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Cohomology of Crystalline Smooth Sheaves

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Abstract. We define and study a candidate for 'arithmetic' cohomology theory with values in crystalline p-adic smooth sheaves. We show that it injects into arithmetic étale cohomology and prove a duality result.

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1. Introduction

Let V denote a complete discrete valuation ring with a fraction field K of characteristic 0 and a perfect residue field k of characteristic p. Let X be a smooth and proper scheme over V.

Recall [3] that crystalline p-adic smooth sheaves are $\mathbf{Q}_p$-adic étale smooth sheaves on X_K characterized by associating to them certain filtered convergent F-isocrystals on the special fiber X_k: $\mathcal{O}_{\widehat{X}} \otimes K$ – vector bundles equipped with a convergent in the open unit disc connection, a filtration, Griffiths transversal to the above connection, and an action of the Frobenius on the corresponding convergent isocrystal. Étale cohomology tends to form such sheaves.

In various arithmetic applications (for example, in [11]) the need for 'arithmetic' cohomology theory over X with values in crystalline p-adic smooth sheaves arises. We define and study in this paper a candidate for such a cohomology theory. The construction is similar to that of the generalized syntomic cohomology of crystalline local systems on X_K from [10].

It has turned out (cf., [11]) to be important to understand the relation between our cohomology and the étale one. We show here that the cohomology we define maps to the arithmetic étale cohomology of X_K and that this map is injective defining a subspace of étale cohomology classes 'coming from the integral model'. We prove this injectivity by a detailed study of the behaving of the descent (from the geometric to the arithmetic cohomologies) spectral sequence over X with the corresponding one over X_K – a study which was also important in our proof of the f-ness of K-theory classes in [11].

Finally, we prove that, in the case k is finite, our cohomology satisfies a duality. This generalizes the results of Bloch and Kato [1] for $X = V$ to higher dimensions and agrees with thinking about the cohomology introduced in this paper as a 'good' arithmetic p-adic cohomology of X.

Throughout the paper p will be a fixed prime, for a field K, $\overline{K}$ will denote a fixed algebraic closure of K, and, for a scheme X, $\mathcal{X}$ will denote the associated formal scheme.

2. Preliminaries

We gather in this section all the basic properties of rings of periods and crystalline smooth sheaves we will need.

2.1. RINGS OF PERIODS

2.1.1. *Definitions*

Let V denote a complete discrete valuation ring with a fraction field K of characteristic 0 and a perfect residue field k of characteristic p. Choose a uniformizer π of V. Let $V_0 = W(k)$ be the ring of Witt vectors with coefficients in k and K_0 its fraction field. Let R be a smooth V-algebra such that $R/pR \neq 0$. Consider the p-adic completion $\widehat{R}$. For simplicity, we will assume that $\mathrm{Spec}(R/pR)$ is connected, which implies that $\widehat{R}$ is a normal domain. In general, $\widehat{R}$ is a product of normal domains and what follows applies to each factor. We will also require R to be small, i.e., that there is an étale map $V[T_1^{\pm 1}, \dots, T_d^{\pm 1}] \to R$. It implies that Frobenius is surjective on $\overline{\widehat{R}}/p$.

We will now briefly recall the construction and properties of the rings $B^+(\widehat{R})$, $B(\widehat{R})$, and $B_{dR}(\widehat{R})$ defined by Faltings in [3]. Denote by $\overline{\widehat{R}}$ the normalization of $\widehat{R}$ in the maximal étale extension of $\widehat{R}[1/p]$. Let $S(\widehat{R}) = \operatorname{proj\,lim} \overline{\widehat{R}}/p\overline{\widehat{R}}$, where the maps in the projective system are the pth power maps. With addition and multiplication defined coordinatewise $S(\widehat{R})$ is a ring of characteristic p. There is a homomorphism θ from the ring of Witt vectors $W(S(\widehat{R}))$ to $\overline{\widehat{R}}^{\wedge}$: θ maps $(x_0, x_1, \dots) \in W(S(\widehat{R}))$, $x_n = (x_{nm}) \in S(\widehat{R})$, to the limit over m of $\hat{x}_{0m}^{p^m} + p\hat{x}_{1m}^{p^{m-1}} + \cdots + p^m \hat{x}_{mm}$, where $\widehat{\ }$ means a lift from $\overline{\widehat{R}}/p\overline{\widehat{R}}$ to $\overline{\widehat{R}}^{\wedge}$. The map θ is surjective. Its kernel is generated by $\xi = [(p)] + p[(-1)]$, where $(p), (-1) \in S(\widehat{R})$ are the reductions mod p of sequences of p-roots of p and -1, respectively (if $p \neq 2$ we may and will choose $(-1) = -1$).

The ring $B^+(\widehat{R})$ is defined as the completion of the divided power envelope $D_\xi(W(S(\widehat{R})))$ of the ideal $\xi W(S(\widehat{R}))$ in $W(S(\widehat{R}))$ with respect to the topology defined by the ideals $(I^{[m]} + p^n W(S(\widehat{R})))$, where I is the PD ideal of $D_\xi(W(S(\widehat{R})))$. It is an algebra over $B^+(V)$. The Frobenius automorphism on $S(\widehat{R})$ induces an automorphisms ϕ on $W(S(\widehat{R}))$ and $B^+(\widehat{R})$. $B^+(\widehat{R})$ is equipped with a decreasing separated filtration $F^n B^+(\widehat{R})$ such that $\phi(F^n B^+(\widehat{R})) \subset p^n B^+(\widehat{R})$ and $\mathrm{gr}_F^0(B^+(\widehat{R})) = \overline{\widehat{R}}^{\wedge}$ (in fact, if we denote by $\mathrm{Fil}^{\cdot} B^+(\widehat{R})$ the filtration by the closures of $I^{[n]}$, then $F^n B^+(\widehat{R})$ is the ideal consisting of those elements of $\mathrm{Fil}^n B^+(\widehat{\overline{R}})$ whose ϕ-image is divisible by p^n). The Galois group $\mathrm{Gal}(\overline{\widehat{R}}/\widehat{R})$ acts on $B^+(\widehat{R})$. The action is continuous, commutes with ϕ and preserves the filtration. There exists an element $t \in F^1 B^+(\widehat{R})$ such that $\phi(t) = pt$ and $\mathrm{Gal}(\overline{\widehat{R}}/\widehat{R})$ acts on t via the cyclotomic character: if we fix $\varepsilon \in S(\widehat{R})$ – a sequence of nontrivial p-roots of unity, then $t = \log([\varepsilon])$.

The ring $B(\widehat{R})$ is defined as the ring $B^+(\widehat{R})[p^{-1}, t^{-1}]$ with the induced topology, filtration, Frobenius and the Galois action. Its associated graded is isomorphic to

$$\oplus_{n\in\mathbf{Z}} \mathrm{gr}^0_F(B^+(\widehat{R}))[1/p](n) = \oplus_{n\in\mathbf{Z}} \overline{\overline{R}}^\wedge[1/p](n).$$

Remark 1. The reader will notice that the ring $B(V)$ we use here differs slightly from the one introduced by Fontaine [5] and the one used in [10]. It is easy to check though that the results and constructions from [10] hold if we replace the ring of periods used there with the ring $B(V)$ used in this article.

To construct the ring $B_{dR}(\widehat{R})$, consider again the homomorphism $\theta\colon W(S(\widehat{R})) \to \overline{\widehat{R}}^\wedge$. As a V_0-linear map θ extends to a V-linear homomorphism $\theta\colon V \otimes_{V_0} W(S(\widehat{R})) \to \overline{\widehat{R}}^\wedge$. The kernel of this homomorphism is a principal ideal in $V \otimes_{V_0} W(S(\widehat{R}))$ generated by $\xi_\pi = [(\pi)] + \pi[(-1)]$. Set $W_V(S(\widehat{R})) = V \otimes_{V_0} W(S(\widehat{R}))$. Then

$$B^+_{dR}(\widehat{R}) = \operatorname{proj\,lim} W_V(S(\widehat{R}))/(\xi_\pi W_V(S(\widehat{R})))^n[1/p].$$

It is equipped with the projective limit topology, where every quotient $W_V(S(\widehat{R}))/(\xi_\pi W_V(S(\widehat{R})))^n[1/p]$ has the p-adic topology. $B^+_{dR}(\widehat{R})$ has a natural filtration defined by the closures of the powers of the ideal $(\xi_\pi W_V(S(\widehat{R})))[1/p] = tW_V(S(\widehat{R})))[1/p]$ and a continuous Galois action.

The ring $B_{dR}(\widehat{R})$ is defined as the fraction field of $B^+_{dR}(\widehat{R})$ with induced structures. The associated graded is isomorphic to the one associated to $B(\widehat{R})$.

2.1.2. *Fundamental Exact Sequences*

LEMMA 2.1. *Let $k \geqslant 0$. Fix $n \geqslant 0$. There exists an arbitrarily large number m making the following sequence of Galois modules*

$$0 \to \mathbf{Z}/p^n t^{\{k\}} \to F^k B^+(\widehat{R})_{n,m} \xrightarrow{1-p^{-k}\phi} B^+(\widehat{R})_{n,m} \to 0$$

exact. Here, if we set $0 \leqslant r(k) < p-1$ and $q(k)$ by the equality $k = (p-1)q(k) + r(k)$, then $t^{\{k\}} = t^{r(k)}\gamma_{q(k)}(t^{p-1}/p)$ and $B^+(\widehat{R})_{n,m} = B^+(\widehat{R})/(p^n B^+(\widehat{R}) + \mathrm{Fil}^m B^+(\widehat{R}))$.

Proof. By [10, prop. 5.1] we have an exact sequence

$$0 \to \mathbf{Z}_p t^{\{k\}} \to F^k D_\xi(W(S(\widehat{R})))^\wedge \xrightarrow{1-p^{-k}\phi} D_\xi(W(S(\widehat{R})))^\wedge \to 0.$$

Since

$$F^k D_\xi(W(S(\widehat{R}))^\wedge/(p^n D_\xi(W(S(\widehat{R}))^\wedge + I^{[m]}) \cap F^k D_\xi(W(S(\widehat{R}))^\wedge$$
$$= F^k B^+(\widehat{R})/(p^n B^+(\widehat{R}) + \mathrm{Fil}^m B^+(\widehat{R})) \cap F^k B^+(\widehat{R}),$$

it suffices to show that the map

$$1 - p^{-k}\phi\colon (p^n D_\xi(W(S(\widehat{R}))^\wedge + I^{[m]}) \cap F^k D_\xi(W(S(\widehat{R}))^\wedge$$
$$\to (p^n D_\xi(W(S(\widehat{R}))^\wedge + I^{[m]})$$

is surjective and that $\mathbf{Z}_p t^{\{k\}} \cap (p^n D_\xi(W(S(\widehat{R}))^\wedge + I^{[m]}) = p^n \mathbf{Z}_p t^{\{k\}}$. If we take $m \geqslant k+1$, the last fact easily follows since t/ξ goes under the map into $\widehat{\overline{R}}^\wedge$ to a product of $p^{1/(p-1)}$ and a unit.

Concerning the surjection, note that $\phi(\xi^{[r]}) \subset p^{[r]} D_\xi(W(S(\widehat{R})))$. For $p \neq 2$, this immediately gives that, for m big enough, $\phi(I^{[m]}) \subset p^{n+k} D_\xi(W(S(\widehat{R})))$. Making sure that $m \geqslant k$, we get a surjection. For $p = 2$, we can not do as well. The problem arises because $ord_2(2^{[m]}) = \sum a_i$ if $m = \sum a_i 2^i$. Still, we can make it bigger than $n+k$ for arbitrarily large m, as wanted. □

PROPOSITION 2.1. *There are exact sequences of* $\mathrm{Gal}(\overline{\widehat{R}}/\widehat{R})$*-modules*

(i) $0 \to \mathbf{Q}_p(k) \to F^k B^+(\widehat{R})[1/p] \xrightarrow{1-p^{-k}\phi} B^+(\widehat{R})[1/p] \to 0, \quad \text{for} \quad k \geqslant 0,$

(ii) $0 \to \mathbf{Q}_p(k) \to F^k B(\widehat{R}) \xrightarrow{1-p^{-k}\phi} B(\widehat{R}) \to 0, \quad \text{for} \quad k \in \mathbf{Z}.$

Moreover, the surjections admit $\mathbf{Q}_p$*-linear continuous sections.*

Proof. Fix $k \geqslant 0$. For every n choose $m(n)$ as in Lemma 2.1 such that there are inclusions $(p^n B^+(\widehat{R}) + \mathrm{Fil}^{m(n)} B^+(\widehat{R})) \supset (p^{n+1} B^+(\widehat{R}) + \mathrm{Fil}^{m(n+1)} B^+(\widehat{R}))$. Passing to the limit with the exact sequences from Lemma 2.1 over the pairs $(n, m(n))$ one gets an exact sequence

$$0 \to \mathbf{Z}_p t^{\{k\}} \to F^k B^+(\widehat{R}) \xrightarrow{1-p^{-k}\phi} B^+(\widehat{R}) \to 0.$$

The existence of a $\mathbf{Z}_p$-linear continuous section follows now from the fact that $B^+(\widehat{R})_{n,m(n)}$ is $\mathbf{Z}/p^n$-free. The proposition easily follows from that. □

The map

$$D_\xi(W(S(\widehat{R}))) \to W_V(S(\widehat{R}))[1/p] \to B^+_{dR}(\widehat{R})$$

can be extended to a continuous Galois equivariant map from $B^+(\widehat{R})$ to $B^+_{dR}(\widehat{R})$. This map is injective. Indeed, every element of $B^+_{dR}(\widehat{R})$ can be written as an infinite sum $\sum a_n \xi^n/n!$, for $a_n \in W_V(S(\widehat{R}))[1/p]$. It belongs to $F^m B^+_{dR}(\widehat{R})$ if and only if $\theta(a_n) = 0$ for $n < m$. On the other hand, every element of $D_\xi(W(S(\widehat{R})))$ can be written as a finite sum $\sum a_n \xi^n/n!$, for $a_n \in W(S(\widehat{R}))$. One can also assume that if $a_n \neq 0$, then $\theta(a_n) \neq 0$. One computes now easily that the injection $D_\xi(W(S(\widehat{R}))) \hookrightarrow B^+_{dR}(\widehat{R})$

extends to the completion. The filtration induced on $B^+(\widehat{R})$ via the above embedding into $B^+_{dR}(\widehat{R})$ is by the closures of $I^{[n]}$, hence induces the correct filtration on $B^+(\widehat{R})[1/p]$ and $F^0B(\widehat{R})$.

