ON p-ADIC ABSOLUTE HODGE COHOMOLOGY AND SYNTOMIC COEFFICIENTS, I.

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Abstract. We interpret syntomic cohomology defined in [50] as a p-adic absolute Hodge cohomology. This is analogous to the interpretation of Deligne-Beilinson cohomology as an absolute Hodge cohomology by Beilinson [8] and generalizes the results of Bannai [6] and Chiarellotto, Ciccioni, Mazzari [15] in the good reduction case. This interpretation yields a simple construction of the syntomic descent spectral sequence and its degeneration for projective and smooth varieties. We introduce syntomic coefficients and show that in dimension zero they form a full triangulated subcategory of the derived category of potentially semistable Galois representations.

Along the way, we obtain p-adic realizations of mixed motives including p-adic comparison isomorphisms. We apply this to the motivic fundamental group generalizing results of Olsson and Vologodsky [56], [71].

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1. Introduction

In [8], Beilinson gave an interpretation of Deligne-Beilinson cohomology as an *absolute Hodge cohomology*, i.e., as derived Hom in the derived category of mixed Hodge structures. This approach is

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advantageous: absolute Hodge cohomology allows coefficients. It follows that Deligne-Beilinson cohomology can be interpreted as derived Hom between Tate twists in the derived category of Saito's mixed Hodge modules [38, A.2.7].

Syntomic cohomology is a p-adic analog of Deligne-Beilinson cohomology. The purpose of this paper is to give an analog of the above results for syntomic cohomology. Namely, we will show that the syntomic cohomology introduced in [50] is a p-adic absolute Hodge cohomology, i.e., it can be expressed as derived Hom in the derived category of p-adic Hodge structures, and we will begin the study of syntomic coefficients - an approximation of p-adic Hodge modules. This generalizes the results of Bannai [6] and Chiarellotto, Ciccioni, Mazzari [15] in the good reduction case.

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field k. Let $G_K=\mathrm{Gal}(\overline{K}/K)$ be the Galois group of K. For the category of p-adic Hodge structures we take the abelian category DF_K of (weakly) admissible filtered (φ,N,G_K)-modules defined by Fontaine. For a variety X over K, we construct a complex $\mathrm{R}\Gamma_{DF_K}(X_{\overline{K}},r)\in D^b(DF_K)$, $r\in\mathbf{Z}$. The absolute Hodge cohomology of X is then by definition

$$\mathrm{R}\Gamma_{\mathscr{H}}(X,r):=\mathrm{R}\operatorname{Hom}_{D^b(DF_K)}(K(0),\mathrm{R}\Gamma_{DF_K}(X_{\overline{K}},r)),\quad r\in\mathbf{Z}.$$

For $r\geq 0$, it coincides with the syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)$ defined in [50]. Recall that the latter was defined as the following mapping fiber

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)=[\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)^{\varphi=p^r,N=0}\xrightarrow{\iota_{\mathrm{dR}}}\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r],$$

where $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)$ is the Beilinson-Hyodo-Kato cohomology from [10], $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$ is the Deligne de Rham cohomology, and the map ι_{dR} is the Beilinson-Hyodo-Kato map.

We present two approaches to the definition of the complex $\mathrm{R}\Gamma_{DF_K}(X_{\overline{K}},r)$. In the first one, we follow Beilinson's construction of the complex of mixed Hodge structures associated to a variety [8]. Thus, we build the dg category $\mathscr{D}_{pH}$ of *p-adic Hodge complexes* (an analog of Beilinson's mixed Hodge complexes) which is obtained by gluing two dg categories, one, corresponding morally to the special fiber, whose objects are equipped with an action of a Frobenius and a monodromy operator, and the other one, corresponding to the generic fiber, whose objects are equipped with a filtration thought of as the Hodge filtration on de Rham cohomology. It contains a dg subcategory of *admissible p-adic Hodge complexes* with cohomology groups belonging to DF_K. The category $\mathscr{D}^{\mathrm{ad}}_{pH}$ admits a natural t-structure whose heart is the category DF_K and $\mathscr{D}^{\mathrm{ad}}_{pH}$ is equivalent to the derived category of its heart. That is, we have the following equivalences of categories

$$\theta: DF_K\xrightarrow{\sim}\mathscr{D}^{\mathrm{ad},\heartsuit}_{pH},\quad \theta:\mathscr{D}^b(DF_K)\xrightarrow{\sim}\mathscr{D}^{\mathrm{ad}}_{pH}.$$

The interest of the category $\mathscr{D}^{\mathrm{ad}}_{pH}$ lies in the fact that, for $r\in\mathbf{Z}$, a variety X over K gives rise to the admissible p-adic Hodge complex

$$\mathrm{R}\Gamma_{pH}(X_{\overline{K}},r):=(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}},r),(\mathrm{R}\Gamma_{\mathrm{dR}}(X),F^{\bullet+r}),\iota_{\mathrm{dR}})\in\mathscr{D}^{\mathrm{ad}}_{pH}$$

We define $\mathrm{R}\Gamma_{DF_K}(X_{\overline{K}},r):=\theta^{-1}\mathrm{R}\Gamma_{pH}(X_{\overline{K}},r)$.

Since the category DF_K is equivalent to that of potentially semistable representations [20], i.e., we have a functor $V_{\mathrm{pst}}: DF_K\xrightarrow{\sim}\mathrm{Rep}_{\mathrm{pst}}(G_K)$, we can also write

$$\mathrm{R}\Gamma_{\mathscr{H}}(X,r)=\operatorname{Hom}_{\mathscr{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))}(\mathbf{Q}_p,\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r)),$$

for $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r):=V_{\mathrm{pst}}\mathrm{R}\Gamma_{DF_K}(X_{\overline{K}},r)$. Using Beilinson's comparison theorems [10] we prove that $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r)\simeq\mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p(r))$ as Galois modules. It follows that there is a functorial syntomic descent spectral sequence (constructed originally by a different, more complicated, method in [50])

$${}^{\mathscr{H}}E_2^{i,j}:=H^i_{\mathrm{st}}(G_K,H^j_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p(r)))\Rightarrow H^{i+j}_{\mathscr{H}}(X,r),$$

where $H^i_{\mathrm{st}}(G_K,\cdot):=\operatorname{Ext}^i_{\mathrm{Rep}_{\mathrm{pst}}(G_K)}(\mathbf{Q}_p,\cdot)$. By a classical argument of Deligne [25], it follows from Hard Lefschetz Theorem, that it degenerates at E_2 for X projective and smooth.

A more direct definition of the complex $\mathrm{R}\Gamma_{DF_K}(X_{\overline{K}},r)$, or, equivalently, of the complex $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r)$ of potentially semistable representations associated to a variety was proposed by Beilinson [11] using

Beilinson's Basic Lemma. This lemma allows one to associate a potentially semistable analog of a cellular complex (of a CW-complex) to an affine variety X over K: one stratifies the variety by closed subvarieties such that consecutive relative geometric étale cohomology is concentrated in the top degree (and is a potentially semistable representation). For a general X one obtains Beilinson's potentially semistable complex by a Čech gluing argument.

All the p-adic cohomologies mentioned above (de Rham, étale, Hyodo-Kato, and syntomic) behave well, hence they lift to realizations of both Nori's abelian and Voevodsky's triangulated category of mixed motives. We also lift the comparison maps between them, thus obtaining comparison theorems for mixed motives. We illustrate this construction by two applications. The first one is a p-adic realization of the motivic fundamental group including a potentially semistable comparison theorem. We rely on Cushman's motivic (in the sense of Nori) theory of the fundamental group [22]. This generalizes results obtained earlier for curves and proper varieties with good reduction [37], [1], [71], [56]. The second is a compatibility result. We show that Beilinson's p-adic comparison theorems (with compact support or not) are compatible with Gysin morphisms and (possibly mixed) products.

To define a well-behaved notion of syntomic coefficients (i.e., coefficients for syntomic cohomology) we use Morel-Voevodsky *motivic homotopy theory*, and more precisely the concept of modules over (motivic) ring spectra. Recall that objects of motivic stable homotopy theory, called spectra, represent cohomology theories with suitable properties. A multiplicative structure on the cohomology theory corresponds to a monoid structure on the representing spectrum, which is then called a *ring spectrum*. These objects should be be thought of as a generalization of (h-sheaves[1] of) differential graded algebras. In fact, as we will only consider ordinary cohomology theories (as opposed to K-theory or algebraic cobordism with integral coefficients), we will always restrict to this later concept. Therefore modules over ring spectra should be understood as the more familiar concept of modules over differential graded algebras.

One of the basic examples of a representable cohomology theory is de Rham cohomology in characteristic 0. Denote the corresponding motivic ring spectrum by $\mathscr{E}_{\mathrm{dR}}$. By [18], [28], working relatively to a fixed complex variety X, modules over $\mathscr{E}_{\mathrm{dR},X}$ satisfying a suitable finiteness condition correspond naturally to (regular holonomic) $\mathscr{D}_X$-modules of geometric origin.

In [50] it is shown that syntomic cohomology can be represented by a motivic dg algebra $\mathscr{E}_{\mathrm{syn}}$, i.e., we have

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,r) = \mathrm{R}\operatorname{Hom}_{DM_h(K,\mathbf{Q}_p)}(M(X),\mathscr{E}_{\mathrm{syn}}(r)), \tag{1.1}$$

where $M(X)$ is the Voevodsky's motive associated to X and $DM_h(K,\mathbf{Q}_p)$ is the category of h-motives. So we have the companion notion of *syntomic modules*, that is, modules over the motivic dg-algebra $\mathscr{E}_{\mathrm{syn}}$. The main advantage of this definition is that the link with mixed motives is rightly given by the construction and, most of all, the 6 functors formalism follows easily from the motivic one.

Now the crucial question is to understand how the category of syntomic modules is related to the category of filtered (φ, N, G_K)-modules, the existing candidates for syntomic smooth sheaves [30], [31], [66], [62], and the category of syntomic coefficients introduced in [24] by a method analogous to the one we use but based on Gros-Besser's version of syntomic cohomology. In this paper we study this question only in dimension zero, i.e., for syntomic modules over the base field. With a suitable notion of finiteness for syntomic modules, called *constructibility*, we prove the following theorem.

Theorem (Theorem 5.13). The triangulated monoidal category of constructible syntomic modules over a p-adic field K is equivalent to a full subcategory of the derived category of admissible filtered (φ, N, G_K)-modules.

It implies, by adjunction from (1.1), that p-adic absolute Hodge cohomology coincides with derived Hom in the (homotopy) category of syntomic modules, i.e., we have

$$\mathrm{R}\Gamma_{\mathscr{H}}(X,r) = \mathrm{R}\operatorname{Hom}_{\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_X}(\mathscr{E}_{\mathrm{syn},X},\mathscr{E}_{\mathrm{syn},X}(r)).$$

[1] An h-sheaf is a sheaf for the h-topology. The h-topology is the Grothendieck topology generated by universal topological epimorphisms (see [69, 3.1.2]).

In the conclusion of the paper, we use syntomic modules to introduce new notions of p-adic Galois representations (Definition 5.20). We define *geometric representations* which correspond to the common intuition of representations associated to (mixed) motives, and *constructible representations*, corresponding to cohomology groups of Galois realizations of syntomic modules.

We expect that the categories of geometric, constructible, and potentially semistable representations are not the same. This is at least what is predicted by the current general conjectures. Note that this is in contrast to the case of number fields where the analogs of these notions are conjectured to coincide with the known definition of "representations coming from geometry" [34].

1.0.1. *Notation.* Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic 0, with perfect residue field k of characteristic p. Let $\overline{K}$ be an algebraic closure of K. Let $W(k)$ be the ring of Witt vectors of k with fraction field K_0 and denote by K_0^{nr} the maximal unramified extension of K_0. Set $G_K = \mathrm{Gal}(\overline{K}/K)$ and let I_K denote its inertia subgroup. Let φ be the absolute Frobenius on K_0^{nr}. We will denote by $\mathscr{O}_K$, $\mathscr{O}_K^{\times}$, and $\mathscr{O}_K^{0}$ the scheme $\mathrm{Spec}(\mathscr{O}_K)$ with the trivial, canonical (i.e., associated to the closed point), and $(\mathbf{N} \to \mathscr{O}_K, 1 \mapsto 0)$ log-structure respectively. For a scheme X over $W(k)$, X_n will denote its reduction mod p^n, X_0 will denote its special fiber. Let $\mathscr{V}ar_K$ denote the category of varieties over K, i.e., reduced, separated, K-schemes of finite type.

For a dg category $\mathscr{C}$ with a t-structure, we will denote by $\mathscr{C}^{\heartsuit}$ the heart of the t-structure. We will use a shorthand for certain homotopy limits. Namely, if $f : C \to C'$ is a map in the dg derived category of abelian groups , we set

$$[\, C \xrightarrow{f} C' \,] := \mathrm{holim}(C \to C' \leftarrow 0).$$

And, if

$$\begin{array}{ccc} C_1 & \xrightarrow{f} & C_2 \\ \downarrow & & \downarrow \\ C_3 & \xrightarrow{g} & C_4 \end{array}$$

is a commutative diagram in the dg derived category of abelian groups , we set

$$\left[\begin{array}{ccc} C_1 & \xrightarrow{f} & C_2 \\ \downarrow & & \downarrow \\ C_3 & \xrightarrow{g} & C_4 \end{array}\right] := [[C_1 \stackrel{f}{\to} C_2] \to [C_3 \stackrel{g}{\to} C_4]].$$

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2. A p-adic absolute Hodge cohomology, I

2.1. The derived category of admissible filtered (φ, N, G_K)-modules.

2.1. For a field K, let V_K denote the category of K-vector spaces. It is an abelian category. We will denote by $\mathscr{D}^b(V_K)$ its bounded derived dg category and by $D^b(V_K)$ – its bounded derived category. Let V_{dR}^K denote the category of K-vector spaces with a descending exhaustive separated filtration $F^{\bullet}$. The category V_{dR}^K (and the category of bounded complexes $C^b(V_{\mathrm{dR}}^K)$) is additive but not abelian. It is an exact category in the sense of Quillen [57], where short exact sequences are exact sequences of K-vector spaces with *strict morphisms* (recall that a morphism $f : M \to N$ is *strict* if $f(F^iM) = F^iN \cap \mathrm{im}(f)$). It is also a quasi-abelian category in the sense of [61] (see [60, 2] for a quick review). Thus its derived category can be studied as usual (see [12]).

An object $M \in C^b(V^K_{\mathrm{dR}})$ is called a *strict complex* if its differentials are strict. There are canonical truncation functors on $C^b(V^K_{\mathrm{dR}})$:

$$\tau_{\leq n}M := \cdots \to M^{n-2} \to M^{n-1} \to \ker(d^n) \to 0 \to \cdots$$
$$\tau_{\geq n}M := \cdots \to 0 \cdots \to \mathrm{coim}(d^{n-1}) \to M^n \to M^{n+1} \to \cdots$$

with cohomology objects

$$\tau_{\leq n}\tau_{\geq n}(M) = \cdots \to 0 \to \mathrm{coim}(d^{n-1}) \to \ker(d^n) \to 0 \to \cdots$$

We will denote the bounded derived dg category of V^K_{dR} by $\mathscr{D}^b(V^K_{\mathrm{dR}})$. It is defined as the dg quotient [29] of the dg category $C^b(V^K_{\mathrm{dR}})$ by the full dg subcategory of strictly exact complexes [48]. A map of complexes is a quasi-isomorphism if and only if it is a quasi-isomorphism on the grading. The homotopy category of $\mathscr{D}^b(V^K_{\mathrm{dR}})$ is the bounded filtered derived category $D^b(V^K_{\mathrm{dR}})$.

For $n \in \mathbf{Z}$, let $D^b_{\leq n}(V^K_{\mathrm{dR}})$ (resp. $D^b_{\geq n}(V^K_{\mathrm{dR}})$) denote the full subcategory of $D^b(V^K_{\mathrm{dR}})$ of complexes that are strictly exact in degrees $k > n$ (resp. $k < n$)[2]. The above truncation maps extend to truncations functors $\tau_{\leq n} : D^b(V^K_{\mathrm{dR}}) \to D^b_{\leq n}(V^K_{\mathrm{dR}})$ and $\tau_{\geq n} : D^b(V^K_{\mathrm{dR}}) \to D^b_{\geq n}(V^K_{\mathrm{dR}})$. The pair $(D^b_{\leq n}(V^K_{\mathrm{dR}}), D^b_{\geq n}(V^K_{\mathrm{dR}}))$ defines a t-structure on $D^b(V^K_{\mathrm{dR}})$ by [61]. The heart $D^b(V^K_{\mathrm{dR}})^{\heartsuit}$ is an abelian category $LH(V^K_{\mathrm{dR}})$. We have an embedding $V^K_{\mathrm{dR}} \hookrightarrow LH(V^K_{\mathrm{dR}})$ that induces an equivalence $D^b(V^K_{\mathrm{dR}}) \xrightarrow{\sim} D^b(LH(V^K_{\mathrm{dR}}))$. This t-structure pulls back to a t-structure on the derived dg category $\mathscr{D}^b(V^K_{\mathrm{dR}})$.

2.2. Let the field K be again as at the beginning of this article. A φ-module over K_0 is a pair (D, φ), where D is a K_0-vector space and the Frobenius $\varphi = \varphi_D$ is a φ-semilinear endomorphism of D. We will usually write D for (D, φ). The category $M_{K_0}(\varphi)$ of φ-modules over K_0 is abelian and we will denote by $\mathscr{D}^b_{K_0}(\varphi)$ its bounded derived dg category.

For $D_1, D_2 \in M_{K_0}(\varphi)$, let $\mathrm{Hom}_{K_0,\varphi}(D_1, D_2)$ denote the group of Frobenius morphisms. We have the exact sequence

$$0 \to \mathrm{Hom}_{K_0,\varphi}(D_1, D_2) \to \mathrm{Hom}_{K_0}(D_1, D_2) \to \mathrm{Hom}_{K_0}(D_1, \varphi_* D_2), \tag{2.1}$$

where the last map is $\delta : x \mapsto \varphi_{D_2} x - \varphi_*(x)\varphi_{D_1}$. Set $\mathrm{Hom}^{\sharp}_{K_0,\varphi}(D_1, D_2) := \mathrm{Cone}(\delta)[-1]$. Beilinson proves the following lemma.

Lemma 2.3. *([10, 1.13,1.14])*
For $D_1, D_2 \in \mathscr{D}^b_{K_0}(\varphi)$, the map $\mathrm{R}\,\mathrm{Hom}_{K_0,\varphi}(D_1, D_2) \to \mathrm{Hom}^{\sharp}_{K_0,\varphi}(D_1, D_2)$ is a quasi-isomorphism, i.e.,

$$\mathrm{R}\,\mathrm{Hom}_{K_0,\varphi}(D_1, D_2) = \mathrm{Cone}(\mathrm{Hom}_{K_0}(D_1, D_2) \xrightarrow{\delta} \mathrm{Hom}_{K_0}(D_1, \varphi_* D_2))[-1]$$

Proof. Note that, for $D_1, D_2 \in \mathscr{D}^b_{K_0}(\varphi)$, from the exact sequence (2.1), we get a map

$$\alpha : \mathrm{R}\,\mathrm{Hom}_{K_0,\varphi}(D_1, D_2) \to \mathrm{Cone}(\mathrm{R}\,\mathrm{Hom}_{K_0}(D_1, D_2) \xrightarrow{\delta} \mathrm{R}\,\mathrm{Hom}_{K_0}(D_1, \mathrm{R}\varphi_* D_2))[-1]$$

Since

$$\mathrm{R}\,\mathrm{Hom}_{K_0}(D_1, D_2) \simeq \mathrm{Hom}_{K_0}(D_1, D_2), \quad \mathrm{R}\,\mathrm{Hom}_{K_0}(D_1, \mathrm{R}\varphi_* D_2) \simeq \mathrm{Hom}_{K_0}(D_1, \varphi_* D_2)$$

it suffices to show that the map α is a quasi-isomorphism.

The forgetful functor $M_{K_0}(\varphi) \to V_{K_0}$ has a right adjoint $M \to M_{\varphi}$, where the φ-module $M_{\varphi} := \prod_{n \geq 0} \varphi^n_* M$ with Frobenius $\varphi_{M_{\varphi}} : (x_0, x_1, \ldots,) \to (x_1, x_2, \ldots)$. The functor $M \to M_{\varphi}$ is left exact and preserves injectives. Since all K_0-modules are injective, the map $M \to M_{\varphi}$, $m \mapsto (m, \varphi(m), \varphi^2(m), \ldots)$, embeds M into an injective φ-module. It suffices thus to check that the map α is a quasi-isomorphism for D_1 any φ-module and $D_2 = G_{\varphi}$. We calculate

$$\begin{aligned}\mathrm{R}\,\mathrm{Hom}_{K_0,\varphi}(D_1, G_{\varphi}) \xleftarrow{\sim} \mathrm{Hom}_{K_0,\varphi}(D_1, G_{\varphi}) &\xrightarrow{\sim} \mathrm{Cone}(\mathrm{Hom}_{K_0}(D_1, G_{\varphi}) \xrightarrow{\delta} \mathrm{Hom}_{K_0}(D_1, \varphi_* G_{\varphi}))[-1]\\ &\xrightarrow{\sim} \mathrm{Cone}(\mathrm{R}\,\mathrm{Hom}_{K_0}(D_1, G_{\varphi}) \xrightarrow{\delta} \mathrm{R}\,\mathrm{Hom}_{K_0}(D_1, \mathrm{R}\varphi_* G_{\varphi}))[-1]\end{aligned}$$

This proves the lemma. $\square$

[2]Recall [61, 1.1.4] that a sequence $A \xrightarrow{e} B \xrightarrow{f} C$ such that $fe = 0$ is called *strictly exact* if the morphism e is strict and the natural map $\mathrm{im}\, e \to \ker f$ is an isomorphism.

2.4. A (φ, N)-module is a triple (D, φ_D, N) (abbreviated often to D), where (D, φ_D) is a finite rank φ-module over K_0 and φ_D is an automorphism, and N is a K_0-linear endomorphism of D such that $N\varphi_D = p\varphi_D N$ (hence N is nilpotent). The category $M_{K_0}(\varphi, N)$ of (φ, N)-modules is naturally a Tannakian tensor $\mathbf{Q}_p$-category and $(M, \varphi_M, N) \mapsto M$ is a fiber functor over K_0. Denote by $\mathscr{D}^b_{\varphi,N}(K_0)$ and $D^b_{\varphi,N}(K_0)$ the corresponding bounded derived dg category and bounded derived category, respectively.

For (φ, N)-modules M, T, let $\mathrm{Hom}_{\varphi,N}(M,T)$ be the group of (φ, N)-module morphisms. Let $\mathrm{Hom}^\sharp_{\varphi,N}(M,T)$ be the complex [10, 1.15]

$$\mathrm{Hom}_{K_0}(M,T) \to \mathrm{Hom}_{K_0}(M, \varphi_* T) \oplus \mathrm{Hom}_{K_0}(M,T) \to \mathrm{Hom}_{K_0}(M, \varphi_* T)$$

beginning in degree 0 and with the following differentials

$$\begin{aligned} d_0 : \quad & x \mapsto (\varphi_2 x - x\varphi_1, N_2 x - x N_1); \\ d_1 : \quad & (x,y) \mapsto (N_2 x - pxN_1 - p\varphi_2 y + y\varphi_1) \end{aligned}$$

Clearly, we have $\mathrm{Hom}_{\varphi,N}(M,T) = H^0 \mathrm{Hom}^\sharp_{\varphi,N}(M,T)$. Complexes $\mathrm{Hom}^\sharp_{\varphi,N}$ compose naturally and supply a dg category structure on the category of bounded complexes of (φ, N)-modules.

Beilinson states the following fact.

