

ON DE RHAM FLIP-FLOPPING IN DUAL TOWERS

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ABSTRACT. We prove a version of de Rham and Hyodo-Kato flip-flopping for dual towers of rigid analytic spaces including those coming from dual basic local Shimura varieties. The main tool are comparison theorems expressing the two cohomologies as pro-étale cohomology of corresponding relative period sheaves that, by definition, satisfy pro-étale descent. As an application, we show that de Rham and Hyodo-Kato cohomologies of finite level coverings of the Drinfeld space of any dimension d over K are admissible as representations of $\mathrm{GL}_{d+1}(K)$.

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1. INTRODUCTION

It is a classical result (due to Faltings, Fargues and extended by Scholze, see [35, Prop.5.4]) that, for $\ell \neq p$, the ℓ -adic cohomology with compact support of the (uncompleted) Lubin-Tate tower is isomorphic to the one for the (uncompleted) Drinfeld tower. This can be deduced more or less formally from the fact that there is a perfectoid space such that the two towers are de completions of this space. The same argument works for p -torsion étale cohomology with compact support, but fails very badly for p -adic étale or p -adic pro-étale cohomologies.

One of the key results in [10] was a similar flip-flopping theorem [10, Th.0.6] for compactly supported de Rham cohomology of the Drinfeld and Lubin-Tate towers in dimension 1. This result was somewhat surprising since, contrary to ℓ -adic or p -adic étale cohomology, de Rham cohomology does not really make sense for perfectoid spaces (meanwhile the situation changed, thanks to the introduction of de Rham stacks, see [3]). This comparison theorem was used in loc. cit. to deduce admissibility of compactly supported de Rham cohomology on the Drinfeld side from the one on the Lubin-Tate side, which was known. This paper reports on a generalization of this result to any dimension and then to other dual towers.

G.D. and W.N.'s research was partially supported by the project ANR-19-CE40-0015-02 COLOSS. W.N.'s research was also supported in part by the Simons Collaboration on Perfection in Algebra, Geometry, and Topology and G.D.'s research was also supported by the PNRR project "Schémas en groupes, systèmes de racines et représentations".

1.1. Motivation and strategy. Let p be a prime number. Let K be a finite extension of \mathbf{Q}_p and let $C := \widehat{\mathbf{Q}}_p$. Let $d > 1$ and let D^\times be the invertible elements in the central division algebra D of invariant $1/d$ over K . Let $\mathcal{M}_\infty^\varpi$ and $\mathrm{LT}_\infty^\varpi$ be the (uncompleted) Drinfeld and Lubin-Tate spaces at infinity¹ of dimension $d - 1$ over K . Let $H_{\mathrm{dR},c}^\bullet(\mathcal{M}_{\infty,C}^\varpi)$ be the compactly supported de Rham cohomology of $\mathcal{M}_\infty^\varpi$, i.e. the colimit of compactly supported cohomologies at finite levels in the Drinfeld tower. Define similarly the compactly supported Hyodo-Kato cohomology $H_{\mathrm{HK},c}^\bullet(\mathcal{M}_{\infty,C}^\varpi)$. Identical considerations apply to the Lubin-Tate tower.

We prove the following result that motivated our research:

Theorem 1.1. (Drinfeld and Lubin-Tate flip-flopping) *There are $\mathrm{GL}_d(K) \times D^\times$ -equivariant isomorphisms in solid C -modules and solid $(\varphi, N, \mathcal{G}_{\check{C}})$ -modules² over \check{C} , respectively*

$$H_{\mathrm{dR},c}^i(\mathcal{M}_{\infty,C}^\varpi) \simeq H_{\mathrm{dR},c}^i(\mathrm{LT}_{\infty,C}^\varpi), \quad H_{\mathrm{HK},c}^i(\mathcal{M}_{\infty,C}^\varpi) \simeq H_{\mathrm{HK},c}^i(\mathrm{LT}_{\infty,C}^\varpi).$$

Moreover all the above representations are smooth and admissible already as $\mathrm{GL}_d(K)$ -representations.

As in [10], the main difficulty in proving Theorem 1.1 lies in the fact that the differential forms on the base of the towers do not come, by descent, from differential forms on the completed tower at infinity, and only the completed towers compare naturally as perfectoid spaces. Hence, what objects should represent the differentials at infinity? The inspiration came from the arguments used in [10]. Namely, what we proved in loc. cit. could be called "Hodge flip-flopping": for $d = 2$ in the above theorem, we constructed an isomorphism

$$H_{\mathrm{dR},c}^1(\mathcal{M}_{\infty,C}^\varpi) \simeq H_{\mathrm{dR},c}^1(\mathrm{LT}_{\infty,C}^\varpi)$$

as the colimit, over $m, n \geq 0$, of the compositions of isomorphisms:

$$H_{\mathrm{dR},c}^i(\mathcal{M}_{n,C}^\varpi)^{G_m} \xleftarrow{\sim} H_c^i(\widehat{\mathcal{M}}_{\infty,C}^\varpi, \mathcal{O})^{G_m \times \check{G}_n} \simeq H_c^i(\widehat{\mathrm{LT}}_{\infty,C}^\varpi, \mathcal{O})^{G_m \times \check{G}_n} \xleftarrow{\sim} H_{\mathrm{dR},c}^i(\mathrm{LT}_{m,C}^\varpi)^{\check{G}_n}.$$

Here G_m, \check{G}_n are level m, n congruence subgroups of G and \check{G} , respectively. The middle isomorphism follows from the duality of the Drinfeld and Lubin-Tate towers at infinity, but the other isomorphisms require non-trivial arguments using the p -adic period maps, which are not easily extendable to higher dimensions. Hence the sheaf \mathcal{O} is the avatar of the Hodge cohomology: it plays the role of Hodge cohomology on perfectoid spaces³.

In this paper, the avatar of the filtered de Rham cohomology will be the filtered period sheaf \mathbb{B}_{dR} . More specifically, we derive the de Rham flip-flopping from a de Rham comparison theorem of Bosco [7] that expresses de Rham cohomology twisted by the period ring \mathbf{B}_{dR} as the pro-étale cohomology of the relative period sheaf \mathbb{B}_{dR} . The latter can be defined for adic spaces more general than rigid analytic spaces, for example for perfectoid spaces and, for trivial reasons, descends along the two towers and hence satisfies flip-flopping in dual towers. This flip-flopping then can be transferred to de Rham cohomology via the de Rham comparison theorem, but with a twist by \mathbf{B}_{dR} , which can be killed by taking Galois fixed points on the level of cohomology. On the derived level the same can be achieved by replacing \mathbb{B}_{dR} with Fontaine's almost de Rham period sheaf $\mathbb{B}_{\mathrm{pdR}}$. Similarly, for the avatar of the Hyodo-Kato cohomology we take the period sheaf \mathbb{B} (the relative version of the Fargues-Fontaine curve analog of the period ring \mathbf{B}_{cr}). We derive the Hyodo-Kato flip-flopping from a \mathbf{B} -comparison theorem of Colmez-Gilles-Nizioł [12] that expresses Hyodo-Kato cohomology twisted by the period ring \mathbf{B} as the pro-étale cohomology of the relative period sheaf \mathbb{B} , which satisfies flip-flopping in dual towers for trivial reasons. On the derived level the sheaf $\mathbb{B}_{\mathrm{pdR}}$ is replaced by the period sheaf $\mathbb{B}_{\mathrm{pFF}}$ introduced in [13].

¹See Section 7.1.1 for precise definitions.

²We write \check{C} for the fraction field of the Witt vectors $W(\overline{\mathbf{F}}_p)$.

³However, this turned out to be a wrong Hodge avatar in higher dimensions. We do prove a version of Hodge flip-flopping but this follows from a filtered de Rham flip-flopping (see Corollary 6.8).

That takes care of the flip-flopping part of Theorem 1.1. The admissibility part is now proved by looking at the Lubin-Tate side and using global methods (it would be very interesting to prove this more directly, in particular using purely local arguments).

1.1.1. *Dual basic local Shimura varieties.* The Drinfeld and Lubin-Tate towers are examples of dual towers of Rapoport-Zink spaces (see [34]) and one would naturally expect that a flip-flopping as in Theorem 1.1 holds in this more general context. This is indeed the case (see Theorem 1.2 below) and the proof is basically the same as in the Drinfeld and Lubin-Tate case.

To state the theorem, let $(G, [b], \{\mu\})$ be a basic local Shimura datum over \mathbf{Q}_p . Thus G is a connected reductive group over \mathbf{Q}_p , $\mu : \mathbb{G}_{m, \overline{\mathbf{Q}}_p} \rightarrow G_{\overline{\mathbf{Q}}_p}$ is a (conjugacy class of a) minuscule cocharacter, and b is the unique basic element in the set $B(G, \mu)$ of neutral acceptable elements. Let E be the reflex field, i.e. the field of definition of (the conjugacy class of) μ and let \check{E} be the completion of its maximal unramified extension. Attached to this datum there is a canonical tower $(\mathrm{Sh}(G, b, \mu)_K)_K$ (indexed by sufficiently small compact open subgroups K of $G(\mathbf{Q}_p)$) of local Shimura varieties, which are smooth rigid analytic spaces over \check{E} (this is one of the main results of [33]). As for the Drinfeld tower, we define the groups $H_{\mathrm{dR}, c}^\bullet(\mathrm{Sh}(G, b, \mu)_\infty)$ and $H_{\mathrm{HK}, c}^\bullet(\mathrm{Sh}(G, b, \mu)_\infty)$ by taking the colimit of the de Rham (respectively Hyodo-Kato) cohomology groups with compact support of the various $\mathrm{Sh}(G, b, \mu)_K$. There is also a natural dual local Shimura datum $(\check{G}, [\check{b}], \{\check{\mu}\})$, and so a dual tower of local Shimura varieties, for which the above considerations apply.

Theorem 1.2. (Local Shimura varieties flip-flopping) *There are $G \times \check{G}$ -equivariant isomorphisms in solid C -modules and solid $(\varphi, \mathcal{G}_{\check{C}})$ -modules⁴ over \check{C} , respectively*

$$H_{\mathrm{dR}, c}^i(\mathrm{Sh}(\check{G}, \check{b}, \check{\mu})_\infty) \simeq H_{\mathrm{dR}, c}^i(\mathrm{Sh}(G, b, \mu)_\infty), \quad H_{\mathrm{HK}, c}^i(\mathrm{Sh}(\check{G}, \check{b}, \check{\mu})_\infty) \simeq H_{\mathrm{HK}, c}^i(\mathrm{Sh}(G, b, \mu)_\infty).$$

If these representations of $G \times \check{G}$ are admissible the Hyodo-Kato isomorphism lies in the category of solid $(\varphi, N, \mathcal{G}_{\check{C}})$ -modules.

1.2. **The basic flip-flopping results.** The set-up for our main technical flip-flopping result is the following. Let K be a complete discretely valued field of mixed characteristic $(0, p)$ with perfect residue field. Consider a diagram

$$(1.3) \quad \begin{array}{ccc} & & G \times \check{G} \\ & & \curvearrowright \\ & T & \\ & \swarrow \pi & \searrow \tilde{\pi} \\ \check{G} & & G \\ \curvearrowright & & \curvearrowright \\ X = [T/G] & & \check{X} = [T/\check{G}] \end{array}$$

where

- X and \check{X} are smooth rigid analytic spaces over K
- T is a diamond over $\mathrm{Spd}(K)$, endowed with a continuous action of $G \times \check{G}$, where G and \check{G} are profinite groups.
- The maps⁵ $\pi : T \rightarrow X$ and $\tilde{\pi} : T \rightarrow \check{X}$ are pro-étale G -torsors and \check{G} -torsors respectively.

The following theorem is proved using the strategy sketched in Section 1.1:

Theorem 1.4. *Assume that X, \check{X} are partially proper.*

(1) *There is a natural quasi-isomorphism in $\mathcal{D}(K_\square)$*

$$\mathrm{RF}(\check{G}, F^r \mathrm{RF}_{\mathrm{dR}, *}(X)) \simeq \mathrm{RF}(G, F^r \mathrm{RF}_{\mathrm{dR}, *}(X)), \quad r \in \mathbf{Z}, * = \emptyset, c.$$

⁴We write \check{C} for the fraction field of the Witt vectors $W(\overline{\mathbf{F}}_p)$.

⁵We abuse notation and write X for the diamond attached to X .

(2) If G, \check{G} are compact p -adic Lie groups with Lie algebras $\mathfrak{g}, \check{\mathfrak{g}}$, then there are natural quasi-isomorphisms in $\mathcal{D}(K_{\square})$

$$\mathrm{R}\Gamma(\check{\mathfrak{g}}, K) \otimes_K^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\infty}) \simeq \mathrm{R}\Gamma(\mathfrak{g}, K) \otimes_K^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{dR},c}(\check{X}_{\infty}).$$

(3) Parts (1) and (2) are also valid for Hyodo-Kato cohomology.

Here the cohomology $\mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\infty})$ is defined as the colimit of compactly supported de Rham cohomologies from the finite levels of the π -tower in diagram (1.3); similarly, $\mathrm{R}\Gamma_{\mathrm{dR},c}(\check{X}_{\infty})$ is defined using the $\tilde{\pi}$ -tower. The second claim follows from the first one because the compactly supported cohomology is a smooth representation in the sense of [29], [30].

Assume now on that k is algebraically closed⁶. In the case of compactly supported cohomology we have the following underived version of flip-flopping if we are willing to pass to infinity.

Corollary 1.5. *Assume that X, \check{X} are partially proper and that G, \check{G} are p -adic Lie groups with*

$$(1.6) \quad \dim_K H^i(\mathfrak{g}, K) = \dim_K H^i(\check{\mathfrak{g}}, K), \quad i \geq 0.$$

Then:

(1) *There exist compact open subgroups $H \subset G, \check{H} \subset \check{G}$, such that there is a $H \times \check{H}$ -equivariant isomorphism in K_{\square} and $\mathcal{D}_{\varphi, \mathcal{G}_K}(\check{C}_{\square})$, respectively*

$$\begin{aligned} H_{\mathrm{dR},c}^i(X_{\infty}) &\simeq H_{\mathrm{dR},c}^i(\check{X}_{\infty}), \quad i \geq 0, \\ H_{\mathrm{HK},c}^i(X_{\infty}) &\simeq H_{\mathrm{HK},c}^i(\check{X}_{\infty}), \quad i \geq 0. \end{aligned}$$

(2) *If moreover in (1) the action of $H \times \check{H}$ is admissible, then the second isomorphism in (6.24) could also be made N -equivariant.*

We can choose H, \check{H} to be any subgroups acting trivially on all Lie algebra cohomology groups. We do not know a case when the condition (1.6) is not satisfied. The de Rham case of Corollary 1.5 is obtained from claim (2) of Theorem 1.4 by untwisting the Lie algebra cohomology using complete decomposition results for smooth admissible representations of compact or reductive groups. Similar arguments work for Hyodo-Kato cohomology, where however it is more difficult to get rid of the twists by the Lie algebra cohomology due to the presence of possibly nontrivial monodromy operator on Hyodo-Kato cohomology. Hence the admissibility assumption.

Theorem 1.2 follows directly from Corollary 1.5 since the condition (1.6) is satisfied in this case (the groups G, \check{G} being inner forms) and the groups H, \check{H} could be taken to be maximal compact open subgroups of G, \check{G} , respectively. Moreover, we have enough of functoriality of the flip-flop isomorphism to allow us to treat the Hecke correspondences lifting this action to the full groups G, \check{G} .

Remark 1.7. (Related work.)

- (1) An alternative approach to the de Rham part of Theorem 1.1 for dual towers of local Shimura varieties was found by the second author and Rodriguez Camargo (see [17, Th. 1.7]). It is a byproduct of a flip-flopping result for locally analytic vectors in the pro-étale cohomology at infinite level of relative period sheaves. Roughly speaking, working with locally analytic vectors instead of the whole period sheaves allows them to get rid of the Lie algebra cohomology computations in this paper that force us to use various "total decomposibility" conditions. Hence their flip-flop isomorphism has stronger functoriality properties.
- (2) For the Drinfeld and Lubin-Tate towers, yet another approach (though similar in spirit to that of [17]) to the de Rham part of Theorem 1.1 was found by Su [36].

⁶This is not necessary in the case of de Rham cohomology and probably not necessary in the case of Hyodo-Kato cohomology either

- (3) These results can also be obtained using the machinery of de Rham and Hyodo-Kato stacks developed in [3].

Acknowledgments. We would like to thank Piotr Achinger, Grigory Andreychev, Francesco Baldassari, Laurent Fargues, Juan Esteban Rodriguez Camargo, and Olivier Taibi for helpful conversations related to the content of this paper. We have also profited from many conversations with Guido Bosco. Special thanks go to Pierre Colmez for endless conversations and immense optimism: without him this paper would have never been finished.

Parts of this paper were written during the second author's stay at the Banach Center in Warsaw in 2019 and the authors' stay at the Hausdorff Research Institute for Mathematics in Bonn in 2023. We would like to thank these institutes for support and hospitality. The first author was supported by the project "Group schemes, root systems, and related representations" funded by the European Union - NextGenerationEU through Romania's National Recovery and Resilience Plan (PNRR) call no. PNRR-III- C9-2023-I8, Project CF159/31.07.2023, and coordinated by the Ministry of Research, Innovation and Digitalization (MCID) of Romania. He would like to thank the Institute of Mathematics "Simion Stoilow" of the Romanian Academy for the wonderful working conditions.

Notation and conventions Let \mathcal{O}_K be a complete discrete valuation ring with fraction field K of characteristic 0 and perfect residue field k of characteristic p . Let \overline{K} be an algebraic closure of K and let $\mathcal{O}_{\overline{K}}$ denote the integral closure of \mathcal{O}_K in \overline{K} . Set $\mathcal{G}_K = \text{Gal}(\overline{K}/K)$. Let $C = \widehat{\overline{K}}$ be the p -adic completion of \overline{K} . Let $W(k)$ be the ring of Witt vectors of k with fraction field F (i.e., $W(k) = \mathcal{O}_F$). Let C^{nr} denote the maximal unramified extension of F in C . Let \check{C} denote the fraction field of $W(\overline{k})$ and let φ be the absolute Frobenius on $W(\overline{k})$.

We will denote by $\mathbf{B}_{\text{cr}}, \mathbf{B}_{\text{st}}, \mathbf{B}_{\text{dR}}$ the (absolute) crystalline, semistable, and de Rham period rings of Fontaine, respectively. We will also use the (absolute) period rings $\mathbf{B}^+, \mathbf{B} := \mathbf{B}^+[1/t]$, which are analogs of $\mathbf{B}_{\text{cr}}^+, \mathbf{B}_{\text{cr}}$ tailored to the Fargues-Fontaine curve. We changed the standard notation here a bit replacing \mathbf{B} with \mathbf{B}^+ etc to make it uniform with the \mathbf{B}_{dR} -notation.

All rigid analytic spaces and dagger spaces considered will be over K or C . We assume that they are separated, taut, and countable at infinity. Pro-étale cohomology of rigid analytic spaces is the quasi-pro-étale cohomology of the associated diamond. Group cohomology is always (inner) condensed.

2. PRELIMINARIES

We gather here the facts we need concerning period rings and their Galois cohomology⁷.

2.1. Condensed and solid abelian groups. Let Cond (resp. CondAb) be the category of condensed sets (resp. abelian groups), i.e., accessible sheaves of sets (resp. abelian groups) on the site⁸ PS of profinite sets with finite jointly surjective families of maps as coverings. The category CondAb is closed symmetric monoidal and the forgetful functor $\text{CondAb} \rightarrow \text{Cond}$ has a left adjoint denoted by $X \mapsto \mathbf{Z}[X]$, the free condensed abelian group on X .

There is a functor $\underline{(-)} : \text{Top}_1 \rightarrow \text{Cond}$ from the category of T1 topological spaces (i.e., all points are closed) to Cond , given by $\underline{T}(S) := \mathcal{C}^0(S, T)$, for $S \in \text{PS}$. This functor is fully faithful on compactly generated weakly Hausdorff spaces, which cover all topological spaces we deal with in this paper, so we will sometimes be sloppy and simply write T instead of \underline{T} , especially when T is a discrete topological space (such as \mathbf{Z} or $\mathcal{C}(S, \mathbf{Z})$, for $S \in \text{PS}$) or a profinite set. The functor $T \mapsto \underline{T}$ restricts to a functor $\text{TopAb} \rightarrow \text{CondAb}$, where TopAb is the category of T1 topological abelian groups.

⁷Always taken in the condensed mathematics setting.

⁸This is also the pro-étale site of the geometric point $\text{Spa}(C, \mathcal{O}_C)$.

Let Solid be the full subcategory of CondAb generated under (small) colimits by the objects $\mathbf{Z}^I := \prod_I \mathbf{Z}$, for arbitrary sets I . A theorem of Clausen and Scholze ensures that Solid is stable under (small) limits, colimits, extensions (in particular it is abelian), the \mathbf{Z}^I (as I varies) form a class of compact projective generators, and there is a closed symmetric monoidal structure \otimes^\square on Solid for which $\mathbf{Z}^I \otimes^\square \mathbf{Z}^J = \mathbf{Z}^{I \times J}$, for all sets I, J , and such that \otimes^\square commutes with colimits separately in each variable. Moreover, the inclusion $\text{Solid} \subset \text{CondAb}$ has a symmetric monoidal left adjoint $M \mapsto M^\square$ sending $\mathbf{Z}[S]$ (with $S \in \text{PS}$) to $\mathbf{Z}[S]^\square := \underline{\text{Hom}}(\mathcal{C}^0(S, \mathbf{Z}), \mathbf{Z})$, and $\mathbf{Z}_\square[S] := \mathbf{Z}[S]^\square$ is compact projective in Solid . For any condensed ring A we thus have $A[S]^\square \simeq \mathbf{Z}_\square[S] \otimes^\square A^\square$.

The category Solid contains all discrete abelian groups (seen as objects of CondAb), thus it also contains \mathbf{Z}_p and \mathbf{Q}_p . It easily follows (see [7, Prop. A.31]) that Solid contains all \underline{V} , with V a complete locally convex \mathbf{Q}_p -vector space (for instance a \mathbf{Q}_p -Banach space or a Fréchet space).

2.2. Solid and Fréchet K -vector spaces. If A is a solid \mathbf{Q}_p -algebra⁹, we let

$$A_\square := \text{Mod}_A(\text{Solid})$$

be the category of A -modules in Solid . Setting $A_\square[S] := A[S]^\square = \mathbf{Z}_\square[S] \otimes^\square A \in A_\square$ (where $A[S] := \mathbf{Z}[S] \otimes_{\mathbf{Z}} A \in \text{CondAb}$), the category A_\square is generated under colimits by the $A_\square[S]$, which are compact projective, and A_\square has a natural closed symmetric monoidal structure \otimes_A^\square for which $A_\square[S] \otimes_A^\square A_\square[S'] \simeq A_\square[S \times S']$, for $S, S' \in \text{PS}$. If $A = \underline{B}$, with B a complete locally convex \mathbf{Q}_p -algebra, we simply write $B_\square := A_\square$ and $B_\square[S] := A_\square[S]$.

Let Fr_K be the category of Fréchet spaces over K (i.e., countable inverse limits of K -Banach spaces). By the above discussion, the functor $V \mapsto \underline{V}$ induces a fully faithful embedding $\text{Fr}_K \subset K_\square$. Recall the following facts:

Proposition 2.1. *a) The functor $\text{Fr}_K \rightarrow K_\square$ is exact and symmetric monoidal.*¹⁰

b) For all $V \in \text{Fr}_K$, the object \underline{V} of K_\square is a filtered colimit of (images of) K -Banach spaces, and is flat for \otimes_K^\square , i.e., $\underline{V} \otimes_K^\square (-)$ is exact on K_\square .

