

K-THEORY OF LOG-SCHEMES II: LOG-SYNTOMIC K-THEORY

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ABSTRACT. We prove that suitably truncated topological log-syntomic-étale homotopy K -theory of proper semistable schemes in mixed characteristic surjects onto the Kummer log-étale p -adic K -theory of the generic fiber. As a corollary we get that the log-syntomic-étale cohomology is a direct factor of the log-syntomic-étale cohomology of homotopy K -theory sheaves. The proofs use p -adic Hodge theory computations of p -adic nearby cycles.

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1. INTRODUCTION

We understand reasonably well now the relationship between l -adic K -theory and l -adic étale cohomology. We ask in this paper whether there exists (and, if so, what form it takes) a similar relationship between p -adic K -theory and systemic cohomology.

In more concrete terms, the paper is a sequel to [18]. It addresses two (related) questions in p -adic K -theory of log-schemes. The first one concerns the p -adic version of Theorem 5.14 from [18]. This theorem states that, for a log-regular, regular log-scheme X satisfying a mild condition and a prime l invertible on X , the l -adic Kummer log-étale K -theory of X computes the l -adic étale K -theory of the largest open set of X on which the log-structure is trivial. More precisely, we have the following theorem.

Theorem 1.1. ([18, Theorem 5.14]) *Let X be a log-regular, regular scheme satisfying condition (*) stated in Section 3 below. Let l be a prime invertible on X . Then the open immersion $j : U \hookrightarrow X$, where $U = X_{tr}$ is the maximal open set of X on which the log-structure is trivial, induces an isomorphism*

$$j^* : K_m(X_{\text{két}}, \mathbf{Z}/l^n) \xrightarrow{\sim} K_m(U_{\text{ét}}, \mathbf{Z}/l^n), \quad m \geq 0.$$

Here $K_m(X_{\text{két}}, \mathbf{Z}/l^n)$, resp. $K_m(U_{\text{ét}}, \mathbf{Z}/l^n)$, denote K -theory mod l^n fibrant in the Kummer log-étale, resp. étale, topology.

We have attempted to find the analog of the above theorem for p -adic K -theory of log-schemes in mixed characteristic $(0, p)$. The main result of this paper is not as satisfying as in the l -adic case but suggests that perhaps we are on the right track. A natural analog of Kummer log-étale K -theory to consider is Kummer log-syntomic K -theory. That turned out to be a very difficult cohomology to work with and we were not able to get a handle on it. However when we replace K -theory with homotopy K -theory

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we are able to show that for proper semistable schemes the p -adic étale K -theory of the largest open set on which the log-structure is trivial is a natural direct factor of the p -adic log-syntomic-étale homotopy K -theory of X itself. The proof uses p -adic Hodge theory, more precisely, the Fontaine-Messing [2], [26] construction of the p -adic period morphism and Kato and Tsuji computations of the p -adic nearby cycles [9], [27]. The key property that makes homotopy K -theory work as opposed to K -theory itself is the excision for finite morphisms.

To state our main result, let V be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p . Let \overline{V} denote the integral closure of V in \overline{K} and let $\zeta_n \in \overline{V}$ denote a primitive p^n 'th root of unity.

Theorem 1.2. *Let X be a proper semistable scheme over V equipped with its natural log-structure. Assume that $p^n > 2$, $\zeta_n \in V$. Then, for any $m \geq \max(2d, 1)$, where $d = \dim X_K$, and for any $i \geq 2d + m/2$, the cokernel of the restriction map*

$$j^* : KH_m^{\leq i}(X_{\text{sét}}, \mathbf{Z}/p^n) \rightarrow K_m^{\leq i}(U_{\text{ét}}, \mathbf{Z}/p^n)$$

is annihilated by a constant $T(p, d, i, m)$ depending only on p , d , m , and i .

Here $KH_m(X_{\text{sét}}, \mathbf{Z}/p^n)$ denotes the homotopy K -theory mod p^n fibrant in the log-syntomic-étale topology of X . The superscripts i refer to the truncations of K -theory spaces that we take to be able to control the cohomological dimension of the log-syntomic-étale site.

Passing to sheaves of homotopy groups we get

Corollary 1.3. *Let X be a proper semistable scheme over V equipped with its natural log-structure. Then, for any $0 \leq i \leq r$, the surjection*

$$j^* : \text{proj} \lim_n H^i(X_{\overline{V}, \text{sét}}, \tilde{\pi}_{2r}(KH/p^n)) \otimes \mathbf{Q} \rightarrow H^i(U_{\overline{K}, \text{ét}}, \mathbf{Q}_p(r)), \quad r \geq 1,$$

has a natural section.

The second question we address concerns the identification of the p -adic K -theory sheaves for the log-syntomic-étale topology of log-schemes in mixed characteristic. Recall that the mod p^n K -theory sheaves for the étale topology of smooth schemes over local fields of mixed characteristic yield Tate twists. That is, the sheaf $\tilde{\pi}_{2r}(K/p^n)$ is $\mathbf{Z}/p^n(r)$ for positive r and trivial otherwise [24, Theorem 3.1]. If we do the same for the Zariski topology on a smooth scheme over a perfect field of characteristic p we get that $\tilde{\pi}_{2r}(K/p^n) \simeq \nu_n^r$ [3, Theorem 8.3], where ν_n^r is the logarithmic de Rham Witt sheaf of Milne and Illusie [5].

Similarly, in [18, Prop. 5.12], we have computed that the sheaves $\tilde{\pi}_{2r}(K/p^n)$ of homotopy groups of mod p^n K -theory presheaves on the Kummer étale site of a smooth scheme over a local field of mixed characteristic are isomorphic to $\mathbf{Z}/p^n(r)$ for r positive and trivial otherwise. It is natural then to ask whether the mod p^n K -theory sheaves for the log-syntomic-étale topology yield coefficient sheaves of arithmetic interest. In particular, whether they relate to the log-syntomic-étale sheaves. The methods we described above allow us to show that the log-syntomic-étale cohomology of the log-syntomic sheaves is a direct factor of the log-syntomic-étale cohomology of the homotopy K -theory sheaves. More precisely we have the following

Corollary 1.4. *Assume that $\zeta_n \in V$. Let X be a proper semistable scheme over V^\times . Then, for any $0 \leq i \leq r - 2 \leq p - 2$, there exists a natural map*

$$\gamma_r : H^i(X_{\text{sét}}, \mathcal{S}_n(r)) \rightarrow H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n)),$$

where $H^i(X_{\text{sét}}, \mathcal{S}_n(r))$ denotes the log-syntomic-étale cohomology. This map is injective and has a natural section.

Throughout the paper, let p be a fixed prime, let \overline{K} denote a chosen algebraic closure of a field K , and, for a scheme X , let $X_n = X \otimes \mathbf{Z}/p^n$. All the log-schemes are assumed to be fine and saturated unless otherwise stated.

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2. LOG-SYNTOMIC COHOMOLOGY

We collect in this section the necessary facts about the syntomic construction of the p -adic period morphism and the log-syntomic-étale cohomology of degree zero nearby cycles.

2.1. Log-syntomic period morphisms. Let V be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p . Let \overline{V} denote the integral closure of V in \overline{K} . Let $W(k)$ be the ring of Witt vectors of k with fraction field K_0 . Set $G_K = \text{Gal}(\overline{K}/K)$, and let σ be the absolute Frobenius on $W(\overline{k})$. For a V -scheme X , let X_0 denote the special fiber of X . We will denote by V , V^\times , and V^0 the scheme $\text{Spec}(V)$ with the trivial, canonical (i.e., associated to the closed point), and $(\mathbf{N} \rightarrow V, 1 \mapsto 0)$ log-structure respectively. Otherwise we keep the notation and terminology from [18].

For now we will be working with fine log-schemes [8]. For a log-scheme X we denote by X_{syn} the small log-syntomic site of X defined as follows. The objects are morphisms $f : Y \rightarrow Z$ that are log-syntomic in the sense of Kato [9, 2.5] (see also [1, 6.1]), i.e., the morphism f is integral, the underlying morphism of schemes is flat and locally of finite presentation, and, étale locally on Y , there is a factorization $Y \xrightarrow{i} W \xrightarrow{h} Z$ where h is log-smooth and i is an exact closed immersion that is transversally regular over Z . We also require f to be locally quasi-finite on the underlying schemes and the cokernel of the map $(f^*M_Z)^{\text{gp}} \rightarrow M_Y^{\text{gp}}$ to be torsion. We will call such morphisms *small log-syntomic*.

For a log-scheme X log-syntomic over $\text{Spec}(W(k))$, define

$$\mathcal{O}_n^{\text{cr}}(X) = H_{\text{cr}}^0(X_n/W_n(k), \mathcal{O}_{X_n/W_n(k)}), \quad \mathcal{J}_n^{[r]}(X) = H_{\text{cr}}^0(X_n/W_n(k), \mathcal{J}_{X_n/W_n(k)}^{[r]}),$$

where $\mathcal{O}_{X_n/W_n(k)}$ is the structure sheaf of the log-crystalline site, $\mathcal{J}_{X_n/W_n(k)} = \text{Ker}(\mathcal{O}_{X_n/W_n(k)} \rightarrow \mathcal{O}_{X_n})$, and $\mathcal{J}_{X_n/W_n(k)}^{[r]}$ is its r 'th divided power of $\mathcal{J}_{X_n/W_n(k)}$. Set $\mathcal{J}_{X_n/W_n(k)}^{[r]} = \mathcal{O}_{X_n/W_n(k)}$ if $r \leq 0$. We know [2, II.1.3] that the presheaves $\mathcal{J}_n^{[r]}$ are sheaves on $X_{n,\text{syn}}$, flat over \mathbf{Z}/p^n , and that $\mathcal{J}_{n+1}^{[r]} \otimes \mathbf{Z}/p^n \simeq \mathcal{J}_n^{[r]}$. There is a canonical isomorphism

$$H^*(X_{\text{syn}}, \mathcal{J}_n^{[r]}) \simeq H_{\text{cr}}^*(X_n/W_n(k), \mathcal{J}_{X_n/W_n(k)}^{[r]})$$

that is compatible with Frobenius. It is easy to see that $\phi(\mathcal{J}_n^{[r]}) \subset p^r \mathcal{O}_n^{\text{cr}}$ for $0 \leq r \leq p-1$. This fails in general and we modify $\mathcal{J}_n^{[r]}$:

$$\mathcal{J}_n^{<r>} := \{x \in \mathcal{J}_{n+s}^{[r]} \mid \phi(x) \in p^r \mathcal{O}_{n+s}^{\text{cr}}\} / p^n,$$

for some $s \geq r$. This definition is independent of s . We check that $\mathcal{J}_n^{<r>}$ is flat over \mathbf{Z}/p^n and $\mathcal{J}_{n+1}^{<r>} \otimes \mathbf{Z}/p^n \simeq \mathcal{J}_n^{<r>}$. This allows us to define the divided Frobenius $\phi_r = \text{"}\phi/p^r\text{"} : \mathcal{J}_n^{<r>} \rightarrow \mathcal{O}_n^{\text{cr}}$. Set

$$\mathcal{S}_n(r) := \text{Ker}(\mathcal{J}_n^{<r>} \xrightarrow{\phi_r} \mathcal{O}_n^{\text{cr}}).$$

In the same way we can define log-syntomic sheaves $\mathcal{S}_n(r)$ on $X_{m,\text{syn}}$ for $m \geq n$. Abusing notation, we have $\mathcal{S}_n(r) = i_* \mathcal{S}_n(r)$ for the natural map $i : X_{m,\text{syn}} \rightarrow X_{\text{syn}}$. Since i_* is exact, $H^*(X_{m,\text{syn}}, \mathcal{S}_n(r)) = H^*(X_{\text{syn}}, \mathcal{S}_n(r))$. Because of that we will write $\mathcal{S}_n(r)$ for the log-syntomic sheaves on $X_{m,\text{syn}}$ as well as on X_{syn} . We will also need the "undivided" version of log-syntomic sheaves:

$$\mathcal{S}'_n(r) := \text{Ker}(\mathcal{J}_n^{[r]} \xrightarrow{\phi_r} \mathcal{O}_n^{\text{cr}}).$$

There is a natural map $\mathcal{S}'_n(r) \rightarrow \mathcal{S}_n(r)$ whose kernel and cokernel are killed by p^r . If it does not cause confusion, we will also write $\mathcal{S}_n(r)$, $\mathcal{S}'_n(r)$ for $R\varepsilon_* \mathcal{S}_n(r)$, $R\varepsilon_* \mathcal{S}'_n(r)$, respectively, where $\varepsilon : X_{n,\text{syn}} \rightarrow X_{n,\text{ét}}$ is the canonical projection.

Proposition 2.1. ([2, III.1.1]) *The following sequence is exact for $r \geq 0$*

$$0 \longrightarrow \mathcal{S}_n(r) \longrightarrow \mathcal{J}_n^{<r>} \xrightarrow{\phi_r} \mathcal{O}_n^{\text{cr}} \longrightarrow 0.$$

Let X be a log-smooth scheme over V . For $0 \leq r \leq p-2$, there is a natural homomorphism on the étale site of X_n

$$(2.1) \quad \alpha_r : R\varepsilon_* \mathcal{S}_n(r) \rightarrow \tau_{\leq r} i^* Rj_* \mathbf{Z}/p^n(r),$$

for the natural maps $i : X_{n,\text{ét}} \rightarrow X_{\text{ét}}$ and $j : X_{\text{tr},\text{ét}} \rightarrow X_{\text{ét}}$. Here X_{tr} is the locus of X_K where the log-structure is trivial. Recall how this map is defined (see [2], [11] for details). We need the logarithmic version of the syntomic-étale site of Fontaine-Messing [2]. We say that a morphism $\mathcal{Z} \rightarrow \mathcal{Y}$ of p -adic formal log-schemes over $\text{Spf}(W(k))$ is (small) log-syntomic if every $Z_n \rightarrow Y_n$ is (small) log-syntomic. For a formal log-scheme \mathcal{Z} the log-syntomic-étale site $\mathcal{Z}_{\text{sé}}$ is defined by taking as objects morphisms $f : \mathcal{Y} \rightarrow \mathcal{Z}$ that are small log-syntomic and have log-étale generic fiber. This last condition means that, étale locally on \mathcal{Y} , f has a factorization $\mathcal{Y} \xrightarrow{i} \mathcal{X} \xrightarrow{g} \mathcal{Z}$ with \mathcal{X} affine, i an exact closed immersion, and g log-smooth such that the map $K_0 \otimes_{W(k)} \Gamma(\mathcal{Y}, I/I^2) \rightarrow K_0 \otimes_{W(k)} \Gamma(\mathcal{Y}, i^* \Omega_{\mathcal{X}/\mathcal{Z}}^1)$ is an isomorphism, where I is the ideal of $\mathcal{O}_{\mathcal{X}}$ defining \mathcal{Y} . For a log-scheme Z , we also have the log-syntomic-étale site $Z_{\text{sé}}$. Here the objects are morphisms $U \rightarrow Z$ that are small log-syntomic with the generic fiber U_K log-étale over Z_K . Morphisms are given by all maps that are compatible with the maps to Z .

