Compositio Mathematica **109:** 67–97, 1997.

Duality in the cohomology of crystalline local systems

WIESŁAWA NIZIOL
The University of Chicago, Department of Mathematics, 5734 University Avenue, Chicago, Illinois 60637. e-mail: niziol@math.uchicago.edu

Received 16 July 1995; accepted in final form 3 August 1996

Abstract. Let k be a perfect field of a positive characteristic p, K – the fraction field of the ring of Witt vectors $W(k)$. Let X be a smooth and proper scheme over $W(k)$. We present a candidate for a cohomology theory with coefficients in crystalline local systems: p-adic étale local systems on X_K characterized by associating to them so called Fontaine-crystals on the crystalline site of the special fiber X_k. We show that this cohomology satysfies a duality theorem.

Mathematics Subject Classifications (1991): 14F30, 11G25.

Key words: Crystalline local systems, duality

1. Introduction

In this article we prove a duality theorem in the cohomology of crystalline local systems.

Let k be a perfect field of a positive characteristic p, K – the fraction field of the ring of Witt vectors $W(k)$. Let X be a smooth and proper scheme over $W(k)$. In [5] Faltings introduced the notion of crystalline local systems: p-adic étale local systems on X_K characterized by associating to them so called Fontaine-crystals on the crystalline site of the special fiber X_k. Étale cohomology sheaves, generic fibers of finite flat p-group schemes on X, and Tate-twists tend to form such systems.

We present here a candidate for a cohomology theory with such coefficients. In the particular case of the Tate-twists $\mathbf{Z}/p^n(r)$, $r \geqslant 0$, it is equal to the syntomic cohomology introduced by Fontaine and Messing [7]. In general, it should be thought of as a p-torsion analogue of the arithmetic étale cohomology of X with coefficients in locally constant sheaves with torsion different from p.

Pursuing this analogy one would expect that, in the case k is finite, these cohomology groups would satisfy certain duality. We show here that this is indeed the case. This is done by a careful study of a map from our cohomology to the étale cohomology of X_K and follows from Faltings comparison theorem between crystalline and étale cohomologies, and the crystalline, étale, Galois, and Bloch–Kato dualities. As an interesting byproduct of our computations we get a degeneration of the Hochschild–Serre spectral sequence of crystalline local systems.

Throughout the paper p will be a fixed odd prime, for a field K, $\overline{K}$ will denote a fixed algebraic closure of K, and, for a scheme X, $\mathcal{X}$ will denote the associated formal scheme. Locally constant étale sheaf on X will mean, depending on a context, an element of the Ind-category of finite étale covers of X or the Ind-category of finite étale commutative group schemes over X.

2. The categories $\mathcal{MF}^{\nabla}_{[a,b]}(X)$

Let V denote a complete discrete valuation ring with a fraction field K of characteristic 0 and a perfect residue field k of characteristic p. Assume that V is absolutely unramified.

Let R be a smooth V-algebra. Fix a semilinear endomorphism $\phi\colon \widehat{R} \to \widehat{R}$ lifting the Frobenius on R/pR. For all integers a, b, $a \leqslant b$, we have the following category $\mathcal{MF}^{\nabla}_{[a,b]}(R,\phi)$ [5]: an object of $\mathcal{MF}^{\nabla}_{[a,b]}(R,\phi)$ is a p-torsion, finitely generated R-module M with a descending filtration F^iM such that $F^aM = M$, $F^{b+1}M = 0$ and R-linear maps $\phi^i\colon F^iM \otimes_{R\phi} R \to M$ such that $\phi^i(x) = p\phi^{i+1}(x)$ for $x \in F^{i+1}M$. Let R-module $\widetilde{M}$ be the colimit of the following diagram

$$F^{i-1} \leftarrow MF^i \xrightarrow{p} MF^i \leftarrow MF^{i+1} \xrightarrow{p} MF^{i+1}M.$$

The above condition is equivalent to the fact that the maps ϕ^i induce an R-linear homomorphism $\phi\colon \widetilde{M} \otimes_{R\phi} R \to M$. One additionaly requires this homomorphism to be an isomorphism.

M is also equipped with an integrable nilpotent connection $\nabla\colon M \to M \otimes_R \Omega^1_{R/V}$ satisfying Griffiths transversality, i.e., $\nabla(F^iM) \subset F^{i-1}M \otimes_R \Omega^1_{R/V}$. Moreover the maps ϕ^i are parallel with respect to the map $d\phi_*/p\colon \Omega^1_{R/V} \otimes_{R\phi} R \to \Omega^1_{R/V}$. In a more convenient form, the connection ∇ induces a connection (integrable and nilpotent) on $\widetilde{M} \otimes_{R\phi} R$ and the above condition is equivalent to the map $\phi\colon \widetilde{M} \otimes_{R\phi} R \xrightarrow{\sim} M$ being parallel in the usual sense.

Category $\mathcal{MF}^{\nabla}_{[a,b]}(R,\phi)$ has many nice properties:

(1) it is Abelian;
(2) the filtration is by direct summonds;
(3) F^iM is locally a direct sum of modules of the form R/p^eR;
(4) for $b - a \leqslant p - 1$, it is independent of the choice of the Frobenius lift ϕ, i.e., if ϕ_1 is another Frobenius lift, then there is an equivalence of the categories $\mathcal{MF}^{\nabla}_{[a,b]}(R,\phi)$ and $\mathcal{MF}^{\nabla}_{[a,b]}(R,\phi_1)$ satisfying a cocycle condition. In fact, there is a well defined parallel transition map

$$\alpha_{\phi,\phi_1}\colon \widetilde{M} \otimes_{R\phi} R \xrightarrow{\sim} \widetilde{M} \otimes_{R\phi_1} R.$$

such that $\alpha_{\phi_1,\phi_2}\alpha_{\phi,\phi_1} = \alpha_{\phi,\phi_2}$.

From now on we will assume that $0 \leqslant b - a \leqslant p - 2$.

Let X be a smooth and separated scheme over V. To globalize the above construction one covers X with affines $U_i = \mathrm{Spec}(R_i)$ and chooses Frobenius lifts ϕ_i: $\widehat{R}_i \to \widehat{R}_i$. $\mathcal{MF}^{\nabla}_{[a,b]}(X)$ is defined as the glueing of the categories $\mathcal{MF}^{\nabla}_{[a,b]}(R_i, \phi_i)$ via the maps α_{ϕ_i,ϕ_j}. Thus $\mathcal{MF}^{\nabla}_{[a,b]}(X)$ consists of filtered $\mathcal{O}_X$-modules M equipped with an integrable, quasi-nilpotent and Griffiths transversal connection, and, for every i, a structure of an $\mathcal{MF}^{\nabla}_{[a,b]}(R_i, \phi_i)$-object on M_{U_i} such that on U_{ij} the two structures glue well under α_{ϕ_i,ϕ_j}. It easily follows that $\mathcal{MF}^{\nabla}_{[a,b]}(X)$ does not depend on the choice of the data $\{(R_i), (\phi_i)\}$ and that it is an Abelian category.

Recall [3] that given a filtered $\mathcal{O}_X$-module M equipped with an integrable, quasi-nilpotent and Griffiths transversal connection, there is a unique filtration on the associated crystal $\mathcal{M}$ whose value on $\mathcal{X}$ is the given filtration and such that for every thickening $U \hookrightarrow T$ in $\mathrm{Cris}(\mathcal{X}/\mathrm{Spf}(V))$, and for every k,

$$J_T \cap F^k\mathcal{M}_T = J_T^{[1]}F^{k-1}\mathcal{M}_T + J_T^{[2]}F^{k-2}\mathcal{M}_T + \cdots,$$

where J_T is the ideal of U in T. In addition, we also know that, for every morphism $f: T' \to T$ in $\mathrm{Cris}(\mathcal{X}/\mathrm{Spf}(V))$,

$$F^k\mathcal{M}_{T'} = F^k_f\mathcal{M}_{T'} + J_{T'}F^{k-1}_f\mathcal{M}_{T'} + \cdots,$$

where $F^i_f\mathcal{M}_{T'} = \mathrm{Im}(f^*F^i\mathcal{M}_T \to \mathcal{M}_{T'})$.

The maps ϕ^i acting on objects of $\mathcal{MF}^{\nabla}_{[a,b]}(X)$ can also be lifted to some thickenings. Take a smooth V-algebra R and an embedding $\mathrm{Spec}(R) \hookrightarrow W$ into a smooth V-scheme W. Choose Frobenius lifts ϕ on $\widehat{R}$ and ψ on $\mathcal{W}$ – the p-adic formal scheme associated to W. Let D be the p-adic completion of the divided power envelope algebra of $\mathrm{Spec}(R)$ in W. Denote by ψ_D the extension of ψ to D. D being p-torsion free, we can give it a structure of an $\mathcal{MF}$-object: set D^i equal to D if $i \leqslant 0$, to the closure of the ideal $J_D^{[i]}$ if $0 \leqslant i \leqslant p-1$, and to 0 otherwise, and define $\psi^i_D: D^i \to D$ as the divided Frobenius ψ_D/p^i.

LEMMA 2.1. *For any $M \in \mathcal{MF}^{\nabla}_{[a,b]}(R, \phi)$, there is a canonical D-linear map*

$$\phi_D: \widetilde{\mathcal{M}_D} \otimes_{D\psi_D} D \to \mathcal{M}_D,$$

where, in the definition of $\widetilde{\mathcal{M}_D}$, the filtration on $\mathcal{M}_D$ is cut by setting $F^i\mathcal{M}_D = 0$ for $i > p - 1 + a$. Moreover, the map ϕ_D is independent of the choice of ϕ.

Proof. Fix a retraction $h: \widehat{R} \to D$. Define $\widetilde{\mathcal{M}^h_D}$ as $\sim$-object associated to the filtration

$$\mathcal{M}^i_D = \sum_{i-k \leqslant p-1} D^{i-k}h^*F^k\mathcal{M}_{\widehat{R}}$$

on $\mathcal{M}_D$. Since $\mathcal{M}_{\widehat{R}} \simeq \mathrm{gr}^{\cdot}_F(\mathcal{M}_{\widehat{R}})$, it easily follows that the natural map ω: $\widetilde{h^*\mathcal{M}_{\widehat{R}}} \otimes_D \widetilde{D} \to \widetilde{\mathcal{M}^h_D}$ sending $x^k \otimes y^j$, $x^k \in h^*F^k\mathcal{M}_{\widehat{R}}$, $y^j \in D^j$ to x^ky^j in $\mathcal{M}^{k+j}_D$ is an isomorphism.

Define ϕ_D: $\widetilde{\mathcal{M}}_D \otimes_{D\psi_D} D \to \mathcal{M}_D$ as the composition

$$
\begin{aligned}
&\widetilde{\mathcal{M}}_D \otimes_{D\psi_D} D \to \widetilde{\mathcal{M}^h_D} \otimes_{D\psi_D} D \xleftarrow[\sim]{\omega} (\widetilde{h^*\mathcal{M}_{\widehat{R}}} \otimes_D \widetilde{D}) \otimes_{D\psi_D} D \\
&\simeq h^*\widetilde{\mathcal{M}}_{\widehat{R}} \otimes_{D\psi_D} D \otimes_D D_{\psi_D} \otimes_D \widetilde{D} \\
&= \widetilde{\mathcal{M}}_{\widehat{R}} \otimes_{\widehat{R}} D \otimes_{D\psi_D} D \otimes_D D_{\psi_D} \otimes_D \widetilde{D} \\
&\xrightarrow[\sim]{\alpha_{\psi_D,\phi}} \widetilde{\mathcal{M}}_{\widehat{R}} \otimes_{\widehat{R}\phi} \widehat{R} \otimes_{\widehat{R}} D \otimes_D D_{\psi_D} \otimes_D \widetilde{D} \\
&\xrightarrow{\phi_{\widehat{R}} \otimes (\psi^i_D)} \mathcal{M}_{\widehat{R}} \otimes_{\widehat{R}} D \otimes_D D \simeq \mathcal{M}_D.
\end{aligned}
$$

Here, $\phi_{\widehat{R}}$ is the structural map $\phi_{\widehat{R}}$: $\widetilde{\mathcal{M}}_{\widehat{R}} \otimes_{\widehat{R}\phi} \widehat{R} \simeq \mathcal{M}_{\widehat{R}}$, and the map $\alpha_{\psi_D,\phi}$ is the transition map of Faltings [5]: in local coordinates $t_1, \dots, t_d$ on R and for $m \in F^i\mathcal{M}_{\widehat{R}}$,

$$
\alpha_{\psi_D,\phi}(m \otimes 1) = \sum_I \nabla(\partial_I)(m) \otimes \left(\frac{(\psi_D h(t) - h\phi(t))^I}{(I\,!\, p^{\min(i-a,|I|)})} \right),
$$

where, for any multindex $I = (i_1, \dots, i_d)$, $\nabla(\partial_I)$ is an endomorphism of $\mathcal{M}_{\widehat{R}}$ corresponding to the PD-differential operator ∂_I ($\partial_i = \partial/\partial t_i$). Here, $\nabla(\partial_I)(m)$ is considered as an element of $F^{\max(a,i-|I|)}\mathcal{M}_{\widehat{R}}$.

As expected, in the case when ϕ and ψ commute, the map ϕ_D is the obvious composition of $\phi_{\widehat{R}}$ and ψ_D.

A transition map between the constructions corresponding to two different choices of h can be induced from the transition map between two retractions of $\mathcal{M}_{\widehat{R}}$. That it commutes with our maps follows from the fact that $\widetilde{\mathcal{M}}$ is a Frobenius twisted crystal, that the Frobeniuses acting on $\mathcal{M}$ are parallel and that the transition maps α are compatible with the change of the retraction.

Independence of the construction from the choice of ϕ is easily seen. □

For any $M \in \mathcal{MF}^{\nabla}_{[a,b]}(R, \phi)$, the D-module $\mathcal{M}_D$ is equipped with an $\mathcal{O}_W$-connection which is integrable, quasi-nilpotent, and compatible with the natural connection on D. One easily checks that it is also Griffiths transversal. We will need the following fact.

LEMMA 2.2. *The maps* ϕ^i_D: $F^i\mathcal{M}_D \otimes_{D\phi_D} D \to \mathcal{M}_D$ *are parallel, i.e., the following diagram commutes*

$$\begin{array}{ccc} F^i\mathcal{M}_D & \xrightarrow{\nabla} & F^{i-1}\mathcal{M}_D\otimes_{\mathcal{O}_W}\Omega^1_{W/V} \\ \downarrow{\scriptstyle \phi^i_D} & & \downarrow{\scriptstyle 1\otimes d\psi_*/p} \\ \mathcal{M}_D & \xrightarrow{\nabla} & \mathcal{M}_D\otimes_{\mathcal{O}_W}\Omega^1_{W/V}. \end{array}$$

Proof. Define a connection, that is compatible with the connection on D, $\tilde{\nabla}$: $\widetilde{\mathcal{M}}_D\otimes_{D\phi_D}D\to\widetilde{\mathcal{M}}_D\otimes_{D\phi_D}D\otimes_{\mathcal{O}_W}\Omega^1_{W/V}$ by sending $m\otimes 1$ to $(1\otimes d\psi_*/p)(\nabla(m))$. Both connections on $\mathcal{M}_D$ and D being integrable, we get an integrable connection. Griffiths transversality of ∇ and the fact that $\widetilde{\mathcal{M}_D/p}\simeq \mathrm{gr}^{\cdot}_F(\mathcal{M}_D/p)$ yield also that $\tilde{\nabla}$ is nilpotent. We can thus look at the corresponding hyperstratifications ε_{∇} and $\varepsilon_{\tilde{\nabla}}$ and to see that they commute with ϕ_D it suffices to use that ϕ itself is parallel and that the maps α (being parallel) exhibit certain compatibility with the transition maps between different retractions. R being smooth, the computations are tedious but easy. □

In what follows, we will denote by $\mathcal{MF}^{\nabla}_{[a,b]}(X_n)$ the subcategory of objects from $\mathcal{MF}^{\nabla}_{[a,b]}(X)$ annihilated by p^n.