Lemma 2.5. *([10, 1.15]) For $D_1, D_2 \in \mathscr{D}^b_{K_0}(\varphi, N)$, the map $\mathrm{R}\,\mathrm{Hom}_{\varphi,N}(D_1, D_2) \to \mathrm{Hom}^\sharp_{\varphi,N}(D_1, D_2)$ is a quasi-isomorphism, i.e.,*

$$\mathrm{R}\,\mathrm{Hom}_{\varphi,N}(D_1, D_2) = \left[\begin{array}{ccc} \mathrm{Hom}_{K_0}(D_1, D_2) & \xrightarrow{\delta_1} & \mathrm{Hom}_{K_0}(D_1, \varphi_* D_2)) \\ \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta_2'} \\ \mathrm{Hom}_{K_0}(D_1, D_2) & \xrightarrow{\delta_1'} & \mathrm{Hom}_{K_0}(D_1, \varphi_* D_2)) \end{array} \right]$$

Here

$$\begin{aligned} \delta_1 : x \mapsto \varphi_2 x - x\varphi_1, \quad & \delta_1' : x \mapsto p\varphi_2 x - x\varphi_1; \\ \delta_2 : x \mapsto N_2 x - xN_1, \quad & \delta_2' : x \mapsto N_2 x - pxN_1 \end{aligned}$$

Proof. By devissage we may assume that D_1, D_2 are just (φ, N)-modules placed in degree 0. By devissage on m such that $N^m D_1 = 0$, we may assume that $N = 0$ on D_1. Fix such a D_1. Since, clearly, the map in the lemma induces an isomorphism on H^0, it suffices to show that, as a functor of D_2, the cohomology groups H^i, $i = 1, 2$, are effaceable in the category of (φ, N)-modules.

We will start with H^2. To kill $H^2 \mathrm{Hom}^\sharp_{\varphi,N}(D_1, D_2)$ by an injection $D_2 \hookrightarrow D_3$, take m such that $N^m = 0$ on D_2 and define

$$\begin{aligned} & L(m) := K_0(0) \oplus K_0(1) \oplus \cdots \oplus K_0(m), \quad N : (a_0, \ldots, a_m) \mapsto (a_1, \ldots, a_m, 0), \\ & f : D_2 \to D_3 := D_2 \otimes_{K_0} L(m), \quad a \mapsto (a, 0, \ldots, 0). \end{aligned}$$

It is easy to check (by induction on m) that, for every $x \in D_2$ there exists a $y \in D_3$, $y = (0, y_1, \ldots, y_m)$ such that $Ny = x$ in D_3. It follows that the same property holds for the map $\delta_2' : \mathrm{Hom}_{K_0}(D_1, \varphi_* D_2) \to \mathrm{Hom}_{K_0}(D, \varphi_* D_2)$, killing H^2 as wanted.

To treat H^1, pass first to D_3 as above so that, for every class $x \in H^1 \mathrm{Hom}^\sharp_{\varphi,N}(D_1, D_2)$, $f(x)$ can be represented by an element $y \in \mathrm{Hom}_{K_0}(D_1, \varphi_* D_3^{N=0})$. By Lemma 2.3, there exists a finite φ-module M, where φ is an isomorphism, and an embedding $D_3^{N=0} \hookrightarrow M$ such that the image of $\mathrm{Hom}_{K_0}(D_1, \varphi_* D_3^{N=0}) \to \mathrm{Hom}_{K_0}(D_1, \varphi_* M)$ is in the image of $\mathrm{Hom}_{K_0}(D_1, M)$ by the map δ_1. Note that $\delta_2 = 0$ on $\mathrm{Hom}_{K_0}(D_1, M)$. It follows that the pushout $D_4 = D_3 \coprod_{D_3^{N=0}} M$ kills $H^1 \mathrm{Hom}^\sharp_{\varphi,N}(D_1, D_2)$, i..e, that the image of the map $H^1 \mathrm{Hom}^\sharp_{\varphi,N}(D_1, D_2) \to H^1 \mathrm{Hom}^\sharp_{\varphi,N}(D_1, D_4)$ is zero. This concludes our proof.

∎

2.6. A filtered (φ, N)-module is a tuple $(D_0, \varphi, N, F^\bullet)$, where (D_0, φ, N) is a (φ, N)-module and $F^\bullet$ is a decreasing finite filtration of $D_K := D_0 \otimes_{K_0} K$ by K-vector spaces. There is a notion of a *(weakly) admissible* filtered (φ, N)-module [20]. Denote by

$$MF_K^{\mathrm{ad}}(\varphi, N) \subset MF_K(\varphi, N) \subset M_{K_0}(\varphi, N)$$

the categories of admissible filtered (φ, N)-modules, filtered (φ, N)-modules, and (φ, N)-modules, respectively. We know [20] that the pair of functors

$$D_{\mathrm{st}}(V) = (\mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{G_K}, \quad D_K(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K};$$
$$V_{\mathrm{st}}(D) = (\mathbf{B}_{\mathrm{st}} \otimes_{K_0} D_0)^{\varphi = \mathrm{Id}, N=0} \cap F^0(\mathbf{B}_{\mathrm{dR}} \otimes_K D_K)$$

defines an equivalence of categories $MF_K^{\mathrm{ad}}(\varphi, N) \simeq \mathrm{Rep}_{\mathrm{st}}(G_K) \subset \mathrm{Rep}(G_K)$, where the last two categories denote the subcategory of semistable Galois representations [32] of the category of finite dimensional $\mathbf{Q}_p$-linear representations of the Galois group G_K. The rings $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{dR}}$ are the semistable and de Rham period rings of Fontaine [32]. The category $MF_K^{\mathrm{ad}}(\varphi, N)$ is naturally a Tannakian tensor $\mathbf{Q}_p$-category and $(D_0, \varphi, N, F^\bullet) \mapsto D_0$ is a fiber functor over K_0.

A filtered (φ, N, G_K)-module is a tuple $(D_0, \varphi, N, \rho, F^\bullet)$, where

1. D_0 is a finite dimensional K_0^{nr}-vector space;
2. $\varphi : D_0 \to D_0$ is a Frobenius map;
3. $N : D_0 \to D_0$ is a K_0^{nr}-linear monodromy map such that $N\varphi = p\varphi N$;
4. ρ is a K_0^{nr}-semilinear G_K-action on D (hence $\rho|I_K$ is linear) that is *smooth*, i.e., all vectors have open stabilizers, and that commutes with φ and N;
5. $F^\bullet$ is a decreasing finite filtration of $D_K := (D \otimes_{K_0^{\mathrm{nr}}} \overline{K})^{G_K}$ by K-vector spaces.

Morphisms between filtered (φ, N, G_K)-modules are K_0^{nr}-linear maps preserving all structures. There is a notion of a *(weakly) admissible* filtered (φ, N, G_K)-module [20], [33]. Denote by

$$DF_K := MF_K^{\mathrm{ad}}(\varphi, N, G_K) \subset MF_K(\varphi, N, G_K) \subset M_K(\varphi, N, G_K)$$

the categories of admissible filtered (φ, N, G_K)-modules (DF stands for Dieudonné-Fontaine), filtered (φ, N, G_K)-modules, and (φ, N, G_K)-modules, respectively. The last category is built from tuples (D_0, φ, N, ρ) having properties 1, 2, 3, 4 above. We know [20] that the pair of functors

$$D_{\mathrm{pst}}(V) = \operatorname{inj\,lim}_H (\mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^H, \quad H \subset G_K - \text{an open subgroup}, \quad D_K(V) := (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}})^{G_K};$$
$$V_{\mathrm{pst}}(D) = (\mathbf{B}_{\mathrm{st}} \otimes_{K_0^{\mathrm{nr}}} D_0)^{\varphi = \mathrm{Id}, N=0} \cap F^0(\mathbf{B}_{\mathrm{dR}} \otimes_K D_K)$$

define an equivalence of categories $MF_K^{\mathrm{ad}}(\varphi, N, G_K) \simeq \mathrm{Rep}_{\mathrm{pst}}(G_K)$, where the last category denotes the category of potentially semistable Galois representations [32]. We have the abstract period isomorphisms

$$\rho_{\mathrm{pst}} : D_{\mathrm{pst}}(V) \otimes_{K_0^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}} \simeq V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}}, \quad \rho_{\mathrm{dR}} : D_K(V) \otimes_K \mathbf{B}_{\mathrm{dR}} \simeq V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}, \tag{2.2}$$

where the first one is compatible with the action of φ, N, and G_K, and the second one is compatible with filtration. The category MF_K^{pst} is naturally a Tannakian tensor $\mathbf{Q}_p$-category and $(D_0, \varphi, N, \rho, F^\bullet) \mapsto D_0$ is a fiber functor over K_0^{nr}. We will denote by $\mathscr{D}^b(DF_K)$ and $D^b(DF_K)$ its bounded derived dg category and bounded derived category, respectively.

The category $M_K(\varphi, N, G_K)$ is abelian. We will denote by $\mathscr{D}^b_K(\varphi, N, G_K)$ and $D^b_K(\varphi, N, G_K)$ its bounded derived dg category and bounded derived category, respectively. For (φ, N, G_K)-modules M, T, let $\mathrm{Hom}_{\varphi, N, G_K}(M, T)$ be the group of (φ, N, G_K)-module morphisms and let $\mathrm{Hom}_{G_K}(M, T)$ be the group of K_0^{nr}-linear and G_K- equivariant morphisms. Let $\mathrm{Hom}^\sharp_{\varphi, N, G_K}(M, T)$ be the complex

$$\mathrm{Hom}_{G_K}(M, T) \to \mathrm{Hom}_{G_K}(M, \varphi_* T) \oplus \mathrm{Hom}_{G_K}(M, T) \to \mathrm{Hom}_{G_K}(M, \varphi_* T).$$

This complex is supported in degrees $0, 1, 2$ and the differentials are as above for (φ, N)-modules. Clearly, we have $\mathrm{Hom}_{\varphi, N, G_K}(M, T) = H^0 \mathrm{Hom}^\sharp_{\varphi, N, G_K}(M, T)$. Complexes $\mathrm{Hom}^\sharp_{\varphi, N, G_K}$ compose naturally. Arguing as in the proof of Lemma 2.5, we can show that, for $M, T \in \mathscr{D}^b_K(\varphi, N, G_K)$,

$$\mathrm{R}\,\mathrm{Hom}_{\varphi, N, G_K}(M, T) \simeq \mathrm{Hom}^\sharp_{\varphi, N, G_K}(M, T). \tag{2.3}$$

Let M, T be two complexes in $C^b(MF_K(\varphi,N,G_K))$. Define the complex $\operatorname{Hom}^\flat(M,T)$ as the following homotopy fiber

$$\operatorname{Hom}^\flat(M,T) := \operatorname{Cone}(\operatorname{Hom}^\sharp_{\varphi,N,G_K}(M_0,T_0)\oplus \operatorname{Hom}_{\mathrm{dR}}(M_K,T_K)\xrightarrow{\mathrm{can}-\mathrm{can}}\operatorname{Hom}_{G_K}(M_{\overline{K}},T_{\overline{K}}))[-1],$$

where $\operatorname{Hom}_{\mathrm{dR}}(M_K,T_K)$ is the group of filtered K-linear morphisms and $\operatorname{Hom}_{G_K}(M_{\overline{K}},T_{\overline{K}})$ is the group of G_K-equivariant, $\overline{K}$-linear morphisms. Complexes $\operatorname{Hom}^\flat$ compose naturally.

Proposition 2.7. *We have* $\mathrm{R}\operatorname{Hom}_{DF_K}(M,T)\simeq \operatorname{Hom}^\flat(M,T)$.

Proof. We follow the method of proof of Beilinson and Bannai [8, Lemma 1.7], [6, Prop. 1.7]. Denote by $f_{M,T}$ the morphism in the cone defining $\operatorname{Hom}^\flat(M,T)$. We have the distinguished triangle

$$\ker(f_{M,T})\to \operatorname{Hom}^\flat(M,T)\to \operatorname{coker}(f_{M,T})[-1]$$

We also have the functorial isomorphism

$$\operatorname{Hom}_{K^b(DF_K)}(M,T[i])\xrightarrow{\sim} H^i(\ker(f_{M,T}))$$

Hence a long exact sequence

$$\to H^{i-2}(\operatorname{coker}(f_{M,T}))\to \operatorname{Hom}_{K^b(DF_K)}(M,T[i])\to H^i(\operatorname{Hom}^\flat(M,T))\to H^{i-1}(\operatorname{coker}(f_{M,T}))\to$$

Let I_T be the category whose objects are quasi-isomorphisms $s:T\to L$ in $K^b(DF_K)$ and whose morphisms are morphisms $L\to L'$ in $K^b(DF_K)$ compatible with s. Since $\operatorname{inj\,lim}_{I_T}\operatorname{Hom}_{K^b(DF_K)}(M,L[i]) = \operatorname{Hom}_{D(DF_K)}(M,T[i])$, it suffices to show that $\operatorname{inj\,lim}_{I_T} H^i(\operatorname{Hom}^\flat(M,L)) = H^i(\operatorname{Hom}^\flat(M,T))$ and that $\operatorname{inj\,lim}_{I_T} H^i(\operatorname{coker}(f_{M,L}))=0$. The first fact follows from Lemma 2.5 and the second one from the Lemma 2.8 below. □

Lemma 2.8. *Let* $u\in \operatorname{Hom}^j_{G_K}(M_{\overline{K}},T_{\overline{K}})$. *There exists a complex* $E\in C^b(DF_K)$ *and a quasi-isomorphism* $T\to E$ *such that the image of* u *in the cokernel of the map* f *is zero.*

Proof. We will construct an extension

$$0\to T\to E\to \operatorname{Cone}(M\xrightarrow{\mathrm{Id}} M)[-j-1]\to 0$$

in the category of filtered (φ,N,G_K)-modules. Since the category of admissible modules is closed under extension, E will be admissible. The underlying complex of K_0^{nr}-vector spaces is

$$E_0 := \operatorname{Cone}(M_0[-j-1]\xrightarrow{(0,\mathrm{Id})} T_0\oplus M_0[-j-1])$$

The Frobenius, monodromy operator, and Galois action are defined on $E_0^{i+j} := T_0^{i+j}\oplus M_0^{i-1}\oplus M_0^i$ coordinatewise. The filtration on $E_K^{i+j} := E_0^{i+j}\otimes_{K_0^{\mathrm{nr}}}\overline{K}$ is defined as

$$\begin{aligned} F^nE_K^{i+j} &= F^nT_K^j\oplus\{(u^i(x),0,x)\mid x\in F^nM_K^i\}\\ &\qquad\oplus\{(d_T(u^{i-1}(x)),-x,-d_M(x))\mid x\in F^nM_K^{i-1}\}\end{aligned}$$

Now take $\xi=(0,0,\mathrm{Id})+(u^i,0,\mathrm{Id})\in \operatorname{Hom}^\sharp_{\varphi,N,G_K}(M_0^i,E_0^{i+j})\oplus\operatorname{Hom}_{\mathrm{dR}}(M_K^i,E_K^{i+j})$. We have $f(\xi) = (u^i,0,0)$, as wanted. □

2.2. **The category of p-adic Hodge complexes.**

2.9. Let $V^G_{\overline{K}}$ be the category of $\overline{K}$-vector spaces with a smooth $\overline{K}$-semilinear action of G_K. It is a Grothendieck abelian category. We will consider the following functors:

- $F_{\mathrm{dR}}: V^K_{\mathrm{dR}}\to V^G_{\overline{K}}$, which to a filtered K-vector space $(E,F^\bullet)$ associates the $\overline{K}$-vector space $E\otimes_K\overline{K}$ with its natural action of G_K.
- $F_0: M_K(\varphi,N,G_K)\to V^G_{\overline{K}}$, which to a (φ,N,G_K)-module M associates the $\overline{K}$-vector space $M\otimes_{K_0^{\mathrm{nr}}}\overline{K}$ whose G_K-action is induced by the given G_K-action on M.

Both functors are exact and monoidal. Note in particular that they induces functors on the respective categories of complexes which are dg-functors.

2.10. Let $\mathscr{D}^b(V_K^G)$ and $D^b(V_K^G)$ denote the bounded derived dg category and the bounded derived category of V_K^G, respectively. We define the dg category $\mathscr{D}_{pH}$ of p-adic Hodge complexes as the homotopy limit

$$\mathscr{D}_{pH} := \operatorname{holim}(\mathscr{D}^b(M_K(\varphi, N, G_K)) \stackrel{F_0}{\to} \mathscr{D}^b(V_K^G) \stackrel{F_{\mathrm{dR}}}{\leftarrow} \mathscr{D}^b(V_{\mathrm{dR}}^K))$$

We denote by D_{pH} the homotopy category of $\mathscr{D}_{pH}$. By [63, Def. 3.1], [13, 4.1], an object of $\mathscr{D}_{pH}$ consists of objects $M_0 \in \mathscr{D}^b(M_K(\varphi, N, G_K))$, $M_K \in \mathscr{D}^b(V_{\mathrm{dR}}^K)$, and a quasi-isomorphism

$$F_0(M_0) \stackrel{a_M}{\to} F_{\mathrm{dR}}(M_K)$$

in $\mathscr{D}(V_K^G)$. We will denote the object above by $M = (M_0, M_K, a_M)$. The morphisms are given by the complex $\operatorname{Hom}_{\mathscr{D}_{pH}}((M_0, M_K, a_M), (N_0, N_K, a_N))$:

(2.4)

$$\begin{aligned}
&\operatorname{Hom}^i_{\mathscr{D}_{pH}}((M_0, M_K, a_M), (N_0, N_K, a_N)) \\
&\quad = \operatorname{Hom}^i_{\mathscr{D}^b(M_K(\varphi,N,G_K))}(M_0, N_0) \oplus \operatorname{Hom}^i_{\mathscr{D}^b(V_{\mathrm{dR}}^K)}(M_K, N_K) \oplus \operatorname{Hom}^{i-1}_{\mathscr{D}^b(V_K^G)}(F_0(M_0), F_{\mathrm{dR}}(N_K))
\end{aligned}$$

The differential is given by

$$d(a, b, c) = (da, db, dc + a_N F_0(a) - (-1)^i F_{\mathrm{dR}}(b) a_M)$$

and the composition

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{D}_{pH}}((N_0, N_K, a_N),(T_0, T_K, a_T)) \otimes \operatorname{Hom}_{\mathscr{D}_{pH}}((M_0, M_K, a_M), (N_0, N_K, a_N)) \\
\to \operatorname{Hom}_{\mathscr{D}_{pH}}((M_0, M_K, a_M), (T_0, T_K, a_T))
\end{aligned} \tag{2.5}$$

is given by

$$(a', b', c')(a, b, c) = (a'a, b'b, c'F_0(a) + F_{\mathrm{dR}}(b')c)$$

It now follows easily that a (closed) morphism $(a, b, c) \in \operatorname{Hom}_{\mathscr{D}_{pH}}((M_0, M_K, a_M), (N_0, N_K, a_N))$ is a quasi-isomorphism if and only so are the morphisms a and b (see [13, Lemma 4.2]).

By definition, we get a commutative square of dg categories over $\mathbf{Q}_p$:

$$\begin{array}{ccc}
\mathscr{D}_{pH} & \xrightarrow{T_{\mathrm{dR}}} & \mathscr{D}^b(V_{\mathrm{dR}}^K) \\
{\scriptstyle T_0}\downarrow & & \downarrow{\scriptstyle F_{\mathrm{dR}}} \\
\mathscr{D}^b(M_K(\varphi, N, G_K)) & \xrightarrow{F_0} & \mathscr{D}^b(V_K^G).
\end{array} \tag{2.6}$$

Given a p-adic Hodge complex M, we will call $T_{\mathrm{dR}}(M)$ (resp. $T_0(M)$) the *generic fiber* (resp. *special fiber*) of M. As pointed out above, a morphism f of p-adic Hodge complexes is a quasi-isomorphism if and only if $T_{\mathrm{dR}}(f)$ and $T_0(f)$ are quasi-isomorphisms.

2.11. Let us recall that, since the category $\mathscr{D}_{pH}$ is obtained by gluing, it has a canonical t-structure [36, Prop. 4.1.12]. We will denote by $\mathscr{D}_{pH,\leq 0}$ (resp. $\mathscr{D}_{pH,\geq 0}$) the full dg subcategory of $\mathscr{D}_{pH}$ made of non-positive (resp. non negative) p-adic Hodge complexes. Let M be a p-adic Hodge complex. We define its non positive truncation $\tau_{\leq 0}(M)$ according to the following formula:

$$\tau_{\leq 0}(M) := (\tau_{\leq 0} M_0, \tau_{\leq 0} M_K, \tau_{\leq 0} a_M).$$

The functors F_{dR} and F_0 being exact, this is indeed a p-adic Hodge module. The non negative truncation is obtained using the same formula. According to this definition, we get a canonical morphism of p-adic Hodge complexes:

$$\tau_{\leq 0}(M) \to M$$

whose cone is positive. This is all we need to get that the pair $(\mathscr{D}_{pH,\leq 0}, \mathscr{D}_{pH,\geq 0})$ forms a t-structure on $\mathscr{D}_{pH}$.

Definition 2.12. The t-structure $(\mathscr{D}_{pH,\leq 0}, \mathscr{D}_{pH,\geq 0})$ defined above will be called the canonical t-structure on $\mathscr{D}_{pH}$.

2.13. Let $M\in C^b(MF_K(\varphi,N,G_K))$. Define $\theta(M)\in\mathscr{D}_{pH}$ to be the object

$$\theta(M):=(M_0,M_K,\mathrm{Id}_M:M_{\overline{K}}\simeq M_{\overline{K}})$$

Through this functor we can regard $MF_K(\varphi,N,G_K)$ as a subcategory of the heart of the t-structure on $\mathscr{D}_{pH}$.

Lemma 2.14. *The natural functor*

$$\theta:\quad MF_K(\varphi,N,G_K)\to\mathscr{D}_{pH}^{\heartsuit}$$

is fully faithful.

Proof. Analogous to [36, Prop. 4.1.12], [61, 1.2.27]. □

Definition 2.15. We will say that a strict p-adic Hodge complex M is *admissible* if its cohomology filtered (φ,N,G_K)-modules $H^n(M)$ are (weakly) admissible. Denote by $\mathscr{D}_{pH}^{\mathrm{ad}}$ the full dg subcategory of $\mathscr{D}_{pH}$ of admissible p-adic Hodge complexes. It carries the induced t-structure.

Since θ preserves quasi-isomorphisms, it induces a canonical functor:

$$\theta:\quad\mathscr{D}^b(DF_K)\to\mathscr{D}_{pH}^{\mathrm{ad}}.$$

This is a functor between dg categories compatible with the t-structures.

Lemma 2.16. *The natural functor*

$$\theta:\quad DF_K\xrightarrow{\sim}\mathscr{D}_{pH}^{\mathrm{ad},\heartsuit}$$

is an equivalence of abelian categories.