Let \widehat{X} be the p -adic completion of X . We have the following commutative diagram of topoi (here, and in what follows, we will denote in the same way sites and their associated topoi)

$$\begin{array}{ccccc} \widehat{X}_{\text{sé}} & \xrightarrow{i_{\text{sé}}} & X_{\text{sé}} & \xleftarrow{j_{\text{sé}}} & X_{K,\text{sé}} \\ \widehat{\varepsilon} \downarrow & & \varepsilon \downarrow & & \varepsilon_K \downarrow \\ \widehat{X}_{\text{ét}} & \xrightarrow{i_{\text{ét}}} & X_{\text{ét}} & \xleftarrow{j_{\text{ét}}} & X_{K,\text{ét}} \end{array}$$

Abusively, let $\mathcal{S}_n(r)$ denote also the direct image of $\mathcal{S}_n(r)$ under the canonical morphism $X_{n,\text{syn}} \rightarrow \widehat{X}_{\text{sé}}$. Since this morphism is exact [2, III.4.1], we have $R\widehat{\varepsilon}_* \mathcal{S}_n(r) = R\varepsilon_* \mathcal{S}_n(r)$. By [2, III.5], there is a canonical homomorphism

$$\alpha_r : \mathcal{S}_n(r) \rightarrow i_{\text{sé}}^* j_{\text{sé}}^* j'_* \mathbf{Z}/p^n(r),$$

for $j' : X_{\text{tr},\text{ét}} \rightarrow X_{K,\text{sé}}$ the map of topoi induced by the the open immersion $X_{\text{tr}} \hookrightarrow X_K$. We apply $R\widehat{\varepsilon}_*$ to the induced map $\mathcal{S}_n(r) \rightarrow i_{\text{sé}}^* Rj_{\text{sé}}^* Rj'_* \mathbf{Z}/p^n(r)$ and get

$$\varepsilon_r : R\varepsilon_* \mathcal{S}_n(r) = R\widehat{\varepsilon}_* \mathcal{S}_n(r) \rightarrow R\widehat{\varepsilon}_* i_{\text{sé}}^* Rj_{\text{sé}}^* Rj'_* \mathbf{Z}/p^n(r) = i_{\text{ét}}^* R\varepsilon_* Rj_{\text{sé}}^* Rj'_* \mathbf{Z}/p^n(r) = i^* Rj_* \mathbf{Z}/p^n(r).$$

The second equality was proved in [11, 2.5], [26, 5.2.3]. Since $R^q \varepsilon_* \mathcal{S}_n(r) = 0$ for $q > r$, the map α_r factors through $\tau_{\leq r} i^* Rj_* \mathbf{Z}/p^n(r)$. We have obtain the period morphism. One checks that so obtained period map α_r is compatible with products.

Similarly, for any $r \geq 0$, we get a natural map

$$\alpha_r : \mathcal{S}_n(r) \rightarrow i^* Rj_* \mathbf{Z}/p^n(r)',$$

where $\mathbf{Z}/p^n(r)' = (p^a a!)^{-1} \mathbf{Z}(r)/p^n$ for $r = (p-1)a + b$, $a, b \in \mathbf{Z}$, $0 \leq b < p-1$ [2, III.5]. Composing with the map $\mathcal{S}'_n(r) \rightarrow \mathcal{S}_n(r)$ we get a natural, compatible with products, morphism

$$(2.2) \quad \alpha_r : \mathcal{S}'_n(r) \rightarrow i^* Rj_* \mathbf{Z}/p^n(r)'.$$

We have the following comparison theorem proved by Tsuji [28, Theorem 5.1], [27, 3.3.4].

Theorem 2.2. (1) *Let X be a fine and saturated, log-smooth scheme over V^\times . Then for $0 \leq r \leq p-2$, the period map α_r (2.1) induces a quasi-isomorphism*

$$\alpha_r : \mathcal{S}_n(r)_{X_n} \xrightarrow{\sim} \tau_{\leq r} i^* Rj_* \mathbf{Z}/p^n(r)_{X_{\text{tr}}}.$$

(2) *Let X be semistable over V^\times or a finite base change of such, then for any $0 \leq i \leq r$, the kernel and cokernel of the period map (2.2)*

$$\alpha_r : \mathcal{H}^i(\mathcal{S}'_n(r)_{X_n}) \rightarrow i^* R^i j_* \mathbf{Z}/p^n(r)'_{X_{\text{tr}}},$$

are annihilated by p^N for an integer N which depends only on p , r , and i .

For X fine, saturated, log-smooth and proper over V^\times the above implies that

$$(2.3) \quad \alpha_{i,r} : H^i(X_{\text{ét}}, \mathcal{S}_n(r)) \xrightarrow{\sim} H^i(X_{tr,\text{ét}}, \mathbf{Z}/p^n(r)), \quad \text{for } p-2 \geq r \geq i.$$

For higher twists r and X proper and semistable over V^\times or a finite base change of such it implies that the morphism

$$(2.4) \quad \alpha_{i,r} : H^i(X_{\text{ét}}, \mathcal{S}'_n(r)) \rightarrow H^i(X_{tr,\text{ét}}, \mathbf{Z}/p^n(r)'), \quad \text{for } r \geq i,$$

has kernel and cokernel annihilated by p^M for an integer M depending only on r, i, p .

2.2. Log-syntomic-étale cohomology of degree zero nearby cycles.

Corollary 2.3. (1) *Let X be a fine and saturated, log-smooth, proper scheme over V^\times . Then for $0 \leq i \leq r \leq p-2$, the natural map*

$$H^i(X_{\text{sé}}, j_{\text{sé}*} j'_* \mathbf{Z}/p^n(r)_{X_{tr}}) \rightarrow H^i(X_{tr,\text{ét}}, \mathbf{Z}/p^n(r))$$

is surjective.

(2) *Let X be a fine and saturated, proper scheme over V^\times that is semistable over V^\times or a finite base change of such. Then for any $0 \leq i \leq r$, the cokernel of the natural map*

$$H^i(X_{\text{sé}}, j_{\text{sé}*} j'_* \mathbf{Z}/p^n(r)_{X_{tr}}) \rightarrow H^i(X_{tr,\text{ét}}, \mathbf{Z}/p^n(r))$$

is annihilated by p^N for an integer N which depends only on p, r , and i .

Proof. Recall [2, III.4.4], [26, 5.2.2] that the functor $\mathcal{F} \mapsto (i_{\text{sé}}^* \mathcal{F}, j_{\text{sé}}^* \mathcal{F}, i_{\text{sé}}^* \mathcal{F} \rightarrow i_{\text{sé}}^* j_{\text{sé}*} j_{\text{sé}}^* \mathcal{F})$ from the category of sheaves on $X_{\text{sé}}$ to the category of triples $(\mathcal{G}, \mathcal{H}, \mathcal{G} \rightarrow i_{\text{sé}}^* j_{\text{sé}*} \mathcal{H})$, where \mathcal{G} (resp. \mathcal{H}) are sheaves on $\widehat{X}_{\text{sé}}$ (resp. $X_{K,\text{sé}}$) is an equivalence of categories. It follows that we can glue $\mathcal{S}_n(r)$ and $j'_* \mathbf{Z}/p^n(r)'$ by the map α_r and obtain a sheaf $\mathcal{S}_n(r)$ on $X_{\text{sé}}$.

Let $0 \leq i \leq r \leq p-2$. In this language, our map $\alpha_r : \mathcal{S}_n(r) \rightarrow i_{\text{sé}}^* j_{\text{sé}*} j'_* \mathbf{Z}/p^n(r)$ is induced (by taking $i_{\text{sé}}^*$) from the natural map $\alpha_r : \mathcal{S}_n(r) \rightarrow j_{\text{sé}*} j'_* \mathbf{Z}/p^n(r)$. Hence the period map

$$\alpha_{i,r} : H^i(X_{\text{ét}}, \mathcal{S}_n(r)) \xrightarrow{\sim} H^i(X_{tr,\text{ét}}, \mathbf{Z}/p^n(r))$$

arises via the proper base change theorems from the map

$$H^i(X_{\text{sé}}, \mathcal{S}_n(r)) \xrightarrow{\alpha_r} H^i(X_{\text{sé}}, j_{\text{sé}*} j'_* \mathbf{Z}/p^n(r)) \rightarrow H^i(X_{tr,\text{ét}}, \mathbf{Z}/p^n(r)).$$

Since, by (2.3), this is an isomorphism we get the first part of the corollary.

For the second part it suffices to show that the induced map $H^i(X_{\text{sé}}, j_{\text{sé}*} j'_* \mathbf{Z}/p^n(r)) \rightarrow H^i(X_{tr,\text{ét}}, \mathbf{Z}/p^n(r)')$ has cokernel annihilated by p^M for an integer M dependent only on p, r , and i . Arguing as above, we get that the period map

$$\alpha_{i,r} : H^i(X_{\text{ét}}, \mathcal{S}_n(r)) \rightarrow H^i(X_{tr,\text{ét}}, \mathbf{Z}/p^n(r)')$$

arises via the proper base change theorems from the map

$$H^i(X_{\text{sé}}, \mathcal{S}_n(r)) \xrightarrow{\alpha_r} H^i(X_{\text{sé}}, j_{\text{sé}*} j'_* \mathbf{Z}/p^n(r)') \rightarrow H^i(X_{tr,\text{ét}}, \mathbf{Z}/p^n(r)').$$

Since, by (2.4), its composition with the map $H^i(X_{\text{sé}}, \mathcal{S}'_n(r)) \rightarrow H^i(X_{\text{sé}}, \mathcal{S}_n(r))$ has cokernel annihilated by p^M for an integer M dependent only on p, r , and i , we are done. \square

2.3. Log-syntomic cohomology revisited. We are now back to fine and saturated log-schemes. For a log-scheme X we denote by X_{ksyn} the Kummer log-syntomic site of X as defined in [18, 2.1]. From now on we will omit the term "Kummer" from this terminology if there is no risk of confusion. To study the relationship of log-syntomic cohomology with topological log-syntomic K -theory, we need to redefine the former using site X_{ksyn} .

First we will show that we have the expected relationship with crystalline cohomology. We will introduce (large) log-syntomic-crystalline sites. For the base we take a tuple (S, I, γ) , where S is a log-scheme such that \mathcal{O}_S is killed by a nonzero integer, I is a principal ideal on S with a PD-structure γ . Let X be a log-scheme. We define the log-syntomic-crystalline site $(X/S)_{\text{SynCr}}$ as follows. An object is a tuple (U, T, i, δ) , where U is a log-scheme over X , T is a log-scheme over S , $i : U \hookrightarrow T$ is an exact immersion, and δ is a PD-structure on the ideal of \mathcal{O}_T defining U , which is compatible with γ . Morphisms

are defined in the obvious way. Coverings are families of morphisms $\{f_\alpha : (U'_\alpha \hookrightarrow T'_\alpha) \rightarrow (U \hookrightarrow T)\}_\alpha$ such that $T = \bigcup_\alpha f_\alpha(T'_\alpha)$ and every map f_α is a composition of morphisms $(U_1 \hookrightarrow D) \rightarrow (U \hookrightarrow T)$, where D is the PD-envelope of certain $U_1 \hookrightarrow T_1$ and we have a cartesian diagram

$$\begin{array}{ccc} U_1 & \longrightarrow & T_1 \\ \downarrow & & \downarrow \\ U & \longrightarrow & T, \end{array}$$

where the map $T_1 \rightarrow T$ is Kummer log-syntomic. The morphism $D \rightarrow T$ is Kummer. Note that if T_1/T is flat then $D = T_1$. By [18, Lemma 2.7] and [18, Lemma 2.14], coverings are stable under base changes.

Let the sheaves $\mathcal{O}_{X/S}$ (the structure sheaf) and $J_{X/S}$ be the sheaves associated to the presheaves

$$\mathcal{O}_{X/S} : (U \hookrightarrow T) \mapsto \Gamma(T, \mathcal{O}_T); \quad J_{X/S} : (U \hookrightarrow T) \mapsto \Gamma(T, J_T),$$

where J_T is the ideal sheaf of U in T . The sheaf $J_{X/S}$ is a PD-ideal in $\mathcal{O}_{X/S}$. We will denote by $J_{X/S}^{[r]}$ its PD-filtration.

Proposition 2.4. *Let $U \hookrightarrow T$ be an object of $(X/S)_{\text{Synchr}}$ such that,*

$$(2.5) \quad \text{for every point } x \in U, \quad (M_U/\mathcal{O}_U^*)_{\bar{x}} \simeq \mathbf{N}^{r(x)}.$$

Then on the Kummer log-syntomic site of T the sheaf $J_{X/S}^{[r]}$ is given by $T' \mapsto \Gamma(T', J_{T'})^{[r]}$. Moreover, for every Kummer log-syntomic morphism $f : T' \rightarrow T$, $f^ J_T^{[r]} \xrightarrow{\sim} J_{T'}^{[r]}$.*

Proof. Denote by $J^{[r]}$ the presheaf $(U \hookrightarrow T) \mapsto \Gamma(T, J_T)^{[r]}$ on $(X/S)_{\text{Synchr}}$. We have to show that the sheafification process does not change $J^{[r]}|_{T_{\text{ksyn}}}$ for T as above. Let $f : T' \rightarrow T$ be a Kummer log-syntomic map. We claim that it is flat. Indeed, by [18, Lemma 2.8] we can find a chart $(\mathbf{N}^{r(x)} \rightarrow M_T, P \rightarrow M_{T'}, \mathbf{N}^{r(x)} \rightarrow P)$ of f such that, for $y \in T'$, $x = f(y)$, $\mathbf{N}^{r(x)} \simeq (M_X/\mathcal{O}_X^*)_x$, $P/G \simeq (M_{T'}/\mathcal{O}_{T'}^*)_y$, for a group G , and the map $h : \mathbf{N}^{r(x)} \rightarrow P$ is an injection. It suffices to show that the morphism h is flat. Since it is injective, this would follow, via the criterion [10, 4.1], from the fact that $\mathbf{N}^{r(x)} \rightarrow P/G$ is flat, i.e., that the morphism $\text{Spec}(\mathbf{Z}[P']) \rightarrow \text{Spec}(\mathbf{Z}[\mathbf{N}^{r(x)}])$ is flat, where $P' = P/G$. This follows from the fact that $\text{Spec}(\mathbf{Z}[P'])$ is Cohen-Macaulay, $\text{Spec}(\mathbf{Z}[\mathbf{N}^{r(x)}])$ is regular and the morphism $\text{Spec}(\mathbf{Z}[P']) \rightarrow \text{Spec}(\mathbf{Z}[\mathbf{N}^{r(x)}])$ is finite (being Kummer).