PROPOSITION 2.2. *The following sequence of* $\mathrm{Gal}(\overline{\overline{R}}/\widehat{R})$*-modules is exact*

$$0 \to \mathbf{Q}_p \xrightarrow{\alpha} B(\widehat{R}) \oplus F^0 B_{dR}(\widehat{R}) \xrightarrow{\beta} B(\widehat{R}) \oplus B_{dR}(\widehat{R}) \to 0,$$

where $\alpha(x) = (x, x)$ *and* $\beta(x, y) = (x - \phi(x), x - y)$. *Moreover, the morphism* β *admits a* $\mathbf{Q}_p$*-linear continuous right inverse.*

Proof. Since $B(\widehat{R})^{\phi=1} \cap F^0 B_{dR}(\widehat{R}) = \mathbf{Q}_p$, we get the exactness on the left. Concerning the surjectivity of β, by Proposition 2.1 it suffices to show that $B(\widehat{R})^{\phi=1} + F^0 B_{dR}(\widehat{R}) = B_{dR}(\widehat{R})$ or that $t^{-n}F^0 B_{dR}(\widehat{R}) \subset B(\widehat{R})^{\phi=1} + F^0 B_{dR}(\widehat{R})$ for $n \geqslant 0$. Since $\mathrm{gr}^0_F(B_{dR}(\widehat{R})) = \mathrm{gr}^0_F(B^+(\widehat{R})[1/p])$, this last fact follows from the inclusion

$$B^+(\widehat{R})[1/p] \subset F^1 B^+(\widehat{R})[1/p] + \{a \in B^+(\widehat{R})[1/p] | \phi(a) = p^r a\} \quad \text{for } r \geqslant 1,$$

which could be proved paraphrasing [1, 1.17.3]. The key points are the surjection of the projection $\mathrm{proj\,lim}\, \overline{\overline{R}}^\wedge \to \overline{\overline{R}}^\wedge$, $(u_n) \to u_0$, onto the units of $\overline{\overline{R}}^\wedge$, which follows from Lemma 2.2 below; and the fact that $\mathrm{gr}^0_F(B^+(\widehat{R})[1/p]) = \overline{\overline{R}}^\wedge[1/p]$. One gets in fact somewhat more, namely that $t^{-n}F^0 B_{dR}(\widehat{R})$ is already contained in $(t^{-n}B^+(\widehat{R}))^{\phi=1} + F^0 B_{dR}(\widehat{R})$.

To show that β has a continuous section it suffices to show the same for the surjections

$$t^{-k}B^+(\widehat{R})[1/p] \oplus F^0 B_{dR}(\widehat{R}) \xrightarrow{\beta} t^{-k}B^+(\widehat{R})[1/p] \oplus t^{-k}F^0 B_{dR}(\widehat{R}), \quad k \geqslant 0.$$

That, in turn, follows from the fact that the surjections

$$F^k B^+(\widehat{R})[1/p] \xrightarrow{1-p^{-k}\phi} B^+(\widehat{R})[1/p],$$
$$(t^{-k}B^+(\widehat{R})[1/p])^{\phi=1} \oplus F^0 B_{dR}(\widehat{R}) \to t^{-k}F^0 B_{dR}(\widehat{R}), \quad (x, y) \to x - y,$$

admit continuous sections by Proposition 2.1 and the argument of [1, 1.18] (here one uses Lemma 2.2 again and the fact that $\mathrm{gr}^{\cdot}_F(B_{dR}(\widehat{R})) = \oplus_{n \in \mathbf{Z}} \overline{\overline{R}}^\wedge(n)[1/p]$). $\square$

LEMMA 2.2. *Let* y *be a unit in* $\overline{\overline{R}}^\wedge$. *There exists a unit* $x \in \overline{\overline{R}}^\wedge$ *such that* $x^p = y$.

Proof. We can find a unit $y' \in \overline{\overline{R}}$ such that $y' \equiv y \bmod p^{1+2/(p-1)}\overline{\overline{R}}^\wedge$, and then, from the definition of $\overline{\overline{R}}$, a unit $x_2 \in \overline{\overline{R}}$ such that $x_2^p = y'$. Thus we get a solution of the congruence $x^p \equiv y \bmod p^{1+2/(p-1)}\overline{\overline{R}}^\wedge$. Now we apply Newton's method: having a solution $x_n \in \overline{\overline{R}}^\wedge$ of the above congruence mod $p^{1+n/(p-1)}\overline{\overline{R}}^\wedge$, $n \geqslant 2$, we can lift it to a solution $x_{n+1} \in \overline{\overline{R}}^\wedge$ mod $p^{1+(n+1)/(p-1)}\overline{\overline{R}}^\wedge$ such that $x_{n+1} \equiv x_n \bmod p^{n/(p-1)}\overline{\overline{R}}^\wedge$. In the limit we get $x \in \overline{\overline{R}}^\wedge$ such that $x^p = y$. It is cleary a unit. $\square$

2.2. CRYSTALLINE SMOOTH SHEAVES

2.2.1. *Convergent F-Isocrystals*

Let X be a k-scheme of finite type. An affine enlargement of X/V is a triple (A, I, z), where A is a flat V-algebra, I is an ideal of A, nilpotent modulo p, A is I-adically complete and $z\colon \mathrm{Spec}(A/I) \to X$ is a V-morphism. Endowing the category of affine enlargements of X/V with the Zariski topology and taking sheaves on this category one obtains the convergent topos $(X/V)_{\mathrm{conv}}$ of Ogus [13]. In particular, we get a sheaf $\mathcal{K}_{X/V}$ in $(X/V)_{\mathrm{conv}}$ by setting $\mathcal{K}_{X/V}(A) := A \otimes K$ for any enlargement (A, I, z) of X/V. Now, a convergent isocrystal $\mathcal{E}$ is a crystal of $\mathcal{K}_{X/V}$-modules in $(X/V)_{\mathrm{conv}}$, i.e., a $\mathcal{K}_{X/V}$-sheaf such that each morphism of enlargements $(A, I, z_A) \to (B, J, z_B)$ gives rise to an isomorphism $\mathcal{E}(B) \xleftarrow{\sim} \mathcal{E}(A) \otimes_A B$ satisfying expected compatibilities. A convergent isocrystal $\mathcal{E}$ is called of finite type if all $\mathcal{E}(A)$'s are finite $A \otimes K$-modules. This yields that each $\mathcal{E}(A)$ is a projective $A \otimes K$-module. Also, in this case $\mathcal{E}(A)$ has a natural p-adic topology. A convergent F-isocrystal on X/V_0 is a convergent isocrystal $\mathcal{E}$ on X/V_0, together with an isomorphism $\phi^*(\mathcal{E}) \to \mathcal{E}$, where ϕ is the Frobenius on X.

It is convenient to consider more general objects than enlargements, namely widenings, where the p-nilpotency condition on the ideal I is dropped. One advantage is that while products do not exist for general enlargements, they do exist for widenings. Every widening T gives rise to a sheaf h_T in $(X/V)_{\mathrm{conv}}$, which associates to an enlargement T' of X/V the set of morphisms $T' \to T$ of widenings of X/V. It is canonically a direct limit of n'th level enlargements $h_T \simeq \mathrm{inj\,lim}\, T_n$. If $T = (A, I, z)$, then T_n is given by the p-adic completion of the algebra $A[I^n/\pi]/(p - \mathrm{torsion})$.

Fundamental enlargements are gotten by taking local embeddings $X \supset U = \mathrm{Spec}(A/I) \subset \mathrm{Spf}(A)$ into formally smooth V-algebras A and forming T_n's of the widenings $T = (A, I, U \hookrightarrow X)$. If X is smooth over k, it is even sufficient to consider only local liftings. In that case, a convergent isocrystal of finite type corresponds to locally free finite modules on these liftings with integrable, convergent in the open unit disk connections. More generally, any sheaf $\mathcal{E}$ in $(X/V)_{\mathrm{conv}}$ can be evaluated on a widening $T = (A, I, z)$: set $\mathcal{E}_T \simeq \mathrm{Mor}(h_T, \mathcal{E}) \simeq \mathrm{proj\,lim}\{\mathrm{Mor}(T_n, \mathcal{E})\} \simeq \mathrm{proj\,lim}\mathcal{E}_{T_n}$, and in the case $\mathcal{E}$ is a convergent isocrystal and A/V is formally smooth, $\mathcal{E}_T$ is equipped with an integrable connection as an A-module. This connection comes from finite level connections $\nabla\colon \mathcal{E}_{T_{n+1}} \to \mathcal{E}_{T_n} \otimes_A \Omega_{A/V}$. Note that in the case $\mathrm{Spf}(A)$ is a lifting of $\mathrm{Spec}(A/I)$, $\mathcal{E}_{T_n} \simeq \mathcal{E}_T$, for every n.

2.2.2. *Filtered Convergent F-Isocrystals*

Assume now X to be a smooth, separated scheme of finite type over V. Let X_k denote the special fiber of X. Let $\mathcal{E}$ be a convergent F-isocrystal on X_k/V_0, which is filtered on the formal completion $\mathcal{X}$ of X along X_k, i.e., for enlargements coming from open subschemes $\mathrm{Spf}(A)$ of $\mathcal{X}$, $\mathcal{E}(A)$ has a filtration compatible with open immersions and Griffiths transversal to the connection. We will assume from now on all isocrystals to be of finite type.

Griffiths transversality allows this filtration to be lifted to certain widenings.

PROPOSITION 2.3. *Let* $\mathrm{Spf}(A) \hookrightarrow \mathrm{Spf}(B)$ *be a closed immersion of noetherian formally smooth p-adic formal finite type* V*-schemes and let* $\mathcal{E}$ *be a filtered convergent isocrystal of finite type on* $(A/\pi)/V$. *Then the value* $\mathcal{E}_B$ *of* $\mathcal{E}$ *on the formal completion of* $\mathrm{Spf}(B)$ *along* $\mathrm{Spec}(A/\pi)$ *is canonically endowed with a Griffiths transversal filtration.*

Proof. Recall [12, 2.8], that $\mathcal{E}_A$ has a canonical structure of a crystal of $K \otimes \mathcal{O}_{A/V}$-modules, i.e., there is a crystal $\mathcal{E}_A^{\mathrm{an}}$ of $K \otimes \mathcal{O}_{A/V}$-modules on the nilpotent site of $\mathrm{Spf}(A)$ such that $(\mathcal{E}_A^{\mathrm{an}})_A \simeq \mathcal{E}_A$. Set $\hat{B}^{\mathrm{an}} = \mathrm{proj}\lim(K \otimes B/K \otimes J^n)$, where $J \subset B$ is the ideal defining A. Equip it with the projective limit topology, where every quotient $K \otimes B/K \otimes J^n$ has the p-adic topology. The crystal $\mathcal{E}_A^{\mathrm{an}}$ gives rise to a topological $\hat{B}^{\mathrm{an}}$-module $\hat{\mathcal{E}}_B^{\mathrm{an}} = \mathrm{proj}\lim \mathcal{E}_{B/J^n}$ endowed with a continuous isomorphism $p_1^*\hat{\mathcal{E}}_B^{\mathrm{an}} \to p_2^*\hat{\mathcal{E}}_B^{\mathrm{an}}$, where p_i^* are the projections $\hat{B}^{\mathrm{an}} \to ((B\hat{\otimes}B^\wedge))^{\mathrm{an}}$, satisfying the usual cocycle condition. $\hat{\mathcal{E}}_B^{\mathrm{an}}$ can be filtered: set

$$F^k\hat{\mathcal{E}}_B^{\mathrm{an}} = \sum (\hat{J}^{\mathrm{an}})^i F_h^{k-i}\hat{\mathcal{E}}_B^{\mathrm{an}},$$

where, for a continuous retraction $h: A \to B^\wedge$ ($B^\wedge$ is the J-adic completion of B equipped with the projective limit topology, where every quotient has the p-adic topology) and the induced map $\tilde{h}: K \otimes A \to \hat{B}^{\mathrm{an}}$, $F_h^j\hat{\mathcal{E}}_B^{\mathrm{an}} = \mathrm{Im}(\tilde{h}^*F^j\mathcal{E}_A \to \hat{\mathcal{E}}_B^{\mathrm{an}})$. That this definition is independent of the choice of h follows, just as in the case of usual crystals [14, 3.1.2], from Griffiths transversality of the connection.