LEMMA 2.3. *If*

$$0\to L\to M\to N\to 0$$

is exact in $\mathcal{MF}^{\nabla}_{[a,b]}(X_n)$*, then, for every fundamental thickening* $U\hookrightarrow T$ *in* $\mathrm{Cris}(X_n/V_n)$*, the sequence*

$$0\to\mathcal{L}_T\to\mathcal{M}_T\to\mathcal{N}_T\to 0$$

is exact in the filtered sense.

Proof. It is enough to argue locally. Assume thus that U is affine and that there is a retraction $h\colon T\to U$. We want an exactness of

$$0\to\sum_k J_T^{[k]}F_h^{i-k}\mathcal{L}_T\to\sum_k J_T^{[k]}F_h^{i-k}\mathcal{M}_T\xrightarrow{f}\sum_k J_T^{[k]}F_h^{i-k}\mathcal{N}_T\to 0,$$

where J_T is the ideal of U in T. First, we claim that, for every i,k, the sequence

$$0\to J_T^{[k]}F_h^i\mathcal{L}_T\to J_T^{[k]}F_h^i\mathcal{M}_T\to J_T^{[k]}F_h^i\mathcal{N}_T\to 0$$

is exact. Note that $J_T^{[k]}\otimes_{\mathcal{O}_T}F_h^i\mathcal{M}_T\simeq J_T^{[k]}F_h^i\mathcal{M}_T$. Indeed, since $F_h^i\mathcal{M}_T\simeq h^*F^iM$, it suffices to show that $\mathrm{Tor}_1{}^{\mathcal{O}_U}(F^iM,\mathcal{O}_T/J_T^{[k]})=0$, which follows by devissage on $\mathcal{O}_T/J_T^{[k]}$, $J_T^{[j]}/J_T^{[j+1]}$ being locally free on U. This yields the exact sequence

$$J_T^{[k]}F_h^i\mathcal{L}_T\to J_T^{[k]}F_h^i\mathcal{M}_T\to J_T^{[k]}F_h^i\mathcal{N}_T\to 0.$$

Since $\mathcal{L}_T \hookrightarrow \mathcal{M}_T$, we also get an injection. Returning to the previous sequence we see that the only nontrivial fact is the exactness in the middle. We will argue by induction on $a \leqslant n \leqslant b$ such that $x \in \Sigma_{k\geqslant n} J_T^{[i-k]} F_h^k \mathcal{M}_T$, $f(x) = 0$. The case $n = b$ follows from the above. Assume now that we know the exactness for the elements of $\Sigma_{k>n} J_T^{[i-k]} F_h^k \mathcal{M}_T$. Take $x \in \Sigma_{k\geqslant n} J_T^{[i-k]} F_h^k \mathcal{M}_T$ such that $f(x) = 0$. Write $x = y + z, y \in \Sigma_{k>n} J_T^{[i-k]} F_h^k \mathcal{M}_T, z \in J_T^{[i-n]} F_h^n \mathcal{M}_T$. Since $f(x) = f(y) + f(z) = 0$, $f(y) \in J_T^{[i-n]} F_h^n \mathcal{N}_T \cap (\Sigma_{k>n} J_T^{[i-k]} F_h^k \mathcal{N}_T)$. But $\mathcal{N}_T \simeq \mathrm{gr}^{\cdot}_{F_h} \mathcal{N}_T$, locally, so $f(y) \in J_T^{[i-n]} F_h^{n+1} \mathcal{N}_T$. Take $y' \in J_T^{[i-n]} F_h^{n+1} \mathcal{M}_T$ such that $f(y') = f(y)$. Set $x' = y' - y$. We have that $f(x') = 0$ and $x' \in \Sigma_{k>n} J_T^{[i-k]} F_h^k \mathcal{M}_T$. By induction, x' comes from $\Sigma_{k>n} J_T^{[i-k]} F_h^k \mathcal{L}_T$. Since $x = y + z = y' - x' + z = -x' + (y' + z)$, $f(y' + z) = 0$ and $y' + z \in J_T^{[i-n]} F_h^n \mathcal{M}_T$, $y' + z$ comes from $J_T^{[i-n]} F_h^n \mathcal{L}_T$ and we are done. □

3. Cohomology of $\mathcal{MF}^{\nabla}_{[a,b]}(X)$-crystals

Let $V = W(k)$ be the ring of Witt vectors over a perfect field k of positive characteristic p. Let X be a smooth, separated scheme over V of relative dimension d.

Choose a covering of X by a finite number of $U_i = \mathrm{Spec}(R_i)$, $i \in I$, and embeddings $U_i \hookrightarrow W_i$ into affine, smooth V-schemes $W_i = \mathrm{Spec}(T_i)$. For every $J \subset I$, set

$$U_J = \mathrm{Spec}(R_J) = \bigcap_{j \in J} U_j,$$

$$W_J = \mathrm{Spec}(T_J) = \prod_{j \in J} W_j, \qquad D_J = D_{\widehat{R_J}(T_J)}.$$

Fix n. Let $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X_n)$, $b - a \leqslant p - 2$. We will reduce everything above mod p^n but, as long as this does not cause confusion, we will try to omit the indices.

Define $\Omega(\mathcal{M}_J)^{\cdot}$ as the complex $\mathcal{M}_{D_J} \otimes_{T_J} \Omega^{\cdot}_{T_J/V}$. Filter the D_J-modules $\mathcal{M}_{D_J} \otimes_{T_J} \Omega^i_{T_J/V}$ by submodules $F^k(\mathcal{M}_{D_J} \otimes_{T_J} \Omega^i_{T_J/V}) := F^{k-i}\mathcal{M}_{D_J} \otimes_{T_J} \Omega^i_{T_J/V}$. Griffiths transversality gives that, for fixed k, the submodules $F^k(\mathcal{M}_{D_J} \otimes_{T_J} \Omega^i_{T_J/V})$ form a subcomplex $F^k\Omega(\mathcal{M}_J)^{\cdot}$ of $\Omega(\mathcal{M}_J)^{\cdot}$.

Choose Frobenius lifts ϕ_J on $\widehat{R}_J$ and ψ_i on $\widehat{T}_i$ and set $\psi_J = \Pi\psi_i$. For $k \leqslant p - 1 + a$, the maps

$$\phi_J^{k,i} \colon F^{k-i}\mathcal{M}_{D_J} \otimes_{T_J} \Omega^i_{T_J/V} \to \mathcal{M}_{D_J} \otimes_{T_J} \Omega^i_{T_J/V},$$

$\phi_J^{k,i} = \phi_{D_J}^{k-i} \otimes d\psi_{J*}/p^i$, glue (Lemma 2.2) to a Frobenius $\phi^k \colon F^k\Omega(\mathcal{M}_J)^{\cdot} \to \Omega(\mathcal{M}_J)^{\cdot}$.

Assume now that $p-1+a \geqslant 0$. Set

$$\mathcal{S}(\mathcal{M}_J)^{\cdot} := \mathrm{Cone}(F^0\Omega(\mathcal{M}_J)^{\cdot} \xrightarrow{\phi^0-1} \Omega(\mathcal{M}_J)^{\cdot})[-1].$$

This is independent of the choice of n such that $p^n\mathcal{M}=0$.

Now we globalize. Take an index set $K \subset J$. There is the obvious restriction map $\mathrm{res}_{K,J}\colon \Omega(\mathcal{M}_K)^{\cdot} \to \Omega(\mathcal{M}_J)^{\cdot}$. It clearly preserves the filtrations and it is easy to see that it behaves well with respect to the connections. Also, since the following diagram commutes

$$\begin{array}{ccc} \widetilde{\mathcal{M}_{D_K}} \otimes_{D_K \psi_K} D_K & \longrightarrow & \widetilde{\mathcal{M}_{D_J}} \otimes_{D_J \psi_J} D_J \\ \downarrow{\scriptstyle \phi_{D_K}} & & \downarrow{\scriptstyle \phi_{D_J}} \\ \mathcal{M}_{D_K} & \longrightarrow & \mathcal{M}_{D_J} \end{array}$$

(use the transition maps α), it behaves well with respect to the Frobenius.

The restriction maps satisfy the usual compatibilities, hence varying J we get from the complexes $\mathcal{S}(\mathcal{M}_J)^{\cdot}$ a double complex. Denote by $\mathcal{S}(\mathcal{M})^{\cdot}$ the associated simple complex.

LEMMA 3.1. *For any two choices of the covering, there is a canonical quasi isomorphism between the corresponding complexes* $\mathcal{S}(\mathcal{M})^{\cdot}$.

Proof. Assume that we have two choices $\mathcal{A}=(U_i,\phi_I,W_i,\psi_i)$, $\mathcal{B}=(V_j,\beta_J,Z_j,\gamma_j)$. Consider two new choices

$$\mathcal{C}_1=(U_i\cap V_j,\phi_I,W_i\times Z_j,\psi_i\times\gamma_j),$$

$$\mathcal{C}_2=(U_i\cap V_j,\beta_J,W_i\times Z_j,\psi_i\times\gamma_j).$$

The complexes associated to $\mathcal{C}_1$, $\mathcal{C}_2$ are in fact identical (Lemma 2.1). In studying the pairs $(\mathcal{A},\mathcal{C}_1)$ and $(\mathcal{B},\mathcal{C}_2)$ we may disregard the Frobenius and then the required quasi isomorphisms follow from filtered cohomological descent for crystalline cohomology. □

For $\mathcal{M}\in\mathcal{MF}^{\nabla}_{[a,b]}(X)$, define

$$H^*_{f,a,b}(X,\mathcal{M}) := H^*(\mathcal{S}(\mathcal{M})^{\cdot}).$$

It follows from Lemma 2.3 that, for every n, $H^*_{f,a,b}(X,\cdot)$ is a cohomology theory on $\mathcal{MF}^{\nabla}_{[a,b]}(X_n)$.

Remark 1. Various generalizations of the syntomic cohomology of Fontaine and Messing appear in the work of many people. In particular, we believe, although we didn't check the details, that our construction agrees with that of [14].

Remark 2. When the relative dimension of X over V is 0, our cohomology theory $H^*_{f,a,b}(X,\cdot)$ agrees with that of Bloch and Kato [3].

PROPOSITION 3.1. *If* $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$, then $H^i_{f,a,b}(X, \mathcal{M}) = 0$ for $i > -a + d + 1$.

Proof. We can assume that $p\mathcal{M} = 0$. Consider a set of data $(U_i, \phi_J, W_i, \psi_i)$, $i \in I$, $J \subset I$ as above, and the associated complex $\mathcal{S}(\mathcal{M})^{\cdot}$. Denote by $\mathcal{S}_{\text{ét}}(\mathcal{M})^{\cdot}$ the induced complex of étale sheaves on the special fiber X_k. By the last lemma $\mathcal{S}_{\text{ét}}(\mathcal{M})^{\cdot}$, modulo quasi isomorphisms, is independent of all choices. We clearly have that $H^*(X_k, \mathcal{S}_{\text{ét}}(\mathcal{M})^{\cdot}) \simeq H^*(\mathcal{S}(\mathcal{M})^{\cdot})$. From the spectral sequence

$$H^p(X_k, \mathcal{H}^q(\mathcal{S}_{\text{ét}}(\mathcal{M})^{\cdot})) \Rightarrow H^{p+q}(X_k, \mathcal{S}_{\text{ét}}(\mathcal{M})^{\cdot})$$

and the fact that $\text{cd}_p(X_k) \leqslant d+1$, we see that it suffices to show that $\mathcal{H}^q(\mathcal{S}_{\text{ét}}(\mathcal{M})^{\cdot}) = 0$ for $q > -a$.

We can now assume that $X = \text{Spec}(R)$ and $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(R, \phi)$ for some choice of the Frobenius lift ϕ. We claim that the map $\phi^0 - 1 \colon F^0\Omega(\mathcal{M})^{\cdot} \to \Omega(\mathcal{M})^{\cdot}$ is an isomorphism in degrees strictly bigger than $-a$ and a surjection in degree $-a$. Indeed, in degree $k > -a$, $\phi^0 - 1 \colon F^{-k}\mathcal{M} \otimes_{R/pR} \Omega^k_{R/pR} \to \mathcal{M} \otimes_{R/pR} \Omega^k_{R/pR}$ is equal to $\phi^{-k} \otimes d\phi_*/p^k - 1$. But $-k < a$, so $F^{-k}\mathcal{M} = F^a\mathcal{M} = \mathcal{M}$ and $\phi^{-k} = p^{a+k}\phi^a = 0$. Hence $\phi^0 - 1 = -1$. In degree $k = -a$, $\phi^0 - 1 \colon \mathcal{M} \otimes_{R/pR} \Omega^{-a}_{R/pR} \to \mathcal{M} \otimes_{R/pR} \Omega^{-a}_{R/pR}$ is equal to $\phi^a \otimes d\phi_*/p^{-a} - 1$. Looking at logarithmic differentials we see that it suffices to prove that the morphism $\phi^a - 1 \colon \mathcal{M} \to \mathcal{M}$ is surjective in the étale topology of R/pR. That easily follows from the fact that $\mathcal{M} \simeq \oplus R/pR$. □

PROPOSITION 3.2. *If* $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$*, then the morphism*

$$H^k_{\text{cr}}(X/V, \mathcal{M}) \to H^{k+1}_f(X, \mathcal{M})$$

is an isomorphism for $-b > d$ *or* $-b > k + 1$*. It is an injection for* $-b > k$*.*

Proof. From the definition of $\mathcal{S}(\mathcal{M})^{\cdot}$ we get the long exact sequence

$$\to H^k_f(X, \mathcal{M}) \to H^k_{\text{cr}}(X/V, F^0(\mathcal{M}))$$
$$\xrightarrow{1-\phi^0} H^k_{\text{cr}}(X/V, \mathcal{M}) \to H^{k+1}_f(X, \mathcal{M}) \to .$$

Let $\text{Gr}^i_F\Omega^{\cdot}_{X/V}(\mathcal{M})$ be the complex

$$\text{Gr}^i_F\mathcal{M}_X \xrightarrow{\nabla} \text{Gr}^{i-1}_F\mathcal{M}_X \otimes \Omega^1_{X/V} \xrightarrow{\nabla} \cdots$$

We get the 'Hodge spectral sequence'

$$E_1^{ij} = H^{i+j}(X, \text{Gr}^i_F\Omega^{\cdot}_{X/V}(\mathcal{M})) \Rightarrow H^{i+j}_{\text{cr}}(X/V, F^0(\mathcal{M})).$$

Since $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$, this thus yields that $H^i_{\text{cr}}(X/V, F^0(\mathcal{M})) = 0$ for $-b > \min(i, d)$, from which the proposition follows. □

4. The ring B^+

In the next two sections p is allowed to be equal to 2. Let V be a complete discrete valuation ring of mixed characteristic $(0, p)$ with a perfect residue field k. Let R be a smooth V-algebra such that $R/pR \neq 0$. Consider the p-adic completion $\widehat{R}$. For simplicity, we will assume that Spec(R/pR) is connected, which implies that $\widehat{R}$ is a normal domain. In general, $\widehat{R}$ is a product of normal domains and what follows applies to each factor.

We will briefly recall the construction and properties of the ring $B^+(\widehat{R})$ [5]. Denote by $\bar{\widehat{R}}$ the normalization of $\widehat{R}$ in the maximal étale extension of $\widehat{R}[1/p]$. We will write $x = p^{m/n}$ if $x^n = p^m$ and this does not cause problems. Let $S = \text{proj}\lim \bar{\widehat{R}}/p\bar{\widehat{R}}$, where the maps in the projective system are the pth power maps. With addition and multiplication defined coordinatewise S is a ring of characteristic p. We will also find useful the projective limit $\widehat{S} = \text{proj}\lim \bar{\widehat{R}}^{\wedge}$, where the transition maps are the pth power maps and multiplication is defined coordinatewise. The projection $\widehat{S} \to S$ is a multiplicative isomorphism: the inverse $(x^{(n)})$ of x is given by setting $x^{(n)} = \lim_{m\to\infty} \hat{x}_{n+m}^{p^m}$, where $\hat{}$ means a lift from $\bar{\widehat{R}}/p$ to $\bar{\widehat{R}}^{\wedge}$.