Proof. By Lemma 2.14, it suffices to prove essential subjectivity. Note that a strict p-adic Hodge complex M is in the heart of the t-structure if and only if M is isomorphic to $\tau_{\leq 0}\tau_{\geq 0}(M)$. According to the formula for this truncation, we get that M is isomorphic to an object $\widetilde{M}$ such that $\widetilde{M}_0$ is a (φ,N,G_K)-module, $\widetilde{M}_K$ is a filtered K-vector space, and one has a G_K-equivariant isomorphism

$$\widetilde{M}_0\otimes_{K_0^{\mathrm{nr}}}\overline{K}\simeq\widetilde{M}_K\otimes_K\overline{K}.$$

In particular, $\widetilde{M}_0$ has the structure of a filtered (φ,N,G_K)-module, as wanted. □

Theorem 2.17. *The functor θ induces an equivalence of dg categories*

$$\theta:\quad\mathscr{D}^b(DF_K)\xrightarrow{\sim}\mathscr{D}_{pH}^{\mathrm{ad}}$$

Proof. Since, by Lemma 2.16, we have the equivalence of abelian categories

$$\theta:\quad DF_K=\mathscr{D}^b(DF_K)^{\heartsuit}\xrightarrow{\sim}\mathscr{D}_{pH}^{\mathrm{ad},\heartsuit}$$

and we work with bounded complexes, it suffices to show that, given two complexes M, M' of $C^b(MF_K^{pst})$, the functor θ induces a quasi-isomorphism:

$$\theta:\quad\mathrm{Hom}_{\mathscr{D}^b(DF_K)}(M,M')\to\mathrm{Hom}_{\mathscr{D}_{pH}}(\theta(M),\theta(M'))$$

By (2.3) and Proposition 2.7, since $F_0(M_0)=F_{\mathrm{dR}}(M_K)=M_{\overline{K}},F_0(M_0')=F_{\mathrm{dR}}(M_K')=M_{\overline{K}}'$, we have the following sequence of quasi-isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathscr{D}_{pH}}(\theta(M),\theta(M'))&=\mathrm{Hom}_{\mathscr{D}_{pH}}((M_0,M_K,\mathrm{Id}_M),(M_0',M_K',\mathrm{Id}_{M'}))\\&\simeq(\mathrm{Hom}_{\mathscr{D}^b(M_K(\varphi,N,G_K))}(M_0,M_0')\xrightarrow{F_0}\mathrm{Hom}_{\mathscr{D}^b(V_{\overline{K}}^G)}(M_{\overline{K}},M_{\overline{K}}')\xleftarrow{F_{\mathrm{dR}}}\mathrm{Hom}_{\mathscr{D}^b(V_{\mathrm{dR}}^K)}(M_K,M_K'))\\&\simeq(\mathrm{Hom}^{\sharp}_{\varphi,N,G_K}(M_0,M_0')\xrightarrow{F_0}\mathrm{Hom}_{G_K}(M_{\overline{K}},M_{\overline{K}}')\xleftarrow{F_{\mathrm{dR}}}\mathrm{Hom}_{\mathrm{dR}}(M_K,M_K'))\\&\simeq\mathrm{Hom}_{\mathscr{D}^b(DF_K)}(M,M').\end{aligned}$$

This concludes our proof. □

2.3. **The absolute p-adic Hodge cohomology.**

2.18. Any potentially semistable p-adic representation is a p-adic Hodge complex. Therefore, we can define the Tate twist in $\mathscr{D}_{pH}$ as follows: given any integer $r \in \mathbf{Z}$, we let $K(r)$ be the p-adic Hodge complex

$$K(-r) = (K_0^{\mathrm{nr}}, K, \mathrm{Id}_K : \overline{K} \xrightarrow{\sim} \overline{K})$$

that is equal to K_0^{nr} and K concentrated in degree 0; the Frobenius is $\varphi_{K(-r)}(a) = p^r\varphi(a)$, the Galois action is canonical and the monodromy operator is zero; the filtration is $F^i = K$ for $i \leq r$ and zero otherwise.

As usual, given any p-adic Hodge complex M, we put $M(r) := M \otimes K(r)$. In other words, twisting a p-adic Hodge complex r-times divides the Frobenius by p^r, leaves unchanged the monodromy operator, and shifts the filtration r-times.

Example 2.19. Given any p-adic Hodge complex M, by formula (2.5) and by (2.3), we have the quasi-isomorphism of complexes of $\mathbf{Q}_p$-vector spaces

$$\mathrm{Hom}_{\mathscr{D}_{pH}}(K(0), M(r)) \simeq \mathrm{Cone}(M_0^\sharp \oplus F^r M_K \xrightarrow{a_M - \mathrm{can}} F_{\mathrm{dR}}(M_K)^{G_K})[-1],$$

where $M_0^\sharp$ is defined as the following homotopy limit (we set $\varphi_i := \varphi/p^i$)

$$M_0^\sharp := \left[\begin{array}{ccc} M_0^{G_K} & \xrightarrow{1-\varphi_r} & M_0^{G_K} \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} \\ M_0^{G_K} & \xrightarrow{1-\varphi_{r-1}} & M_0^{G_K} \end{array}\right]$$

2.20. Let X be a variety over K. Consider the following complex in $\mathscr{D}_{pH}^{\mathrm{ad}}$

$$\mathrm{R}\Gamma_{pH}(X_{\overline{K}}, 0) := (\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}}), (\mathrm{R}\Gamma_{\mathrm{dR}}(X), F^\bullet), \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}}) \xrightarrow{\iota_{\mathrm{dR}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}}))$$

Here $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}})$ is the (geometric) Beilinson-Hyodo-Kato cohomology [10], [50, 3.4]; by definition it is a bounded complex of (φ, N, G_K)-modules. The filtered complex $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$ is the Deligne de Rham cohomology. The map ι_{dR} is the Beilinson-Hyodo-Kato map [10] that induces a quasi-isomorphism

$$\iota_{\mathrm{dR}}: \quad \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{K_0^{\mathrm{nr}}} \overline{K} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}}).$$

The comparison theorems of p-adic Hodge theory (proved in [31], [65], [53], [10], [14]) imply that the p-adic Hodge complex $\mathrm{R}\Gamma_{pH}(X_{\overline{K}}, 0)$ is admissible.

We will denote by

$$\mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r) := \mathrm{R}\Gamma_{pH}(X_{\overline{K}}, 0)(r) \in \mathscr{D}_{pH}^{\mathrm{ad}}$$

the r'th Tate twist of $\mathrm{R}\Gamma_{pH}(X_{\overline{K}}, 0)$. We will call it *the geometric p-adic Hodge cohomology of X*. Since the Beilinson-Hyodo-Kato map is a map of dg K_0^{nr}-algebras, the assignment $X \mapsto \mathrm{R}\Gamma_{pH}(X_{\overline{K}}, *)$ is a presheaf of dg algebras on $\mathscr{V}ar_K$. Moreover, we also have the external product $\mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r) \otimes \mathrm{R}\Gamma_{pH}(Y_{\overline{K}}, s)$ in $\mathscr{D}_{pH}^{\mathrm{ad}}$.

Lemma 2.21 (Künneth formula)**.** *The natural map*

$$\mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r) \otimes \mathrm{R}\Gamma_{pH}(Y_{\overline{K}}, s) \xrightarrow{\sim} \mathrm{R}\Gamma_{pH}(X_{\overline{K}} \times Y_{\overline{K}}, r+s)$$

is a quasi-isomorphism.

Proof. This follows easily from the Künneth formulas in the filtered de Rham cohomology and the Hyodo-Kato cohomology (use the Hyodo-Kato map to pass to de Rham cohomology). $\square$

Set

$$\mathrm{R}\Gamma_{DF_K}(X_{\overline{K}}, r) := \theta^{-1}\mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r) \in \mathscr{D}^b(DF_K),$$
$$\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r) := V_{\mathrm{pst}}\theta^{-1}\mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r) \in \mathscr{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K)).$$

Lemma 2.22. *There exists a canonical quasi-isomorphism in* $\mathscr{D}^b(\mathrm{Rep}(G_K))$

$$\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r)\simeq \mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p(r)).$$

Proof. To start, we note that we have the following commutative diagram of dg categories.

$$\begin{tikzcd}
\mathscr{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K)) \arrow[rr, "\mathrm{can}"] \arrow[d, shift left, "D_{\mathrm{pst}}"] & & \mathscr{D}^b(\mathrm{Rep}(G_K)) \arrow[d, "\mathrm{can}"] \\
\mathscr{D}^b(DF_K) \arrow[r, "\theta"] \arrow[u, shift left, "V_{\mathrm{pst}}"] & \mathscr{D}^{\mathrm{ad}}_{pH} \arrow[r, "r_{\text{ét}}"] & \mathscr{D}(\mathrm{Spec}(K)_{\text{proét}})
\end{tikzcd}$$

Here the functor

$$r_{\text{ét}}:\quad \mathscr{D}^{\mathrm{ad}}_{pH}\to \mathscr{D}(\mathrm{Spec}(K)_{\text{proét}})$$

associates to a p-adic Hodge complex $(M_0,M_K,a_M:F_0(M_0)\to F_{\mathrm{dR}}(M_K))$ the complex

$$\begin{aligned}
&[[M_0\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}]^{\varphi=\mathrm{Id},N=0}\oplus F^0(M_K\otimes_K\mathbf{B}_{\mathrm{dR}})\xrightarrow{a_M\otimes\iota-\mathrm{can}\otimes\iota}F_{\mathrm{dR}}(M_K)\otimes_{\overline{K}}\mathbf{B}_{\mathrm{dR}}]\\
&=[[M_0\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}]^{\varphi=\mathrm{Id},N=0}\xrightarrow{a_M\otimes\iota}(F_{\mathrm{dR}}(M_K)\otimes_{\overline{K}}\mathbf{B}_{\mathrm{dR}})/F^0]
\end{aligned}$$

where $\iota:\mathbf{B}_{\mathrm{st}}\hookrightarrow\mathbf{B}_{\mathrm{dR}}$ is the canonical map of period rings[3]. To see that the diagram commutes, recall that we have the fundamental exact sequence

$$0\to\mathbf{Q}_p(r)\to\mathbf{B}_{\mathrm{st}}^{\varphi=p^r,N=0}\oplus F^r\mathbf{B}_{\mathrm{dR}}\xrightarrow{\iota}\mathbf{B}_{\mathrm{dR}}\to 0,\quad r\in\mathbf{N}. \tag{2.7}$$

It follows that, for $V\in\mathscr{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))$, we have a canonical morphism

$$\begin{aligned}
V&\simeq[V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}}^{\varphi=\mathrm{Id},N=0}\oplus V\otimes_{\mathbf{Q}_p}F^0\mathbf{B}_{\mathrm{dR}}\xrightarrow{\mathrm{Id}\otimes\iota-\mathrm{can}\otimes\iota}V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}]\\
&\simeq[[V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}}]^{\varphi=\mathrm{Id},N=0}\oplus V\otimes_{\mathbf{Q}_p}F^0\mathbf{B}_{\mathrm{dR}}\xrightarrow{\mathrm{Id}\otimes\iota-\mathrm{can}\otimes\iota}V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}]\\
&\xrightarrow{(\rho_{\mathrm{pst}}\oplus\rho_{\mathrm{dR}},\rho_{\mathrm{dR}})}[[D_{\mathrm{pst}}(V)\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}]^{\varphi=\mathrm{Id},N=0}\oplus F^0(D_K(V)\otimes_K\mathbf{B}_{\mathrm{dR}})\xrightarrow{a_M\otimes\iota-\mathrm{can}\otimes\iota}D_K(V)\otimes_K\mathbf{B}_{\mathrm{dR}}]\\
&\simeq r_{\text{ét}}\theta D_{\mathrm{pst}}(V).
\end{aligned}$$

Since the abstract period morphisms $\rho_{\mathrm{pst}},\rho_{\mathrm{dR}}$ from (2.2) are isomorphisms, the above morphism is a quasi-isomorphism and we have the commutativity we wanted.

The above diagram gives us the first quasi-isomorphism in the formula

$$\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r)\simeq r_{\text{ét}}\mathrm{R}\Gamma_{pH}(X_{\overline{K}},r)\simeq\mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p(r)). \tag{2.8}$$

It suffices now to prove the second quasi-isomorphism. But, we have

$$\mathrm{R}\Gamma_{pH}(X_{\overline{K}},r)=(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}},r),(\mathrm{R}\Gamma_{\mathrm{dR}}(X),F^{\bullet+r}),\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}},r)\otimes_{K_0^{\mathrm{nr}}}\overline{K}\xrightarrow{\iota_{\mathrm{dR}}}\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})),$$

where we twisted the Beilinson-Hyodo-Kato cohomology to remember the Frobenius twist. Recall that Beilinson has constructed period morphisms (of dg-algebras) [9, 3.6], [10, 3.2] [4]

$$\begin{aligned}
\rho_{\mathrm{pst}}:&\quad \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}})\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}\simeq\mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}},\\
\rho_{\mathrm{dR}}:&\quad \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})\otimes_{\overline{K}}\mathbf{B}_{\mathrm{dR}}\simeq\mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}.
\end{aligned}$$

The first morphism is compatible with Frobenius, monodromy, and G_K-action; the second one - with filtration. These morphisms allow us to define a quasi-isomorphism

$$\beta:\quad r_{\text{ét}}\mathrm{R}\Gamma_{pH}(X_{\overline{K}},r)\simeq\mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p(r))$$

[3] For an explanation why we work with the pro-étale site as well as the technicalities involved in the passage between continuous Galois cohomology and pro-étale cohomology see [50, proof of Theorem 4.8].

[4] We will be using consistently Beilinson's definition of the period maps. It is likely that the uniqueness criterium stated in [54] can be used to show that these maps coincide with the other existing ones.

in $\mathscr{D}(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}})$ as the composition

$$\begin{aligned}\beta: \quad r_{\mathrm{\acute{e}t}}\mathrm{R}\Gamma_{pH}(X_{\overline{K}},r) &= [[\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}})\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}]^{\varphi=p^r,N=0}\xrightarrow{\iota_{\mathrm{dR}}}(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})\otimes_{\overline{K}}\mathbf{B}_{\mathrm{dR}})/F^r]\\ &\xrightarrow{(\rho_{\mathrm{HK}},\rho_{\mathrm{dR}})}[\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}}^{\varphi=p^r,N=0}\xrightarrow{\iota}\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}(\mathbf{B}_{\mathrm{dR}})/F^r]\\ &\stackrel{\sim}{\leftarrow}\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbf{Q}_p(r))\end{aligned}$$

Here the last quasi-isomorphism follows from the fundamental exact sequence (2.7).

To finish we note that the quasi-isomorphism in (2.8) come from quasi-isomorphisms between complexes of continuous representations of G_K on (locally convex) $\mathbf{Q}_p$-vector spaces. □

Remark 2.23. The geometric p-adic Hodge cohomology $\mathrm{R}\Gamma_{pH}(X_{\overline{K}},r)$ we work with here is not the same as the geometric syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h},r)$ defined in [50]. While the first one, by the above lemma, represents the étale cohomology $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbf{Q}_p(r))$, the second one represents only its piece, i.e., we have $\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h},r)\simeq\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbf{Q}_p(r))$.

2.24. The *p-adic absolute Hodge cohomology of X* (also called *syntomic cohomology* of X if this does not cause confusion) is defined as

$$\mathrm{R}\Gamma_{\mathscr{H}}(X,r)=\mathrm{R}\Gamma_{\mathrm{syn}}(X,r):=\mathrm{Hom}_{\mathscr{D}_{pH}}(K(0),\mathrm{R}\Gamma_{pH}(X_{\overline{K}},r)). \tag{2.9}$$

By Theorem 2.17, we have

$$\begin{aligned}\mathrm{R}\Gamma_{\mathscr{H}}(X,r)&\simeq\mathrm{Hom}_{\mathscr{D}^b(DF_K)}(K(0),\mathrm{R}\Gamma_{DF_K}(X_{\overline{K}},r))\\ &\simeq\mathrm{Hom}_{\mathscr{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))}(\mathbf{Q}_p,\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r)).\end{aligned}$$

The assignment $X\mapsto\mathrm{R}\Gamma_{\mathscr{H}}(X,r)=\mathrm{R}\Gamma_{\mathrm{syn}}(X,*)$ is a presheaf of dg $\mathbf{Q}_p$-algebras on $\mathscr{V}ar_K$.

Set $H^i_{\mathrm{syn}}(X,r):=H^i\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)$.

Theorem 2.25. (1) *There is a functorial syntomic descent spectral sequence*

$${}^{\mathrm{syn}}E^{i,j}:=H^i_{\mathrm{st}}(G_K,H^j_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbf{Q}_p(r)))\Rightarrow H^{i+j}_{\mathrm{syn}}(X,r), \tag{2.10}$$

where $H^i_{\mathrm{st}}(G_K,\cdot)$ is the group of (potentially) semistable extensions $\mathrm{Ext}^i_{\mathrm{Rep}_{\mathrm{pst}}(G_K)}(\mathbf{Q}_p,\cdot)$ as defined in [35, 1.19].

(2) *There is a functorial syntomic period morphism*

$$\rho_{\mathrm{syn}}:\quad\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\to\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X,\mathbf{Q}_p(r)).$$

(3) *The syntomic descent spectral sequence is compatible with the Hochschild-Serre spectral sequence*

$${}^{\mathrm{\acute{e}t}}E_2^{i,j}=H^i(G_K,H^j_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbf{Q}_p(r)))\Rightarrow H^{i+j}_{\mathrm{\acute{e}t}}(X,\mathbf{Q}_p(r)). \tag{2.11}$$

More specifically, there is a natural map ${}^{\mathrm{syn}}E_2^{i,j}\to{}^{\mathrm{\acute{e}t}}E_2^{i,j}$ that is compatible with the syntomic period map ρ_{syn}.

Proof. From the definition (2.9) of $\mathrm{R}\Gamma_{pH}(X_{\overline{K}},r)$ we obtain the following spectral sequence

$$E_2^{i,j}=\mathrm{Ext}^i_{\mathrm{Rep}_{\mathrm{pst}}(G_K)}(\mathbf{Q}_p,H^j\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r))\Rightarrow H^{i+j}\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)$$

Since, by Lemma 2.22, we have $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r))\simeq\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbf{Q}_p(r))$, the first statement of our theorem follows.

We define the syntomic period map $\rho_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\to\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X,\mathbf{Q}_p(r))$ as the composition

$$\begin{aligned}\rho_{\mathrm{syn}}:\quad\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)=\mathrm{Hom}_{\mathscr{D}_{pH}}(K(0),\mathrm{R}\Gamma_{pH}(X_{\overline{K}},r)))&\stackrel{r_{\mathrm{\acute{e}t}}}{\to}\mathrm{Hom}_{\mathscr{D}(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}})}(\mathbf{Q}_p,r_{\mathrm{\acute{e}t}}\mathrm{R}\Gamma_{pH}(X_{\overline{K}},r)))\\ &\stackrel{\beta}{\to}\mathrm{Hom}_{\mathscr{D}(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}})}(\mathbf{Q}_p,\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbf{Q}_p(r)))=\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X,\mathbf{Q}_p(r)).\end{aligned}$$

The second statement of the theorem follows.

Finally, since the Hochschild-Serre spectral sequence

$${}^{\mathrm{\acute{e}t}}E_2^{i,j}:=H^i(G_K,H^j(X_{\overline{K}},\mathbf{Q}_p(r)))\Rightarrow H^{i+j}(X,\mathbf{Q}_p(r))$$

can be identified with the spectral sequence

$$^{\text{ét}}E_2^{i,j} := H^i(\operatorname{Spec}(K)_{\text{proét}}, H^j(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H^{i+j}(X, \mathbf{Q}_p(r))$$

we get that the syntomic descent spectral sequence is compatible with the Hochschild-Serre spectral sequence via the map ρ_{syn}, as wanted. □

Theorem 2.26. *Let* $\mathrm{R}\Gamma_{\text{syn}}(X_h, r)$ *be the syntomic cohomology defined in* [50, 3.3]. *There exists a natural quasi-isomorphism (in the classical derived category)*

$$\mathrm{R}\Gamma_{\text{syn}}(X_h, r) \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{syn}}(X, r), \quad r \geq 0.$$

It is compatible with syntomic period morphisms and the syntomic as well as the étale descent spectral sequences.

Proof. Let $r \geq 0$. Recall that we have a natural quasi-isomorphism [50, Prop. 3.20]

$$\mathrm{R}\Gamma_{\text{syn}}(X_h, r) \simeq \operatorname{Cone}(\mathrm{R}\Gamma^B_{\text{HK}}(X)^{\varphi,N} \oplus F^r\mathrm{R}\Gamma_{\text{dR}}(X) \xrightarrow{\iota_{\text{dR}}-\text{can}} \mathrm{R}\Gamma_{\text{dR}}(X))[-1],$$

where

$$\mathrm{R}\Gamma^B_{\text{HK}}(X_h)^{\varphi,N} := \left[\begin{array}{ccc} \mathrm{R}\Gamma^B_{\text{HK}}(X) & \xrightarrow{1-\varphi_r} & \mathrm{R}\Gamma^B_{\text{HK}}(X) \\ \downarrow N & & \downarrow N \\ \mathrm{R}\Gamma^B_{\text{HK}}(X) & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma^B_{\text{HK}}(X) \end{array}\right]$$

and the complex $\mathrm{R}\Gamma^B_{\text{HK}}(X)$ is the (arithmetic) Beilinson-Hyodo-Kato cohomology [10] that comes equipped with the Beilinson-Hyodo-Kato map $\iota_{\text{dR}} : \mathrm{R}\Gamma^B_{\text{HK}}(X) \to \mathrm{R}\Gamma_{\text{dR}}(X)$ [50, 3.3].

Since $\mathrm{R}\Gamma^B_{\text{HK}}(X) \simeq \mathrm{R}\Gamma^B_{\text{HK}}(X_{\overline{K}})^{G_K}$ and $\mathrm{R}\Gamma_{\text{dR}}(X) \simeq \mathrm{R}\Gamma_{\text{dR}}(X_{\overline{K}})^{G_K}$ by [50, Prop. 3.22], Example 2.19 and Theorem 2.17 yield

$$\mathrm{R}\Gamma_{\text{syn}}(X_h, r) \simeq \operatorname{Hom}_{\mathscr{D}_{pH}}(K(0), \mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r))) \simeq \operatorname{Hom}_{\mathscr{D}^b(DF_K)}(K(0), \mathrm{R}\Gamma_{DF_K}(X_{\overline{K}}, r))) \simeq \mathrm{R}\Gamma_{\text{syn}}(X, r),$$

as wanted. The last claim of the theorem is now clear. □

Remark 2.27. The above theorems gives an alternative construction of the syntomic descent spectral sequence from [50, 4.2] (that construction used the geometric syntomic cohomology mentioned in Remark 2.23) and an alternative proof of its compatibility with the Hochschild-Serre spectral sequence [50, Theorem 4.8]. In the present approach the syntomic descent spectral sequence is a genuine descent spectral sequence: from geometric étale cohomology to syntomic cohomology. In the approach of [50] this sequence appears as a piece of a larger descent spectral sequence that remains to be understood.

Remark 2.28. In everything above, the variety X can be replaced by a finite simplicial scheme or a finite diagram of schemes. In particular, we obtain statements about cohomology with compact support: use resolutions of singularities to get a compactification of the variety with a divisor with normal crossing at infinity and then represent cohomology with compact support as a cohomology of a finite simplicial scheme built from the closed strata. In particular, we get the syntomic descent spectral sequence with compact support:

$$^{\text{syn},c}E_2^{i,j} := H^i_{\text{st}}\big(G_K, H^j_{\text{ét},c}(X_{\overline{K}}, \mathbf{Q}_p(r))\big) \Rightarrow H^{i+j}_{\text{syn},c}(X, r)$$

that is compatible with the Hochschild-Serre spectral sequence for étale cohomology with compact support.

Corollary 2.29. *For X smooth and projective over K, the syntomic descent spectral sequence (2.10)*

$$^{\text{syn}}E_2^{i,j}(r) = H^i_{\text{st}}(G_K, H^j_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H^{i+j}_{\text{syn}}(X, r)$$

degenerates at E_2.

Proof. The argument proceeds along standard lines [25, Thm 1.5]. Let X be a smooth and projective variety over K, of equal dimension d. Recall that we have the Hard Lefschetz Theorem [26, Thm 4.1.1]: if $L \in H^2(X_{\overline{K}}, \mathbf{Q}_p(1))$ is the class of a hyperplane, then for $i \leq d$, the map $L^i : H^{d-i}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \to H^{d+i}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i))$, $a \mapsto a \cup L^i$, is an isomorphism. This gives us the Lefschetz primitive decomposition

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r)) = \oplus_{k\geq 0} L^k H^{i-2k}_{\text{prim}}(X_{\overline{K}}, \mathbf{Q}_p(r-k)), \tag{2.12}$$

where

$$H^a_{\text{prim}}(X_{\overline{K}}, \mathbf{Q}_p(b)) := \operatorname{Ker} L^{d-a+1} \subset H^a_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(b)).$$

Moreover, we get a morphism of spectral sequences

$$L : {}^{\text{syn}}E_2^{i,j}(r) \to {}^{\text{syn}}E_2^{i,j+2}(r+1).$$

Take $s \geq 2$. Assume that the differentials of our spectral sequence $d_2 = \cdots = d_{s-1} = 0$. We want to show that $d_s = 0$. This assumption is trivially true for $s = 2$. By the inductive assumption ${}^{\text{syn}}E_s^{i,j} = {}^{\text{syn}}E_2^{i,j}$. We note that Hard Lefschetz gives us that the differentials

$$d_s : \quad H^j_{\text{st}}(G_K, H^{i-2k}_{\text{prim}}(X_{\overline{K}}, \mathbf{Q}_p(r-k))) \to H^{j+s}_{\text{st}}(G_K, H^{i-2k-s+1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r-k))) \tag{2.13}$$

are trivial. Indeed, we have the following commutative diagram (we set $q = i - 2k$, $t = r - k$, a=d-q+1)

$$\begin{tikzcd}
H^j_{\text{st}}(G_K, H^q_{\text{prim}}(X_{\overline{K}}, \mathbf{Q}_p(t))) \arrow[r, "d_s"] \arrow[d, "L^a=0"] & H^{j+s}_{\text{st}}(G_K, H^{q-s+1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(t))) \arrow[d, "L^a"] \arrow[dd, bend left=60, "\simeq"] \\
H^j_{\text{st}}(G_K, H^{q+2a}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(t+a))) \arrow[r, "d_s"] & H^{j+s}_{\text{st}}(G_K, H^{q-s+1+2a}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(t+a))) \arrow[d, "L^{s-2}"] \\
& H^{j+s}_{\text{st}}(G_K, H^{q+2a+s-3}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(t+a+s-2)))
\end{tikzcd}$$

which implies that the top map d_s is zero. Applying L^k to the differentials in (2.13) we obtain that the differentials

$$d_s : \quad H^j_{\text{st}}(G_K, L^k H^{i-2k}_{\text{prim}}(X_{\overline{K}}, \mathbf{Q}_p(r-k))) \to H^{j+s}_{\text{st}}(G_K, H^{i-s+1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r)))$$

are trivial as well. By (2.12), this gives that $d_s = 0$, as wanted.