Remark 2.2. (1) As S varies in PS , the $V_S := \underline{\text{Hom}}_{\text{CondAb}}(\mathbf{Z}[S], \mathbf{Q}_p) = \underline{\text{Hom}}_{\text{Solid}}(\mathbf{Z}_\square[S], \mathbf{Q}_p)$ describe the full subcategory of Solid spanned by the \mathbf{Q}_p -Banach spaces, and they satisfy $V_S \otimes_{\mathbf{Q}_p}^\square V_{S'} \simeq V_{S \times S'}$ (see [7, Prop. A.49, Lemma A.30]).

Let A be a solid \mathbf{Q}_p -algebra. Proposition A.29 in [7] shows that there is a natural analytic ring structure $\mathcal{A} = (A, \mathbf{Z})_\square$ with underlying condensed ring A and functor of measures $S \mapsto \mathcal{A}[S] := A_\square[S]$ (for S in the category ED of extremely disconnected profinite sets, i.e., projective objects of PS). Moreover the category A_\square is the same as the full subcategory $\text{Mod}_{\mathcal{A}}$ of $\text{Mod}_A(\text{CondAb})$ consisting of \mathcal{A} -complete modules, and the tensor product \otimes_A^\square is nothing but the corresponding closed symmetric monoidal structure $\otimes_{\mathcal{A}}$ on $\text{Mod}_{\mathcal{A}}$.

2.3. Condensed group cohomology. Let G be a profinite group and let $M \in \mathcal{D}(\mathbf{Z}[G])$. We define

$$\text{R}\Gamma(G, M) := \text{R}\underline{\text{Hom}}_{\mathbf{Z}[G]}(\mathbf{Z}, M) \in \mathcal{D}(\text{CondAb}).$$

If moreover $M \in \mathcal{D}(\text{Solid})$, by [7, Prop. B2], there is a natural quasi-isomorphism

$$\text{R}\Gamma(G, M) \simeq [M \rightarrow \underline{\text{Hom}}(\mathbf{Z}[G], M) \rightarrow \underline{\text{Hom}}(\mathbf{Z}[G^2], M) \rightarrow \dots],$$

the differentials being the usual ones in the bar resolution of \mathbf{Z} by $\mathbf{Z}[G]$ -modules. If M is a \underline{G} -module in K_\square , we can replace \mathbf{Z} by K_\square in the above formulae.

⁹That is, a commutative unital \mathbf{Q}_p -algebra object in Solid .

¹⁰We use the completed projective tensor product on Fr_K .

Let $\text{Nuc}_K \subset K_\square$ be the full subcategory of nuclear solid K -vector spaces. This subcategory is generated under colimits by (images of) K -Banach spaces and it is stable under countable products; it contains all images of K -Fréchet spaces (see [7, Prop. A.57, Prop. A.64]).

Proposition 2.3. *Let G be a profinite group and let $W \in \text{Nuc}_K$ be endowed with the trivial G -action. For any $V \in \mathcal{D}(K_\square[G])$ there is a natural quasi-isomorphism*

$$\text{R}\Gamma(G, V) \otimes_K^{\text{L}\square} W \simeq \text{R}\Gamma(G, V \otimes_K^{\text{L}\square} W).$$

Proof. Since G acts trivially on W , we obtain a natural map $W = H^0(G, W) \rightarrow \text{R}\Gamma(G, W) \rightarrow \text{R}\Gamma(G, V \otimes_K^{\text{L}\square} W)$ and hence a natural map $\text{R}\Gamma(G, V) \otimes_K^{\text{L}\square} W \rightarrow \text{R}\Gamma(G, V \otimes_K^{\text{L}\square} W)$. To see that it is a quasi-isomorphism we may assume (by dévissage) that V is concentrated in degree 0. Since everything commutes with colimits in W , we may also assume that W is (the image of) a K -Banach space, in particular it is flat. But then $\text{R}\Gamma(G, V) \otimes_K^{\text{L}\square} W$ is quasi-isomorphic to the complex $\underline{\text{Hom}}_K(K_\square[G^\bullet], V) \otimes_K^\square W$, while $\text{R}\Gamma(G, V \otimes_K^{\text{L}\square} W)$ is quasi-isomorphic to the complex $\underline{\text{Hom}}_K(K_\square[G^\bullet], V \otimes_K^\square W)$. Now, [2, Prop. 5.35] provides a natural isomorphism $\underline{\text{Hom}}_K(K_\square[S], V) \otimes_K^\square W \simeq \underline{\text{Hom}}_K(K_\square[S], V \otimes_K^\square W)$, for any profinite set S , yielding the result. \square

Proposition 2.4. *Let $V \in \text{Fr}_K$ be a K -Fréchet space endowed with a continuous K -linear action of a profinite group G and let $n \geq 0$. If $H_{\text{cont}}^n(G, V)$ is a K -Fréchet space, then there is a natural isomorphism in K_\square*

$$H^n(G, V) \simeq \underline{H_{\text{cont}}^n(G, V)}.$$

Proof. If $W \in \text{Fr}_K$ and H is a profinite group, then $\mathcal{C}(H, W)$ is naturally in Fr_K and it is not difficult to check that $\underline{\text{Hom}}(H, W) = \mathcal{C}(H, W)$. Let $D^n = \mathcal{C}(G^n, V) \in \text{Fr}_K$, then $\text{R}\Gamma(G, \underline{V})$ is quasi-isomorphic to the complex \underline{D}^\bullet , and $\text{R}\Gamma(G, V)$ is quasi-isomorphic to the complex D^\bullet . Thus it suffices to see that if D^\bullet is a complex of K -Fréchet spaces and if $H^n(D^\bullet)$ is itself Fréchet, then $H^n(\underline{D}^\bullet) = \underline{H^n(D^\bullet)}$. This follows easily from the exactness of the functor $\underline{(-)}$ on the category Fr_K . \square

2.4. Period rings and their Galois cohomology. We pass now to Galois cohomology of some period rings.

2.4.1. Galois cohomology of C . For $j \in \mathbf{Z}$, let $C(j)$ be the \mathcal{G}_K -module twist of C by χ_{cycl}^j , where $\chi_{\text{cycl}} : \mathcal{G}_K \rightarrow \mathbf{Z}_p^*$ is the cyclotomic character.

Proposition 2.5. (Tate) *We have isomorphisms in K_\square*

$$H^i(\mathcal{G}_K, C(j)) \simeq \begin{cases} K & \text{if } j = 0, i = 0, \\ K \cdot \log \chi_{\text{cycl}} & \text{if } j = 0, i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is a classical computation of Tate [37] in the context of continuous cohomology. To see that the same result holds in the condensed setting it suffices to use Proposition 2.4. \square

We will now modify the period ring C to kill the Galois cohomology classes in degree 1 in the above formulas. We define $\log t$ as a transcendental variable over C , and we equip $C[\log t]$ with the action of \mathcal{G}_K given by $g(\log t) = \log t + \log \chi_{\text{cycl}}(g)$, $g \in \mathcal{G}_K$.

Lemma 2.6. *We have isomorphisms in K_\square*

$$H^i(\mathcal{G}_K, C[\log t](j)) \simeq \begin{cases} K & \text{if } i = 0, j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $A = C[\log t]$ and let $A_k = C[\log t]^{\leq k} := \sum_{i=0}^k C(\log t)^i$, so that $H^i(\mathcal{G}_K, A) = \varinjlim_k H^i(\mathcal{G}_K, A_k)$.

We start by proving that the natural map $K \rightarrow H^0(\mathcal{G}_K, A_k)$ is an isomorphism for all k . It suffices to check that, for $k > 0$, the natural inclusion $H^0(\mathcal{G}_K, A_{k-1}) \subset H^0(\mathcal{G}_K, A_k)$ is an equality. Let $P = \sum_{j=0}^k v_j (\log t)^j \in H^0(\mathcal{G}_K, A_k)$ and let $g \in \mathcal{G}_K$. Identifying the coefficients of $(\log t)^k$ in $g(P) = P$ yields $v_k \in H^0(\mathcal{G}_K, C) \simeq K$. Identifying the coefficients of $(\log t)^{k-1}$ yields $-kv_k \log \chi_{\text{cycl}}(g) = (g-1)v_{k-1}$. The isomorphism $H^1(\mathcal{G}_K, C) = K \log \chi_{\text{cycl}}$ forces $v_k = 0$ and so $P \in H^0(\mathcal{G}_K, A_{k-1})$, as desired.

Next, we observe that A_k is stable under the action of \mathcal{G}_K and we have an exact sequence of \mathcal{G}_K -modules, for all $j \in \mathbf{Z}$:

$$0 \rightarrow A_k(j) \rightarrow A_{k+1}(j) \rightarrow C(j) \rightarrow 0.$$

Consider the long exact sequence in cohomology associated to this exact sequence. If $j \neq 0$, Proposition 2.5 immediately yields (by induction on k) the vanishing of all $H^i(\mathcal{G}_K, A_k(j))$, so we are done (by passing to the colimit over k). Suppose now that $j = 0$. Since the cohomological dimension of \mathcal{G}_K is 2, the long exact sequence yields (again by induction on k) $H^i(\mathcal{G}_K, A_k) = 0$ for $i \geq 2$, hence the long exact sequence reduces to the following exact sequence

$$0 \rightarrow H^0(\mathcal{G}_K, C) \simeq K \rightarrow H^1(\mathcal{G}_K, A_k) \rightarrow H^1(\mathcal{G}_K, A_{k+1}) \rightarrow H^1(\mathcal{G}_K, C) \simeq K \log \chi_{\text{cycl}} \rightarrow 0.$$

An induction on k show that $\dim H^1(\mathcal{G}_K, A_k) = 1$ for all k , thus the maps on the left and on the right of the exact sequence must be isomorphisms and the middle arrow is 0, in particular $H^1(\mathcal{G}_K, A) = \varinjlim_k H^1(\mathcal{G}_K, A_k) = 0$. \square

2.4.2. Almost de Rham period ring. Let $\mathbf{B}_{\text{pdR}}^+$ be the algebra $\mathbf{B}_{\text{dR}}^+[\log t]$ defined in [21, Sec. 4.3] and let $\mathbf{B}_{\text{pdR}} := \mathbf{B}_{\text{pdR}}^+[1/t]$. The Galois group \mathcal{G}_K acts on $\mathbf{B}_{\text{pdR}}^+$ via ring homomorphisms extending its usual action on \mathbf{B}_{dR}^+ and such that $g(\log t) = \log t + \log(\chi_{\text{cycl}}(g))$, $g \in \mathcal{G}_K$. This action naturally extends to \mathbf{B}_{pdR} . The filtration on \mathbf{B}_{pdR} is given by powers of t .

Proposition 2.7. *We have natural quasi-isomorphisms in $\mathcal{D}(K_{\square})$ ($r \in \mathbf{Z}$)*

$$\text{R}\Gamma(\mathcal{G}_K, t^r \mathbf{B}_{\text{pdR}}^+) \simeq \begin{cases} K[0] & \text{if } r \leq 0, \\ 0 & \text{if } r > 0, \end{cases}$$

and

$$\text{R}\Gamma(\mathcal{G}_K, \mathbf{B}_{\text{pdR}}^+) \simeq \text{R}\Gamma(\mathcal{G}_K, \mathbf{B}_{\text{pdR}}) \simeq K[0].$$

Proof. Let $A_n = C[\log t]^{\leq n} = \sum_{i=0}^n C(\log t)^i$. The proof of Lemma 2.6 shows that, for $j \neq 0$, we have $H^i(\mathcal{G}_K, A_n(j)) = 0$, for all $i, n \geq 0$. Moreover, the inductive system $(H^i(\mathcal{G}_K, A_n))_{n \geq 0}$ is zero for $i \geq 1$ (more precisely $H^i(\mathcal{G}_K, A_n) = 0$ for $i \geq 2$, and the map $H^1(\mathcal{G}_K, A_n) \rightarrow H^1(\mathcal{G}_K, A_{n+1})$ is 0) and it is constant of value K , for $i = 0$.

Let

$$M_r := t^r \mathbf{B}_{\text{pdR}}^+ = \text{colim}_n M_r^{\leq n}, \quad M_r^{\leq n} := t^r \mathbf{B}_{\text{dR}}^+[\log t]^{\leq n} = \sum_{i=0}^n t^r \mathbf{B}_{\text{dR}}^+(\log t)^i.$$

We claim that, for any $i \geq 0, r \in \mathbf{Z}$, we have an isomorphism of inductive systems

$$(H^i(\mathcal{G}_K, M_r^{\leq n}))_n \simeq \begin{cases} (H^i(\mathcal{G}_K, A_n(r)))_n & \text{if } r \geq 0, \\ (H^i(\mathcal{G}_K, A_n))_n & \text{if } r < 0. \end{cases}$$

The claim implies the result for $H^*(\mathcal{G}_K, M_r) = \text{colim}_n H^*(\mathcal{G}_K, M_r^{\leq n})$, using the description of the inductive system $(H^i(\mathcal{G}_K, A_n(r)))_n$ in the first paragraph. With this in hand, the second part of the proposition is immediate, since $H^i(\mathcal{G}_K, \mathbf{B}_{\text{pdR}}) = \text{colim}_{r \geq 0} H^i(\mathcal{G}_K, t^{-r} \mathbf{B}_{\text{pdR}}^+)$.

Fix $n \geq 0$ and $r \in \mathbf{Z}$, we define

$$X_m := t^r (\mathbf{B}_{\text{dR}}^+ / t^m) [\log t]^{\leq n},$$

so that $M_r^{\leq n} = \varprojlim_m X_m$, $X_1 \simeq A_n(r)$ and we have an exact sequence

$$0 \rightarrow \mathbf{R}^1 \lim_m H^{i-1}(\mathcal{G}_K, X_m) \rightarrow H^i(\mathcal{G}_K, M_r^{\leq n}) \rightarrow \lim_m H^i(\mathcal{G}_K, X_m) \rightarrow 0.$$

The natural projection $\mathbf{B}_{\text{dR}}^+/t^{m+1} \rightarrow \mathbf{B}_{\text{dR}}^+/t^m$ induces an exact sequence of \mathcal{G}_K -modules

$$0 \rightarrow A_n(m+r) \rightarrow X_{m+1} \rightarrow X_m \rightarrow 0.$$

Suppose that $r \geq 0$. Then $H^i(\mathcal{G}_K, A_n(m+r)) = 0$, for all $m \geq 1$, thus the projective system $(H^i(\mathcal{G}_K, X_m))_m$ is constant with value $H^i(\mathcal{G}_K, A_n(r))$, for all $i \geq 0$, and the result follows.

Suppose that $r = -j$ with $j > 0$. Then the above exact sequence and the fact that $H^i(\mathcal{G}_K, X_1) = H^i(\mathcal{G}_K, A_n(r)) = 0$, for all i , imply that $H^i(\mathcal{G}_K, X_k) = 0$, for $k \leq j$ and all i , that the projective system $(H^i(\mathcal{G}_K, X_m))_m$ is constant starting with $m = j+1$, with value $H^i(\mathcal{G}_K, M_r^{\leq n})$, and we have $H^i(\mathcal{G}_K, A_n) \simeq H^i(\mathcal{G}_K, X_{j+1}) \simeq H^i(\mathcal{G}_K, M_r^{\leq n})$, for all i . By construction, these isomorphisms are compatible with the variation of n , yielding the result. \square

Remark 2.8. (1) Lemma 2.6 and Proposition 2.7 are folklore. Somewhat different proofs to the ones presented above can be found in [13, Prop. 1.2, Th. 1.4].

(2) Proposition 2.3 and the above results yield the following more general version of Proposition 2.7: if $W \in \text{Nuc}_K$ is equipped with the trivial \mathcal{G}_K -action, then there are natural quasi-isomorphisms in $\mathcal{D}(K_\square)$

$$\mathbf{R}\Gamma(\mathcal{G}_K, W \otimes_K^\square C[\log t](j)) \simeq \begin{cases} W[0] & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that we have natural quasi-isomorphisms in $\mathcal{D}(K_\square)$ ($r \in \mathbf{Z}$)

$$\mathbf{R}\Gamma(\mathcal{G}_K, W \otimes_K^\square t^r \mathbf{B}_{\text{pDR}}^+) \simeq \begin{cases} W[0] & \text{if } r \leq 0, \\ 0 & \text{if } r > 0, \end{cases}$$

and

$$\mathbf{R}\Gamma(\mathcal{G}_K, W \otimes_K^\square \mathbf{B}_{\text{pDR}}^+) \simeq \mathbf{R}\Gamma(\mathcal{G}_K, W \otimes_K^\square \mathbf{B}_{\text{pDR}}) \simeq W[0].$$

2.4.3. Period ring \mathbf{B} . Our main reference here is [13].

Let Y_{FF} be the Fargues-Fontaine curve obtained by removing from $\text{Spa}(\mathbf{A}_{\text{inf}})$ the divisors $p = 0$ and $[p^b] = 0$. Let I be an interval of \mathbf{R}_+^* . Let $\mathbf{B}_I = \mathcal{O}(Y_{\text{FF}, I})$, where $Y_{\text{FF}, I}$ is the locus of v such that $v([p^b])/v(p) \in I$. If $I = [r, s]$, with $r, s \in \mathbf{Q}$, then $\mathbf{B}_I = \mathbf{A}_{[r, s]}[\frac{1}{p}]$, where $\mathbf{A}_{[r, s]}$ is the p -adic completion of $\mathbf{A}_{\text{inf}}[\frac{[p^b]^{1/r}}{p}, \frac{p}{[p^b]^{1/s}}]$. Set

$$\mathbf{B}^+ := \mathbf{B}^{0, \infty[} = \mathcal{O}(Y_{\text{FF}}) = \varprojlim \mathbf{B}_{[r, s]}, \quad \mathbf{B} := \mathbf{B}^+[1/t].$$

(Usually, the ring \mathbf{B}^+ is denoted by \mathbf{B} .) The natural Frobenius automorphism φ on \mathbf{A}_{inf} induces isomorphisms $\varphi : \mathbf{B}_I \simeq \mathbf{B}_{I/p}$ and, in the limit, an automorphism φ of \mathbf{B}^+ and \mathbf{B} . We denote by $x_{\text{dR}} \in Y_{\text{FF}}$ the point defined by $[p^b] = p$; the completion of the local ring at this point is \mathbf{B}_{dR}^+ , which yields a \mathcal{G}_K -equivariant injective morphism $\mathbf{B}^+ \hookrightarrow \mathbf{B}_{\text{dR}}^+$.

Define $U = \log[p^b]$ as a transcendental element over \mathbf{B}^+ , with

$$\varphi(U) = pU \quad \text{and} \quad g(U) = U + \log[g(p^b)/p^b].$$

We set:

$$\mathbf{B}_{\text{FF}}^+ := \mathbf{B}^+[U], \quad \mathbf{B}_{\text{FF}} := \mathbf{B}_{\text{FF}}^+[\frac{1}{t}], \quad \mathbf{B}_{\text{pFF}}^+ := \mathbf{B}_{\text{FF}}^+[\log t], \quad \mathbf{B}_{\text{pFF}} := \mathbf{B}_{\text{pFF}}^+[\frac{1}{t}]$$

The ring \mathbf{B}_{FF}^+ is endowed with a surjective monodromy operator $N = -\frac{d}{dU}$, commuting with the action of \mathcal{G}_K , such that $N\varphi = p\varphi N$ and $\mathbf{B}^+ = \mathbf{B}_{\text{FF}}^{+, N=0}$. These operators extend to $\mathbf{B}_{\text{pFF}}^+$ by setting $\varphi(\log t) = \log t, N \log t = 0$.

We extend the canonical injection $\mathbf{B}^+ \hookrightarrow \mathbf{B}_{\text{dR}}^+$ to a ring morphism $\mathbf{B}_{\text{FF}}^+ \otimes_F K \rightarrow \mathbf{B}_{\text{dR}}^+$ by sending U to $\log \frac{[p]}{p}$. This morphism is \mathcal{G}_K -equivariant and, by [19, Prop. 10.3.15], injective. Similarly, the natural morphism $\mathbf{B}_{\text{pFF}} \otimes_F K \rightarrow \mathbf{B}_{\text{pDR}}$ is \mathcal{G}_K -equivariant and injective. Moreover, we have $H^0(\mathcal{G}_K, \mathbf{B}_{\text{pFF}}) \simeq F$.

Theorem 2.9. ([13, Th. 2.4]) *For $\Lambda = \mathbf{B}_{\text{pFF}}^+, \mathbf{B}_{\text{pFF}}$, we have*

$$\mathrm{R}\Gamma(\mathcal{G}_K, \Lambda) \simeq \begin{cases} F[0] & \text{if } i = 0, \\ 0 & \text{if } i \geq 1. \end{cases}$$

Remark 2.10. Proposition 2.3 and Theorem 2.9 yield the following more general version of Theorem 2.9: if $W \in \mathrm{Nuc}_F$ is equipped with the trivial \mathcal{G}_K -action, then there is a natural quasi-isomorphisms in $\mathcal{D}(F_{\square})$

$$\mathrm{R}\Gamma(\mathcal{G}_K, W \otimes_F^{\square} \mathbf{B}_{\text{pFF}}^+) \simeq \mathrm{R}\Gamma(\mathcal{G}_K, W \otimes_F^{\square} \mathbf{B}_{\text{pFF}}) \simeq W[0].$$

2.4.4. *Derived \mathbf{B}_e -formula.* Set $\mathbf{B}_e := \mathbf{B}^{\varphi=1}$. We quote the following result¹¹:

Proposition 2.11. ([19, Prop. 10.3.20])

(1) *Let M be a finite rank (φ, N) -module over F . We have a canonical isomorphism¹² of $(\varphi, N, \mathcal{G}_K)$ -modules over \mathbf{B}_{FF} :*

$$(2.12) \quad (M \otimes_F \mathbf{B}_{\text{FF}})^{N=0, \varphi=1} \otimes_{\mathbf{B}_e} \mathbf{B}_{\text{FF}} \xrightarrow{\sim} M \otimes_F \mathbf{B}_{\text{FF}}.$$

It follows that the canonical morphism

$$M \rightarrow ((M \otimes_F \mathbf{B}_{\text{FF}})^{N=0, \varphi=1} \otimes_{\mathbf{B}_e} \mathbf{B}_{\text{FF}})^{\mathcal{G}_K}$$

is an isomorphism of (φ, N) -modules over F .

(2) *Let M be a finite rank $(\varphi, N, \mathcal{G}_K)$ -module over C^{nr} . The canonical morphism*

$$M \rightarrow ((M \otimes_{C^{\text{nr}}} \mathbf{B}_{\text{FF}})^{N=0, \varphi=1} \otimes_{\mathbf{B}_e} \mathbf{B}_{\text{FF}})^{\mathcal{G}_K - \text{sm}}$$

is an isomorphism of $(\varphi, N, \mathcal{G}_K)$ -modules over C^{nr} . Here $(-)^{\mathcal{G}_K - \text{sm}} := \mathrm{colim}_{H \subset \mathcal{G}_K} (-)^H$, where the index set runs over open subgroups H of \mathcal{G}_K .

The following Corollary of Theorem 2.9 is a derived version of Proposition 2.11.

Corollary 2.13. (1) *Let M be a bounded complex of finite rank (φ, N) -modules over F . The canonical morphism in $\mathcal{D}_{\varphi, N}(F)$*

$$M \rightarrow \mathrm{R}\Gamma(\mathcal{G}_K, [M \otimes_F^{\mathrm{L}\square} \mathbf{B}_{\text{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\text{pFF}})$$

is a quasi-isomorphism.