The Frobenius of S is bijective, so that the ring of Witt vectors $W(S)$ is p-torsion free, complete and separated for the p-adic topology.

There is a homomorphism θ from $W(S)$ to $\bar{\widehat{R}}^{\wedge}$: θ maps $(x_0, x_1, \ldots) \in W(S)$, $x_n = (x_{nm}) \in S$, to the limit over m of $\hat{x}_{0m}^{p^m} + p\hat{x}_{1m}^{p^{m-1}} + \cdots + p^m \hat{x}_{mm}$. This is a surjection if Frobenius is surjective on $\bar{\widehat{R}}/p$. The kernel of θ is being generated by $\xi = [(p)] + p[(-1)]$, where $(p), (-1) \in S$ are the reductions mod p of sequences of p-roots of p and (-1) respectively (if $p \neq 2$ we may and will choose $(-1) = -1$). In what follows we fix $p^{1/(p-1)} \in S$ – a sequence of p-roots of a fixed element x such that $x^{p-1} = p$ and (p) is equal to $(p^{1/(p-1)})^{p-1}$.

We will need

LEMMA 4.1. *Let* $0 < \varepsilon \leqslant 1$ *be a rational number. Let* p^ε *be a sequence of* p*-roots of* p^ε. *The map* $S/(p^\varepsilon) \to \bar{\widehat{R}}/p^\varepsilon$ *sending* x *to* x_0 *is always injective. It is surjective if Frobenius on* $\bar{\widehat{R}}/p$ *is surjective.*

Proof. It suffices to show that, if $x, y \in \bar{\widehat{R}}^{\wedge}$, $y^p = x$, $x \in p^\delta \bar{\widehat{R}}^{\wedge}$, then $y \in p^{\delta/p} \bar{\widehat{R}}^{\wedge}$. Set $n = \lfloor \delta/p \rfloor + 1$, and if $x = p^\delta r$, $r \in \bar{\widehat{R}}^{\wedge}$, take $\bar{r}, \bar{y} \in \bar{\widehat{R}}$ such that $\bar{r} \equiv r \bmod p^n \bar{\widehat{R}}^{\wedge}$, $\bar{y} \equiv y \bmod p^n \bar{\widehat{R}}^{\wedge}$. Then $\bar{y}^p \equiv p^\delta \bar{r} \bmod p^n \bar{\widehat{R}}^{\wedge}$, so $\bar{y}^p = p^\delta \bar{r} + p^n a$, $a \in \bar{\widehat{R}}$ and $(\bar{y}/p^{\delta/p})^p = \bar{r} + p^{n-\delta/p} a$. $\bar{\widehat{R}}$ being normal, $\bar{y}/p^{\delta/p} \in \bar{\widehat{R}}$, hence $y \in p^{\delta/p} \bar{\widehat{R}}^{\wedge}$ as wanted. □

The ring $B^+(\widehat{R})$ is defined as the p-adic completion of the divided power envelope $D_\xi(W(S))$ of the ideal $\xi W(S)$ in $W(S)$. Let J denote the PD ideal of

$D_\xi(W(S))$. $B^+(\widehat{R})$ is an algebra over $B^+(V)$ having the following four properties:

(1) the Frobenius automorphism on S induces an automorphisms ϕ on $W(S)$ and $B^+(\widehat{R})$;
(2) $B^+(\widehat{R})$ is equipped with a decreasing separated filtration $F^nB^+(\widehat{R})$ such that $\phi(F^nB^+(\widehat{R})) \subset p^nB^+(\widehat{R})$ (in fact, $F^nB^+(\widehat{R})$ is the closure of the ideal consisting of those elements in the n-th divided power of J whose ϕ-image is divisible by p^n);
(3) the Galois group $\mathrm{Gal}(\bar{\widehat{R}}/\widehat{R})$ acts on $B^+(\widehat{R})$; the action is continuous, commutes with ϕ and preserves the filtration;
(4) there exists an element $t \in F^1B^+(\widehat{R})$ such that $\phi(t) = pt$ and $\mathrm{Gal}(\bar{\widehat{R}}/\widehat{R})$ acts on t via the cyclotomic character: if we fix $\varepsilon \in S$ – a sequence of nontrivial p-roots of unity, then $t = \log([\varepsilon])$.

5. The fundamental exact sequence

Recall that R is called small if there is an étale map $V[T_1^{\pm 1}, \dots, T_d^{\pm 1}] \to R$. If R is small, Frobenius is surjective on $\bar{\widehat{R}}/p$. For $n \geqslant 0$, write $n = r(n) + (p-1)q(n)$, $0 \leqslant r(n) < p-1$ and set $t^{\{n\}} = t^{r(n)}\gamma_{q(n)}(t^{p-1}/p)$.

PROPOSITION 5.1. *For small R, there is an exact sequence of* $\mathrm{Gal}(\bar{\widehat{R}}/\widehat{R})$-*modules*

$$0 \to \mathbf{Z}_p t^{\{r\}} \to F^rB^+(\widehat{R}) \xrightarrow{p^{-r}\phi - 1} B^+(\widehat{R}) \to 0, \quad \textit{for } r \geqslant 0.$$

Proof. The proof follows closely that of Fontaine for the case $R = V$. We advice the reader to consult [9, 5.3.6] for details. The main point is that, assuming R small, we can solve certain polynomials involving Frobenius already in $\bar{\widehat{R}}/p$.

Set $\nu = p^{-r}\phi - 1$. We will first show that $\ker\nu = \mathbf{Z}_p t^{\{r\}}$. Clearly $\mathbf{Z}_p t^{\{r\}} \in \ker\nu$. Assume that $x \in \ker\nu$. Then

$$x \in I^{[r]} \stackrel{\mathrm{df}}{=} \{x \in B^+(\widehat{R}) \mid \phi^n(x) \in \widehat{J^{[r]}}, n \in \mathbf{N}\}.$$

To proceed we will need few more facts about the structure of the ring $B^+(\widehat{R})$. Set $\pi_\varepsilon = [\varepsilon] - 1 \in W(S)$, $q = \Sigma_{a \in \mathbf{F}_p}[\varepsilon]^{[a]}$ if $p \neq 2$ and $q = [\varepsilon] + [\varepsilon]^{-1}$ if $p = 2$, and, for $x \in W(S)$, set $x' = \phi^{-1}(x)$. One easily checks [8, 2.4] that the element $\Sigma_{n \geqslant 0} p^n[u_n] \in W(S)$ generates the kernel of θ if $u_1^{(0)}$ is a unit in $\bar{\widehat{R}}^{\wedge}$. In particular, that is true of $\Sigma_{a \in \mathbf{F}_p}[\varepsilon']^{[a]}$.

Having that, the arguments of Fontaine [9, 5.2.7] suffice to prove that every $a \in B^+(\widehat{R})$ can be written as $a = \Sigma_{n \geqslant 0} a_n\gamma_n(t^{p-1}/p)$, where the coefficients $a_n \in W(S)$ converge p-adically to 0. We also have (cf., [9, 5.1.4]).

LEMMA 5.1. *The ideal* $I = I^{[1]} \cap W(S)$ *is generated by* $\pi_\varepsilon = [\varepsilon] - 1$.

Proof. Set $\sigma = p/(p-1)$. One checks as in [8, 4.16] that if $x = (x_0, x_1, \ldots) \in I$, then $x_0^{(0)} \in p^{1+\cdots+p^{-r}} \hat{\bar{R}}^{\wedge}$ for every $r \geqslant 0$. This gives that $x_0^{(0)} \in p^\sigma \hat{\bar{R}}^{\wedge}$: take $\alpha \in \hat{\bar{R}}$ such that $\alpha \equiv x_0^{(0)} \bmod p^\sigma \hat{\bar{R}}^{\wedge}$ and consider the element $\alpha/p^\sigma \in K(\hat{\bar{R}})$. Since $\alpha/p^\sigma \in p^{-\varepsilon} \hat{\bar{R}}$ for arbitrarily small ε, it lies in all the localizations of $\hat{\bar{R}}$ at height one primes. Hence $\alpha/p^\sigma \in \hat{\bar{R}}$, as wanted.

We claim that $I \subset (\pi_\varepsilon, p)$. Take $x = (x_0, x_1, \ldots) \in I$. We want to see that $x_0 \in (\varepsilon - 1)S$ or, since $\varepsilon - 1 = (p^\sigma)$ unit [9, 5.1.2], that $x_0 \in (p^\sigma)S$. We already know that $x_0^{(0)} \in p^\sigma \hat{\bar{R}}^{\wedge}$. Since the composition

$$S/(p^\sigma)S \xrightarrow[\phi^{-1}]{\sim} S/(p^{1/(p-1)})S \to \hat{\bar{R}}/p^{1/(p-1)}\hat{\bar{R}},$$

where the last map sends $u = (u_n)$ to u_0, is injective, it suffices to prove that $x_0^{(1)} \in p^{1/(p-1)} \hat{\bar{R}}^{\wedge}$. But that follows from the proof of Lemma 4.1. Now, since $\theta(\phi^n(\pi_\varepsilon)) = \theta([\varepsilon]^{p^n} - 1) = \varepsilon^{p^n} - 1 = 0$, $\pi_\varepsilon \in I$. Thus, to finish it would suffice to know that if $px \in I$, then $x \in I$: consider the map $\omega: W(S) \to (\hat{\bar{R}}^{\wedge})^{\mathbf{N}}$ sending α to $(\theta(\phi^n(\alpha)))_{n \in \mathbf{N}}$. Since $px \in I$, we have $0 = \omega(px) = p\omega(x)$. But $(\hat{\bar{R}}^{\wedge})^{\mathbf{N}}$ is p-torsion free, so $\omega(x) = 0$ as wanted. □

Using the above lemma one can prove as in [9, 5.3.1] that $I^{[r]}$ is the closure of the $W(S)$-module generated by $t^{\{s\}}$, $s \geqslant r$. Hence we can write our x as $x = \Sigma_{s \geq r} a_s t^{\{s\}}$, where the coefficients $a_s \in W(S)$ converge p-adically to 0. For $n \in \mathbf{N}$, $(p^{-r}\phi)^n(x) \equiv \phi^n(a_r)t^{\{r\}} \bmod p^n B^+(\hat{R})$. So, since $x \in \ker \nu$, $x \equiv \phi^n(a_r)t^{\{r\}} \bmod p^n B^+(\hat{R})$. If we set $b = \lim_{n\to\infty} \phi^n(a_r)$, $b \in W(S)$, we have that $x = bt^r$, $\phi(b) = b$. $\hat{\bar{R}}$ being a henselian domain (with respect to the ideal $p\hat{\bar{R}}$), we get that $b \in \mathbf{Z}_p$.

Remains to prove that ν is surjective. Define N as the closure of the $W(S)$-submodule of $B^+(\hat{R})$ generated by $q'^j \gamma_n(t^{p-1}/p)$ with $j + n(p-1) \geqslant r$. Clearly $N \subset F^r B^+(\hat{R})$. Also, for $p \neq 2$, $t/q' \in B^+(\hat{R})$, thus $\mathbf{Z}_p t^{\{r\}} \subset N$. Since both N and $B^+(\hat{R})$ are p-adically complete and separated, it suffices to show that the induced map $N/p \xrightarrow{\nu} B^+(\hat{R})/p$ is surjective. Take $a \in B^+(\hat{R})$ and write it as $a = \Sigma_{n \geqslant 0} a_n \gamma_n(t^{p-1}/p)$, $a_n \in W(S)$. If $a = \Sigma_{n > r/(p-1)} a_n \gamma_n(t^{p-1}/p)$, then $n(p-1) > r$, $a \in N$, and we can take $x = -a$ to get $\nu(x) \equiv a \bmod pB^+(\hat{R})$. Remains to show that, for every $i \in \mathbf{N}$ such that $i(p-1) \leqslant r$ and all $b \in W(S)$, there is an $x \in N$ such that the element $\nu(x) - b\gamma_i(t^{p-1}/p)$, modulo $pB^+(\hat{R})$, belongs to the $W(S)$-submodule generated by $\gamma_n(t^{p-1}/p)$, $n > i$.

First, let us prove the following

LEMMA 5.2. *Let* $b \in S$ *and write* q' *also for the reduction* $\bmod\, p$ *of* $q' \in W(S)$. *Let* $k \geqslant 0$ *be an integer. Then, for* n *big enough, the polynomial* $P(X) = X^p - q'^k_n X - b_n \in \hat{\bar{R}}/p[X]$, *has a solution in* $\hat{\bar{R}}/p$.

Proof. Lift q'_n to $\bar{V}$ and b_n to $\hat{\bar{R}}$. Consider the $\hat{\bar{R}}$-algebra $A = \hat{\bar{R}}[X]/(X^p - q_n^{'k}X - b_n)$. It is a finite, flat algebra with the discriminant $\delta(A[1/p]/\hat{\bar{R}}[1/p]) = (X^{p-1} - q_n^{'k}/p)$. Easy computations all show that the element $(q_n^{'k}/p)^p - (b_n/(p-1))^{p-1}$ belongs to $\delta(A[1/p]/\hat{\bar{R}}[1/p])$. Now, the reader will note [9, 5.1.2] that, if $n > 1$, then $v(q'_n) = 1/p^{n-1}$ if $p \neq 2$ and $v(q'_n) = 1/p^{n-2}$ if $p = 2$, where v is the p-adic valuation on $\bar{V}$ normalized by $v(p) = 1$. In particular, $q'_n \neq 0$. Also, $v(p/q_n^{'k}) = 1 - k/p^{n-1}$ if $p \neq 2$ and $v(p/q_n^{'k}) = 1 - k/p^{n-2}$ if $p = 2$. Thus, for n big enough, $v(p/q_n^{'k}) > 0$. Hence $1 - (b_n/(p-1))^{p-1}(p/q_n^{'k})^p$ is a unit in $\hat{\bar{R}}$ and, since $q_n^{'kp}$ is a unit up to p-powers, we get that $A[1/p]$ is étale over $\hat{\bar{R}}[1/p]$. The lemma follows now from the definition of $\hat{\bar{R}}$. □

Consider now the composition

$$S/(p)^{p^n} \xrightarrow[\phi^{-n}]{\sim} S/(p) \to \hat{\bar{R}}/p,$$

where the last map sends x to x_0. Since Frobenius is surjective on $\hat{\bar{R}}/p$, this is an isomorphism. Take the polynomial $P(Y) = Y^p - q^{'k}Y - b \in S[Y]$, $k = r - (p-1)i$. By the above lemma and the isomorphism there are $y, s \in S$ such that $P(y) = (p)^{p^n}s$ for some big $n \geqslant 2$. Set $x = [y]q^{'k}\gamma_i(t^{p-1}/p)$. Compute

$$\begin{aligned}
&\nu(x) - b\gamma_i(t^{p-1}/p)\\
&\quad = p^{-r}\phi(x) - x - b\gamma_i(t^{p-1}/p)\\
&\quad = p^{-r}\phi([y])q^k p^{i(p-1)}\gamma_i(t^{p-1}/p) - [y]q^{'k}\gamma_i(t^{p-1}/p) - b\gamma_i(t^{p-1}/p)\\
&\quad = [y^p]q^k p^{-k}\gamma_i(t^{p-1}/p) - [y]q^{'k}\gamma_i(t^{p-1}/p) - b\gamma_i(t^{p-1}/p).
\end{aligned}$$

Fontaine [9, 5.2.5] computed that, in the case $p \neq 2$, q/p can be written as $1 + u\gamma_1(t^{p-1}/p)$ for some unit $u \in B^+(\hat{R})$ of the form $u = \Sigma_{n\geqslant 0}a_n\gamma_n(t^{p-1}/p)$ for $a_n \in W(k)$ converging p-adically to 0. The case $p = 2$ is simpler: $(q/p - 1) \in pB^+(\hat{R})$. Thus, if we set u_1 equal to u or 0 depending on the characteristic, we get

$$\begin{aligned}
&\nu(x) - b\gamma_i(t^{p-1}/p)\\
&\quad \equiv (q^{'k}[y] + b + [(p)^{p^n}][s])(1 + u_1\gamma_1(t^{p-1}/p))^k\gamma_i(t^{p-1}/p)\\
&\qquad -[y]q^{'k}\gamma_i(t^{p-1}/p) - b\gamma_i(t^{p-1}/p) \quad \operatorname{mod} pB^+(\hat{R})\\
&\quad = \{(1 + u_1\gamma_1(t^{p-1}/p))^k - 1\}(b + q^{'k})\gamma_i(t^{p-1}/p)\\
&\qquad +[(p)]^{p^n}[s](1 + u_1\gamma_1(t^{p-1}/p))^k\gamma_i(t^{p-1}/p)\\
&\quad = \{(1 + u_1\gamma_1(t^{p-1}/p))^k - 1\}(b + q^{'k})\gamma_i(t^{p-1}/p)\\
&\qquad +p^n!\gamma_{p^n}([(p)])[s](1 + u_1\gamma_1(t^{p-1}/p))^k\gamma_i(t^{p-1}/p),
\end{aligned}$$

which, modulo $pB^+(\widehat{R})$, belongs to the $W(S)$-submodule generated by $\gamma_j(t^{p-1}/p)$, $j > i$, as wanted. □

Remark 3. Since we are dealing in this paper only with the integral theory, that is, our r is never greater than $p-2$, the above proposition states more than we will need here. Proof of the proposition in the case $r \leqslant p-2$ simplifies considerably and can already be found in [5].

