map $\pi_{\mathrm{HT}} : \mathrm{LT}_\infty \to \Omega$ is a proétale $D^\times$-torsor and we let $\mathrm{Dr}_n = \mathrm{LT}_\infty/(1 + p^n\mathcal{O}_D)$. As n increases we obtain a tower of $G \times D^\times$-equivariant finite étale coverings of $\coprod_{n\in\mathbf{Z}} \Omega$, defined over the completion of $\mathbf{Q}_p^{\mathrm{nr}}$, and endowed with a Weil descent datum. For simplicity, we will pretend that they are defined over $\mathbf{Q}_p$ (the spaces $\mathrm{Dr}_n/p^{\mathbf{Z}}$ are defined over $\mathbf{Q}_p$ and one can always reduce our problems to these spaces, by twisting by unramified characters). In our previous work [1] we proved that for any $V \in \mathrm{Irr}(\mathrm{Gal}_{\mathbf{Q}_p})$ which is nice (i.e. de Rham, with Hodge-Tate weights $0, 1$ and non trianguline) we have

$$\mathrm{Hom}_{\mathrm{Gal}_{\mathbf{Q}_p}}(V, \varinjlim_n H^1_{\mathrm{et}}(\mathrm{Dr}_n, L)) \simeq \mathrm{JL}^{\mathrm{cl}}(V) \otimes_L \Pi(V)^*,$$

for some (finite dimensional) irreducible smooth L-representation $\mathrm{JL}^{\mathrm{cl}}(V)$ depending only on $D_{\mathrm{pst}}(V)$ and constructed using the classical Jacquet-Langlands and local Langlands correspondence.

We would like to have a similar picture for $\Pi(V)$ for *any* $V \in \mathrm{Irr}(\mathrm{Gal}_{\mathbf{Q}_p})$. Let

$$H_\infty = H^1_{\mathrm{et}}(\mathrm{LT}_\infty, L).$$

Work of Scholze shows that there is a functor

$$S^1 : \mathrm{Ban}(G) \to \mathrm{Ban}(D^\times)^{\mathrm{Gal}_{\mathbf{Q}_p}},\ \Pi \mapsto S^1(\Pi) = \mathrm{Hom}_G(\Pi^*, H_\infty).$$

This functor has obvious integral and torsion versions that we will use implicitly.

Many of the results in the next theorem (in joint work with Juan Esteban Rodriguez Camargo) were known under mild hypotheses on the reduction mod p of Π, by work of Hansen-Mann [3], Hu-Wang [4], [5] and Ludwig [6]. In the statement of the theorem "almost" means "up to finite dimensional representations" (i.e. we implicitly work with the composition of S^1 and the natural functor to the quotient of $\mathrm{Ban}(D^\times)^{\mathrm{Gal}_{\mathbf{Q}_p}}$ by the category of finite dimensional representations). Most results hold for $\mathrm{GL}_2(F)$, with $F/\mathbf{Q}_p$ finite. The proof heavily uses the work of Mann [7] on 6-functor formalisms for rigid analytic varieties.

Theorem 1. (1) *The functor S^1 is almost exact, in particular almost compatible with reduction modulo p.*
(2) *There is a natural almost isomorphism $S^1(\Pi)^* \simeq \mathrm{Hom}_G(\Pi, H_\infty(1))$.*
(3) *Letting* GK *be the Gelfand-Kirillov (or canonical) dimension, we have*

$$\mathrm{GK}(S^1(\Pi)) \leq \mathrm{GK}(\Pi),$$

with equality for Π Cohen-Macaulay, in particular $\mathrm{GK}(S^1(\Pi)) = 1$ for $\Pi \in \widehat{G}_{\mathrm{ss}}$ (and so $S^1(\Pi)$ is infinite-dimensional).
(4) *There is a natural almost isomorphism*

$$S^1(\Pi)^{\mathrm{la}} \simeq \mathrm{Hom}_G((\Pi^{\mathrm{la}})^*, H^1_{\mathrm{proet}}(\mathrm{LT}_\infty, L)).$$

For $\Pi \in \widehat{G}_{\mathrm{ss}}$ the representations $S^1(\Pi)^{\mathrm{la}}$ and Π^{la} have the same infinitesimal character.

The next result (joint with Colmez and Nizioł) is proved using results of Boston, Lenstra and Ribet, as well as an "analytic continuation" argument using patching. Again, the result was known in many cases by previous work of many people.

Theorem 2. *For any $\Pi \in \widehat{G}_{\mathrm{ss}}$ there is a $\mathrm{JL}(\Pi) \in \mathrm{Ban}(D^\times)$ such that*

$$S^1(\Pi) \simeq V(\Pi) \otimes_L \mathrm{JL}(\Pi),$$

in other words $S^1(\Pi)$ is $V(\Pi)$-isotypic as $\mathrm{Gal}_{\mathbf{Q}_p}$-representation.

Let

$$\tilde{H}_{\mathrm{Dr}} = (\varprojlim_k \varinjlim_n H^1_{\mathrm{et}}(\mathrm{Dr}_n, \mathcal{O}_L/p^k))[1/p]$$

be the completed cohomology of the tower $(\mathrm{Dr}_n)_{n\geq 1}$. Contrary to the completed cohomology of the modular curves this is much smaller than $H^1_{\mathrm{et}}(\mathrm{LT}_\infty, L)$ and does not have any reasonable finiteness property since each $H^1_{\mathrm{et}}(\mathrm{Dr}_n, \mathcal{O}_L/p^k)$ is huge (while $H^1_{\mathrm{et}}(X_n \otimes \mathbf{C}_p, \mathcal{O}_L/p^k)$ is finite!). Another key difficulty compared to modular curves is that we don't know whether $H^2_{\mathrm{et}}(\mathrm{Dr}_n, \mathcal{O}_L/p^k)$ is 0 or huge. One can still compare $\tilde{H}_{\mathrm{Dr}}$ and $Z := H^1_{\mathrm{et}}(\mathrm{LT}_\infty, \mathcal{O}_L)$ as follows:

$$\tilde{H}_{\mathrm{Dr}} = (\varprojlim_k (Z/p^k Z)^{D^\times-\mathrm{sm}})[1/p].$$

Our most delicate result (joint with Colmez and Niziol̀) is the following factorisation theorem, where we say that V is generic if its reduction modulo p (semi-simplified) is not the direct sum of two characters whose quotient is 1 or the mod p cyclotomic character (there is a more technical notion of being very generic which we don't recall here).

Theorem 3. (1) *For any generic* $V \in \mathrm{Irr}(\mathrm{Gal}_{\mathbf{Q}_p})$ *we have a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Gal}_{\mathbf{Q}_p}}(V, \tilde{H}_{\mathrm{Dr}}) \simeq \Pi(V)^* \widehat{\otimes}_L \mathrm{JL}(\Pi(V)),$$

the tensor product being p-adically completed.

(2) *If* V *is very generic then* $\mathrm{JL}(\Pi(V))$ *is irreducible or has finite dimensional locally algebraic vectors (this happens if and only if* V *is nice), the quotient being irreducible. Moreover there is a natural isomorphism*

$$\mathrm{Hom}_{D^\times \times \mathrm{Gal}_{\mathbf{Q}_p}}(S^1(\Pi(V)), H_\infty) \simeq \Pi(V)^*.$$

Towards the end of the talk we discussed variants modulo p of these results as well as some speculations on what happens for $\mathrm{GL}_2(F)$.

References

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