TORIC SINGULARITIES: LOG-BLOW-UPS AND GLOBAL RESOLUTIONS

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1. Introduction

It is well-known that singularities of toroidal embeddings over complex numbers can be resolved (globally) by equivariant blow-ups. This can be done in an elementary way in the case of the divisor at infinity not having self-intersections (cf. [15]) and generally (like in Theorem 5.10 in this paper) by evoking the canonical resolution of Bierstone and Milman [1].

We show in this paper that singularities of Kato's toric varieties (a base-free analogue of toroidal embeddings) introduced and studied in [13] can be resolved in a similar way. In the language of log-geometry, Kato's toric singularities are called log-regular schemes. Since many of our results hold for more general log-schemes than that, we have chosen to work here in the setting of fine and saturated étale log-schemes, restricting to the smaller class of log-regular schemes only when it seemed necessary.

The content of the paper is as follows. In section 2 we clarify the relationship between Zariski and étale log-structures (i.e., between the "without self-intersections" and "with self-intersections" cases) and describe the behaviour of local toric models under cospecializations. Section 3 describes a correspondence between saturated ideals of the log-structure and global sections of certain sheaves that is similar to the classical correspondence between equivariant complete fractional ideals and certain convex functions (cf. [15, Theorem 9]).

We use this correpondence in section 4 to study log-blow-ups (log-geometry version of equivariant blow-ups). We state their universal property, relate them to subdivisions of classical fans, and show that they are stable under compositions and (surprisingly) under base changes. The reader will notice that this is similar to the behaviour of admissible formal blow-ups in rigid geometry (cf. [2, 2.4,2.5]).

Finally, in section 5 we resolve the singularities adapting the classical algorithms to log-schemes using the log-language introduced in section 3 and the properties of log-blow-ups proved in section 4. We would like to stress here that the resolutions we obtain are global.

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Throughout the paper, unless otherwise stated, log-structures are defined using the étale topology. An fs log-scheme is a log-scheme whose log-structure is fine with saturated stalks. We will assume all the schemes to be locally noetherian.

2. Preliminaries on log-structures

2.1. **Zariski log-structures.** Although we are primerily interested in étale log-structures, Zariski log-structures will appear naturally: they are the local version of étale log-structures and are often easier to work with. We will now study a little bit the relationship between the two.

2.1.1. *Basics.* Recall the following

Definition 2.1. *(*[13, 5.1,5.2]*) Let P be a monoid. A subset $I \subset P$ is said to be an ideal if $IP \subset I$. An ideal $\mathfrak{p} \subset P$ is prime if $P \setminus \mathfrak{p}$ is a submonoid of P. We denote by $\mathrm{Spec}(P)$ the set of all prime ideals of P.*

For a submonoid $S \subset P$, $S^{-1}P$ is the monoid $\{s^{-1}a | a \in P, s \in S\}$, where $s_1^{-1}a_1 = s_2^{-1}a_2$ if and only if $ts_1a_2 = ts_2a_1$ for some $t \in S$. In particular, for a prime ideal $\mathfrak{p} \in \mathrm{Spec}(P)$, we set $P_{\mathfrak{p}} = S^{-1}P$, where $S = P \setminus \mathfrak{p}$.

Definition 2.2. *(*[17, 4.4.5]*) An fs log-scheme (X, M_X) is called log-regular at $x \in X$ if $\mathcal{O}_{X,\overline{x}}/I_{\overline{x}}\mathcal{O}_{X,\overline{x}}$ is regular and $\dim(\mathcal{O}_{X,\overline{x}}) = \dim(\mathcal{O}_{X,\overline{x}}/I_{\overline{x}}\mathcal{O}_{X,\overline{x}}) + \mathrm{rank}_{\mathbf{Z}}((M_X^{gp}/\mathcal{O}_X^*)_{\overline{x}})$, where $I_{\overline{x}} = M_{X,\overline{x}} \setminus \mathcal{O}^*_{X,\overline{x}}$. We say that (X, M_X) is log-regular if (X, M_X) is log-regular at every point $x \in X$.*