Write B_n for the p-adic completion of the algebra $B[J^n/\pi]/(p-\text{torsion})$. By definition $\mathcal{E}_B = \mathrm{proj}\lim \mathcal{E}_{B_n}$. We filter $\mathcal{E}_{B_n}$ via the canonical continuous embedding $\mathcal{E}_{B_n} \hookrightarrow \hat{\mathcal{E}}_B^{\mathrm{an}}$ and let $F^i\mathcal{E}_B = \mathrm{proj}\lim F^i\mathcal{E}_{B_n}$. That this is independent of the chosen retraction h follows from the convergent isocrystal structure on $\mathcal{E}$.

It remains to show that this filtration is Griffiths transversal to the connection on $\mathcal{E}_B$. Again it is simplest to do the computations first on the nilpotent site of $\mathrm{Spf}(A)$. Think about $\hat{\mathcal{E}}_B^{\mathrm{an}}$ as coming from $\mathcal{E}_A$ via the retraction $\tilde{h}$. Then the integrable connection $\hat{\mathcal{E}}_B^{\mathrm{an}} \to \hat{\mathcal{E}}_B^{\mathrm{an}} \otimes_B \Omega^1_{B/V}$ provided by the structure of crystal of $K \otimes \mathcal{O}_{A/V}$-modules is induced from the one on $\mathcal{E}_A$ and compatible with the canonical connection on $\hat{B}^{\mathrm{an}}$. Since the last two connections are Griffiths transversal, so is the connection $\hat{\mathcal{E}}_B^{\mathrm{an}} \to \hat{\mathcal{E}}_B^{\mathrm{an}} \otimes_B \Omega^1_{B/V}$. Finally, since the connection $\mathcal{E}_{B_n} \to \mathcal{E}_{B_{n-1}} \otimes_B \Omega^1_{B/V}$ is compatible with the connection $\hat{\mathcal{E}}_B^{\mathrm{an}} \to \hat{\mathcal{E}}_B^{\mathrm{an}} \otimes_B \Omega^1_{B/V}$, we are done. □

2.2.3. *Crystalline Smooth Sheaves*

Let $\mathcal{E}$ be a filtered convergent F-isocrystal on X_k/V_0 and let $\mathrm{Spec}(R)$ be a small open of X. We can evaluate $\mathcal{E}$ on $B(\widehat{R})$, $B^+_{dR}(\widehat{R})$, and $B_{dR}(\widehat{R})$. First, consider the

V_0-enlargement $B_t(\widehat{R})$ (with the ideal $(\xi, [(\pi)])$) of R/π equal to the p-adic completion of $W(S(\widehat{R}))[\xi^p/p]$. It maps into $B(\widehat{R})$, $B^+_{dR}(\widehat{R})$, and $B_{dR}(\widehat{R})$. We evaluate $\mathcal{E}$ on $B_t(\widehat{R})$ and then pullback it to get $\mathcal{E}_{B(\widehat{R})}$, $\mathcal{E}_{B^+_{dR}(\widehat{R})}$, and $\mathcal{E}_{B_{dR}(\widehat{R})}$. $\mathcal{E}_{B(\widehat{R})}$ inherits Frobenius action. To filter $\mathcal{E}_{B^+_{dR}(\widehat{R})}$ and $\mathcal{E}_{B_{dR}(\widehat{R})}$, first note that $B^+_{dR}(\widehat{R}) = W_V(S(\widehat{R}))^\wedge_{\mathrm{an}} = \operatorname{proj\,lim}(K \otimes W_V(S(\widehat{R}))/K \otimes J^n)$, where J is the ideal defining $\overline{\widehat{R}}^\wedge$ in $W_V(S(\widehat{R}))$. Next, choose a continuous retraction $h\colon \widehat{R} \to B^+_{dR}(\widehat{R})$ over $\overline{\widehat{R}}^\wedge$. Then $\mathcal{E}_{B^+_{dR}(\widehat{R})} \simeq h^*\mathcal{E}_{\widehat{R}}$, and $\mathcal{E}_{B_{dR}(\widehat{R})} \simeq h^*\mathcal{E}_{\widehat{R}} \otimes_{B^+_{dR}(\widehat{R})} B_{dR}(\widehat{R})$. Endow them with the tensor product filtrations. Griffiths transversality yields that these definitions are independent of the choice of the retraction h (use the usual diagonal argument and the fact that $\mathcal{E}$ induces a crystal of $K \otimes \mathcal{O}_{\widehat{R}/V}$-modules on the nilpotent site of $\mathrm{Spf}(\widehat{R})$). In particular, we get that $\mathcal{E}_{B(\widehat{R})}$, $\mathcal{E}_{B^+_{dR}(\widehat{R})}$, and $\mathcal{E}_{B_{dR}(\widehat{R})}$ inherit $\pi_1(\widehat{R}[1/p])$ action compatible with all the other structures.

Recall that a filtered convergent F-isocrystal $\mathcal{E}$ on X_k/V_0 is associated to a smooth $\mathbf{Q}_p$-adic étale sheaf $\mathbf{L}$ on X_K if there is an isomorphism (functorial in R's as above) $\mathcal{E}_{B(\widehat{R})} \simeq \mathbf{L}_{\hat{R}} \otimes B(\widehat{R})$ preserving Galois action, Frobenius, and, after extension to $B_{dR}(\widehat{R})$, filtration. Associated isocrystals form an abelian category. There exists a functor from this category to the category of smooth $\mathbf{Q}_p$-adic étale sheaves on X_K: an associated isocrystal $\mathcal{E}$ is sent to a smooth sheaf $\mathbf{L}(\mathcal{E})$ on X_K such that $\mathbf{L}(\mathcal{E})_{\hat{R}}$ is functorially isomorphic to $\mathbf{L}(\mathcal{E}_{\widehat{R}})\colon = (\mathcal{E}_{B(\widehat{R})} \cap F^0\mathcal{E}_{B_{dR}(\widehat{R})})^{\phi=1}$. This functor is fully faithful, exact, preserves tensor products and internal $\mathcal{H}om$'s. Its image is called the category of crystalline smooth sheaves.

3. Cohomology of Crystalline Smooth Sheaves

Let X be a smooth, separated scheme of finite type over V and let $\mathcal{E}$ be an associated isocrystal. Choose a covering of X by small affine open sets $\mathrm{Spec}(R''_i)$, $i \in I$. The special fiber X_k is then covered by $U_i = \mathrm{Spec}(R''_i/\pi R''_i)$. Choose close embeddings of every U_i into a smooth V_0-scheme $\mathrm{Spf}(R'_i)$ and of every $\mathrm{Spec}(R''_i)$ into a smooth V-scheme $\mathrm{Spf}(R_i)$, and a V_0-morphism $\alpha_i\colon R'_i \to R_i$ over the identity on U_i. Fix also a Frobenius lift ϕ_i on $\widehat{R}'_i$. We also require there to be étale maps from $\mathrm{Spec}(R'_i)$ and $\mathrm{Spec}(R_i)$ into affine spaces over V_0 and V respectively.

For every subset J of the index set, set

$$R_J = \otimes_{j\in J} R_j, \quad R''_J = \otimes_{j\in J} R'_j, \qquad U_J = \bigcap_{j\in J} U_j, \quad \phi_J = \otimes_{j\in J}\phi_j.$$

Let R''_J be the algebra of $\bigcap_{j\in J} \mathrm{Spec}(R''_j)$. Consider the widenings

$$T_J = (R^\wedge_J, I^\wedge_J, U_J \hookrightarrow X_k), \qquad T'_J = (R'^\wedge_J, I'^\wedge_J, U_J \hookrightarrow X_k),$$

where I_J is the ideal of U_J in $\mathrm{Spf}(\widehat{R}_J)$ and $R^\wedge_J$ denotes the I_J-adic completion of $\widehat{R}_J$. Similarly for T'_J.

We can evaluate $\mathcal{E}$ on the widenings T'_J, T_J getting, by the Proposition 2.3, $\widehat{R}'_J[1/p]$-modules $\mathcal{E}_{T'_J}$ equipped with an integrable connection and a parallel Frobenius action, and $\widehat{R}_J[1/p]$-modules $\mathcal{E}_{T_J}$ equipped with an integrable connection and a filtration satisfying Griffiths transversality.

Define

$$\Omega(\mathcal{E}_J)^{\cdot} = \mathcal{E}_{T'_J} \otimes_{R'_J} \Omega^{\cdot}_{R'_J/V_o}, \qquad \Omega(\mathcal{E}_J)^{\cdot}_V = \mathcal{E}_{T_J} \otimes_{R_J} \Omega^{\cdot}_{R_J/V}.$$

Filter the modules $\mathcal{E}_{T_J} \otimes_{R_J} \Omega^i_{R_J/V}$ by submodules $F^k(\mathcal{E}_{T_J} \otimes_{R_J} \Omega^i_{R_J/V}) := F^{k-i}\mathcal{E}_{T_J} \otimes_{R_J} \Omega^i_{R_J/V}$. By Griffiths transversality, for fixed k, the submodules $F^k(\mathcal{E}_{T_J} \otimes_{R_J} \Omega^i_{R_J/V})$ form a subcomplex $F^k\Omega(\mathcal{E}_J)^{\cdot}_V$ of $\Omega(\mathcal{E}_J)^{\cdot}_V$.

Concerning the Frobenius, the maps

$$\phi^i_J\colon \mathcal{E}_{T'_J} \otimes_{R'_J} \Omega^i_{R'_J/V_o} \to \mathcal{E}_{T'_J} \otimes_{R'_J} \Omega^i_{R'_J/V_o},$$

$\phi^i_J = \phi_J \otimes d\phi_J$ glue to a Frobenius $\phi_J\colon \Omega(\mathcal{E}_J)^{\cdot} \to \Omega(\mathcal{E}_J)^{\cdot}$.

Set

$$\mathcal{S}(\mathcal{E}_J)^{\cdot} := \mathrm{Cone}\big(\Omega(\mathcal{E}_J)^{\cdot} \oplus F^0\Omega(\mathcal{E}_J)^{\cdot}_V \xrightarrow{\beta} \Omega(\mathcal{E}_J)^{\cdot} \oplus \Omega(\mathcal{E}_J)^{\cdot}_V\big)[-1],$$

where $\beta(x, y) = (x - \phi_J(x), x - y)$ and the map $\Omega(\mathcal{E}_J)^{\cdot} \to \Omega(\mathcal{E}_J)^{\cdot}_V$ is induced by the V_0-morphisms $\alpha_i\colon R'_i \to R_i$ we have chosen.

Varying J we get from the complexes $\mathcal{S}(\mathcal{E}_J)^{\cdot}$ a double complex. Denote by $\mathcal{S}(\mathcal{E})^{\cdot}$ the associated simple complex. One easily checks that up to quasi-isomorphism nothing depends on the choices made. Set

$$H^*_f(X, \mathcal{E}) := H^*(\mathcal{S}(\mathcal{E})^{\cdot}).$$

The groups $H^*_f(X, \cdot)$ clearly yield functors on the category of crystalline smooth sheaves. The following lemma shows that they give us a cohomology theory on this category.

LEMMA 3.1. *If* $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ *is an exact sequence of associated isocrystals, then for every closed immersion* $\mathrm{Spf}(\widehat{R}) \hookrightarrow \mathrm{Spf}(A)$, $\mathrm{Spec}(R) \subset X$, *into a noetherian formally smooth p-adic formal finite type V-scheme, the sequence* $0 \to \mathcal{E}_1(A) \to \mathcal{E}_2(A) \to \mathcal{E}_3(A) \to 0$ *is exact in the filtered sense.*

Proof. An exact sequence of associated isocrystals $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ yields, for every widening T and every $n \geqslant 0$, an exact sequence $0 \to \mathcal{E}_{1,T_n} \to \mathcal{E}_{2,T_n} \to \mathcal{E}_{3,T_n} \to 0$. Since $\mathcal{E}_{\widehat{R}}$ is locally free of finite type, we get an exact sequence $0 \to \mathcal{E}_{1,T} \to \mathcal{E}_{2,T} \to \mathcal{E}_{3,T} \to 0$. Remains to show that, for widenings of the type appearing in the statement of the lemma, this sequence is exact in the filtered sense as well. In the notation of the proof of Proposition 2.3, set $J^{(i)}_n = (K \otimes A_n) \cap (\hat{J}^{\mathrm{an}})^i$ and $J^{(i)} = \mathrm{proj\,lim} J^{(i)}_n$. Since, for an associated isocrystal $\mathcal{E}$, the associated graded is locally free, we get

$$F^k\mathcal{E}(A) = \sum J^{(i)} F^{k-i}_{\hat{R}}\mathcal{E}(A),$$

where the map $\bar{h}: \widehat{R} \to \operatorname{proj}\lim A_n$ is induced by a continuous retraction $h: \widehat{R} \to \operatorname{proj}\lim A/J^n$. The lemma follows now easily from the fact that any homomorphism between associated isocrystals is strict for the filtration. $\square$

Remark 2. In the case of Tate twists $\mathcal{E} = \mathcal{K}_{X_k/V}\{n\}$ one can give a definition of $H_f^*(X, \mathcal{E})$ avoiding making any of the above choices of coverings of X (cf., [11]).