COROLLARY 5.1. *Fix* $n \geqslant 0$ *and* $0 \leqslant r \leqslant p-1$. *Let* R *be small and* $p > 2$. *There exists an* M *such that for all* $m \geqslant M$, *there is an exact sequence of* $\mathrm{Gal}(\bar{\widehat{R}}/\widehat{R})$-*modules*

$$0 \to \mathbf{Z}/p^n t^{\{r\}} \to F^r B^+(\widehat{R})_{n,m} \xrightarrow{p^{-r}\phi - 1} B^+(\widehat{R})_{n,m} \to 0,$$

where $B^+(\widehat{R})_{n,m} = B^+(\widehat{R})/(p^n B^+(\widehat{R}) + J^{[m]})$.

Proof. We have $\phi(\xi^{[r]}) = p^{[r]}([(p)]^{[p]}(p-1)! + [(-1)]^p)^r$. For $p \neq 2$, this immediately gives that, for m big enough, $\phi(J^{[m]}) \subset p^{r+n} B^+(\widehat{R})$. Since $F^r B^+(\widehat{R})$ is equal to the closure of $J^{[r]}$ itself, the exactness of the above sequence follows from the last proposition. □

6. Étale cohomology and Galois cohomology

Let V be a complete discrete valuation ring of mixed characteristic $(0, p)$ with a perfect residue field k and a fraction field K. We will now introduce, after Faltings [5], two auxiliary topoi, topoi of ‘sheaves of local systems’. Let X be a smooth, separated scheme of finite type over V or a strict henselization of such.

Let $\widetilde{X}$ be the following category. An object of $\widetilde{X}$ is a collection $L = ((L_U), (r_{U_1U_2}))$ of locally constant étale sheaves L_U on U_K, for every étale open U of X and, for every pair $U_2 \to U_1$, a morphism $r_{U_1U_2}\colon L_{U_1} \mid (U_1)_K \to L_{U_2}$ such that $r_{U_2U_3} r_{U_1U_2} = r_{U_1U_3}$ and $r_{UU} = \mathrm{id}$. One also requires that for every tranquated étale hypercovering $U_1 \rightrightarrows U_0 \to U$, L_U is the maximal locally constant subsheaf of $\ker(j_{0*}L_{U_0} \rightrightarrows j_{1*}L_{U_1})$, where $j_i\colon (U_i)_K \to U_K$. Morphism $f\colon L \to M$ in $\widetilde{X}$ is a collection of morphisms of locally constant sheaves $f_U\colon L_U \to M_U$ compatible with $r_{U_1U_2}$.

The category $\widetilde{X}$ is a topos. We will also denote by $\widetilde{X}$ the equivalent topos, where all U’s are assumed to be affine.

The following notation will be useful. A presheaf on $\widetilde{X}$ is a collection $L = ((L_U), (r_{U_1U_2}))$ satisfying the usual compatibilities. Every presheaf L has an associated sheaf. First, define

$$(L^+)_U = \operatorname{inj\,lim} \ker(j_{0*}L_{U_0} \rightrightarrows j_{1*}L_{U_1})),$$

where the limit is over tranquated étale hypercoverings $\mathcal{U} = (U_1 \rightrightarrows U_0 \to U)$, ker refers to the maximal locally constant subsheaf of the sheaf kernel, and

$j_i\colon (U_i)_K \to (U)_K$. It is a separated preasheaf (cf., [1]). We get a sheaf by taking L_U^{++}. This construction is functorial and has the expected adjointness property.

For a map of schemes $f\colon X \to Y$, we get an associated map of topoi $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$: the pushforward of L associates to $U \to Y$ the local system $f_{U*}L_U$, where $f_U\colon \mathcal{B}((U_X)_K \supset \mathcal{B}(U_K)$, and, for a noetherian K-scheme Z, $\mathcal{B}(Z)$ is the topos of locally constant sheaves on Z. The pullback is the sheaf associated to the presheaf $(\tilde{f}^*L)_U = \operatorname{inj\,lim} f^*_{Z,K}L_Z$, where the limit is over the diagrams

$$\begin{array}{ccc} U & \xrightarrow{f_Z} & Z \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y\,. \end{array}$$

There is a canonical map ρ from the étale topos of X_K to $\widetilde{X}$. First, one equippes every étale and irreducible $U \to X$ with a geometric generic point, and every map $U_1 \to U_2$ between two such étales with a path between the chosen points. Then the inverse image of L by ρ is the direct limit over all tranquated hypercoverings $U_1 \rightrightarrows U_0 \to X_K$ of $\ker(j_{0*}L_{U_0} \rightrightarrows j_{1*}L_{U_1})$; the direct image of $\mathcal{F}$ associates to U the locally constant subsheaf corresponding to the global sections of $\mathcal{F}$ on the universal covering of U_K. While computing cohomology of $\widetilde{X}$ it is often convenient to use the left exact functor ψ from $\widetilde{X}$ to the étale topos of X, sending L to the sheaf $U \mapsto L_U(U_K)$. Since $H^0(\widetilde{X}, L) = L_X(X_K) = H^0(X, \psi L)$, we have $H^*(\widetilde{X}, L) = H^*(X, R\psi L)$.

We also have a projection $\pi\colon \widetilde{X} \to \mathcal{B}(X_K)$. The inverse image π^*L associates to $j\colon U \to X$, the local system j_K^*L, the direct image π_*L is equal to L_X.

One checks [5] that, for a locally constant sheaf L on X_K, there is an isomorphism $H^*(\widetilde{X}, \rho_*L) \simeq H^*(X_K, L)$: it suffices to show that $R^k\rho_*L = 0$ for $k > 0$. But $(R^k\rho_*L)_U = R^k\omega_*L_{U_K}$, where $\omega\colon U_K \to \mathcal{B}(U_K)$. Hence it is trivial in the case U is a $K(\pi, 1)$ space. Since such U's form a base for the topology of X, we are done.

Faltings also defines the geometric cohomology $H^*(\widetilde{X}_{\bar{K}}, L) := (R^*\widetilde{\pi}_*L)_{\bar{K}}$, where $\widetilde{\pi}\colon \widetilde{X} \to \mathcal{B}(K)$. Since $\mathcal{B}(K)$ is equivalent to the étale topos of Spec(K), we have the following commutative diagram

$$\begin{array}{ccc} X_K & \xrightarrow{\rho} & \widetilde{X} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\widetilde{\pi}} & \\ & K & \end{array}$$

Hence, by the above, we get an isomorphism $H^*(\widetilde{X}_{\bar{K}}, \rho_*L) \simeq H^*(X_{\bar{K}}, L)$, for any locally constant sheaf L on X_K.

The cohomology $H^*(\widetilde{X},L)$ (resp. $H^*(\widetilde{X}_{\bar{K}},L)$) can be computed as the limit over the hypercoverings $U.$ of X of the generalized Čech complexes of $\mathcal{B}(U_K)$-cohomology (resp. $\mathcal{B}(U_{\bar{K}})$-cohomology) complexes of L_U. Also the construction of $\widetilde{X}$ can be done with the Zariski topology on X instead of the étale topology.

For X as before we also have a category $\widetilde{\mathcal{X}}$. An object of $\widetilde{\mathcal{X}}$ is a collection $L=((L_U),(r_{U_1U_2}))$ of locally constant étale sheaves L_U on Spec$(\mathcal{A}_K)$, for every $U=$ Spf$(\mathcal{A})$ – an étale open of $\mathcal{X}$ with $\mathcal{A}$ – a p-adically separated, complete V-algebra, and, for every pair $U_2\to U_1=$ Spf$(\mathcal{A}_2)\to$ Spf$(\mathcal{A}_1)$, a morphism $r_{U_1U_2}$: $f^*_{U_1U_2}L_{U_1}\to L_{U_2}$, where $f^*_{U_1U_2}$ is the induced map Spec$(\mathcal{A}_{2,K})\to$ Spec$(\mathcal{A}_{1,K})$ satisfying the usual compatibilities. Further the definition is analogous to that of $\widetilde{X}$. In particular, we equip every irreducible $\mathcal{A}$ as above with a $\overline{K(\mathcal{A})}$-point, and every map $\mathcal{A}\to\mathcal{A}'$ between two such algebras with a map between $\overline{K(\mathcal{A})}$ and $\overline{K(\mathcal{A}')}$.

The category $\widetilde{\mathcal{X}}$ is a topos. As before, to an affine morphism of schemes $f\colon X\to Y$, we can associate a map of topoi $\hat{f}\colon\widetilde{\mathcal{X}}\to\widetilde{\mathcal{Y}}$, and, for an affine scheme X, we can define a projection $\widehat{\pi}\colon\widetilde{\mathcal{X}}\to\mathcal{B}(\mathcal{A}(\mathcal{X})_K)$. Also, there is a map $\bar{\imath}\colon\widetilde{\mathcal{X}}\to\widetilde{X}$. The inverse image $\bar{\imath}^*L$ is the sheaf associated to the presheaf sending Spf$\mathcal{B}\to\mathcal{X}$ to the direct limit over the diagrams

$$\begin{array}{ccc}\mathrm{Spf}\,\mathcal{B} & \xrightarrow{f} & \mathrm{Spec}\,A\\ \downarrow & & \downarrow\\ \mathcal{X} & \longrightarrow & X\end{array}$$

of the local systems $f_K^*L_A$, where $f_K\colon$ Spec$(\mathcal{B}_K)\to$ Spec(A_K). The direct image $\bar{\imath}_*L$ therefore associates to Spec$(A)\to X$ the induced local system under the map Spec$(\widehat{A}_K)\to$ Spec(A_K).

PROPOSITION 6.1. *For any proper, smooth scheme X over V and any sheaf L on $\widetilde{X}$, the inverse image induces an isomorphism $H^*(\widetilde{\mathcal{X}},\bar{\imath}^*L)\overset{\sim}{\leftarrow}H^*(\widetilde{X},L)$. In particular, for a local system L on X_K, $H^*(X_K,L)$ can be computed on $\widetilde{\mathcal{X}}$.*

Proof. We can write

$$H^*(\widetilde{\mathcal{X}},\bar{\imath}^*L)\simeq H^*(\widetilde{X},R\bar{\imath}_*\bar{\imath}^*L)\simeq H^*(X,R\psi R\bar{\imath}_*\bar{\imath}^*L).$$

Similarly, $H^*(\widetilde{X},L)\overset{\sim}{\to}H^*(X,R\psi L)$. It suffices thus to study the composition

$$H^*(X,R\psi L)\xrightarrow{f}H^*(X,\bar{R}\psi L)\xrightarrow{g}H^*(X,R\psi R\bar{\imath}_*\bar{\imath}^*L).$$

Here $\bar{R}\psi L$ is the complex of sheaves associated to the complex of presheaves $R\psi L\circ\phi$, where ϕ maps U to the union of these connected components whose special fiber is nontrivial. We will prove that both f and g are isomorphisms (f by a global argument, g – by a local one).

First, consider the map f. It fits into the sequence of morphisms

$$H^*(X, R\psi L) \xrightarrow{f} H^*(X, \bar{R}\psi L) \xrightarrow{t} H^*(X, i_*i^*R\psi L),$$

with $i\colon X_k \hookrightarrow X$. By proper base change theorem, the composition tf is an isomorphism. Also, the map t is an isomorphism. In fact, the morphism $\bar{R}\psi L \to i_*i^*R\psi L$ is an isomorphism, as one can easily check looking at stalks. Thus f itself is an isomorphism.

Consider now the map g. We claim that the morphism $\bar{R}\psi L \to R\psi R\bar{\imath}_*\bar{\imath}^*L$ is a quasi isomorphism. Take a geometric point $\bar{x}$ over the special fiber. We have to show that the map

$$\text{inj lim}\, H^q(\widetilde{U}, L) \to \text{inj lim}\, H^q(\widetilde{\mathcal{U}}, \bar{\imath}^*L),$$

where the limit is over affine connected étale neighborhoods of $\bar{x}$ in X, is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{U}} & \xrightarrow{\quad\bar{\imath}\quad} & \widetilde{U} \\ \uparrow{\scriptstyle \tilde{\hat{p}}_U} & & \uparrow{\scriptstyle \tilde{p}_U} \\ \text{Spf}(\mathcal{O}^{\wedge}_{X,\bar{x}})^{\sim} & \xrightarrow{\bar{\imath}_{\bar{x}}} & \text{Spec}(\mathcal{O}_{X,\bar{x}})^{\sim}, \end{array}$$

and the induced commutative diagram of maps

$$\begin{array}{ccc} \text{inj lim}\, H^q(\widetilde{U}, L) & \xrightarrow{\qquad\bar{\imath}^*\qquad} & \text{inj lim}\, H^q(\widetilde{\mathcal{U}}, \bar{\imath}^*L) \\ \downarrow{\scriptstyle \text{inj lim}\, \tilde{p}_U^*} & & \downarrow{\scriptstyle \text{inj lim}\, \tilde{\hat{p}}_U^*} \\ H^q(\text{Spec}(\mathcal{O}_{X,\bar{x}})^{\sim}, \widetilde{p}^*L) & \xrightarrow{\bar{\imath}_{\bar{x}}^*} & H^q(\text{Spf}(\mathcal{O}^{\wedge}_{X,\bar{x}})^{\sim}, \widetilde{\hat{p}}^*\bar{\imath}^*L) \simeq H^q(\text{Spf}(\mathcal{O}^{\wedge}_{X,\bar{x}})^{\sim}, \bar{\imath}_{\bar{x}}^*\widetilde{p}^*L), \end{array}$$

where $\widetilde{p}\colon \text{Spec}(\mathcal{O}_{X,\bar{x}})^{\sim} \to \widetilde{X}$, $\widetilde{\hat{p}}\colon \text{Spf}(\mathcal{O}^{\wedge}_{X,\bar{x}})^{\sim} \to \widetilde{\mathcal{X}}$. By Elkik's theorem [4, Theorem 5] $\bar{\imath}_{\bar{x}}^*$ is an isomorphism (both cohomologies being isomorphic to the corresponding Galois group cohomologies). Remains to show that both $\text{inj lim}\, \widetilde{\hat{p}}_U^*$ and $\text{inj lim}\, \widetilde{p}_U^*$ are isomorphisms. The arguments being similar, we present here only the one for $\text{inj lim}\, \widetilde{\hat{p}}_U^*$. We claim that for a sheaf $\mathcal{F}$ on $\widetilde{\mathcal{X}}$, there is an isomorphism

$$\text{inj lim}\, H^q(\widetilde{\mathcal{U}}, \mathcal{F}) \to H^q(\text{Spf}(\mathcal{O}^{\wedge}_{X,\bar{x}})^{\sim}, \widetilde{\hat{p}}^*\mathcal{F}).$$

Since both sides define cohomological functors it suffices to check their behaving for $q = 0$ and for injectives. We thus have $\text{inj lim}\, H^0(\widetilde{\mathcal{U}}, \mathcal{F}) = \text{inj lim}\, \mathcal{F}_{\mathcal{U}}(\mathcal{A}(\mathcal{U})_K)$. On the other hand

$$\begin{aligned} H^0(\text{Spf}(\mathcal{O}^{\wedge}_{X,\bar{x}})^{\sim}, \widetilde{\hat{p}}^*\mathcal{F}) &= H^0(\text{Spec}(\mathcal{O}^{\wedge}_{X,\bar{x}}[1/p]), (\widetilde{\hat{p}}^*\mathcal{F})_{\mathcal{O}^{\wedge}_{X,\bar{x}}}) \\ &= H^0(\text{Spec}(\mathcal{O}^{\wedge}_{X,\bar{x}}[1/p]), \text{inj lim}\, f_{\mathcal{U}}^*\mathcal{F}_{\mathcal{U}}) \\ &= H^0(\mathcal{B}(\mathcal{O}^{\wedge}_{X,\bar{x}}[1/p]), \text{inj lim}\, f_{\mathcal{U}}^*\mathcal{F}_{\mathcal{U}}), \end{aligned}$$

where $f_{\mathcal{U}}\colon \mathrm{Spec}(\mathcal{O}^{\wedge}_{X,\bar{x}}[1/p]) \to \mathrm{Spec}(\mathcal{A}(\mathcal{U})_K)$.