Remark 2.30. In fact, we have the Decomposition Theorem, i.e., there is a natural quasi-isomorphism in $D^b(\operatorname{Rep}_{\text{pst}}(G_K))$

$$\bigoplus_i H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)[-i] \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{pst}}(X_{\overline{K}}, \mathbf{Q}_p).$$

Our corollary follows immediately from that.

□

3. A p-adic absolute Hodge cohomology, II: Beilinson's definition

In this section we will describe the definition of p-adic absolute Hodge cohomology due to Beilinson [11]. Beilinson associates to any variety over K a canonical complex of potentially semistable representations of G_K representing the geometric étale cohomology of the variety as a Galois module. Then he defines p-adic absolute Hodge cohomology of this variety as the derived Hom in the category of potentially semistable representations from the trivial representation to this complex.

3.1. **Potentially semistable complex of a variety.**

3.1.1. *Potentially semistable cellular complexes.* The Basic Lemma of Beilinson [7, Lemma 3.3] allows one, in analogy with the cellular complex for CW-complexes, to associate a canonical complex of potentially semistable representations of G_K to any affine variety over K. Recall that the cellular complex associated to a CW-complex X is a complex of singular homology groups

$$\cdots \to H_2^B(X^2, X^1) \stackrel{d_2}{\to} H_1^B(X^1, X^0) \stackrel{d_1}{\to} H_0^B(X^0, \emptyset) \stackrel{d_0}{\to} 0 \tag{3.1}$$

where X^j denotes the j-skeleton of X. The homology of the above complex computes the singular homology of X: we have $H_j^B(X^j/X^{j-1}) \simeq H_j^B(\vee_{|I|} S^j) \simeq \sum_{i\in I} e_i\mathbf{Z}$, I being the index set of j-cells in X.

We will briefly sketch the construction of potentially semistable (cohomological) cellular complexes and we refer interested reader for details to [43], [55], [39].

Definition 3.1.
(1) A *pair* is a triple (X, Y, n), for a closed K-subvariety $Y \subset X$ of a K-variety X and an integer n.
(2) Pair (X, Y, n) is called a *good pair* if the relative geometric étale cohomology
$$H^j(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p) = 0, \quad \text{unless } j \neq n.$$
(3) A good pair is called *very good* if X is affine and $X \setminus Y$ is smooth and either X is of dimension n and Y of dimension $n-1$ or $X = Y$ is of dimension less than n.

Lemma 3.2. *(Basic Lemma) Let X be an affine variety over K and let $Z \subset X$ be a closed subvariety such that* $\dim(Z) < \dim(X)$. *Then there is a closed subvariety $Y \supset Z$ such that* $\dim(Y) < \dim(X)$ *and* (X, Y, n), $n := \dim(X)$, *is a good pair, i.e.,*
$$H^j(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p) = 0, \quad j \neq n.$$
Moreover, $X \setminus Y$ can be chosen to be smooth.

Proof. See [7, Lemma 3.3] (a result in any characteristic), [55], [40, 7], [43]. □

Corollary 3.3.
(1) *Every affine variety X over K has a* cellular stratification
$$F_\bullet X: \quad \emptyset = F_{-1}X \subset F_0X \subset \cdots \subset F_{d-1}X \subset F_dX = X$$
That is, a stratification by closed subvarieties such that the triple $(F_jX, F_{j-1}X, j)$ is very good.
(2) *Celullar stratifications of X form a filtered system.*
(3) *Let $f : X \to Y$ be a morphism of affine varieties over K. Let $F_\bullet X$ be a cellular stratification on X. Then there exists a cellular stratification $F_\bullet Y$ such that $f(F_iX) \subset F_iY$.*

Proof. See Corollary D.11, Corollary D.12 in [39]. □

Having the above facts it is easy to associate a potentially semistable analog of the cellular complex (3.1) to an affine variety X over K [39, Appendix D]. We just pick a cellular stratification
$$F_\bullet X: \quad \emptyset = F_{-1}X \subset F_0X \subset \cdots \subset F_{d-1}X \subset F_dX = X$$
and take the complex
$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, F_\bullet X) := 0 \to H^0(F_0X_{\overline{K}}, \mathbf{Q}_p) \to \cdots \to H^j(F_jX_{\overline{K}}, F_{j-1}X_{\overline{K}}, \mathbf{Q}_p)\\ \stackrel{d_j}{\to} H^{j+1}(F_{j+1}X_{\overline{K}}, F_jX_{\overline{K}}, \mathbf{Q}_p) \stackrel{d_{j+1}}{\to} \cdots \to H^d(X_{\overline{K}}, F_{d-1}X_{\overline{K}}, \mathbf{Q}_p) \to 0\end{aligned}$$
This is a complex of Galois modules that, by p-adic comparison theorems, are potentially semistable. To get rid of the choice we take the homotopy colimit over all cellular stratifications, i.e., we set
$$\mathrm{R}\Gamma^T_{\mathrm{pst}}(X_{\overline{K}}) := \mathrm{hocolim}_{F_\bullet X}\, \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, F_\bullet X)$$
It is a complex in $\mathscr{D}(\mathrm{Ind}-\mathrm{Rep}_{\mathrm{pst}}(G_K))$ whose cohomology groups are in $\mathrm{Rep}_{\mathrm{pst}}(G_K)$ hence we can think of it as being in $\mathscr{D}(\mathrm{Rep}_{\mathrm{pst}}(G_K))$.

The complex $\mathrm{R}\Gamma^T_{\mathrm{pst}}(X_{\overline{K}})$ computes the étale cohomology groups $H^*(X_{\overline{K}}, \mathbf{Q}_p)$ as Galois modules. More precisely, we have the following proposition.

Proposition 3.4. *(* [43, Prop. 2.1]*)*

(1) *Let $F_\bullet X$ be a cellular stratification of X. There is a natural quasi-isomorphism*
$$\kappa_{(X,F_\bullet X)}:\quad \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},F_\bullet X)\simeq \mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p)$$
that is compatible with the action of G_K.

(2) *Let $f:Y\to X$ be a map of affine schemes and let $F_\bullet Y$ be a cellular stratification of Y such that, for all i, $F_iY\subset F_iX$. Then the following diagram commutes (in the dg derived category)*
$$\begin{tikzcd} \mathrm{R}\Gamma_{\mathrm{pst}}(Y_{\overline{K}},F_\bullet Y) \arrow[r, "\kappa_{(Y,F_\bullet Y)}", "\sim"'] & \mathrm{R}\Gamma_{\text{ét}}(Y_{\overline{K}},\mathbf{Q}_p) \\ \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},F_\bullet X) \arrow[r, "\kappa_{(X,F_\bullet X)}", "\sim"'] \arrow[u, "f^*"] & \mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p) \arrow[u, "f^*"'] \end{tikzcd}$$

(3) *There exists a natural quasi-isomorphism*
$$\kappa_X:\quad \mathrm{R}\Gamma^T_{\mathrm{pst}}(X_{\overline{K}})\simeq \mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p)$$
that is compatible with the action of G_K.

Proof. We have the following commutative diagram of Galois equivariant morphisms
$$\begin{tikzcd} H^0(F_0X_{\overline{K}},\mathbf{Q}_p) \arrow[r] \arrow[d, "\wr"] & \cdots \arrow[r] & H^k(F_kX_{\overline{K}},F_{k-1}X_{\overline{K}},\mathbf{Q}_p) \arrow[r] \arrow[d, "\wr"] & \cdots \arrow[r] & H^d(X_{\overline{K}},F_{d-1}X_{\overline{K}},\mathbf{Q}_p) \arrow[d, "\wr"] \\ \mathrm{R}\Gamma_{\text{ét}}(F_0X_{\overline{K}},\mathbf{Q}_p) \arrow[r] \arrow[d] & \cdots \arrow[r] & {[\mathrm{R}\Gamma_{\text{ét}}(F_kX_{\overline{K}},\mathbf{Q}_p)\to \mathrm{R}\Gamma_{\text{ét}}(F_{k-1}X_{\overline{K}},\mathbf{Q}_p)][k]} \arrow[r] \arrow[d] & \cdots \arrow[r] & {[\mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p)\to \mathrm{R}\Gamma_{\text{ét}}(F_{d-1}X_{\overline{K}},\mathbf{Q}_p)][d]} \arrow[d] \\ 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & \mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p)[d] \end{tikzcd}$$
The first vertical maps are the truncations $\tau_{\leq d}\tau_{\geq d}$. We obtain the map $\kappa_{(X,F_\bullet X)}$ from the first statement of the proposition by taking homotopy fibers of the rows of the diagram. Second statement is now clear. The third one is an immediate corollary of the first statement and Corollary 3.3. □

3.1.2. *Potentially semistable complex of a variety.* For a general variety X over K, one (Zariski) covers it with (rigidified) affine varieties defined over K, takes the associated Čech covering, and applies the above construction to each level of the covering [39, D.5-D.10]. Then, to make everything canonical, one goes to limit over such coverings.

Proposition 3.4 implies now the following result [39, Prop. D.3].

Theorem 3.5. *Let X be a variety over K. There is a canonical complex $\mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}})\in\mathscr{D}^b(\mathrm{Rep}_{\mathrm{pst}})$ which represents the étale cohomology $\mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p)$ of $X_{\overline{K}}$ together with the action of G_K, i.e., there is a natural quasi-isomorphism*
$$\kappa_X:\quad \mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}})\simeq \mathrm{R}\Gamma_{\text{ét}}X_{\overline{K}},\mathbf{Q}_p),$$
that is compatible with the action of G_K.

3.2. **Beilinson's p-adic absolute Hodge cohomology.** Beilinson [11] uses the above construction of the potentially semistable complexes to define his syntomic complexes.

Definition 3.6. ([11]) Let X be a variety over K, $r\in\mathbf{Z}$. Set $\mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}},\mathbf{Q}_p(r)):=\mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}})(r)$ and
$$\mathrm{R}\Gamma^B_{\mathscr{H}}(X,r)=\mathrm{R}\Gamma^B_{\mathrm{syn}}(X,r):=\mathrm{Hom}_{\mathscr{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))}(\mathbf{Q}_p,\mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}},\mathbf{Q}_p(r))),\quad H^i_{\mathrm{syn}}(X,r):=H^i\mathrm{R}\Gamma^B_{\mathrm{syn}}(X,r).$$

Immediately from this definition we obtain that

(1) For $X=\mathrm{Spec}(K)$, we have $\mathrm{R}\Gamma^B_{\mathrm{syn}}(X,r)=\mathrm{Hom}_{\mathscr{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))}(\mathbf{Q}_p,\mathbf{Q}_p(r))$.

(2) There is a natural syntomic descent spectral sequence
$$ {}^{\mathrm{syn}}E_2^{i,j}:=H^i_{\mathrm{st}}(G_K,H^j(X_{\overline{K}},\mathbf{Q}_p(r)))\Rightarrow H^{i+j}_{\mathrm{syn}}(X,r) \tag{3.2}$$

(3) We have a natural period map
$$\rho^B_{\mathrm{syn}}:\quad \mathrm{R}\Gamma^B_{\mathrm{syn}}(X,r)\to \mathrm{R}\Gamma_{\text{ét}}(X,\mathbf{Q}_p(r))$$
defined as the composition
$$\begin{aligned}\mathrm{R}\Gamma^B_{\mathrm{syn}}(X,r)&=\mathrm{Hom}_{\mathscr{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))}(\mathbf{Q}_p,\mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}},\mathbf{Q}_p(r)))\to \mathrm{Hom}_{\mathscr{D}^b(\mathrm{Spec}(K)_{\text{proét}})}(\mathbf{Q}_p,\mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}},\mathbf{Q}_p(r)))\\ &\stackrel{\kappa_X}{\to}\mathrm{Hom}_{\mathscr{D}^b(\mathrm{Spec}(K)_{\text{proét}})}(\mathbf{Q}_p,\mathrm{R}f^X_*\mathbf{Q}_p(r))=\mathrm{R}\Gamma_{\text{ét}}(X,\mathbf{Q}_p(r))\end{aligned}$$
It follows that the syntomic descent spectral sequence is compatible with the Hochschild-Serre spectral sequence via the map ρ^B_{syn}.

3.3. **Comparison of the two constructions of syntomic cohomology.** We will show now that the syntomic complexes defined in 2.24 and by Beilinson are naturally quasi-isomorphic.

Corollary 3.7. (1) *There is a canonical quasi-isomorphism in* $\mathscr{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))$
$$\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r)\stackrel{\sim}{\to}\mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}},\mathbf{Q}_p(r)).$$
(2) *There is a canonical quasi-isomorphism*
$$\rho^B_{\mathrm{syn}}:\mathrm{R}\Gamma^B_{\mathrm{syn}}(X,r)\simeq \mathrm{R}\Gamma_{\mathrm{syn}}(X,r),\quad r\in\mathbf{Z}.$$
It is compatible with period maps to étale cohomology and the syntomic as well as the étale descent spectral sequences.

Proof. The second statement follows immediately from the first one. To prove the first statement, consider the complex $\mathrm{R}\Gamma^B_{DF_K}(X_{\overline{K}},r)$ in $\mathscr{D}^b(DF_K)$ defined by a procedure analogous to the one we used in Proposition 3.4 to define $\mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}},\mathbf{Q}_p(r))$(but starting with cohomology $\mathrm{R}\Gamma_{pH}(Y_{\overline{K}},r)$ of good pairs Y instead of pst-representations $\mathrm{R}\Gamma_{\text{ét}}(Y_{\overline{K}},\mathbf{Q}_p(r))$ of such pairs). This is possible since, for a good pair (X,Y,j), we have
$$\mathrm{R}\Gamma_{pH}(X_{\overline{K}},Y_{\overline{K}},r)\simeq(H^j_{\mathrm{HK}}(X_{\overline{K}},Y_{\overline{K}},r),(H^j_{\mathrm{dR}}(X,Y),F^{\bullet+r}),H^j_{\mathrm{HK}}(X_{\overline{K}},Y_{\overline{K}})\xrightarrow{\iota_{\mathrm{dR}}}H^j_{\mathrm{dR}}(X_{\overline{K}},Y_{\overline{K}})),$$
and, by p-adic comparison theorems, this is an element of DF_K. Proceeding as in the proof of Proposition 3.4, we get a functorial quasi-isomorphism in $\mathscr{D}^b(DF_K)$:
$$\kappa_X:\quad \mathrm{R}\Gamma^B_{DF_K}(X_{\overline{K}},r)\simeq \mathrm{R}\Gamma_{DF_K}(X_{\overline{K}},r).$$
For good pairs (X,Y,j), the Beilinson period maps $\rho_{\mathrm{HK}},\rho_{\mathrm{dR}}$ [9, 3.6], [10, 3.2] induce the period isomorphism $V_{\mathrm{pst}}\mathrm{R}\Gamma_{pH}(X_{\overline{K}},Y_{\overline{K}},r)\stackrel{\sim}{\to}H^j(X_{\overline{K}},Y_{\overline{K}},\mathbf{Q}_p(r))$. This period map lifts to a period map
$$V_{\mathrm{pst}}\mathrm{R}\Gamma^B_{DF_K}(X_{\overline{K}},r)\stackrel{\sim}{\to}\mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}},\mathbf{Q}_p(r)).$$
We define the map $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r)\stackrel{\sim}{\to}\mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}},\mathbf{Q}_p(r))$ as the following composition
$$\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}},r)\stackrel{\kappa_X^{-1}}{\to}V_{\mathrm{pst}}\mathrm{R}\Gamma^B_{DF_K}(X_{\overline{K}},r)\simeq \mathrm{R}\Gamma^B_{\mathrm{pst}}(X_{\overline{K}},\mathbf{Q}_p(r)).$$
□

3.4. **The Bloch-Kato exponential and the syntomic descent spectral sequence.** Let V be a potentially semistable representation. Let $D=D_{\mathrm{pst}}(V)\in DF_K$. The Bloch-Kato exponential
$$\exp_{\mathrm{BK}}:D_K/F^0\to H^1(G_K,V)$$
is defined as the composition [50, 2.14]
$$D_K/F^0\to C(G_K,C_{\mathrm{pst}}(D)[1])\to C(G_K,C(D)[1])\stackrel{\sim}{\leftarrow}C(G_K,V[1]),$$
where $C(G_K,\cdot)$ denotes the continuous cochains cohomology of G_K. The complexes $C_{\mathrm{pst}}(D)$, $C(D)$ are defined as follows
$$\begin{aligned}&C_{\mathrm{pst}}(D):\quad D_{\mathrm{st}}\xrightarrow{(N,1-\varphi,\iota)}D_{\mathrm{st}}\oplus D_{\mathrm{st}}\oplus D_K/F^0\xrightarrow{(1-p\varphi)-N}D_{\mathrm{st}},\\ &C(D):\quad D\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}\xrightarrow{(N,1-\varphi,\iota)}D\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}\oplus D\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}\oplus(D_{\overline{K}}\otimes_{\overline{K}}\mathbf{B}_{\mathrm{dR}})/F^0\xrightarrow{(1-p\varphi)-N}D\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}\end{aligned}$$

We have $C_{\mathrm{pst}}(D) = C(D)^{G_K}$.

The following compatibility result is used in the study of special values of L-functions. Its f-analog was proved in [52, Theorem 5.2][5].

Proposition 3.8. *Let $i \geq 0, r \geq 0$. The composition*

$$H^{i-1}_{\mathrm{dR}}(X)/F^r \xrightarrow{\partial} H^i_{\mathrm{syn}}(X_h, r) \xrightarrow{\rho_{\mathrm{syn}}} H^i_{\text{ét}}(X, \mathbf{Q}_p(r)) \to H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$$

is the zero map. The induced (from the syntomic descent spectral sequnce) map

$$H^{i-1}_{\mathrm{dR}}(X)/F^r \to H^1(G_K, H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r)))$$

is equal to the Bloch-Kato exponential associated with the Galois representation $H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$.

Proof. By the compatibility of the syntomic descent spectral sequence and the Hochschild-Serre spectral sequence [50, Theorem 4.8], we have the commutative diagram

$$\begin{tikzcd}
H^i\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)_0 \arrow[r, "\rho_{\mathrm{syn}}"] \arrow[d, "\delta_1"] & H^i_{\text{ét}}(X, \mathbf{Q}_p(r))_0 \arrow[d, "\delta_1"] \\
H^1_{\mathrm{st}}(G_K, H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))) \arrow[r, "\mathrm{can}"] & H^1(G_K, H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))),
\end{tikzcd}$$

where

$$\begin{aligned}
H^i\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, j)_0 &:= \ker(H^i\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \to H^0_{\mathrm{st}}(G_K, H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r)))),\\
H^i_{\text{ét}}(X, \mathbf{Q}_p(r))_0 &:= \ker(H^i_{\text{ét}}(X, \mathbf{Q}_p(r)) \to H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))).
\end{aligned}$$

It suffices thus to show that the dotted arrow in the following diagram

$$\begin{tikzcd}
H^i\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) & H^i\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)_0 \arrow[l] \arrow[d, "\delta_1"] \\
H^{i-1}_{\mathrm{dR}}(X)/F^r \arrow[u, "\partial"] \arrow[ur, dotted, "\partial"] \arrow[r] & H^1_{\mathrm{st}}(G_K, H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))))
\end{tikzcd}$$

exists and that this diagram commutes.

To do that, we will use freely the notation from the proof of Corollary 3.7. Set

$$\widetilde{\mathrm{R}}\Gamma^B_{\mathrm{syn}}(X, r) = \mathrm{Hom}_{\mathscr{D}^b(DF_K)}(K(0), \mathrm{R}\Gamma^B_{DF_K}(X_{\overline{K}}, r)) = \operatorname{holim} C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{DF_K}(X_{\overline{K}}, r)).$$

Arguing as in the proof of Proposition 3.4, we get the following commutative diagram (we denoted by $(H^*_{\mathrm{HK}}(X, r), H^*_{\mathrm{dR}}(X, r))$ the r'th twist of the canonical Dieudonné-Fontaine modules associated to X)

$$\begin{tikzcd}
H^i\widetilde{\mathrm{R}}\Gamma^B_{\mathrm{syn}}(X, r)_0 \arrow[r, "\kappa_X", "\sim"'] \arrow[d, "\delta_1'"] \arrow[dd, bend right=60, "\delta_1"'] & H^i\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)_0 \arrow[dl, "\delta_1'"] \arrow[ddl, bend left, "\delta_1"] \\
H^1(C_{\mathrm{pst}}(H^{i-1}_{\mathrm{dR}}(X, r))) \arrow[d, "(\rho_{\mathrm{HK}}, \rho_{\mathrm{dR}})", "\wr"'] & \\
H^1_{\mathrm{st}}(G_K, H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))) &
\end{tikzcd}$$

[5] There the exponential $\exp_{\mathrm{st}}$ is called l.

Moreover the comparison map κ_X is compatible with the boundary maps ∂ from the de Rham cohomology complexes $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$ and $\mathrm{R}\Gamma^B_{\mathrm{dR}}(X)$. It suffices thus to show that the dotted arrow in the following diagram

$$\begin{tikzcd}
H^i\widetilde{\mathrm{R}}\Gamma_{\mathrm{syn}}(X,r) & H^i\widetilde{\mathrm{R}}\Gamma_{\mathrm{syn}}(X,r)_0 \arrow[l] \arrow[d, "\delta_1'"] \\
H^{i-1}_{\mathrm{dR}}(X)/F^r \arrow[u, "\partial"] \arrow[r] \arrow[ur, dotted, "\partial"] & H^1(C_{\mathrm{pst}}(H^{i-1}_{\mathrm{dR}}(X,r)))
\end{tikzcd}$$

exists and that this diagram commutes.

Let

$$\mathrm{R}\Gamma^B_{DF_K}(X_{\overline{K}},r) = D^\bullet = D^0 \xrightarrow{d^0} D^1 \xrightarrow{d^1} D^2 \xrightarrow{d^2} \cdots$$

Then $\operatorname{holim} C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{DF_K}(X_{\overline{K}},r))$ is the total complex of the double complex below.

$$\begin{tikzcd}
\cdots & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \\
C_{\mathrm{pst}}(D^2): \arrow[u, "d^2"] & D^2_{\mathrm{st}} \arrow[u, "d^2"] \arrow[r, "{(N,1-\varphi,\iota)}"] & D^2_{\mathrm{st}} \oplus D^2_{\mathrm{st}} \oplus D^2_K/F^0 \arrow[u, "d^2"] \arrow[r, "(1-p\varphi)-N"] & D^2_{\mathrm{st}} \arrow[u, "d^2"] \\
C_{\mathrm{pst}}(D^1): \arrow[u, "d^1"] & D^1_{\mathrm{st}} \arrow[u, "d^1"] \arrow[r, "{(N,1-\varphi,\iota)}"] & D^1_{\mathrm{st}} \oplus D^1_{\mathrm{st}} \oplus D^1_K/F^0 \arrow[u, "d^1"] \arrow[r, "(1-p\varphi)-N"] & D^1_{\mathrm{st}} \arrow[u, "d^1"] \\
C_{\mathrm{pst}}(D^0): \arrow[u, "d^0"] & D^0_{\mathrm{st}} \arrow[u, "d^0"] \arrow[r, "{(N,1-\varphi,\iota)}"] & D^0_{\mathrm{st}} \oplus D^0_{\mathrm{st}} \oplus D^0_K/F^0 \arrow[u, "d^0"] \arrow[r, "(1-p\varphi)-N"] & D^0_{\mathrm{st}} \arrow[u, "d^0"]
\end{tikzcd}$$

We note that $D^\bullet_{\mathrm{st}} = \mathrm{R}\Gamma^B_{\mathrm{HK}}(X,r)$, $D^\bullet_K = \mathrm{R}\Gamma^B_{\mathrm{dR}}(X)$. The following facts are easy to check.