4. Map to Étale Cohomology

We will now define a map of cohomology theories

$$l: H_f^*(X, \cdot) \to H^*(X_K, \mathbf{L}(\cdot))$$

on the category of associated isocrystals.

4.1. ÉTALE AND GALOIS COHOMOLOGIES

Recall ([3], [10]) that one can associate with X a topos of locally constant sheaves $\widetilde{\mathcal{X}}$. A sheaf in $\widetilde{\mathcal{X}}$ associates to every open $\mathrm{Spf}(\mathcal{A})$ of $\mathcal{X}$ a locally constant étale sheaf on $\mathrm{Spec}(\mathcal{A}_K)$. This association behaves well with respect to the restriction maps and satisfies certain sheaf condition. In particular, every locally constant sheaf L on X_K defines a sheaf in $\widetilde{\mathcal{X}}$, and one proves [10] that, for X proper and smooth over V, the étale cohomology $H^*(X_K, L)$ is isomorphic to $H^*(\widetilde{\mathcal{X}}, L)$. There is also the geometric cohomology $H^*(X_{\overline{K}}, \cdot)$, which, for a proper and smooth X/V and a locally constant sheaf on X_K, computes $H^*(X_{\overline{K}}, L)$.

The cohomology $H^*(X_K, L)$ is thus given by certain canonical complexes. Assume that X is connected. First, one fixes an algebraic closure of the p-adic completion of the function field of X. Next, for every open $\mathrm{Spec}(R)$ of X and a locally constant sheaf $\mathcal{F}$ on $\mathrm{Spec}(R_K^\wedge)$, one takes the standard resolution $S^\cdot(\mathcal{F})$ of $\mathcal{F}$, i.e., the standard resolution of the discrete Galois module induced by $\mathcal{F}$ via the above choice of the base point. It is a complex of discrete representations of the fundamental group, acyclic for the group cohomology. For a sheaf $\mathcal{F}$ in $\widetilde{\mathcal{X}}$, the cohomology $H^*(\widetilde{\mathcal{X}}, \mathcal{F})$ can be computed by the inductive limit, over affine Zariski hypercoverings $V_\cdot$ of X, of the cohomology of complexes $C^\cdot(\mathcal{F}_\mathcal{V}) = S^\cdot(\mathcal{F}_\mathcal{V})(\mathcal{A}(\mathcal{V})_{K^\cdot})$.

Using continuous Galois cohomology one can extend the above construction to smooth, $\mathbf{Q}_p$-adic sheaves on X_K or, more generally, to any well behaved system $(\mathcal{F}_\mathcal{A})$, $\mathrm{Spf}(\mathcal{A})$ – an open of $\mathcal{X}$, of continuous representations of the fundamental groups of $\mathrm{Spec}(\mathcal{A}_K)$. Concerning the geometric cohomology, we set $H^*(\widetilde{\mathcal{X}}_{\overline{K}}, \mathcal{F}) := \operatorname{inj}\lim H^*(\widetilde{\mathcal{X}_V}, \mathcal{F}_V)$, where the limit is over the rings of integers of finite field extensions of K in $\overline{K}$. For a smooth, $\mathbf{Q}_p$-adic sheaf L on X_K, we get $H^*(X_K, L) \simeq H^*(\widetilde{\mathcal{X}}, L)$ and $H^*(X_{\overline{K}}, L) \simeq H^*(\widetilde{\mathcal{X}}_{\overline{K}}, L)$.

4.2. POINCARÉ LEMMAS

To construct the map l to étale cohomology we will need the existence of various resolutions.

4.2.1. *Rigid Poincaré Lemma*

Let $\mathcal{B}(\widehat{R''_J})_n$ be the algebra of the nth enlargement associated to the widening given by the algebra $B_t(\widehat{R''_J})\hat{\otimes}_{V_0}R''_J$ completed along $B_t(\widehat{R''_J})/(\xi, [(\pi)])$. We equip it with the Frobenius induced by that on $B_t(\widehat{R''_J})$ and $\widehat{R}'_J$, and with an action of the fundamental group of $\widehat{R''_J}[1/p]$. We have the following rigid Poincaré lemma.

LEMMA 4.1. *The complex*

$$0 \to B_t(\widehat{R''_J})[1/p] \to (\operatorname{proj}\lim \mathcal{B}(\widehat{R''_J})_n[1/p]) \otimes_{R''_J} \Omega^{\cdot}_{R''_J/V_0}$$

is a Frobenius equivariant resolution of $B_t(\widehat{R''_J})[1/p]$.

Proof. We will exhibit a continuous $B_t(\widehat{R''_J})[1/p]$-linear homotopy contracting the above complex. Choose a p-adically complete formally smooth V_0-algebra $\widetilde{R}_J$ lifting R''_J/π, a retraction $h: \widetilde{R}_J \to B_t(\widehat{R''_J})$ lifting the obvious map $R''_J/\pi \to B_t(\widehat{R''_J})/(\xi, [(\pi)])$ (note that $B_t(\widehat{R''_J})/(\xi, [(\pi)]) \simeq \widehat{R}_J/\pi[X]$), and a V_0-map $\omega: \widehat{R}'_J \to \widetilde{R}_J$ lifting the map $R''_J/p \to R''_J/\pi$, and consider the algebras $B_t(\widehat{R''_J})\hat{\otimes}_{\widehat{R'_{J,n}}} \widehat{R}'_J(1)_n$, where $\widehat{R}'_{J,n}$ and $\widehat{R}'_J(1)_n$ are the algebras of the nth enlargements associated to the widenings given respectively by $\widehat{R}'_J$ and $R''_J\hat{\otimes}_{V_0}R''_J$ completed along U_J. For $n \geqslant p(e+1)$, where e is the ramification index of V over V_0, the retraction h and the map ω give a map $B_t(\widehat{R''_J})\hat{\otimes}_{\widehat{R'_{J,n}}} \widehat{R}'_J(1)_n \to \mathcal{B}(\widehat{R''_J})_n$. Set $n' = n - p(e+1)$ for $n \geqslant p(e+1)$. We claim that there is also a map $\mathcal{B}(\widehat{R''_J})_n \to B_t(\widehat{R''_J})\hat{\otimes}_{\widehat{R'_{J,n'}}} \widehat{R}'_J(1)_{n'}$. Indeed, since $B_t(\widehat{R''_J})\hat{\otimes}_{\widehat{R'_{J,n'}}} \widehat{R}'_J(1)_{n'}$ is p-torsion free (by [12] $\widehat{R}'_J(1)_{n'}$ is flat over $\widehat{R}'_{J,n'}$), by the universal property of $\mathcal{B}(\widehat{R''_J})_n$, it suffices to check that the map $B_t(\widehat{R''_J})\hat{\otimes}_{V_0}\widehat{R}'_J \to B_t(\widehat{R''_J})\hat{\otimes}_{\widehat{R'_{J,n'}}} \widehat{R}'_J(1)_{n'}$ induced by the second projection $R'_J \to R'_J\hat{\otimes}_{V_0}R'_J$ maps the nth power of the ideal of $B_t(\widehat{R''_J})/(\xi, [(\pi)])$ into the ideal generated by p. If we write $B_t(\widehat{R''_J})\hat{\otimes}_{V_0}\widehat{R}'_J$ as $B_t(\widehat{R''_J})\hat{\otimes}_{\widehat{R'_J}} (R'_J\hat{\otimes}_{V_0}R'_J)$, then this ideal is equal to the image of the ideal $B_t(\widehat{R''_J})\hat{\otimes}_{\widehat{R'_J}} I_{U_J} + (\xi, [(\pi)])\hat{\otimes}_{\widehat{R'_J}} (R'_J\hat{\otimes}_{V_0}R'_J)$. But in $B_t(\widehat{R''_J})$, $(\xi, [(\pi)])^{p(e+1)} \subset pB_t(\widehat{R''_J})$, hence we are done. Since the composition of the maps $B_t(\widehat{R''_J})\hat{\otimes}_{\widehat{R'_{J,n}}}$

$\widehat{R}'_J(1)_n \to \mathcal{B}(\widehat{R''_J})_n$ and $\mathcal{B}(\widehat{R''_J})_n \to B_t(\widehat{R''_J})\hat{\otimes}_{\widehat{R'_{J,n'}}} \widehat{R}'_J(1)_{n'}$ either way is the obvious transition morphism, we get an isomorphism of projective limits

$$\operatorname{proj\,lim}(B_t(\widehat{R''_J})\hat{\otimes}_{\widehat{R'_{J,n}}} \widehat{R}'_J(1)_n)[1/p] \xrightarrow{\sim} \operatorname{proj\,lim}\mathcal{B}(\widehat{R''_J})_n[1/p].$$

Let now the ideal $(x_1, \dots, x_r) \subset \widehat{R}'_J$ be the kernel of ω and let $\xi_1, \dots, \xi_m$ in $\widehat{R}'_J(1) = R'_J \hat{\otimes}_{V_0} R'_J$ be the standard generators of the ideal of the diagonal corresponding to the chosen coordinates of R'_J. We have $\widehat{R}'_{J,n} = C^{\wedge}_{J,n}$, where $C_{J,n} = \widehat{R}'_J[T_I]/((pT_I - x_I) + p - \text{torsion})$, for $|I| = n$, and $\widehat{R}'_J(1)_n = C_J(1)^{\wedge}_n$, where $C_J(1)_n = \widehat{R}'_J(1)[S_I R_{I'}]/((pS_I R_{I'} - x_I\xi_{I'}) + p - \text{torsion})$, for $|I| + |I'| = n$. Since under the map $\widehat{R}'_J \to B_t(\widehat{R''_J})$, x_i goes to 0, $S_I R_{I'}$, $|I| > 0$, becomes p-torsion in the tensor product $(B_t(\widehat{R''_J})\hat{\otimes}_{\widehat{R'_{J,n}}} \widehat{R}'_J(1)_n)[1/p]$.

Hence

$$\begin{aligned}(B_t(\widehat{R}''_J)\hat{\otimes}_{\widehat{R'_{J,n}}} \widehat{R}'_J(1)_n)[1/p] &\simeq (B_t(\widehat{R}''_J)\hat{\otimes}_{\widehat{R'_J}} \widehat{R}'_J(1)[S_I]/(pS_I - \xi_I))[1/p]\\ &\simeq (B_t(\widehat{R}''_J)\hat{\otimes}_{\widehat{R'_J}} \widehat{R}'_J[[\xi_1, \dots, \xi_m]][S_I]/(pS_I - \xi_I))[1/p]\\ &\simeq (B_t(\widehat{R}''_J)[[\xi_1, \dots, \xi_m]][S_I]/(pS_I - \xi_I))^{\wedge}[1/p]\\ &\simeq B_t(\widehat{R}''_J)[\xi_1, \dots, \xi_m]\{\eta_1, \dots, \eta_m\}/(p\eta_i - \xi_i^n)[1/p].\end{aligned}$$

Our complex is thus isomorphic to the complex

$$\begin{aligned}&0 \to B_t(\widehat{R}''_J)[1/p] \to\\ &\operatorname{proj\,lim} B_t(\widehat{R}''_J)[\xi_1, \dots, \xi_m]\{\eta_1, \dots, \eta_m\}/(p\eta_i - \xi_i^n)[1/p]\otimes\\ &\quad \otimes_{B_t(\widehat{R''_J})} \Omega^{\cdot}_{B_t(\widehat{R''_J})[\xi_1,\dots,\xi_m]/B_t(\widehat{R''_J})}.\end{aligned}$$

Since the standard integration (say with respect to ξ_1) preserves the convergence condition encoded in the above projective limit, we can use it to construct the required homotopy. □

COROLLARY 4.1. *Let $\mathcal{E}$ be an associated isocrystal on X. The complex*

$$0 \to \mathcal{E}_{B(\widehat{R''_J})} \to \mathcal{E}_{B(\widehat{R''_J})} \otimes_{B_t(\widehat{R''_J})[1/p]} (\operatorname{proj\,lim}\mathcal{B}(\widehat{R''_J})_n[1/p]) \otimes_{R'_J} \Omega^{\cdot}_{R'_J/V_0}$$

is a Frobenius equivariant resolution of $\mathcal{E}_{B(\widehat{R''_J})}$.