We claim now that there is an equivalence of topoi

$$\mathcal{B}(\mathcal{O}^{\wedge}_{X,\bar{x}}[1/p]) \xrightarrow{\sim} \mathrm{proj\,lim}\,\mathcal{B}(\mathcal{A}(\mathcal{U})_K).$$

Indeed, consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{proj\,lim}\,\mathcal{B}(\mathcal{A}(\mathcal{U})_K) & \xrightarrow{\mathcal{B}(i)} & \mathrm{proj\,lim}\,\mathcal{B}(\mathcal{A}(U)^h_K) \\ \uparrow{\scriptstyle \mathcal{B}(\hat{p})} & & \uparrow{\scriptstyle \mathcal{B}(p^h)} \\ \mathcal{B}(\mathcal{O}^{\wedge}_{X,\bar{x}}[1/p]) & \xrightarrow{\mathcal{B}(i_{\bar{x}})} & \mathcal{B}(\mathcal{O}_{X,\bar{x}}[1/p]), \end{array}$$

where h denotes the henselization at p. In view of the fact that all the rings in sight are noetherian, we can again use Elkik's theorem to conclude that both $\mathcal{B}(i)$ and $\mathcal{B}(i_{\bar{x}})$ are equivalences. Also, since $\mathcal{O}_{X,\bar{x},K} \xrightarrow{\sim} \mathrm{inj\,lim}\,\mathcal{A}(U)^h_K$, the same is true of $\mathcal{B}(p^h)$ and finally of $\mathcal{B}(\widehat{p})$.

The above yields

$$\begin{aligned} &H^0(\mathcal{B}(\mathcal{O}^{\wedge}_{X,\bar{x}}[1/p]), \mathrm{inj\,lim}\, f^*_{\mathcal{U}}\mathcal{F}_{\mathcal{U}}) \\ &\quad\simeq \mathrm{inj\,lim}\, H^0(\mathcal{B}(\mathcal{A}(\mathcal{U})_K), \mathcal{F}_{\mathcal{U}}) = \mathrm{inj\,lim}\,\mathcal{F}_{\mathcal{U}}(\mathcal{A}(\mathcal{U})_K) \end{aligned}$$

as wanted.

Now, let I be an injective sheaf on $\widetilde{\mathcal{X}}$ and $q > 0$. Clearly, $\mathrm{inj\,lim}\, H^q(\widetilde{\mathcal{U}}, I) = 0$. On the other hand, we have

$$\begin{aligned} H^q(\mathrm{Spf}(\mathcal{O}^{\wedge}_{X,\bar{x}})^{\sim}, \widetilde{\widehat{p}}^{\,*} I) &= H^q(\mathcal{B}(\mathcal{O}^{\wedge}_{X,\bar{x}}[1/p]), (\widetilde{\widehat{p}}^{\,*} I)_{\mathcal{O}^{\wedge}_{X,\bar{x}}}) \\ &\simeq H^q(\mathcal{B}(\mathcal{O}^{\wedge}_{X,\bar{x}}[1/p]), \mathrm{inj\,lim}\, f^*_{\mathcal{U}} I_{\mathcal{U}}) \\ &\simeq \mathrm{inj\,lim}\, H^q(\mathcal{B}(\mathcal{A}(\mathcal{U})_K), I_{\mathcal{U}}) = 0 \end{aligned}$$

as well. □

There is also the geometric cohomology $H^*(\widetilde{\mathcal{X}}_{\bar{K}}, L) := (R^*\hat{\pi}_* L)_{\bar{K}}$, where $\widehat{\pi}$: $\widetilde{\mathcal{X}} \to \mathrm{Spec}(K)$, $\widehat{\pi} = \widetilde{\pi}\bar{\imath}$. The above then yields, for a proper X, an isomorphism $H^*(\widetilde{\mathcal{X}}_{\bar{K}}, \bar{\imath}^* L) \xleftarrow{\sim} H^*(\widetilde{X}_{\bar{K}}, L)$.

Since the cohomology $H^*(\widetilde{\mathcal{X}}, L)$ can be computed as the limit over the hypercoverings $U.$ of X of the generalized Čech complexes of $\mathcal{B}(\mathcal{A}(\mathcal{U})_K)$-cohomology complexes of $L_{\mathcal{U}}$, the above proposition asserts that, in the case X is proper, in computing $H^*(\widetilde{X}, L)$ we can use, instead of the $\mathcal{B}(U_K)$-group cohomology, the 'completed' $\mathcal{B}(\mathcal{A}(\mathcal{U})_K)$-group cohomology. Same for the geometric cohomology $H^*(\widetilde{\mathcal{X}}_{\bar{K}}, L)$.

As before we can use the Zariski topology on $\mathcal{X}$ instead of the étale topology.

PROPOSITION 6.2. *Let X be a smooth, separated scheme over V. Let L be a local system on X_K. Then there is an isomorphism $H^*(\widetilde{\mathcal{X}}_{\text{Zar}}, L) \xrightarrow{\sim} H^*(\widetilde{\mathcal{X}}_{\text{ét}}, L)$.*

Proof. The functor ψ reduces the question to a local one, namely, that, for every point $x \in X$, the map inj lim $H^*(\widetilde{\mathcal{U}}_{\text{Zar}}, L) \to$ inj lim $H^*(\widetilde{\mathcal{U}}_{\text{ét}}, L)$, where the limit is over affine, Zariski neighborhoods of x, is an isomorphism. We will prove that both limits are isomorphic (via the inverse images of the projections $\widehat{\pi}$) to

$$\operatorname{inj\,lim} H^*(\mathcal{B}(\mathcal{A}(\mathcal{U})[1/p]), L) \simeq H^*(\mathcal{B}(\mathcal{O}_{X,x}^{\wedge}[1/p]), L).$$

For the Zariski limit, since the ring $\mathcal{O}_{X,x}^{\wedge}$ is local, one can argue as in the proof of Proposition 6.1. For the étale one, note that the inverse image $\bar{\imath}^*$ induces an isomorphism inj lim $H^*(\widetilde{\mathcal{U}}_{\text{ét}}, L) \xleftarrow{\sim}$ inj lim $H^*(\widetilde{U}^h_{\text{ét}}, L)$. Indeed, the arguments from the proof of Proposition 6.1 will work as soon as we know that a 'proper' base change theorem holds for U^h. But this was proved by Gabber in [10]. Now, since, by Elkik, $H^*(\mathcal{B}(\mathcal{A}(U)^h[1/p]), L) \xrightarrow{\sim} H^*(\mathcal{B}(\mathcal{A}(\mathcal{U})[1/p]), L)$, and also, by a $K(\pi, 1)$ argument, $H^*(\widetilde{U}^h_{\text{ét}}, L) \xrightarrow{\sim} H^*(U^h[1/p], L)$, it suffices to show that the inverse image of the projection ω induces an isomorphism $H^*(\mathcal{B}(\mathcal{O}^h_{X,x}[1/p]), L) \xrightarrow{\sim} H^*(\mathcal{O}^h_{X,x}[1/p], L)$. But the ring $\mathcal{O}^h_{X,x}$ can be represented as a direct limit of rings of the same form as $\mathcal{O}_{X,x}$. It is thus a $K(\pi, 1)$ space and we are done. □

7. Cohomology supported on the special fiber

Assume now that V is a complete, absolutely unramified discrete valuation ring of mixed characteristic $(0, p)$ with a perfect residue field k and a fraction field K. Let X be a smooth, separated V-scheme. We choose, once and for all, for every irreducible, étale Spf$(\mathcal{A})/\mathcal{X}$, a $\overline{K(\mathcal{A})}$-point, and for every map $\mathcal{A} \to \mathcal{A}'$ between two such étales a map between $\overline{K(\mathcal{A})}$ and $\overline{K(\mathcal{A}')}$. Everything below is independent of this choice.

Consider a set of data $(U_i, \phi_I, W_i, \psi_i)$, $i \in L$, $I \subset L$, where $U_i =$ Spec(R_i) is assumed to be small. For every $I \subset L$, we have the following commutative diagram

$$\begin{array}{ccc} \widehat{\widehat{\mathcal{U}}}_I & \longrightarrow & D_\xi(\widehat{W(S_I)}) \\ \big\downarrow & & \big\downarrow \\ U_I & \longrightarrow & V \end{array}$$

Take $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$. We can evaluate $\mathcal{M}$ on $B_I^+ := B^+(\widehat{R}_I) = D_\xi(\widehat{W(S_I)})$ Choose a retraction $h\colon \widehat{R}_I \to B_I^+$ (such a retraction always exists, U_I/p^n being V/p^n-projective). Set $\mathcal{M}_{B_I^+} = h^*\mathcal{M}_{\widehat{R}_I}$. Filter it by the saturation of the filtration defined by $h^*F^i\mathcal{M}_{\widehat{R}_I}$, i.e., $F^i\mathcal{M}_{B_I^+} := \Sigma_k J^{[k]}_{B_I^+} h^* F^{i-k}\mathcal{M}_{\widehat{R}_I}$, where $J_{B_I^+}$ is the PD ideal of B_I^+.

As in [13] one checks that Griffiths transversality yields that these definitions are, up to canonical isomorphism, independent of the choice of the retraction h.

Concerning the Frobenius, from the structural maps $\phi_I\colon \widetilde{\mathcal{M}_{\widehat{R}_I}} \otimes_{\widehat{R}_I \phi_I} \widehat{R}_I \xrightarrow{\sim} \mathcal{M}_{\widehat{R}_I}$ and the divided Frobenius $\phi_{B_I^+}/p^i\colon F^i B_I^+ \to B_I^+$, $i \leqslant p-1$, we can induce, as we did before (Lemma 2.1), a canonical compatible family of maps $\phi^i_{B_I^+}\colon F^i\mathcal{M}_{B_I^+} \otimes_{B_I^+ \phi_{B_I^+}} B_I^+ \to \mathcal{M}_{B_I^+}, i \leqslant p-1+a$. Again, up to canonical isomorphism, this does not depend on h and the Frobenius lift on $\widehat{R}_I$.

LEMMA 7.1. *If*

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

is exact in $\mathcal{MF}^{\nabla}_{[a,b]}(X_n)$, *then*

$$0 \to \mathcal{L}_{B_{I,n}^+} \to \mathcal{M}_{B_{I,n}^+} \to \mathcal{N}_{B_{I,n}^+} \to 0$$

is exact in the filtered sense.

Proof. First, we claim that for any retraction $h\colon \widehat{R}_{I,n} \to B_{I,n}^+$ the sequence

$$0 \to h^*\mathcal{L}_{R_{I,n}} \to h^*\mathcal{M}_{R_{I,n}} \to h^*\mathcal{N}_{R_{I,n}} \to 0$$

is exact. Suffices to prove that $\mathrm{Tor}_1{}^{R_{I,n}}(\mathcal{N}_{R_{I,n}}, B_{I,n}^+) = 0$. Since, locally, $\mathcal{N} \simeq \oplus R_{I,k}$ this reduces us to proving that $\mathrm{Tor}_1{}^{R_{I,n}}(R_{I,k}, B_{I,n}^+) = 0$. But, $B_{I,n}^+$ being flat V_n-module, we get the exact sequence

$$0 \to R_{I,k'} \otimes_{R_{I,n} h} B_{I,n}^+ \to R_{I,n} \otimes_{R_{I,n} h} B_{I,n}^+ \to R_{I,k} \otimes_{R_{I,n} h} B_{I,n}^+ \to 0,$$

where $k' = n - k$. In particular, $\mathrm{Tor}_1{}^{R_{I,n}}(R_{I,k}, B_{I,n}^+) = 0$ as wanted.

Next, we will need the fact that, for any k there is an exact sequence

$$0 \to J^{[k]}_{B_{I,n}^+} h^* F^i\mathcal{L}_{R_{I,n}} \to J^{[k]}_{B_{I,n}^+} h^* F^i\mathcal{M}_{R_{I,n}} \to J^{[k]}_{B_{I,n}^+} h^* F^i\mathcal{N}_{R_{I,n}} \to 0.$$

Since, by the above (and the fact that, locally, $\mathcal{L}_{R_{I,n}} \simeq \mathrm{gr}^{\cdot}_F\mathcal{L}_{R_{I,n}}$), the injection is clear, it suffices to show that $J^{[k]}_{B_{I,n}^+} h^* F^i\mathcal{M}_{R_{I,n}} \simeq J^{[k]}_{B_{I,n}^+} \otimes_{B_{I,n}^+} h^* F^i\mathcal{M}_{R_{I,n}}$, or, that $\mathrm{Tor}_1{}^{R_{I,n}}(F^i\mathcal{M}_{R_{I,n}}, B_{I,n}^+/J^{[k]}_{B_{I,n}^+}) = 0$. By devissage on $B_{I,n}^+/J^{[k]}_{B_{I,n}^+}$, we reduce the question to the computation of $\mathrm{Tor}_1{}^{R_{I,n}}(F^i\mathcal{M}_{R_{I,n}}, J^{[j]}_{B_{I,n}^+}/J^{[j+1]}_{B_{I,n}^+})$. Since $J^{[j]}_{B_{I,n}^+}/J^{[j+1]}_{B_{I,n}^+} \simeq \bar{\widehat{R}}_{I,n}$ is a flat $R_{I,n}$-module, the last group is clearly 0. The rest of the argument follows the proof of Lemma 2.3. $\square$

Fix now n. Consider the resolution of $\mathcal{O}_{U_{I,n}/V_n}$ by the linearization of the de Rham complex $\Omega^{\cdot}_{W_{I,n}/V_n}$

$$0 \to \mathcal{O}_{U_{I,n}/V_n} \to \mathcal{L}(\Omega^{\cdot}_{W_{I,n}/V_n}).$$

It is a locally free resolution, acyclic for the projection into the Zariski topos of $U_{I,n}$.