Hence divided powers extend from T to any log-scheme that is Kummer and log-syntomic over T . This and the flatness imply the last statement of the proposition. To show now that the two iterations of the sheafification process do not change $J^{[r]}$ it suffices to show that $T' \mapsto \Gamma(T', J_{T'})^{[r]}$ is a sheaf for the Kummer log-syntomic topology. By [18, Corollary 2.16] and faithfully flat descent this follows from the lemma below. \square

Lemma 2.5. *Let T be an affine log-scheme equipped with a chart $P \rightarrow M_T$. Let $P \rightarrow Q$ be a homomorphism of Kummer type, let $T' = T \times_{\mathbf{Z}[P]} \mathbf{Z}[Q]$, and let $T'' = T' \times_T T'$. Then*

$$\Gamma(T, J_T)^{[r]} \rightarrow \Gamma(T', J_{T'})^{[r]} \rightarrow \Gamma(T'', J_{T''})^{[r]}$$

is exact.

Proof. Consider first the case $r = 0$. Set $A = \Gamma(T, \mathcal{O}_T)$. Then the above diagram is isomorphic to

$$A \rightarrow A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q] \rightarrow A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q \oplus (Q^{gp}/P^{gp})].$$

The exactness of this sequence is proved by exhibiting a contracting A -linear homotopy (see the proof of [18, Lemma 3.28]). The wanted exactness for general r is now shown by tensoring this complex with $\Gamma(T, J_T)^{[r]}$ (use the last statement of Proposition 2.4). \square

The above lemma implies immediately the following corollary

Corollary 2.6. *For any log-scheme X over S , the presheaf*

$$\mathcal{O}_X/S : (U \hookrightarrow T) \mapsto \Gamma(T, \mathcal{O}_T)$$

is a sheaf.

Let $(X/S)_{\text{Crr}}$ denote the large crystalline topos (defined as above with coverings built from étale maps). There is a map of topoi $p : (X/S)_{\text{Crr}} \rightarrow (X/S)_{\text{cr}}$, where $(X/S)_{\text{cr}}$ is the (small) crystalline topos of Kato [10, 5].

Proposition 2.7. *There is a morphism of topoi $\pi : (X/S)_{\text{Synchr}} \rightarrow (X/S)_{\text{Crr}}$.*

(1) *For any log-scheme X , π induces a quasi-isomorphism $\mathcal{O}_{X/S} \xrightarrow{\sim} R(p\pi)_* \mathcal{O}_{X/S}$.*

(2) *If X is such that, for every point $x \in X$, $(M_X/\mathcal{O}_X^*)_{\bar{x}} \simeq \mathbf{N}^{r(x)}$, then π induces a quasi-isomorphism $J_{X/S}^{[r]} \xrightarrow{\sim} R(p\pi)_* J_{X/S}^{[r]}$.*

Proof. Let $\mathcal{F} \in (X/S)_{\text{Synchr}}$ and let $U \hookrightarrow T$ be an object of $(X/S)_{\text{Crr}}$. We set $\pi_* \mathcal{F}(U \hookrightarrow T) = \mathcal{F}(U \hookrightarrow T)$. Let $\mathcal{F} \in (X/S)_{\text{Crr}}$, we define $\pi^*(\mathcal{F})$ as the sheaf associated to the presheaf: $(U \hookrightarrow T) \mapsto \mathcal{F}(U \hookrightarrow T)$. It is easy to check that π^* and π_* are adjoints and that π^* commutes with finite inverse limits.

Set $v = p\pi$. First, note that Proposition 2.4 and Corollary 2.6 yield an isomorphism $v_* J_{X/S}^{[r]} \xrightarrow{\sim} J_{X/S}^{[r]}$. Next, we have to check that for $i \geq 1$, $R^i v_* J_{X/S}^{[r]} = 0$. The sheaf $R^i v_* J_{X/S}^{[r]}$ is the sheaf associated to the presheaf $(U \hookrightarrow T) \mapsto H^i(v^* \tilde{T}, J_{X/S}^{[r]})$, where \tilde{T} is the sheaf in $(X/S)_{\text{cr}}$ associated to $U \hookrightarrow T$. We easily check that $v^* \tilde{T} \simeq \tilde{T}$, where “the second” \tilde{T} is the sheaf in $(X/S)_{\text{Synchr}}$ associated to $U \hookrightarrow T$. Also,

$$H^i(\tilde{T}, J_{X/S}^{[r]}) \simeq H^i((X/S)_{\text{Synchr}}/\tilde{T}, J_{X/S}^{[r]}) \simeq H^i((X/S)_{\text{Synchr}}/T, J_{X/S}^{[r]}).$$

It suffices thus to show that our sheaf $J_{X/S}^{[r]}$ is flasque in the topos associated to the site $(X/S)_{\text{Synchr}}/T$. We will show that for every covering $Y \rightarrow T$ from some cofinal system of coverings the Čech cohomology groups $H^i(Y/T, J_{X/S}^{[r]}) = H^i(C(Y/T))$, $i \geq 1$, are trivial. We may assume that T is an affine scheme equipped with a chart $P \rightarrow M_T$, where $P^* \simeq \{1\}$ ($P = \mathbf{N}^{r(x)}$ in the second case of the proposition). Since our coverings are log-syntomic and of Kummer type, by [18, Corollary 2.16] we may also assume that there exists factorization of $Y \rightarrow T$ into $f : Y \rightarrow Y_1$ and $g : Y_1 \rightarrow T$, where f is affine, strictly syntomic and a covering and $Y_1 = Y \times_{\text{Spec}(\mathbf{Z}[P])} \text{Spec}(\mathbf{Z}[Q])$, for a Kummer morphism $u : P \rightarrow Q$. Using Proposition 2.4, we can now practically verbatim quote the proof of [18, Proposition 3.27] for the log-syntomic site and $\mathcal{F} = J_{X/S}^{[r]}$. \square

Let X_{Ksyn} denote the large Kummer log-syntomic site of X .

Proposition 2.8. *There is a morphism of topoi $u : (X/S)_{\text{Synchr}} \rightarrow X_{\text{Ksyn}}$.*

Proof. For $\mathcal{F} \in (X/S)_{\text{Synchr}}$ and a morphism of log-schemes $j : U \rightarrow X$, we set $u_*(\mathcal{F})(U) = \Gamma((U/S)_{\text{Synchr}}, \mathcal{F})$. For $\mathcal{F} \in X_{\text{Ksyn}}$ and a thickening $U \hookrightarrow T$, we set $u^*(\mathcal{F})(U \hookrightarrow T) = \mathcal{F}(U)$. We claim that $u_*(\mathcal{F})$ is a sheaf. Let U be a log-scheme over X and let $(U_i \rightarrow U)_i$ be a Kummer log-syntomic covering of U . We need to show exactness of

$$0 \rightarrow H^0((U/S)_{\text{Synchr}}, \mathcal{F}) \rightarrow \prod_i H^0((U_i/S)_{\text{Synchr}}, \mathcal{F}) \rightarrow \prod_{i,j} H^0((U_i \times_U U_j/S)_{\text{Synchr}}, \mathcal{F}).$$

Let $s \in H^0((U/S)_{\text{Synchr}}, \mathcal{F})$. The section s is represented by a compatible family (s_T) , where $V \hookrightarrow T$ is a thickening over U and $s_T \in \Gamma(T, \mathcal{F})$. Assume that (s_T) goes to 0 in $\prod_i H^0((U_i/S)_{\text{Synchr}}, \mathcal{F})$. Take a thickening $V \hookrightarrow T$ over U and set $V_i = U_i \times_U V$. Using the lifting property ([18, Lemma 2.9]) we can find étale coverings (W_i) of V_i 's and liftings $W_i \hookrightarrow T_i$ of $V \hookrightarrow T$, the log-schemes T_i forming a Kummer log-syntomic covering of T . Let D_i be the PD-envelope of W_i in T_i . Now the maps $(W_i \hookrightarrow D_i) \rightarrow (V \hookrightarrow T)$ form a covering. Since $s_T \mapsto 0$ in $H^0((U_i/S)_{\text{Synchr}}, \mathcal{F})$, we have $\text{res}_{D_i, T}(s_T) = 0$. Hence $s_T = 0$. Since $V \hookrightarrow T$ was arbitrary, we get $s = 0$, as wanted. The argument for the exactness in the middle is very similar.

It is now easy to check that the functors u^* and u_* are adjoints and that u^* commutes with finite inverse limits. \square

Proposition 2.9. *Let X be such that, for every point $x \in X$, $(M_X/\mathcal{O}_X^*)_{\bar{x}} \simeq \mathbf{N}^{r(x)}$. Then, for u as above, $R^i(pu)_* J_{X/S}^{[r]} = 0$ for $i \geq 1$, where $p : X_{\text{Ksyn}} \rightarrow X_{\text{ksyn}}$.*

Proof. The sheaf $R^i(pu)_* J_{X/S}^{[r]}$ on X_{ksyn} is associated to the presheaf $U \mapsto H^i((U/S)_{\text{SynCr}}, J_{U/S}^{[r]})$. Let $s \in H^i((U/S)_{\text{SynCr}}, J_{U/S}^{[r]})$. We will show that there exists a Kummer log-syntomic covering $T \rightarrow U$ such that s restricts to 0 in $H^i((T/S)_{\text{SynCr}}, J_{T/S}^{[r]})$.

By [18, Corollary 2.16] we may assume that, for every point $u \in U$, $(M_U/\mathcal{O}_U^*)_{\bar{u}} \simeq \mathbf{N}^{r(u)}$. By Propositions 2.4, 2.7, $H^i((U/S)_{\text{SynCr}}, J_{U/S}^{[r]}) = H^i((U/S)_{\text{cr}}, J_{U/S}^{[r]})$. Hence the group $H^i((U/S)_{\text{SynCr}}, J_{U/S}^{[r]})$ can be computed as the Zariski hypercohomology of the complex

$$J_D^{[r]} \rightarrow J_D^{[r-1]} \otimes \Omega_{Y/S}^1 \rightarrow J_D^{[r-2]} \otimes \Omega_{Y/S}^2 \rightarrow \dots$$

for a closed embedding $U \hookrightarrow Y$ over S of U into a log-smooth scheme Y over S . Here D is the PD-envelope of U in Y . We may localize on Y and assume that we have a chart $P \rightarrow \mathcal{O}_Y$ such that $P^* = \{1\}$. Let m be an integer annihilating \mathcal{O}_S . Set $Y_m = Y \times_{\text{Spec}(\mathbf{Z}[P])} \text{Spec}(\mathbf{Z}[P^{1/m}])$. Notice that the above complex pullbacks to 0 in degrees strictly larger than 0 under the projection $Y_m \rightarrow Y$. Since $U_m = U \times_Y Y_m$ is Kummer log-syntomic over U and s vanishes in $H^i((U_m/S)_{\text{SynCr}}, J_{U_m/S}^{[r]})$, we are done. \square

Let now S in the above be $\text{Spec}(W_n(k))$ with the trivial log-structure. Let X be a log-scheme over $W(k)$. Define

$$\mathcal{O}_n^{\text{cr}} = (pu)_* \mathcal{O}_{X_n/W_n(k)}, \quad \mathcal{J}_n^{[r]} = (pu)_* J_{X_n/W_n(k)}^{[r]}.$$

These are sheaves on $X_{n,\text{ksyn}}$ associated to the presheaves

$$Y \mapsto \Gamma((Y/W_n(k))_{\text{SynCr}}, \mathcal{O}_{Y/W_n(k)}), \quad Y \mapsto \Gamma((Y/W_n(k))_{\text{SynCr}}, J_{Y/W_n(k)}^{[r]}).$$

By Propositions 2.7 and 2.9, we always have

$$H^*(X_{n,\text{ksyn}}, \mathcal{O}_n^{\text{cr}}) \simeq H^*((X_n/W_n(k))_{\text{cr}}, \mathcal{O}_{X_n/W_n(k)})$$

and, if X has property (2.5), then

$$H^*(X_{n,\text{ksyn}}, \mathcal{J}_n^{[r]}) \simeq H^*((X_n/W_n(k))_{\text{cr}}, J_{X_n/W_n(k)}^{[r]}).$$

It follows that $\mathcal{O}_n^{\text{cr}}$ is naturally endowed with a Frobenius endomorphism. Define

$$\mathcal{S}'_n(r) := \text{Ker}(\mathcal{J}_n^{[r]} \xrightarrow{\phi - p^r} \mathcal{O}_n^{\text{cr}}).$$

We have a map of topoi $\varepsilon_n : X_{n,\text{ksyn}} \rightarrow X_{n,\text{syn}}$ induced by the map of sites sending small log-syntomic schemes to their saturations. We get the induced pullback maps

$$\varepsilon_n^* : H^i(X_{n,\text{syn}}, \mathcal{J}_n^{[r]}) \rightarrow H^i(X_{n,\text{ksyn}}, \mathcal{J}_n^{[r]}), \quad r \geq 0.$$

They are isomorphisms for $r = 0$ and, for log-schemes X satisfying property (2.5), for all r . These maps are compatible with Frobenius hence induce the pullback map

$$\varepsilon_n^* : H^i(X_{n,\text{syn}}, \mathcal{S}'_n(r)) \rightarrow H^i(X_{n,\text{ksyn}}, \mathcal{S}'_n(r)), \quad r \geq 0.$$

To understand it we need to study flatness of log-syntomic sheaves more carefully. For a log-scheme X log-syntomic over $\text{Spec}(W(k))$, define

$$\mathcal{O}_n^{\text{cr}}(X) = H_{\text{cr}}^0(X_n/W_n(k), \mathcal{O}_{X_n/W_n(k)}), \quad J_n^{[r]}(X) = H_{\text{cr}}^0(X_n/W_n(k), J_{X_n/W_n(k)}^{[r]}).$$

The presheaves $J_n^{[r]}$ on $X_{n,\text{syn}}$ are flat over \mathbf{Z}/p^n and $J_{n+1}^{[r]} \otimes \mathbf{Z}/p^n \simeq J_n^{[r]}$. This follows from Section 2.1 and the following proposition.