Proof. Tensor the Poincaré resolution of $B(\widehat{R''_J})[1/p]$ from Lemma 4.1 with $\mathcal{E}_{B(\widehat{R''_J})}$.

Remark 3. Later on we will need to compute continuous cochain (Galois) cohomology of $\Omega(\mathcal{E}_{B(\widehat{R''_J})})^{\cdot}$. It is thus necessary to know that $\Omega(\mathcal{E}_{B(\widehat{R''_J})})^{\cdot}$ resolves $\mathcal{E}_{B(\widehat{R''_J})}$ in the strong sense (cf., [2]). That follows here from the existence of a continuous contracting homotopy, which we have shown in the proof of Lemma 4.1.

4.2.2. *Filtered Rigid Poincaré Lemmas*

Let $\mathcal{E}$ be an associated isocrystal on X. To treat the filtrations, consider the algebra

$$E(\widehat{R''_J}) = (W_V(S(\widehat{R''_J}))\hat{\otimes}_V R_J)^{\wedge}_{\mathrm{an}} = \mathrm{proj\,lim} K \otimes (W_V(S(\widehat{R''_J}))\hat{\otimes}_V R_J)/K \otimes I^n,$$

where I is the ideal of $\overline{\widehat{R''_J}}^{\wedge}$ in $W_V(S(\widehat{R''_J}))\hat{\otimes}_V R_J$, equipped with the projective limit topology with every quotient having the p-adic topology. It is filtered with the closures of the ideals $K \otimes I^n$. The fundamental group of $\widehat{R''_J}[1/p]$ acts continuously on $E(\widehat{R''_J})$ via its action on $W_V(S(\widehat{R''_J}))$. We can evaluate $\mathcal{E}$ on $E(\widehat{R''_J})$: choose a continuous retraction h: $\widehat{R''_J} \to \mathrm{proj\,lim}(W_V(S(\widehat{R''_J}))\hat{\otimes}_V R_J)/I^n$ over the natural map $\widehat{R''_J} \to \overline{\widehat{R''_J}}^{\wedge}$, take the induced retraction $\hbar$: $R''_J K \otimes \widehat{R''_J} \to E(\widehat{R''_J})$, and set $\mathcal{E}_{E(\widehat{R''_J})} = \hbar^* \mathcal{E}_{\widehat{R''_J}}$. Define the filtration on $\mathcal{E}_{E(\widehat{R''_J})}$ as the tensor product filtration. Griffiths transversality yields that this definition is independent of the choice of h (look at the crystal induced by $\mathcal{E}$ on the nilpotent site of $\mathrm{Spf}(\widehat{R''_J})$). Think now about $\mathcal{E}_{E(\widehat{R''_J})}$ as coming from $\hat{\mathcal{E}}^{\mathrm{an}}_{\widehat{R_J}} = \mathrm{proj\,lim} \mathcal{E}_{\widehat{R_J/I^n}}$, where I is the ideal defining $\widehat{R''_J}$ in $\widehat{R}_J$. Equip it with the integrable $K \otimes \widehat{R}_J$- connection induced from the one on $\hat{\mathcal{E}}^{\mathrm{an}}_{\widehat{R_J}}$ and compatible with the canonical $K \otimes \widehat{R}_J$- connection on $E(\widehat{R''_J})$.

LEMMA 4.2. *The complex*

$$0 \to \mathcal{E}_{B^+_{dR}(\widehat{R''_J})} \to \mathcal{E}_{E(\widehat{R''_J})} \otimes_{R_J} \Omega^{\cdot}_{R_J/V}$$

is a filtered, Galois equivariant resolution of $\mathcal{E}_{B^+_{dR}(\widehat{R''_J})}$.

Proof. The argument follows the one in the integral case [10]: we only need to substitute the infinitisimal site of $\mathrm{Spf}(\widehat{R''_J})$ for the crystalline site appearing there. First, we write $\mathcal{E}_{E(\widehat{R''_J})} \simeq \mathcal{E}_{B^+_{dR}(\widehat{R''_J})} \otimes_{B^+_{dR}(\widehat{R''_J})} E(\widehat{R''_J})$. Then the above complex is isomorphic (preserving the filtration and the Galois action) to the complex

$$0 \to B^+_{dR}(\widehat{R''_J}) \to E(\widehat{R''_J}) \otimes_{R_J} \Omega^{\cdot}_{R_J/V}$$

tensored with $\mathcal{E}_{B^+_{dR}(\widehat{R''_J})}$. Next, choosing a continuous retraction $\widehat{R''_J} \to B^+_{dR}(\widehat{R''_J})$ over the natural map $\widehat{R''_J} \to \overline{\widehat{R''_J}}^{\wedge}$, we have a filtered, continuous, Galois equivariant and horizontal isomorphism $E(\widehat{R''_J}) \simeq B^+_{dR}(\widehat{R''_J}) \hat{\otimes}_{K \otimes \widehat{R}_J} R_J(1)^{\wedge}_{\mathrm{an}}$, where $R_J(1)^{\wedge}_{\mathrm{an}}$ is the rigid analytic completion of $\widehat{R}_J \otimes \widehat{R}_J$ along the diagonal and $\hat{\otimes}$ refers to the completion of $B^+_{dR}(\widehat{R''_J}) \otimes_{K \otimes \widehat{R}_J} R_J(1)^{\wedge}_{\mathrm{an}}$ with respect to the ideal $\mathcal{K}$ defining $\overline{\widehat{R''_J}}^{\wedge}$ (the quotients $(B^+_{dR}(\widehat{R''_J}) \otimes_{K \otimes \widehat{R}_J} R_J(1)^{\wedge}_{\mathrm{an}})/\mathcal{K}^n$ have natural p-adic topology coming from that of $B^+_{dR}(\widehat{R''_J})$; they are complete in this topology). It suffices now to show that the complex

$$0 \to B^+_{dR}(\widehat{R''_J}) \to B^+_{dR}(\widehat{R''_J})\hat{\otimes}_{K\otimes\widehat{R_J}} R_J(1)^{\wedge}_{\mathrm{an}} \otimes_{R_J} \Omega^{\cdot}_{R_J/V}$$

tensored with $\mathcal{E}_{B^+_{dR}(\widehat{R''_J})}$ is exact in the filtered sense. Consider the complex

$$0 \to B^+_{dR}(\widehat{R''_J})/t^n \to (B^+_{dR}(\widehat{R''_J}) \otimes_{K\otimes\widehat{R_J}} R_J(1)^{\wedge}_{\mathrm{an}})/\mathcal{K}^n \to$$
$$\to (B^+_{dR}(\widehat{R''_J}) \otimes_{K\otimes\widehat{R_J}} R_J(1)^{\wedge}_{\mathrm{an}})/\mathcal{K}^{n-1} \otimes_{R_J} \Omega^1_{R_J}/V \to \dots$$

That it is a filtered resolution of $B^+_{dR}(\widehat{R''_J})/t^n$ is standard: write $R_J(1)^{\wedge}_{\mathrm{an}}$ as $(K \otimes \widehat{R}_J)[[\xi_1, \dots, \xi_k]]$ and use integration to construct a contracting continuous filtered homotopy. The homotopies for various n glue and contract continuously the complex

$$0 \to B^+_{dR}(\widehat{R''_J}) \to B^+_{dR}(\widehat{R''_J})\hat{\otimes}_{K\otimes\widehat{R_J}} R_J(1)^{\wedge}_{\mathrm{an}} \otimes_{R_J} \Omega^{\cdot}_{R_J/V}.$$

The same homotopy contracts this complex tensored with $\mathcal{E}_{B^+_{dR}(\widehat{R''_J})}$ (also in the filtered sense). □

Consider now $E^+(\widehat{R''_J}) = \bigcup_{n\geqslant 0} t^{-n}F^nE(\widehat{R''_J})$ as a $B^+_{dR}(\widehat{R''_J})$-subalgebra of $E(\widehat{R''_J})[t^{-1}]$ with the induced structures. Define $\mathcal{B}^+_{dR}(\widehat{R''_J}) = \mathrm{proj\,lim} E^+(\widehat{R''_J})/t^nE^+(\widehat{R''_J})$ with the projective limit topology. Filter it by powers of t. Finally, set $\mathcal{B}_{dR}(\widehat{R''_J}) = \mathcal{B}^+_{dR}(\widehat{R''_J})[t^{-1}]$ with the induced structures (put t^{-1} in degree -1).

Set $\mathcal{E}_{\mathcal{B}_{dR}(\widehat{R''_J})} = \mathcal{E}_{E(\widehat{R''_J})} \otimes_{E(\widehat{R''_J})} \mathcal{B}_{dR}(\widehat{R''_J})$. Endow it with the tensor product filtration.

LEMMA 4.3. *The complex*

$$0 \to \mathcal{E}_{\mathcal{B}_{dR}(\widehat{R''_J})} \to \mathcal{E}_{\mathcal{B}_{dR}(\widehat{R''_J})} \otimes_{R_J} \Omega^{\cdot}_{R_J/V}$$

is a filtered, Galois equivariant resolution of $\mathcal{E}_{\mathcal{B}_{dR}(\widehat{R''_J})}$.

Proof. Folows from the fact that the homotopy we have constructed in the proof of the above lemma was $B^+_{dR}(\widehat{R''_J})$-linear. □

Remark 4. As before, the contracting homotopy is continuous.

Remark 5. A reader familiar with the proof of the de Rham conjecture will recognize these resolutions as the good reduction incarnations of the resolutions of B_{dR} appearing there ([3, 15]).

4.3. MAP TO ÉTALE COHOMOLOGY

We are now ready to define a functorial map from $\mathcal{S}(\mathcal{E})^{\cdot}$ to a complex computing the étale cohomology groups $H^*(X_K, \mathbf{L}(\mathcal{E}))$. Set

$$\Omega(\mathcal{E}_{B(\widehat{R''_J})})^{\cdot} := \mathcal{E}_{B(\widehat{R''_J})} \otimes_{B_t(\widehat{R''_J})[1/p]} (\operatorname{proj}\lim \mathcal{B}(\widehat{R''_J})_n[1/p]) \otimes R'_J \Omega R'_J / V_0;$$
$$\Omega(\mathcal{E}_{B_{dR}}(\widehat{R''_J}))^{\cdot} := \mathcal{E}_{\mathcal{B}_{dR}}(\widehat{R''_J}) \otimes R_J \Omega R_J / V.$$

Everything above is equipped with an action of the fundamental group of $\widehat{R''_J}[1/p]$ and the resolutions, Frobenius and filtrations behave well with respect to this action.

LEMMA 4.4. *There is a Galois equivariant map of complexes* $\Omega(\mathcal{E}_{B(\widehat{R''_J})})^{\cdot} \to \Omega(\mathcal{E}_{B_{dR}(\widehat{R''_J})})^{\cdot}$ *compatible with the natural map* $\mathcal{E}_{B(\widehat{R''_J})} \to \mathcal{E}_{B_{dR}(\widehat{R''_J})}$.

Proof. It suffices to construct a $B_t(\widehat{R''_J})$-linear map $\operatorname{proj}\lim \mathcal{B}(\widehat{R''_J})_n[1/p] \to \mathcal{B}_{dR}(\widehat{R''_J})$ compatible with the natural morphism $B_t(\widehat{R''_J}) \to B^+_{dR}(\widehat{R''_J})$. The map $\alpha_J: R'_J \to R_J$ we have chosen induces a continuous morphism from

$$\operatorname{proj}\lim \mathcal{B}(\widehat{R''_J})_n \simeq \operatorname{proj}\lim (B_t(\widehat{R''_J}) \hat{\otimes}_{V_0} R'_J)_n$$

to $\operatorname{proj}\lim (D_\xi(W(S(\widehat{R}''_J))) \hat{\otimes}_{V_0} R_J)_n$. Let now $\widetilde{I}$ be the ideal of $\overline{\widehat{R}}{}''^{\wedge}_J$ in $D_\xi(W(S(\widehat{R}''_J))) \hat{\otimes}_{V_0} R_J$. Since it is a finitely generated ideal, we have a natural map

$$(D_\xi(W(S(\widehat{R}''_J))) \hat{\otimes}_{V_0} R_J)_n \to$$
$$\operatorname{proj}\lim K \otimes D_\xi(W(S(\widehat{R}''_J))) \hat{\otimes}_{V_0} R_J / K \otimes \widetilde{I}^n \qquad [14, 2.6.4].$$