(1) The map $\partial : \mathrm{R}\Gamma^B_{\mathrm{dR}}(X)/F^r \to \widetilde{\mathrm{R}}\Gamma^B_{\mathrm{syn}}(X,r)[1]$ is given by the canonical morphism

$$D^\bullet_K/F^0 \to [D^\bullet_{\mathrm{st}} \longrightarrow D^\bullet_{\mathrm{st}} \oplus D^\bullet_{\mathrm{st}} \oplus D^\bullet_K/F^0 \longrightarrow D^\bullet_{\mathrm{st}}][1]$$

Similarly, the map $H^{i-1}_{\mathrm{dR}}(X)/F^r \to H^1(C_{\mathrm{pst}}(H^{i-1}_{\mathrm{dR}}(X,r)))$ is given by the canonical morphism

$$H^{i-1}_{\mathrm{dR}}(X)/F^r \to [H^{i-1}_{\mathrm{HK}}(X,r) \to H^{i-1}_{\mathrm{HK}}(X,r) \oplus H^{i-1}_{\mathrm{HK}}(X,r) \oplus H^{i-1}_{\mathrm{dR}}(X,r)/F^0 \to H^{i-1}_{\mathrm{HK}}(X,r)][1].$$

(2) The map $H^i\widetilde{\mathrm{R}}\Gamma^B_{\mathrm{syn}}(X,r) \to H^0(C_{\mathrm{pst}}(H^i_{\mathrm{dR}}(X,r)))$ is induced by $(a,b,c) \mapsto a$.

(3) The map $\delta_1' : H^i\widetilde{\mathrm{R}}\Gamma^B_{\mathrm{syn}}(X,r)_0 \to H^1(C_{\mathrm{pst}}(H^{i-1}_{\mathrm{dR}}(X,r)))$ is induced by $(a,b,c) \mapsto b - d_0a'$, where a' is such that $d^ia' = a$.

(4) As a corollary of the above, we get that the composition

$$H^{i-1}_{\mathrm{dR}}(X)/F^r \to H^i\widetilde{\mathrm{R}}\Gamma^B_{\mathrm{syn}}(X,r)_0 \xrightarrow{\delta_1'} H^1(C_{\mathrm{pst}}(H^{i-1}_{\mathrm{dR}}(X,r)))$$

is induced by the map $b \mapsto (0,b,0) \mapsto b$.

This proves our proposition. □

4. p-adic realizations of motives

4.1. **p-adic realizatons of Nori's motives.** We start with a quick review of Nori's motives. We follow [39], [44], [5], and [2, 2].

Take an embedding $K \hookrightarrow \mathbf{C}$ and a field $F \supset \mathbf{Q}$. A diagram Δ is a directed graph. A representation $T : \Delta \to V_F$ assigns to every vertex in Δ an object in V_F and to every edge e from v to v' a homomorphism $T(e) : T(v) \to T(v')$. Let $\mathscr{C}(\Delta, T)$ be its associated diagram category [44, Thm 41], [2, 2.1]: the category of finite dimentional right $\mathrm{End}^\vee(T)$-comodules. It is the universal F-linear abelian category together with a unique representation $\widetilde{T} : \Delta \to \mathscr{C}(\Delta, T)$ and a faithful, exact, F-linear functor $T : \mathscr{C}(\Delta, T) \to V_F$ extending the original representation T. If Δ is an abelian category then we have an equivalence $\Delta \simeq \mathscr{C}(\Delta, T)$.

More specifically we have the following result of Nori.

Proposition 4.1. *(Nori,* [2, Cor. 2.2.10], [2, Cor. 2.2.11]*)*

(1) *Let $\mathscr{R}$ be an F-linear abelian category with a faithful exact functor $\rho : \mathscr{R} \to V_F$. Assume that the representation $T : \Delta \to V_F$ factors, up to natural equivalence, as $T_1\rho$. Let $\mathscr{A}$ be an F-linear abelian category equipped with a faithful exact functor $U : \mathscr{A} \to \mathscr{R}$. If $G : \Delta \to \mathscr{A}$ is a morphism of directed graphs such that T_1 is equivalent to UG, then there exist functors $\mathscr{C}(\Delta, T) \to \mathscr{R}$, $\widetilde{G} : \mathscr{C}(\Delta, T) \to \mathscr{A}$ such that the following diagram*

$$\begin{tikzcd}
\Delta \arrow[r, "G"] \arrow[d, "\widetilde{T}"] \arrow[dr, "T_1"] & \mathscr{A} \arrow[d, "U"] \\
\mathscr{C}(\Delta, T) \arrow[ur, dashed, "\widetilde{G}"] \arrow[r, dashed] \arrow[dr, "T"] & \mathscr{R} \arrow[d, "\rho"] \\
& V_F
\end{tikzcd}$$

commutes up to natural equivalence.

(2) *For a commutative (up to natural equivalence) diagram*

$$\begin{tikzcd}
\Delta \arrow[r, "G"] \arrow[dd, "\pi"] \arrow[drr, "T"] & \mathscr{A} \arrow[dr, "U"] \arrow[dd] & \\
& & V_F \\
\Delta' \arrow[r, "G'"] \arrow[urr, "T'"] & \mathscr{A}' \arrow[ur, "U'"] &
\end{tikzcd}$$

we have a commutative (up to natural equivalence) diagram

$$\begin{tikzcd}
\mathscr{C}(\Delta, T) \arrow[r, "\widetilde{G}"] \arrow[d, "\pi"] & \mathscr{A} \arrow[d] \\
\mathscr{C}(\Delta', T') \arrow[r, "\widetilde{G}'"] & \mathscr{A}'
\end{tikzcd}$$

Example 4.2. The following diagrams appear in the construction of Nori's motives.

(1) The diagram Δ^{eff} of effective pairs consists of pairs (X, Y, i) and two types of edges:
   (a) (functoriality) for every morphism $f : X \to X'$, with $f(Y) \subset Y'$, an edge $f^* : (X', Y', i) \to (X, Y, i)$.
   (b) (coboundary) for every chain $X \supset Y \supset Z$ of closed K-subvarieties of X, an edge $\partial : (Y, Z, i) \to (X, Y, i+1)$.

(2) The diagram Δ^{eff}_g (resp. Δ^{eff}_{vg}) of effective good (resp. of effective very good) pairs is the full subdiagram of Δ^{eff} with vertices good (resp. very good) pairs (X, Y, i).

(3) The diagrams Δ of pairs, Δ_g of good pairs, and Δ_{vg} of very good pairs are obtained by localization with respect to the pair $(\mathbb{G}_m, \{1\}, 1)$ [39, B.18].

Let $H^* : \Delta_g \to V_F$ be the representation which assigns to (X, Y, i) the relative singular cohomology $H^i(X(\mathbf{C}), Y(\mathbf{C}), F)$.

Definition 4.3. The category of (reps. effective) Nori motives $\mathrm{MM}(K)_F$ (resp. $\mathrm{MM}(K)_F$) is defined as the diagram category $\mathscr{C}(\Delta_g, H^*)$ (resp. $\mathscr{C}(\Delta^{\text{eff}}_g, H^*)$). For a good pair (X, Y, i), we denote by $H^i_{\text{mot}}(X, Y)$ the object of $\mathrm{EMM}(K)_F$ (resp. $\mathrm{MM}(K)_F$) corresponding to it and we define the Tate object as

$$\mathbb{1}(-1) := H^1_{\text{mot}}(\mathbb{G}_{m,K}, \{1\}) \in \mathrm{EMM}(K)_F, \quad \mathbb{1}(-n) := \mathbb{1}(-1)^{\otimes n}.$$

We have [39, Thm 1.6, Cor. 1.7]

- $\mathrm{EMM}(K)_F \simeq \mathrm{EMM}(K)_{\mathbf{Q}} \otimes_{\mathbf{Q}} F$ and $\mathrm{MM}(K)_F \simeq \mathrm{MM}(K)_{\mathbf{Q}} \otimes_{\mathbf{Q}} F$.
- As an abelian category $\mathrm{EMM}(K)_F$ is generated by Nori motives of the form $H^i_{\mathrm{mot}}(X,Y)$ for good pairs (X,Y,i); every object of $\mathrm{EMM}(K)_F$ is a subquotient of a finite direct sum of objects of the form $H^i_{\mathrm{mot}}(X,Y)$.
- $\mathrm{EMM}(K)_F \subset \mathrm{MM}(K)_F$ are commutative tensor categories [44, p. 466].
- $\mathrm{MM}(K)_F$ is obtained from $\mathrm{EMM}(K)_F$ by $\otimes$-inverting $\mathbb{1}(-1)$.
- The diagram categories of Δ^{eff} and of $\Delta^{\mathrm{eff}}_{vg}$ with respect to singular cohomology with coefficients in F are equivalent to $\mathrm{EMM}(K)_F$ as abelian categories. The diagram categories of Δ and of Δ_{vg} are equivalent to $\mathrm{MM}(K)_F$.[6] In particular, any pair (X,Y,i) defines a Nori motive $H^i_{\mathrm{mot}}(X,Y)$.
- Nori shows that these categories are independent of the embedding $K \hookrightarrow \mathbf{C}$.

From the universal property of the category $\mathrm{EMM}(K)_F$ it is easy to construct realizations. We will describe the ones coming from p-adic Hodge Theory.

Construction 4.4. (*Galois realization*) Consider the map $\Delta^{\mathrm{eff}} \to \mathrm{Rep}(G_K)$:

$$(X,Y,i) \mapsto H^i(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p).$$

We have $H^i(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p) \simeq H^i(X(\mathbf{C}), Y(\mathbf{C}), \mathbf{Q}_p)$. Thus, by Proposition 4.1, we obtain an extension which is the exact étale realization functor

$$\mathrm{R}_{\mathrm{\acute{e}t}}: \quad \mathrm{EMM}(K)_{\mathbf{Q}_p} \to \mathrm{Rep}(G_K).$$

Note that $\mathrm{R}_{\mathrm{\acute{e}t}}(\mathbb{1}(-1)) = H^1(\mathbb{G}_{m,\overline{K}}, \{1\}, \mathbf{Q}_p) = \mathbf{Q}_p(-1)$. Hence the functor $\mathrm{R}_{\mathrm{\acute{e}t}}$ lifts to $\mathrm{MM}(K)_{\mathbf{Q}_p}$.

In analogous way we obtain the exact potentially semistable realization

$$\mathrm{R}_{\mathrm{pst}}: \quad \mathrm{MM}(K)_{\mathbf{Q}_p} \to \mathrm{Rep}_{\mathrm{pst}}(G_K).$$

It factors $\mathrm{R}_{\mathrm{\acute{e}t}}$ via the natural functor $\mathrm{Rep}_{\mathrm{pst}}(G_K) \to \mathrm{Rep}(G_K)$.

Construction 4.5. (*Filtered* (φ, N, G_K) *realization*) Consider the map $\Delta^{\mathrm{eff}} \to DF_K$:

$$(X,Y,i) \mapsto H^i_{DF}(X,Y) := (H^i_{\mathrm{HK}}(X_{\overline{K}}, Y_{\overline{K}}), (H^i_{\mathrm{dR}}(X,Y), F^\bullet), \iota_{\mathrm{dR}}: H^i_{\mathrm{HK}}(X_{\overline{K}}, Y_{\overline{K}}) \otimes_{K_0^{\mathrm{nr}}} \overline{K} \xrightarrow{\sim} H^i_{\mathrm{dR}}(X_{\overline{K}}, Y_{\overline{K}})).$$

By p-adic comparison theorems, we have

$$D_{\mathrm{pst}}(H^i_{DF}(X,Y)) \simeq H^i(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p) \simeq H^i(X(\mathbf{C}), Y(\mathbf{C}), \mathbf{Q}_p).$$

Thus, by Proposition 4.1, we obtain an extension which is the exact filtered (φ, N, G_K) realization functor

$$\mathrm{R}_{DF_K}: \quad \mathrm{EMM}(K)_{\mathbf{Q}_p} \to DF_K.$$

Since $\mathrm{R}_{DF}(\mathbb{1}(-1)) = K(-1)$, the functor R_{DF} lifts to $\mathrm{MM}(K)_{\mathbf{Q}_p}$.

Projections yield faithful exact functors from DF_K to the categories $M_K(\varphi, N, G_K)$ and V^K_{dR}. Composing them with the realization R_{DF} we get

- the exact Hyodo-Kato realization

$$\mathrm{R}_{\mathrm{HK}}: \quad \mathrm{MM}(K)_{\mathbf{Q}_p} \to M_K(\varphi, N, G_K),$$

- the exact de Rham realization

$$\mathrm{R}_{\mathrm{dR}}: \quad \mathrm{MM}(K)_{\mathbf{Q}_p} \to V^K_{\mathrm{dR}}.$$

Composing R_{DF_K} with the projection on the third factor of the filtered (φ, N, G_K)-module, we obtain the Hyodo-Kato natural equivalence

$$\iota_{\mathrm{dR}}: \quad \mathrm{R}_{\mathrm{HK}} \otimes_{K_0^{\mathrm{nr}}} \overline{K} \simeq \mathrm{R}_{\mathrm{dR}} \otimes_K \overline{K}: \quad \mathrm{MM}(K)_{\mathbf{Q}_p} \to V_{\overline{K}}, \tag{4.1}$$

where the tensor product is taken pointwise.

[6] This is shown by an argument analogous to the one we have used in the construction of Beilinson's potentially semistable complex of a variety in Section 3.1.2 : via cellular complexes and Čech coverings one lifts the representation H^* from very good pairs to all pairs to a representation that canonically computes relative singular cohomology.

Construction 4.6. (*Realization of period isomorphism*) To realize period isomorphisms, we define the category of realizations $\mathscr{R}(K)$. An object of $\mathscr{R}(K)$ is a tuple $M := (M_{DF}, M_{\mathrm{pst}}, \rho_{\mathrm{pst}})$ consisting of $M_{DF} \in DF_K$, $M_{\mathrm{pst}} \in \mathrm{Rep}_{\mathrm{pst}}(G_K)$, and a comparison isomorphism $\rho_{\mathrm{pst}} : V_{\mathrm{pst}} M \simeq M_{\mathrm{pst}}$ of Galois modules. It is a abelian category (it is naturally equivalent to the category $\mathrm{Rep}_{\mathrm{pst}}(G_K)$). Projections yield faithful exact functors from $\mathscr{R}(K)$ to the categories DF_K and $\mathrm{Rep}_{\mathrm{pst}}(G_K)$.

Consider the following map $\Delta^{\mathrm{eff}} \to \mathscr{R}(K)$:

$$(X, Y, i) \mapsto (H^i_{DF}(X, Y), H^i(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p), \rho_{\mathrm{pst}} : V_{\mathrm{pst}} H^i_{DF}(X, Y) \simeq H^i(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p)).$$

Since the functor $\mathscr{R}(K) \to \mathrm{Rep}_{\mathrm{pst}}(G_K) \to V_{\mathbf{Q}_p}$ is faithful and exact, Proposition 4.1 gives us an extension $\mathrm{EMM}(K)_{\mathbf{Q}_p} \to \mathscr{R}(K)$ that is compatible with the étale realization. Since

$$(\mathbb{G}_m, \{1\}, 1) \mapsto (K(-1), \mathbf{Q}_p(-1), V_{\mathrm{pst}} K(-1) \simeq \mathbf{Q}_p(-1)),$$

again by Proposition 4.1, we obtain the exact realization

$$\mathrm{R}_{\mathscr{R}} : \quad \mathrm{MM}(K)_{\mathbf{Q}_p} \to \mathscr{R}(K).$$

Projecting on the first two factors we get back the realizations R_{DF_K} and $\mathrm{R}_{\mathrm{pst}}$ and projecting on the third factor we get that the above two realizations are related via a period morphism, i.e., we have a natural equivalence

$$\rho_{\mathrm{pst}} : \quad V_{\mathrm{pst}} \mathrm{R}_{DF_K} \simeq \mathrm{R}_{\mathrm{pst}} : \quad \mathrm{MM}(K)_{\mathbf{Q}_p} \to \mathrm{Rep}_{\mathrm{pst}}(G_K).$$

To sum up, we have a potentially semistable comparison theorem for Nori's motives.

Corollary 4.7. *For $M \in \mathrm{MM}(K)_{\mathbf{Q}_p}$, there is a functorial isomorphism*

$$\rho_{\mathrm{pst}} : \quad \mathrm{R}_{\mathrm{HK}}(M) \otimes_{K_0^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}} \simeq \mathrm{R}_{\text{ét}}(M) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}}$$

that is compatible with Galois action, Frobenius, and the monodromy operator. Moreover, after passing to $\mathbf{B}_{\mathrm{dR}}$ via the Hyodo-Kato map (4.1), it yields a functorial isomorphism

$$\rho_{\mathrm{dR}} : \quad \mathrm{R}_{\mathrm{dR}}(M) \otimes_K \mathbf{B}_{\mathrm{dR}} \simeq \mathrm{R}_{\text{ét}}(M) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}$$

that is compatible with filtration.

We can illustrate the above constructions by the following, essentially commutative, diagram of exact functors

$$\begin{tikzcd}
 & & \mathrm{Rep}(G_K) \arrow[dr, "\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}}"] \arrow[drr, bend left, "\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}"] & & \\
 & \mathrm{Rep}_{\mathrm{pst}}(G_K) \arrow[ur, "\iota"] & & M_{\mathbf{B}_{\mathrm{st}}}(\varphi, N, G_K) \arrow[r] & MF_{\mathbf{B}_{\mathrm{dR}}} \\
\mathrm{MM}(K)_{\mathbf{Q}_p} \arrow[uurr, bend left, "\mathrm{R}_{\text{ét}}"] \arrow[ur, "\mathrm{R}_{\mathrm{pst}}"] \arrow[rr, "\mathrm{R}_{\mathrm{HK}}"] \arrow[dr, "R_{DF_K}"'] \arrow[ddrr, bend right, "\mathrm{R}_{\mathrm{dR}}"'] & & M_K(\varphi, N, G_K) \arrow[ur, "\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}"'] \arrow[dr, "F_0"] & & \\
 & DF_K \arrow[uu, "V_{\mathrm{pst}}"] \arrow[ur] \arrow[dr] & & V_K^G & \\
 & & V_{\mathrm{dR}}^K \arrow[ur, "F_{\mathrm{dR}}"] \arrow[uuurr, bend right, "\otimes_K\mathbf{B}_{\mathrm{dR}}"'] & &
\end{tikzcd}$$

Here $M_{\mathbf{B}_{\mathrm{st}}}(\varphi, N, G_K)$ is the exact category of free finite rank $\mathbf{B}_{\mathrm{st}}$-modules equipped with an action of φ, N, G_K (φ is an isomorphism, N is nilpotent, and G_K-action is continuous - everything being compatible in the usual way and compatible with the same structures on $\mathbf{B}_{\mathrm{st}}$). $MF_{\mathbf{B}_{\mathrm{dR}}}$ is the exact category of filtered finite rank $\mathbf{B}_{\mathrm{dR}}$-modules equipped with a continuous action of G_K compatible with its action on $\mathbf{B}_{\mathrm{dR}}$; $M_{\mathbf{B}_{\mathrm{st}}} \to M_{\mathbf{B}_{\mathrm{dR}}}$ is the natural functor.

4.2. p-adic realizations of Voevodsky's motives.

Recall 4.8. The category of Voevodsky's motives $DM(K, \mathbf{Q}_p)$ with rational coefficients admits several equivalent constructions, each interesting in its own. In this section, we will be using the one of Morel (see [47]) for a review of which we refer the reader to [24, §1].

By construction, the triangulated category $DM(K, \mathbf{Q}_p)$ is stable under taking arbitrary coproducts. In this category, each smooth K-scheme X admits a homological motive $M(X)$, covariant with respect to morphism of K-schemes (and even finite correspondences). Each motive can be twisted by an arbitrary integer power of the Tate object $\mathbf{Q}_p(1)$, and as a triangulated category stable under taking coproducts, $DM(K, \mathbf{Q}_p)$ is generated by motives of the form $M(X)(n)$, X/K smooth, and $n \in \mathbf{Z}$.

The category of *constructible* motives (see also 5.4) is the thick[7] triangulated subcategory of $DM(K, \mathbf{Q}_p)$ generated by the motives $M(X)(n)$, X/K smooth, and $n \in \mathbf{Z}$, without requiring stability by infinite coproducts. It is equivalent to Voevodsky's category of geometric motives $DM_{gm}(K, \mathbf{Q}_p)$ ([70, chap. 5]) and can also be described in an elementary way as follows. Let $\mathbf{Q}_p[\mathrm{Sm}_K^{aff}]$ be the $\mathbf{Q}_p$-linearization of the category of smooth affine K-varieties, $K^b(\mathbf{Q}_p[\mathrm{Sm}_K^{aff}])$ its bounded homotopy category. This is a triangulated monoidal category, the tensor structure being induced by cartesian products of K-schemes. First we get the geometric $\mathbb{A}^1$-derived category $D_{\mathbb{A}^1,gm}(K, \mathbf{Q}_p)$ out of $K^b(\mathbf{Q}_p[\mathrm{Sm}_K^{aff}])$ by the following operations:

(1) Take the Verdier quotient with respect to the triangulated subcategory generated by complexes of the form:
   - (*homotopy*) $\ldots \to 0 \to \mathbb{A}^1_X \xrightarrow{p} X \to 0 \ldots$, for $X \in \mathrm{Sm}_K^{aff}$, p canonical projection;
   - (*excision*) $\ldots \to 0 \to W \xrightarrow{q-k} U \oplus V \xrightarrow{j+p} X \ldots$, for any cartesian square $\begin{array}{ccc} W & \xrightarrow{k} & V \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{j} & X \end{array}$ in Sm_K^{aff}

     such that j is an open immersion, p is étale and an isomorphism above the complement of j.

(2) Formally invert the Tate object $\mathbf{Q}_p(1)$, which is the cokernel of $\{1\} \to \mathbb{G}_m$ placed in cohomological degree $+1$.

(3) Take the pseudo-abelian envelope.

Let τ be the automorphism of $\mathbf{Q}_p(1)[1] \otimes \mathbf{Q}_p(1)[1]$ in $D_{\mathbb{A}^1,gm}(K, \mathbf{Q}_p)$ which permutes the factors. Because $\mathbf{Q}_p(1)$ is invertible, it induces an automorphism ε of $\mathbf{Q}_p$ in $D_{\mathbb{A}^1,gm}(K, \mathbf{Q}_p)$ such that $\varepsilon^2 = 1$. Then we can define complementary projectors: $p_+ = (1-\varepsilon)/2, p_- = (\varepsilon - 1)/2$, which cut the objects, and therefore the category, into two pieces:

$$D_{\mathbb{A}^1,gm}(K, \mathbf{Q}_p)_+ = \mathrm{Im}(p_+), \quad D_{\mathbb{A}^1,gm}(K, \mathbf{Q}_p)_- = \mathrm{Im}(p_-).$$

Then, according to a theorem of Morel (cf. [18, 16.2.13]), $DM_{gm}(K, \mathbf{Q}_p) \simeq D_{\mathbb{A}^1,gm}(K, \mathbf{Q}_p)_+$.