Lemma 2.3. *(*[17, 4.4.6]*) Let (X, M_X) be an fs log-scheme equipped with a chart $P_X \to M_X$, where P is a saturated monoid. Let M_X^{Zar} be the log-structure on the Zariski site associated to $P \to \Gamma(X, M_X) \to \Gamma(X, \mathcal{O}_X)$. Then, for any $x \in X$, (X, M_X) is log-regular at x if and only if (X, M_X^{Zar}) is log-regular at x in the sense of* [13, 2.1].

Lemma 2.4. *In the situation of the above lemma, the natural morphism $\pi_1 : M_X^{\mathrm{Zar}} \to \varepsilon_* M_X$, where $\varepsilon : X_{\text{ét}} \to X_{\mathrm{Zar}}$ is the projection from the étale to the Zariski topos of X, is an isomorphism.*

Proof. We easily check that the sheaf of monoids $\varepsilon_* M_X$ is a logarithmic structure. Take now any point $x \in X$. It suffices to show that the map $\overline{\pi}_1 : M_{X,x}^{\mathrm{Zar}}/\mathcal{O}^*_{X,x} \to (\varepsilon_* M_X)_x/\mathcal{O}^*_{X,x}$ is an isomorphism.

Consider the composition

$$M_{X,x}^{\mathrm{Zar}}/\mathcal{O}^*_{X,x} \xrightarrow{\overline{\pi}_1} (\varepsilon_* M_X)_x/\mathcal{O}^*_{X,x} \xrightarrow{\pi_2} M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}}.$$

Let $\mathfrak{p}$ be the inverse image of the maximal ideal of $\mathcal{O}_{X,\overline{x}}$ in P. Then $\mathfrak{p}$ is also the inverse image of the maximal ideal of $\mathcal{O}_{X,x}$ in P. By [17, 4.1.4] the natural morphisms $P_X \to M_X$ and $P_X \to M_X^{\mathrm{Zar}}$ induce isomorphisms $P_{\mathfrak{p}}/P^*_{\mathfrak{p}} \simeq M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$ and $P_{\mathfrak{p}}/P^*_{\mathfrak{p}} \simeq M_{X,x}^{\mathrm{Zar}}/\mathcal{O}^*_{X,x}$. Hence the composition $\overline{\pi}_1\pi_2$ is an isomorphism.

To show that $\overline{\pi}_1$ is an isomorphism, it suffices now to show that π_2 is injective (this holds for general log-structure M_X). But, by Hilbert's Theorem 90, π_2 factors as

$$(\varepsilon_* M_X)_x/\mathcal{O}^*_{X,x} \xrightarrow{\sim} \varepsilon_*(M_X/\mathcal{O}^*_X|\,\mathrm{Spec}(\mathcal{O}_{X,x})_{\text{ét}}) \to M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}},$$

and the second isomorphism is clearly injective. □

Following the above lemma, in the rest of the paper we will identify the log-structures M_X^{Zar} and $\varepsilon_* M_X$.

Let M_X be an étale fs log-structure on a scheme X. The sheaf of monoids $M_X^{\mathrm{Zar}} = \varepsilon_* M_X$ (this is consistent with the notation used earlier), where $\varepsilon : X_{\text{ét}} \to X_{\mathrm{Zar}}$ is the natural projection, is easily checked to be a log-structure. We say that M_X is *Zariski* if $M_X = \varepsilon^* M_X^{\mathrm{Zar}}$ (ε^* is the pullback of log-structures). Clearly M_X is Zariski if and only if, for every point $x \in X$, the natural map $\varepsilon_x : P_x = M_{X,x}^{\mathrm{Zar}}/\mathcal{O}_{X,x}^* \to M_{X,\overline{x}}/\mathcal{O}_{X,\overline{x}}^* = P_{\overline{x}}$ is an isomorphism.