(2) *Let M be a bounded complex of finite rank $(\varphi, N, \mathcal{G}_K)$ -modules over C^{nr} , with \mathcal{G}_K acting through a finite quotient. The canonical morphism in $\mathcal{D}_{\varphi, N, \mathcal{G}_K}(C^{\text{nr}})$*

$$M \rightarrow ([M \otimes_{C^{\text{nr}}}^{\mathrm{L}\square} \mathbf{B}_{\text{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\text{pFF}})^{\mathrm{R}\mathcal{G}_K - \text{sm}}$$

is a quasi-isomorphism. Here we set $(-)^{\mathrm{R}\mathcal{G}_K - \text{sm}} := \mathrm{colim}_{H \subset \mathcal{G}_K} (-)^{\mathrm{RH}}$.

Proof. For the first claim, we may assume that $M = M[0]$ for a (φ, N) -module over F . We need to show that the canonical morphism

$$M \rightarrow \mathrm{R}\Gamma(\mathcal{G}_K, [M \otimes_F^{\mathrm{L}\square} \mathbf{B}_{\text{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\text{pFF}})$$

is a quasi isomorphism. Since we have Proposition 2.11, it suffices to show that

$$(2.14) \quad H^i(\mathcal{G}_K, [M \otimes_F^{\mathrm{L}\square} \mathbf{B}_{\text{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\text{pFF}}) = 0, \quad i \geq 1.$$

¹¹The second claim in the proposition is obtained from the first one by passing to a colimit.

¹²Perhaps the quickest way to see this is to equip M with an admissible filtration (this is always possible) and to use the associated Galois representation.

But, by (2.12), we have

$$\begin{aligned} [M \otimes_F^{\mathbb{L}\square} \mathbf{B}_{\text{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathbb{L}\square} \mathbf{B}_{\text{pFF}} &\simeq (M \otimes_F^{\square} \mathbf{B}_{\text{FF}})^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\square} \mathbf{B}_{\text{pFF}} \\ &\simeq (M \otimes_F^{\square} \mathbf{B}_{\text{FF}} \otimes_{\mathbf{B}_{\text{FF}}}^{\square} \mathbf{B}_{\text{pFF}} \simeq M \otimes_F^{\square} \mathbf{B}_{\text{pFF}} \end{aligned}$$

and the Galois action on the $M \otimes_F^{\square} \mathbf{B}_{\text{pFF}}$ is via the action on \mathbf{B}_{pFF} . Hence, for $i \geq 1$, by Theorem 2.9

$$H^i(\mathcal{G}_K, [M \otimes_F^{\mathbb{L}\square} \mathbf{B}_{\text{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathbb{L}\square} \mathbf{B}_{\text{pFF}}^+) \simeq H^i(\mathcal{G}_K, M \otimes_F^{\square} \mathbf{B}_{\text{pFF}}) = 0,$$

as wanted.

The second claim follows from the first one by a colimit argument. \square

3. COMPARISON THEOREMS

We recall here the de Rham and Hyodo-Kato comparison theorems from [7], [8] [15], [16], generalize them slightly, for instance by adapting them to compactly supported cohomology. For $L = K, C$ let Sm_L be the category of smooth rigid analytic varieties over L .

3.1. Period sheaves. Let X be an analytic adic space over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. On the pro-étale site¹³ $X_{\text{proét}}$ of X one has the following sheaves of rings:

- the integral structure sheaf $\widehat{\mathcal{O}}_X^+$ and its tilted version $\widehat{\mathcal{O}}_X^{b,+} = \varprojlim_{s \rightarrow s^p} \widehat{\mathcal{O}}_X^+ / p$. For any perfectoid space $U \in X_{\text{proét}}$ we have $\widehat{\mathcal{O}}_X^{b,+}(U) = \widehat{\mathcal{O}}_{U^\sharp}^+(U^\sharp)$, where U^\sharp is the untilt of U induced by the map $U \rightarrow X^\diamond \rightarrow \text{Spd}(\mathbf{Z}_p)$.

- the sheaf (of p -typical Witt vectors) $\mathbb{A}_{\text{inf}} = W(\widehat{\mathcal{O}}_X^+)$ endowed with Fontaine's map $\theta : \mathbb{A}_{\text{inf}} \rightarrow \widehat{\mathcal{O}}_X^+$, which extends to a map $\theta : \mathbb{B}_{\text{inf}} := \mathbb{A}_{\text{inf}}[1/p] \rightarrow \widehat{\mathcal{O}}_X^+ := \widehat{\mathcal{O}}_X^+[1/p]$.

- the relative de Rham period sheaves $\mathbb{B}_{\text{dR}}^+ = \varprojlim_n \mathbb{B}_{\text{inf}} / (\ker \theta)^n$, endowed with the filtration $F^n \mathbb{B}_{\text{dR}}^+ = (\ker \theta)^n \mathbb{B}_{\text{dR}}^+$. Locally on $X_{\text{proét}}$ there is a generator t of $F^1 \mathbb{B}_{\text{dR}}^+$, it is a non-zero divisor and is unique up to a unit, which allows us to define the sheaf $\mathbb{B}_{\text{dR}} = \mathbb{B}_{\text{dR}}^+[1/t]$, with the filtration $F^n \mathbb{B}_{\text{dR}} = t^n \mathbb{B}_{\text{dR}}^+$.

- If $I = [s, r] \subset (0, \infty)$ is a compact interval with rational endpoints, one defines the sheaf $\mathbb{A}_{\text{inf}, I} = \mathbb{A}_{\text{inf}}[\frac{p}{[\alpha]}, \frac{[\beta]}{p}]$, where $\alpha, \beta \in \mathcal{O}_{C^b}$ satisfy $v(\alpha) = 1/r$ and $v(\beta) = 1/s$ (here v is the standard valuation on C^b). The sheaf \mathbb{B}_I is obtained from $\mathbb{A}_{\text{inf}, I}$ by p -adic completion, followed by inverting p . Finally, the period sheaves \mathbb{B}^+ and \mathbb{B} are defined by $\mathbb{B}^+ = \varprojlim_I \mathbb{B}_I$ and $\mathbb{B} = \mathbb{B}^+[1/t]$. The natural Frobenius automorphism φ on \mathbb{A}_{inf} induces isomorphisms $\varphi : \mathbb{B}_I \simeq \mathbb{B}_{I/p}$ and, in the limit, an automorphism φ of \mathbb{B}^+ and \mathbb{B} .

- The sheaves

$$\mathbb{B}_{\text{pdR}}^+ := \mathbb{B}_{\text{dR}}^+[\log t], \quad \mathbb{B}_{\text{pdR}} := \mathbb{B}_{\text{pdR}}^+[1/t]$$

will play a crucial role in this paper. They are endowed with natural filtrations induced by that on \mathbb{B}_{dR}^+ . We endow them with a Galois action by setting $g(\log t) = \log t + \log(\chi_{\text{cycl}}(g))$, $g \in \mathcal{G}_K$.

- We finally define $\mathbb{B}_{\text{FF}}^+ := \mathbb{B}^+[U]$, $U = \log[p^b]$ endowed with a Frobenius φ and an action of \mathcal{G}_K extending the ones on \mathbb{B}^+ and such that $\varphi(U) = pU$ and $g(U) = U + \log[g(p^b)/p^b]$ for $g \in \mathcal{G}_K$. The sheaf \mathbb{B}_{FF}^+ is endowed with a surjective monodromy operator $N = -\frac{d}{dU}$, commuting with the action of \mathcal{G}_K , such that $N\varphi = p\varphi N$ and $\mathbb{B}^+ = \mathbb{B}_{\text{FF}}^{+, N=0}$. We set $\mathbb{B}_{\text{FF}} := \mathbb{B}_{\text{FF}}^+[1/t]$.

- The sheaves

$$\mathbb{B}_{\text{pFF}}^+ := \mathbb{B}^+[\log t], \quad \mathbb{B}_{\text{pFF}} := \mathbb{B}_{\text{pFF}}^+[1/t]$$

will play a crucial role in this paper. We endow them with a Frobenius, monodromy, and a Galois action by setting

$$\varphi(\log t) = \log t, \quad N \log t = 0, \quad g(\log t) = \log t + \log(\chi_{\text{cycl}}(g)), \quad g \in \mathcal{G}_K.$$

¹³Recall that, by definition, this is the quasi-pro-étale site of the associated diamond X^\diamond .

For $X = \mathrm{Spa}(C, \mathcal{O}_C)$, the above sheaves define condensed rings $\mathbf{A}_{\mathrm{inf}}, \mathbf{B}, \mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{pdR}}, \mathbf{B}_{\mathrm{FF}}, \mathbf{B}_{\mathrm{pFF}}$, etc, which are condensed versions of Fontaine's classical period rings.

3.2. Condensed enhancement of sheaf cohomology. Let X be a rigid analytic variety over K and let \mathcal{B} be the basis of affinoid open subspaces of X . If F is a TopAb -valued sheaf on \mathcal{B} (for instance F could be a coherent sheaf on X , in which case $F(U)$ is naturally a K -Banach space for all $U \in \mathcal{B}$), there is a unique CondAb -valued sheaf F_{cond} such that $F_{\mathrm{cond}}(U) = F(U)$ for all affinoid open subspaces U of X . If $F(U) \in \mathrm{Fr}_K$ for all $U \in \mathcal{B}$, then F_{cond} is a K_{\square} -valued sheaf on X . We thus obtain a complex $\mathrm{R}\Gamma_{\mathrm{cond}}(X, F) := \mathrm{R}\Gamma(X, \underline{F}) \in \mathcal{D}(\mathrm{CondAb})$ with cohomology groups $H_{\mathrm{cond}}^i(X, F) \in \mathrm{CondAb}$, which are enhancements of the usual $\mathrm{R}\Gamma(X, F) \in \mathcal{D}(\mathrm{Ab})$ and $H^i(X, F) \in \mathrm{Ab}$ in the sense that $\mathrm{R}\Gamma_{\mathrm{cond}}(X, F)(*) = \mathrm{R}\Gamma(X, F)$ and similarly for cohomology groups. Moreover, if F is Fr_K -valued on \mathcal{B} then $\mathrm{R}\Gamma_{\mathrm{cond}}(X, F) \in \mathcal{D}(K_{\square})$. A similar discussion applies if we replace F with a bounded below complex of such sheaves.

If X is an analytic adic space over C (i.e. over $\mathrm{Spa}(C, \mathcal{O}_C)$), we can replace in the above discussion the analytic site of X with its pro-étale site $X_{\mathrm{pro\acute{e}t}}$ and \mathcal{B} with the basis of $X_{\mathrm{pro\acute{e}t}}$ consisting of affinoid perfectoid $U \in X_{\mathrm{pro\acute{e}t}}$. Thus any TopAb -valued sheaf F on \mathcal{B} yields an enhanced cohomology complex $\mathrm{R}\Gamma_{\mathrm{cond}}(X_{\mathrm{pro\acute{e}t}}, F) := \mathrm{R}\Gamma(X_{\mathrm{pro\acute{e}t}}, F_{\mathrm{cond}}) \in \mathcal{D}(\mathrm{CondAb})$, and if F is Fr_K -valued then $\mathrm{R}\Gamma_{\mathrm{cond}}(X_{\mathrm{pro\acute{e}t}}, F) \in \mathcal{D}(K_{\square})$. A similar discussion applies when F is a bounded below complex of such sheaves.

Remark 3.1. For any $X \in \mathrm{Sm}_K$, we obtain a bounded complex

$$\mathrm{R}\Gamma_{\mathrm{dR}, \mathrm{cond}}(X) := \mathrm{R}\Gamma_{\mathrm{cond}}(X, \Omega_X^{\bullet}) \in \mathcal{D}(K_{\square}),$$

whose cohomology groups are denoted $H_{\mathrm{dR}, \mathrm{cond}}^i(X) \in K_{\square}$. It has a natural enhancement as object of the filtered ∞ -category $\mathcal{F}\mathcal{D}(K_{\square})$ of $\mathcal{D}(K_{\square})$, where $F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X) = \mathrm{R}\Gamma_{\mathrm{cond}}(X, \Omega_X^{\geq r})$. If X is either Stein or proper, then $H_{\mathrm{dR}}^i(X) = \underline{H}_{\mathrm{dR}}^i(X)$, with $H_{\mathrm{dR}}^i(X) \in \mathrm{Fr}_K$ (even finite dimensional if X is proper). If X is an affinoid of positive dimension, then $H_{\mathrm{dR}, \mathrm{cond}}^i(X) \in K_{\square}$ is *not* in (the image of) Fr_K . Since we will always work in the condensed world, we will simply write $\mathrm{R}\Gamma_{\mathrm{dR}}(X) \in \mathcal{D}(K_{\square})$ instead of $\mathrm{R}\Gamma_{\mathrm{dR}, \mathrm{cond}}(X)$ and $H_{\mathrm{dR}}^i(X) \in K_{\square}$ instead of $H_{\mathrm{dR}, \mathrm{cond}}^i(X)$.

If X is an analytic adic space over C and if \mathcal{F} is one of the sheaves $\mathbb{B}_{\mathrm{dR}}^+, \mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\mathrm{pdR}}, \mathbb{B}_I, \mathbb{B}^+$, then \mathcal{F} defines a TopAb -valued sheaf on the basis \mathcal{B} of affinoid perfectoids $U \in X_{\mathrm{pro\acute{e}t}}$. More precisely, if $\mathcal{F} \in \{\mathbb{B}_{\mathrm{dR}}^+, \mathbb{B}_I, \mathbb{B}^+\}$ then $\mathcal{F}(U)$ is naturally in Fr_K for $U \in \mathcal{B}$, while if $\mathcal{F} \in \{\mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\mathrm{pdR}}\}$ then $\mathcal{F}(U)$ is a countable colimit of Fréchet spaces, with closed immersions as transitions maps. It follows that the associated sheaf $\mathcal{F}_{\mathrm{cond}}$ is K_{\square} -valued and $\mathrm{R}\Gamma_{\mathrm{cond}}(X_{\mathrm{pro\acute{e}t}}, \mathcal{F}) \in \mathcal{D}(K_{\square})$. Moreover, it is nuclear. In what follows, unless this causes confusion, we will simply write $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathcal{F})$ for $\mathrm{R}\Gamma_{\mathrm{cond}}(X_{\mathrm{pro\acute{e}t}}, \mathcal{F})$.

The following result will be very useful later (it is also instrumental in the proof of Lemma 3.4):

Lemma 3.2. ([7, proof of Prop. 4.12]) *Let \mathcal{F} be one of the sheaves $\mathbf{Q}_p, \mathbb{B}_{\mathrm{dR}}^+, \mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\mathrm{pdR}}, \mathbb{B}_I, \mathbb{B}^+$. For any profinite set S we have a natural isomorphism in $\mathcal{D}(\mathrm{Solid})$*

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X \times S, \mathcal{F}) \simeq \mathrm{R}\underline{\mathrm{Hom}}(\mathbf{Z}[S], \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathcal{F})).$$

3.3. Hochschild-Serre spectral sequences. We will now construct Hochschild-Serre spectral sequences for a variety of period sheaves. For that, we will use the following definition of torsors:

Definition 3.3. Let G be a profinite group. A map $f : Y \rightarrow X$ of analytic adic spaces over \mathbf{Z}_p is called a *pro-étale G -torsor* if the associated map of diamonds $f^{\diamond} : Y^{\diamond} \rightarrow X^{\diamond}$ is endowed with an action of G on Y^{\diamond} over X^{\diamond} and, quasi-pro-étale locally on X^{\diamond} , there is a G -equivariant isomorphism of diamonds $Y^{\diamond} \simeq X^{\diamond} \times G^{\diamond}$. In such case, we will call G the Galois group of f .

Lemma 3.4. (Hochschild-Serre spectral sequences) *Let \mathcal{F} be as in Lemma 3.2. Then:*

(1) *There is a natural Hochschild-Serre quasi-isomorphism in $\mathcal{D}(B_\square)$, for $B = \mathcal{F}(C)$.*

$$\mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(Y, \mathcal{F})) \simeq \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathcal{F}).$$

(2) *There is a natural Hochschild-Serre spectral sequence*

$$E_2^{i,j} = H^i(G, H_{\mathrm{pro\acute{e}t}}^j(Y, \mathcal{F})) \Rightarrow H_{\mathrm{pro\acute{e}t}}^{i+j}(X, \mathcal{F}).$$

Proof. The case of \mathbf{Q}_p was treated in [11, Lemma 4.4]. The cases of $\mathbb{B}_{\mathrm{dR}}, \mathbb{B}^+, \mathbb{B}_I$ were treated in [7, Prop. 4.12]. The case of $\mathbb{B}_{\mathrm{pdR}}$ follows by the same type of argument since the analog of [7, Prop. 4.7] holds for it by a colimit argument starting from $\mathbb{B}_{\mathrm{dR}}^+$. \square

3.4. De Rham comparison theorem. Let $X \in \mathrm{Sm}_K$. See Remark 3.1 for our conventions concerning the object $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$ of the filtered ∞ -category $\mathcal{F}\mathcal{D}(K_\square)$ of $\mathcal{D}(K_\square)$.

Let $*$ \in $\{\mathrm{dR}, \mathrm{pdR}\}$. The \mathbf{B}_* -cohomology of X ([5], [23] for $*$ = dR) is the object of the filtered ∞ -category $\mathcal{F}\mathcal{D}(\mathbf{B}_{*,\square})$ of $\mathcal{D}(\mathbf{B}_{*,\square})$ defined by

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_*) := \mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_K^{\mathrm{L}\square} \mathbf{B}_*.$$

It is endowed with the tensor product filtration ($r \in \mathbf{Z}$)

$$F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbf{B}_*) := \mathrm{colim}_{j+k \geq r} (F^j \mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_K^{\mathrm{L}\square} F^k \mathbf{B}_*).$$

Example 3.5. If X is Stein or affinoid, thanks to the vanishing of higher coherent cohomology on X , we have natural quasi-isomorphisms in $\mathcal{D}(\mathbf{B}_{*,\square})$

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_*) &\simeq (\mathcal{O}(X) \otimes_K^{\square} \mathbf{B}_* \rightarrow \Omega^1(X) \otimes_K^{\square} \mathbf{B}_* \rightarrow \dots) \\ F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_*) &\simeq (\mathcal{O}(X) \otimes_K^{\square} t^r \mathbf{B}_*^+ \rightarrow \Omega^1(X) \otimes_K^{\square} t^{r-1} \mathbf{B}_*^+ \rightarrow \dots) \end{aligned}$$

Here we were able to replace the derived tensor product by the tensor product itself because the spaces $\Omega^i(X)$ are Fréchet, hence flat (\mathbf{B}_* is also flat, as a filtered colimit of Fréchet spaces).

Similarly, the object $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_*) \in \mathcal{D}(\mathbf{B}_{*,\square})$ can be promoted to an object of $\mathcal{F}\mathcal{D}(\mathbf{B}_{*,\square})$ by setting

$$F^r \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_*) := \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, F^r \mathbb{B}_*) \in \mathcal{D}(\mathbf{B}_{*,\square}^+).$$

We quote the following de Rham comparison theorem:

Theorem 3.6. (Bosco [7, Th. 1.8, Rem. 6.14]) *Let $X \in \mathrm{Sm}_K$.*

(1) *We have a natural quasi-isomorphism in $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$*

$$(3.7) \quad F^r \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_{\mathrm{dR}}) \simeq F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}), \quad r \in \mathbf{Z}.$$

(2) *If X is quasi-compact we also have a natural quasi-isomorphism in $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square})$*

$$(3.8) \quad \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_{\mathrm{dR}}) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}).$$

Remark 3.9. We warn the reader that the first claim in the above theorem yields a natural quasi-isomorphism in $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$

$$(3.10) \quad \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_{\mathrm{dR}}^+) \simeq F^0(\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_K^{\mathrm{L}\square} \mathbf{B}_{\mathrm{dR}}),$$

That is, the filtration on the right is twisted! For an untwisted version (see [8, Th. 5.1]) we need to change the group $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_{\mathrm{dR}}^+)$ by using the décalage operator: setting¹⁴

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}^\eta(X_C, \mathbb{B}_{\mathrm{dR}}^+) := \mathrm{R}\Gamma_{\acute{e}t}(X_C, \mathrm{L}\eta_t \mathrm{R}\nu_* \mathbb{B}_{\mathrm{dR}}^+) \in \mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+),$$

there is a natural quasi-isomorphism in $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}^\eta(X_C, \mathbb{B}_{\mathrm{dR}}^+) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_K^{\mathrm{L}\square} \mathbf{B}_{\mathrm{dR}}^+.$$

¹⁴Here $\nu : X_{C,\mathrm{pro\acute{e}t}} \rightarrow X_{C,\acute{e}t}$ is the canonical projection of sites.

We will need the following series of facts later on.