Since

$$E(\widehat{R}''_J) = \operatorname{proj}\lim (K \otimes W_V(S(\widehat{R}''_J)) \hat{\otimes}_{V_0} R_J / K \otimes I^n)$$
$$\overset{\sim}{\leftarrow} \operatorname{proj}\lim (K \otimes D_\xi(W(S(\widehat{R}''_J))) \hat{\otimes}_{V_0} R_J / K \otimes \widetilde{I}^n),$$

we are done. □

For each subset J of the index set, we have sequences of morphisms between complexes of sheaves of $\pi_1(\widehat{R''_J}[1/p])$-modules

$$\Omega(\mathcal{E}_J)^{\cdot} \to \Omega(\mathcal{E}_{B(\widehat{R''_J})})^{\cdot} \to S^{\cdot}(\Omega(\mathcal{E}_{B(\widehat{R''_J})})^{\cdot}) \overset{(1)}{\longleftarrow} S^{\cdot}(\mathcal{E}_{B(\widehat{R''_J})}) \overset{\sim}{\to} S^{\cdot}(\mathbf{L}(\mathcal{E}_{\widehat{R''_J}}) \otimes B(\widehat{R}''_J));$$
$$\Omega(\mathcal{E}_J)^{\cdot}_V \to \Omega(\mathcal{E}_{B_{dR}(\widehat{R''_J})})^{\cdot} \to S^{\cdot}(\Omega(\mathcal{E}_{B_{dR}(\widehat{R''_J})})^{\cdot}) \overset{(2)}{\longleftarrow} S^{\cdot}(\mathcal{E}_{B_{dR}(\widehat{R''_J})})$$
$$\overset{\sim}{\to} S^{\cdot}(\mathbf{L}(\mathcal{E}_{\widehat{R''_J}}) \otimes B_{dR}(\widehat{R}''_J)),$$

where, by the above, (1) and (2) are quasi-isomorphisms. The first sequence is compatible with the Frobenius, the second one with the filtration. These sequences yield a morphism

$$\begin{aligned}\mathcal{S}(\mathcal{E}_J)^{\cdot} &\to \mathrm{Cone}(C^{\cdot}(\mathbf{L}(\mathcal{E}_{\widehat{R''_J}}) \otimes B(\widehat{R}''_J)) \oplus C^{\cdot}(\mathbf{L}(\mathcal{E}_{\widehat{R''_J}}) \otimes F^0 B_{dR}(\widehat{R}''_J))\\ &\xrightarrow{\beta} C^{\cdot}(\mathbf{L}(\mathcal{E}_{\widehat{R''_J}}) \otimes B(\widehat{R}''_J)) \oplus C^{\cdot}(\mathbf{L}(\mathcal{E}_{\widehat{R''_J}}) \otimes B_{dR}(\widehat{R}''_J)))[-1],\end{aligned}$$

where $\beta(x, y) = (x - \phi_J(x), x - y)$. Now, our fundamental exact sequence (Proposition 2.2) yields a quasi-isomorphism

$$\begin{aligned}C^{\cdot}(\mathbf{L}(\mathcal{E}_{\hat{R}''_J})) &\to \mathrm{Cone}(C^{\cdot}(\mathbf{L}(\mathcal{E}_{\widehat{R''_J}}) \otimes B(\widehat{R}''_J)) \oplus C^{\cdot}(\mathbf{L}(\mathcal{E}_{\widehat{R''_J}}) \otimes F^0 B_{dR}(\widehat{R}''_J))\\ &\xrightarrow{\beta} C^{\cdot}(\mathbf{L}(\mathcal{E}_{\widehat{R''_J}}) \otimes B(\widehat{R}''_J)) \oplus C^{\cdot}(\mathbf{L}(\mathcal{E}_{\widehat{R''_J}}) \otimes B_{dR}(\widehat{R}''_J)))[-1].\end{aligned}$$

Denote by $U_{\cdot} \to X$ the hypercovering of X induced by our chosen covering of X. Assume that $U_{\cdot} \to X$ is rigid (cf. [6, 4.2]). We find the desired morphism $l: \mathcal{S}(\mathcal{E})^{\cdot} \to \mathrm{inj\,lim}_{V_{\cdot} \in \mathrm{RHR}(X)} C^{\cdot}(\mathbf{L}(\mathcal{E})_{\mathcal{V}})$ into the étale cohomology as the composition

$$\begin{aligned}\mathcal{S}(\mathcal{E})^{\cdot} &\to \mathrm{Cone}\big(C^{\cdot}(\mathbf{L}(\mathcal{E}_{\mathcal{U}}) \otimes B(\mathcal{U})) \oplus C^{\cdot}(\mathbf{L}(\mathcal{E}_{\mathcal{U}}) \otimes F^0 B_{dR}(\mathcal{U}))\\ &\xrightarrow{\beta} C^{\cdot}(\mathbf{L}(\mathcal{E}_{\mathcal{U}}) \otimes B(\mathcal{U})) \oplus C^{\cdot}(\mathbf{L}(\mathcal{E}_{\mathcal{U}}) \otimes B_{dR}(\mathcal{U}))\big)[-1]\\ &\simeq \mathrm{Cone}\big(C^{\cdot}(\mathbf{L}(\mathcal{E})_{\mathcal{U}} \otimes B(\mathcal{U})) \oplus C^{\cdot}(\mathbf{L}(\mathcal{E})_{\mathcal{U}} \otimes F^0 B_{dR}(\mathcal{U}))\\ &\xrightarrow{\beta} C^{\cdot}(\mathbf{L}(\mathcal{E})_{\mathcal{U}} \otimes B(\mathcal{U})) \oplus C^{\cdot}(\mathbf{L}(\mathcal{E})_{\mathcal{U}} \otimes B_{dR}(\mathcal{U}))\big)[-1]\\ &\xleftarrow{\sim} C^{\cdot}(\mathbf{L}(\mathcal{E})_{\mathcal{U}}) \to \mathop{\mathrm{inj\,lim}}_{V_{\cdot} \in \mathrm{RHR}(X)} C^{\cdot}(\mathbf{L}(\mathcal{E})_{\mathcal{V}}).\end{aligned}$$

Here $\mathrm{RHR}(X)$ denotes the category of affine Zariski rigid hypercoverings of X. By [6, 4.3], it is a directed category. Since, for an associated isocrystal $\mathcal{E}$, the associated graded is locally free, any homomorphism between associated isocrystals is strict for the filtration, and the passage from hypercoverings to rigid hypercoverings does not change cohomology (cf. [6, 4.5]), l defines a natural transformation of cohomology theories

$$l: H^*_f(X, \cdot) \to H^*(X_K, \mathbf{L}(\cdot)).$$

Everything above is independent of choices.

5. Comparison of 'Descent' Spectral Sequences

Assume that X is proper and smooth over V. Set $G_K = \mathrm{Gal}(\overline{K}/K)$. Recall [3, IV.e] that, for every filtered convergent F-isocrystal $\mathcal{E}$ on X, the crystalline cohomology groups $H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})$, i.e., the cohomology groups of the convergent topos $(X_k/V)_{\mathrm{conv}}$ with coefficients in $\mathcal{E}$, form filtered F-isocrystals on the base V. Moreover [3, 5.6], if $\mathcal{E}$ is associated to a smooth $\mathbf{Q}_p$-adic étale sheaf $\mathbf{L}$ on X_K, then the groups $H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})$ are associated to the G_K-representations $H^i(X_{\overline{K}}, \mathbf{L})$. In particular, we can apply to $H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})$ the results of [1] and [9]. We will need the following two facts.

LEMMA 5.1. *Suppose that $\mathcal{E}$ is an associated isocrystal on X. Then there is a long exact sequence*

$$\to H^i_f(X, \mathcal{E}) \to H^i_{\mathrm{cr}}(X_k/V_0, \mathcal{E}) \oplus F^0 H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})$$
$$\xrightarrow{\beta} H^i_{\mathrm{cr}}(X_k/V_0, \mathcal{E}) \oplus H^i_{\mathrm{cr}}(X_k/V, \mathcal{E}) \to,$$

where $\beta(x, y) = (x - \phi(x), x - y)$.

Proof. From the definition of $H^i_f(X, \mathcal{E})$, we get the long exact sequence

$$\to H^i_f(X, \mathcal{E}) \to H^i_{\mathrm{cr}}(X_k/V_0, \mathcal{E}) \oplus H^i_{\mathrm{cr}}(X_k/V, F^0\mathcal{E})$$
$$\xrightarrow{\beta} H^i_{\mathrm{cr}}(X_k/V_0, \mathcal{E}) \oplus H^i_{\mathrm{cr}}(X_k/V, \mathcal{E}) \to .$$

If $\mathcal{E}$ is associated to a smooth $\mathbf{Q}_p$-adic étale sheaf $\mathbf{L}$ on X_K, it is easy to check that $\mathbf{L}$ is de Rham in the sense of Tsuzuki [15, 3.1.3], hence [15, 3.1.4] Hodge–Tate in the sense of Hyodo [7]. In particular [7, 0.3], we have a Hodge–Tate decomposition for the geometric étale cohomology of $\mathbf{L}$, which yields, by dimension count, the degeneration of the Hodge–de Rham spectral sequence for $\mathcal{E}$. From that we can conclude that the morphism $H^i_{\mathrm{cr}}(X_k/V, F^0\mathcal{E}) \to H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})$ is an injection, i.e., $H^i_{\mathrm{cr}}(X_k/V, F^0\mathcal{E}) \simeq F^0 H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})$, and we are done. □

Recall [9, p.757] that the complex

$$H^i_{\mathrm{cr}}(X_k/V_0, \mathcal{E}) \oplus F^0 H^i_{\mathrm{cr}}(X_k/V, \mathcal{E}) \xrightarrow{\beta} H^i_{\mathrm{cr}}(X_k/V_0, \mathcal{E}) \oplus H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})$$

computes the cohomology groups $H^*_f(V, H^i_{\mathrm{cr}}(X_k/V, \mathcal{E}))$. Hence the above lemma yields the short exact sequences

$$0 \to H^1_f(V, H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})) \to H^i_f(X, \mathcal{E}) \to H^0_f(V, H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})) \to 0. \tag{1}$$

This sequence can be thought of as a crystalline 'descent' spectral sequence – a spectral sequence relating the geometric crystalline cohomology to the arithmetic one. On the level of étale cohomology we have a similar 'descent' spectral sequence, i.e., the Hochschild–Serre spectral sequence

$$H^p(G_K, H^q(X_{\overline{K}}, \mathbf{L}(\mathcal{E}))) \Longrightarrow H^{p+q}(X_K, \mathbf{L}(\mathcal{E})).$$

The question arises how the map $l\colon H^*_f(X, \mathcal{E}) \to H^*(X_K, \mathbf{L}(\mathcal{E})))$ behaves with respect to these two spectral sequences. Theorems 5.1 and 5.2 below address this question. As a corollary (Corollary 5.1) we get that the map l is injective.

THEOREM 5.1. *Suppose that $\mathcal{E}$ is an associated isocrystal on X. Then there is a commutative diagram*

$$\begin{array}{ccc} H^0_f(V, H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})) & \xrightarrow[\sim]{l} & H^0(G_K, H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E}))) \\ \uparrow & & \uparrow \\ H^i_f(X, \mathcal{E}) & \xrightarrow{l} & H^i(X_K, \mathbf{L}(\mathcal{E})). \end{array}$$

Proof. The definition of the map l and the above lemma yield the following commutative diagram

$$\begin{array}{ccc} H^i_{\mathrm{cr}}(X_k/V_0, \mathcal{E}) \oplus F^0 H^i_{\mathrm{cr}}(X_k/V, \mathcal{E}) & \longrightarrow & H^i(\widetilde{\mathcal{X}}, \mathbf{L}(\mathcal{E}) \otimes B \oplus \mathbf{L}(\mathcal{E}) \otimes F^0 B_{dR}) \rightarrow \\ \uparrow & & \uparrow \\ H^i_f(X, \mathcal{E}) & \xrightarrow{l} & H^i(\widetilde{\mathcal{X}}, \mathbf{L}(\mathcal{E}))\longrightarrow \\ \rightarrow H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}(\mathcal{E}) \otimes B \oplus \mathbf{L}(\mathcal{E}) \otimes F^0 B_{dR}) & \xleftarrow{g} & H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E})) \otimes (B(V) \oplus F^0 B_{dR}(V)) \\ \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle f} \\ \longrightarrow H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}(\mathcal{E})) & \xleftarrow{\sim} & H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E})), \end{array}$$

where B and B_{dR} stand for the systems of continuous Galois modules $B(\widehat{R})$ and $B_{dR}(\widehat{R})$, respectively.

By Proposition 2.2, the map f is injective. We claim that so is the map g. To see that write $\mathbf{L}(\mathcal{E}) = \mathcal{L} \otimes \mathbf{Q}_p$ for a locally constant sheaf $\mathcal{L} = (\mathcal{L}_n)_{n \in \mathbf{N}}$. Set

$$H^*(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}(\mathcal{E}) \otimes B^\wedge) := (\operatorname*{proj\,lim}_m \operatorname*{proj\,lim}_n H^*(\widetilde{\mathcal{X}}_{\overline{K}}, \mathcal{L}_n \otimes B/I^{[m]}))[1/p, 1/t];$$
$$H^*(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}(\mathcal{E}) \otimes B^\wedge_{dR}) := (\operatorname*{proj\,lim}_m H^*(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}(\mathcal{E}) \otimes F^0 B_{dR}/F^m B_{dR}))[1/t].$$

Recall that Faltings' theory of almost étale extensions yields isomorphisms

$$H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E})) \otimes_{\mathbf{Q}_p} B(V) \xrightarrow{\sim} H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}(\mathcal{E}) \otimes_{\mathbf{Q}_p} B^\wedge);$$
$$H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E})) \otimes_{\mathbf{Q}_p} B_{dR}(V) \xrightarrow{\sim} H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}(\mathcal{E}) \otimes_{\mathbf{Q}_p} B^\wedge_{dR}).$$

Hence the injectivity of the map g, the map h, and, consequently, the map f.