Take $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X_n)$ and fix $I \subset L$. We get a resolution

$$0 \to \mathcal{M}_{U_{I,n}/V_n} \to \mathcal{M}_{U_{I,n}/V_n} \otimes_{\mathcal{O}_{U_{I,n}/V_n}} \mathcal{L}(\Omega^{\cdot}_{W_{I,n}/V_n}).$$

The complex $\mathcal{L}(\Omega^{\cdot}_{W_{I,n}/V_n})$ is equipped with a canonical filtration. If we induce the tensor product filtration on $\mathcal{M}_{U_{I,n}/V_n} \otimes_{\mathcal{O}_{U_{I,n}/V_n}} \mathcal{L}(\Omega^{\cdot}_{W_{I,n}/V_n})$, the above turns out to be a resolution in the filtered sense as well. Evaluate it on $B^+_{I,n}$. $\mathcal{L}(\Omega^{\cdot}_{W_{I,n}/V_n})$ being flat we get a filtered resolution

$$0 \to \mathcal{M}_{B^+_{I,n}} \to \mathcal{M}_{B^+_{I,n}} \otimes_{B^+_{I,n}} \mathcal{L}(\Omega^{\cdot}_{W_{I,n}/V_n})_{B^+_{I,n}}.$$

To study the Frobenius note that

$$\begin{aligned}\mathcal{L}(\Omega^{\cdot}_{W_{I,n}/V_n})_{B^+_{I,n}} &\simeq B^+_{I,n}\,{}_h\!\otimes_{R_{I,n}} \mathcal{L}(\Omega^{\cdot}_{W_{I,n}/V_n})_{U_{I,n}}\\ &\simeq B^+_{I,n}\,{}_{hi}\!\otimes_{T_{I,n}} D_{W_{I,n}/V_n}(1) \otimes_{T_{I,n}/V_n} \Omega^{\cdot}_{W_{I,n}/V_n},\end{aligned}$$

where h is, say, the reduction mod p^n of a retraction $h\colon \widehat{R}_I \to B^+_I$ and i is the map $i\colon U_I \hookrightarrow W_I$.

A well-known formula [2, p. 275] gives a filtered isomorphism $B^+_{I,n}\,{}_{hi}\!\otimes_{T_{I,n}} \times D_{W_{I,n}/V_n}(1) \xrightarrow{\sim} D_{\bar{\widehat{R}}_{I,n}}(B^+_{I,n} \times_{V_n} T_{I,n})$, where $D_{\bar{\widehat{R}}_{I,n}}(B^+_{I,n} \times_{V_n} T_{I,n})$ is the PD-enveloping algebra of $\bar{\mathcal{U}}_{I,n}$ in $\mathrm{Spec}(B^+_{I,n}) \times_{V_n} W_{I,n}$ compatible with the PD-structure on $J_{B^+_{I,n}} + pB^+_{I,n}$. This yields a filtered resolution

$$0 \to \mathcal{M}_{B^+_{I,n}} \to \mathcal{M}_{B^+_{I,n}} \otimes_{B^+_{I,n}} D_{\bar{\widehat{R}}_{I,n}}(B^+_{I,n} \times_{V_n} T_{I,n}) \otimes_{T_{I,n}/V_n} \Omega^{\cdot}_{W_{I,n}/V_n}.$$

Consider now $\bar{D}_I = \mathrm{proj}\lim_n D_{\bar{\widehat{R}}_{I,n}}(B^+_{I,n} \times_{V_n} T_{I,n})$. It is p-torsion free V-algebra. Take $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$. It can be evaluated on $\bar{D}$: choose a retraction $h\colon \widehat{R}_I \to \bar{D}_I$ and set $\mathcal{M}_{\bar{D}_I} = h^*\mathcal{M}_{\widehat{R}_I}$. Define the filtration as the saturation of the filtration coming from $\mathcal{M}_{\widehat{R}_I}$ and induce, in the usual way, a compatible family of maps $\phi^i_{\bar{D}_I}\colon F^i\mathcal{M}_{\bar{D}_I} \otimes_{\bar{D}_I}\, {}_{\psi_I}\bar{D}_I \to \mathcal{M}_{\bar{D}_I}$, $i \leqslant p-1+a$, from the corresponding maps on $\mathcal{M}_{\bar{\widehat{R}}_I}$ and $\bar{D}_I$. Here $\psi_I\colon \bar{D}_I \to \bar{D}_I$ is the Frobenius coming from $\mathcal{W}_I$ and B^+_I. Up to canonical isomorphism this definition does not depend on the choice of h and the Frobenius lifts.

LEMMA 7.2. *If*

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

is exact in $\mathcal{MF}^{\nabla}_{[a,b]}(X_n)$, *then*

$$0 \to \mathcal{L}_{\bar{D}_{I,n}} \to \mathcal{M}_{\bar{D}_{I,n}} \to \mathcal{N}_{\bar{D}_{I,n}} \to 0$$

is exact in the filtered sense.

Proof. Note that locally $\bar{D}^{[k-1]}_{I,n}/\bar{D}^{[k]}_{I,n}$ is a direct sum of free modules over $J^{[i]}_{B^+_{I,n}}/J^{[i+1]}_{B^+_{I,n}}$, $0 \leqslant i \leq k-1$. Since $J^{[i]}_{B^+_{I,n}}/J^{[i+1]}_{B^+_{I,n}} \cong \bar{\widehat{R}}_{I,n}$, this yields that $\bar{D}^{[k-1]}_{I,n}/\bar{D}^{[k]}_{I,n}$ is a flat $R_{I,n}$-module and the lemma follows as in the case of $B^+_{I,n}$. □

Think now about $\mathcal{M}_{\bar{D}_{I,n}}$ as coming from $\mathcal{M}_{D_{I,n}}$. Equip it with the integrable $T_{I,n}$-connection induced from the one on $\mathcal{M}_{D_{I,n}}$ and compatible with the canonical $T_{I,n}$-connection on $\bar{D}_{I,n}$. Note also that the natural isomorphism $\mathcal{M}_{B^+_{I,n}} \otimes_{B^+_{I,n}} \times \bar{D}_{I,n} \xrightarrow{\sim} \mathcal{M}_{\bar{D}_{I,n}}$ is compatible with the Frobenius, identifies the filtration on $\mathcal{M}_{\bar{D}_{I,n}}$ with the tensor product filtration and yields an isomorphism of filtered, Frobenius equivariant resolutions

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{M}_{B^+_{I,n}} & \longrightarrow & \mathcal{M}_{B^+_{I,n}} \otimes_{B^+_{I,n}} \bar{D}_{I,n} \otimes_{T_{I,n}} \Omega^{\cdot}_{W_{I,n}/V_n} \\ & & \| & & \downarrow \wr \\ 0 & \longrightarrow & \mathcal{M}_{B^+_{I,n}} & \longrightarrow & \mathcal{M}_{\bar{D}_{I,n}} \otimes_{T_{I,n}} \Omega^{\cdot}_{W_{I,n}/V_n}. \end{array}$$

Everything above is equipped with an action of the fundamental group of $\widehat{R}_{I,K}$ and it is easy to check that the Frobenius, filtration and the resolutions behave well with respect to this action (use compatibility of the transition maps α with the change of the retraction). An important thing to note is that though the Galois action on $\mathcal{M}_{B^+_{I,n}}$ involves connection, the Galois action on $\mathcal{M}_{\bar{D}_{I,n}}$ comes only from the action on $\bar{D}_{I,n}$ (the map $D_{I,n} \to \bar{D}_{I,n}$ being Galois equivariant).

Define the following complex of sheaves on $\mathcal{B}(\widehat{R}_{I,K})$

$$\Omega(\mathcal{M}_{B^+_I})^{\cdot} := \mathcal{M}_{\bar{D}_{I,n}} \otimes_{T_{I,n}} \Omega^{\cdot}_{W_{I,n}/V_n}.$$

It is independent of the choice of n such that $p^n\mathcal{M} = 0$.

Choose now a number $m \geqslant p-1$ such that $\phi(J^{[m]}_{B^+_I}) \subset p^{n+(p-1)}B^+_I$. Set $B^+_{I,n,m} = B^+_{I,n}/J^{[m]}_{B^+_{I,n}}$. Since, by the choice of m, for $i \leqslant p-1+a$,

$$F^i\mathcal{M}_{B^+_{I,n,m}} = F^i\mathcal{M}_{B^+_{I,n}}/J^{[m]}_{B^+_{I,n}}\mathcal{M}_{B^+_{I,n}} \quad \text{and} \quad \phi^i_{B^+_{I,n}} \mid J^{[m]}_{B^+_{I,n}}\mathcal{M}_{B^+_{I,n}} = 0,$$

all of the above goes through for $\mathcal{M}_{B^+_{I,n,m}}$ in a manner compatible with $\mathcal{M}_{B^+_{I,n}}$.

Assume now that X is smooth and proper over V. Let a,b be such that $-a \leqslant p-1$ and $b-a \leqslant p-2$. Fix a positive integer n. Take $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X_n)$. We will now define a morphism from $\mathcal{S}(\mathcal{M})^{\cdot}$ to a complex computing the étale cohomology groups $H^*(X_K, \mathbf{L}(\mathcal{M}))$ of the crystalline local system $\mathbf{L}(\mathcal{M})$ associated to $\mathcal{M}$ by Faltings [5]. Recall that the local system $\mathbf{L}(\mathcal{M})$ on X_K is the unique local system inducing a compatible family of local systems $\mathbf{L}(\mathcal{M}_{\widehat{R}_i})$ on Spec$(\widehat{R}_i[1/p])$, where $\mathbf{L}(\mathcal{M}_{\widehat{R}_i}) = \mathrm{Hom}(\mathcal{M}_{B^+(\widehat{R}_i)}, B^+(\widehat{R}_i)\{a\}(a) \otimes \mathbf{Q}_p/\mathbf{Z}_p)^*$. Here, the homomorphisms are supposed to be $B^+(\widehat{R}_i)$-linear, respecting the filtrations and Frobeniuses. The symbols $\{a\}, (a)$ are the $\mathcal{MF}$-twist and the Tate-twist respectively.

First, fix Ω, a sufficiently big algebraically closed field of characteristic 0. In particular, we require that, for all étale Spf$(\mathcal{A})/\mathcal{X}$, the Ω-points of Spec$(\mathcal{A}_K)$ form a conservative family of the associated étale topos. Also, for any étale Spf$(\mathcal{A})$ over $\mathcal{X}$ and a sheaf $\mathcal{F}$ on $\mathcal{B}(\mathcal{A}_K)$, denote by $G^{\cdot}(\mathcal{F})$ the Godement resolution of $\mathcal{F}$. It is a complex of locally constant sheaves on Spec$(\mathcal{A}_K)$, acyclic for $\mathcal{B}(\mathcal{A}_K)$-cohomology.

Next, fix a sufficiently big number m. For each $J \subset L$, we have a sequence of morphisms between complexes of sheaves on $\mathcal{B}(\widehat{R}_{J,K})$.

$$\begin{aligned}\Omega(\mathcal{M}_J)^{\cdot} &\to \Omega(\mathcal{M}_{B^+_{J,n}})^{\cdot} \to G^{\cdot}(\Omega(\mathcal{M}_{B^+_{J,n}})^{\cdot}) \\ &\to G^{\cdot}(\Omega(\mathcal{M}_{B^+_{J,n,m}})^{\cdot}) \xleftarrow{(1)} G^{\cdot}(\mathcal{M}_{B^+_{J,n,m}}) \\ &\to G^{\cdot}(\mathbf{L}(\mathcal{M}_J) \otimes_{\mathbf{Z}_p} B^+_{J,n,m}\{a\}(a)).\end{aligned}$$

By the above, the morphism (1) is a quasi isomorphism. This sequence yields a morphism

$$\begin{aligned}\mathcal{S}(\mathcal{M}_J)^{\cdot} &\to \mathrm{Cone}(G^{\cdot}(\mathbf{L}(\mathcal{M}_J) \otimes_{\mathbf{Z}_p} F^{(-a)}B^+_{J,n,m}(a))(\widehat{R}_{J,K}) \\ &\xrightarrow{p^a\phi-1} G^{\cdot}(\mathbf{L}(\mathcal{M}_J) \otimes_{\mathbf{Z}_p} B^+_{J,n,m}(a))(\widehat{R}_{J,K}))[-1],\end{aligned}$$

which is functorial with respect to the change of J (note that the restriction maps are independent of the choice of the path between the base points).

To proceed, define the sheaf $B^+_{n,m}$ on the topos $\widetilde{\mathcal{X}}$ by associating to Spf$(\widehat{R})$, for Spec$(R) \to X$ small, affine and étale, the locally constant sheaf on Spec$(\widehat{R}_K)$ defined by the Galois module $B^+(\widehat{R})_{n,m} = B^+(\widehat{R})/(p^nB^+(\widehat{R}) + J^{[m]}_{B^+(\widehat{R})})$. For any R as above, we can use the fundamental exact sequence (of locally constant sheaves on Spec$(\widehat{R}_K)$)

$$0 \to \mathbf{L}(\mathcal{M})_{\widehat{R}} \to \mathbf{L}(\mathcal{M})_{\widehat{R}} \otimes_{\mathbf{Z}_p} F^{(-a)}B^+(\widehat{R})_{n,m}(a)$$
$$\xrightarrow{1-p^a\phi} \mathbf{L}(\mathcal{M})_{\widehat{R}} \otimes_{\mathbf{Z}_p} B^+(\widehat{R})_{n,m}(a) \to 0$$

to construct a quasi isomorphism

$$G^{\cdot}(\mathbf{L}(\mathcal{M})_{\widehat{R}})(\widehat{R}_K)$$
$$\to \mathrm{Cone}(G^{\cdot}(\mathbf{L}(\mathcal{M})_{\widehat{R}} \otimes_{\mathbf{Z}_p} F^{(-a)}B^+_{n,m}(a)_{\widehat{R}})(\widehat{R}_K)$$
$$\xrightarrow{p^a\phi-1} G^{\cdot}(\mathbf{L}(\mathcal{M})_{\widehat{R}} \otimes_{\mathbf{Z}_p} B^+_{n,m}(a)_{\widehat{R}})(\widehat{R}_K))[-1].$$

Denote by $U. \to X$ the chosen hypercovering of X. We find the desired morphism l into the étale cohomology as the composition

$$\mathcal{S}(\mathcal{M})^{\cdot} \to \mathrm{Cone}(G^{\cdot}(\mathbf{L}(\mathcal{M}_{\mathcal{U}}) \otimes_{\mathbf{Z}_p} F^{(-a)}B^+_{n,m}(a)_{\mathcal{U}})(\mathcal{A}(\mathcal{U})_{K^{\cdot}})$$
$$\xrightarrow{p^a\phi-1} G^{\cdot}(\mathbf{L}(\mathcal{M}_{\mathcal{U}}) \otimes_{\mathbf{Z}_p} B^+_{n,m}(a)_{\mathcal{U}})(\mathcal{A}(\mathcal{U})_{K^{\cdot}}))[-1]$$
$$\simeq \mathrm{Cone}(G^{\cdot}(\mathbf{L}(\mathcal{M})_{\mathcal{U}} \otimes_{\mathbf{Z}_p} F^{(-a)}B^+_{n,m}(a)_{\mathcal{U}})(\mathcal{A}(\mathcal{U})_{K^{\cdot}})$$
$$\xrightarrow{p^a\phi-1} G^{\cdot}(\mathbf{L}(\mathcal{M})_{\mathcal{U}} \otimes_{\mathbf{Z}_p} B^+_{n,m}(a)_{\mathcal{U}})(\mathcal{A}(\mathcal{U})_{K^{\cdot}}))[-1]$$
$$\xleftarrow{\sim} G^{\cdot}(\mathbf{L}(\mathcal{M})_{\mathcal{U}})(\mathcal{A}(\mathcal{U})_{K^{\cdot}})$$
$$\to \operatorname*{inj\,lim}_{V.\in \mathrm{HR}(X)} G^{\cdot}(\mathbf{L}(\mathcal{M})_{\mathcal{V}})(\mathcal{A}(\mathcal{V})_{K^{\cdot}}).$$

Here $\mathrm{HR}(X)$ denotes the homotopy category of affine hypercoverings of X. By Lemmas 7.1 and 7.2, l defines a natural transformation of cohomology theories

$$l\colon H^*_{f,a,b}(X,\cdot) \to H^*(X_K, \mathbf{L}(\cdot)).$$

Everything above is independent of choices.