Example 4.9. Let F be an extension field of $\mathbf{Q}_p$ and $\mathscr{A}$ be a Tannakian F-linear category with a fiber functor $\omega : \mathscr{A} \to V_F$. Consider a contravariant functor:

$$R : \quad (\mathrm{Sm}_K^{aff})^{op} \to C^b(\mathscr{A}).$$

It automatically extends to a contravariant functor $R' : K^b(\mathbf{Q}_p[\mathrm{Sm}_K^{aff}])^{op} \to D^b(\mathscr{A})$. The conditions for R' to induce a contravariant functor defined on $DM_{gm}(K, \mathbf{Q}_p)$ are easy to state given the description of DM_{gm} given above. We will use the following simpler criterion:

We now suppose that the functor R takes its values in the bigger category $C^b(\mathrm{Ind}-\mathscr{A})$ but we assume that there exists a functorial isomorphism

$$H^i \omega R(X) \simeq H^i(X(\mathbf{C}), F)$$

and that the product map $H^i(X(\mathbf{C}), F) \otimes H^j(Y(\mathbf{C}), F) \to H^{i+j}(X(\mathbf{C}) \times Y(\mathbf{C}), F)$ can be lifted to a map $R(X) \otimes R(Y) \to R(X \times_K Y)$ in $C^b(\mathscr{A})$.

Then the functor R' uniquely extends to a realization functor

$$\tilde{R}^\vee : \quad DM_{gm}(K, \mathbf{Q}_p)^{op} \to D^b(\mathscr{A})$$

[7] *i.e.* stable by direct factors [49, Definition 2.1.6].

which is monoidal and such that $H^i(\tilde{R}^\vee(M(X))) = H^i(R(X))$.[8] After composing this functor with the canonical duality endofunctor of the (rigid) triangulated monoidal category $D^b(\mathscr{A})$, we get a covariant realization:

$$\tilde{R}: \quad DM_{gm}(K, \mathbf{Q}_p) \to D^b(\mathscr{A})$$

such that $H^i\tilde{R}(M(X)) = H^i(R(X))^\vee$. Note also that, by construction, the preceding identification can be extended to closed pairs. Also, because $DM_{gm}(K, \mathbf{Q}_p)$ satisfies h-descent (see section 5.5), it can be extended to singular K-varieties and pairs of such.

Using this example we can easily build realizations:

Proposition 4.10. *Let F be an extension field of $\mathbf{Q}_p$ and $\mathscr{A}$ be a Tannakian F-linear category with a fiber functor $\omega : \mathscr{A} \to V_F$. Consider a representation $A^* : \Delta_g \to \mathscr{A}$ such that ωA^* is isomorphic to the singular representation (see Definition 4.3).*

Then there exists a canonical covariant monoidal realization:

$$R_A: \quad DM_{gm}(K, \mathbf{Q}_p) \to D^b(\mathscr{A})$$

such that for any good pair (X, Y, i), $H^iR_A(M(X, Y)) = A^i(X, Y)^\vee$ and this identification is functorial in (X, Y, i) – including with respect to boundaries.

Moreover, this construction is funtorial with respect to exact morphisms of representations.

Proof. Let X be a smooth affine K-scheme. To any cellular stratification of X (cf. Corollary 3.3) $F_\bullet X$, we can associate the complex

$$R'_A(F_\bullet X) := 0 \to A^0(F_0X) \to A^1(F_1X, F_0X) \to \ldots \to A^d(X, F_{d-1}X) \to 0.$$

We put: $R'_A(X) := \operatorname{colim}_{F_\bullet X} R'_A(F_\bullet X)$. This defines a contravariant functor:

$$R'_A: \quad (\mathrm{Sm}_K^{aff})^{op} \to C^b(\mathrm{Ind}-\mathscr{A})$$

which satisfies the assumptions of the previous example. Hence we get the proposition by applying the construction of this example. □

Remark 4.11. Consider again a fiber functor $\omega : \mathscr{A} \to V_F$ and a contravariant functor

$$R: \quad (\mathrm{Sch}_K)^{op} \to C^b(\mathrm{Ind}-\mathscr{A})$$

such that for any K-variety X, one has a functorial isomorphism $H^i\omega R(X) \simeq H^i(X(\mathbf{C}), F)$. Then we can apply the preceding example to $R|_{\mathrm{Sm}_K^{aff}}$ and also the preceding proposition to the unique representation A^* induced by R such that $A^i(X, Y) = H^i(\mathrm{Cone}(R(X) \to R(Y)[-1])$. By applying the construction of the preceding proof, we get for any smooth affine K-scheme a canonical map of complexes

$$R(X) \to R_A(X)$$

which is a quasi-isomorphism. By the functoriality of the construction of the previous example, we thus get a canonical isomorphism between the two realizations of any Voevodsky's motive M:

$$\tilde{R}(M) \xrightarrow{\sim} R_A(M)$$

Remark 4.12. Voevodsky's motives $M(X)$ are homological: they are covariant in X. In fact, the monoidal category $DM_{gm}(K, \mathbf{Q}_p)$ is rigid: any object has a strong dual; this follows from [58] and from the existence of the monoidal triangulated functor $SH(K) \to DM(K, \mathbf{Q}_p)$ [18, 5.3.35] (here $SH(K)$ denotes the stable homotopy category of Morel-Voyevodsky over K). Then for any smooth K-variety X, $M(X)^\vee$ is the cohomological motive of X/K. Using the notations of the previous proposition, because R_A is monoidal and therefore commutes with strong duals, we get: $H^iR_A(M(X)^\vee) = A^i(X)$.

Recall that the category $DM_{gm}(K, \mathbf{Q}_p)$ can be extended to any base and satisfies the 6 functors formalism (cf. [18], in particular 16.1.6). According to *loc. cit.*, 15.2.4, $M(X)^\vee = f_*(\mathbb{1}_X)$ where $f : X \to \mathrm{Spec}(K)$ is the structural morphism. The preceding relation can be rewritten:

$$H^iR_A(f_*(\mathbb{1}_X)) = A^i(X).$$

[8] Note in particular that the permutation ε acts by -1 on singular cohomology.

Note finally that f_* exists for any K-variety X. One can extend the above identification to this general case using De Jong resolution of singularities and h-descent, which is true for Voevodsky'srational motives ([18, 14.3.4]) and for Betti cohomology.

There is fully faithful monoidal functor

$$\mathrm{CHM}(K)^{op}_{\mathbf{Q}_p} \to DM_{gm}(K, \mathbf{Q}_p), \quad h(X) \mapsto M(X)$$

from the category of Chow motives (X is smooth projective over K) [70, Chapter 5, Prop. 2.1.4, Cor. 2.4.6.]. Applying duality on the right hand side, we get a covariant fully faithful monoidal functor:

$$\mathrm{CHM}(K)_{\mathbf{Q}_p} \to DM_{gm}(K, \mathbf{Q}_p), \quad h(X) \mapsto M(X)^\vee = f_*(\mathbb{1}_X).$$

In view of this embedding, it is convenient to identify the Chow motive $h(X)$ with the Voevodsky's(cohomological) motive $M(X)^\vee$.

Let us also state the following corollary which follows from the preceding proposition and [27]:

Corollary 4.13. *In the assumptions of the previous proposition, for any smooth projective K-scheme X of dimension d, the complex $R_A(h(X)) = R_A(M(X)^\vee)$ is split: there exists a canonical isomorphism:*

$$R_A(h(X)) = \bigoplus_{i=0}^{2d} H^i\big(R_A(h(X))\big)[-i] = \bigoplus_{i=0}^{2d} A^i(X)[-i].$$

This decomposition statement follows simply from *loc. cit.* as the derived category $D^b(\mathscr{A})$ satisfies the assumptions of *loc. cit.* and the object $R_A(h(X))$ satisfies the assumption (L.V.) for the map $h(X) \to h(X)(1)[2]$ given by multiplication by the (motivic) first Chern class of an ample invertible bundle on X.

Example 4.14. In particular, applying the preceding proposition to the functor $\Delta_g \to \mathrm{MM}(K)_{\mathbf{Q}_p}$ coming from the singular representation, we get the classical realization,[9] due to Nori, of (cohomological) Nori's motives:

$$\Gamma : \quad DM_{gm}(K, \mathbf{Q}_p) \to D^b(\mathrm{MM}(K)_{\mathbf{Q}_p}).$$

By definition, and applying the preceding remark, we get for any smooth projective (resp. smooth, any) K-variety $f : X \to \mathrm{Spec}(K)$:

$$H^i\Gamma(h(X)) = H^i_{\mathrm{mot}}(X), \text{ resp. } H^i\Gamma(M(X)) = H^i_{\mathrm{mot}}(X)^\vee, \ H^i\Gamma(f_*(\mathbb{1}_X)) = H^i_{\mathrm{mot}}(X).$$

When X is smooth projective of dimension d, we also get by the above corollary the decomposition:

$$\Gamma(h(X)) = \bigoplus_{i=0}^{2d} H^i_{\mathrm{mot}}(X)[-i]$$

Moreover, because of the functorialility statement of the proposition, this realization of Voevodsky's motives is the universal (initial) one.

4.15. More interestingly, using either Example 4.9 or Proposition 4.10, we can get various p-adic realizations of Voevodsky's motives, and extend the de Rham p-adic comparison theorem to the derived situation as summarized in the following essentially commutative diagram of triangulated monoidal functors:

$$\begin{tikzcd}
 & & D^b\big(\mathrm{Rep}(G_K)\big) \arrow[dr, "\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}"] & \\
DM_{gm}(K, \mathbf{Q}_p) \arrow[r, "\mathrm{R}\Gamma_{\mathrm{pst}}"] \arrow[urr, bend left, "\mathrm{R}\Gamma_{\text{ét}}"] \arrow[dr, "\mathrm{R}\Gamma_{DF_K}"'] \arrow[drr, bend right, "\mathrm{R}\Gamma_{\mathrm{dR}}"'] & D^b\big(\mathrm{Rep}_{\mathrm{pst}}(G_K)\big) \arrow[ur, "\iota"'] & & D^b(MF_{\mathbf{B}_{\mathrm{dR}}}) \\
 & D^b(DF_K) \arrow[u, "V_{\mathrm{pst}}"] \arrow[r] & D^b(V^K_{\mathrm{dR}}) \arrow[ur, "\otimes_K \mathbf{B}_{\mathrm{dR}}"'] &
\end{tikzcd}$$

[9]Conjecturally, this is more than a realization: it is thought to be an equivalence of categories!

where ι is the canonical functor.[10] The functors $\mathrm{R}\Gamma_{\text{ét}}$, $\mathrm{R}\Gamma_{\text{pst}}$ and $\mathrm{R}\Gamma_{DF_K}$ are obtained either from 4.9 or equivalently from 4.10 (according to Remark 4.11) by considering respectively the following functors:

- $X \in \mathrm{Sm}_K^{aff}, f : X \to \mathrm{Spec}(K) \mapsto \mathrm{R}\, f_*(\mathbf{Q}_p)$ and $(X, Y, i) \mapsto H^i_{\text{ét}}(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p)$;
- $X \in \mathrm{Sm}_K^{aff} \mapsto \mathrm{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r) \simeq \mathrm{R}\Gamma^B_{\text{pst}}(X_{\overline{K}}, \mathbf{Q}_p(r))$ and $(X, Y, i) \mapsto H^i_{\text{ét}}(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p) \in \mathrm{Rep}_{\text{pst}}(G_K)$;
- $(X, Y, i) \mapsto H^i_{DF}(X, Y)$ (see Construction 4.5).

The functor $\mathrm{R}\Gamma_{\text{dR}}$ is obtained by composing $\mathrm{R}\Gamma_{DF_K}$ with the canonical functor $DF_K \to V^K_{\text{dR}}$.

For $\varepsilon = \text{ét}, \text{pst}, DF_K$, one has defined in the preceding section an analoguous exact monoidal realization functor R_ε from the category of Nori's motives $\mathrm{MM}(K)_{\mathbf{Q}_p}$. This functor being exact induces a functor on the (bounded) derived categories and according to the functoriality in Proposition 4.10, one gets for any Voevodsky motive $M \in DM_{gm}(K, \mathbf{Q}_p)$:

$$\mathrm{R}\Gamma_\varepsilon(M) = \mathrm{R}_\varepsilon\big(\Gamma(M)\big). \tag{4.2}$$

Same for the de Rham realizations: we have $\mathrm{R}\Gamma_{\text{dR}}(M) = \mathrm{R}_{\text{dR}}\big(\Gamma(M)\big)$. Therefore, the essential commutativity of the previous diagram simply follows from the de Rham comparison theorem for Nori's motives. More precisely, it yields, for any Voevodsky's motive M, the de Rham comparison isomorphism:

$$\rho_{\text{dR}} : \quad \mathrm{R}\Gamma_{\text{dR}}(M) \otimes_K \mathbf{B}_{\text{dR}} \simeq \mathrm{R}\Gamma_{\text{ét}}(M) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}}$$

which is a quasi-isomorphism of complexes of filtered finite rank $\mathbf{B}_{\text{dR}}$-modules equipped with an action of G_K (continuous and compatible with the canonical action on $\mathbf{B}_{\text{dR}}$).

This comparison can be made more precise through the Hyodo-Kato realization, as illustrated in the essentially commutative diagram:

$$\begin{tikzcd}
 & & D^b\big(\mathrm{Rep}(G_K)\big) \arrow[dr, "\otimes_{\mathbf{Q}_p}\mathbf{B}_{\text{st}}"] \arrow[drr, bend left, "\otimes_{\mathbf{Q}_p}\mathbf{B}_{\text{dR}}"] & & \\
 & D^b\big(\mathrm{Rep}_{\text{pst}}(G_K)\big) \arrow[ur, "\iota"] & (1) & D^b\big(M_{\mathbf{B}_{\text{st}}}(\varphi, N, G_K)\big) \arrow[r] & D^b(MF_{\mathbf{B}_{\text{dR}}}) \\
DM_{gm}(K, \mathbf{Q}_p) \arrow[uurr, bend left, "\mathrm{R}\Gamma_{\text{ét}}"] \arrow[ur, "\mathrm{R}\Gamma_{\text{pst}}"] \arrow[rr, "\mathrm{R}\Gamma_{\text{HK}}"] \arrow[dr, "\mathrm{R}\Gamma_{DF_K}"'] \arrow[ddrr, bend right, "\mathrm{R}\Gamma_{\text{dR}}"'] & & D^b\big(M_K(\varphi, N, G_K)\big) \arrow[ur, "\otimes_{K_0^{\text{nr}}}\mathbf{B}_{\text{st}}"'] \arrow[dr, "F_0"] & & \\
 & D^b(DF_K) \arrow[uu, "V_{\text{pst}}"] \arrow[ur] \arrow[dr] & (2) & D^b\big(V^G_{\overline{K}}\big). & \\
 & & D^b(V^K_{\text{dR}}) \arrow[ur, "F_{\text{dR}}"] \arrow[uuurr, bend right, "\otimes_K\mathbf{B}_{\text{dR}}"'] & &
\end{tikzcd}$$

The Hyodo-Kato realization $\mathrm{R}\Gamma_{\text{HK}}$ is obtained by composing $\mathrm{R}\Gamma_{DF_K}$ with the projection $DK_F \to M_K(\varphi, N, G_K)$. Then the essential commutativity of the part (1) and (2) of the above diagram corresponds, respectively, for any Voevodsky's motive M, to the potentially semistable comparison theorem and to the Hyodo-Kato quasi-isomorphism:

$$\begin{aligned}
\rho_{\text{pst}} : &\quad \mathrm{R}\Gamma_{\text{HK}}(M) \otimes_{K_0^{\text{nr}}} \mathbf{B}_{\text{st}} \simeq \mathrm{R}\Gamma_{\text{ét}}(M) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}}, \\
\iota_{\text{dR}} : &\quad \mathrm{R}\Gamma_{\text{HK}}(M) \otimes_{K_0^{\text{nr}}} \overline{K} \simeq \mathrm{R}\Gamma_{\text{dR}}(M) \otimes_K \overline{K}.
\end{aligned}$$

Again, the identification (4.2) holds when $\varepsilon = \text{HK}$ and the above canonical comparison quasi-isomorphisms correspond to the comparison isomorphisms obtained in the previous section.

Remark 4.16. By construction, for any (smooth) K-variety $f : X \to \mathrm{Spec}(K)$, one has a canonical identification: $\mathrm{R}\Gamma_{\text{ét}}(f_*(\mathbb{1}_X)) = \mathrm{R}\, f_*(\mathbf{Q}_p)$ where the right hand side denotes the right derived functor of the direct image for étale p-adic sheaves.

[10] One should be careful that though ι is induced by a fully faithful functor on the corresponding abelian categories, it is a *non full* faithful functor.

This implies that the realization functor $\mathrm{R}\Gamma_{\text{ét}}$ constructed above coincides with that of [19, 7.2.24], denoted by ρ_p^*, and equivalently to the one defined in [4]. In particular, it can be extended to any base and commutes with the six functors formalism. This explains the preceding relation and why we have prefered the covariant realization rather than the contravariant one (see the end of Example 4.9).[11]

Example 4.17. The above realizations allow us to define syntomic cohomology of a motive M in $DM_{gm}(K, \mathbf{Q}_p)$ as

$$\mathrm{R}\Gamma_{\mathrm{syn}}(M) := \mathrm{R}\operatorname{Hom}_{D(\mathrm{Rep}_{\mathrm{pst}})}(\mathbf{Q}_p, \mathrm{R}\Gamma_{\mathrm{pst}}(M)) = \mathrm{R}\operatorname{Hom}_{D(DF_K)}(K(0), \mathrm{R}\Gamma_{DF_K}(M)).$$

In particular, we have the syntomic descent spectral sequence

$${}^{\mathrm{syn}}E_2^{i,j} := H^i_{\mathrm{st}}\big(G_K, H^j\mathrm{R}\Gamma_{\text{ét}}(M)\big) \Rightarrow H^{i+j}\mathrm{R}\Gamma_{\mathrm{syn}}(M).$$

If we apply it to the cohomological Voevodsky's motive $M(X)^\vee = f_*(\mathbb{1}_X)$ of any K-variety X with structural morphism f, we get back the results of Theorem 2.25.

An interesting case is obtained by using the (homological) motive with compact support $M^c(X)$ in $DM_{gm}(K, \mathbf{Q}_p)$ of Voevodsky for any K-variety X, and its dual $M^c(X)^\vee = \underline{\operatorname{Hom}}(M^c(X), \mathbf{Q}_p)$ which belongs to $DM_{gm}(K, \mathbf{Q}_p)$. Then $\mathrm{R}\Gamma_{\mathrm{syn}}(M^c(X)^\vee(r))$ is the *n-th twisted syntomic complex with compact support* and we recover the syntomic descent spectral sequence with compact support from Remark 2.28:

$${}^{\mathrm{syn},c}E_2^{i,j} := H^i_{\mathrm{st}}\big(G_K, H^j_{\text{ét},c}(X_{\overline{K}}, \mathbf{Q}_p(r))\big) \Rightarrow H^{i+j}_{\mathrm{syn},c}(X, r).$$

Indeed, in terms of the 6 functors formalism, $M^c(X)^\vee(r) = f_!(\mathbb{1}_X)(r)$ and the identification relevant to compute the above E_2-term follows from the previous remark.

4.3. **Example I: p-adic realizations of the motivic fundamental group.** Let $\mathrm{EHM}(K)_{\mathbf{Q}_p}$ denote the category of effective homological Nori's motives, i.e., the diagram category $\mathscr{C}(\widetilde{\Delta}_g^{\mathrm{eff}}, H_*)$, $H_* := (H^*)^* := \operatorname{Hom}(H^*, \mathbf{Q}_p)$, where the diagram $\widetilde{\Delta}_g^{\mathrm{eff}}$ is obtained from the diagram Δ_g^{eff} by reversing the edge f^* to $f_* : (X, Y, i) \to (X', Y', i)$ and changing ∂ to $\partial : (X, Y, i) \to (Y, Z, i-1)$. There is a duality functor $\vee : \mathrm{EHM}(K)_{\mathbf{Q}_p} \to \mathrm{EMM}(K)^{\mathrm{op}}_{\mathbf{Q}_p}$ respecting the representations H_* and H^* via the usual duality that sends a good pair (X, Y, i) to (X, Y, i). This induces an equivalence on the derived categories $\vee : D^b(\mathrm{EHM}(K)_{\mathbf{Q}_p}) \xrightarrow{\sim} D^b(\mathrm{EMM}(K)_{\mathbf{Q}_p})^{\mathrm{op}}$.

In [21] Cushman developed a motivic theory of the fundamental group, i.e., he showed that the unipotent completion of the fundamental group of varieties over complex numbers carries a motivic structure in the sense of Nori. We will recall his main theorem.

• Let $\mathscr{V}ar_K^*$ be the category of pairs (X, x), where X is a variety defined over K and x is a K-rational base point; morphism between such pairs are morphisms between the corresponding varieties defined over K that are compatible with the base points.

• Let $\mathscr{V}ar_K^{**}$ be the category of triples $(X; x_1, x_2)$, where X is defined over K and x_1, x_2 are K-rational base points.

For a variety X over $\mathbf{C}$, let $\pi_1(X, x)$ be the fundamental group of X with base point x and let $\pi_1(X; x_1, x_2)$ be the space of based paths up to homotopy from x_1 to x_2. Denote by I_{x_2} – the augmentation ideal in $\mathbf{Q}_p[\pi_1(X, x_2)]$ (i.e., the kernel of the augmentation map $\mathbf{Q}_p[\pi_1(X, x_2)] \to \mathbf{Q}_p$) which acts on the right on $\pi_1(X; x_1, x_2)$. The following theorem [22, Thm 3.1] shows that the quotient $\mathbf{Q}_p[\pi_1(X; x_1, x_2)]/I^n_{x_2}$, $n \in \mathbf{N}$, has motivic version $\Pi^n(X; x_1, x_2)$ (in the sense of Nori).

Theorem 4.18. *For every $n \in \mathbf{N}$, there are functors*

$$\Pi^n : \mathscr{V}ar_K^{**} \to \mathrm{EHM}(K)_{\mathbf{Q}_p}, \quad \Pi^n : \mathscr{V}ar_K^{*} \to \mathrm{EHM}(K)_{\mathbf{Q}_p}.$$

These functors have the following properties.

(1) *There is a natural transformation*

$$\Pi^{n+1}(X; x_1, x_2) \to \Pi^n(X; x_1, x_2).$$

[11]In Section 5, we will similarly extend the realization functor $\mathrm{R}\Gamma_{\mathrm{pst}}$ to arbitrary K-bases (see more precisely Rem. 5.16).

(2) *We have a natural isomorphism of $\mathbf{Q}_p$-vector spaces*

$$\widetilde{H}_*(\Pi^n(X(\mathbf{C});x_1,x_2)) \simeq \mathbf{Q}_p[\pi_1(X(\mathbf{C});x_1,x_2)]/I^n_{x_2}.$$

(3) *There are natural transformations*

$$\Pi^n(X;x_1,x_2)\otimes\Pi^n(X;x_2,x_3)\to\Pi^n(X;x_1,x_3)$$
$$\Pi^{n+m+1}(X,x_2)\to\Pi^{m+1}(X,x_2)\otimes\Pi^{n+1}(X,x_2)$$

Via the natural isomorphisms in (2), these transformations are compatible with the product and coproduct structures as well as with the inversion in the path space.

*This data is equivalent to giving a pro-*EHM *structure on the inverse limit $\mathbf{Q}_p[\pi_1(X(\mathbf{C});x_1,x_2)]/I^n_{x_2}$ such that all the obvious maps are motivic, and the completed ideal $I^{\wedge}_{x_2}$ is a sub-motive.*

Dualizing the realization functors of Nori's motives used in Constructions 4.4, 4.5 we obtain the following functors

$$\Pi^n_{\text{ét}}:\quad \mathscr{V}ar^{**}_K\to \operatorname{Rep}(G_K),\quad \Pi^n_{\text{ét}}:=\mathrm{R}_{\text{ét}}\Pi^n;$$
$$\Pi^n_{\mathrm{HK}}:\quad \mathscr{V}ar^{**}_K\to M_K(\varphi,N,G_K),\quad \Pi^n_{\mathrm{HK}}:=\mathrm{R}_{\mathrm{HK}}\Pi^n;$$
$$\Pi^n_{\mathrm{dR}}:\quad \mathscr{V}ar^{**}_K\to V^K_{\mathrm{dR}},\quad \Pi^n_{\mathrm{dR}}:=\mathrm{R}_{\mathrm{dR}}\Pi^n.$$

These realizations are compatible with change of the index n and with the structure maps that endow these realizations with Hopf algebra structures.

From Constructions 4.5,4.6 (again dualizing) we obtain also the following comparison isomorphisms.