Recall now [9, p. 757] that the map

$$l\colon H^0_f(V, H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})) \rightarrow H^0(G_K, H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E})))$$

is induced by the comparison morphisms of Faltings, i.e., by the compositions

$$H^i_{\mathrm{cr}}(X_k/V_0, \mathcal{E}) \otimes_{V_0} B(V) \xrightarrow{\sim} H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}(\mathcal{E}) \otimes_{\mathbf{Q}_p} B^\wedge) \xleftarrow{\sim} H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E})) \otimes_{\mathbf{Q}_p} B(V),$$
$$H^i_{\mathrm{cr}}(X_k/V, \mathcal{E}) \otimes_V B_{dR}(V)$$
$$\xrightarrow{\sim} H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}(\mathcal{E}) \otimes_{\mathbf{Q}_p} B^\wedge_{dR}) \xleftarrow{\sim} H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E})) \otimes_{\mathbf{Q}_p} B_{dR}(V).$$

The statement of the proposition follows now easily from the above diagram. □

THEOREM 5.2. *Let $\mathcal{E}$ be an associated isocrystal on X. Then there is a commutative diagram.*

$$\begin{array}{ccc} H^i_f(X, \mathcal{E})_0 & \overset{l}{\hookrightarrow} & H^i(X_K, \mathbf{L}(\mathcal{E}))_0 \\ \uparrow \wr & & \downarrow \rho_1 \\ H^1_f(V, H^{i-1}_{\mathrm{cr}}(X_k/V, \mathcal{E})) & \overset{l}{\longrightarrow} & H^1(G_K, H^{i-1}(X_{\overline{K}}, \mathbf{L}(\mathcal{E})) \end{array}$$

Here

$$H^i(X_K, \mathbf{L}(\mathcal{E}))_0 = \ker(H^i(X_K, \mathbf{L}(\mathcal{E})) \to H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E}))),$$

and

$$H^i_f(X, \mathcal{E})_0 = \ker(H^i_f(X, \mathcal{E}) \to H^0_f(V, H^i_{\mathrm{cr}}(X_k/V, \mathcal{E}))),$$

and ρ_1 comes from the Hochschild–Serre spectral sequence

$$H^p(G_K, H^q(X_{\overline{K}}, \mathbf{L}(\mathcal{E}))) \Longrightarrow H^{p+q}(X_K, \mathbf{L}(\mathcal{E})).$$

Proof. Proposition 5.1 and the description of the comparison morphisms given in its proof together with the long exact sequence (1) and Proposition 2.2 reduce the proof to the below lemma. □

LEMMA 5.2. *Let $\mathbf{L}$ be any smooth $\mathbf{Q}_p$-adic étale sheaf on X_K. Then the following diagram*

$$\begin{array}{ccc} H^i(\widetilde{\mathcal{X}}, \mathbf{L} \otimes (B \oplus B_{dR}/F^0 B_{dR})) & \longrightarrow & H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes (B \oplus B_{dR}/F^0 B_{dR}))^{G_K} \\ \Big\downarrow \delta & & \Big\downarrow \delta \\ H^{i+1}(X_K, \mathbf{L})_0 & \xrightarrow{\rho_1} & H^1(G_K, H^i(X_{\overline{K}}, \mathbf{L})) \end{array}$$

commutes, where the connecting morphisms δ are induced by the exact sequence from Proposition 2.2.

Remark 6. The fact that the left morphism δ factors through $H^{i+1}(X_K, \mathbf{L})_0$ follows from the commutative diagram

$$\begin{array}{ccc}
H^i(\widetilde{\mathcal{X}}, \mathbf{L} \otimes (B \oplus B_{dR}/F^0 B_{dR})) & \longrightarrow & H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes (B \oplus B_{dR}/F^0 B_{dR})) \\
\Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle \delta=0} \\
H^{i+1}(X_K, \mathbf{L}) & \longrightarrow & H^{i+1}(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}),
\end{array}$$

where the right δ is zero by the theory of almost étale extensions (see the proof of Theorem 5.1).

Proof. Since $B(\widehat{R}) = \mathrm{inj}\lim t^{-k} B^+(\widehat{R})[1/p]$ and $B_{dR}(\widehat{R}) = \mathrm{inj}\lim t^{-k} F^0 B_{dR}(\widehat{R})$, we can argue on finite levels. Recall (Proposition 2.1 and the proof of Proposition 2.2) that, for $k \geqslant 0$, we have the following two exact sequences

$$0 \to \mathbf{Q}_p \to F^k B^+(\widehat{R})[1/p](-k) \xrightarrow{1-p^{-k}\phi} B^+(\widehat{R})[1/p](-k) \to 0, \quad \text{for } k \geqslant 0,$$
$$0 \to \mathbf{Q}_p \to (t^{-k} B^+(\widehat{R})[1/p])^{\phi=1} \to t^{-k} F^0 B_{dR}(\widehat{R})/F^0 B_{dR}(\widehat{R}) \to 0, \quad \text{for } k \geqslant 0.$$

Since $F^0 B_{dR}(\widehat{R})/F^k B_{dR}(\widehat{R}) \overset{\sim}{\leftarrow} B^+(\widehat{R})[1/p]/F^k B^+(\widehat{R})[1/p]$, the second one can be written as

$$0 \to \mathbf{Q}_p \to (B^+(\widehat{R})[1/p])^{\phi=p^k}(-k) \to B^+(\widehat{R})[1/p]/F^k B^+(\widehat{R})[1/p](-k) \to 0.$$

We will now pass to the integral setting, where we have at our disposal the topos of locally constant systems $\widetilde{\mathcal{X}}$ and all the associated cohomological machinery.

LEMMA 5.3. *Let $k \geqslant 0$. Fix $n \geqslant 0$. Let $\mathbf{L}$ be a locally constant sheaf on X_K such that $p^n \mathbf{L} = 0$. Then, for an arbitrarily large number m, there exists an exact sequence of G_K-modules*

$$0 \to H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes \mathbf{Z}_p t^{\{k\}}) \to H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes F^k B^+_{n,m}) \to H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes B^+_{n,m}) \to 0,$$

where $F^k B^+_{n,m}$ and $B^+_{n,m}$ are the sheaves on $\widetilde{\mathcal{X}}$ given by $F^k B^+(\widehat{R})_{n,m}$ and $B^+(\widehat{R})_{n,m}$.

Proof. The numbers m and the sequence come from Lemma 2.1. It remains to prove that the morphism $H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes \mathbf{Z}_p t^{\{k\}}) \to H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes F^k B^+_{n,m})$ is injective for large enough m. Consider the following commutative diagram

$$\begin{array}{ccccc}
H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes \mathbf{Z}_p t^{\{k\}}) & \longrightarrow & \mathrm{proj}\lim_m H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes F^k B^+_{n,m}) & \longrightarrow & \mathrm{proj}\lim_m H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes \mathrm{Fil}^k B^+_{n,m}) \\
\Big\uparrow{\scriptstyle \wr} & & \Big\uparrow & & \Big\uparrow \\
H^i(X_{\overline{K}}, \mathbf{L}) \otimes \mathbf{Z}_p t^{\{k\}} & \longrightarrow & H^i(X_{\overline{K}}, \mathbf{L}) \otimes F^k B^+(V) & \longrightarrow & H^i(X_{\overline{K}}, \mathbf{L}) \otimes \mathrm{Fil}^k B^+(V).
\end{array}$$

Since the right vertical map is by Faltings an injection (cf., the proof of Lemma 8.1 in [10]), it suffices to show that so is the map $H^i(X_{\overline{K}}, \mathbf{L}) \otimes \mathbf{Z}_p t^{\{k\}} \to H^i(X_{\overline{K}}, \mathbf{L}) \otimes$

$\mathrm{Fil}^k B^+(V)$. This follows from the exact sequence

$$0 \longrightarrow \mathbf{Z}_p t^{\{k\}} \longrightarrow \mathrm{Fil}^k B^+(\widehat{R}) \xrightarrow{p^k - \phi} B^+(\widehat{R}). \qquad \square$$

LEMMA 5.4. *Let $k \geqslant 0$. Fix $n \geqslant 0$. Let $\mathbf{L}$ be a locally constant sheaf on X_K such that $p^n\mathbf{L} = 0$. Then, there exists an exact sequence of G_K-modules*

$$0 \to H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes \mathbf{Z}_p t^{\{k\}}) \to H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes (B^+)^{\phi=p^k}) \to H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes B_k^+) \to 0,$$

where $(B^+)^{\phi=p^k}$, B_k^+ are the sheaves on $\widetilde{X}$ associated to the presheaves

$$B^+(\widehat{R})^{\phi=p^k} \quad \text{and} \quad B^+(\widehat{R})_k = \mathrm{Im}(B^+(\widehat{R})^{\phi=p^k} \to B^+(\widehat{R})/F^k B^+(\widehat{R})).$$

Proof. Since the ring $B^+(\widehat{R})_k$ injects into $B^+(\widehat{R})/F^k B^+(\widehat{R})$, it is $\mathbf{Z}_p$-flat and we get a sequence as in the lemma. Remains to prove that the map $H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes \mathbf{Z}_p t^{\{k\}}) \to H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes (B^+)^{\phi=p^k})$ is injective. For large enough m, we have a commutative diagram

$$\begin{array}{ccccc}
H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes \mathbf{Z}_p t^{\{k\}}) & \longrightarrow & H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes (B^+)^{\phi=p^k}) & \longrightarrow & \mathrm{proj\,lim}_m H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes B^+_{n,m}) \\
\wr\big\uparrow & & \big\uparrow & & \big\uparrow \\
H^i(X_{\overline{K}}, \mathbf{L}) \otimes \mathbf{Z}_p t^{\{k\}} & \longrightarrow & H^i(X_{\overline{K}}, \mathbf{L}) \otimes B^+(V)^{\phi=p^k} & \longrightarrow & H^i(X_{\overline{K}}, \mathbf{L}) \otimes B^+(V).
\end{array}$$

Since the right vertical map is by Faltings an injection and $B^+(V)_k$ is $\mathbf{Z}_p$-flat, it suffices to show that so is the map $H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}) \otimes B^+(V)^{\phi=p^k} \to H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L}) \otimes B^+(V)$. But that follows from $B^+(V)^{\phi=p^k}/B^+(V)$ being $\mathbf{Z}_p$-flat. $\square$

The above two lemmas allow us to reduce the proof of the proposition to the question of showing that, for $k \geqslant 0$, $n \geqslant 0$, any locally constant sheaf $\mathbf{L}$ on X_K annihilated by p^n, and a well chosen arbitralily large m, the following two diagrams commute.

$$\begin{array}{ccc}
H^i(\widetilde{\mathcal{X}}, \mathbf{L} \otimes B^+_{n,m}(-k)) & \longrightarrow & H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes B^+_{n,m}(-k))^{G_K} \\
{\scriptstyle\delta}\big\downarrow & & {\scriptstyle\delta}\big\downarrow \\
H^{i+1}(X_K, \mathbf{L} \otimes \mathbf{Z}_p t^{\{k\}}(-k))_0 & \xrightarrow{\rho_1} & H^1(G_K, H^i(X_{\overline{K}}, \mathbf{L} \otimes \mathbf{Z}_p t^{\{k\}}(-k))) \\
H^i(\widetilde{\mathcal{X}}, \mathbf{L} \otimes B^+_k(-k)) & \longrightarrow & H^i(\widetilde{\mathcal{X}}_{\overline{K}}, \mathbf{L} \otimes B^+_k(-k))^{G_K} \\
{\scriptstyle\delta}\big\downarrow & & {\scriptstyle\delta}\big\downarrow \\
H^{i+1}(X_K, \mathbf{L} \otimes \mathbf{Z}_p t^{\{k\}}(-k))_0 & \xrightarrow{\rho_1} & H^1(G_K, H^i(X_{\overline{K}}, \mathbf{L} \otimes \mathbf{Z}_p t^{\{k\}}(-k)))
\end{array}$$

Here the connecting morphisms come from the exact sequences in the above lemmas. We finish by evoking Lemma 5.5 below. $\square$

The following lemma is a variation on a cohomological lemma of Jannsen [8, 9.5].