We will now treat products.

PROPOSITION 7.1. *If $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$, $\mathcal{N} \in \mathcal{MF}^{\nabla}_{[c,d]}(X)$, then there exists a canonical product*

$$\cup\colon H^p_{f,a,b}(X,\mathcal{M}) \otimes H^q_{f,c,d}(X,\mathcal{N}) \to H^{p+q}_{f,a+c,b+d}(X,\mathcal{M}\otimes\mathcal{N})$$

which is anticommutative and associative. Moreover, it commutes with the morphism $l\colon H^*_f(X,\cdot) \to H^*(X_K,\mathbf{L}(\cdot))$.

Proof. Consider a set of data $(U_i, \phi_I, W_i, \psi_i)$, $i \in L$, $I \subset L$, where U_i's are assumed to be small. For a given $J \subset L$, we can use [12, Prop. 3.1] and the de Rham product $\Omega(\mathcal{M}_J)^\cdot \otimes \Omega(\mathcal{N}_J)^\cdot \to \Omega((\mathcal{M} \otimes \mathcal{N})_J)^\cdot$ to define a homotopic family of maps of complexes

$$\bigcup_\alpha : \mathcal{S}(\mathcal{M}_J)^\cdot \otimes \mathcal{S}(\mathcal{N}_J)^\cdot \to \mathcal{S}(\mathcal{M}_J \otimes \mathcal{N}_J)^\cdot, \quad \alpha \in \mathbf{Z}_p,$$

$$(x_1, x_2) \bigcup_\alpha (y_1, y_2) = (x_1 \cup y_1, x_2 \cup (\alpha\phi^0(y_1) + (1-\alpha)y_1) + (-1)^{\deg x_1}((1-\alpha)\phi^0(x_1) + \alpha x_1) \cup y_2).$$

Moreover, the maps $\cup_0, \cup_1$ are associative and $\cup_\alpha, \cup_{1-\alpha}$ anticommute.

Everything behaving well with respect to the change of the index set J, we can combine, by [12, Prop. 3.1], the above family and Čech products to define a homotopic family of maps of complexes $\cup_\alpha: \mathcal{S}(\mathcal{M})^\cdot \otimes \mathcal{S}(\mathcal{N})^\cdot \to \mathcal{S}(\mathcal{M} \otimes \mathcal{N})^\cdot$, $\alpha \in \mathbf{Z}_p$. The maps $\cup_0$, $\cup_1$ are associative and $\cup_\alpha$, $\cup_{1-\alpha}$ anticommute modulo a homotopic to the identity transposition operator. This induces a cup product

$$\cup: H^p_{f,a,b}(X, \mathcal{M}) \otimes H^q_{f,c,d}(X, \mathcal{N}) \to H^{p+q}_{f,a+c,b+d}(X, \mathcal{M} \otimes \mathcal{N}).$$

We can now follow step by step the definition of the map l, use the de Rham, Čech and Godement products, and [12, Prop. 3.1] to induce, at every step, compatible families of homotopic pairings. This will show compatibility of the map l with products. □

For $\mathcal{M} \in \mathcal{MF}^\nabla_{[a,b]}(X)$, define the cohomology with support on the special fiber X_k as

$$_sH^*_{f,a,b}(X, \mathcal{M}) := H^*(\mathrm{Cone}(\mathcal{S}(\mathcal{M})^\cdot \xrightarrow{l} \operatorname*{inj\,lim}_{V^\cdot \in \mathrm{HR}(X)} G^\cdot(\mathbf{L}(\mathcal{M})_\mathcal{V})(\mathcal{A}(\mathcal{V})_{K^\cdot}))[-1]).$$

From this definition we get the long exact sequence

$$\to H^{i-1}(X_K, \mathbf{L}(\mathcal{M})) \to {}_sH^i_{f,a,b}(X, \mathcal{M}) \to H^i_{f,a,b}(X, \mathcal{M}) \xrightarrow{l} H^i(X_K, \mathbf{L}(\mathcal{M})) \to .$$

As we will see in the next section, for sufficiently big p, this long exact sequence splits into short exact sequences.

Remark 4. Note that the cohomology groups $H^*_{f,a,b}(X, \mathcal{M})$ and ${}_sH^*_{f,a,b}(X, \mathcal{M})$, and the map l do not really depend on the a and b chosen.

PROPOSITION 7.2. *If* $\mathcal{M} \in \mathcal{MF}^\nabla_{[a,b]}(X)$, *then*

$$_sH^i_{f,a,b}(X, \mathcal{M}) \xleftarrow{\sim} H^{i-1}(X_K, \mathbf{L}(\mathcal{M})) \quad \textit{for} \quad i > -a + \dim X + 2.$$

Proof. It follows from the fact that $H^i_{f,a,b}(X,\mathcal{M})=0$ for $i>-a+\dim X+1$ (Proposition 3.1). □

8. Duality

Assume that X is proper and smooth over V, of pure relative dimension d. Set $G_K=\mathrm{Gal}(\bar{K}/K)$ and $B^+(V)_m=B^+(V)/J^{[m]}_{B^+(V)}$.

LEMMA 8.1. *For any $\mathcal{M}\in\mathcal{MF}^{\nabla}_{[a,b]}(X)$ annihilated by p^n, $b-a\leqslant p-2$, $-a\leqslant p-1$, and sufficiently big m, the morphism*

$$H^i(\widetilde{\mathcal{X}}_{\bar{K}},\mathbf{L}(\mathcal{M})),\to H^i(\widetilde{\mathcal{X}}_{\bar{K}},\mathbf{L}(\mathcal{M})\otimes_{\mathbf{Z}_p}F^{(-a)}B^+_{n,m}(a))$$

is an injection.

Proof. We have the following commutative diagram

$$\begin{array}{ccccc} & & H^i(\widetilde{\mathcal{X}}_{\bar{K}},\mathbf{L}(\mathcal{M})) & \longrightarrow & H^i(\widetilde{\mathcal{X}}_{\bar{K}},\mathbf{L}(\mathcal{M})\otimes_{\mathbf{Z}_p}F^{(-a)}B^+_{n,m}(a)) \\ & & \wr\uparrow & & \uparrow \\ 0 & \longrightarrow & H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M})) & \longrightarrow & H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M}))\otimes_{\mathbf{Z}_p}F^{(-a)}B^+(V)_m(a). \end{array}$$

The left vertical morphism is an isomorphism by Proposition 6.1. The right one is an almost isomorphism by [5, Th. 3.3], i.e., its kernel and cokernel are killed by a power of m_B, where m_B is the preimage of the maximal ideal of $\bar{V}^\wedge$ via the map $B^+(V)\to\bar{V}^\wedge$. It is, in fact, an injection: it suffices to check that there is no elements x in $H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M}))\otimes_{\mathbf{Z}_p}F^{(-a)}B^+(V)_m(a)$ annihilated by m_B^k for some k, or, since $H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M}))$ is finitely generated, that that is the case for $F^{(-a)}B^+(V)_m/p^j=(J^{[-a]}_{B^+(V)}/J^{[m]}_{B^+(V)})/p^j$ for any j. Filtering $(J^{[-a]}_{B^+(V)}/J^{[m]}_{B^+(V)})/p^j$ with divided powers of $J_{B^+(V)}$, we reduce to showing that if $x\in\bar{V}$ and $p^\varepsilon x\in p^j\bar{V}$ for every ε, then $x\in p^j\bar{V}$, which is clear. □

Hence, the short exact sequence of sheaves on $\widetilde{\mathcal{X}}$, for n such that $p^n\mathbf{L}(\mathcal{M})=0$, for sufficiently big m, and $-a\leqslant p-1$,

$$\begin{aligned}0\to\mathbf{L}(\mathcal{M})&\to\mathbf{L}(\mathcal{M})\otimes_{\mathbf{Z}_p}F^{(-a)}B^+_{n,m}(a)\\ &\xrightarrow{1-p^a\phi}\mathbf{L}(\mathcal{M})\otimes_{\mathbf{Z}_p}B^+_{n,m}(a)\to 0,\end{aligned}$$

yields the short exact sequences of G_K-modules

$$\begin{aligned}0\to H^i(\widetilde{\mathcal{X}}_{\bar{K}},\mathbf{L}(\mathcal{M}))&\to H^i(\widetilde{\mathcal{X}}_{\bar{K}},\mathbf{L}(\mathcal{M})\otimes_{\mathbf{Z}_p}F^{(-a)}B^+_{n,m}(a))\\ &\xrightarrow{1-p^a\phi}H^i(\widetilde{\mathcal{X}}_{\bar{K}},\mathbf{L}(\mathcal{M})\otimes_{\mathbf{Z}_p}B^+_{n,m}(a))\to 0.\end{aligned}$$

Recall that, by Faltings [5], for every $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$, $b-a+d \leqslant p-2$, the crystalline cohomology groups $H^i_{\mathrm{cr}}(X/V,\mathcal{M})$ are in $\mathcal{MF}_{[a,b+d]}(V)$. Moreover [5, 5.3], the crystalline G_K-representation $\mathbf{L}(H^i_{\mathrm{cr}}(X/V,\mathcal{M}))$ associated to $H^i_{\mathrm{cr}}(X/V,\mathcal{M})$ is canonically isomorphic to $H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M}))$. In particular, we can apply to $H^i_{\mathrm{cr}}(X/V,\mathcal{M})$ the results of [3] and [12].

LEMMA 8.2. *For* $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$, $b-a+d \leqslant p-2$, $-a \leqslant p-1$, *there is a long exact sequence*

$$\begin{aligned} &\to H^i_{f,a,b}(X,\mathcal{M}) \to F^0 H^i_{\mathrm{cr}}(X/V,\mathcal{M}) \\ &\xrightarrow{1-\phi^0} H^i_{\mathrm{cr}}(X/V,\mathcal{M}) \to H^{i+1}_{f,a,b}(X,\mathcal{M}) \to . \end{aligned}$$

Proof. From the definition of $\mathcal{S}(\mathcal{M})^{\cdot}$ we get the long exact sequence

$$\begin{aligned} &\to H^i_{f,a,b}(X,\mathcal{M}) \to H^i_{\mathrm{cr}}(X/V,F^0(\mathcal{M})) \\ &\xrightarrow{1-\phi^0} H^i_{\mathrm{cr}}(X/V,\mathcal{M}) \to H^{i+1}_{f,a,b}(X,\mathcal{M}) \to , \end{aligned}$$

and we know, from [5], that the morphism $H^i_{\mathrm{cr}}(X/V,F^0(\mathcal{M})) \to H^i_{\mathrm{cr}}(X/V,\mathcal{M})$ is an injection, i.e., that $H^i_{\mathrm{cr}}(X/V,F^0(\mathcal{M})) \xrightarrow{\sim} F^0 H^i_{\mathrm{cr}}(X/V,\mathcal{M})$. □

Let $c = \max(b+d,0)$. The complex $F^0 H^i_{\mathrm{cr}}(X/V,\mathcal{M}) \xrightarrow{1-\phi^0} H^i_{\mathrm{cr}}(X/V,\mathcal{M})$ computes the cohomology groups $H^*_{f,a,c}(V,H^i_{\mathrm{cr}}(X/V,\mathcal{M}))$. Hence, the above lemma yields the short exact sequences

$$\begin{aligned} 0 &\to H^1_{f,a,c}(V,H^{i-1}_{\mathrm{cr}}(X/V,\mathcal{M})) \to H^i_{f,a,b}(X,\mathcal{M}) \\ &\to H^0_{f,a,c}(V,H^i_{\mathrm{cr}}(X/V,\mathcal{M})) \to 0\,. \end{aligned}$$

PROPOSITION 8.1. *For* $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$, $b-a+d \leqslant p-2$, *and* $-a \leqslant p-2$, *there is a commutative diagram*

$$\begin{array}{ccc} H^0_{f,a,c}(V,H^i_{\mathrm{cr}}(X/V,\mathcal{M})) & \xrightarrow[\sim]{l} & H^0(G_K,H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M}))) \\ \uparrow & & \uparrow \\ H^i_{f,a,b}(X,\mathcal{M}) & \xrightarrow{l} & H^i(X_K,\mathbf{L}(\mathcal{M})). \end{array}$$

Proof. For n such that $p^n\mathcal{M}=0$ and sufficiently big m, we have the following commutative diagram

$$\begin{array}{ccccc}
F^0H^i_{\mathrm{cr}}(X/V,\mathcal{M}) & \longrightarrow & H^i(\widetilde{\mathcal{X}},\mathbf{L}(\mathcal{M})\otimes F^{(-a)}B^+_{n,m}(a)) & \longrightarrow & \\
\uparrow & & \uparrow & & \\
H^i_{f,a,b}(X,\mathcal{M}) & \xrightarrow{\ l\ } & H^i(\widetilde{\mathcal{X}},\mathbf{L}(\mathcal{M})) & \longrightarrow &
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow H^i(\widetilde{\mathcal{X}}_{\bar{K}},\mathbf{L}(\mathcal{M})\otimes F^{(-a)}B^+_{n,m}(a)) & \longleftarrow & H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M}))\otimes F^{(-a)}B^+(V)_m(a) \\
\uparrow & & \uparrow \\
\longrightarrow H^i(\widetilde{\mathcal{X}}_{\bar{K}},\mathbf{L}(\mathcal{M})) & \xleftarrow{\ \sim\ } & H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M}))\,.
\end{array}$$

Recall [12] that the map $l\colon H^0_f(V,H^i_{\mathrm{cr}}(X/V,\mathcal{M}))\to H^0(G_K,H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M})))$ is induced from the morphism

$$H^i_{\mathrm{cr}}(X/V,\mathcal{M})\otimes_V B^+(V)_m(a)\to H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M}))\otimes_{\mathbf{Z}_p} B^+(V)_m(a)$$

fitting into the commutative diagram

$$\begin{array}{ccc}
H^i_{\mathrm{cr}}(X/V,\mathcal{M})\otimes_V B^+(V)_m(a) & \longrightarrow & H^i(\widetilde{\mathcal{X}}_{\bar{K}},\mathbf{L}(\mathcal{M})\otimes_{\mathbf{Z}_p} B^+_{n,m}(a)) \\
\searrow & & \nearrow \\
& H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M}))\otimes_{\mathbf{Z}_p} B^+(V)_m(a) &
\end{array}$$

The statement of the proposition follows now easily from the above commutative diagram and the injectivity of the map

$$H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M}))\otimes_{\mathbf{Z}_p} F^{(-a)}B^+(V)_m(a)$$
$$\to H^i(\widetilde{\mathcal{X}}_{\bar{K}},\mathbf{L}(\mathcal{M})\otimes_{\mathbf{Z}_p} F^{(-a)}B^+_{n,m}(a)).$$
□

COROLLARY 8.1. *Let* $\mathcal{M}\in\mathcal{MF}^{\nabla}_{[a,b]}(X)$, $b-a+d\leqslant p-2$, *and* $-a\leqslant p-2$. *If the residue field of* V *is finite, then the Hochschild–Serre spectral sequence*

$$H^i(G_K,H^j(X_{\bar{K}},\mathbf{L}(\mathcal{M})))\Rightarrow H^{i+j}(X_K,\mathbf{L}(\mathcal{M}))$$

degenerates.