Proposition 2.10. *If a log-scheme $X \rightarrow \text{Spec}(W(k))$ is log-syntomic in the above sense then it is log-syntomic in the sense of Kato.*

Proof. By [18, Lemma 2.8] and [14, Prop. A.2], the map $X \rightarrow \text{Spec}(W(k))$ has a chart (locally in the syntomic topology on X)

$$\begin{array}{ccc} U & \xrightarrow{h} & Y[Q] \\ \downarrow f & & \downarrow g \\ Y & \xlongequal{\quad} & Y, \end{array}$$

where $Y = \text{Spec}(W(k))$, such that the monoid Q is torsion free and the map h is classically syntomic. Note that the map g is log-smooth hence log-syntomic in Kato's sense. It follows that the map $f : U \rightarrow Y$ is log-syntomic in Kato's sense as well. It remains to show that the morphism of log-schemes $X \rightarrow \text{Spec}(W(k))$ is log-syntomic in Kato's sense if it is so locally in the syntomic topology on X . Étale locally on X we have a chart

$$\begin{array}{ccc} X & \xrightarrow{i} & S_Y[P] \\ \downarrow & & \downarrow \\ Y & \xlongequal{\quad} & Y, \end{array}$$

such that S_Y is a smooth scheme over Y , P is a torsion free monoid, and the map i is an exact closed immersion. Since $S_Y[P]$ is log-smooth over Y , it suffices to show that i is $W(k)$ -regular. Take a point $x \in X$. By assumption there exists a syntomic map $U \rightarrow X$, surjective in the neighbourhood of x , such that the composition $U \rightarrow Y$ is log-syntomic. Take a point $u \in U$ mapping to x . After étale localization on U around u , we can find a X -regular immersion $a : U \hookrightarrow S_X$ into a log-scheme S_X classically smooth over X . Let $s \in S_X$ be the image of the point u under this immersion. Then étale locally on S_X around s we can lift S_X to a smooth scheme S over $S_Y[P]$. We have the following cartesian diagram of log-schemes

$$\begin{array}{ccc} S_X & \xrightarrow{b} & S \\ \downarrow & & \downarrow \\ X & \xrightarrow{i} & S_Y[P] \end{array}$$

Since U is log-syntomic over Y in the sense of Kato and S is log-smooth over $\text{Spec}(W(k))$, the composition $ba : U \hookrightarrow S$ is $W(k)$ -regular at u [1, Prop. 6.1.1.2]. But X being flat over Y , the immersion a is also $W(k)$ -regular at u . It implies [32, 19.2.7] that b is $W(k)$ -regular at s . Since the schemes X and $S_Y[P]$ are flat over $W(k)$ we can use the fiberwise criterium of transversal regularity [32, 19.2.4] and assume that Y is a spectrum of a field. Now, the faithfully flat descent of regular sequences [32, 19.1.5] yields that b being regular at s implies that the immersion i is regular at x , as wanted. \square

We have a natural map of presheaves $J_n^{[r]} \rightarrow \mathcal{J}_n^{[r]}$ that is an isomorphism for $r = 0$ and, for log-schemes X having property (2.5), it induces an isomorphism on the associated sheaves for all r . As before $\phi(J_n^{[r]}) \subset p^r \mathcal{O}_n^{\text{cr}}$ for $0 \leq r \leq p-1$. This fails in general and we modify $J_n^{[r]}$:

$$J_n^{<r>} := \{x \in J_{n+s}^{[r]} \mid \phi(x) \in p^r \mathcal{O}_{n+s}^{\text{cr}}\} / p^n,$$

for some $s \geq r$. This definition is independent of s . Again $J_n^{<r>}$ is flat over \mathbf{Z}/p^n and $J_{n+1}^{<r>} \otimes \mathbf{Z}/p^n \simeq J_n^{<r>}$. This allows us to define the divided Frobenius $\phi_r = \phi/p^{rn} : J_n^{<r>} \rightarrow \mathcal{O}_n^{\text{cr}}$. Set

$$S_n(r) := \text{Ker}(J_n^{<r>} \xrightarrow{\phi_r - 1} \mathcal{O}_n^{\text{cr}}), \quad S'_n(r) := \text{Ker}(J_n^{[r]} \xrightarrow{\phi - p^r} \mathcal{O}_n^{\text{cr}})$$

Denote by $\mathcal{S}_n(r)$ and $\mathcal{S}'_n(r)$ the associated sheaves. There are natural maps $S'_n(r) \rightarrow S_n(r)$, $\mathcal{S}'_n(r) \rightarrow \mathcal{S}_n(r)$ whose kernels and cokernels are killed by p^r .

Lemma 2.11. *The following sequence of sheaves is exact for $r \geq 0$*

$$0 \longrightarrow \mathcal{S}_n(r) \longrightarrow aJ_n^{<r>} \xrightarrow{\phi_r - 1} \mathcal{O}_n^{\text{cr}} \longrightarrow 0,$$

where $aJ_n^{<r>}$ is the sheaf associated to the presheaf $J_n^{<r>}$.

Proof. We only need to prove the right exactness. Let $T \rightarrow X$ by small log-syntomic. Take $a \in \mathcal{O}_n^{\text{cr}}(T)$. We have $\mathcal{O}_n^{\text{cr}}(T) = \Gamma((T_n/W_n(k))_{\text{cr}}, \mathcal{O}_{T_n/W_n(k)})$. By Proposition 2.10 and Proposition 2.1, there exists a covering $Z \rightarrow T$ for Kato's log-syntomic topology such that the pullback $a \in \Gamma((Z_n/W_n(k))_{\text{cr}}, \mathcal{O}_{Z_n/W_n(k)})$ comes from an element $b \in J_n^{\langle r \rangle}(Z)$. By saturating Z we get a covering of T on which the pullback of a is in the image of the map $\phi_r - 1$, as wanted. \square

It follows that, for a log-scheme X having property (2.5), for $0 \leq r \leq p-1$, we have a short exact sequence

$$0 \longrightarrow \mathcal{S}_n(r) \longrightarrow \mathcal{J}_n^{[r]} \xrightarrow{\phi_r - 1} \mathcal{O}_n^{\text{cr}} \longrightarrow 0;$$

and for general r we get that in the exact sequence

$$0 \longrightarrow \mathcal{S}'_n(r) \longrightarrow \mathcal{J}_n^{[r]} \xrightarrow{\phi - p^r} \mathcal{O}_n^{\text{cr}}$$

the cokernel is annihilated by p^r .

In the same way we can define log-syntomic sheaves $\mathcal{S}_n(r)$ on $X_{m, \text{ksyn}}$ for $m \geq n$. Abusing notation, we have $\mathcal{S}_n(r) = i_* \mathcal{S}_n(r)$ for the natural map $i : X_{m, \text{ksyn}} \rightarrow X_{\text{ksyn}}$. Since i_* is exact, $H^*(X_{m, \text{ksyn}}, \mathcal{S}_n(r)) = H^*(X_{\text{ksyn}}, \mathcal{S}_n(r))$. Because of that we will write $\mathcal{S}_n(r)$ for the log-syntomic sheaves on $X_{m, \text{ksyn}}$ as well as on X_{ksyn} .

Let now X be a log-smooth scheme over V^\times . Again, we need the logarithmic version of the syntomic-étale site of Fontaine-Messing [2]. We say that a morphism $\mathcal{Z} \rightarrow \mathcal{Y}$ of p -adic formal log-schemes over $\text{Spf}(W(k))$ is (small) log-syntomic if every $Z_n \rightarrow Y_n$ is (small) log-syntomic. For a formal log-scheme \mathcal{Z} the log-syntomic-étale site $\mathcal{Z}_{\text{sét}}$ is defined by taking as objects morphisms $f : \mathcal{Y} \rightarrow \mathcal{Z}$ that are small log-syntomic and have log-étale generic fiber in the sense mentioned above. For a log-scheme Z , we also have the log-syntomic-étale site $Z_{\text{sét}}$. Here the objects are morphisms $U \rightarrow Z$ that are small log-syntomic with the generic fiber U_K log-étale over Z_K .

Let \widehat{X} be the p -adic completion of X . We have the following diagram of topoi

$$\widehat{X}_{\text{sét}} \xrightarrow{i_{\text{sét}}} X_{\text{sét}} \xleftarrow{j_{\text{sét}}} X_{K, \text{két}}$$

Abusively, we will denote by $\mathcal{S}_n(r)$ the direct image of $\mathcal{S}_n(r)$ under the canonical morphism $i_n : X_{n, \text{ksyn}} \rightarrow \widehat{X}_{\text{sét}}$. Since this morphism is exact (use Proposition 2.10 and [2, III.4.1]) we have $H^*(X_{n, \text{ksyn}}, \mathcal{S}_n(r)) \simeq H^*(\widehat{X}_{\text{sét}}, \mathcal{S}_n(r))$. There is a canonical homomorphism

$$\alpha_r : \mathcal{S}_n(r) \rightarrow i_{\text{sét}}^* j_{\text{sét}}^* j'_* \mathbf{Z}/p^n(r), \quad 0 \leq r \leq p-2,$$

where $j' : X_{\text{tr}, \text{ét}} \hookrightarrow X_{K, \text{két}}$ is the map of topoi induced by the open immersion $X_{\text{tr}} \hookrightarrow X_K$. More generally, for any $r \geq 0$, we get a natural map

$$\alpha_r : \mathcal{S}_n(r) \rightarrow i_{\text{sét}}^* j_{\text{sét}}^* j'_* \mathbf{Z}/p^n(r)',$$

where $\mathbf{Z}/p^n(r)' = (p^a a!)^{-1} \mathbf{Z}(r)/p^n$ for $r = (p-1)a + b$, $a, b \in \mathbf{Z}$, $0 \leq b < p-1$. Composing with the map $\mathcal{S}'_n(r) \rightarrow \mathcal{S}_n(r)$ we get a natural morphism

$$\alpha_r : \mathcal{S}'_n(r) \rightarrow i_{\text{sét}}^* j_{\text{sét}}^* j'_* \mathbf{Z}/p^n(r)'.$$

The functor $\mathcal{F} \mapsto (j_{\text{sét}}^* \mathcal{F}, j_{\text{sét}}^* \mathcal{F}, i_{\text{sét}}^* \mathcal{F} \rightarrow i_{\text{sét}}^* j_{\text{sét}}^* j'_* \mathcal{F})$ from the category of sheaves on $X_{\text{sét}}$ to the category of triples $(\mathcal{G}, \mathcal{H}, \mathcal{G} \rightarrow i_{\text{sét}}^* j_{\text{sét}}^* j'_* \mathcal{H})$, where \mathcal{G} (resp. \mathcal{H}) are sheaves on $\widehat{X}_{\text{sét}}$ (resp. $X_{K, \text{sét}}$) is an equivalence of categories. It follows that we can glue $\mathcal{S}_n(r)$ and $j'_* \mathbf{Z}/p^n(r)'$ by the map α_r and obtain a sheaf $\mathcal{S}_n(r)$ on $X_{\text{sét}}$.

We have maps of topoi $\varepsilon : X_{\text{sét}} \rightarrow X_{\text{sé}}$, $\widehat{\varepsilon} : \widehat{X}_{\text{sét}} \rightarrow \widehat{X}_{\text{sé}}$, $\varepsilon_n : X_{n, \text{syn}} \rightarrow X_{n, \text{ksyn}}$ induced by the maps of sites sending small log-syntomic-étale and log-syntomic schemes to their saturations. Consider the pullback maps

$$\widehat{\varepsilon}^* : H^i(\widehat{X}_{\text{sé}}, \mathcal{S}_n(r)) \xrightarrow{\sim} H^i(\widehat{X}_{\text{sét}}, \mathcal{S}_n(r)), \quad \varepsilon_n^* : H^i(X_{n, \text{syn}}, \mathcal{S}_n(r)) \xrightarrow{\sim} H^i(X_{n, \text{ksyn}}, \mathcal{S}_n(r)).$$

Lemma 2.12. *Let X be a log-smooth scheme over V^\times satisfying property (2.5).*

- (1) *If $0 \leq r \leq p-1$ then the above pullbacks are isomorphisms.*
- (2) *The kernel and cokernel of the above pullbacks are annihilated by p^{2r} .*

Moreover, if X is proper over V the same is true of the pullback $\varepsilon^ : H^i(X_{\text{sé}}, \mathcal{S}_n(r)) \rightarrow H^i(X_{\text{sét}}, \mathcal{S}_n(r))$.*

Proof. Assume first that $0 \leq r \leq p-1$. For X proper over V we have the following commutative diagram

$$\begin{array}{ccc} H^i(X_{\text{s}\acute{e}}, \mathcal{S}_n(r)) & \xrightarrow{\varepsilon^*} & H^i(X_{\text{s}\acute{e}\acute{t}}, \mathcal{S}_n(r)) \\ \wr \downarrow i_{\text{s}\acute{e}}^* & & \wr \downarrow i_{\text{s}\acute{e}\acute{t}}^* \\ H^i(\widehat{X}_{\text{s}\acute{e}}, \mathcal{S}_n(r)) & \xrightarrow{\widehat{\varepsilon}^*} & H^i(\widehat{X}_{\text{s}\acute{e}\acute{t}}, \mathcal{S}_n(r)) \end{array}$$

Since the vertical maps are isomorphisms by the proper base change theorem, it suffices to show that the lower horizontal map is an isomorphism. But we have the isomorphisms

$$H^i(\widehat{X}_{\text{s}\acute{e}}, \mathcal{S}_n(r)) \simeq H^i(X_{n, \text{syn}}, \mathcal{S}_n(r)), \quad H^i(\widehat{X}_{\text{s}\acute{e}\acute{t}}, \mathcal{S}_n(r)) \simeq H^i(X_{n, \text{ksyn}}, \mathcal{S}_n(r))$$

so it suffices to show that the map

$$\varepsilon_n^* : H^i(X_{n, \text{syn}}, \mathcal{S}_n(r)) \rightarrow H^i(X_{n, \text{ksyn}}, \mathcal{S}_n(r))$$

is an isomorphism. For that use the following map of long exact sequences

$$\begin{array}{ccccccc} H^{i-1}(X_{n, \text{s}\acute{e}}, \mathcal{O}_n^{\text{cr}}) & \longrightarrow & H^i(X_{n, \text{s}\acute{e}}, \mathcal{S}_n(r)) & \longrightarrow & H^i(X_{n, \text{s}\acute{e}}, \mathcal{J}_n^{[r]}) & \xrightarrow{\phi_{r-1}} & H^i(X_{n, \text{s}\acute{e}}, \mathcal{O}_n^{\text{cr}}) \longrightarrow \\ \wr \downarrow \varepsilon^* & & \downarrow \varepsilon^* & & \wr \downarrow \varepsilon^* & & \wr \downarrow \varepsilon^* \\ H^{i-1}(X_{n, \text{s}\acute{e}\acute{t}}, \mathcal{O}_n^{\text{cr}}) & \longrightarrow & H^i(X_{n, \text{s}\acute{e}\acute{t}}, \mathcal{S}_n(r)) & \longrightarrow & H^i(X_{n, \text{s}\acute{e}\acute{t}}, \mathcal{J}_n^{[r]}) & \xrightarrow{\phi_{r-1}} & H^i(X_{n, \text{s}\acute{e}\acute{t}}, \mathcal{O}_n^{\text{cr}}) \longrightarrow \end{array}$$

For the second part argue similarly using Lemma 2.11. \square

We finish this section with an analog of Corollary 2.3 for the "new" log-syntomic-étale site that we will need later.