Lemma 3.11. *Let G be a profinite group acting (continuously) on $X \in \text{Sm}_K$. There are natural quasi-isomorphisms in $\mathcal{D}(\mathbf{B}_{\text{dR}, \square}^+)$ and $\mathcal{D}(\mathbf{B}_{\text{dR}, \square})$*

$$\begin{aligned} \text{R}\Gamma(G, \text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{dR}})) &\simeq \text{R}\Gamma(G, \text{R}\Gamma_{\text{dR}}(X)) \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}}, \\ \text{R}\Gamma(G, F^r \text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{dR}})) &\simeq \text{colim}_{j+k \geq r} \text{R}\Gamma(G, F^j \text{R}\Gamma_{\text{dR}}(X)) \otimes_K^{\text{L}\square} F^k \mathbf{B}_{\text{dR}}. \end{aligned}$$

Proof. The first quasi-isomorphism follows directly from Proposition 2.3, since $\mathbf{B}_{\text{dR}} \in \text{Nuc}_K$, as a filtered colimit of K -Fréchet spaces. For the second one, we use that $\text{R}\Gamma(G, -)$ commutes with filtered colimits and that $F^r \mathbf{B}_{\text{dR}} \in \text{Fr}_K \subset \text{Nuc}_K$, so by Proposition 2.3 we can write

$$\begin{aligned} \text{R}\Gamma(G, F^r \text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{dR}})) &\simeq \text{R}\Gamma(G, \text{colim}_{j+k \geq r} F^j \text{R}\Gamma_{\text{dR}}(X) \otimes_K^{\text{L}\square} F^k \mathbf{B}_{\text{dR}}) \\ &\simeq \text{colim}_{j+k \geq r} \text{R}\Gamma(G, F^j \text{R}\Gamma_{\text{dR}}(X) \otimes_K^{\text{L}\square} F^k \mathbf{B}_{\text{dR}}) \\ &\simeq \text{colim}_{j+k \geq r} \text{R}\Gamma(G, F^j \text{R}\Gamma_{\text{dR}}(X)) \otimes_K^{\text{L}\square} F^k \mathbf{B}_{\text{dR}}, \end{aligned}$$

as wanted. \square

Corollary 3.12. *If $X \in \text{Sm}_K$ is quasi-compact, there are natural quasi-isomorphisms in $\mathcal{D}(\mathbf{B}_{\text{pdR}, \square}^+)$ and $\mathcal{D}(\mathbf{B}_{\text{pdR}, \square})$, respectively:*

$$(3.13) \quad F^r \text{R}\Gamma_{\text{proét}}(X_C, \mathbb{B}_{\text{pdR}}) \simeq F^r \text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{pdR}}), \quad r \in \mathbf{Z}.$$

$$(3.14) \quad \text{R}\Gamma_{\text{proét}}(X_C, \mathbb{B}_{\text{pdR}}) \simeq \text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{pdR}}).$$

Proof. Since X is quasi-compact, $\text{R}\Gamma_{\text{proét}}(X_C, -)$ commutes with filtered colimits, we have a natural quasi-isomorphism

$$\text{R}\Gamma_{\text{proét}}(X_C, \mathbb{B}_{\text{pdR}}) \simeq \text{R}\Gamma_{\text{proét}}(X_C, \mathbb{B}_{\text{dR}})[\log t].$$

Since $\text{R}\Gamma_{\text{dR}}(X) \otimes_K^{\text{L}\square} (-)$ commutes with colimits, we also have a natural quasi-isomorphism

$$\text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{pdR}}) \simeq \text{R}\Gamma_{\text{dR}}(X) \otimes_K^{\text{L}\square} \mathbf{B}_{\text{pdR}} \simeq (\text{R}\Gamma_{\text{dR}}(X) \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}})[\log t].$$

Hence the result is a direct consequence of Theorem 3.6. \square

Lemma 3.15. *For any $X \in \text{Sm}_K$ there are natural quasi-isomorphisms in $\mathcal{D}(K_{\square})$ ($r \in \mathbf{Z}$)*

$$\begin{aligned} \text{R}\Gamma_{\text{dR}}(X) &\simeq \text{R}\Gamma(\mathcal{G}_K, \text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{pdR}})), \\ F^r \text{R}\Gamma_{\text{dR}}(X) &\simeq \text{R}\Gamma(\mathcal{G}_K, F^r \text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{pdR}})). \end{aligned}$$

Proof. For simplicity write $A = \mathbf{B}_{\text{pdR}}$ and $M = \text{R}\Gamma_{\text{dR}}(X)$. Note that M and $F^k M$ belong to Nuc_K , as filtered colimits of K -Fréchet spaces. Thus by Proposition 2.3 (and the fact that $\text{R}\Gamma(\mathcal{G}_K, -)$ commutes with filtered colimits) we obtain natural quasi-isomorphisms

$$\begin{aligned} \text{R}\Gamma(\mathcal{G}_K, \text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{pdR}})) &\simeq \text{R}\Gamma(\mathcal{G}_K, M \otimes_K^{\text{L}\square} A) \simeq \text{R}\Gamma(\mathcal{G}_K, A) \otimes_K^{\text{L}\square} M, \\ \text{R}\Gamma(\mathcal{G}_K, F^r \text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{pdR}})) &\simeq \text{R}\Gamma(\mathcal{G}_K, \text{colim}_{j+k \geq r} F^j M \otimes_K^{\text{L}\square} F^k A) \\ &\simeq \text{colim}_{j+k \geq r} \text{R}\Gamma(\mathcal{G}_K, F^j M \otimes_K^{\text{L}\square} F^k A) \simeq \text{colim}_{j+k \geq r} \text{R}\Gamma(\mathcal{G}_K, F^k A) \otimes_K^{\text{L}\square} F^j M. \end{aligned}$$

The result follows then by the computation of $\text{R}\Gamma(\mathcal{G}_K, A)$ and $\text{R}\Gamma(\mathcal{G}_K, F^k A)$ in Proposition 2.7. \square

Corollary 3.16. *If $X \in \text{Sm}_K$ is quasi-compact and endowed with a continuous action of a profinite group G , there are natural quasi-isomorphism in $\mathcal{D}(K_{\square})$ ($r \in \mathbf{Z}$)*

$$\begin{aligned} \text{R}\Gamma(G, F^r \text{R}\Gamma_{\text{dR}}(X)) &\simeq \text{R}\Gamma(G \times \mathcal{G}_K, F^r \text{R}\Gamma_{\text{proét}}(X_C, \mathbb{B}_{\text{pdR}})), \\ \text{R}\Gamma(G, \text{R}\Gamma_{\text{dR}}(X)) &\simeq \text{R}\Gamma(G \times \mathcal{G}_K, \text{R}\Gamma_{\text{proét}}(X_C, \mathbb{B}_{\text{pdR}})) \end{aligned}$$

Proof. It suffices to write $\mathrm{R}\Gamma(G \times \mathcal{G}_K, -) = \mathrm{R}\Gamma(G, \mathrm{R}\Gamma(\mathcal{G}_K, -))$ and use Corollary 3.12, Lemma 3.15 and Lemma 3.11. \square

3.5. De Rham cohomology with compact support. Let $L = K, C$. For any functor \mathcal{F} defined on Sm_L , with values in some ∞ -category with small colimits, and any $X \in \mathrm{Sm}_L$ we define

$$F(\partial X) = \mathrm{colim}_{Z \in \Phi(X)} F(X \setminus Z),$$

where $\Phi(X)$ is the set of quasi-compact open subsets of X . In particular, for $X \in \mathrm{Sm}_K$, we obtain objects $\mathrm{R}\Gamma_{\mathrm{dR}}(\partial X) \in \mathcal{F}\mathcal{D}(K_{\square})$, $\mathrm{R}\Gamma(\partial X_C/\mathbf{B}_{\mathrm{dR}}) \in \mathcal{F}\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square})$, as well as $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\partial X_C, \mathcal{F}) \in \mathcal{D}(\mathrm{CondAb})$ for any pro-étale abelian sheaf \mathcal{F} on X_C .

Using these constructions we can define various compactly supported cohomology groups (the brackets $[\dots]$ denote the mapping fiber)

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{dR},c}(X) &:= [\mathrm{R}\Gamma_{\mathrm{dR}}(X) \rightarrow \mathrm{R}\Gamma_{\mathrm{dR}}(\partial X)] \in \mathcal{F}\mathcal{D}(K_{\square}), \\ F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X) &:= [F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X) \rightarrow F^r \mathrm{R}\Gamma_{\mathrm{dR}}(\partial X)], \quad r \in \mathbf{Z}. \end{aligned}$$

The same construction applied to the functor $X \mapsto \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}})$ yields $\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}) \in \mathcal{F}\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square})$ and, for any abelian pro-étale sheaf \mathcal{F} on $X_{C,\mathrm{pro\acute{e}t}}$, an object $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(X_C, \mathcal{F}) \in \mathcal{D}(\mathrm{CondAb})$. In particular, we get $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(X_C, \mathbb{B}_{\mathrm{dR}}) \in \mathcal{F}\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square})$ by setting

$$F^r \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(X_C, \mathbb{B}_{\mathrm{dR}}) := \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(X_C, F^r \mathbb{B}_{\mathrm{dR}}) \simeq [F^r \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_{\mathrm{dR}}) \rightarrow F^r \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\partial X_C, \mathbb{B}_{\mathrm{dR}})].$$

Remark 3.17. Let X be a smooth partially proper rigid analytic variety over K and let $\{X_n\}_{n \in \mathbf{N}}$, be an increasing covering of X by quasi-compact opens. We can write:

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\partial X_C/\mathbf{B}_{\mathrm{dR}}) \simeq \mathrm{colim}_n \mathrm{R}\Gamma_{\mathrm{dR}}((X \setminus X_n)_C/\mathbf{B}_{\mathrm{dR}})$$

and similarly with other cohomology theories.

Example 3.18. If X is Stein, since $H_c^i(X, \Omega_X^j) = 0$, for $i \neq d$, we compute in $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square})$:

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}) &\simeq (H_c^d(X, \mathcal{O}_X) \otimes_K^{\mathrm{L}\square} \mathbf{B}_{\mathrm{dR}} \rightarrow H_c^d(X, \Omega_X^1) \otimes_K^{\mathrm{L}\square} \mathbf{B}_{\mathrm{dR}} \rightarrow \dots)[-d] \\ &\simeq (H_c^d(X, \mathcal{O}_X) \otimes_K^{\square} \mathbf{B}_{\mathrm{dR}} \rightarrow H_c^d(X, \Omega_X^1) \otimes_K^{\square} \mathbf{B}_{\mathrm{dR}} \rightarrow \dots)[-d]. \end{aligned}$$

Here we were able to remove the derived tensor product because \mathbf{B}_{dR} is flat (as filtered colimit of K -Fréchet spaces). For similar reasons we have in $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$, for $r \in \mathbf{Z}$:

$$(3.19) \quad \begin{aligned} F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}) &\simeq (H_c^d(X, \mathcal{O}_X) \otimes_K^{\mathrm{L}\square} F^r \mathbf{B}_{\mathrm{dR}} \rightarrow H_c^d(X, \Omega_X^1) \otimes_K^{\mathrm{L}\square} F^{r-1} \mathbf{B}_{\mathrm{dR}} \rightarrow \dots)[-d] \\ &\simeq (H_c^d(X, \mathcal{O}_X) \otimes_K^{\square} t^r \mathbf{B}_{\mathrm{dR}}^+ \rightarrow H_c^d(X, \Omega_X^1) \otimes_K^{\square} t^{r-1} \mathbf{B}_{\mathrm{dR}}^+ \rightarrow \dots)[-d], \end{aligned}$$

and a natural quasi-isomorphism in $\mathcal{D}(K_{\square})$

$$(3.20) \quad F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X) \simeq (H_c^d(X, \Omega_X^r) \rightarrow H_c^d(X, \Omega_X^{r+1}) \rightarrow \dots)[-d-r].$$

Proposition 3.21. *If $X \in \mathrm{Sm}_K$ is partially proper, then $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(X_C, -)$ commutes with filtered colimits of pro-étale abelian sheaves.*

Proof. Let $(\mathcal{F}_i)_{i \in I}$ be a filtered direct system of pro-étale sheaves and let $\mathcal{F} = \mathrm{colim}_{i \in I} \mathcal{F}_i$. Set

$$P(X) = [\mathrm{colim}_i \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathcal{F}_i) \rightarrow \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathcal{F})].$$

It is not difficult to see that the quasi-isomorphism we want to establish is equivalent to proving that the natural map $P(X) \rightarrow P(\partial X) := \mathrm{colim}_{Z \in \Phi(X_C)} P(X_C \setminus Z)$ is a quasi-isomorphism. This follows using the same arguments as in [1, Lemma 3.21]. \square

Proposition 3.22. *If $X \in \text{Sm}_K$ is partially proper, then there are natural quasi-isomorphisms in $\mathcal{D}(\mathbf{B}_{\text{dR}, \square})$ and $\mathcal{D}(\mathbf{B}_{\text{dR}, \square}^+)$, respectively:*

$$\begin{aligned} \text{R}\Gamma_{\text{proét}, c}(X_C, \mathbb{B}_{\text{dR}}) &\simeq \text{R}\Gamma_{\text{dR}, c}(X) \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}}, \\ F^r \text{R}\Gamma_{\text{proét}, c}(X_C, \mathbb{B}_{\text{dR}}) &\simeq F^r(\text{R}\Gamma_{\text{dR}, c}(X) \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}}), \quad r \in \mathbf{Z}. \end{aligned}$$

Proof. To simplify the notation, set

$$M = \text{R}\Gamma_{\text{dR}}(X), \quad \partial M = \text{R}\Gamma_{\text{dR}}(\partial X), \quad M_c = \text{R}\Gamma_{\text{dR}, c}(X) = [M \rightarrow \partial M].$$

Note that $F^r((-) \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}}) = \text{colim}_{j+k \geq r} F^j((-) \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}})$ commutes with mapping fibers, so using Theorem 3.6 we obtain natural quasi-isomorphisms

$$\begin{aligned} F^r \text{R}\Gamma_{\text{proét}, c}(X_C, \mathbb{B}_{\text{dR}}) &\simeq [F^r \text{R}\Gamma_{\text{proét}}(X_C, \mathbb{B}_{\text{dR}}) \rightarrow F^r \text{R}\Gamma_{\text{proét}}(\partial X_C, \mathbb{B}_{\text{dR}})] \\ &\simeq [F^r(M \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}}) \rightarrow F^r(\partial M \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}})] \simeq F^r(M_c \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}}). \end{aligned}$$

This shows the second quasi-isomorphism of the proposition.

Concerning the first quasi-isomorphism, for $T \in \mathcal{D}(\mathbf{B}_{\text{dR}, \square}^+)$, define

$$T[1/t] := T \otimes_{\mathbf{B}_{\text{dR}}^+}^{\text{L}\square} \mathbf{B}_{\text{dR}} \in \mathcal{D}(\mathbf{B}_{\text{dR}, \square}).$$

Proposition 3.21 yields a natural quasi-isomorphism

$$\text{R}\Gamma_{\text{proét}, c}(X_C, \mathbb{B}_{\text{dR}}^+)[1/t] \xrightarrow{\sim} \text{R}\Gamma_{\text{proét}, c}(X_C, \mathbb{B}_{\text{dR}}).$$

By what we have already established we have natural quasi-isomorphisms

$$\text{R}\Gamma_{\text{proét}, c}(X_C, \mathbb{B}_{\text{dR}}^+) \simeq F^0 \text{R}\Gamma_{\text{proét}, c}(X_C, \mathbb{B}_{\text{dR}}) \simeq F^0(M_c \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}}).$$

Thus it remains to see that we have a canonical quasi-isomorphism

$$F^0(M_c \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}})[1/t] \xrightarrow{\sim} M_c \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}}.$$

Since we can pass to colimits on both sides we may assume that X is Stein. But, since by (3.19), we have

$$\begin{aligned} F^0((-) \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}})[1/t] &\simeq (H_c^d(X, \mathcal{O}_X) \otimes_K^{\square} \mathbf{B}_{\text{dR}}^+ \rightarrow H_c^d(X, \Omega_X^1) \otimes_K^{\square} t^{-1} \mathbf{B}_{\text{dR}}^+ \rightarrow \dots \rightarrow H_c^d(X, \Omega_X^d) \otimes_K^{\square} t^{-d} \mathbf{B}_{\text{dR}}^+)[-d][1/t] \\ &\xrightarrow{\sim} (H_c^d(X, \mathcal{O}_X) \otimes_K^{\square} \mathbf{B}_{\text{dR}} \rightarrow H_c^d(X, \Omega_X^1) \otimes_K^{\square} \mathbf{B}_{\text{dR}} \rightarrow \dots \rightarrow H_c^d(X, \Omega_X^d) \otimes_K^{\square} \mathbf{B}_{\text{dR}})[-d] \\ &\simeq M_c \otimes_K^{\text{L}\square} \mathbf{B}_{\text{dR}}, \end{aligned}$$

as wanted. \square

3.6. Hyodo-Kato comparison theorem. Assume that k is algebraically closed¹⁵. Let $\mathcal{D}_{\varphi, N}(\mathbf{B}_{\square})$ be the ∞ -derived category of solid (φ, N) -modules over \mathbf{B} ; similarly for various variants.

We quote the following Hyodo-Kato comparison theorem:

Proposition 3.23. *Let $X \in \text{Sm}_K$. Then we have a natural, \mathcal{G}_K -equivariant quasi-isomorphism in $\mathcal{D}_{\varphi}(\mathbf{B}_{\square})$*

$$(3.24) \quad \text{R}\Gamma_{\text{proét}}(X_C, \mathbb{B}^+) \otimes_{\mathbf{B}^+}^{\text{L}\square} \mathbf{B} \simeq [\text{R}\Gamma_{\text{HK}}(X_C) \otimes_{\mathcal{C}}^{\text{L}\square} \mathbf{B}_{\text{FF}}]^{N=0}.$$

Proof. Let $r \geq 2d$. By [12, Cor. 4.9], we have a natural \mathcal{G}_K -equivariant comparison quasi-isomorphism in $\mathcal{D}_{\varphi}(\mathbf{B}_{\square})$ (see [12, Cor. 4.9] for an explanation of the notation)

$$(3.25) \quad \text{R}\Gamma_{\text{proét}}(X_C, \mathbb{B}^+) \simeq [[\text{R}\Gamma_{\text{HK}}(X_C)\{r\} \otimes_{\mathcal{C}}^{\text{L}\square} \mathbf{B}_{\text{FF}}^+]^{N=0} \xrightarrow{t_{\text{HK}}^{\mathbf{B}}} \text{R}\Gamma_{\text{dR}}^{\mathbf{B}^+}(X_C, r)](-r).$$

The de Rham-type complex $\text{R}\Gamma_{\text{dR}}^{\mathbf{B}^+}(X_C, r)$ is annihilated by t^{r-d} . The wanted quasi-isomorphism (3.24) follows. \square

¹⁵This assumption could probably be removed if one is more careful.

Remark 3.26. (1) Bosco in [8, Th. 4.1, Rem. 6.13] proved a version of the comparison quasi-isomorphism (3.25), where the torsion on the right-hand side is incorporated to the left-hand side via the $L\eta_t$ operator. This also yields the quasi-isomorphism (3.24).

(2) Let $I \subset (0, \infty)$ be a compact interval with rational endpoints such that t has one zero on $\mathrm{Spa}(\mathbf{B}_I)$. Then, still for $r \geq 2d$, we have the following analog of (3.25)

$$(3.27) \quad \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_I) \simeq [[\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{I, \log}]^{N=0} \xrightarrow{t_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X_C, r)](-r).$$

Tensoring both sides of (3.27) with $\mathbf{B}_I/t^n \simeq \mathbf{B}_{\mathrm{dR}}^+/t^n$, $n \in \mathbf{N}$, passing to limit over n , and dropping the twist by r since we do not care about the Frobenius anymore, we get a quasi-isomorphism in $\mathcal{D}(\mathbf{B}_{\mathrm{dR}, \square}^+)$

$$(3.28) \quad \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_{\mathrm{dR}}^+) \simeq F^0(\mathrm{R}\Gamma_{\mathrm{dR}}(X_K) \otimes_K^{\mathrm{L}\square} \mathbf{B}_{\mathrm{dR}}),$$

which recovers (3.7). This proof of (3.28) is however more involved than the one of Bosco in [7] that follows directly from the \mathbb{B}_{dR} -Poincaré Lemma.

Corollary 3.29. *Let $X \in \mathrm{Sm}_K$. Assume that it has finite rank de Rham cohomology. Then there are natural \mathcal{G}_K -equivariant quasi-isomorphisms in $\mathcal{D}_{\varphi, N}(\check{C}_{\square})$*

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{HK}}(X_C) &\xrightarrow{\sim} ([\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}})^{\mathrm{R}\mathcal{G}_K - \mathrm{sm}}, \\ \mathrm{R}\Gamma_{\mathrm{HK}}(X_C) &\xrightarrow{\sim} ([\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B})[1/t]]^{\varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}})^{\mathrm{R}\mathcal{G}_K - \mathrm{sm}}. \end{aligned}$$

Proof. For the first quasi-isomorphism, we can represent $\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)$ by a bounded complex of finite rank $(\varphi, N, \mathcal{G}_K)$ -modules over \check{C} . Then it suffices to evoke Corollary 2.13. The second quasi-isomorphism follows from the first one and Proposition 3.23. \square

4. GROUP ACTIONS AND COMPARISON THEOREMS

In this chapter we study the behaviour of de Rham and Hyodo-Kato cohomologies and comparison theorems in the presence of locally profinite groups acting on analytic varieties.

4.1. Smoothness of group actions on De Rham and Hyodo-Kato cohomology. We start by showing that the group action on compactly supported de Rham and Hyodo-Kato cohomologies tends to be smooth.

4.1.1. Smooth representations. Before we proceed, let us make a small digression on smooth representations as defined in [30, Sec. 5.1].

Let G be a locally profinite group. Recall that a representation of G on a solid K -vector space V is a map of condensed sets $G \times V \rightarrow V$ having the usual properties. The Iwasawa algebra of G with coefficients in K is defined as

$$K_{\square}[G] := (\lim_{H \subset G} \mathcal{O}_K[G/H])[1/p],$$

where H runs over all open normal subgroups of G . The category of G -representations on solid K -vector spaces is equivalent to the category $\mathrm{Mod}_{K_{\square}}^{\mathrm{solid}}[G]$ of solid $K_{\square}[G]$ -modules. We will write $\mathcal{D}(K_{\square}[G])$ for its ∞ -derived category. Similarly, for a solid K_{\square} -algebra A we write $\mathcal{D}(A_{\square}[G])$ for the ∞ -derived category of solid $A \otimes_K^{\square} K_{\square}[G]$ -modules.

Recall that [30, Prop. 2.2.5], if G is compact, the algebra of smooth K -valued distributions of G can be described as

$$\mathcal{D}^{\mathrm{sm}}(G, K) \simeq \lim_{H \subset G} K_{\square}[G/H].$$

In general, for any compact open subgroup $H \subset G$, we have

$$\mathcal{D}^{\mathrm{sm}}(G, K) \simeq \mathcal{D}^{\mathrm{sm}}(H, K) \otimes_{K_{\square}[H]} K_{\square}[G].$$

We quote the following:

Definition 4.1. ([30, Def. 5.1.1])

(1) Let $V \in \text{Mod}_{K_\square}^{\heartsuit}(\mathcal{D}^{\text{sm}}(G, K))$. The smooth vectors of V are defined by

$$V^{\text{sm}} := \text{colim}_{H \subset G} V^H = \text{colim}_{H \subset G} \underline{\text{Hom}}_{\mathcal{D}^{\text{sm}}(G, K)}(K_\square[G/H], V),$$

where H runs over all compact open subgroups of G . We say that V is a *smooth representation* of G if the natural map $V^{\text{sm}} \rightarrow V$ is an isomorphism.

(2) We let $(-)^{\text{Rsm}} : \text{Mod}_{K_\square}(\mathcal{D}^{\text{sm}}(G, K)) \rightarrow \text{Mod}_{K_\square}(\mathcal{D}^{\text{sm}}(G, K))$ be the functor of derived smooth vectors

$$V^{\text{Rsm}} := \text{colim}_{H \subset G} V^{\text{RH}} = \text{colim}_{H \subset G} \underline{\text{RHom}}_{\mathcal{D}^{\text{sm}}(G, K)}(K_\square[G/H], V).$$

We say that an object in $\text{Mod}_{K_\square}(\mathcal{D}^{\text{sm}}(G, K))$ is *smooth* if the natural arrow $V^{\text{Rsm}} \rightarrow V$ is an equivalence. We let $\text{Rep}_{K_\square}^{\text{sm}}(G) \subset \text{Mod}_{K_\square}(\mathcal{D}^{\text{sm}}(G, K))$ be the full subcategory consisting of smooth objects.

We will often use the following result:

Proposition 4.2. ([30, Prop. 5.1.7]) *An object $V \in \text{Mod}_{K_\square}(\mathcal{D}^{\text{sm}}(G, K))$ is smooth if and only if $H^i(V)$ is smooth, for all $i \in \mathbf{Z}$. Hence, the natural t -structure of $\text{Mod}_{K_\square}(\mathcal{D}^{\text{sm}}(G, K))$ induces a t -structure on $\text{Rep}_{K_\square}^{\text{sm}}(G)$. Moreover, $\text{Rep}_{K_\square}^{\text{sm}, \heartsuit}(G)$ is a Grothendieck abelian category and $\text{Rep}_{K_\square}^{\text{sm}}(G)$ is the derived category of its heart.*

4.1.2. *Smoothness of group actions on De Rham and Hyodo-Kato cohomology.* We pass now to the group action on de Rham and Hyodo-Kato cohomologies.

Proposition 4.3. *For a locally profinite group G acting (continuously) on a smooth dagger analytic space Y over K , we have*

(1) *The G -action on $\text{R}\Gamma_{\text{dR}, c}(Y)$ extends to a canonical structure of solid $\mathcal{D}^{\text{sm}}(G, K)$ -module, i.e.*

$$\text{R}\Gamma_{\text{dR}, c}(Y) \in \text{Mod}_{K_\square}(\mathcal{D}^{\text{sm}}(G, K)).$$

(2) *The G -representation $H_{\text{dR}, c}^i(Y)$, $i \geq 0$, is smooth. Hence $\text{R}\Gamma_{\text{dR}, c}(Y) \in \text{Rep}_{K_\square}^{\text{sm}}(G)$.*

Similarly for Hyodo-Kato cohomology $\text{R}\Gamma_{\text{HK}, c}(Y_C)$.