LEMMA 5.5. *Let $\mathcal{A}$ be an abelian category with enough injectives and let $A^{\cdot} \xrightarrow{v} B^{\cdot} \to C^{\cdot} \xrightarrow{u} A^{\cdot}[1]$, where $\mathcal{H}(v)$ is injective, be a distinguished triangle in the derived category of the bounded below complexes in $\mathcal{A}$. Let $f\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor into another abelian category and denote by F^i the decreasing filtration induced on the limit terms by the hypercohomology spectral sequence $R^p f\mathcal{H}^q(K^{\cdot}) \Longrightarrow R^{p+q}f(K^{\cdot})$ for a bounded below complex $K^{\cdot}$ in $\mathcal{A}$. Then the diagram*

$$\begin{tikzcd} R^n f(C^{\cdot}) \arrow[r] \arrow[d, "\delta"'] & f\mathcal{H}^n(C^{\cdot}) \arrow[d, "\delta"] \\ F^1R^{n+1}f(A^{\cdot}) \arrow[r, "\rho_1"] & R^1 f\mathcal{H}^n(A^{\cdot}) \end{tikzcd}$$

commutes, where ρ_1 is induced by the hypercohomology spectral sequence.

Proof. We can assume that $A^{\cdot}$ and $B^{\cdot}$ consists of injectives, and $C^{\cdot} = \mathrm{Cone}(v)$. There is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}^n(A^{\cdot}) \arrow[r] & A^n/\mathrm{Im}\, d_A^{n-1} \arrow[r, "d_A^n"] & \mathrm{Im}\, d_A^n \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{H}^n(A^{\cdot}) \arrow[r] \arrow[u] \arrow[d] & E \arrow[r] \arrow[u] \arrow[d] & \ker d_C^n \arrow[r] \arrow[u, "u"] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{H}^n(A^{\cdot}) \arrow[r] & \mathcal{H}^n(B^{\cdot}) \arrow[r] & \mathcal{H}^n(C^{\cdot}) \arrow[r] & 0, \end{tikzcd}$$

where $E = \ker(d_A^n - u)$ and the map $E \to \mathcal{H}^n(B^{\cdot})$ sending $([a], (a', b'))$, $a \in A^n$, $a' \in A^{n+1}$, $b' \in B^n$, $(a', b') \in \ker d_C^n$ to $b' + v(a)$. The morphism $u\colon \ker d_C^n \to A^{n+1}$ factors through $\mathrm{Im}\, d_A^n$ because of the injectivity of $\mathcal{H}^{n+1}(v)$.

We get the following commutative diagram

$$\begin{tikzcd} & R^1 f\mathcal{H}^n(A^{\cdot}) & \\ f\mathrm{Im}\, d_A^n \arrow[ur] \arrow[r] & f\ker d_C^n \arrow[u] \arrow[r] & f\mathcal{H}^n(C^{\cdot}) \arrow[ul] \end{tikzcd}$$

Since $R^n f(C^{\cdot}) = f\ker d_C^n/\mathrm{Im} f d_C^{n-1}$ and $F^1R^{n+1}f(C^{\cdot}) \xleftarrow{\sim} f\mathrm{Im}\, d_A^n/\mathrm{Im} f(d_A^n)$, it suffices now to show that the diagram

$$\begin{tikzcd} f\mathrm{Im}\, d_A^n \arrow[rr, "\delta"] \arrow[dr, "\omega"'] & & R^1 f\mathcal{H}^n(A^{\cdot}) \\ & F^1R^{n+1}f(C^{\cdot}), \arrow[ur, "\rho_1"'] & \end{tikzcd}$$

where the map ω is surjective, commutes. But this was already proved by Jannsen. □

COROLLARY 5.1. *For an associated isocrystal $\mathcal{E}$ on X, the morphism $l\colon H^i_f(X, \mathcal{E}) \to H^i(X_K, \mathbf{L}(\mathcal{E})))$ is an injection.*

Proof. Let $x \in H^i_f(X, \mathcal{E})$ map to zero in $H^i(X_K, \mathbf{L}(\mathcal{E})))$. By Theorem 5.1 the image of x in $H^i_{\mathrm{cr}}(X_k/V_0, \mathcal{E}) \oplus F^0H^i_{\mathrm{cr}}(X_k/V, \mathcal{E})$ is zero. Hence $x \in H^i_f(X, \mathcal{E})_0$. By Theorem 5.2,

x comes from an element in the kernel of the map $l\colon H^1_f(V, H^{i-1}_{\mathrm{cr}}(X_k/V, \mathcal{E})) \to H^1(G_K, H^{i-1}(X_{\overline{K}}, \mathbf{L}(\mathcal{E}))$. Since this map is injective, we are done. □

6. Duality

6.1. PRODUCTS

PROPOSITION 6.1. *If $\mathcal{M}$ and $\mathcal{N}$ are associated isocrystals, then there exists a canonical product*

$$\cup\colon H^p_f(X, \mathcal{M}) \otimes H^q_f(X, \mathcal{N}) \to H^{p+q}_f(X, \mathcal{M} \otimes \mathcal{N})$$

which is anticommutative and associative. Moreover, it commutes with the morphism $l\colon H^*_f(X, \cdot) \to H^*(X_K, \mathbf{L}(\cdot))$.

Proof. The de Rham product $\Omega(\mathcal{M}_J)^\cdot \otimes \Omega(\mathcal{N}_J)^\cdot \to \Omega((\mathcal{M} \otimes \mathcal{N})_J)^\cdot$ can be used [9, prop. 3.1] to define a homotopic family of maps of complexes

$$\cup_\alpha\colon \mathcal{S}(\mathcal{M}_J)^\cdot \otimes \mathcal{S}(\mathcal{N}_J)^\cdot \to \mathcal{S}((\mathcal{M} \otimes \mathcal{N})_J)^\cdot, \quad \alpha \in \mathbf{Z}_p,$$

$$\begin{aligned}
&(x_1, x_2, x_3, x_4) \cup_\alpha (y_1, y_2, y_3, y_4) \\
&\quad = (x_1 \cup y_1, x_2 \cup y_2, x_3 \cup (\alpha y_1 + (1-\alpha)\phi_J(y_1)) + \\
&\qquad + (-1)^{\circ x_1}((1-\alpha)x_1 + \alpha\phi_J(x_1)) \cup y_3, x_4 \cup (\alpha y_1 + (1-\alpha)y_2) + \\
&\qquad + (-1)^{\circ x_1}((1-\alpha)x_1 + \alpha x_2) \cup y_4).
\end{aligned}$$

The maps $\cup_0$, $\cup_1$ are associative and $\cup_\alpha$, $\cup_{1-\alpha}$ anticommute. The rest of the proof follows the one for the integral crystals [10, 7.1]. □

6.2. COHOMOLOGY SUPPORTED ON THE SPECIAL FIBER

For an associated isocrystal $\mathcal{E}$ on X, define the groups with support on the special fiber X_k as

$${}_sH^*_f(X, \mathcal{E}) := H^*\big(\mathrm{Cone}(\mathcal{S}(\mathcal{E})^\cdot \xrightarrow{l} \operatorname*{inj\,lim}_{V.\in \mathrm{RHR}(X)} C^\cdot(\mathbf{L}(\mathcal{E})_{\mathcal{V}}))[-1]\big).$$

Since, for an associated isocrystal $\mathcal{E}$, the associated graded is locally free and any homomorphism between associated isocrystals is strict for the filtration, they form a cohomology theory. From the definition we get the long exact sequence

$$\to H^{i-1}(X_K, \mathbf{L}(\mathcal{E})) \to {}_sH^i_f(X, \mathcal{E}) \to H^i_f(X, \mathcal{E}) \xrightarrow{l} H^i(X_K, \mathbf{L}(\mathcal{E})) \to$$

Assume that X is proper and smooth over V. We have shown Corollary 5.1 that the map $l\colon H^i_f(X, \mathcal{E}) \to H^i(X_K, \mathbf{L}(\mathcal{E})))$ is injective. In particular, we have an exact sequence

$$0 \to H^i_f(X, \mathcal{E}) \xrightarrow{l} H^i(X_K, \mathbf{L}(\mathcal{E}))) \to {}_sH^{i+1}_f(X, \mathcal{E}) \to 0.$$

6.3. DUALITY

Having all the constructions out of the way, the duality theorem follows now just as in the integral case [10, Theorem 8.1]. Let X be a proper and smooth scheme over V, of pure relative dimension d. Assume that the residue field of V is finite. For an associated isocrystal $\mathcal{E}$ define $\mathcal{E}^D := \mathcal{E}^*\{-d-1\}$, where $\mathcal{E}^*$ is the internal $\mathcal{H}om$ into the trivial filtered convergent F-isocrystal $\mathcal{K}_{X/V}$.

THEOREM 6.1. *For any associated isocrystal $\mathcal{E}$ on X, there is a perfect pairing*

$${}_sH^i_f(X, \mathcal{E}) \otimes H^{2d+3-i}_f(X, \mathcal{E}^D) \to {}_sH^{2d+3}_f(X, \mathcal{K}_{X/V}\{-d-1\}) \xrightarrow{\text{tr}} \mathbf{Q}_p.$$

Proof. Concerning the trace map, we have

$$\begin{aligned} &{}_sH^{2d+3}_f(X, \mathcal{K}_{X/V}\{-d-1\}) \xleftarrow{\sim} H^{2d+2}(X_K, \mathbf{Q}_p(d+1)) \\ &\qquad \xleftarrow{\sim} H^2(G_K, H^{2d}(X_{\overline{K}}, \mathbf{Q}_p(d+1))) \xrightarrow{\text{tr}} H^2(G_K, \mathbf{Q}_p(1)) \xrightarrow{\text{inv}} \mathbf{Q}_p, \end{aligned}$$

where the first isomorphism follows from the fact that $H^i_f(X, \mathcal{K}_{X/V}\{-d-1\}) = 0$ for $i > 2d+1$ (cf., [10, Theorem 8.1].

To define the product notice that the commutative diagram

$$\begin{array}{ccc} H^i_f(X, \mathcal{E}) \otimes H^{2d+2-i}_f(X, \mathcal{E}^D) & \longrightarrow & H^{2d+2}_f(X, \mathcal{K}_{X/V}\{-d-1\}) = 0 \\ \Big\downarrow{\scriptstyle l\otimes l} & & \Big\downarrow \\ H^i(X_K, \mathbf{L}(\mathcal{E})) \otimes H^{2d+2-i}(X_K, \mathbf{L}(\mathcal{E}^D)) & \longrightarrow & H^{2d+2}(X_K, \mathbf{Q}_p(d+1))) \end{array}$$

shows that $H^i_f(X, \mathcal{E})$ and $H^{2d+2-i}_f(X, \mathcal{E}^D)$ annihilate each other. Since the morphism $l\colon H^i_f(X, \mathcal{E}) \to H^i(X_K, \mathbf{L}(\mathcal{E}))$ is injective, this diagram and the products on étale and f-cohomologies induce a product

$${}_sH^i_f(X, \mathcal{E}) \otimes H^{2d+3-i}_f(X, \mathcal{E}^D) \to {}_sH^{2d+3}_f(X, \mathcal{K}_{X/V}\{-d-1\}).$$

Consider the complex

$$0 \to H^{i-1}_f(X, \mathcal{E}) \xrightarrow{l} H^{i-1}(X_K, \mathbf{L}(\mathcal{E})) \xrightarrow{t} H^{2d+3-i}_f(X, \mathcal{E}^D) \to 0,$$

where the map t is induced by the étale product. Using étale duality $(H^{i-1}(X_K, \mathbf{L}(\mathcal{E})) \simeq H^{2d+3-i}(X_K, \mathbf{L}(\mathcal{E}^D))^*)$ we check that $t = l^*$. Hence, t is surjective. That reduces checking that we have the perfect pairing we want to showing that the $\mathbf{Q}_p$-rank of $H^{i-1}(X_K, \mathbf{L}(\mathcal{E}))$ is equal to the sum of the $\mathbf{Q}_p$-ranks of $H^{i-1}_f(X, \mathcal{E})$ and $H^{2d+3-i}_f(X, \mathcal{E}^D)$. This follows exactly like in the integral case [10] from the crystalline, étale, and Galois dualities, and the degeneration of the Hochschild–Serre spectral sequence shown in the Proposition 6.2 below. □

PROPOSITION 6.2. *Let $\mathcal{E}$ be an associated isocrystal on X. If the residue field of V is finite, then the Hochschild-Serre spectral sequence*

$$H^i(G_K, H^j(X_{\overline{K}}, \mathbf{L}(\mathcal{E}))) \Rightarrow H^{i+j}(X_K, \mathbf{L}(\mathcal{E}))$$

degenerates.

Proof. Since $H^0_f(V, H^i_{\mathrm{cr}}(X/V, \mathcal{E}) \xrightarrow{l} H^0(G_K, H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E})))$ and $H^i_f(X, \mathcal{E}) \twoheadrightarrow H^0_f(V, H^i_{\mathrm{cr}}(X/V, \mathcal{E})$, Theorem 5.1 gives that $H^i(X_K, \mathbf{L}(\mathcal{E})) \twoheadrightarrow H^0(G_K, H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E})))$. By Poincaré duality this yields that $H^2(G_K, H^i(X_{\overline{K}}, \mathbf{L}(\mathcal{E}))) \hookrightarrow H^{i+2}(X_K, \mathbf{L}(\mathcal{E}))$. The proposition follows now from the fact that the group G_K has cohomological dimension 2. □

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