Corollary 2.13. (1) *Let X be a log-smooth, proper scheme over V^\times . Then for $0 \leq i \leq r \leq p-2$, the natural map*

$$H^i(X_{\text{s}\acute{e}\acute{t}}, j_{\text{s}\acute{e}\acute{t}} * \mathbf{Z}/p^n(r)) \rightarrow H^i(X_{K, \text{k}\acute{e}\acute{t}}, \mathbf{Z}/p^n(r))$$

is surjective. Here $j_{\text{s}\acute{e}\acute{t}} : X_{K, \text{k}\acute{e}\acute{t}} \hookrightarrow X_{\text{s}\acute{e}\acute{t}}$.

(2) *Let X be a proper scheme over V^\times that is semistable over V^\times or a base change of such. Then for any $0 \leq i \leq r$, the cokernel of the natural map*

$$H^i(X_{\text{s}\acute{e}\acute{t}}, j_{\text{s}\acute{e}\acute{t}} * \mathbf{Z}/p^n(r)) \rightarrow H^i(X_{K, \text{k}\acute{e}\acute{t}}, \mathbf{Z}/p^n(r))$$

is annihilated by p^N for an integer N which depends only on p , r , and i .

Proof. Recall that [14, Theorem 0.2] $\mathbf{Z}/p^n(r)_{X_{K, \text{k}\acute{e}\acute{t}}} \xrightarrow{\sim} Rj'_* \mathbf{Z}/p^n(r)$, where $j' : X_{tr, \acute{e}\acute{t}} \hookrightarrow X_{K, \text{k}\acute{e}\acute{t}}$. The map in the corollary becomes $H^i(X_{\text{s}\acute{e}\acute{t}}, j_{\text{s}\acute{e}\acute{t}} * Rj'_* \mathbf{Z}/p^n(r)) \rightarrow H^i(X_{tr, \acute{e}\acute{t}}, \mathbf{Z}/p^n(r))$. It suffices thus to prove the corollary for the induced map

$$H^i(X_{\text{s}\acute{e}\acute{t}}, j_{\text{s}\acute{e}\acute{t}} * j'_* \mathbf{Z}/p^n(r)) \rightarrow H^i(X_{tr, \acute{e}\acute{t}}, \mathbf{Z}/p^n(r))$$

Recall that we have a map of topoi $\varepsilon : X_{\text{s}\acute{e}\acute{t}} \rightarrow X_{\text{s}\acute{e}}$ induced by the map of sites sending small log-syntomic-étale scheme $U \rightarrow X$ to its saturation $U^{\text{sat}} \rightarrow X$. Note that $U_{tr} = U_{tr}^{\text{sat}}$. Hence the identity map

$$\Gamma(U, j_{\text{s}\acute{e}} * j'_* \mathbf{Z}/p^n(r)) = \Gamma(U_{tr}, \mathbf{Z}/p^n(r)) \rightarrow \Gamma(U_{tr}^{\text{sat}}, \mathbf{Z}/p^n(r)) = \Gamma(U^{\text{sat}}, j_{\text{s}\acute{e}} * j'_* \mathbf{Z}/p^n(r))$$

induces a map of sheaves $\varepsilon^* : j_{\text{s}\acute{e}} * j'_* \mathbf{Z}/p^n(r) \rightarrow \varepsilon_* j_{\text{s}\acute{e}\acute{t}} * j'_* \mathbf{Z}/p^n(r)$ and we get the following commutative diagram

$$\begin{array}{ccc} H^i(X_{\text{s}\acute{e}\acute{t}}, j_{\text{s}\acute{e}\acute{t}} * j'_* \mathbf{Z}/p^n(r)) & \longrightarrow & H^i(X_{tr, \acute{e}\acute{t}}, \mathbf{Z}/p^n(r)) \\ \varepsilon^* \uparrow & & \parallel \\ H^i(X_{\text{s}\acute{e}}, j_{\text{s}\acute{e}} * j'_* \mathbf{Z}/p^n(r)) & \longrightarrow & H^i(X_{tr, \acute{e}\acute{t}}, \mathbf{Z}/p^n(r)) \end{array}$$

Our corollary now follows from the previous one. \square

3. TOPOLOGICAL LOG-SYNTOMIC-ÉTALE HOMOTOPY K-THEORY

Consider the following proposition.

Proposition 3.1. *Let X be a classically smooth log-scheme over V^\times . For $i \geq 2 \dim X_K$, the restriction map*

$$j^* : K_i(X_{\overline{V}, \text{sét}}, \mathbf{Z}/p^n) \rightarrow K_i(X_{\overline{K}, \text{ét}}, \mathbf{Z}/p^n)$$

is surjective with a natural section.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} K_i(X_{\overline{V}}, \mathbf{Z}/p^n) & \xrightarrow[\sim]{j^*} & K_i(X_{\overline{K}}, \mathbf{Z}/p^n) \\ \downarrow & & \downarrow \rho_i \\ K_i(X_{\overline{V}, \text{sét}}, \mathbf{Z}/p^n) & \xrightarrow{j^*} & K_i(X_{\overline{K}, \text{ét}}, \mathbf{Z}/p^n). \end{array}$$

By [15, Lemma 3.1], the upper restriction map is an isomorphism. By the Quillen-Lichtenbaum Conjecture for fields (as shown in [23], [4], [12], a corollary to, now proved by Rost and Voevodsky, Bloch-Kato Conjecture [31]), the change of topology map ρ_i is an isomorphism for $i \geq 2 \dim X_K$. Our proposition now follows. \square

In this section we will investigate to what degree the above statement holds for more general log-schemes.

3.1. Preliminaries. For a site C , we equip the category of presheaves of spectra on C with Jardine's model structure [6], [7, Thm. 2.8]. Recall that a map of presheaves of spectra $E \rightarrow F$ is called a weak equivalence if it induces an isomorphism $\tilde{\pi}_*(E) \rightarrow \tilde{\pi}_*(F)$ on the sheaves of stable homotopy groups. For a presheaf of spectra F we will denote by F^f a fibrant replacement of F . That is, we have a map $F \rightarrow F^f$ to a fibrant presheaf of spectra F^f that is a weak equivalence. We will denote by

$$\{F\langle n \rangle\} := \dots \rightarrow F\langle m+1 \rangle \rightarrow F\langle m \rangle \rightarrow F\langle m-1 \rangle \rightarrow \dots$$

a Postnikov tower of F . We have $\pi_q F\langle m \rangle = 0$ if $q > m$ and the map $\pi_q F \rightarrow \pi_q F\langle m \rangle$ is an isomorphism for $q \leq m$. We will denote by $F(m)$ the fiber of the map $F \rightarrow F\langle m \rangle$. Then $\pi_q F(m) \rightarrow \pi_q F$ is an isomorphism for $q > m$ and $\pi_q F(m) = 0$ for $q \leq m$. Note that both $F \rightarrow F^f$ and $\{F\langle n \rangle\}$ can be chosen to be functorial.

We define the cohomology of C with values in F [7, p. 746]

$$H^{-m}(C, F) := [\Sigma^n S^0, F] = \pi(\Sigma^n S^0, F^f).$$

Here $\Sigma^n S^0$ denotes the n 'th suspension of the sphere spectrum. The bracket $[\]$ denotes maps in the homotopy category of presheaves of spectra and $\pi(\)$ stands for the set of pointed homotopy classes of maps.

For a scheme X , write $K(X)$ for the Thomason-Throbaugh spectrum of nonconnective K-theory as defined in [25, 6.4]. Write $KH(X)$ for the Weibel homotopy invariant K-theory spectrum [29] as defined in [25, 9.11]:

$$KH(X) = \text{hocolim}_{\Delta^{op}} (i \mapsto K(X_i)), \quad X_i := X[T_0, T_1, \dots, T_i]/(T_0 + \dots + T_i = 1).$$

Here Δ^{op} is the opposite category of the category of simplicial sets. Write $K/n(X)$, $KH/n(X)$ for the corresponding mod- n spectra. We will denote by $K_i(X)$, $KH_i(X)$, $K_i(X, \mathbf{Z}/n)$, $KH_i(X, \mathbf{Z}/n)$ their i 'th homotopy groups.

For a site C built from schemes, denote by KH and KH/n the presheaves of spectra

$$KH : X \mapsto KH(X), \quad KH/n : X \mapsto KH/n(X).$$

Define similarly the presheaves of spectra K and K/n . Set

$$\begin{aligned} KH_m(C) &:= H^{-m}(C, KH), & KH_m(C, \mathbf{Z}/p^n) &:= H^{-m}(C, KH/n), \\ K_m(C) &:= H^{-m}(C, K), & K_m(C, \mathbf{Z}/p^n) &:= H^{-m}(C, K/n). \end{aligned}$$

We also have the truncated versions $KH_m^{\leq j}(C) := H^{-m}(C, KH\langle j \rangle)$, etc.

To control the cohomological dimension of some schemes we will often impose the following property

(*) S is separated, Noetherian and regular. The prime number p is invertible on S and $\sqrt{(-1)} \in \mathcal{O}_X$ if $p = 2$. S has finite Krull dimension and a uniform bound on p -torsion étale cohomological dimension of all residue fields. Each residue field of S has a finite Tate-Tsen filtration.

Recall that the last condition means that, for each residue field k , there exists a sequence of subextensions

$$k \subset L_1 \subset \dots \subset L_j = \bar{k}$$

of \bar{k}/k with each L_i/L_{i-1} Galois and having $\text{cd}_p(L_i/L_{i-1}) \leq 1$.

In what follows, choose a sequence of nontrivial p -roots of unity $\zeta_n \in \overline{\mathbf{Q}}_p$, $\zeta_n^{p^n} = 1$, $\zeta_{n+1}^p = \zeta_n$. Let Y be a scheme on which p is invertible, satisfying condition (*) such that $\Gamma(Y, \mathcal{Y})$ contains a primitive p^n 'th root of unity, Let $\mu_{p^n}(Y)$ denote the group of p^n 'th roots of unity in $\Gamma(Y, \mathcal{O}_Y)$. Recall [30, 2.7, 2.7.2] that there is a functorial Bott element homomorphism

$$\beta : \mu_{p^n}(Y) \rightarrow K_2(Y, \mathbf{Z}/p^n),$$

which is also compatible with the transition maps $\mathbf{Z}/p^{n+1} \rightarrow \mathbf{Z}/p^n$. Let K_1 be a finite field extension of K containing ζ_n . Define the Bott class $\beta_n \in K_2(K_1, \mathbf{Z}/p^n)$ as $\beta(\zeta_n)$. These classes form a compatible sequence with respect to n . Let now V_1 be the ring of integers in K_1 . We have the localization exact sequence

$$\rightarrow K_2(k_1, \mathbf{Z}/p^n) \rightarrow K_2(V_1, \mathbf{Z}/p^n) \rightarrow K_2(K_1, \mathbf{Z}/p^n) \rightarrow K_2(X_{k_1}, \mathbf{Z}/p^n) \rightarrow,$$

where k_1 is the residue field of V_1 . Recall that, k_1 being perfect, $K_i(k_1, \mathbf{Z}/p^n) = 0$ for $i \geq 1$ (see [3, Theorem 8.4]) and the above exact sequence yields the isomorphism

$$K_2(V_1, \mathbf{Z}/p^n) \xrightarrow{\sim} K_2(K_1, \mathbf{Z}/p^n).$$

We take for the class $\tilde{\beta}_n \in K_2(V_1, \mathbf{Z}/p^n)$ the unique element that maps to β_n via the above isomorphism. For a scheme Y over V_1 , we will denote by $\tilde{\beta}_n$ and β_n the images of $\tilde{\beta}_n$ and β_n in $K_2(Y, \mathbf{Z}/p^n)$ and $K_2(Y_{K_1}, \mathbf{Z}/p^n)$, respectively.