Proof. Let us start with the first claim. We may assume that G is profinite. Assume first that Y is a dagger affinoid. Since $\text{R}\Gamma_{\text{dR}, c}(Y)$ is (functorially) dual to $\text{R}\Gamma_{\text{dR}}(Y)$ (see [1, Th. 5.29]), our claim follows from Lemma 4.4 below. In the general case, by writing $Y = \cup_{i \in I} Y_i$ for an increasing sequence of G -stable dagger quasi-compact opens $Y_i \subset Y$ we may assume that Y is quasi-compact. Using (co-)Meyer-Vietoris for a (finite) dagger affinoid covering of each Y_i and passing to an open subgroup that stabilizes every affinoid in the covering, we reduce to the case of a dagger affinoid treated above. By allowing all coverings as above this can be done functorially yielding the first claim of the proposition.

The first part of the second claim follows from the first claim. The second part of that claim follows now from Proposition 4.2. \square

The following result in the case of dagger affinoids is a ∞ -categorical version of a result of van der Put (see [38, Prop. 1.9]).

Lemma 4.4. *Let G be a locally profinite group. Let Y be a rigid analytic or a dagger affinoid over K with a continuous action of G . Then $\text{R}\Gamma_{\text{dR}}(Y)$ has a canonical (functorial in Y and G) structure of solid $\mathcal{D}^{\text{sm}}(G, K)$ -module (i.e. object of $\text{Mod}_{K_\square}(\mathcal{D}^{\text{sm}}(G, K))$) and similarly for $\text{R}\Gamma_{\text{HK}}(Y_C)$.*

Proof. Let us look first at de Rham cohomology. In the rigid analytic case, the claim follows from the fact that the de Rham cohomology "depends only on the reduction mod p of Y ". To make this precise (see [14, Sec. 4.2.2] for the notation and the details), take the natural quasi-isomorphism in $\mathcal{D}(K_\square[G])$

$$\text{R}\Gamma_{\text{dR}}(Y) \simeq \text{R}\Gamma_{\text{conv}}(Y)$$

between de Rham cohomology and convergent cohomology. Here

$$\mathrm{R}\Gamma_{\mathrm{conv}}(Y) := \mathrm{colim}_{\mathcal{Y}_\bullet \rightarrow Y} \mathrm{R}\Gamma_{\mathrm{conv}}(\mathcal{Y}_{\bullet,1})$$

and the colimit is over η -étale hypercoverings of Y from $\mathcal{M}_K^{\mathrm{ss}}$ (we may assume that in every degree we have a quasi-compact formal scheme). Hence the connected components of \mathcal{Y}_n are semistable formal schemes over \mathcal{O}_L for a finite extension L of K . As the notation suggest, the convergent cohomology depends only on the reduction mod p of \mathcal{Y}_n . It follows that there exists an open subgroup $G_n \subset G$ that acts trivially on $\mathrm{R}\Gamma_{\mathrm{conv}}(\mathcal{Y}_{n,1})$. Since the de Rham cohomology of Y is of bounded degree, we may assume that our simplicial scheme \mathcal{Y}_\bullet is truncated and then there is an open subgroup $G_{\mathcal{Y}_\bullet} \subset G$ that acts trivially on $\mathrm{R}\Gamma_{\mathrm{conv}}(\mathcal{Y}_{\bullet,1})$. We used here that we can choose strictly (!) functorial complexes representing convergent cohomology. It follows that $\mathrm{R}\Gamma_{\mathrm{conv}}(\mathcal{Y}_{\bullet,1}) \in \mathcal{D}^{\mathrm{sm}}(G_{\mathcal{Y}_\bullet}, K)$, functorially in the hypercoverings and the actions of the groups. This yields that $\mathrm{R}\Gamma_{\mathrm{dR}}(Y) \in \mathrm{Mod}_{K_\square}(\mathcal{D}^{\mathrm{sm}}(G, K))$, as wanted.

The dagger case follows from the rigid analytic case by taking a dagger presentation $\{Y_h\}$ of Y and writing $\mathrm{R}\Gamma_{\mathrm{dR}}(Y) \simeq \mathrm{colim}_h \mathrm{R}\Gamma_{\mathrm{dR}}(Y_h)$.

For Hyodo-Kato cohomology we argue similarly using the fact that the Hyodo-Kato cohomology "depends only on the reduction mod p of Y_C ". Namely, by definition, we have in $\mathcal{D}(\check{C}_\square[G])$

$$\mathrm{R}\Gamma_{\mathrm{HK}}(Y_C) := \mathrm{colim}_{\mathcal{Y}_\bullet \rightarrow Y_C} \mathrm{R}\Gamma_{\mathrm{HK}}(\mathcal{Y}_{\bullet,1}).$$

See [14, Sec. 4.3] for details and notation. \square

4.2. Equivariant properties of comparison theorems. We show here that comparison morphisms behave well with respect to group actions on the underlying variety. We already know that they are functorial hence equivariant for the action of the discrete groups; what we need to show is that the comparison maps are actually in the relevant topological categories.

4.2.1. Group action on cohomology of period sheaves. Let X be an analytic adic space over C . Assume that it is equipped with a continuous action of a profinite group G . We have the following useful corollary of Lemma 3.2:

Lemma 4.5. (1) Let \mathcal{F} be as in Lemma 3.2. Then $\mathrm{R}\Gamma_{\mathrm{proét}}(X, \mathcal{F})$ has a natural structure of an object from $\mathcal{D}(\mathbf{Q}_{p,\square}[G])$.

(2) Let $X \in \mathrm{Sm}_K$ and let $\mathcal{F} \in \{F^j \mathcal{O}_{\mathrm{dR}} \otimes_{\mathcal{O}_X} \Omega_X^i, \mathcal{O}_{\mathrm{dR}} \otimes_{\mathcal{O}_X} \Omega_X^i\}$, $i \geq 0, j \in \mathbf{Z}$. Then $\mathrm{R}\Gamma_{\mathrm{proét}}(X_C, \mathcal{F})$ has a natural structure of an object from $\mathcal{D}(\mathbf{Q}_{p,\square}[G])$.

Proof. For claim (1), from Lemma 3.2, we have the quasi-isomorphism:

$$(4.6) \quad \mathrm{R}\Gamma_{\mathrm{proét}}(X \times G, \mathcal{F}) \simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_{p,\square}}(\mathbf{Q}_{p,\square}[G], \mathrm{R}\Gamma_{\mathrm{proét}}(X, \mathcal{F})).$$

Hence the action map $a : X \times G \rightarrow X$ together with (4.6) yields the action map

$$a^* : \mathbf{Q}_{p,\square}[G] \otimes_{\mathbf{Q}_p}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{proét}}(X, \mathcal{F}) \rightarrow \mathrm{R}\Gamma_{\mathrm{proét}}(X, \mathcal{F}),$$

which satisfies all the higher coherences by the same argument.

For claim (2), we can argue as in claim (1) provided that we know that Lemma 3.2 holds for \mathcal{F} . But this can be argued exactly as in the case of $\mathbb{B}_{\mathrm{dR}}^+, \mathbb{B}_{\mathrm{dR}}$. \square

4.2.2. Equivariant properties of comparison theorems.

Proposition 4.7. Let $X \in \mathrm{Sm}_K$ and let G be a profinite group acting continuously on X . Then:

(1) The quasi-isomorphisms (3.7), (3.8) and (3.14), (3.13) are G -equivariant, i.e., they are in $\mathcal{D}(B_\square[G])$, for $B = \mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{pdR}}^+, \mathbf{B}_{\mathrm{pdR}}$, respectively.

(2) The quasi-isomorphism (3.24) is G -equivariant, i.e., it is in $\mathcal{D}(\mathbf{B}_\square[G])$.

Proof. For the first claim, it suffices to argue for the quasi-isomorphisms (3.7), (3.8). We treat the case of $F^r \mathbb{B}_{\mathrm{dR}}$; the other period sheaves could be treated similarly.

Recall that we have an exact sequence of sheaves of periods on $X_{C, \mathrm{pro\acute{e}t}}$:

$$(4.8) \quad 0 \rightarrow F^r \mathbb{B}_{\mathrm{dR}} \rightarrow F^r \mathcal{O}_{\mathrm{dR}} \xrightarrow{\nabla} F^{r-1} \mathcal{O}_{\mathrm{dR}}^+ \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{\nabla} \dots$$

The quasi-isomorphism (3.7) for $F^r \mathbb{B}_{\mathrm{dR}}$ is induced by the map

$$F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}) \rightarrow \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, F^r \mathbb{B}_{\mathrm{dR}}),$$

which in turn, is induced by the maps

$$\nu^* : \mathrm{R}\Gamma_{\acute{e}t}(X_C, F^{r-i} \mathbf{B}_{\mathrm{dR}} \otimes_K \Omega_X^i) \rightarrow \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, F^{r-i} \mathcal{O}_{\mathrm{dR}} \otimes_{\mathcal{O}_X} \Omega_X^i),$$

where $\nu : X_{C, \mathrm{pro\acute{e}t}} \rightarrow X_{C, \acute{e}t}$ is the canonical projection. To check that the map ν^* is $\mathbf{B}_{\mathrm{dR}, \square}^+[G]$ -equivariant it is enough to check that it is $\mathbf{Z}_p[G]$ -equivariant but this is clear.

For the second claim, we need to work harder. Let $r \geq 2d$. The proof of the comparison quasi-isomorphism has two steps (see the proof of [15, Th. 7.3, Prop. 7.35] for details).

Step 1 (*Pro-étale cohomology of \mathbb{B}^+ vs Fontaine-Messing syntomic cohomology*) That passage is basically clear so we will just briefly sketch the argument (see the proof of [15, Th. 7.3] for details). Via a limit-argument we may replace \mathbb{B}^+ with \mathbb{B}_I , for a compact interval $I \subset (0, \infty)$ with rational end points (assuming functoriality with respect to I). We claim that the period quasi-isomorphism (for $r \geq 0$)

$$\alpha_{r, I} : \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_I(r)) \simeq \mathrm{R}\Gamma'_{\mathrm{syn}}(X_C, \mathbf{Q}_p(r)),$$

where

$$\mathrm{R}\Gamma'_{\mathrm{syn}}(X_C, \mathbf{Q}_p(r)) := [\mathrm{R}\Gamma_{\mathrm{cr}}(X_C)\{r\} \rightarrow \bigoplus_{\Psi(I)} \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}^+)/F^r],$$

with the index set $\Psi(I)$ running over zeros of t in $\mathrm{Spa}(\mathbf{B}_I)$, is the Frobenius untwisted syntomic cohomology. This quasi-isomorphism is actually defined modulo p^n as a functorial zigzag (see the definition [15, (7.34)]), where all the maps are quasi-isomorphisms modulo a universal constant.

We work modulo p^n now. Both domain and target of the period map $\alpha_{r, I}$ are functorial $\mathbf{Z}_{p, \square}[G]$ -modules: see [17, Prop. 3.1.1] for an argument for the domain. For the target, it suffices to show this locally and functorially, for an étale G -equivariant map $U \rightarrow X$ with a coordinate system (also G -equivariant). Since we work modulo p^n , the morphism $\alpha_{r, I}$ depends only on moduli p^n reduction of this system. But now, the action of G on this moduli p^n reduction factors through G/H_U for an open subgroup H_U . Hence, for all practical purposes we might assume that G is finite and then our claim follows from the functoriality of the zig-zag defining the period morphism (that also defines the $\mathbf{Z}_{p, \square}[G]$ -structure on syntomic cohomology). There is one subtle point here: a choice of a coordinate system lands $\alpha_{r, I}$ in $\mathrm{R}\Gamma(\mathbf{Z}_p[\Gamma_\infty], \mathbb{B}_{I, n}(U_\infty))$ on which G acts through a finite quotient. The passage to the pro-étale cohomology is now via the quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_{I, n}^+) \xrightarrow{\sim} \mathrm{R}\Gamma(\mathbf{Z}_p[\Gamma_\infty], \mathbb{B}_{I, n}^+(U_\infty)),$$

which is clearly $\mathbf{Z}_{p, \square}[G]$ -equivariant.

Step 2 (*Fontaine-Messing syntomic cohomology vs Bloch-Kato syntomic cohomology*) This step is harder and, alas, requires us to dig deeper into the structure of the comparison morphism. We need to show that the quasi-isomorphism (see [15, (7.36)])

$$[\mathrm{R}\Gamma_{\mathrm{cr}}(X_C) \otimes_{\mathbf{B}_{\mathrm{cr}}^+}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{log}}]^{N=0} \simeq [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{log}}]^{N=0}$$

is $\mathbf{Q}_{p, \square}[G]$ -equivariant. Again, assuming enough functoriality, one can argue étale locally on X . That is, we need to show that the Hyodo-Kato section (see [15, Sec. 2] for the notation)

$$\iota : \mathrm{R}\Gamma_{\mathrm{HK}}(\mathcal{X}_1^0)_{\mathbf{Q}} \rightarrow \mathrm{R}\Gamma_{\mathrm{cr}}(\mathcal{X}_1/R)_{l, \mathbf{Q}}$$

of the canonical projection

$$(4.9) \quad \mathrm{R}\Gamma_{\mathrm{cr}}(\mathcal{X}_1/R)_{l, \mathbf{Q}} \rightarrow \mathrm{R}\Gamma_{\mathrm{HK}}(\mathcal{X}_1^0)_{\mathbf{Q}}$$

is $\mathbf{Q}_{p, \square}[G]$ -equivariant. Here \mathcal{X}_1 is a semistable formal scheme over $\mathcal{O}_{K,1}$ equipped with a G -action.

We note that the action of G on the domain and the target of the projection (4.9) factors through $G/H_{\mathcal{X}}$, where $H_{\mathcal{X}}$ is an open subgroup of G acting trivially on \mathcal{X}_1 . Hence it is enough to show that the map ι_l is $\mathbf{Z}_{p, \square}[G/H_{\mathcal{X}}]$ -equivariant in a functorial way.

But for this we can argue as in the proof of [15, Prop. 2.14], where one had to show that the Hyodo-Kato section, a priori just Frobenius-equivariant by [15, Th. 2.12], is actually also compatible with monodromy. Then one needed to lift the monodromy action to an action of \mathbb{G}_m^{\natural} on \mathcal{X}_1 and run the argument for the Frobenius-equivariance from the proof of [15, Th. 2.12] again but for the simplicial variety representing $[\mathcal{X}_1/\mathbb{G}_m^{\natural}]$. Here the relevant simplicial variety is $[\mathcal{X}_1/(G/H_{\mathcal{X}})]$, hence even simpler since $G/H_{\mathcal{X}}$ is finite. So the same argument works but is now basically trivial. \square

4.2.3. Local analyticity. Let X be a smooth rigid analytic variety over K . Let $I \subset (0, \infty)$ be a compact interval with rational endpoints and let $\Psi(I)$ be the set of zeros of t on $\mathrm{Spa}(\mathbf{B}_I)$. Then, by the proof of Proposition 4.7, for $r \geq 2d$, we have the following quasi-isomorphism

$$(4.10) \quad \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_I) \simeq [[\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{I, \log}]^{N=0} \xrightarrow{\iota_{\mathrm{HK}}} \bigoplus_{\Psi(I)} \mathrm{R}\Gamma_{\mathrm{dR}}(X_C, r)](-r).$$

Moreover, if a profinite group G acts continuously on X this quasi-isomorphism is in $\mathcal{D}(\mathbf{Q}_{p, \square}[G])$. The following result was proved in [17, Cor.1.1]. Our argument using the quasi-isomorphism (4.10) yields a conceptual explanation of this phenomena.

Corollary 4.11. *Assume that G is a compact p -adic Lie group and $X \in \mathrm{Sm}_K$ is quasi-compact. Then the solid G -representation $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}_I)$ is G -locally analytic. If t is a unit in $\mathrm{Spa}(B_I)$ this representation is G -smooth.*

Proof. We use the quasi-isomorphism 4.10. But the action on the de Rham part of the right-hand side of (4.10) is G -locally analytic and, by Lemma 4.4, the action on Hyodo-Kato part is actually G -smooth. \square

5. REPRESENTATION THEORETIC INPUT

In this short chapter we state and prove some basic representation-theoretic results which will be used later on in the proof of the main results.

5.1. Stable cohomology. One could obtain more refined versions of the following, but the one below suffices for our purposes.

Proposition 5.1. *Let G be a locally profinite group and let $\pi \in \mathrm{Rep}_{K_{\square}}^{\mathrm{sm}}(G)$ be a complex of smooth solid representations of H . There is a natural isomorphism (the colimit being taken over the compact open subgroups H of G)*

$$\mathrm{colim}_H \mathrm{R}\Gamma(H, \pi) \simeq (\mathrm{colim}_H \mathrm{R}\Gamma(H, K)) \otimes_K^{\mathrm{L}\square} \pi.$$

If G is a p -adic Lie group with Lie algebra \mathfrak{g} this is further naturally isomorphic to $\mathrm{R}\Gamma(\mathfrak{g}, K) \otimes_K^{\mathrm{L}\square} \pi$.

Proof. The second claim follows directly from the first one and classical results of Lazard. For the first one, by dévissage we may assume that π is in the heart of $\mathrm{Rep}_{K_{\square}}^{\mathrm{sm}}(G)$. Recall that $\mathrm{R}\Gamma(H, -)$ and the solid tensor product commute (separately in each variable for the tensor product) with filtered colimits. By writing $\pi = \mathrm{colim}_H \pi^H$ we are thus reduced to the case when some open subgroup H of G acts trivially on π , then (replacing G by H) to the case when G acts trivially on π . But then $\mathrm{R}\Gamma(H, \pi) \simeq \mathrm{R}\Gamma(H, K) \otimes_K^{\mathrm{L}\square} \pi$ for all H and we are done. \square

5.2. Cancellation results. Let G be a locally profinite group. We say that $\pi \in \text{Rep}_{K\Box}^{\text{sm}}(G)$ is *admissible* if π is concentrated in degree 0 and if, for all compact open subgroups H of G , the solid K -vector space π^H is of the form \underline{V} for some finite dimensional K -vector space V (endowed with its natural topology). This is equivalent to saying that $\pi = \underline{\sigma}$ for an admissible smooth (in the usual sense) K -representation σ of G (endowed with its natural inductive limit topology).

Remark 5.2. The above definition is ad-hoc, but suitable for our applications. The "right" definition would be to drop the assumption that π is concentrated in degree 0, impose that $H^i(\pi)$ is admissible in the above sense for all i , and moreover that for each compact open subgroup G of H the solid K -vector space $(H^i(\pi))^G$ vanishes for all but finitely many i .

Proposition 5.3. *Let G be a profinite group and let $\pi, \pi_1, \pi_2 \in \text{Rep}_{K\Box}^{\text{sm}}(G)$ be admissible representations of G such that*

$$\pi_1 \oplus \pi \simeq \pi_2 \oplus \pi$$

Then $\pi_1 \simeq \pi_2$.

Proof. We may assume that π, π_1, π_2 are admissible smooth non-condensed representations. Since G is profinite, π, π_1, π_2 are semi-simple and each irreducible smooth representation σ of G is finite dimensional and has finite multiplicity in π, π_1, π_2 . Since $\pi_1 \oplus \pi \simeq \pi_2 \oplus \pi$, the multiplicities of σ in π_1 and π_2 must be equal, so $\pi_1 \simeq \pi_2$. \square

While the above result is very elementary, the next one requires more serious input:

Proposition 5.4. *Let G be the group of K -rational points of a reductive group over K .*

a) If $\pi, \pi_1, \pi_2 \in \text{Rep}_{K\Box}^{\text{sm}}(G)$ are admissible representations of G such that $\pi \oplus \pi_1 \simeq \pi \oplus \pi_2$, then $\pi_1 \simeq \pi_2$.

a) If $\pi, \pi_1, \pi_2 \in \text{Rep}_{K\Box}^{\text{sm}}(G)$ are admissible representations of G on $(\varphi, N, \mathcal{G}_{\check{C}})$ -modules over \check{C} such that $\pi \oplus \pi_1 \simeq \pi \oplus \pi_2$, then $\pi_1 \simeq \pi_2$.

Proof. a) Using the Bernstein decomposition of the category of smooth representations of G , we may assume that π, π_1, π_2 all belong to a block of this category. Bernstein also proved (see [28, Prop. VI.10.6]) the existence of a compact open subgroup H of G so that the functor $V \mapsto V^H$ defined on the block, with values in modules over the associated Hecke algebra $H(G, H)$ is fully faithful. Thus it suffices to show that $\pi_1^H \simeq \pi_2^H$ as $H(G, H)$ -modules. But $\pi^H \oplus \pi_1^H \simeq \pi^H \oplus \pi_2^H$ as $H(G, H)$ -modules. By assumption π^H, π_1^H, π_2^H are finite dimensional over the field of coefficients, thus artinian and noetherian $H(G, H)$ -modules. The Krull-Schmidt theorem therefore applies and shows that $\pi_1^H \simeq \pi_2^H$ as $H(G, H)$ -modules, finishing the proof.

b) Since the $(\varphi, N, \mathcal{G}_{\check{C}})$ -module structure commutes with the action of G , the same argument as in the proof of part a) applies, by replacing the ring $H(G, H)$ with the ring

$$R = H(G, H)[\varphi, \varphi^{-1}, N, \mathcal{G}_{\check{C}}]/(N\varphi - p\varphi N; \varphi c - \varphi(c)\varphi | c \in \check{C}).$$

\square

Remark 5.5. Let M be a finite rank (φ, N) -module over \check{C} . Since the category of φ -crystals over \check{C} is semisimple we can write

$$M = NM \oplus W$$

in the category of φ -crystals. Now write (in the same category)

$$W = W^k \oplus W^{k-1} \oplus \dots \oplus W^1,$$

where $W^1 = \ker N$, W^2 is a complementary φ -crystal of W^1 in $\ker N^2$ (i.e. $\ker N^2 = W^1 \oplus W^2$), etc. Then we can write

$$(5.6) \quad M = (W^k \oplus NW^k \oplus \dots \oplus N^{k-1}W^k) \oplus (W^{k-1} \oplus NW^{k-1} \oplus \dots \oplus N^{k-2}W^{k-1}) \oplus \dots$$

We note that we have isomorphisms

$$W^k \xrightarrow[\sim]{N} NW^k \xrightarrow[\sim]{N} \dots \xrightarrow[\sim]{N} N^{k-1}W^k$$

which are Frobenius equivariant up to a twist (since we have $N\varphi = p\varphi N$). Similarly for W^{k-1}, \dots, W^1 .

This computation is a very explicit version of Krull-Schmidt theory for finite length modules over the ring $R := \check{C}[\varphi, \varphi^{-1}, N]/(N\varphi - p\varphi N; \varphi c - \varphi(c)\varphi | c \in \check{C})$. The maximal indecomposable modules are of the form $W^k \oplus NW^k \oplus \dots \oplus N^{k-1}W^k$, where W is an irreducible φ -crystal, as above.

6. FLIP-FLOPPING

We will now state and prove de Rham and Hyodo-Kato flip-flopping. For the set-up, consider a diagram

$$\begin{array}{ccc} & & G \times \check{G} \\ & & \curvearrowright \\ & T & \\ \swarrow \pi & & \searrow \tilde{\pi} \\ \check{G} & & G \\ \curvearrowright & & \curvearrowright \\ X = [T/G] & & \check{X} = [T/\check{G}] \end{array}$$

where

- (a) X and \check{X} are smooth rigid analytic spaces over K ; T is a diamond over K ;
- (b) G and \check{G} are profinite groups;
- (c) $G \times \check{G}$ acts on T continuously and the map $\pi : T \rightarrow X$ (respectively $\tilde{\pi} : T \rightarrow \check{X}$) is a pro-étale G -torsor (respectively a pro-étale \check{G} -torsor).