Corollary 4.19. (1) *There exists the Hyodo-Kato natural equivalence*

$$\iota_{\mathrm{dR}}:\quad \Pi^n_{\mathrm{HK}}(X;x_1,x_2)\otimes_{K_0^{\mathrm{nr}}}\overline{K}\simeq\Pi^n_{\mathrm{dR}}(X;x_1,x_2)\otimes_K\overline{K}.$$

(2) *There exists a natural equivalence (potentially semistable period isomorphism)*

$$\rho_{\mathrm{pst}}:\quad \Pi^n_{\mathrm{HK}}(X;x_1,x_2)\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}\simeq\Pi^n_{\text{ét}}(X;x_1,x_2)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}}$$

that is compatible with Galois action, Frobenius, the monodromy operator. Extending to $\mathbf{B}_{\mathrm{dR}}$ and using the Hyodo-Kato equivalence, we get the de Rham period isomorphism

$$\rho_{\mathrm{dR}}:\quad \Pi^n_{\mathrm{dR}}(X;x_1,x_2)\otimes_K\mathbf{B}_{\mathrm{dR}}\simeq\Pi^n_{\text{ét}}(X;x_1,x_2)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}$$

that is compatible with filtrations.

These comparison isomorphisms are compatible with change of the index n and with Hopf algebra structures.

The above comparison statements were proved before in the case of curves in [37], [1], for varieties with good reduction over slightly ramified base in [71], and for varieties with good reduction over an unramified base in [56]. The various realizations appearing in these constructions should be naturally isomorphic with ours but we did not check it.

4.4. **Example II: p-adic comparison maps with compact support and compatibilities.**

4.20. When $\varepsilon=\mathrm{HK},\text{ét},\mathrm{dR},DF_K,\mathrm{pst}$, we get from the preceding section, for any K-variety, a complex

$$\mathrm{R}\Gamma_\varepsilon(X):=\mathrm{R}\Gamma_\varepsilon(M(X)^\vee)=\mathrm{R}\Gamma_\varepsilon(M(X))^*$$

which computes the ε-cohomology with enriched coefficients. When $\varepsilon=\text{ét},\mathrm{HK},\mathrm{dR}$ this is the usual complex, respectively, of Galois representations, (φ,N,G_K)-modules, filtered K-vector spaces which computes, respectively, geometric étale cohomology, Hyodo-Kato cohomology and De Rham cohomology with their natural algebraic structures. These complexes are related by the comparison isomorphisms ρ_{dR}, ρ_{pst}, and ι_{dR}.

An interesting point is that these complexes, as well as the comparison isomorphisms are contravariantly functorial in the homological motive $M(X)$. Recall Voevodsky's motives are equipped with special covariant functorialities.

Let X and Y be K-varieties. A *finite correspondence* α from X to Y is an algebraic cycle in $X \times_K Y$ whose support is finite equidimensional over X and which is *special* over X in the sense of [18, 8.1.28].[12] Then by definition, α induces a map $\alpha_* : M(X) \to M(Y)$.

Assume now that X and Y are smooth. Let $f : X \to Y$ be any morphism of schemes of constant relative dimension d. Then we have the Gysin maps $f^* : M(Y) \to M(X)(d)[2d]$ (cf. [23]).

Corollary 4.21. *Consider the notations above.*

Then $\mathrm{R}\Gamma_\varepsilon(X)$ is contravariant with respect to finite correspondences and covariant with respect to morphisms of smooth K-varieties.

Moreover, the comparison isomorphisms ρ_{dR}, ρ_{pst}, ι_{dR} are natural with respect to these functorialities.

Remark 4.22.
(1) Note in particular that covariance with respect to finite correspondences implies the existence of transfer maps f_* for any finite equidimensional morphism $f : X \to Y$ which is special (eg. flat, or X is geometrically unibranch).
(2) The syntomic descent spectral sequence and the syntomic period map of Example 4.17 are natural with respect to the functorialities of the corollary.
(3) We can deduce from [23] the usual good properties of covariant funtoriality (compatibility with composition, projection formulas, excess of intersection formulas,...)

4.23. *Products.* Consider again the notations of the Paragraph 4.20. As said previously, from the Künneth formula, $\mathrm{R}\Gamma_\varepsilon$ is a monoidal functor and the comparison isomorphisms are isomorphisms of monoidal functors.

Consider a K-variety X with structural morphism f. Recall from Remark 4.12 that $M(X)^\vee = f_*(\mathbb{1}_X)$. The functor f_* is left adjoint to a monoidal functor. Therefore it is weakly monoidal and we get a pairing:

$$\mu : \quad M(X)^\vee \otimes M(X)^\vee = f_*(\mathbb{1}_X) \otimes f_*(\mathbb{1}_X) \to f_*(\mathbb{1}_X) = M(X)^\vee$$

in $DM_{gm}(K, \mathbf{Q}_p)$. This induces a cup-product on the ε-complexes:

$$\begin{aligned}\mathrm{R}\Gamma_\varepsilon(X) \otimes \mathrm{R}\Gamma_\varepsilon(X) = \mathrm{R}\Gamma_\varepsilon\big(f_*(\mathbb{1}_X)\big) \otimes \mathrm{R}\Gamma_\varepsilon\big(f_*(\mathbb{1}_X)\big) \overset{K}{\simeq} \mathrm{R}\Gamma_\varepsilon(f_*(\mathbb{1}_X) \otimes f_*(\mathbb{1}_X)) \xrightarrow{\mu_*} \mathrm{R}\Gamma_\varepsilon(f_*(\mathbb{1}_X) \\ = \mathrm{R}\Gamma_\varepsilon(X),\end{aligned} \tag{4.3}$$

where the isomorphism labelled K stands for the structural morphism of the monoidal functor $\mathrm{R}\Gamma_\varepsilon$ – and corresponds to the Künneth formula in ε-cohomology. When $\varepsilon = \text{ét}, \mathrm{HK}, \mathrm{dR}$, we deduce from the definition of this structural isomorphism that these products correspond to the natural products on the respective cohomology. As the comparison isomorphisms are isomorphisms of monoidal functors, we deduce that they are compatible with the above cup-products.

From the end of Example 4.17, we can also define the ε-complex of X with compact support:

$$\mathrm{R}\Gamma_{\varepsilon,c}(X) = \mathrm{R}\Gamma_\varepsilon(f_!(\mathbb{1}_X)).$$

Because we have a natural map $f_* \to f_!$ of functors ([18, 2.4.50(2)]), we also deduce, as usual, a natural map:

$$\mathrm{R}\Gamma_{\varepsilon,c}(X) \to \mathrm{R}\Gamma_\varepsilon(X).$$

From the 6 functors formalism, we get a pairing in $DM_{gm}(K, \mathbf{Q}_p)$:

$$\mu_c : \quad f_*(\mathbb{1}_X) \otimes f_!(\mathbb{1}_X) \overset{(1)}{\simeq} f_!(f^* f_*(\mathbb{1}_X) \otimes \mathbb{1}_X) = f_!(f^* f_*(\mathbb{1}_X)) \xrightarrow{(2)} f_!(\mathbb{1}_X)$$

where the isomorphism (1) stands for the projection formula ([18, 2.4.50(5)]) and the map (2) is the unit map of the adjunction (f^*, f_*). Then, using μ_c instead of μ in formula (4.3), we get the pairing between cohomology and cohomology with compact support:

$$\mathrm{R}\Gamma_\varepsilon(X) \otimes \mathrm{R}\Gamma_{\varepsilon,c}(X) \to \mathrm{R}\Gamma_{\varepsilon,c}(X). \tag{4.4}$$

Using again the fact that the comparison isomorphisms ρ_{dR}, ρ_{pst}, ι_{dR} are isomorphisms of monoidal functor, we deduce that they are compatible with this pairing. Let us summarize:

[12] If X is geometrically unibranch, every α whose support is finite equidimensional over X is special (cf. [18, 8.3.27]). If Z is a closed subset of $X \times_K Y$ which is flat and finite over X, the cycle associated with Z is a finite correspondence (cf. [18, 8.1.31]).

Proposition 4.24. *For $* = \emptyset, c$, we have comparison isomorphisms*

$$\begin{aligned} \iota_{\mathrm{HK},*}: &\quad \mathrm{R}\Gamma_{\mathrm{HK},*}(X)\otimes_{K_0^{\mathrm{nr}}}\overline{K}\simeq \mathrm{R}\Gamma_{\mathrm{dR},*}(X)\otimes_K\overline{K},\\ \rho_{\mathrm{pst},*}: &\quad \mathrm{R}\Gamma_{\mathrm{HK},*}(X)\otimes_{K_0^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}\simeq \mathrm{R}\Gamma_{\mathrm{\acute{e}t},*}(X_{\overline{K}})\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}},\\ \rho_{\mathrm{dR},*}: &\quad \mathrm{R}\Gamma_{\mathrm{dR},*}(X)\otimes_K\mathbf{B}_{\mathrm{dR}}\simeq \mathrm{R}\Gamma_{\mathrm{\acute{e}t},*}(X_{\overline{K}})\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}, \end{aligned}$$

that are compatible with cup-products (4.3) *and with the pairing* (4.4).

5. Syntomic modules

5.1. **Definition.** In this section we use the dg-algebra $\mathscr{E}_{\mathrm{syn},K}$, which represents syntomic cohomology of varieties over K [50, Appendix] to define a category of syntomic modules over any such variety. This is our candidate for coefficients systems (of geometric origin) for syntomic cohomology. We prove that in the case of $\operatorname{Spec} K$ itself the category of syntomic coefficients is (via the period map) a subcategory of potentially semistable representations that is closed under extensions. We call such representations *constructible representations.*

Let us first recall the setting of Voevodsky's h-motives, with coefficients in a given ring R and over any noetherian base scheme S. We let $\mathrm{Sh}(S,R)$ be the category of h-sheaves of R-modules on Sch_S – the category of separated schemes of finite type over S. This is a monoidal Grothendieck abelian category with generators the free R-linear h-sheaves represented by any X in Sch_S; we denote them by $R^h_S(X)$. In particular, its derived category $\mathscr{D}(\mathrm{Sh}(S,R))$ has a canonical structure of a stable monoidal ∞-category in the sense of [59, Def. 3.5] (see also [46]).[13] Moreover, it admits infinite direct sums. Let us define the Tate object as the following complex of R-sheaves: $R_S(1) := R^h_S(\mathbb{P}^1_S)/R^h_S(\{\infty\})[-2]$.

The following theorem is an ∞-categorical summary of a classical construction phrased in terms of model categories in [19]:

Theorem 5.1. *There exists a universal monoidal ∞-category $\underline{\mathscr{D}M_h}(S,R)$ which admits infinite direct sums and is equipped with a monoidal ∞-functor*

$$\Sigma^\infty: \quad \mathscr{D}(\mathrm{Sh}(S,R))\to \underline{\mathscr{D}M}_h(S,R)$$

such that:

- $\mathbb{A}^1$*-Homotopy: for any scheme X in Sch_S, the induced map $\Sigma^\infty R^h_S(\mathbb{A}^1_X)\to\Sigma^\infty R^h_S(X)$ is an isomorphism;*
- $\mathbb{P}^1$*-stability: the object $\Sigma^\infty R_S(1)$ is $\otimes$-invertible.*

Moreover, the monoidal ∞-category $\underline{\mathscr{D}M}_h(S,R)$ is stable and presentable.

Concerning the first point, the statement follows from the existence of localization for monoidal ∞-categories. The statement for the second point follows from [59, 4.16] and the fact that, up to $\mathbb{A}^1$-homotopy, the cyclic permutation on $R_S(1)^{\otimes,3}$ is the identity.

Remark 5.2. According to [19] and [59], the ∞-category $\underline{\mathscr{D}M}_h(S,R)$ is associated with an underlying symmetric monoidal model category – this also implies it can be described by a canonical R-linear dg-category. According to the description of this model category, up to quasi-isomorphism, the objects of $\underline{\mathscr{D}M_h}(S,R)$ can be understood as $\mathbb{N}$-graded complexes of R-linear h-sheaves $(\mathscr{E}_r)_{r\in\mathbb{N}}$ which satisfy the following properties:

- (*Homotopy invariance*) for any integer r, the h-cohomology presheaves $H^*_h(-,\mathscr{E}_r)$ are $\mathbb{A}^1$-invariant;
- (*Tate twist*) there exists a (structural) quasi-isomorphism $\mathscr{E}_r\to\underline{\mathrm{Hom}}(R_S(1),\mathscr{E}_{r+1})$.

One should be careful however that, in order to get the right *symmetric* monoidal structure on the underlying model category, one has to consider in addition an action of the symmetric group of order r on $\mathscr{E}_r$, in a way compatible with the structural isomorphism associated with Tate twists. The corresponding objects are called *symmetric Tate spectra.*[14]

[13] Actually, this follows from the existence of a closed monoidal category structure on the category of complexes of $\mathrm{Sh}(S,R)$ (cf. [16] or [19]) and from [59, Sec. 3.9.1].

[14] See [18, Sec. 5.3] for the construction in motivic homotopy theory.

Example 5.3. Let $S = \mathrm{Spec}(K)$ and $R = \mathbf{Q}_p$. Consider the h-sheaf associated with the presheaf of dg-$\mathbf{Q}_p$-algebras

$$X \mapsto (\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(X, r))$$

defined in 2.9 (see Theorem 2.26 for the isomorphism). Because of [50], it satisfies the homotopy invariance and Tate twist properties stated above; thus as explained in Appendix B of [50], it canonically defines an object $\mathscr{E}_{syn}$ of $\underline{\mathscr{D}M}_h(K, \mathbf{Q}_p)$. Moreover, the dg-structure allows us to put a canonical ring structure on this object, which corresponds to a strict structure (the diagrams encoding commutativity and associativity are commutative not only up to homotopy).

For any scheme X in Sch_S, we put $M_S(X) := \Sigma^\infty R^h_S(X)$, called the (homological) h-motive associated with X/S.

Definition 5.4. We define the stable monoidal ∞-category of h-motives $\mathscr{D}M_h(S, R)$ (resp. constructible h-motives $\mathscr{D}M_{h,c}(S, R)$) over S with coefficients in R as the smallest stable sub-∞-category[15] of $\underline{\mathscr{D}M}_h(S, R)$ containing arbitrary direct sums of objects of the form $M_S(X)(n)[i]$ (resp. objects of the form $M_S(X)(n)[i]$) for a smooth S-scheme X and integers $(n, i) \in \mathbf{Z}^2$.

We let $DM_h(S, R)$ (resp. $DM_{h,c}(S, R)$) be the associated homotopy category, as a triangulated monoidal category.

Example 5.5. When R is a $\mathbf{Q}$-algebra (resp. R is a $\mathbf{Z}/n$-algebra where n is invertible on S), $DM_h(S, R)$ is equivalent to the triangulated monoidal category of rational mixed motives (resp. derived category of R-sheaves on the small étale site of S): see [19], Th. 5.2.2 (resp. Cor. 5.5.4). In particular, $\mathscr{D}M_h(S, R)$ is presentable by a monoidal model category.

The justification of the axioms of $\mathbb{A}^1$-homotopy and $\mathbb{P}^1$-stability added to the derived category of h-sheaves comes from the following theorem:

Theorem 5.6 ([19])**.** *The triangulated categories $DM_h(S, R)$ for various schemes S are equipped with Grothendieck 6 functors formalism and satisfy the absolute purity property. If one restricts to quasi-excellent schemes S and morphisms of finite type, the subcategories $DM_{h,c}(S, R)$ are stable under the 6 operations, and satisfy Grothendieck-Verdier duality.*

We refer the reader to [18, A.5] or [19, Appendix A] for a summary of Grothendieck 6 functors formalism and Grothendieck-Verdier duality.

Let us now take the notations of Example 5.3. We view $\mathscr{E}_{syn}$ in the model category underlying $\underline{\mathscr{D}M}_h(K, \mathbf{Q}_p)$, equiped with its structure of (commutative) dg-algebra. According to [18, 7.1.11(d)], one can assume that $\mathscr{E}_{syn}$ is cofibrant (by taking a cofibrant resolution in the category of dg-algebras according to *loc. cit.*). Given any morphism $f : S \to \mathrm{Spec}(K)$, we put

$$\mathscr{E}_{syn,S} := \mathrm{L}f^*(\mathscr{E}_{syn})$$

which is again a dg-algebra because f^* is monoidal. According to the construction of [18, Sec. 7.2], the category $\mathscr{E}_{\mathrm{syn}}-\underline{\mathscr{M}\mathrm{od}}_S$ of modules over this dg-algebra is endowed with a monoidal model structure, and therefore with a structure of monoidal ∞-category. The free $\mathscr{E}_{\mathrm{syn}}$-module functor induces an adjunction of ∞-categories:

$$R_{\mathrm{syn}} : \underline{\mathscr{D}M}_h(S, \mathbf{Q}_p) \leftrightarrows \mathscr{E}_{\mathrm{syn}}-\underline{\mathscr{M}\mathrm{od}}_S : \mathscr{O}_{\mathrm{syn}}.$$

Given any S-scheme X, and any integer $n \in \mathbf{Z}$, we put $\mathscr{E}_{\mathrm{syn},S}(X)(n) := R_{\mathrm{syn}}(M_S(X)(n))$.

Definition 5.7. Using the above notations, we define the ∞-category of *syntomic modules* (resp. *constructible syntomic modules*) over S as the smallest stable ∞-subcategory of $\mathscr{E}_{\mathrm{syn}}-\underline{\mathscr{M}\mathrm{od}}_S$ containing arbitrary direct sums of modules of the form $\mathscr{E}_{\mathrm{syn},S}(X)(n)[i]$ for a smooth S-scheme X and integers $(n, i) \in \mathbf{Z}^2$.

We denote it by $\mathscr{E}_{\mathrm{syn}}-\mathscr{M}\mathrm{od}_S$ (resp. $\mathscr{E}_{\mathrm{syn}}-\mathscr{M}\mathrm{od}_{c,S}$) and let $\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_S$ (resp. $\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_{c,S}$) be its associated homotopy category. This is a monoidal triangulated category.

[15]Here, and later in Definition 5.7, a *stable sub-∞-category* of an ∞-category $\mathscr{D}$ means a sub-∞-category $\mathscr{D}_0$ of $\mathscr{D}$ in the sense of [45, 1.2.11] such that the associated homotopy category $h\mathscr{D}_0$ is a full triangulated subcategory of the associated homotopy category $h\mathscr{D}$.

In particular, we get an adjunction of triangulated categories:

$$R_{\mathrm{syn}} : DM_h(S, \mathbf{Q}_p) \leftrightarrows \mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_S : \mathscr{O}_{\mathrm{syn}}, \tag{5.1}$$

such that R_{syn}, called the realization functor, is monoidal and sends constructible motives to constructible syntomic modules.

Remark 5.8. By definition, the triangulated category $\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_S$ (resp. $DM_h(S, \mathbf{Q}_p)$) is generated by the objects of the form $\mathscr{E}_{\mathrm{syn},S}(X)(n)$ (resp. $M_S(X)(n)$) for a smooth S-scheme X and an integer $n \in \mathbf{Z}$. By construction, the functor $\mathscr{O}_{\mathrm{syn}}$ commutes with arbitrary direct sums.[16] Thus, because $M_S(X)(n)$ is compact[17] in $DM_h(S, \mathbf{Q}_p)$ (see [18, 15.1.4]), we deduce that $\mathscr{E}_{\mathrm{syn},S}(X)(n)$ is compact. This implies that a syntomic module is constructible if and only if it is compact.[18]

Note also that $\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_S$ is a *compactly generated* triangulated category.

Essentially using the previous theorem and the good properties of the forgetful functor $\mathscr{O}_{\mathrm{syn}}$, we get the following result:

Theorem 5.9. *The triangulated categories $\mathscr{E}_{\mathrm{syn}}$–$\mathrm{mod}_S$ for various schemes S are equipped with Grothendieck 6 functors formalism and satisfy the absolute purity property. If one restricts to quasi-excellent K-schemes S and morphisms of finite type, the subcategories $\mathscr{E}_{\mathrm{syn}}$–$\mathrm{mod}_{c,S}$ are stable under the 6 operations, and satisfy Grothendieck-Verdier duality.*

If one restricts to K-varieties S, the syntomic (pre-)realization functors:

$$R'_{\mathrm{syn}} : \quad DM_{h,c}(S, \mathbf{Q}_p) \to \mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_{c,S},$$

for various S, commute with the 6 operations and in particular with duality.

See Corollary 5.15 for the computation of this functor over the base field K.

Proof. All the reference in this proof refer to [18]. According to 7.2.18, the fibred triangulated category $\mathscr{E}_{\mathrm{syn}}$–$\mathrm{mod} : S \mapsto \mathscr{E}_{\mathrm{syn}}$–$\mathrm{mod}_S$ is a motivic triangulated category (Definition 2.4.45) because $DM_h(-, \mathbf{Q}_p) : S \mapsto DM_h(S, \mathbf{Q}_p)$ is such a category. Besides, it is oriented in the sense of 2.4.38 as the same facts hold for $DM_h(-, \mathbf{Q}_p)$. Thus it satisfies the six functors formalism as explained in 2.4.50.

Applying again 7.2.18, we also deduce that $S \mapsto \mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_S$ is separated (see Def. 2.1.7) and satisfies the absolute purity property (as stated in 14.4.1). This implies in particular that $\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}$ is τ-compatible (see Definition 4.2.20 and Example 4.2.22). Thus the assertion about the stability of constructible syntomic modules under the 6 operations is an application of Theorem 4.2.29.

Besides, the absolute purity property also implies that $\mathscr{E}_{\mathrm{syn}}$–$\mathrm{mod}$ is τ-dualizable (see Definition 4.4.13 and Example 4.4.14). Thus the assertion about duality comes from Theorem 4.4.21 and its Corollary 4.4.24.

The last assertion follows from what was said about $\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}$ and Theorem 4.4.25 applied to the adjunction (5.1). ☐

Remark 5.10. To get a feeling for the category $\mathscr{E}_{\mathrm{syn}}$–$\mathrm{mod}_{c,S}$ the reader might want to recall a more classical case of coefficients defined by de Rham cohomology. Let $K = \mathbf{C}$ be the field of complex numbers; let $\mathscr{E}_{\mathrm{dR}}$ be the commutative ring spectrum representing de Rham cohomology $X \mapsto \mathrm{R}\Gamma_{\mathrm{dR}}(X)$, for varieties X over K. We have

$$H^n_{\mathrm{dR}}(X) = \mathrm{R}\,\mathrm{Hom}_{DM_h(K,\mathbf{C})}(M(X), \mathscr{E}_{\mathrm{dR}}[n]).$$

We can define, in a way analogous to what we have done above, the category of constructible de Rham coefficients $\mathscr{E}_{\mathrm{dR}} - \mathrm{mod}_{c,S}$, for varieties S that are smooth over K. By [18, Example 17.2.22] (using the Riemann-Hilbert correspondence) or by [28, Theorem 3.3.20] (more directly, using the isomorphism

[16] This follows from the fact it is the derived functor of a left Quillen functor, more precisely the functor which forgets the structure of $\mathscr{E}_{\mathrm{syn}}$-module in the category of symmetric spectra which trivially commutes with arbitrary direct sums. See [18], proof of 7.2.14.

[17] Recall an object M of a triangulated category $\mathscr{T}$ is compact when the functor $\mathrm{Hom}_{\mathscr{T}}(M, -)$ commutes with arbitrary direct sums.

[18] This corresponds to the description of perfect complexes of a ring as compact objects of the derived category.

between Betti and de Rham cohomologies) this category is equivalent to the bounded derived category of analytic regular holonomic $\mathscr{D}$-modules on S that are constructible, of geometric origin.

5.11. Recall the Grothendieck-Verdier duality property means that for any regular K-scheme S and any separated morphism of finite type $f : X \to S$, the syntomic module $M_X = f^!(\mathscr{E}_{\mathrm{syn},S})$ is dualizing for the category of constructible syntomic modules over X. In other words, the functor

$$D_X := \underline{\mathrm{Hom}}(-, M_X) : \quad (\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_{c,X})^{op} \to \mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_{c,X} \tag{5.2}$$

is an anti-equivalence of monoidal triangulated categories. Moreover, it exchanges usual functors with exceptional functors: given any separated morphism of finite type $p : Y \to X$, one has: $D_Y p^* = p^! D_X$ and $D_X p_* = p_! D_Y$.