Represent $\tilde{\beta}_n \in K_2(V_1, \mathbf{Z}/p^n)$ by a map $\tilde{\beta}_n : \Sigma^2 S^0 \rightarrow K/p^n(V_1)$. For a scheme $f : Y \rightarrow V_1$ this induces a functorial map

$$\tilde{\beta}_n : \Sigma^2 S^0 \wedge K/p^n(Y) \xrightarrow{\tilde{\beta}_n \wedge 1} K/p^n(V_1) \wedge K/p^n(Y) \xrightarrow{f^* \wedge 1} K/p^n(Y) \wedge K/p^n(Y) \xrightarrow{\text{mult}} K/p^n(Y).$$

Since $K_2(k_1, \mathbf{Z}/p^n) = 0$, if Y is a k_1 -scheme, this map is homotopic to the null map. The above map extends to a functorial map

$$\tilde{\beta}_n : \Sigma^2 S^0 \wedge KH/p^n(Y) \rightarrow KH/p^n(Y)$$

defined by the composition

$$\begin{aligned} \tilde{\beta}_n : \Sigma^2 S^0 \wedge KH/p^n(Y) &= \Sigma^2 S^0 \wedge \text{hocolim}_{\Delta^{op}}(i \mapsto K/p^n(Y_i)) \\ &\simeq \text{hocolim}_{\Delta^{op}}(i \mapsto \Sigma^2 S^0 \wedge K/p^n(Y_i)) \xrightarrow{\tilde{\beta}_n} \text{hocolim}_{\Delta^{op}}(i \mapsto K/p^n(Y_i)) = KH/p^n(Y). \end{aligned}$$

As before, if Y is a k_1 -scheme, this map is homotopic to the null map.

3.2. The main theorem. We are now ready to prove the main theorem of this paper. Let X be a proper semistable scheme over V^\times . Let $KH_m(X_{\text{sét}}, \mathbf{Z}/p^n)$, resp. $KH_m(X_{K, \text{két}}, \mathbf{Z}/p^n)$, denote the K-groups defined in the previous section (with $C = X_{\text{sét}}$, resp. $C = X_{K, \text{két}}$). Denote by $j : X_{K, \text{két}} \rightarrow X_{\text{sét}}$ the map of topoi induced by the natural immersion $j : X_K \hookrightarrow X$.

Theorem 3.2. *Assume that $\zeta_n \in V$. Then, for any $m \geq \max(2d, 1)$, where $d = \dim X_K$, and for any $i \geq 2d + m/2$, the cokernel of the restriction map*

$$j^* : KH_m^{\leq i}(X_{\text{sét}}, \mathbf{Z}/p^n) \rightarrow KH_m^{\leq i}(X_{K, \text{két}}, \mathbf{Z}/p^n)$$

is annihilated by a constant $T(p, d, i, m)$ depending only on p, d, m , and i .

Proof. Let $Z \rightarrow X$ be a log-syntomic-étale scheme. By Kato [10, 4.1] Z_K is a disjoint union of its irreducible (normal) components. Write Z as a union of its irreducible components with their reduced scheme structures $Z = \bigcup Z_i$. Each scheme Z_i is integral. Hence it is flat over V or maps to the special fiber of X . Let $\tilde{Z} = \coprod Z_i$ be the disjoint union of the flat components. The map $Z \mapsto \tilde{Z}$ is a functor on the log-syntomic-étale site of X .

Consider the presheaf of spectra $KH'/p^n : Z \mapsto KH/p^n(\tilde{Z})$ on X . We have a natural map $KH/p^n \rightarrow KH'/p^n$. We claim that there exists a map $\omega : KH'/p^n \rightarrow \Omega^2 KH/p^n$ in the homotopy category such that the composition

$$KH'/p^n \xrightarrow{\omega} \Omega^2 KH/p^n \rightarrow \Omega^2 KH'/p^n$$

is equal (in the homotopy category) to the map $KH'/p^n \xrightarrow{\tilde{\beta}_n} \Omega^2 KH'/p^n$. Indeed, since the maps $\tilde{Z} \rightarrow Z$ are finite, we have a homotopy cartesian square [29, 4.9]

$$\begin{array}{ccc} KH/p^n(Z) & \longrightarrow & KH/p^n(\tilde{Z}) \\ \downarrow & & \downarrow \\ KH/p^n(Z_0) & \longrightarrow & KH/p^n(\tilde{Z}_0). \end{array}$$

This induces a homotopy cartesian square

$$\begin{array}{ccc} KH/p^n & \longrightarrow & KH'/p^n \\ \downarrow & & \downarrow \\ KH^0/p^n & \longrightarrow & KH'^0/p^n. \end{array}$$

Here KH^0/p^n is the presheaf $KH^0/p^n : Z \mapsto KH/p^n(Z_0)$ and KH'^0/p^n is the presheaf $KH'^0/p^n : \tilde{Z} \mapsto KH/p^n(\tilde{Z}_0)$. The Bott element $\tilde{\beta}_n$ acts on this square. The schemes Z_0, \tilde{Z}_0 are in characteristic p , so the action on the bottom row is trivial. On the level of maps in the homotopy category we get the following commutative diagram of abelian groups (here we write K_n for KH'/p^n)

$$\begin{array}{ccccccc} \longrightarrow & [K_n, KH/p^n] & \longrightarrow & [K_n, KH^0/p^n] \oplus [K_n, KH'/p^n] & \longrightarrow & [K_n, KH'^0/p^n] & \longrightarrow \\ & \downarrow \tilde{\beta}_n & & \downarrow \tilde{\beta}_n & & \downarrow \tilde{\beta}_n & \\ \longrightarrow & [K_n, \Omega^2 KH/p^n] & \longrightarrow & [K_n, \Omega^2 KH^0/p^n] \oplus [K_n, \Omega^2 KH'/p^n] & \longrightarrow & [K_n, \Omega^2 KH'^0/p^n] & \longrightarrow \end{array}$$

Since the rightmost vertical map is zero diagram chase gives us the map $\omega : KH'/p^n \rightarrow \Omega^2 KH/p^n$ compatible with the maps $\Omega^2 KH/p^n \rightarrow \Omega^2 KH'/p^n$ and $\tilde{\beta}_n : KH'/p^n \rightarrow \Omega^2 KH'/p^n$ (start with an element $(0, 1) \in [KH'/p^n, KH^0/p^n] \oplus [KH'/p^n, KH'/p^n]$). Passing to truncations, we get a map $\omega : KH'/p^n\langle i \rangle \rightarrow \Omega^2 KH/p^n\langle i+2 \rangle$ and the following commutative diagram

$$\begin{array}{ccccc} H^{-m}(X_{\text{sét}}, KH'/p^n\langle i \rangle) & \xlongequal{\quad} & H^{-m}(X_{\text{sét}}, KH'/p^n\langle i \rangle) & \longrightarrow & H^{-m}(X_{K, \text{két}}, KH'/p^n\langle i \rangle) \\ \omega \downarrow & & \tilde{\beta}_n \downarrow & & \beta_n \downarrow \wr \\ H^{-m-2}(X_{\text{sét}}, KH/p^n\langle i+2 \rangle) & \longrightarrow & H^{-m-2}(X_{\text{sét}}, KH'/p^n\langle i+2 \rangle) & \longrightarrow & H^{-m-2}(X_{K, \text{két}}, KH'/p^n\langle i+2 \rangle). \end{array}$$

We claim that the rightmost horizontal map is an isomorphism. Indeed, since p is invertible on the generic fiber, the natural map $K/p^n \rightarrow KH/p^n$ is a weak equivalence on $X_{K, \text{két}}$ [29, Prop. 1.6]. Hence we have an isomorphism $H^{-m}(X_{K, \text{két}}, K/p^n) \xrightarrow{\sim} H^{-m}(X_{K, \text{két}}, KH'/p^n)$ and need now to show that the Bott map

$$\beta_n : H^{-m}(X_{K, \text{két}}, K/p^n\langle i \rangle) \rightarrow H^{-m}(X_{K, \text{két}}, K/p^n\langle i+2 \rangle)$$

is an isomorphism. For that we factorize it

$$\beta_n : H^{-m}(X_{K, \text{két}}, K/p^n\langle i \rangle) \xrightarrow{\beta_n} H^{-m-2}(X_{K, \text{két}}, K/p^n\langle 2 \rangle\langle i+2 \rangle) \rightarrow H^{-m-2}(X_{K, \text{két}}, K/p^n\langle i+2 \rangle)$$

In [18, Prop. 5.12], we have computed the sheaves of homotopy groups on $X_{K,\text{két}}$

$$\tilde{\pi}_k(K/p^n) = \begin{cases} \mathbf{Z}/p^n(k/2) & \text{for } k \geq 0, \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the map $\beta_n : K/p^n\langle i \rangle \rightarrow K/p^n\langle 2 \rangle\langle i+2 \rangle$ is a weak equivalence and the first map in the above factorization is an isomorphism. The second map is clearly an isomorphism for $m \geq 1$ and so is our Bott map.

It suffices now to show that for $m \geq \max(2d, 1)$ and $i \geq 2d + m/2$, the cokernel of the map

$$j^* : H^{-m}(X_{\text{sét}}, KH'/p^n\langle i \rangle) \rightarrow H^{-m}(X_{K,\text{két}}, KH'/p^n\langle i \rangle)$$

is annihilated by a constant depending only on p, d, m , and i .

For a pointed simplicial set A , let $\Sigma^\infty A$ denote the suspension spectrum of A and $\Sigma^\infty/p^n A$ the corresponding Moore spectrum. On $X_{K,\text{két}}$, let $\Sigma^\infty/p^n B\mu_{p^\infty+}$ denote the presheaf of spectra $Z \mapsto \Sigma^\infty/p^n B\mu_{p^\infty}(\Gamma(Z, \mathcal{O}_Z))_+$. We will need the following computation

Lemma 3.3. *The sheaves of homotopy groups*

$$\tilde{\pi}_k(j_*\Sigma^\infty/p^n B\mu_{p^\infty+}) = \begin{cases} j_*\mathbf{Z}/p^n(k) & \text{for } k \geq 0, \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Proof. First, recall that $\pi_{2i}(\Sigma^\infty/p^n B\mu_{p^\infty}(\overline{K})_+) = \mathbf{Z}/p^n(i)$ for $i \geq 0$ and is trivial otherwise. Indeed, any imbedding $\overline{K} \hookrightarrow \mathbf{C}$ into complex numbers induces a map $\mu_{p^\infty}(\overline{K}) \hookrightarrow U(1) \hookrightarrow U(\infty)$ hence a morphism $\Sigma^\infty/p^n B\mu_{p^\infty}(\overline{K})_+ \rightarrow bu/p^n$, where bu is the -1 -connected cover of the complex K-theory spectrum. By Suslin [22] this map is a weak equivalence and we are done. It follows easily that, our sheaves of homotopy groups are trivial for $k \leq 0$, that, for $k \geq 0$,

$$\tilde{\pi}_k(\Sigma^\infty/p^n B\mu_{p^\infty+}) = \begin{cases} \mathbf{Z}/p^n(k/2) & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd,} \end{cases}$$

and that the Bott element map

$$\beta_n^k : \mathbf{Z}/p^n(k) \xrightarrow{\sim} \tilde{\pi}_{2k}(\Sigma^\infty/p^n B\mu_{p^\infty+})$$

is an isomorphism.

Now, we will show that, for $k \geq 0$, $\tilde{\pi}_{2k+1}(j_*\Sigma^\infty/p^n B\mu_{p^\infty+}) = 0$. Take any log-syntomic-étale scheme $Z \rightarrow X$. Write $Z_K = \coprod Y_i$ for connected (normal) components Y_i . Let K_i be the field of constants of Y_i , i.e., the maximal algebraic extension of K contained in the function field of Y_i . Then $\mu_{p^\infty}\Gamma(Y_i, \mathcal{O}_{Y_i}) = \mu_{p^\infty}(K_i)$ and we have

$$\pi_{2k+1}(j_*\Sigma^\infty/p^n B\mu_{p^\infty}(Z)_+) = \oplus_i \pi_{2k+1}(\Sigma^\infty/p^n B\mu_{p^\infty}(K_i)_+).$$

It suffices to show that any class $a \in \pi_{2k+1}(\Sigma^\infty/p^n B\mu_{p^\infty}(K_i)_+)$ goes to zero on some covering of Z . Since $\pi_{2k+1}(\Sigma^\infty/p^n B\mu_{p^\infty}(\overline{K})_+) = 0$ there exists a finite extension K_2 of K_i such that a pullbacks to zero in $\pi_{2k+1}(\Sigma^\infty/p^n B\mu_{p^\infty}(K_2)_+)$. Let V_2 be the ring of integers in K_2 . Consider the scheme $Z_{V_2^\times} \rightarrow Z$. Since the map $V_2^\times \rightarrow V^\times$ is log-syntomic-étale, $Z_{V_2^\times} \rightarrow Z$ is a covering. But the fields of constants of all the components of the generic fiber of $Z_{V_2^\times}$ contain K_2 hence the class a goes to zero when pullbacked to $\pi_{2k+1}(j_*\Sigma^\infty/p^n B\mu_{p^\infty}(Z_{V_2^\times})_+)$.