6.1. De Rham flip-flopping. In this section we collect a number of de Rham flip-flopping results.

6.1.1. *Usual de Rham cohomology.* The first main theorem of this paper is the following:

Theorem 6.1. (*De Rham flip-flopping*)

(1) *There are natural quasi-isomorphisms in $\mathcal{D}(K_{\square})$ ($r \in \mathbf{Z}$):*

$$(6.2) \quad \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X)) \simeq \mathrm{R}\Gamma(G, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(\check{X})).$$

(2) *There are natural quasi-isomorphisms in $\mathcal{D}(\mathbf{B}_{\mathrm{dR}, \square}^+)$ and $\mathcal{D}(\mathbf{B}_{\mathrm{dR}, \square})$, respectively:*

$$(6.3) \quad \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}})) \simeq \mathrm{R}\Gamma(G, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(\check{X}_C/\mathbf{B}_{\mathrm{dR}})), \quad r \in \mathbf{Z},$$

and

$$(6.4) \quad \mathrm{R}\Gamma(\check{G}, \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}})) \simeq \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{dR}}(\check{X}_C/\mathbf{B}_{\mathrm{dR}})).$$

Proof. Using the natural quasi-isomorphisms (see Lemma 3.11)

$$\begin{aligned} \mathrm{R}\Gamma(\check{G}, \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}})) &\simeq \mathrm{R}\Gamma(\check{G}, \mathrm{R}\Gamma_{\mathrm{dR}}(X)) \otimes_K^{\mathrm{L}\square} \mathbf{B}_{\mathrm{dR}}, \\ \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}})) &\simeq \mathrm{colim}_{j+k \geq r} \mathrm{R}\Gamma(\check{G}, F^j \mathrm{R}\Gamma_{\mathrm{dR}}(X)) \otimes_K^{\mathrm{L}\square} F^k \mathbf{B}_{\mathrm{dR}}, \end{aligned}$$

we see that the theorem follows once we construct the natural quasi-isomorphisms in (6.2).

First, suppose that X and \check{X} are quasi-compact. Then Corollary 3.16 and Galois descent (see Lemma 3.4) yield natural quasi-isomorphisms in $\mathcal{D}(K_{\square})$

$$\begin{aligned} \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X)) &\simeq \mathrm{R}\Gamma(\check{G} \times \mathcal{G}_K, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, F^r \mathbb{B}_{\mathrm{pdR}})) \simeq \mathrm{R}\Gamma(\check{G} \times \mathcal{G}_K, \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(T_C, F^r \mathbb{B}_{\mathrm{pdR}}))) \\ &\simeq \mathrm{R}\Gamma(G \times \check{G} \times \mathcal{G}_K, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(T_C, F^r \mathbb{B}_{\mathrm{pdR}})) \end{aligned}$$

and similarly with the roles of G and \check{G} (as well as X and \check{X}) exchanged. Combining these quasi-isomorphisms we obtain the desired relation (6.2) when X and \check{X} are quasi-compact.

To treat the general case of (6.2), let $\{X_n\}$, $n \in \mathbf{N}$, be an increasing covering of X by \check{G} -stable quasi-compact opens. Since \check{G} is compact, such a covering always exists by [32, Lemma 2.2]. Moreover, if X is Stein we can even take the X_n 's to be affinoid and the covering can be chosen to be strictly increasing. Let $\{\check{X}_n\}$, $n \in \mathbf{N}$, be the corresponding covering of \check{X} , i.e., $\check{X}_n := \tilde{\pi}(\pi^{-1}(X_n))$. This is an increasing covering of \check{X} by G -stable quasi-compact opens (again, strictly increasing if necessary). We compute in $\mathcal{D}(K_{\square})$:

$$\begin{aligned} \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X)) &\xrightarrow{\sim} \mathrm{R}\Gamma(\check{G}, \mathrm{R}\lim_n F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_n)) \xrightarrow{\sim} \mathrm{R}\lim_n \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_n)) \\ &\simeq \mathrm{R}\lim_n \mathrm{R}\Gamma(G, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(\check{X}_n)) \xleftarrow{\sim} \mathrm{R}\Gamma(G, \mathrm{R}\lim_n F^r \mathrm{R}\Gamma_{\mathrm{dR}}(\check{X}_n)) \xleftarrow{\sim} \mathrm{R}\Gamma(G, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(\check{X})). \end{aligned}$$

This finishes the proof of (6.2) and our theorem. \square

6.1.2. *Compactly supported de Rham cohomology.* Theorem 6.1 has a version for de Rham cohomology with compact support.

Corollary 6.5. (*Compactly supported de Rham flip-flopping*) *Assume that X, \check{X} are partially proper.*

(1) *There are natural quasi-isomorphisms in $\mathcal{D}(\mathbf{B}_{\mathrm{dR}, \square}^+)$*

$$(6.6) \quad \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}, c}(X_C/\mathbf{B}_{\mathrm{dR}})) \simeq \mathrm{R}\Gamma(G, F^r \mathrm{R}\Gamma_{\mathrm{dR}, c}(\check{X}_C/\mathbf{B}_{\mathrm{dR}})), \quad r \in \mathbf{Z}.$$

(2) *There are natural quasi-isomorphisms in $\mathcal{D}(K_{\square})$*

$$(6.7) \quad \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}, c}(X)) \simeq \mathrm{R}\Gamma(G, F^r \mathrm{R}\Gamma_{\mathrm{dR}, c}(\check{X})), \quad r \in \mathbf{Z}.$$

Proof. Let $r \in \mathbf{Z}$. We compute in $\mathcal{D}(\mathbf{B}_{\mathrm{dR}, \square}^+)$ using the quasi-isomorphism (6.3):

$$\begin{aligned} \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}, c}(X_C/\mathbf{B}_{\mathrm{dR}})) &\simeq \mathrm{R}\Gamma(\check{G}, [F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}) \rightarrow F^r \mathrm{R}\Gamma_{\mathrm{dR}}(\partial X_C/\mathbf{B}_{\mathrm{dR}})]) \\ &\simeq [\mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}})) \rightarrow \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(\partial X_C/\mathbf{B}_{\mathrm{dR}}))] \\ &\simeq [\mathrm{R}\Gamma(G, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(\check{X}_C/\mathbf{B}_{\mathrm{dR}})) \rightarrow \mathrm{R}\Gamma(G, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(\partial \check{X}_C/\mathbf{B}_{\mathrm{dR}}))] \\ &\simeq \mathrm{R}\Gamma(G, F^r \mathrm{R}\Gamma_{\mathrm{dR}, c}(\check{X}_C/\mathbf{B}_{\mathrm{dR}})). \end{aligned}$$

Here we used the fact that in $\mathcal{D}(\mathbf{B}_{\mathrm{dR}, \square})$

$$\begin{aligned} \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}}(\partial X_C/\mathbf{B}_{\mathrm{dR}})) &\simeq \mathrm{R}\Gamma(\check{G}, \mathrm{colim}_n F^r \mathrm{R}\Gamma_{\mathrm{dR}}((X \setminus X_n)_C/\mathbf{B}_{\mathrm{dR}})) \\ &\simeq \mathrm{colim}_n \mathrm{R}\Gamma(\check{G}, F^r \mathrm{R}\Gamma_{\mathrm{dR}}((X \setminus X_n)_C/\mathbf{B}_{\mathrm{dR}})), \end{aligned}$$

where the covering $\{X_n\}$ is chosen as in the proof of Theorem 6.1. The second quasi-isomorphism follows from the fact that $\mathbf{Z}_{\square}[\check{G}]$, $\mathbf{Z}_{\square}[G]$ are internally compact objects in solid abelian groups. And similarly for the pair G, \check{X} and the compatible covering $\{\check{X}_n\}$. This proves claim (1) of our corollary.

The above argument can be easily adapted to yield quasi-isomorphism (6.7) using quasi-isomorphism (6.2), hence proving claim (2) of our corollary. \square

6.1.3. *Hodge and Lie algebra cohomologies.* The following two results are simple corollaries of Theorem 6.1 and Corollary 6.5.

Corollary 6.8. (*Hodge flip-flopping*) *Assume that X, \check{X} are Stein of dimension d . Let $j \in \mathbf{N}$. We have natural quasi-isomorphisms in $\mathcal{D}(K_{\square})$*

$$\mathrm{R}\Gamma(\check{G}, \Omega^j(X)) \simeq \mathrm{R}\Gamma(G, \Omega^j(\check{X})), \quad \mathrm{R}\Gamma(\check{G}, H_c^d(X, \Omega^j)) \simeq \mathrm{R}\Gamma(G, H_c^d(\check{X}, \Omega^j)).$$

Proof. Since $\mathrm{gr}_F^j \mathrm{R}\Gamma_{\mathrm{dR}}(Y) \simeq \Omega^j(Y)[-j]$, for $Y = X, \check{X}$, the first quasi-isomorphism follows from the quasi-isomorphism (6.2). Since $\mathrm{gr}_F^j \mathrm{R}\Gamma_{\mathrm{dR}, c}(Y) \simeq H_c^d(Y, \Omega^j)[-d-j]$ (see (3.20)), the second quasi-isomorphism follows from the quasi-isomorphism (6.7). \square

Remark 6.9. One might be tempted to prove the above corollary using the easier period sheaf $\hat{\mathcal{O}}$ and the isomorphism $R^i \nu_* \hat{\mathcal{O}} \simeq \Omega^i(-i)$, where $\nu : X_{C, \text{proét}} \rightarrow X_{C, \text{ét}}$ is the natural projection. Unfortunately, the spectral sequences that arise when passing to cohomology are rather complicated and this approach does not seem to work.

We set in $\mathcal{D}(K_{\square})$

$$\mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\infty}) := \mathrm{colim}_H \mathrm{R}\Gamma_{\mathrm{dR},c}(X_H), \quad \mathrm{R}\Gamma_{\mathrm{dR},c}(\check{X}_{\infty}) := \mathrm{colim}_{\check{H}} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\check{H}}),$$

where H, \check{H} are normal compact open subgroups of G, \check{G} , respectively, and $X_H, X_{\check{H}}$ are the respective quotients of T (representable in rigid analytic spaces). Let us assume that X and \check{X} are partially proper. Then so are all $X_H, X_{\check{H}}$ (as they are finite étale over X and \check{X} respectively). By Proposition 4.3 we have $\mathrm{R}\Gamma_{\mathrm{dR},c}(X_H) \in \mathrm{Rep}_{K_{\square}}^{\mathrm{sm}}(\check{G})$. The natural action of the finite group G/H on X_H over X then shows that $\mathrm{R}\Gamma_{\mathrm{dR},c}(X_H) \in \mathrm{Rep}_{K_{\square}}^{\mathrm{sm}}(G \times \check{G})$ and so $\mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\infty}) \in \mathrm{Rep}_{K_{\square}}^{\mathrm{sm}}(G \times \check{G})$. Similarly, $\mathrm{R}\Gamma_{\mathrm{dR},c}(\check{X}_{\infty}) \in \mathrm{Rep}_{K_{\square}}^{\mathrm{sm}}(G \times \check{G})$.

Corollary 6.10. (Lie algebra flip-flopping) *Assume that X, \check{X} are partially proper and that G, \check{G} are compact p -adic Lie groups over \mathbf{Q}_p . Let $\mathfrak{g}, \check{\mathfrak{g}}$ be their respective Lie algebras. There are $G \times \check{G}$ -equivariant natural quasi-isomorphisms in $\mathcal{D}(K_{\square})$*

$$(6.11) \quad \begin{aligned} \mathrm{R}\Gamma(\check{\mathfrak{g}}, \mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\infty})) &\simeq \mathrm{R}\Gamma(\mathfrak{g}, \mathrm{R}\Gamma_{\mathrm{dR},c}(\check{X}_{\infty})), \\ \mathrm{R}\Gamma(\check{\mathfrak{g}}, K) \otimes_{K_{\square}}^{\mathrm{L}} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\infty}) &\simeq \mathrm{R}\Gamma(\mathfrak{g}, K) \otimes_{K_{\square}}^{\mathrm{L}} \mathrm{R}\Gamma_{\mathrm{dR},c}(\check{X}_{\infty}). \end{aligned}$$

Remark 6.12. The reader will notice that we do not claim that the quasi-isomorphism in the proposition is filtered.

Proof. The second quasi-isomorphism follows from the first one because the complexes $\mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\infty})$ and $\mathrm{R}\Gamma_{\mathrm{dR},c}(\check{X}_{\infty})$ are in $\mathrm{Rep}_{K_{\square}}^{\mathrm{sm}}(G \times \check{G})$, hence, by Proposition 4.2, can be assumed to be represented by complexes of smooth representations (thus the respective actions of \mathfrak{g} and $\check{\mathfrak{g}}$ are trivial).

For the first quasi-isomorphism, by Proposition 5.1, we have natural (hence $G \times \check{G}$ -equivariant) quasi-isomorphisms in $\mathcal{D}(K_{\square})$

$$\mathrm{R}\Gamma(\check{\mathfrak{g}}, \mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\infty})) \simeq \mathrm{colim}_{\check{H}} \mathrm{R}\Gamma(\check{H}, \mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\infty})) \simeq \mathrm{colim}_{\check{H}} \mathrm{colim}_H \mathrm{R}\Gamma(\check{H}, \mathrm{R}\Gamma_{\mathrm{dR},c}(X_H))$$

and similarly

$$\mathrm{R}\Gamma(\mathfrak{g}, \mathrm{R}\Gamma_{\mathrm{dR},c}(\check{X}_{\infty})) \simeq \mathrm{colim}_H \mathrm{colim}_{\check{H}} \mathrm{R}\Gamma(H, \mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\check{H}})).$$

Since, by (6.7), we have natural quasi-isomorphisms $\mathrm{R}\Gamma(\check{H}, \mathrm{R}\Gamma_{\mathrm{dR},c}(X_H)) \simeq \mathrm{R}\Gamma(H, \mathrm{R}\Gamma_{\mathrm{dR},c}(X_{\check{H}}))$, the result follows. \square

6.2. Hyodo-Kato flip-flopping. Assume that k is algebraically closed. Our second main theorem is the following¹⁶:

Theorem 6.13. (Hyodo-Kato flip-flopping) *Let X, \check{X} be partially proper. We have natural \mathcal{G}_K -equivariant quasi-isomorphisms in $\mathcal{D}_{\varphi, N}(\check{C}_{\square})$*

$$(6.14) \quad \mathrm{R}\Gamma(\check{G}, \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)) \simeq \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{HK}}(\check{X}_C)),$$

$$(6.15) \quad \mathrm{R}\Gamma(\check{G}, \mathrm{R}\Gamma_{\mathrm{HK},c}(X_C)) \simeq \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{HK},c}(\check{X}_C)).$$

¹⁶See [1] for the definition of compactly supported Hyodo-Kato cohomology.

Proof. We will first prove a flip-flopping quasi-isomorphism between \mathbb{B}^+ -cohomologies (see (6.16) below). Since $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(-, \mathbb{B}^+)$ satisfies Galois descent (see Lemma 3.4), we have natural quasi-isomorphisms in $\mathcal{D}_\varphi(\mathbf{B}_\square^+)$ compatible with the \mathcal{G}_K -action:

$$\begin{aligned} \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(T_C, \mathbb{B}^+)) &\simeq \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}^+), \\ \mathrm{R}\Gamma(\check{G}, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(T_C, \mathbb{B}^+)) &\simeq \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\check{X}_C, \mathbb{B}^+), \\ \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\check{X}_C, \mathbb{B}^+)) &\simeq \mathrm{R}\Gamma(G \times \check{G}, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(T_C, \mathbb{B}^+)) \simeq \mathrm{R}\Gamma(\check{G}, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}^+)). \end{aligned}$$

In particular, we have a natural quasi-isomorphism in $\mathcal{D}_\varphi(\mathbf{B}_\square^+)$

$$(6.16) \quad \mathrm{R}\Gamma(\check{G}, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}^+)) \simeq \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\check{X}_C, \mathbb{B}^+)),$$

which is compatible with the \mathcal{G}_K -action. It yields a natural quasi-isomorphism in $\mathcal{D}_\varphi(\mathbf{B}_\square)$

$$(6.17) \quad \mathrm{R}\Gamma(\check{G}, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{B}^+) \otimes_{\mathbf{B}^+}^{\mathrm{L}\square} \mathbf{B}) \simeq \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\check{X}_C, \mathbb{B}^+) \otimes_{\mathbf{B}^+}^{\mathrm{L}\square} \mathbf{B}),$$

which is compatible with the \mathcal{G}_K -action.

Now we pass to the Hyodo-Kato flip-flopping. Assume first that the de Rham cohomology of X, \check{X} is of finite rank. Set $Z = X, \check{X}$. Combining quasi-isomorphism from Proposition 3.23 with the quasi-isomorphism (6.17), we get a natural \mathcal{G}_K -equivariant quasi-isomorphism in $\mathcal{D}_\varphi(\mathbf{B}_\square)$

$$\mathrm{R}\Gamma(\check{G}, [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0}) \simeq \mathrm{R}\Gamma(G, [\mathrm{R}\Gamma_{\mathrm{HK}}(\check{X}_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0})$$

By taking φ -invariants, this yields a natural \mathcal{G}_K -equivariant quasi-isomorphism in $\mathcal{D}(\mathbf{B}_{e, \square})$

$$\mathrm{R}\Gamma(\check{G}, [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0, \varphi=1}) \simeq \mathrm{R}\Gamma(G, [\mathrm{R}\Gamma_{\mathrm{HK}}(\check{X}_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0, \varphi=1}).$$

By tensoring it with $\mathbf{B}_{\mathrm{pFF}}$ over \mathbf{B}_e , we get a natural \mathcal{G}_K -equivariant quasi-isomorphism in $\mathcal{D}_{\varphi, N}(\mathbf{B}_{\mathrm{pFF}, \square})$

$$(6.18) \quad \mathrm{R}\Gamma(\check{G}, [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}}) \simeq \mathrm{R}\Gamma(G, [\mathrm{R}\Gamma_{\mathrm{HK}}(\check{X}_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}}),$$

We used here the following computation:

Lemma 6.19. *We have a canonical quasi-isomorphism in $\mathcal{D}_{\varphi, N}(\mathbf{B}_{\mathrm{pFF}, \square})$*

$$\mathrm{R}\Gamma(\check{G}, [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0, \varphi=1}) \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}} \simeq \mathrm{R}\Gamma(\check{G}, [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}}).$$

Similarly for G and \check{X} .

Proof. Since we have assumed the Hyodo-Kato cohomology has finite rank over \check{C} , it suffices to show that, for a finite rank \check{G} representation V over \check{C} , which is a (φ, N) -module, we have in $\mathcal{D}_{\varphi, N}(\mathbf{B}_{\mathrm{pFF}, \square})$

$$\mathrm{R}\Gamma(\check{G}, (V \otimes_{\check{C}}^{\square} \mathbf{B}_{\mathrm{FF}})^{N=0, \varphi=1}) \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}} \simeq \mathrm{R}\Gamma(\check{G}, (V \otimes_{\check{C}}^{\square} \mathbf{B}_{\mathrm{FF}})^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}}).$$

Or that, for a finite free \mathbf{B}_e -module V_e with a \mathbf{B}_e -linear continuous action of \check{G} , we have a quasi-isomorphism in $\mathcal{D}_{\varphi, N}(\mathbf{B}_{\mathrm{pFF}, \square})$

$$\mathrm{R}\Gamma(\check{G}, V_e) \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}} \simeq \mathrm{R}\Gamma(\check{G}, V_e \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}}).$$

By taking the bar resolution, it suffices to show that, for a profinite set S , the canonical map

$$\underline{\mathrm{Hom}}_{\check{C}}(\check{C}_\square[S], V_e) \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}} \rightarrow \underline{\mathrm{Hom}}_{\check{C}}(\check{C}_\square[S], V_e \otimes_{\mathbf{B}_e}^{\square} \mathbf{B}_{\mathrm{pFF}})$$

is an isomorphism in $\mathcal{D}_{\varphi, N}(\mathbf{B}_{\mathrm{pFF}, \square})$. But this is clear since both sides are isomorphic to (see the proof of Lemma 2.3)

$$\underline{\mathrm{Hom}}_{\check{C}}(\check{C}_\square[S], V_{\check{C}}) \otimes_{\check{C}}^{\square} \mathbf{B}_{\mathrm{pFF}},$$

for a \check{C} -vector space such that $V_e \simeq V_{\check{C}} \otimes_{\check{C}}^{\square} \mathbf{B}_e$. □

The quasi-isomorphism (6.18) yields a natural \mathcal{G}_K -equivariant quasi-isomorphism in $\mathcal{D}_{\varphi,N}(\check{C}_{\square})$

$$\mathrm{R}\Gamma(\check{G}, ([\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}})^{\mathrm{R}\mathcal{G}_K\text{-sm}}) \simeq \mathrm{R}\Gamma(G, ([\mathrm{R}\Gamma_{\mathrm{HK}}(\check{X}_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}})^{\mathrm{R}\mathcal{G}_K\text{-sm}}).$$

Now, we can combine it with the \mathcal{G}_K -equivariant quasi-isomorphism in $\mathcal{D}_{\varphi,N}(\check{C}_{\square})$ (see Corollary 3.29)

$$\mathrm{R}\Gamma_{\mathrm{HK}}(Z_C) \xrightarrow{\sim} ([\mathrm{R}\Gamma_{\mathrm{HK}}(Z_C) \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{FF}}]^{N=0, \varphi=1} \otimes_{\mathbf{B}_e}^{\mathrm{L}\square} \mathbf{B}_{\mathrm{pFF}})^{\mathrm{R}\mathcal{G}_K\text{-sm}}, \quad Z = X, \check{X},$$

to get (6.14) in our theorem in the case of finite rank de Rham cohomology.

To treat the general case, let $\{X_n\}$, $n \in \mathbf{N}$, be a strictly increasing covering of X by \check{G} -stable quasi-compact opens. We claim that we can find a covering $\{X_n^0\}$ of X such that: $X_{n-1} \subset X_n^0 \subset X_n$, X_n^0 is partially proper, $H_{\mathrm{dR},c}^i(X_n^0)$ is of finite rank over K , and the covering $\{X_n^0\}$ is \check{G} -equivariant. Indeed, it suffices to find such X_n^0 for a pair of quasi-compact opens $X_{n-1} \Subset X_n$. Arguing as in the proof of [1, Lemma 2.28], we can find U such that $X_{n-1} \subset U \subset X_n$, U is partially proper, and the complement $W := X_n \setminus U$ is quasi-compact. By [22, Th. 3.6] this implies that de Rham cohomology of U has finite rank. We do not know however that U is \check{G} -stable. What we do know though is that there exists an open subgroup $H \subset \check{G}$ that stabilizes U : by continuity of the action of \check{G} on X this is the case for W , which is quasi-compact, and if H stabilizes W it also stabilizes U because U is partially proper. It follows that we can set $X_n^0 := \cup_{g \in \check{G}} gU$ to get the partially proper open that we wanted.

Let $\{\check{X}_n^0\}$, $n \in \mathbf{N}$, be the corresponding covering of \check{X} , i.e., $\check{X}_n^0 := \check{\pi}(\pi^{-1}(X_n^0))$. This is an increasing covering of \check{X} by G -stable partially proper opens \check{X}_n^0 with analogous properties to X_n^0 (we use here [25, Lemma 1.1017 (vii)]). By construction, the coverings $\{X_n^0\}$ and $\{\check{X}_n^0\}$ are both G and \check{G} equivariant.