Proof. Here since $H^0_f(V,H^i_{\mathrm{cr}}(X/V,\mathcal{M}))\ \tilde{\to}\ H^0(G_K,H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M})))$ and $H^i_f(X,\mathcal{M})\twoheadrightarrow H^0_f(V,H^i_{\mathrm{cr}}(X/V,\mathcal{M}))$, the proposition gives that

$$H^i(X_K,\mathcal{M})\twoheadrightarrow H^0(G_K,H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M}))).$$

By Poincaré duality, $H^2(G_K,H^i(X_{\bar{K}},\mathbf{L}(\mathcal{M})))\hookrightarrow H^{i+2}(X_K,\mathcal{M})$. The group G_K having cohomological dimension 2 we are done. □

PROPOSITION 8.2. *For* $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$, $b-a+d \leqslant p-2$, *and* $-a \leqslant p-2$, *the morphism* $l\colon H^i_f(X,\mathcal{M}) \to H^i(X_K, \mathbf{L}(\mathcal{M}))$ *is an injection.*

Proof. Let $x \in H^i_f(X,\mathcal{M})$ map to 0 in $H^i(X_K,\mathbf{L}(\mathcal{M}))$. First, note that, by the last proposition, the image of x in $F^0H^i_{\text{cr}}(X/V,\mathcal{M})$ is trivial. Hence x comes from $H^{i-1}_{\text{cr}}(X/V,\mathcal{M})$. Take n such that $p^n\mathcal{M} = 0$ and sufficiently big m, and consider the following commutative diagram

$$\begin{array}{ccc}
H^i_f(X,\mathcal{M}) & \xrightarrow{\quad l \quad} & H^i(\widetilde{\mathcal{X}}, \mathbf{L}(\mathcal{M})) \\
\uparrow & & \uparrow \\
H^{i-1}_{\text{cr}}(X/V,\mathcal{M}) & \longrightarrow & H^{i-1}(\widetilde{\mathcal{X}}, \mathbf{L}(\mathcal{M}) \otimes B^+_{n,m}(a)) \to \\
 & & \uparrow \\
 & & H^{i-1}(\widetilde{\mathcal{X}}, \mathbf{L}(\mathcal{M}) \otimes F^{(-a)}B^+_{n,m}(a)) \to \\
H^1(G_K, H^{i-1}(\widetilde{\mathcal{X}}_{\bar{K}}, \mathbf{L}(\mathcal{M}))) & \xleftarrow{\quad\sim\quad} & H^1(G_K, H^{i-1}(X_{\bar{K}}, \mathbf{L}(\mathcal{M}))) \\
\uparrow & & \uparrow \\
\longrightarrow H^{i-1}(\widetilde{\mathcal{X}}_{\bar{K}}, \mathbf{L}(\mathcal{M}) \otimes B^+_{n,m}(a))^{G_K} & \longleftarrow & (H^{i-1}(X_{\bar{K}}, \mathbf{L}(\mathcal{M})) \otimes B^+(V)_m(a))^{G_K} \\
\uparrow & & \uparrow \\
\longrightarrow H^{i-1}(\widetilde{\mathcal{X}}_{\bar{K}}, \mathbf{L}(\mathcal{M}) \otimes F^{(-a)}B^+_{n,m}(a))^{G_K} & \longleftarrow & (H^{i-1}(X_{\bar{K}}, \mathbf{L}(\mathcal{M})) \otimes F^{(-a)}B^+(V)_m(a))^{G_K}.
\end{array}$$

A diagram-chase shows that the image of any lifting of x to $H^{i-1}_{\text{cr}}(X/V,\mathcal{M})$ in $H^1(G_K, H^{i-1}(X_{\bar{K}}, \mathbf{L}(\mathcal{M})))$ is zero. Since the map

$$l\colon H^1_f(V, H^{i-1}_{\text{cr}}(X/V,\mathcal{M})) \to H^1(G_K, H^{i-1}(X_K, \mathbf{L}(\mathcal{M})))$$

mapping crystalline extensions to unrestricted extensions is injective, x itself is zero. □

Assume now that the residue field of V is finite, and for $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$ let $\mathcal{M}^D \in \mathcal{MF}^{\nabla}_{[-b-d-1,-a-d-1]}(X)$ denote $\mathcal{M}^*\{-d-1\}$, where $\mathcal{M}^*$ denotes the $\mathcal{MF}$-dual [5] (assuming, of course, that the width of the crystal does not exceed the admissible range).

THEOREM 8.1. *Let* $-(p-2) \leqslant a \leqslant 0, b-a+d \leqslant p-3$. *For any* $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$ *annihilated by* p^n, *there is a perfect pairing*

$$\begin{aligned}
&{}_sH^i_{f,a,b}(X,\mathcal{M}) \otimes H^{2d+3-i}_{f,-b-d-1,-a-d-1}(X,\mathcal{M}^D) \\
&\quad \longrightarrow {}_sH^{2d+3}_{f,a-b-d-1,-d-1}(X, \mathcal{O}_X/p^n\{-d-1\}) \\
&\quad \xrightarrow{\text{tr}} \mathbf{Z}/p^n.
\end{aligned}$$

Proof. Concerning the trace map, we have

$$\begin{aligned}
&{}_sH^{2d+3}{}_{f,a-b-d-1,-d-1}(X,\mathcal{O}_X/p^n\{-d-1\})\\
&\quad \xleftarrow{\sim} H^{2d+2}(X_K,\mathbf{Z}/p^n(d+1)) \xleftarrow{\sim} H^2(G_K,H^{2d}(X_{\bar{K}},\mathbf{Z}/p^n(d+1)))\\
&\quad \xrightarrow{\text{tr}} H^2(G_K,\mathbf{Z}/p^n(1))) \xrightarrow[\sim]{\text{inv}} \mathbf{Z}/p^n.
\end{aligned}$$

Here, the first isomorphism now follows, since $H^i_f(X,\mathcal{O}_X/p^n\{-d-1\})=0$ for $i>2d+1$. This, in turn, holds because we have the long exact sequence

$$\begin{aligned}
&\to H^i_f(X,\mathcal{O}_X/p^n\{-d-1\})\to F^0H^i_{\text{cr}}(X/V,\mathcal{O}_X/p^n\{-d-1\})\\
&\quad\to H^i_{\text{cr}}(X/V,\mathcal{O}_X/p^n\{-d-1\})\to H^{i+1}_f(X,\mathcal{O}_X/p^n\{-d-1\})
\end{aligned}$$

and $H^i_{\text{cr}}(X/V,\mathcal{O}_X/p^n\{-d-1\})=0$ for $i>2d$. We also have the commutative diagram

$$\begin{array}{ccc}
H^i_f(X,\mathcal{M})\otimes H^{2d+2-i}_f(X,\mathcal{M}^D) & \longrightarrow & H^{2d+2}_f(X,\mathcal{O}_X/p^n\{-d-1\})=0\\
\downarrow{\scriptstyle l\otimes l} & & \downarrow\\
H^i(X_K,\mathbf{L}(\mathcal{M}))\otimes H^{2d+2-i}(X_K,\mathbf{L}(\mathcal{M}^D)) & \longrightarrow & H^{2d+2}(X_K,\mathbf{Z}/p^n(d+1)),
\end{array}$$

which shows that $H^i_f(X,\mathcal{M})$ and $H^{2d+2-i}_f(X,\mathcal{M}^D)$ annihilate each other. Since the morphism $l_{\mathcal{M}}\colon H^i_f(X,\mathcal{M})\to H^i(X_K,\mathbf{L}(\mathcal{M}))$ is an injection, this diagram and the products on étale cohomology and on f-cohomology induce a product

$$\begin{aligned}
&{}_sH^i_{f,a,b}(X,\mathcal{M})\otimes H^{2d+3-i}_{f,-b-d-1,-a-d-1}(X,\mathcal{M}^D)\\
&\quad\to {}_sH^{2d+3}_{f,a-b-d-1,-d-1}(X,\mathcal{O}_X/p^n\{-d-1\}).
\end{aligned}$$

We have a complex

$$\begin{aligned}
&0\to H^{i-1}_f(X,\mathcal{M})\xrightarrow{l_{\mathcal{M}}} H^{i-1}(X_K,\mathbf{L}(\mathcal{M}))\\
&\quad\xrightarrow{t} H^{2d+3-i}_f(X,\mathcal{M}^D)^*\to 0,
\end{aligned}$$

where the map t is induced by the étale product. It is a surjection since via étale duality $(H^{i-1}(X_K,\mathbf{L}(\mathcal{M}))\simeq H^{2d+3-i}(X_K,\mathbf{L}(\mathcal{M}^D))^*)$ $t=l^*_{\mathcal{M}^D}$.

It remains to prove that the $\mathbf{Z}_p$-length of $H^{i-1}(X_K,\mathbf{L}(\mathcal{M}))$ is equal to the sum of the $\mathbf{Z}_p$-lengths of $H^{i-1}_f(X,\mathcal{M})$ and $H^{2d+3-i}_f(X,\mathcal{M}^D)$. I hope that the reader will forgive somewhat abusive notation in the following computations.

We have

$$\begin{aligned}
&H^i_f(X,\mathcal{M}) + H^{2d+2-i}_f(X,\mathcal{M}^D)\\
&\quad = H^0_f(V, H^i_{\mathrm{cr}}(X/V,\mathcal{M})) + H^1_f(V, H^{i-1}_{\mathrm{cr}}(X/V,\mathcal{M}))\\
&\qquad + H^0_f(V, H^{2d+2-i}_{\mathrm{cr}}(X/V,\mathcal{M}^D)) + H^1_f(V, H^{2d+1-i}_{\mathrm{cr}}(X/V,\mathcal{M}^D))\\
&\quad = H^0_f(V, H^i_{\mathrm{cr}}(X/V,\mathcal{M})) + H^1(G_K, H^{i-1}(X_{\bar{K}}, \mathbf{L}(\mathcal{M})))\\
&\qquad - H^1_f(V, H^{i-1}_{\mathrm{cr}}(X/V,\mathcal{M})) + H^0_f(V, H^{2d+2-i}_{\mathrm{cr}}(X/V,\mathcal{M}^D))\\
&\qquad + H^1_f(V, H^{i-1}_{\mathrm{cr}}(X/V,\mathcal{M})),
\end{aligned}$$

by crystalline duality [5] ($H^{i-1}_{\mathrm{cr}}(X/V,\mathcal{M}) \cong H^{2d+1-i}_{\mathrm{cr}}(X/V,\mathcal{M}^D)^D$) and the isomorphism [12] $H^1_f(V,\mathcal{N}^D)^* \cong H^1(G_K,\mathcal{N})/H^1_f(V,\mathcal{N})$, for $\mathcal{N} \in \mathcal{MF}_{[i,j]}(V)$ such that $i \leqslant 0$, $j \geqslant 0$ and $j-i \leqslant p-3$.

For a local system L on X_K annihilated by p^n denote by L^D the local system $\mathcal{H}\mathrm{om}(L, \mathbf{Z}/p^n(d+1))$. We get

$$\begin{aligned}
&H^i_f(X,\mathcal{M}) + H^{2d+2-i}_f(X,\mathcal{M}^D)\\
&\quad = H^0_f(V, H^i_{\mathrm{cr}}(X/V,\mathcal{M})) + H^1(G_K, H^{i-1}(X_{\bar{K}}, \mathbf{L}(\mathcal{M})))\\
&\qquad + H^0_f(V, H^{2d+2-i}_{\mathrm{cr}}(X/V,\mathcal{M}^D))\\
&\quad = H^0(G_K, H^i(X_{\bar{K}}, \mathbf{L}(\mathcal{M}))) + H^1(G_K, H^{i-1}(X_{\bar{K}}, \mathbf{L}(\mathcal{M})))\\
&\qquad + H^0(G_K, H^{2d+2-i}(X_{\bar{K}}, \mathbf{L}(\mathcal{M})^D))\\
&\quad = H^0(G_K, H^i(X_{\bar{K}}, \mathbf{L}(\mathcal{M}))) + H^1(G_K, H^{i-1}(X_{\bar{K}}, \mathbf{L}(\mathcal{M})))\\
&\qquad + H^2(G_K, H^{i-2}(X_{\bar{K}}, \mathbf{L}(\mathcal{M})))\\
&\quad = H^i(X_K, \mathbf{L}(\mathcal{M})),
\end{aligned}$$

by étale Poincaré duality, Galois duality and the degeneration of the Hochschild–Serre spectral sequence. □

COROLLARY 8.2. *Let* $-(p-2) \leqslant a \leqslant 0$, $b-a+d \leqslant p-3$. *For* $\mathcal{M} \in \mathcal{MF}^{\nabla}_{[a,b]}(X)$,

$$ {}_sH^i_{f,a,b}(X,\mathcal{M}) = 0 \quad \textit{for} \ \ i < -b+1. $$

In particular, there are isomorphisms

$$ H^{i-1}_{\mathrm{cr}}(X/V,\mathcal{M}) \xrightarrow{\sim} H^i_{f,a,b}(X,\mathcal{M}) \xrightarrow{\sim} H^i(X_K, \mathbf{L}(\mathcal{M})) $$

$$ \textit{for} \quad i < -b. $$

Proof. By the above theorem, we have that

$$ {}_sH^i_{f,a,b}(X,\mathcal{M}) \cong H^{2d+3-i}_{f,-b-d-1,-a-d-1}(X,\mathcal{M}^D)^*. $$

But, by Proposition 3.1., $H^{2d+3-i}_{f,-b-d-1,-a-d-1}(X,\mathcal{M}^D) = 0$ for $2d+3-i > b+d+1+d+1$, or for $i < -b+1$. The last statement of the corollary follows from Proposition 3.2. □

Remark 5. The above result for $\mathbf{L}(\mathcal{M}) = \mathbf{Z}/p^n(b)$, $0 \leqslant b \leqslant p-2$, was obtained earlier by Kurihara [11] as a consequence of his study of the relation between the syntomic sheaves and the sheaves of p-adic vanishing cycles.

Acknowledgements

I would like to thank S. Bloch, G. Faltings, and W. Messing for helpful conversations related to the subject of this paper.

References

1. Artin, M.: *Grothendieck Topologies*, Lecture Notes, Harvard University, Math. Dept, Cambridge, Mass, 1962.
2. Berthelot, P.: *Cohomologie Cristalline des Schémas de Caractéristique* $p > 0$, Lecture Notes in Math. 407, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
3. Bloch, S. and Kato, K.: *L-Functions and Tamagawa Numbers of Motives*, Grothendieck Festschrift, vol. I, Progress in Math., Birkäuser, Boston, Basel and Berlin, 1990, 333–400.
4. Elkik, R.: Solution d'équations a coefficients dans un anneau hensélian, *Ann. Sci. École Norm. Sup.* (4) 6 (1973) 553–604.
5. Faltings, G.: *Crystalline Cohomology and p-adic Galois Representations*, Algebraic analysis, geometry and number theory, J. I. Igusa (ed.), Johns Hopkins University Press, Baltimore, 1989, 25–80.
6. Faltings, G.: p-adic Hodge-theory, *J. of the AMS* 1 (1988) 255–299.
7. Fontaine, J.-M. and Messing, W.: p-adic periods and p-adic étale cohomology, Current Trends in Arithmetical Algebraic Geometry, K. Ribet (ed.), Contemporary Math., Vol. 67, Amer. Math. Soc., Providence, 1987, 179–207.
8. Fontaine, J.-M.: Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti–Tate, *Ann. of Math.* 115 (1982) 529–577.
9. Fontaine, J.-M.: Le corps des périods p-adiques, *Astérisque* 223 (1994) 321–347.
10. Gabber, O.: Affine analogue of the proper base change theorem, *Israel J. Math* 87(1–3) (1994) 325–335.
11. Kurihara, M.: A note on p-adic étale cohomology, *Proc. Japan Acad.* 63 (1987) 275–278.
12. Nizioł, W.: Cohomology of crystalline representations, *Duke Math. J.* 71 (1993) 747–791.
13. Ogus, A.: F-crystals, Griffiths transversality, and the Hodge decomposition, *Astérisque* 221, 1994.
14. Tsuji, T.: Syntomic complexes and p-adic vanishing cycles, Preprint, 1993.