To finish, consider the Bott element map

$$\beta_n^k : j_*\mathbf{Z}/p^n(k) \rightarrow \tilde{\pi}_{2k}(j_*\Sigma^\infty/p^n B\mu_{p^\infty+}).$$

We claim that it is an isomorphism. It is injective since the map

$$\beta_n^k : \mathbf{Z}/p^n(k) \rightarrow \tilde{\pi}_{2k}(\Sigma^\infty/p^n B\mu_{p^\infty+})$$

is an isomorphism on $X_{K,\text{két}}$. For surjectivity, we will argue as above. Take any log-syntomic-étale scheme $Z \rightarrow X$. Write $Z_K = \coprod Y_i$ for connected (normal) components Y_i . Let K_i be the field of constants of Y_i . Then $\mu_{p^\infty}(\Gamma(Y_i, \mathcal{O}_{Y_i})) = \mu_{p^\infty}(K_i)$ and we have

$$\pi_{2k}(j_*\Sigma^\infty/p^n B\mu_{p^\infty}(Z)_+) = \bigoplus_i \pi_{2k}(\Sigma^\infty/p^n B\mu_{p^\infty}(K_i)_+).$$

It suffices to show that any class $a \in \pi_{2k}(\Sigma^\infty/p^n B\mu_{p^\infty}(K_i)_+)$ is in the image of the Bott element map on some covering of Z . Since $\beta_n^k : \mathbf{Z}/p^n(k) \xrightarrow{\sim} \pi_{2k}(\Sigma^\infty/p^n B\mu_{p^\infty}(\overline{K})_+)$ there exists a finite extension K_2 of K_i such that a is in the image of $\beta_n^k : \mathbf{Z}/p^n(k) \rightarrow \pi_{2k}(\Sigma^\infty/p^n B\mu_{p^\infty}(K_2)_+)$. Let V_2 be the ring of integers in K_2 . Consider the covering $Z_{V_2^\times} \rightarrow Z$. Since the fields of constants of all the components of the generic fiber of $Z_{V_2^\times}$ contain K_2 the class a is in the image of β_n^k when pullbacked to $\pi_{2k}(j_*\Sigma^\infty/p^n B\mu_{p^\infty}(Z_{V_2^\times})_+)$. \square

On $X_{\text{sét}}$, let $\Sigma^\infty/p^n B\mu'_{p^\infty+}$ denote the presheaf of spectra $Z \mapsto \bigvee \Sigma^\infty/p^n B\mu_{p^\infty}(\Gamma(Y, \mathcal{O}_Y))_+$, where the wedge is over the connected components Y of \tilde{Z} . We claim that the natural map

$$\Sigma^\infty/p^n B\mu'_{p^\infty+} \xrightarrow{\sim} j_*\Sigma^\infty/p^n B\mu_{p^\infty+},$$

is a weak equivalence. It suffices to show that the natural map of presheaves $\mu'_{p^\infty} \rightarrow j_*\mu_{p^\infty}$, where μ'_{p^∞} is the presheaf $Z \mapsto \mu_{p^\infty}(\tilde{Z})$, induces an isomorphism on the associated sheaves. The injectivity follows from the flatness of \tilde{Z} over V . For surjectivity argue just as in the proof of Lemma 3.3. It follows that

$$\tilde{\pi}_k(\Sigma^\infty/p^n B\mu'_{p^\infty+}) = \begin{cases} j_*\mathbf{Z}/p^n(k/2) & \text{for } k \geq 0, \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

The following corollary gives a useful description of the presheaf $j_*\Sigma^\infty/p^n B\mu_{p^\infty+}$.

Corollary 3.4. *There is a weak equivalence of presheaves of spectra on $X_{\text{sét}}$*

$$(3.1) \quad j_*\Sigma^\infty/p^n B\mu_{p^\infty+} \simeq j_*\pi^*K/p^n,$$

where π is the projection $\pi : X_{\text{két}} \rightarrow \text{Spec}(K)_{\text{ét}}$.

Proof. Recall that, for any scheme Y , we have a natural map

$$\gamma : \Sigma^\infty/p^n B\mu_{p^\infty+}(\mathcal{O}_Y(Y)) \rightarrow K/p^n(\mathcal{O}_Y(Y)),$$

where $K(\mathcal{O}_Y(Y))$ denotes the Waldhausen spectrum of the category of free $\mathcal{O}_Y(Y)$ -modules. The map γ is defined as the composition

$$\Sigma^\infty/p^n B\mu_{p^\infty+}(\mathcal{O}_Y(Y)) \rightarrow \Sigma^\infty/p^n BGL_1(\mathcal{O}_Y(Y)) \rightarrow K/p^n(\mathcal{O}_Y(Y)).$$

This induces a map in the homotopy category of presheaves of spectra on $\text{Spec}(K)_{\text{ét}}$: $\Sigma^\infty/p^n B\mu_{p^\infty+} \rightarrow K/p^n$. By Suslin, it is a weak equivalence. Since weak equivalences are preserved by π^* we get $\pi^*\Sigma^\infty/p^n B\mu_{p^\infty+} \simeq \pi^*K/p^n$. Of course, weak equivalences are not preserved anymore by j_* but now we may argue exactly like in the proof of Lemma 3.3. \square

There is a natural map of presheaves of spectra (in the homotopy category)

$$\Sigma^\infty/p^n B\mu'_{p^\infty+} \rightarrow KH'/p^n.$$

For a log-syntomic-étale scheme $Z \rightarrow X$, where $\tilde{Z} = \coprod Y$ for connected components Y , it is defined by the following sequence of maps

$$\begin{aligned} \Sigma^\infty/p^n B\mu'_{p^\infty+}(Z) &= \bigvee \Sigma^\infty/p^n B\mu_{p^\infty}(\mathcal{O}_Y(Y))_+ \xrightarrow{\sim} \bigvee K/p^n(\mathcal{O}_Y(Y)) \\ &\xleftarrow{h} \bigvee K^Q/p^n(Y) \rightarrow \bigvee K/p^n(Y) \rightarrow \bigvee KH/p^n(Y) = KH'/p^n(Z). \end{aligned}$$

Here $K^Q/p^n(Y)$ denotes the Waldhausen spectrum of the category of locally free sheaves on Y . Since for Y affine, $K^Q/p^n(Y) \xrightarrow{\sim} K/p^n(\mathcal{O}_Y(Y))$, the natural map h induces weak equivalence on the associated presheaves.

Returning to the proof of our theorem, it suffices to show that, for $m \geq \max(2d, 1)$ and $i \geq 2d + m/2$, the composition

$$H^{-m}(X_{\text{sét}}, \Sigma^\infty/p^n B\mu'_{p^\infty+}(i)) \rightarrow H^{-m}(X_{\text{sét}}, KH'/p^n(i)) \rightarrow H^{-m}(X_{K,\text{két}}, KH/p^n(i))$$

has cokernel annihilated by a constant depending only on p, d, i , and m . Since p is invertible on K the natural map $K/p^n \rightarrow KH/p^n$ of presheaves of spectra on $X_{K,\text{két}}$ is a weak equivalence [29, Prop. 1.6] (in fact a pointwise weak equivalence). It suffices thus to prove the above statement for the composition

$$\omega : H^{-m}(X_{\text{sét}}, \Sigma^\infty/p^n B\mu_{p^\infty+}(i)) \rightarrow H^{-m}(X_{\text{sét}}, K'/p^n(i)) \rightarrow H^{-m}(X_{K,\text{két}}, K/p^n(i)),$$

where K'/p^n denotes the presheaf $Z \mapsto \vee K/p^n(Y)$ (where $\tilde{Z} = \coprod Y$) on $X_{\text{sét}}$.

We will do that by studying the induced map on the E_2 -terms of the associated local-global spectral sequences. First, recall the following proposition [18, 5.13]

Proposition 3.5. *There exists a cohomological spectral sequence $E_r^{p,q}$, $r \geq 2$, such that*

$$E_2^{p,q} = \begin{cases} H^p(X_{\text{két}}, \mathbf{Z}/p^n(q/2)) & \text{for } q - p \geq 0 \text{ and } q \text{ even} \\ 0 & \text{for } q - p \geq 0 \text{ and } q \text{ odd.} \end{cases}$$

This spectral sequence converges strongly to $H^{p-q}(X_{\text{két}}, K/p^n)$ for $q - p \geq 1$.

For the log-syntomic-étale site we have similarly the following lemma.

Lemma 3.6. *For each integer $k \geq 0$, there is a spectral sequence with differentials d_r of bidegree $(r, r-1)$:*

$$E_2^{p,q} = \begin{cases} H^p(X_{\text{sét}}, j_* \mathbf{Z}/p^n(q/2)) & \text{for } q - p \geq 0, q \text{ even}, q \leq k \\ 0 & \text{for } q - p \geq 0, q \text{ odd or } q > k. \end{cases}$$

This spectral sequence converges strongly to $H^{p-q}(X_{\text{sét}}, \Sigma^\infty/p^n B\mu'_{p^\infty}(k)_+)$ for $q - p \geq 1$. The differential d_r in the above spectral sequence maps $E_r^{p,q}$ to $E_r^{p+r, q+r-1}$.

Proof. This is just the descent spectral sequence, the crucial point being that by taking the Postnikov truncation we have enough control over the cohomological dimension. Recall the construction [13, 3.4]. Let $\mathcal{E} = \Sigma^\infty/p^n B\mu'_{p^\infty}(k)_+$. We start with a Postnikov tower $\{\mathcal{E}(i)\}$ of \mathcal{E} . Let F_i denote the fiber of $\mathcal{E}(i) \rightarrow \mathcal{E}(i-1)$. By taking fibrant replacements, we get a commutative diagram

$$\begin{array}{ccccc} \dots & \longleftarrow & \mathcal{E}(i-1) & \longleftarrow & \mathcal{E}(i) & \longleftarrow & \dots \\ & & \downarrow & & \downarrow & & \\ \dots & \longleftarrow & \mathcal{E}(i-1)^f & \longleftarrow & \mathcal{E}(i)^f & \longleftarrow & \dots \end{array}$$

where the bottom row is built from fibrant objects and fibrations and the fiber of $\mathcal{E}(i)^f \rightarrow \mathcal{E}(i-1)^f$ is a fibrant replacement of the Eilenberg-McLane presheaf F_i . By Lemma 3.3, we have

$$\pi_l(F_i^f)(Z) = H^{i-l}(Z_{\text{sét}}, \tilde{\pi}_i \mathcal{E}) = \begin{cases} H^{i-l}(Z_{\text{sét}}, j_* \mathbf{Z}/p^n(i/2)) & \text{for } i \geq 0, i \text{ even}, i \leq k \\ 0 & \text{for } i \geq 0, i \text{ odd or } i > k. \end{cases}$$

Clearly the natural map

$$\mathcal{E} = \mathcal{E}(k) = \text{proj} \lim_i \mathcal{E}(i) \rightarrow \text{proj} \lim_i \mathcal{E}(i)^f = \mathcal{E}(k)^f$$

is a weak equivalence. So we can take $\text{proj} \lim_i \mathcal{E}(i)^f$ for a fibrant replacement of \mathcal{E} . The tower of fibrations $\{\mathcal{E}(i)^f\}$, after renumbering, yields the spectral sequence above. Strong convergence follows since $\tilde{\pi}_i \mathcal{E} = 0$ for $i > k$. \square

We need to understand the action of the Adams operations on these spectral sequences. The computations are rather standard and follow [21]. Recall the definition of certain constants [21, 3.4]. Let l be a positive integer, and let w_l be the greatest common divisor of the set of integers $k^N(k^l - 1)$, as k runs over the positive integers and N is large enough with respect to l . Let $M(k)$ be the product of the w_l 's for $2l < k$. An odd prime l divides $M(k)$ if and only if $l < (k/2) + 1$.

Let us look first at the spectral sequence from Lemma 3.6. By Corollary 3.4 we have a weak equivalence

$$\Sigma^\infty/p^n B\mu'_{p^\infty+} \xrightarrow{\sim} j_*\pi^* K/p^n.$$

Via this weak equivalence we transfer the action of the Adams operations ψ^k for k prime to p from the spectrum K/p^n to $\Sigma^\infty/p^n B\mu'_{p^\infty+}$. This action is compatible with our spectral sequence and ψ^k acts on $E_2^{p,2j}$ via its action on $\pi_{2j}(bu/p^n)$. That is by multiplication by k^j , $j \geq 0$.

Now (see [21, 3.3.2]) different eigenvalues of ψ^k on $E_r^{p,2i}$ and $E_r^{p+2j+1,2i+2j}$ must be congruent modulo the order of the differential $d_r = d_{2l+1}$. This forces $w_l d_{2l+1} = 0$. As $d_r = 0$ for $r > i$ we get that $M(i)E_\infty^{p,q}$ is canonically a subobject of $E_2^{p,q}$ containing $M(i)^2 E_\infty^{p,q}$.

Similarly, for the spectral sequence from Proposition 3.5, by studying the action of the Adams operations, we get that $M(2d)E_\infty^{p,q}$ is canonically a subobject of $E_2^{p,q}$ containing $M(2d)^2 E_\infty^{p,q}$.

We are now ready to study the map

$$\omega : H^{-m}(X_{\text{sét}}, \Sigma^\infty/p^n B\mu'_{p^\infty} \langle i \rangle_+) \rightarrow H^{-m}(X_{K,\text{két}}, K/p^n \langle i \rangle)$$

Let $F^s H^{-m}(X_{\text{sét}}, \Sigma^\infty/p^n B\mu'_{p^\infty} \langle i \rangle_+)$ and $F^s H^{-m}(X_{K,\text{két}}, K/p^n \langle i \rangle)$ denote the filtrations defined by the descent spectral sequences. We have

$$\begin{aligned} F^s / F^{s+2} H^{-m}(X_{\text{sét}}, \Sigma^\infty/p^n B\mu'_{p^\infty} \langle i \rangle_+) &= E_\infty^{s,s+m}, \\ F^s / F^{s+2} H^{-m}(X_{K,\text{két}}, K/p^n \langle i \rangle) &= E_\infty^{s,s+m}. \end{aligned}$$

Since $E_\infty^{s,s+m}$ is a subquotient of $E_2^{s,s+m}$ (thus, respectively, of $H^s(X_{\text{sét}}, j_*\mathbf{Z}/p^n((s+m)/2))$ and of $H^s(X_{K,\text{két}}, \mathbf{Z}/p^n((s+m)/2))$) and $s+m \leq i$ the induced gradings are finite. Assume that $s+m$ is even and take a class $x \in F^s / F^{s+2} H^{-m}(X_{K,\text{két}}, K/p^n \langle i \rangle) = E_\infty^{s,s+m}$. Lift x to an element $x' \in K^{s,s+m}$, where $K^{s,s+m} \hookrightarrow H^s(X_{\text{két}}, \mathbf{Z}/p^n((s+m)/2))$ is the kernel of all the differentials d_r , $r \geq 2$. By Corollary 2.13, $p^C x'$, for an integer $C = C(s, m, p)$ depending only on s, m, p , comes from an element $z \in H^s(X_{\text{sét}}, j_*\mathbf{Z}/p^n((s+m)/2))$. The cokernel of the inclusion $K^{s,s+m} \hookrightarrow H^s(X_{\text{sét}}, j_*\mathbf{Z}/p^n((s+m)/2))$ is annihilated by $M(i)$. Clearly, $w = M(i)z$ maps to $M(i)p^C x'$. We get that the cokernel of the map

$$\omega : F^s / F^{s+2} H^{-m}(X_{\text{sét}}, \Sigma^\infty/p^n B\mu'_{p^\infty} \langle i \rangle_+) \rightarrow F^s / F^{s+2} H^{-m}(X_{K,\text{két}}, K/p^n \langle i \rangle)$$

is annihilated by $M(i)p^C$. Varying s from 0 to $2d$ we get the statement of the theorem. \square

The following corollary follows immediately.