Remark 6.20. Let $\pi_Y : Y \rightarrow X$ be a finite étale map from the tower. Then $\pi_Y^{-1}(X_n^0)$ has analogous properties to X_n^0 . Similarly, for the other tower. This follows from [25, Lemma 1.1017 (iv)].

We compute in $\mathcal{D}_{\varphi,N,\mathcal{G}_K}(\check{C}_{\square})$:

$$\begin{aligned} \mathrm{R}\Gamma(\check{G}, \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)) &\xrightarrow{\sim} \mathrm{R}\Gamma(\check{G}, \mathrm{R}\lim_n \mathrm{R}\Gamma_{\mathrm{HK}}(X_{n,C}^0)) \xrightarrow{\sim} \mathrm{R}\lim_n \mathrm{R}\Gamma(\check{G}, \mathrm{R}\Gamma_{\mathrm{HK}}(X_{n,C}^0)) \\ &\simeq \mathrm{R}\lim_n \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{HK}}(\check{X}_{n,C}^0)) \xleftarrow{\sim} \mathrm{R}\Gamma(G, \mathrm{R}\lim_n \mathrm{R}\Gamma_{\mathrm{dR}}(\check{X}_{n,C}^0)) \xleftarrow{\sim} \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{dR}}(\check{X}_C)). \end{aligned}$$

This finishes the proof of (6.14) in our theorem.

For compactly supported cohomology we argue as in the proof of Corollary 6.5 using (6.14) that we have already shown. \square

6.2.1. *Lie algebra flip-flopping.* We set

$$\mathrm{R}\Gamma_{\mathrm{HK},c}(X_{\infty,C}) \simeq \mathrm{colim}_H \mathrm{R}\Gamma_{\mathrm{HK},c}(X_{H,C}), \quad \mathrm{R}\Gamma_{\mathrm{HK},c}(\check{X}_{\infty,C}) \simeq \mathrm{colim}_{\check{H}} \mathrm{R}\Gamma_{\mathrm{HK},c}(X_{\check{H},C}),$$

where H, \check{H} are compact open subgroups of G, \check{G} , respectively, and X_H, \check{X}_H are the respective quotients of T . By Proposition 4.3, $\mathrm{R}\Gamma_{\mathrm{HK},c}(X_{\infty,C}) \in \mathrm{Rep}_{\check{C}_{\square}}^{\mathrm{sm}}(G \times \check{G})$. Similarly for $\mathrm{R}\Gamma_{\mathrm{HK},c}(\check{X}_{\infty,C})$.

Corollary 6.21. (Lie algebra flip-flopping) *Assume that G, \check{G} are compact p -adic Lie groups over \mathbf{Q}_p . Let $\mathfrak{g}, \check{\mathfrak{g}}$ be their respective Lie algebras. There are $G \times \check{G}$ -equivariant natural quasi-isomorphisms in $\mathcal{D}_{\varphi,N,\mathcal{G}_K}(\check{C}_{\square})$*

$$(6.22) \quad \begin{aligned} \mathrm{R}\Gamma(\check{\mathfrak{g}}, \mathrm{R}\Gamma_{\mathrm{HK},c}(X_{\infty,C})) &\simeq \mathrm{R}\Gamma(\mathfrak{g}, \mathrm{R}\Gamma_{\mathrm{HK},c}(\check{X}_{\infty,C})), \\ \mathrm{R}\Gamma(\check{\mathfrak{g}}, \check{C}) \otimes_{\check{C}}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{HK},c}(X_{\infty,C}) &\simeq \mathrm{R}\Gamma(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{HK},c}(\check{X}_{\infty,C}). \end{aligned}$$

Proof. Having the quasi-isomorphism (6.15), we argue exactly as in the proof of Corollary 6.10. \square

6.3. Flip-flopping at infinity. Assume from now on that k is algebraically closed¹⁷. Below, a $(\varphi, N, \mathcal{G}_K)$ -module over \check{C} will always refer to a solid module, where φ an isomorphism and the action of \mathcal{G}_K is smooth. In the case of compactly supported cohomology we have the following underived version of flip-flopping if we are willing to pass to infinity.

Theorem 6.23. *Assume that X, \check{X} are partially proper and that G, \check{G} are p -adic Lie groups with*

$$\dim_K H^i(\mathfrak{g}, K) = \dim_K H^i(\check{\mathfrak{g}}, K), \quad i \geq 0.$$

Then:

(1) *There exist compact open subgroups $H \subset G, \check{H} \subset \check{G}$ and a $H \times \check{H}$ -equivariant isomorphism in K_\square and \check{C}_\square , respectively*

$$(6.24) \quad \begin{aligned} H_{\mathrm{dR},c}^i(X_\infty) &\simeq H_{\mathrm{dR},c}^i(\check{X}_\infty), \quad i \geq 0, \\ H_{\mathrm{HK},c}^i(X_\infty) &\simeq H_{\mathrm{HK},c}^i(\check{X}_\infty), \quad i \geq 0. \end{aligned}$$

(2) *If moreover in (1) the action of $H \times \check{H}$ is admissible, then the second isomorphism in (6.24) could also be made N -equivariant.*

Remark 6.25. (1) The proof of the theorem will show that we can choose H, \check{H} to be any subgroups acting trivially on all Lie algebra cohomology groups.

(2) The second claim of the theorem seems abstractly false without the admissibility assumption.

Proof. We note that the groups $H^i(\check{\mathfrak{g}}, K), H^i(\mathfrak{g}, K)$, for $i \geq 0$, are of finite rank over K . We choose $H \subset G, \check{H} \subset \check{G}$ to be any compact open subgroups acting trivially on all Lie algebra cohomology groups.

(1) *De Rham cohomology.* Let us start with de Rham cohomology. We will first assume that the $G \times \check{G}$ -representations $H_{\mathrm{dR},c}^i(X_\infty), H_{\mathrm{dR},c}^i(\check{X}_\infty)$ are admissible. We will argue by induction on i . We start with $i = 0$, where the claims are clear since both groups $H_{\mathrm{dR},c}^0(X_\infty), H_{\mathrm{dR},c}^0(\check{X}_\infty)$ are trivial. For the inductive step, we assume the isomorphism (6.24) for all degrees $\leq i$ and we want to prove it for $i + 1$.

From (6.11) we get a $G \times \check{G}$ -equivariant isomorphism in K_\square

$$(6.26) \quad H^{i+1}(\mathrm{R}\Gamma(\check{\mathfrak{g}}, K) \otimes_K^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_\infty)) \simeq H^{i+1}(\mathrm{R}\Gamma(\mathfrak{g}, K) \otimes_K^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{dR},c}(\check{X}_\infty)).$$

By Künneth formula, it yields a $G \times \check{G}$ -equivariant isomorphism in K_\square

$$(6.27) \quad \begin{aligned} H^0(\check{\mathfrak{g}}, K) \otimes_K^\square H_{\mathrm{dR},c}^{i+1}(X_\infty) \oplus H^1(\check{\mathfrak{g}}, K) \otimes_K^\square H_{\mathrm{dR},c}^i(X_\infty) \oplus \cdots \oplus H^{i+1}(\check{\mathfrak{g}}, K) \otimes_K^\square H_{\mathrm{dR},c}^0(X_\infty) \\ \simeq H^0(\mathfrak{g}, K) \otimes_K^\square H_{\mathrm{dR},c}^{i+1}(\check{X}_\infty) \oplus H^1(\mathfrak{g}, K) \otimes_K^\square H_{\mathrm{dR},c}^i(\check{X}_\infty) \oplus \cdots \oplus H^{i+1}(\mathfrak{g}, K) \otimes_K^\square H_{\mathrm{dR},c}^0(\check{X}_\infty) \end{aligned}$$

Since the groups $H^j(\check{\mathfrak{g}}, K), H^j(\mathfrak{g}, K)$, for $j \geq 0$, are of finite rank over K and $H^0(\check{\mathfrak{g}}, K) = H^0(\mathfrak{g}, K) = K$, if we want to prove the claim of the theorem for $i + 1$, by the inductive assumption, we find ourselves in the situation where we have an isomorphism of admissible representations of $H \times \check{H}$ over K :

$$\pi_1 \oplus \pi \simeq \pi_2 \oplus \pi$$

and we want to conclude that $\pi_1 \simeq \pi_2$, as $H \times \check{H}$ -representations. This holds by Proposition 5.3.

To treat the general case, choose the coverings $\{X_{n,\infty}^0, \check{X}_{n,\infty}^0\}$, $n \in \mathbf{N}$, as in the proof of Theorem 6.13. Now, let $i \geq 0$. We have $G \times \check{G}$ -equivariant isomorphisms in K_\square :

$$(6.28) \quad H_{\mathrm{dR},c}^i(X_\infty) \xrightarrow{\sim} \mathrm{colim}_n H_{\mathrm{dR},c}^i(X_{n,\infty}^0), \quad H_{\mathrm{dR},c}^i(\check{X}_\infty) \xrightarrow{\sim} \mathrm{colim}_n H_{\mathrm{dR},c}^i(\check{X}_{n,\infty}^0).$$

Moreover, $H_{\mathrm{dR},c}^i(X_{n,\infty}^0)$ and $H_{\mathrm{dR},c}^i(\check{X}_{n,\infty}^0)$ are admissible, hence, by the above, isomorphic as representations of $H \times \check{H}$:

$$\alpha_n : H_{\mathrm{dR},c}^i(X_{n,\infty}^0) \simeq H_{\mathrm{dR},c}^i(\check{X}_{n,\infty}^0).$$

¹⁷Though this is not necessary in the case of de Rham cohomology

Assume first that the transition maps in the inductive systems in (6.28) are injective. Then, we can write

$$\begin{aligned} H_{\mathrm{dR},c}^i(X_\infty) &= H_{\mathrm{dR},c}^i(X_{0,\infty}^0) \oplus W_0 \oplus W_1 \oplus \cdots \\ H_{\mathrm{dR},c}^i(\check{X}_\infty) &= H_{\mathrm{dR},c}^i(\check{X}_{0,\infty}^0) \oplus \check{W}_0 \oplus \check{W}_1 \oplus \cdots \end{aligned}$$

where W_n is a complementary representation of $H_{\mathrm{dR},c}^i(X_{n,\infty}^0)$ in $H_{\mathrm{dR},c}^i(X_{n+1,\infty}^0)$ and, similarly, for \check{W}_n . Starting with the isomorphism α_0 and using the computations in the admissible case we get isomorphisms $\beta_n : W_n \simeq \check{W}_n$. They yield the wanted isomorphism $H_{\mathrm{dR},c}^i(X_\infty) \simeq H_{\mathrm{dR},c}^i(\check{X}_\infty)$.

In the general case, we set, for $n \in \mathbf{N}$,

$$M_n^i := H_{\mathrm{dR},c}^i(X_{n,\infty}^0) / \ker^i_{n,\infty}, \quad \ker^i_{n,\infty} := \mathrm{colim}_m \ker(H_{\mathrm{dR},c}^i(X_{n,\infty}^0) \rightarrow H_{\mathrm{dR},c}^i(X_{m,\infty}^0)).$$

We define similarly, \check{M}_n^i . We have

$$(6.29) \quad H_{\mathrm{dR},c}^i(X_\infty) \xrightarrow{\sim} \mathrm{colim}_n M_n^i, \quad H_{\mathrm{dR},c}^i(\check{X}_\infty) \xrightarrow{\sim} \mathrm{colim}_n \check{M}_n^i.$$

We use now the naturality of the Lie algebra flip-flopping from Corollary 6.10 to conclude that we have the Künneth formula (6.27) with $H_{\mathrm{dR},c}^i(X_{n,\infty})$ replaced by M_n^i and $H_{\mathrm{dR},c}^i(\check{X}_{n,\infty})$ replaced by \check{M}_n^i . The argument below (6.27) goes through and shows that

$$M_n^i \simeq \check{M}_n^i.$$

Having that, we can argue as above to show that $H_{\mathrm{dR},c}^i(X_\infty) \simeq H_{\mathrm{dR},c}^i(\check{X}_\infty)$, as wanted.

(2) *Hyodo-Kato cohomology.* We move now to Hyodo-Kato cohomology. We will first assume that the $G \times \check{G}$ -representations $H_{\mathrm{HK},c}^i(X_\infty)$, $H_{\mathrm{HK},c}^i(\check{X}_\infty)$ are admissible. We will argue by induction on i as in the de Rham case. We start with $i = 0$, where the claims are clear since both groups $H_{\mathrm{HK},c}^0(X_\infty)$, $H_{\mathrm{HK},c}^0(\check{X}_\infty)$ are trivial. For the inductive step, we assume the isomorphism (6.24) for all degrees $\leq i$ and we want to prove it for $i + 1$.

From Corollary 6.21 we get a quasi-isomorphism in $\mathcal{D}_{\varphi,N,\mathcal{G}_K}(\check{C}\square)$

$$H^{i+1}(\mathrm{R}\Gamma(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{HK},c}(X_{\infty,C})) \simeq H^{i+1}(\mathrm{R}\Gamma(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{HK},c}(\check{X}_{\infty,C})).$$

This isomorphism is $\check{C}[G \times \check{G}]$ -equivariant. Using the Künneth formula, we get a $\check{C}[G \times \check{G}]$ -equivariant quasi-isomorphism in $\mathcal{D}_{\varphi,N,\mathcal{G}_K}(\check{C}\square)$

$$(6.30) \quad \begin{aligned} &H^0(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK},c}^{i+1}(X_{\infty,C}) \oplus H^1(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK},c}^i(X_{\infty,C}) \oplus \cdots \oplus H^{i+1}(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK},c}^0(X_{\infty,C}) \\ &\simeq H^0(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK},c}^{i+1}(\check{X}_{\infty,C}) \oplus H^1(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK},c}^i(\check{X}_{\infty,C}) \oplus \cdots \oplus H^{i+1}(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK},c}^0(\check{X}_{\infty,C}). \end{aligned}$$

Since the groups $H^j(\mathfrak{g}, \check{C})$, $H^j(\mathfrak{g}, \check{C})$, for $j \geq 0$, are of finite rank over \check{C} and $H^0(\mathfrak{g}, \check{C}) = H^0(\mathfrak{g}, \check{C}) = \check{C}$, if we want to prove the claim of the theorem for $i + 1$, by the inductive assumption, we find ourselves in the situation where we have an isomorphism of admissible representations of $H \times \check{H}$ on solid $(\varphi, N, \mathcal{G}_K)$ -modules over \check{C} :

$$(6.31) \quad \pi_1 \oplus \pi \simeq \pi_2 \oplus \pi$$

and we want to conclude that $\pi_1 \simeq \pi_2$, as $H \times \check{H}$ -representations on solid $(\varphi, N, \mathcal{G}_K)$ -modules over \check{C} . This follows from Proposition 5.4.

For the general case, we argue as in the de Rham case. Let $i \geq 0$. We have $G \times \check{G}$ -equivariant isomorphisms of $(\varphi, N, \mathcal{G}_K)$ -modules over $\check{C}\square$:

$$(6.32) \quad H_{\mathrm{HK},c}^i(X_{\infty,C}) \xrightarrow{\sim} \mathrm{colim}_n H_{\mathrm{HK},c}^i(X_{n,\infty,C}^0), \quad H_{\mathrm{HK},c}^i(\check{X}_{\infty,C}) \xrightarrow{\sim} \mathrm{colim}_n H_{\mathrm{HK},c}^i(\check{X}_{n,\infty,C}^0).$$

Moreover, $H_{\mathrm{HK},c}^i(X_{n,\infty,C}^0)$ and $H_{\mathrm{HK},c}^i(\check{X}_{n,\infty,C}^0)$ are admissible, hence, by the above, isomorphic as representations of $H \times \check{H}$ on $(\varphi, N, \mathcal{G}_K)$ -modules over $\check{C}\square$:

$$\alpha_n : H_{\mathrm{HK},c}^i(X_{n,\infty}^0) \simeq H_{\mathrm{HK},c}^i(\check{X}_{n,\infty}^0).$$

Arguing exactly as in the de Rham case, we can assume that the transition maps in the inductive systems (6.32) are injective. Then, we can write

$$\begin{aligned} H_{\mathrm{HK},c}^i(X_\infty) &= H_{\mathrm{HK},c}^i(X_{0,\infty}^0) \oplus W_0 \oplus W_1 \oplus \cdots \\ H_{\mathrm{HK},c}^i(\check{X}_\infty) &= H_{\mathrm{HK},c}^i(\check{X}_{0,\infty}^0) \oplus \check{W}_0 \oplus \check{W}_1 \oplus \cdots \end{aligned}$$

where W_n is a complementary $(H \times \check{H}, \varphi, \mathcal{G}_K)$ -representation¹⁸ of $H_{\mathrm{HK},c}^i(X_{n,\infty}^0)$ in $H_{\mathrm{HK},c}^i(X_{n+1,\infty}^0)$ and, similarly, for \check{W}_n . Starting with the isomorphism α_0 and using the computations in the admissible case we get isomorphisms $\beta_n : W_n \simeq \check{W}_n$. They yield the wanted isomorphism $H_{\mathrm{HK},c}^i(X_\infty) \simeq H_{\mathrm{HK},c}^i(\check{X}_\infty)$. This finishes the proof of the theorem. \square

7. FLIP-FLOPPING FOR DUAL TOWERS OF LOCAL SHIMURA VARIETIES

We will now show that the compactly supported de Rham and Hyodo-Kato cohomologies of basic dual towers of local Shimura varieties flip-flop together with the actions of the associated p -adic Lie groups.

7.1. Flip-flopping for Drinfeld and Lubin-Tate towers. We start with the key motivating example of the Drinfeld and Lubin-Tate towers.

Let K be a finite extension of \mathbf{Q}_p , let ϖ be a uniformizer of K , and let $d \geq 1$. Let $G = \mathrm{GL}_{d+1}(K)$ and let $\check{G} = D^\times$ be the invertible elements in the central division algebra D of dimension $(d+1)^2$ and invariant $1/(d+1)$ over K . Let \mathcal{O}_D be the maximal order in D and let ϖ_D be an uniformisant of \mathcal{O}_D .

7.1.1. Drinfeld and Lubin-Tate towers. We start with a quick review of the Drinfeld and Lubin-Tate towers.

(•) *Drinfeld tower.* Recall that the Drinfeld space $\mathbb{P}_K^d \setminus \cup_{H \in \mathcal{H}} H$, where \mathcal{H} is the set of K -rational hyperplanes, admits a natural rigid analytic structure \mathbb{H}_K^d over K and an action of G by homographies, which respects that structure. Drinfeld defined a tower of coverings $\check{\mathcal{M}}_n$, for $n \in \mathbf{N}$, of \mathbb{H}_C^d with the following properties:

- (1) $\check{\mathcal{M}}_n$ is defined over \check{C} and is equipped with an action of the Weil group W_K compatible with its natural action on \check{C} .
- (2) $\check{\mathcal{M}}_n$ is equipped with commuting actions of G and \check{G} which also commute with the action of W_K . The transition maps $\check{\mathcal{M}}_{n+1} \rightarrow \check{\mathcal{M}}_n \rightarrow \mathbb{H}_C^d$ are W_K , \check{G} and G -equivariant (the action of \check{G} on \mathbb{H}_C^d being trivial).
- (3) $\check{\mathcal{M}}_0 = \mathbf{Z} \times \mathbb{H}_C^d$ and, for $n \geq 1$, $\check{\mathcal{M}}_n$ is a Galois covering of $\check{\mathcal{M}}_0$, with Galois group $\mathcal{O}_D^\times / (1 + \varpi_D^n \mathcal{O}_D)$.

We set $\mathcal{M}_n := \check{\mathcal{M}}_n \times_{\check{C}} C$. Let \mathcal{M}_∞ be the (non completed) projective system of \mathcal{M}_n . If $H = G, \check{G}, W_K$, we have a natural morphism $\nu_H : H \rightarrow K^\times$, where $\nu_G = \det$, $\nu_{\check{G}}$ is the reduced norm, and ν_{W_K} is the composition of $W_K \rightarrow W_K^{\mathrm{ab}}$ and the isomorphism $W_K^{\mathrm{ab}} \simeq K^\times$ of local class field theory. The set $\pi_0(\mathcal{M}_\infty)$ of the connected components of \mathcal{M}_∞ is a principal homogenous space under the action of K^\times and $H = G, \check{G}, W_K$ acts on $\pi_0(\mathcal{M}_\infty)$ via $\nu_H : H \rightarrow K^\times$. In particular, \check{G} acts on $\pi_0(\mathcal{M}_\infty)$ via the reduced norm.

Let $\widehat{\mathcal{M}}_\infty$ be the completion of \mathcal{M}_∞ . We denote by $\widehat{\mathcal{M}}_{\infty, \check{C}}, \mathcal{M}_{\infty, \check{C}}$ the analogous spaces over \check{C} .

(•) *Lubin-Tate tower.* Lubin-Tate theory yields a formal scheme $\mathrm{Spf}(A_0)$ over $\mathcal{O}_{\check{C}}$, classifying deformations by quasi-isogenies of the unique \mathcal{O}_K -formal module of dimension 1 and height $d+1$ (relative to K) over $\overline{\mathbf{F}}_p$. Adding level structures to these deformations, Drinfeld constructed a tower of formal schemes $(\mathrm{Spf}(A_n))_{n \geq 1}$ over $\mathcal{O}_{\check{C}}$. The rigid analytic generic fiber $\check{\mathrm{LT}}_n$ of $\mathrm{Spf}(A_n)$ is a finite étale Galois covering of $\check{\mathrm{LT}}_0$ with Galois groupe $\mathrm{GL}_2(\mathcal{O}_K/\varpi^n)$. $\check{\mathrm{LT}}_0$ is a countable disjoint union of open unit discs over \check{C} .

One can define an action of $G \times \check{G}$ on the tower $(\check{\mathrm{LT}}_n)_{n \geq 0}$ (unlike in the case of Drinfeld tower, we do not have an action of $G \times \check{G}$ on every level of the tower). Set $\mathrm{LT}_n := \check{\mathrm{LT}}_n \times_{\check{C}} C$. Let $\widehat{\mathrm{LT}}_\infty$ be the

¹⁸It is here that we lost the monodromy N !

completion of the limit LT_∞ of the tower of $(\mathrm{LT}_n)_{n \geq 0}$. We denote by $\widehat{\mathrm{LT}}_{\infty, \check{C}}, \mathrm{LT}_{\infty, \check{C}}$ the analogous spaces over \check{C} .