5.2. **Comparison theorem.**

5.12. Consider the abelian category $\mathrm{Rep}_{pst}(G_K)$ of potentially semistable representations and the coinvariants functor

$$\omega_! : \quad \mathrm{Rep}_{pst}(G_K) \to V^f_{\mathbf{Q}_p}$$

where the right hand side is the category of finite dimensional $\mathbf{Q}_p$-vector spaces. It admits a right adjoint denoted by $\omega^!$ which to a finite dimensional $\mathbf{Q}_p$-vector space V associates the representation V with trivial action of G_K. It is obviously exact and monoidal. One could also put $\omega^* = \omega^!$ because it also admits a right adjoint ω_* which to a potentially semistable representation V associates the $\mathbf{Q}_p$-vector V^{G_K} of G_K-invariants. The situation can be pictured as follows:

$$\begin{tikzcd} \mathrm{Rep}_{pst}(G_K) \arrow[r, shift left=2, "\omega_!"] \arrow[r, shift right=2, "\omega_*"'] & V^f_{\mathbf{Q}_p} \arrow[l, "\omega^!=\omega^*"'] \end{tikzcd}.$$

It will be convenient for what follows to enlarge the category $\mathrm{Rep}_{pst}(G_K)$. Consider the category

$$\mathrm{Rep}^\infty_{pst}(G_K) := \mathrm{Ind}-\mathrm{Rep}_{pst}(G_K)$$

of ind-objects. Thus, for us, an infinite potentially semistable representation V will be a $\mathbf{Q}_p$-vector space V with an action of G_K which is a filtering union of sub-$\mathbf{Q}_p$-vector spaces stable under the action of G_K which are potentially semistable representations of G_K. The category $\mathrm{Rep}^\infty_{pst}(G_K)$ is an abelian (symmetric closed) monoidal category which contains $\mathrm{Rep}_{pst}(G_K)$ as a full abelian thick subcategory. Moreover, it is a Grothendieck abelian category – it admits infinte direct sums and filtering colimits are exact. The above diagram of functors extends to this larger category. Note in particular that according to this definition, Formula (2.9) can be rewritten:

$$V_{\mathrm{pst}}\theta^{-1} : \quad \mathrm{R}\Gamma_{\mathrm{syn}}(X, r) \xrightarrow{\sim} \mathrm{R}\omega_* \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r). \tag{5.3}$$

Due to the Drew's thesis [28] together with our main construction (§2.24), we get the following computation of syntomic modules over K:

Theorem 5.13. *There exist a canonical pair of adjoints of triangulated categories:*

$$\rho^* : \mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_K \leftrightarrows \mathrm{D}\big(\mathrm{Rep}^\infty_{\mathrm{pst}}(G_K)\big) : \rho_*$$

such that ρ^* is monoidal and which can be promoted to an adjunction of stable ∞-categories. Moreover, the functor ρ^* is fully faithful and induces by restriction a monoidal fully faithfull triangulated functor:

$$\rho^* : \mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_{c,K} \to \mathrm{D}^b\big(\mathrm{Rep}_{\mathrm{pst}}(G_K)\big)$$

such that for any K-variety X with structural morphism f, there exists a canonical quasi-isomorphism of complexes of G_K-representations:

$$\rho^*\left(f_*\mathscr{E}_{\mathrm{syn},X}(r)\right) \simeq \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r). \tag{5.4}$$

Proof. We will apply Theorem 2.2.7 and Proposition 2.2.21 of [28]. To be consistent with the notations of *loc. cit.*, we take $B = \mathrm{Spec}(K)$ and put $\mathscr{T}_0 = \mathrm{Rep}_{\mathrm{pst}}(G_K)$, $\mathscr{T} = \mathrm{Rep}^{\infty}_{\mathrm{pst}}(G_K)$.

Consider the functor $\tilde{\mathscr{E}}_{syn} : X \mapsto \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, 0)$ (recall that $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, 0) \simeq \mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(0))$ as Galois representations). This is a presheaf of dg-$\mathbf{Q}_p$-algebras on K-varieties with values in $\mathscr{T}_0$. Then $\tilde{\mathscr{E}}_{syn}$ satisfies the axioms of a mixed Weil $\mathscr{T}_0$-theory in the sense of [28, 2.1.1]: the axiom (W1) comes from the fact $\tilde{\mathscr{E}}_{syn}$ satisfies h-descent which is stronger than Nisnevich descent, (W2), (W3) comes from homotopy invariance of geometric p-adic Hodge cohomology and the computation of the syntomic cohomology of K, (W4) comes from the projective bundle formula for geometric p-adic Hodge cohomology, and (W5) was proved in Lemma 2.21. Then we can apply 2.2.7 and 2.2.21 of *loc. cit.* to $\tilde{\mathscr{E}}_{syn}$ and this gives the theorem.

Let us explain this in more detail. First, Drew generalizes Theorem 5.1, to the category $SH_{\mathrm{Rep}_{\mathrm{pst}}(G_K)}(S)$ of Nisnevich sheaves with values in the category of ind-representations $\mathscr{T}$, seen as an enriched category over $\mathscr{T}$ – morphisms are not simply sets but ind-representations. This defines the $\mathrm{Rep}_{\mathrm{pst}}(G_K)$-enriched stable homotopy category over any base scheme S. Drew proves that this category is a stable monoidal ∞-category – actually it is defined by a monoidal model category – that we will denote here by $\mathscr{D}_{\mathbb{A}^1}(K, \mathscr{T})$. We will denote by $\mathscr{D}_{\mathbb{A}^1}(K, \mathbf{Q}_p)$ the usual monoidal ∞-category of $\mathbb{A}^1$-homology, obtained by replacing $\mathscr{T}$ with the category of $\mathbf{Q}_p$-vector spaces– and the associated homotopy category still satisfies the 6 functors formalism (cf. *loc. cit.*, Prop. 1.6.7).[19]

Then applying Theorem 2.1.4 of *loc. cit.* to the presheaf $\tilde{\mathscr{E}}_{syn}$ we get that the geometric p-adic Hodge cohomology is representable in $SH_{\mathrm{Rep}_{\mathrm{pst}}(G_K)}(S)$ by a commutative monoid $\tilde{\mathscr{E}}_{syn}$ in the underlying model category – in our case the corresponding object is simply the collections of presheaves $X \mapsto \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)$, as a $\mathbf{N}$-graded dg-algebra indexed by r, seen as presheaves on Sm_K (the category of smooth K-varieties) with values in $\mathscr{T}$.

Then Drew shows that one can define a monoidal ∞-category of modules over the dg-algebra $\tilde{\mathscr{E}}_{syn}$ which is enriched over $\mathscr{T}$, that we will denote here by $\tilde{\mathscr{E}}_{syn}-\mathrm{mod}_K$. It follows that we have the following interpretation of the Künneth formula: by Theorem 2.2.7 of *loc. cit.* the functor

$$\tilde{\rho} : \quad \tilde{\mathscr{E}}_{syn}-\mathrm{mod}_K \xrightarrow{\sim} D(\mathscr{T}), \quad M \mapsto \mathrm{R}\,\mathrm{Hom}^{\mathscr{T}}_{\tilde{\mathscr{E}}_{syn}}(\tilde{\mathscr{E}}_{syn}, M),$$

where $\mathrm{Hom}^{\mathscr{T}}_{?}$ indicates the enriched Hom (with values in complexes of $\mathscr{T}$), is an equivalence of monoidal triangulated categories. Recall that any smooth K-variety X defines a canonical $\tilde{\mathscr{E}}_{syn}$-module $\tilde{\mathscr{E}}_{syn}(X)$. It follows from the construction that, for any smooth K-variety X and any integer $r \in \mathbf{Z}$, there exists a canonical quasi-isomorphism:

$$\mathrm{R}\,\mathrm{Hom}^{\mathscr{T}}_{\tilde{\mathscr{E}}_{syn}}(\tilde{\mathscr{E}}_{syn}(X), \tilde{\mathscr{E}}_{syn}(r)) \simeq \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r) \tag{5.5}$$

functorial in X.

Now we descend. According to *loc. cit.*, 1.6.8, the pair of adjoint functors (ω^*, ω_*) induces an adjunction of stable ∞-categories:

$$\mathrm{L}\,\omega^* : \mathscr{D}_{\mathbb{A}^1}(K, \mathbf{Q}_p) \leftrightarrows \mathscr{D}_{\mathbb{A}^1}(K, \mathscr{T}) : \mathrm{R}\,\omega_*$$

such that $\mathrm{L}\,\omega^*$ is monoidal. Then Drew defines (*loc. cit.*, 2.2.13) the absolute cohomology associated with the enriched mixed Weil cohomology $\tilde{\mathscr{E}}_{\mathrm{syn}}$ as $\mathrm{R}\,\omega_*(\tilde{\mathscr{E}}_{\mathrm{syn}})$, seen as a monoid in $\mathscr{D}_{\mathbb{A}^1}(K, \mathbf{Q}_p)$ – recall $\mathrm{R}\omega_*$ is weakly monoidal. According to this definition, Formula (5.3), and the definition recalled in Example 5.3, we get:

$$\mathscr{E}_{\mathrm{syn}} \simeq \mathrm{R}\omega_*(\tilde{\mathscr{E}}_{syn}),$$

the absolute cohomology associated with $\tilde{\mathscr{E}}_{syn}$. According to this definition, we deduce from the adjunction $(\mathrm{L}\,\omega^*, \mathrm{R}\,\omega_*)$ an adjunction of stable ∞-categories:

$$\mathrm{L}\,\tilde{\omega}^* : \mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_K \leftrightarrows \tilde{\mathscr{E}}_{syn}-\mathrm{mod}_K : \mathrm{R}\tilde{\omega}_*$$

[19] Essentially, its object are graded presheaves on the category of smooth S-scheme with values in $\mathscr{T}$ satisfying homotopy invariance, Tate twist, as in Remark 5.2, but we have to add the Nisnevich descent property.

whose left adjoint, $\mathrm{L}\,\tilde{\omega}^*$, is monoidal. Therefore, one gets the first two statements of the Theorem by putting:

$$\rho^* = \tilde{\rho} \circ \mathrm{L}\,\tilde{\omega}^*, \quad \rho_* = \tilde{\omega}_* \circ \mathrm{R}\,\tilde{\rho}^{-1}.$$

Moreover, Prop. 2.2.21 of *loc. cit.* tells us that $\mathrm{L}\,\tilde{\omega}^*$ is an equivalence of categories if one restricts to constructible objects on both sides (*i.e.*, generated by, respectively, the objects of the form $\mathscr{E}_{syn}(X)(r)$ and $\tilde{\mathscr{E}}_{syn}(X)(r)$ for a smooth K-scheme X and an integer $r \in \mathbf{Z}$). The fact that ρ^* is fully faithful is a formal consequence of this result together with the fact that $\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_K$ is compactly generated (cf. Rem. 5.8).

Recall that, for any smooth K-variety X with structural morphism $f : X \to \mathrm{Spec}(K)$, one gets:

$$\tilde{\mathscr{E}}_{syn}(X) = \mathrm{L}\,\tilde{\omega}^*(\mathscr{E}_{syn}(X)) = \mathrm{L}\,\tilde{\omega}^*(f_!f^!\mathscr{E}_{syn,K}) = \mathrm{L}\,\tilde{\omega}^* D_K(f_*f^*\mathscr{E}_{syn,K}) = \mathrm{L}\,\tilde{\omega}^* D_K(f_*\mathscr{E}_{syn,X}),$$

where D_K is the Grothendieck-Verdier duality operator on constructible syntomic modules over K defined in Paragraph 5.11. Thus, in the case when X is a smooth K-variety, Formula (5.4) follows from this identification, the definition of ρ^*, and (5.5). One removes the assumption that X is smooth using the fact that the quasi-isomorphism (5.4) can be extended to diagrams of smooth K-varieties and that both the left and the right hand side satisfies (by definition) cohomological descent for the h-topology.

□

Remark 5.14. As a consequence, the category of constructible syntomic modules over K can be identified with a full triangulated subcategory $\mathscr{D}$ of the derived category $D^b\big(\mathrm{Rep}_{\mathrm{pst}}(G_K)\big)$.

It is easy to describe this subcategory: using resolution of singularities, all objects of $\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_{c,K}$ are obtained by taking iterated extensions[20] or retracts of syntomic modules of the form $f_*(\mathscr{E}_{\mathrm{syn},X})(r)$ for a smooth projective morphism $f : X \to \mathrm{Spec}(K)$ and an integer $r \in \mathbf{Z}$ (this is an easy case of the general result [18, 4.4.3]). So $\mathscr{D}$ is the full subcategory of $D^b\big(\mathrm{Rep}_{\mathrm{pst}}(G_K)\big)$ whose objects are obtained by taking retract of iterated extensions of complexes of the form $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)$ for X/K smooth projective and $r \in \mathbf{Z}$.

Similarly, the (essential image of the) category of (not necessarily constructible) syntomic modules over K can be identified with the smallest full triangulated subcategory of $D\big(\mathrm{Rep}^{\infty}_{\mathrm{pst}}(G_K)\big)$ stable under taking (infinite) direct sums and which contains complexes of the form $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)$ with the same assumptions as above.

Composing the syntomic realization functor over K with the fully faithful functor ρ^* above, we get:

Corollary 5.15. *The syntomic (pre-)realization functor of Theorem 5.9 in the case* $S = \mathrm{Spec}(K)$ *defines a triangulated monoidal realization functor:*

$$R_{\mathrm{syn}} : \quad DM_{gm}(K, \mathbf{Q}_p) \simeq DM_{h,c}(K, \mathbf{Q}_p) \xrightarrow{R'_{\mathrm{syn}}} \mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_{c,K} \xrightarrow{\rho^*} \mathrm{D}^b\big(\mathrm{Rep}_{\mathrm{pst}}(G_K)\big).$$

It coincides with the functor $\mathrm{R}\Gamma_{\mathrm{pst}}$ *defined in Paragraph 4.15.*

Proof. Only the last statement requires a proof. By definition, $\mathrm{R}\Gamma_{\mathrm{pst}}$ is the functor defined on $DM_{gm}(K, \mathbf{Q}_p)$ applying Example 4.9 to the functor which to a smooth affine K-variety X associates the complex $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)$. Thus the statement follows from the description of the functor ρ^* in the above proof and the identification (5.5). □

Remark 5.16. The corollary means in particular that the realization R'_{syn} of Theorem 5.9 does indeed extends the realization $\mathrm{R}\Gamma_{\mathrm{pst}}$ to arbitrary K-bases in a way compatible with the 6 operations.

Corollary 5.17. *For a variety* $f : X \to \mathrm{Spec}(K)$*, we have a natural quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathscr{H}}(X, r) = \mathrm{R}\,\mathrm{Hom}_{\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_X}(\mathscr{E}_{\mathrm{syn},X}, \mathscr{E}_{\mathrm{syn},X}(r)).$$

[20]Recall: in a triangulated category $\mathscr{T}$, an object M is an extension of M'' by M' if there exists a distinguished triangle $M' \to M \to M'' \to M'[1]$ in $\mathscr{T}$.

Proof. Since, by the above theorem, $\rho^*(f_*\mathscr{E}_{\mathrm{syn},X}(r)) \simeq \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)$, we have

$$\begin{aligned}
\mathrm{R}\operatorname{Hom}_{\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_X}(\mathscr{E}_{\mathrm{syn},X}, \mathscr{E}_{\mathrm{syn},X}(r)) &= \mathrm{R}\operatorname{Hom}_{\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_X}(f^*\mathscr{E}_{\mathrm{syn},K}, \mathscr{E}_{\mathrm{syn},X}(r))\\
&= \mathrm{R}\operatorname{Hom}_{\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_K}(\mathscr{E}_{\mathrm{syn},K}, f_*\mathscr{E}_{\mathrm{syn},X}(r)) = \mathrm{R}\operatorname{Hom}_{D(\mathrm{Rep}_{\mathrm{pst}}(G_K))}(\mathbf{Q}_p, \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r))\\
&\simeq R\Gamma_{\mathscr{H}}(X, r),
\end{aligned}$$

as wanted. □

This means that we can define syntomic cohomology of a syntomic module in the following way.

Definition 5.18. Let X be a variety over K and $\mathscr{M} \in \mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_X$. *Syntomic cohomology of* $\mathscr{M}$ is the complex

$$\mathrm{R}\Gamma_{\mathscr{H}}(X, \mathscr{M}) = \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathscr{M}) := \mathrm{R}\operatorname{Hom}_{\mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_X}(\mathscr{E}_{\mathrm{syn},X}, \mathscr{M}).$$

This definition is compatible with the definition of syntomic cohomology of Voyevodsky's motives from Example 4.17. That is, for $M \in DM_{\mathrm{gm}}(K, \mathbf{Q}_p)$, we have a canonical quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathrm{Spec}(K), R'_{\mathrm{syn}}(M)) \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(M).$$

This follows easily from Theorem 5.13 and Corollary 5.15.

Remark 5.19. Syntomic cohomology with coefficients was studied before in [51], [52], [64], [6]. The coefficients used there could be called "syntomic local systems". They are variants of the crystalline and semistable local systems introduced by Faltings [30], [31]. There exists also a notion of "de Rham local systems". Those were introduced by Tsuzuki in his (unpublished) thesis [66] and later by Scholze [62] in the rigid analytic setting.

In all these cases, syntomic local systems have a de Rham avatar and an étale one. These two avatars are related by relative Fontaine theory and their cohomologies (de Rham, étale, and syntomic) satisfy p-adic comparison isomorphisms. We hope that this is also the case for the syntomic coefficients introduced here and we will discuss it in a forthcoming paper.

5.3. **Geometric and constructible representations.**

Definition 5.20. Keep the notations of the previous section. We define the category $\mathrm{Rep}_{gm}(G_K)$ (resp. $\mathrm{Rep}_{Ngm}(G_K)$, resp. $\mathrm{Rep}_c(G_K)$) of *geometric* (resp. *Nori's geometric*, resp. *constructible*) p-adic representations of G_K as the essential image of the following (composite) functor:

$$\begin{aligned}
&DM_{gm}(K, \mathbf{Q}_p) \xrightarrow{R_{\mathrm{syn}}} \mathrm{D}^b\big(\mathrm{Rep}_{\mathrm{pst}}(G_K)\big) \xrightarrow{H^0} \mathrm{Rep}_{\mathrm{pst}}(G_K),\\
\textit{resp. } \mathrm{R}_{\mathrm{pst}}:\quad &\mathrm{MM}(K)_{\mathbf{Q}_p} \to \mathrm{Rep}_{\mathrm{pst}}(G_K),\\
\textit{resp. } \mathscr{E}_{\mathrm{syn}}-\mathrm{mod}_{c,K} &\xrightarrow{\rho^*} \mathrm{D}^b\big(\mathrm{Rep}_{\mathrm{pst}}(G_K)\big) \xrightarrow{H^0} \mathrm{Rep}_{\mathrm{pst}}(G_K).
\end{aligned}$$

Thus a geometric G_K-representation can be described as the geometric étale p-adic cohomology of a Voevodsky's motive over K with its natural Galois action and Nori's geometric G_K-representation - as the geometric étale p-adic cohomology of a Nori's motive. By Corollary 5.15, a geometric G_K-representation is constructible and by the compatibility of realizations of Nori's and Voevodsky's motives (4.2) geometric representation is Nori's geometric. So we have the following inclusions of categories

(5.6)
$$\begin{tikzcd}
 & \mathrm{Rep}_c(G_K) \arrow[dr, hook] & \\
\mathrm{Rep}_{gm}(G_K) \arrow[ur, hook] \arrow[r, hook] & \mathrm{Rep}_{Ngm}(G_K) \arrow[r, hook] & \mathrm{Rep}_{\mathrm{pst}}(G_K)
\end{tikzcd}$$

We do not know much about these subcategories. Neither do we have a conjectural description of them in purely algebraic terms – this contrasts very much with the case of number fields, see [34].

Here is a few trivial facts:

- All three subcategories are stable under taking tensor products and twists.

- All three categories contain representations of the form $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$ for any integers $i, r \in \mathbf{N} \times \mathbf{Z}$ and any K-variety X (possibly singular). They also contain kernel of projectors of these particular representations when the projector is induced by an algebraic correspondence modulo rational equivalence for X/K projective smooth, and any finite correspondence for an arbitrary X/K.

We do not know if any of these subcategories are stable under taking sub-objects, quotients, or even direct factors.

The following fact is the only nontrivial result about stability.

Proposition 5.21. *The category* $\mathrm{Rep}_c(G_K)$ *contains all potentially semistable extensions of representations of the form* $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$ *for* X/K *smooth and projective,* $i \in \mathbf{N}$*,* $r \in \mathbf{Z}$*.*

Proof. Let $\mathscr{D}$ be the essential image of the functor $\rho^* : \mathscr{E}_{\text{syn}}\text{-mod}_{c,K} \to \mathrm{D}^b\big(\mathrm{Rep}_{\text{pst}}(G_K)\big)$. Note that $\mathscr{D}$ is stable under taking retracts, suspensions, and extensions (see Remark 5.14). We first prove that for any smooth projective morphism $f : X \to \mathrm{Spec}(K)$ and any integer $r \in \mathbf{Z}$, the representation $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$ belongs to $\mathscr{D}$.

The complex or representations $\mathrm{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r) \simeq \mathrm{R}\, f_*(\mathbf{Q}_p)(r)$ belongs to $\mathscr{D}$ (according to the end of Theorem 5.13). Moreover, using [26, 4.1.1] and [25], there exists an isomorphism in $D^b\big(\mathrm{Rep}_{\text{pst}}(G_K)\big)$:

$$\mathrm{R}\, f_*(\mathbf{Q}_p)(r) \simeq \bigoplus_{i \in \mathbf{Z}} \mathrm{R}^i\, f_*(\mathbf{Q}_p)(r)[-i].$$

This means that $\mathrm{R}^i\, f_*(\mathbf{Q}_p)(r)$ is the kernel of a projector of $\mathrm{R}\, f_*(\mathbf{Q}_p)(r)$, thus belongs to $\mathscr{D}$ because the later is stable under taking retracts.

Thus the result follows, using the fact that $\mathscr{D}$ is stable under taking extensions in $D^b\big(\mathrm{Rep}_{\text{pst}}(G_K)\big)$. □

Remark 5.22. The preceding proof shows that the essential image $\mathscr{D}$ of constructible syntomic modules in complexes of pst-representations contains arbitrary truncations of the complexes $\mathrm{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r)$. A natural question would be to determine if, more generally, $\mathscr{D}$ is stable under taking truncation. This would immediately imply that $\mathrm{Rep}_c(G_K)$ is a thick abelian subcategory of $\mathrm{Rep}(G_K)$ (*i.e.*, it is stable under taking sub-objects and quotients) and that $\mathscr{D}$ is the category of bounded complexes of pst-representations whose cohomology groups are constructible in the above sense.

Remark 5.23. In the diagram of inclusions (5.6) we believe that the first bottom one is an equality and the rest are strict. We can support this belief with the following observations. The first bottom inclusion should be an equality since the category of Nori's motives is expected to be the heart of a motivic t-structure on $DM_{gm}(K, \mathbf{Q}_p)$ (see [42, p. 374]). The second bottom and the first skewed inclusions should be strict by the philosophy of weights: by Proposition 5.21, we allow all potentially semistable extensions as extensions of certain geometric representations in the constructible category but in the geometric category such extensions should satisfy a weight filtration condition. For properties of geometric representations coming from abelian varieties over $\mathbf{Q}_p$ see the work of Volkov [67], [68].

For the second skewed inclusion, take $k = F_q$, the finite field with $q = p^s$ elements. Let $V \in \mathrm{Rep}_c(G_K)$ be a constructible representation. Then, by the Conjecture of purity of the weight filtration, the φ-module $D_{\text{pst}}(V)$ is an extension of "pure" φ-modules, i.e., φ-modules such that, for a number $a \geq s$, φ^a has eigenvalues that are p^a - Weil numbers[21] (cf., [41, Conjecture 2.6.5]). But there are crystalline representations that do not have this property. For example, any unramified character $\chi : G_{K_0} \to \mathbf{Q}_p$, $Fr \mapsto \mu \in \mathbf{Q}_p^*$, such that μ is not a p^a -Weil number for any $a \geq 0$ (such a μ exists by the uncountability of $\mathbf{Q}_p$).

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