Corollary 3.7. *Let X be a proper semistable scheme over V^\times . Then, for any $m \geq \max(2d, 1)$, where $d = \dim X_K$, and for any $i \geq 2d + m/2$, the restriction map*

$$j^* : \text{proj} \lim_n KH_m^{\leq i}(X_{\overline{V},\text{sét}}, \mathbf{Z}/p^n) \otimes \mathbf{Q} \rightarrow \text{proj} \lim_n KH_m^{\leq i}(X_{\overline{K},\text{két}}, \mathbf{Z}/p^n) \otimes \mathbf{Q},$$

is surjective.

For any log-scheme X over V^\times , the natural map $KH/p^n \rightarrow j_*KH/p^n$ of presheaves of spectra on $X_{\text{sét}}$, where $j : X_K \hookrightarrow X$ is the natural immersion, induces a restriction map on cohomology of sheaves of homotopy groups

$$j^* : H^*(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n)) \rightarrow H^*(X_{K,\text{két}}, \mathbf{Z}/p^n(r)), \quad r \geq 0.$$

Corollary 3.8. *Assume that $\zeta_n \in V$.*

- (1) *Let X be a proper semistable scheme over V^\times or a finite base change of such. Then, for any $0 \leq i \leq r-1$, the cokernel of the restriction map*

$$j^* : H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n)) \rightarrow H^i(X_{K,\text{két}}, \mathbf{Z}/p^n(r))$$

is annihilated by p^M for an integer M dependent only on p, r , and i . Moreover, for $0 \leq i \leq r-2$, there exists a natural map

$$s : H^i(X_{K,\text{két}}, \mathbf{Z}/p^n(r)) \rightarrow H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n))$$

such that $j^*s = p^N$ for an integer N dependent only on p, r , and i .

(2) Let X be a log-smooth scheme over V^\times . Then, for any $0 \leq i \leq r-1 \leq p-2$, the restriction map

$$j^* : H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n)) \rightarrow H^i(X_{K,\text{két}}, \mathbf{Z}/p^n(r))$$

is surjective and has a natural section for $0 \leq i \leq r-2 \leq p-2$.

Proof. We will keep the notation from the proof of Theorem 3.2. Consider the map $\Sigma^\infty/p^n B\mu'_{p^\infty+} \rightarrow KH'/p^n$ of presheaves of spectra on $X_{\text{sét}}$. Compose it with the map (in the homotopy category) $\omega : KH'/p^n \rightarrow \Omega^2 KH/p^n$. For $r \geq 0$, on the level of sheaves of homotopy groups, we get a map

$$j_* \mathbf{Z}/p^n(r) \rightarrow \tilde{\pi}_{2r}(KH'/p^n) \xrightarrow{\omega} \tilde{\pi}_{2r+2}(KH/p^n).$$

Since $Rj'_* \mathbf{Z}/p^n(r) \simeq \mathbf{Z}/p^n(r)$ for $j'_* : X_{tr,\text{ét}} \hookrightarrow X_{K,\text{két}}$ this yields a map

$$j_* j'_* \mathbf{Z}/p^n(r) \rightarrow \tilde{\pi}_{2r+2}(KH/p^n).$$

Hence a map

$$\begin{aligned} H^i(X_{\text{sét}}, j_* j'_* \mathbf{Z}/p^n(r)) &\xrightarrow{\alpha} H^i(X_{\text{sét}}, \tilde{\pi}_{2r+2}(KH/p^n)) \rightarrow H^i(X_{K,\text{két}}, \tilde{\pi}_{2r+2}(KH/p^n)) \\ &\xleftarrow{\sim} H^i(X_{K,\text{két}}, \mathbf{Z}/p^n(r+1)) \xrightarrow{\sim} H^i(X_{tr,\text{két}}, \mathbf{Z}/p^n(r+1)). \end{aligned}$$

It suffices now to show that the cokernel of this map is annihilated by p^M for an integer M dependent only on p , r , and i . But by construction the above map is equal to the natural map

$$\beta : H^i(X_{\text{sét}}, j_* j'_* \mathbf{Z}/p^n(r)) \rightarrow H^i(X_{K,\text{két}}, \mathbf{Z}/p^n(r)) \xrightarrow{\zeta_n} H^i(X_{K,\text{két}}, \mathbf{Z}/p^n(r+1)),$$

where the last map is clearly an isomorphism. The first parts of our corollary follows now from Corollary 2.13.

Concerning the existence of the natural section, we will show it in the case of (2). For (1) the argument is analogous. We have the following commutative diagram

$$(3.2) \quad \begin{array}{ccccc} & & H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n)) & \xrightarrow{j^*} & H^i(X_{K,\text{két}}, \tilde{\pi}_{2r}(KH/p^n)) \\ & & \tilde{\beta}_n \alpha \uparrow & & \zeta_n \zeta_n \uparrow \wr \\ H^i(X_{\text{sét}}, \mathcal{S}_n(r-2)) & \xrightarrow{\alpha_{r-2}} & H^i(X_{\text{sét}}, j_* j'_* \mathbf{Z}/p^n(r-2)) & \xrightarrow{h} & H^i(X_{tr,\text{ét}}, \mathbf{Z}/p^n(r-2)) \\ \uparrow \varepsilon^* & & \uparrow \varepsilon^* & & \parallel \\ H^i(X_{\text{sé}}, \mathcal{S}_n(r-2)) & \xrightarrow{\alpha_{r-2}} & H^i(X_{\text{sé}}, j_{\text{sé}*} j'_* \mathbf{Z}/p^n(r-2)) & \xrightarrow{h} & H^i(X_{tr,\text{ét}}, \mathbf{Z}/p^n(r-2)) \end{array}$$

By Theorem 2.2, the composition $h\alpha_{r-2}$ is an isomorphism. We take the composition $\tilde{\beta}_n \alpha \alpha_{r-2} \varepsilon^* (h\alpha_{r-2})^{-1} \zeta_n^{-2}$ for the required section. To show that it is a natural section it suffices to show that the composition

$$\tilde{\beta}_n \alpha : H^i(X_{\text{sét}}, j_* j'_* \mathbf{Z}/p^n(r)) \rightarrow H^i(X_{\text{sét}}, \tilde{\pi}_{2r+2}(KH'/p^n)) \xrightarrow{\omega} H^i(X_{\text{sét}}, \tilde{\pi}_{2r+2}(KH/p^n)) \xrightarrow{\tilde{\beta}_n} H^i(X_{\text{sét}}, \tilde{\pi}_{2r+4}(KH/p^n))$$

is natural. This is clear for the first map in the composition. So we need to show that the map $\tilde{\beta}_n \omega$ is natural. Let $Y \rightarrow X$ be a map of two proper log-smooth schemes over V^\times . It suffices to check that for any map $f : T \rightarrow S$ over the map $Y \rightarrow X$ of schemes T, S log-syntomic-étale over Y, X , respectively, in the following diagram the outer square commutes

$$\begin{array}{ccccc} KH'/p^n(S) & \xrightarrow{\omega} & \Omega^2 KH/p^n(S) & \xrightarrow{\tilde{\beta}_n} & \Omega^4 KH/p^n(S) \\ \downarrow f & & \downarrow f & & \downarrow f \\ KH'/p^n(T) & \xrightarrow{\omega} & \Omega^2 KH/p^n(T) & \xrightarrow{\tilde{\beta}_n} & \Omega^4 KH/p^n(T) \end{array}$$

Or that any map $h : KH'/p^n(S) \rightarrow KH'/p^n(T)$ that makes the below diagram commute (in the homotopy category) is unique (in the homotopy category) up to the multiplication by the Bott element $\tilde{\beta}_n$.

$$\begin{array}{ccc} KH'/p^n(S) & \xrightarrow{f} & KH'/p^n(T) \\ \downarrow h & & \downarrow \tilde{\beta}_n \\ \Omega^2 KH/p^n(T) & \longrightarrow & KH'/p^n(T) \end{array}$$

This follows by chasing the below diagram starting with the element $(0, f) \in [A, KH^0/p^n(T)] \oplus [A, KH'/p^n(T)]$, for $A = KH'/p^n(S)$.

$$\begin{array}{ccccccc} \longrightarrow & [A, KH/p^n(T)] & \longrightarrow & [A, KH^0/p^n(T)] \oplus [A, KH'/p^n(T)] & \longrightarrow & [A, KH'^0/p^n(T)] & \longrightarrow \\ & \downarrow \tilde{\beta}_n & & \downarrow \tilde{\beta}_n & & \downarrow \tilde{\beta}_n & \\ \longrightarrow & [A, \Omega^2 KH/p^n(T)] & \longrightarrow & [A, \Omega^2 KH^0/p^n(T)] \oplus [A, \Omega^2 KH'/p^n(T)] & \longrightarrow & [A, \Omega^2 KH'^0/p^n(T)] & \longrightarrow \\ & & & & & & \square \end{array}$$

By a similar argument we get the following

Corollary 3.9. *Let X be a proper log-smooth scheme over V^\times . Then, for any $0 \leq i \leq r-2$, the surjection*

$$j^* : \text{proj} \lim_n H^i(X_{\overline{V}, \text{sét}}, \tilde{\pi}_{2r}(KH/p^n)) \otimes \mathbf{Q} \rightarrow H^i(X_{\overline{K}, \text{két}}, \mathbf{Q}_p(r))$$

has a natural section.

Proof. By [20, Theorem 2.9] there exists a ramified extension V_1 of V such that the base change $X_{V_1^\times}$ has a semistable model. That is, there exists a log-blow-up $\pi : Y \rightarrow X_{V_1^\times}$ such that Y is a semistable scheme (with no multiplicities in the special fiber). It suffices now to show that part (1) of Corollary 3.8 holds with X being the log-scheme $X_{V_1^\times}$ or its finite base change. More precisely, we will show that if $\zeta_n \in V$ and X over V^\times is a proper log-smooth scheme admitting a log-blow-up $Y \rightarrow X$ such that Y is semistable or a finite base change of such then there exists a natural map

$$s : H^i(X_{K, \text{két}}, \mathbf{Z}/p^n(r)) \rightarrow H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n))$$

such that $j^*s = p^N$ for an integer N dependent only on p, r , and i . Here

$$j^* : H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n)) \rightarrow H^i(X_{K, \text{két}}, \mathbf{Z}/p^n(r))$$

is the restriction map.

Consider the diagram (3.2) for X as above. We claim that the bottom composition $\alpha_{r-1}h$ has kernel and cokernel annihilated by p^M for a constant M depending only p, r , and j . Indeed we have the following commutative diagram

$$\begin{array}{ccc} H^i(Y_{\text{sé}}, \mathcal{S}_n(r)) & \xrightarrow{\alpha_{i,r}} & H^i(X_{tr, \text{ét}}, \mathbf{Z}/p^n(r)') \\ \pi^* \uparrow \wr & & \parallel \\ H^i(X_{\text{sé}}, \mathcal{S}_n(r)) & \xrightarrow{\alpha_{i,r}} & H^i(X_{tr, \text{ét}}, \mathbf{Z}/p^n(r)'). \end{array}$$

Since log-blow-ups do not change the log-syntomic cohomology [17, Proposition 2.3] our claim follows from Theorem 2.2.

Now we can argue exactly as in the proof of Corollary 3.8. \square

Let X be a proper semistable scheme over V^\times . Assume that $\zeta_n \in V$. Let $0 \leq i \leq r-2 \leq p-2$ and consider the following composition

$$\begin{aligned} \gamma_r : H^i(X_{\text{sét}}, \mathcal{S}_n(r)) &\xleftarrow{t^2} H^i(X_{\text{sét}}, \mathcal{S}_n(r-2)) \xrightarrow{\alpha_{r-2}} H^i(X_{\text{sét}}, j_* j'_* \mathbf{Z}/p^n(r-2)) \\ &\xrightarrow{\tilde{\beta}_n \alpha} H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n)), \end{aligned}$$

where the last map is from the proof of Corollary 3.8 and $t \in H^0(X_{\text{sét}}, \mathcal{S}_n(1))$ is the element that corresponds to $\zeta_n \in H^0(K_{\text{ét}}, \mathbf{Z}/p^n(1))$ under the period isomorphism $\alpha_{0,1} : H^0(V_{\text{sét}}^\times, \mathcal{S}_n(1)) \simeq H^0(K_{\text{ét}}, \mathbf{Z}/p^n(1))$ [28]. Multiplication by t is an isomorphism. In general, for $0 \leq i \leq r-2$, we have the following rational version of the above composition

$$\begin{aligned} \gamma_r : \quad \text{proj} \lim_n H^i(X_{\text{sét}}, \mathcal{S}_n(r)) \otimes \mathbf{Q} &\xleftarrow{t^2} \text{proj} \lim_n H^i(X_{\text{sét}}, \mathcal{S}_n(r-2)) \otimes \mathbf{Q} \xrightarrow{\alpha_{r-2}} \\ &\text{proj} \lim_n H^i(X_{\text{sét}}, j_* j'_* \mathbf{Z}/p^n(r-2)) \otimes \mathbf{Q} \xrightarrow{\tilde{\beta}_n \alpha} \text{proj} \lim_n H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n)) \otimes \mathbf{Q}. \end{aligned}$$

Again, multiplication by t is an isomorphism.

Corollary 3.10. *Assume that $\zeta_n \in V$. Let X be a proper semistable scheme over V^\times .*

(1) *Then, for any $0 \leq i \leq r-2 \leq p-2$, the map*

$$\gamma_r : H^i(X_{\text{sét}}, \mathcal{S}_n(r)) \rightarrow H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n))$$

is injective and has a natural section.

(2) *Then, for $0 \leq i \leq r-2$, the map*

$$\gamma_r : \text{proj} \lim_n H^i(X_{\text{sét}}, \mathcal{S}_n(r)) \otimes \mathbf{Q} \rightarrow \text{proj} \lim_n H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n)) \otimes \mathbf{Q}$$

is injective and has a natural section.

Proof. For the first part, consider again the diagram (3.2). Note that by Lemma 2.12 the left pullback map ε^* is an isomorphism. It follows that the top composition $h\alpha_{r-2}$ is an isomorphism as well. Define a map $a : H^i(X_{\text{sét}}, \tilde{\pi}_{2r}(KH/p^n)) \rightarrow H^i(X_{\text{sét}}, \mathcal{S}_n(r))$ as the composition $a = t^2(h\alpha_{r-2})^{-1}\zeta_n^{-2}j^*$. We easily check that $a\gamma_r = \text{Id}$. For the second part, argue similarly. \square

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