7.1.2. *Duality of the towers.* The completions $\widehat{\mathrm{LT}}_\infty$ and $\widehat{\mathcal{M}}_\infty$ are equipped with an action of $G \times \check{G}$ and isomorphic as $(G \times \check{G})$ -spaces [34, Th. 7.2.3]. Same holds for their \check{C} -analogous $\widehat{\mathrm{LT}}_{\infty, \check{C}}$ and $\widehat{\mathcal{M}}_{\infty, \check{C}}$. Moreover, the completions $\widehat{\mathrm{LT}}_\infty$ and $\widehat{\mathcal{M}}_\infty$ are perfectoid spaces [34, Th. 6.5.4]. Write $\widehat{\mathrm{LT}}_\infty^\varpi$ and $\widehat{\mathcal{M}}_\infty^\varpi$ (resp. $\widehat{\mathrm{LT}}_{\infty, \check{C}}^\varpi$ and $\widehat{\mathcal{M}}_{\infty, \check{C}}^\varpi$, etc.) for the quotients of $\widehat{\mathrm{LT}}_\infty$ and $\widehat{\mathcal{M}}_\infty$ (resp. $\widehat{\mathrm{LT}}_{\infty, \check{C}}$ and $\widehat{\mathcal{M}}_{\infty, \check{C}}$, etc.) by the action of ϖ seen as an element of the center of G (or, what amounts to the same, of \check{G}). We have duality isomorphisms

$$\widehat{\mathrm{LT}}_{\infty, \check{C}}^\varpi \simeq \widehat{\mathcal{M}}_{\infty, \check{C}}^\varpi, \quad \widehat{\mathrm{LT}}_\infty^\varpi \simeq \widehat{\mathcal{M}}_\infty^\varpi.$$

Set

$$G_n := \begin{cases} \mathrm{GL}_{d+1}(\mathcal{O}_K) & \text{if } n = 0, \\ 1 + \varpi^n \mathrm{M}_{d+1}(\mathcal{O}_K) & \text{if } n \geq 1, \end{cases} \quad \check{G}_n := \begin{cases} \mathcal{O}_D^\times & \text{if } n = 0, \\ 1 + \varpi_D^n \mathcal{O}_D & \text{if } n \geq 1. \end{cases}$$

Let $n, m \geq 0$. We have the following diagram of dual towers

$$\begin{array}{ccc} & & G_n \times \check{G}_m \\ & & \curvearrowright \\ & T & \\ \swarrow \pi & & \searrow \tilde{\pi} \\ G_n \curvearrowright \mathcal{M}_m^\varpi & & \mathrm{LT}_n^\varpi \curvearrowright \check{G}_m \end{array}$$

where we wrote $T = \widehat{\mathrm{LT}}_{\infty, \check{C}}^\varpi \simeq \widehat{\mathcal{M}}_{\infty, \check{C}}^\varpi$.

7.1.3. *Admissibility.* We pass now to the flip-flopping. We start with some preparation. We have the following finiteness property of compactly supported de Rham and Hyodo-Kato cohomologies.

Theorem 7.1. *Let $i \geq 0$. The G -representations $H_{\mathrm{dR}, c}^i(\mathcal{M}_{\infty, \check{C}}^\varpi)$ and $H_{\mathrm{HK}, c}^i(\mathcal{M}_{\infty, \check{C}}^\varpi)$ over \check{C} are admissible.*

Proof. Let $X_\infty := \mathcal{M}_{\infty, \check{C}}^\varpi, \check{X}_\infty := \mathrm{LT}_{\infty, \check{C}}^\varpi$. We write $\mathfrak{g}_K, \check{\mathfrak{g}}_K$ for the K -Lie algebras of G and \check{G} , respectively; $\mathfrak{g}, \check{\mathfrak{g}}$ for their \mathbf{Q}_p -versions. We claim that, for $i \geq 0$, we have

$$(7.2) \quad \dim_{\check{C}} H^i(\mathfrak{g}, \check{C}) = \dim_{\check{C}} H^i(\check{\mathfrak{g}}, \check{C})$$

and, moreover, that the actions of G, \check{G} on the respective Lie algebra cohomologies in (7.2) are trivial. The first claim follows from the fact that we have an isomorphism of Lie algebras

$$\mathfrak{g}_K \otimes_K L \xrightarrow{\sim} \mathfrak{g}_L \simeq \check{\mathfrak{g}}_L \xleftarrow{\sim} \check{\mathfrak{g}}_K \otimes_K L,$$

for a finite field extension L/K , and it suffices to show (7.2) after replacing \check{C} by C . For the second claim we have the following result:

Lemma 7.3. *Let \mathbb{G} be a connected algebraic group over \mathbf{Q}_p and let $\mathfrak{g} = \mathrm{Lie}(\mathbb{G}(\mathbf{Q}_p))$. Then $\mathbb{G}(\mathbf{Q}_p)$ acts trivially on $H^*(\mathfrak{g}, \mathbf{Q}_p)$.*

Proof. Note that $H^*(\mathfrak{g}, \mathbf{Q}_p)$ is finite dimensional and smooth, so fixed by an open subgroup H of $\mathbb{G}(\mathbf{Q}_p)$. Now, H is Zariski dense in \mathbb{G} , since it is open in $\mathbb{G}(\mathbf{Q}_p)$ and $\mathbb{G}(\mathbf{Q}_p)$ is Zariski dense in \mathbb{G} (see [6, Cor. 18.3]). But the action of \mathbb{G} on $H^*(\mathfrak{g}, \mathbf{Q}_p)$ is clearly algebraic (use the explicit Chevalley-Eilenberg complex and the fact that \mathbb{G} acts algebraically on its Lie algebra), so $\mathbb{G}(\mathbf{Q}_p)$ acts trivially on $H^*(\mathfrak{g}, \mathbf{Q}_p)$, as wanted. \square

The above implies that we can use Theorem 6.23 with $H = G_0$, $\check{H} = \check{G}_0$. We get a $G_0 \times \check{G}_0$ -equivariant isomorphism in \check{C}_\square

$$H_{\mathrm{dR},c}^i(X_\infty) \simeq H_{\mathrm{dR},c}^i(\check{X}_\infty).$$

Hence isomorphisms in \check{C}_\square

$$H_{\mathrm{dR},c}^i(X_\infty)^{G_n} \simeq H_{\mathrm{dR},c}^i(\check{X}_\infty)^{G_n} \simeq H_{\mathrm{dR},c}^i(\mathrm{LT}_n^\varpi).$$

Our theorem now follows from the following lemma:

Lemma 7.4. *The cohomology group $H_{\mathrm{dR},c}^i(\mathrm{LT}_n^\varpi)$ is of finite rank over \check{C} .*

Proof. Set $\check{X} := \mathrm{LT}_n^\varpi$. Since \check{X} is Stein, by Poincaré duality (see [9]) it suffices to show that $H_{\mathrm{dR}}^i(\check{X})$ has finite rank. We recall that, by [24, III.4.1], there exists a Shimura variety¹⁹ over \check{C} with a proper flat integral model \mathcal{X} over $\mathcal{O}_{\check{C}}$ such that the tube U in the rigid analytic variety $\widehat{\mathcal{X}}_{\check{C}}$ of any supersingular closed point Z is isomorphic to \check{X} . We note that tubes of locally closed subsets of \mathcal{X}_0 are admissible open of $\widehat{\mathcal{X}}_{\check{C}}$. (For more details concerning tubes see [4, Sec. 5.1.], [26, Sec. 2.1.2].) We want to show that the de Rham cohomology of the tube U is finite dimensional. But $U = \widehat{\mathcal{X}}_{\check{C}} \setminus V$, where V is the tube of $\mathcal{X}_0 \setminus Z$. Since $\mathcal{X}_0 \setminus Z$ is quasi-compact open, the tube V is quasi-compact. Hence we can use [22, Th. 3.6] to finish the argument. \square

Remark 7.5. The above lemma is well-known though we did not find a reference in the literature. The presented argument is a version of an argument communicated to us by Laurent Fargues.

Concerning Hyodo-Kato cohomology, we have a Hyodo-Kato isomorphism in \check{C}_\square

$$\iota_{\mathrm{HK}} : H_{\mathrm{HK},c}^i(\mathcal{M}_\infty^\varpi) \otimes_{\check{C}}^{\mathrm{L}\square} C \xrightarrow{\sim} H_{\mathrm{dR},c}^i(\mathcal{M}_\infty^\varpi).$$

It is G_0 -equivariant. Moreover, we have a G_0 -equivariant isomorphism in \check{C}_\square :

$$H_{\mathrm{HK},c}^i(\mathcal{M}_\infty^\varpi) \otimes_{\check{C}}^{\mathrm{L}\square} C \simeq H_{\mathrm{HK},c}^i(\mathcal{M}_\infty^\varpi) \otimes_{\check{C}}^{\mathrm{L}\square} (\check{C} \oplus W) \simeq H_{\mathrm{HK},c}^i(\mathcal{M}_\infty^\varpi) \oplus H_{\mathrm{HK},c}^i(\mathcal{M}_\infty^\varpi) \otimes_{\check{C}}^{\mathrm{L}\square} W,$$

for a \check{C} -Banach subspace W of C . The wanted admissibility of $H_{\mathrm{HK},c}^i(\mathcal{M}_\infty^\varpi)$ follows now from the one of $H_{\mathrm{dR},c}^i(\mathcal{M}_\infty^\varpi)$ proved above. \square

7.1.4. *Flip-flopping.* We are now ready to prove the main flip-flopping result of this section.

Theorem 7.6. *Let $i \geq 0$. There are $G \times \check{G}$ -equivariant isomorphisms in \check{C}_\square and $(\varphi, N, \mathcal{G}_{\check{C}})$ -modules over \check{C} , respectively:*

$$(7.7) \quad H_{\mathrm{dR},c}^i(\mathcal{M}_{\infty,\check{C}}^\varpi) \simeq H_{\mathrm{dR},c}^i(\mathrm{LT}_{\infty,\check{C}}^\varpi), \quad H_{\mathrm{HK},c}^i(\mathcal{M}_{\infty,C}^\varpi) \simeq H_{\mathrm{HK},c}^i(\mathrm{LT}_{\infty,C}^\varpi).$$

Proof. Let $X_\infty := \mathcal{M}_{\infty,\check{C}}^\varpi$, $\check{X}_\infty := \mathrm{LT}_{\infty,\check{C}}^\varpi$.

(1) *De Rham cohomology.* We will first deal with de Rham cohomology by passing to cohomology groups in (6.11). For the inductive step, we assume the isomorphism (7.7) for $\leq i$ and we want to prove it for $i+1$. We write \mathfrak{g} , $\check{\mathfrak{g}}$ for the \mathbf{Q}_p -Lie algebras of G and \check{G} , respectively. From (6.11) we get an isomorphism in $\mathcal{D}(\check{C}_\square)$

$$(7.8) \quad H^{i+1}(\mathrm{R}\Gamma(\check{\mathfrak{g}}, \check{C}) \otimes_{\check{C}}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_\infty)) \simeq H^{i+1}(\mathrm{R}\Gamma(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{dR},c}(\check{X}_\infty)).$$

A priori, this isomorphism is just $\check{C}[G_0 \times \check{G}_0]$ -equivariant but we have:

Lemma 7.9. *The isomorphism (7.8) is $\check{C}[G \times \check{G}]$ -equivariant.*

¹⁹The relevant Shimura variety is described in [24]; see also [39, 2.4].

Proof. To see this recall that the isomorphism (7.8) is obtained from a $G_0 \times \check{G}_0$ -equivariant quasi-isomorphism in $\mathcal{D}(\check{C}_\square)$

$$(7.10) \quad \text{colim}_{\check{H}} \text{colim}_H \text{R}\Gamma(\check{H}, \text{R}\Gamma_{\text{dR},c}(X_H)) \simeq \text{colim}_H \text{colim}_{\check{H}} \text{R}\Gamma(H, \text{R}\Gamma_{\text{dR},c}(X_{\check{H}})),$$

where H, \check{H} run through the normal compact open subgroups of G_0 and \check{G}_0 , respectively. This quasi-isomorphism is induced by the natural quasi-isomorphisms $\text{R}\Gamma(\check{H}, \text{R}\Gamma_{\text{dR},c}(X_H)) \simeq \text{R}\Gamma(H, \text{R}\Gamma_{\text{dR},c}(X_{\check{H}}))$ from (6.7).

Let $g \in G$ and choose H small enough so that $g^{-1}Hg \subset G_0$. We claim that, for every $\check{H} \subset \check{G}_0$ as in (7.10), we have a commutative diagram

$$(7.11) \quad \begin{array}{ccc} \text{R}\Gamma(g^{-1}Hg, \text{R}\Gamma_{\text{dR},c}(X_{\check{H}})) & \xrightarrow{g^*} & \text{R}\Gamma(H, \text{R}\Gamma_{\text{dR},c}(X_{\check{H}})) \\ \alpha_1 \downarrow \wr & & \alpha_2 \downarrow \wr \\ \text{R}\Gamma(\check{H}, \text{R}\Gamma_{\text{dR},c}(X_{g^{-1}Hg})) & \xrightarrow{g^*} & \text{R}\Gamma(\check{H}, \text{R}\Gamma_{\text{dR},c}(X_{\check{H}})), \end{array}$$

where the quasi-isomorphisms α_1, α_2 come from Corollary 6.5. But this follows from the naturality of the maps α_1, α_2 with respect to the map g acting on the two torsors data:

Now, taking colimit of the diagram (7.11) over H and \check{H} we get an action of g on the isomorphism (7.8), as wanted. A similar argument gives us an action of \check{G} . \square

By Künneth formula, the isomorphism (7.8) yields a $\check{C}[G \times \check{G}]$ -equivariant isomorphism in $\mathcal{D}(\check{C}_\square)$

$$(7.12) \quad \begin{aligned} & H^0(\check{\mathfrak{g}}, \check{C}) \otimes_{\check{C}}^{\square} H_{\text{dR},c}^{i+1}(X_\infty) \oplus H^1(\check{\mathfrak{g}}, \check{C}) \otimes_{\check{C}}^{\square} H_{\text{dR},c}^i(X_\infty) \oplus \cdots \oplus H^{i+1}(\check{\mathfrak{g}}, \check{C}) \otimes_{\check{C}}^{\square} H_{\text{dR},c}^0(X_\infty) \\ & \simeq H^0(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\text{dR},c}^{i+1}(\check{X}_\infty) \oplus H^1(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\text{dR},c}^i(\check{X}_\infty) \oplus \cdots \oplus H^{i+1}(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\text{dR},c}^0(\check{X}_\infty) \end{aligned}$$

Since the groups $H^j(\check{\mathfrak{g}}, \check{C})$, $H^j(\mathfrak{g}, \check{C})$, for $j \geq 0$, are of finite rank over \check{C} , $H^0(\check{\mathfrak{g}}, \check{C}) = H^0(\mathfrak{g}, \check{C}) = \check{C}$, and we have the two properties of Lie algebra cohomologies shown in the proof of Theorem 7.1, if we want to prove the claim of the theorem for $i+1$, by the inductive assumption, we find ourselves in the situation, where we have an isomorphism of admissible representations of $G \times \check{G}$ over \check{C} (by Theorem 7.1):

$$\pi_1 \oplus \pi \simeq \pi_2 \oplus \pi$$

and we want to conclude that $\pi_1 \simeq \pi_2$, as $G \times \check{G}$ -representations. But this follows from Proposition 5.4.

(2) *Hyodo-Kato cohomology.* We will now pass to Hyodo-Kato cohomology. We will argue by induction on i as in the de Rham case starting with $i=0$, where the claim of the theorem is clear by passing to cohomology in (6.22). For the inductive step, we assume the isomorphism (7.7) for $\leq i$ and we want to prove it for $i+1$.

From (6.22) we get an isomorphism in $\mathcal{D}_{\varphi, N, \mathcal{G}_K}(\check{C}_\square)$

$$H^{i+1}(\text{R}\Gamma(\check{\mathfrak{g}}, \check{C}) \otimes_{\check{C}}^{\text{L}\square} \text{R}\Gamma_{\text{HK},c}(X_{\infty, C})) \simeq H^{i+1}(\text{R}\Gamma(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\text{L}\square} \text{R}\Gamma_{\text{HK},c}(\check{X}_{\infty, C})).$$

A priori, this isomorphism is just $\check{C}[G_0 \times \check{G}_0]$ -equivariant but arguing as in the de Rham case we see that it is in fact $\check{C}[G \times \check{G}]$ -equivariant. Using the Künneth formula, we get a $G \times \check{G}$ -equivariant isomorphism

in $\mathcal{D}_{\varphi, N, \mathcal{G}_K}(\check{C}\square)$

$$\begin{aligned} & H^0(\check{\mathfrak{g}}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK}, c}^{i+1}(X_{\infty, C}) \oplus H^1(\check{\mathfrak{g}}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK}, c}^i(X_{\infty, C}) \oplus \cdots \oplus H^{i+1}(\check{\mathfrak{g}}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK}, c}^0(X_{\infty, C}) \\ & \simeq H^0(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK}, c}^{i+1}(\check{X}_{\infty, C}) \oplus H^1(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK}, c}^i(\check{X}_{\infty, C}) \oplus \cdots \oplus H^{i+1}(\mathfrak{g}, \check{C}) \otimes_{\check{C}}^{\square} H_{\mathrm{HK}, c}^0(\check{X}_{\infty, C}) \end{aligned}$$

Since the groups $H^j(\check{\mathfrak{g}}, \check{C})$, $H^j(\mathfrak{g}, \check{C})$, for $j \geq 0$ are of finite rank over \check{C} , $H^0(\check{\mathfrak{g}}, \check{C}) = H^0(\mathfrak{g}, \check{C}) = \check{C}$, and we have the two properties of Lie algebra cohomologies shown in the proof of Theorem 7.1, if we want to prove the claim of the theorem for $i + 1$, by the inductive assumption, we find ourselves in the situation, where we have an isomorphism of admissible representations of $G \times \check{G}$ on $(\varphi, N, \mathcal{G}_{\check{C}})$ -modules over \check{C} (by Theorem 7.1):

$$\pi_1 \oplus \pi \simeq \pi_2 \oplus \pi$$

and we want to conclude that $\pi_1 \simeq \pi_2$, as $G \times \check{G}$ -representations on $(\varphi, N, \mathcal{G}_{\check{C}})$ -modules over \check{C} . But this follows from the representation theoretical Proposition 5.4 \square

The following fact is an immediate corollary of Theorem 7.6 (obtained by taking $G_n \times \check{G}_m$ -invariants of the isomorphisms (7.7)):

Corollary 7.13. *For any $m, n \geq 0$, we have isomorphisms in $\check{C}\square$ and solid $(\varphi, N, \mathcal{G}_{\check{C}})$ -modules over \check{C}*

$$H_{\mathrm{dR}, c}^i(\mathcal{M}_m^{\varpi})^{G_n} \simeq H_{\mathrm{dR}, c}^i(\mathrm{LT}_n^{\varpi})^{\check{G}_m}, \quad H_{\mathrm{HK}, c}^i(\mathcal{M}_m^{\varpi})^{G_n} \simeq H_{\mathrm{HK}, c}^i(\mathrm{LT}_n^{\varpi})^{\check{G}_m},$$

7.2. Flip-flopping for dual towers local Shimura varieties. The above computations for the Drinfeld and Lubin-Tate towers generalize easily to all basic local Shimura varieties.

7.2.1. Local Shimura varieties. We briefly review here a few facts about dual towers of (basic) local Shimura varieties. See [18], [27], [34], [20] for details. Let $(G, [b], \{\mu\})$ be a local Shimura datum over \mathbf{Q}_p with a basic b : G is a connected reductive group over \mathbf{Q}_p , $\mu : \mathbb{G}_{m, \overline{\mathbf{Q}}_p} \rightarrow G_{\overline{\mathbf{Q}}_p}$ is a (conjugacy class of a) minuscule cocharacter, and $b \in B(G, \mu)$ is the unique basic neutral acceptable element. Let E be the field of definition of μ and let $E = E(G, \{\mu\})$ be the reflex field. Let \check{E} be the completion of its maximal unramified extension.

Let $K_0 \subset G(\mathbf{Q}_p)$ be a maximal compact subgroup and let $K \triangleleft K_0$ be an open normal subgroup. Then the finite-level local Shimura variety $\mathrm{Sh}_K(G, b, \mu)$ is a smooth rigid-analytic space (partially proper) over \check{E} , equipped with natural actions of

- (1) K_0/K via Hecke correspondences,
- (2) the group $\check{G} := J_b(\mathbf{Q}_p)$ via quasi-isogenies, and
- (3) $\mathcal{G}_{\check{E}} := \mathrm{Gal}(\check{E}/\check{E})$.

Symmetrically, the finite-level space $\mathrm{Sh}_{\check{K}}(\check{G}, \check{b}, \check{\mu})$, for an open normal subgroup \check{K} of \check{K}_0 – a maximal compact subgroup of \check{G} – carries the action of $G(\mathbf{Q}_p)$ and of the quotient \check{K}_0/\check{K} .

The collection of all levels forms the tower

$$\mathrm{Sh}_{\bullet}(G, b, \mu) := \{\mathrm{Sh}_K(G, b, \mu)\}_K,$$

with commuting actions of $G(\mathbf{Q}_p)$, $\check{G}(\mathbf{Q}_p)$, and $\mathcal{G}_{\check{E}}$. Its completed limit

$$\widehat{\mathrm{Sh}}(G, b, \mu) := \varprojlim_K \mathrm{Sh}_K(G, b, \mu)$$

makes sense as a diamond over $\mathrm{Spd}(\check{E})$, endowed with with continuous actions of the groups $G(\mathbf{Q}_p)$, $\check{G}(\mathbf{Q}_p)$, and $\mathcal{G}_{\check{E}}$. The dual completed space $\widehat{\mathrm{Sh}}(\check{G}, \check{b}, \check{\mu})$ satisfies the same properties.

By the duality theorem, there is a canonical isomorphism of diamonds

$$\widehat{\mathrm{Sh}}(G, b, \mu) \xrightarrow{\sim} \widehat{\mathrm{Sh}}(\check{G}, \check{b}, \check{\mu})$$

which is equivariant with respect to the actions of $G(\mathbf{Q}_p)$, $\check{G}(\mathbf{Q}_p)$, and $\mathcal{G}_{\check{E}}$.

7.2.2. *Flip-flopping.* The main theorem of this section is the following flip-flopping result:

Theorem 7.14. (Local Shimura varieties flip-flopping) *There are $G \times \check{G}$ -equivariant isomorphisms in solid C -modules and solid $(\varphi, \mathcal{G}_{\check{C}})$ -modules²⁰ over \check{C} , respectively*

$$H_{\mathrm{dR},c}^i(\mathrm{Sh}(\check{G}, \check{b}, \check{\mu})_{\infty}) \simeq H_{\mathrm{dR},c}^i(\mathrm{Sh}(G, b, \mu)_{\infty}), \quad H_{\mathrm{HK},c}^i(\mathrm{Sh}(\check{G}, \check{b}, \check{\mu})_{\infty}) \simeq H_{\mathrm{HK},c}^i(\mathrm{Sh}(G, b, \mu)_{\infty}).$$

If these representations of $G \times \check{G}$ are admissible the Hyodo-Kato isomorphism lies in the category of solid $(\varphi, N, \mathcal{G}_{\check{C}})$ -modules.

Proof. Let K_0, \check{K}_0 be maximal compact subgroups of G, \check{G} and let $K \subset K_0, \check{K} \subset \check{K}_0$ be normal open subgroups. We have the following diagram of dual towers

$$\begin{array}{ccc} & \begin{array}{c} \curvearrowright^{K \times \check{K}} \\ \widehat{\mathrm{Sh}}(\check{G}, \check{b}, \check{\mu}) \end{array} & \xrightarrow{\sim} & \begin{array}{c} \curvearrowright^{K \times \check{K}} \\ \widehat{\mathrm{Sh}}(G, b, \mu) \end{array} & \\ & \swarrow^{\pi} & & \searrow_{\tilde{\pi}} & \\ \begin{array}{c} \curvearrowright^K \\ \mathrm{Sh}_{\check{K}}(\check{G}, \check{b}, \check{\mu}) \end{array} & & & & \begin{array}{c} \mathrm{Sh}_K(G, b, \mu) \\ \curvearrowright^{\check{K}} \end{array} \end{array}$$

Starting with this diagram we can run the same argument as in the proof of Theorem 7.1, the Drinfeld-Lubin-Tate case (note that the groups G and \check{G} are inner forms since b is basic) to prove our theorem. \square

8. CONFLICT OF INTEREST STATEMENT

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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²⁰We write \check{C} for the fraction field of the Witt vectors $W(\overline{\mathbb{F}}_p)$.

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