In what follows, we will write $P_{\overline{x}}$ (resp. P_x) for the stalks of $M_X/\mathcal{O}_X^*$ (resp. $M_X^{\mathrm{Zar}}/\mathcal{O}_X^*$).

Proposition 2.5. *For any Zariski log-structure M_X, the log-structure M_X^{Zar} is fs* [13, 1.4], *i.e., it admits charts with fine and saturated monoids.*

Proof. Take any point $x \in X$. Since the monoid P_x is isomorphic to $P_{\overline{x}}$, it is fine and saturated. In particular, P_x^{gp} is a finitely generated free abelian group. Hence the projection $M_{X,x}^{\mathrm{Zar}} \to P_x$ admits a section $s_x : P_x \to M_{X,x}^{\mathrm{Zar}}$. We claim that this section can be extended to a chart $s_{U_1} : P_{x,U_1} \to M_X^{\mathrm{Zar}}|_{U_1}$ for a (Zariski) neighbourhood U_1 of x. Let $s_{\overline{x}} : P_{\overline{x}} \to M_{X,\overline{x}}$ be the map induced by the following commutative diagram

$$\begin{array}{ccccc}
P_{\overline{x}} & \xrightarrow{s_{\overline{x}}} & M_{X,\overline{x}} & \longrightarrow & M_{X,\overline{x}}/\mathcal{O}_{X,\overline{x}}^* = P_{\overline{x}} \\
{\scriptstyle \varepsilon_x}\uparrow\wr & & \uparrow & & {\scriptstyle \varepsilon_x}\uparrow\wr \\
P_x & \xrightarrow{s_x} & M_{X,x}^{\mathrm{Zar}} & \longrightarrow & M_{X,x}^{\mathrm{Zar}}/\mathcal{O}_{X,x}^* = P_x
\end{array}$$

Clearly $P_{\overline{x}} = (s_{\overline{x}}^{gp})^{-1}(M_{X,\overline{x}})$. Hence, by [12, 2.10], the section $s_{\overline{x}}$ extends to a chart $s_U : P_{\overline{x},U} \to M_X|U$ for an étale neighbourhood U of $\overline{x} \to X$. Set $U_1 = \mathrm{Im}(U) \subset X$. We may assume that s_U is induced by an extension $s_{U_1} : P_{x,U_1} \to M_X^{\mathrm{Zar}}|U_1$ of the section s_x.

We claim that s_{U_1} is a chart for $M_X^{\mathrm{Zar}}|U_1$. Indeed, let $y \in U_1$. It suffices to show that

$$(P_{x,U_1})_y^a/\mathcal{O}_{X,y}^* \underset{s_{U_1}}{\xrightarrow{\sim}} M_{X,y}^{\mathrm{Zar}}/\mathcal{O}_{X,y}^* = P_y$$

Let $\mathfrak{p}$ be the inverse image of the maximal ideal of $\mathcal{O}_{X,\overline{y}}$ under the morphism $P_{\overline{x}} \xrightarrow{s_U} M_{X,\overline{y}} \xrightarrow{\alpha_{\overline{y}}} \mathcal{O}_{\overline{y}}$. Then $\varepsilon_x^{-1}(\mathfrak{p})$ is the inverse image of the maximal ideal of $\mathcal{O}_{X,y}$ under the morphism $P_x \xrightarrow{s_{U_1}} M_{X,y}^{\mathrm{Zar}} \xrightarrow{\alpha_y} \mathcal{O}_{X,y}$. By the definition of the associated log-structures, the following diagram of monoids

$$\begin{array}{ccc}
P_x \setminus \varepsilon_x^{-1}(\mathfrak{p}) & \longrightarrow & P_x \\
\downarrow & & \downarrow \\
\mathcal{O}_{X,y}^* & \longrightarrow & (P_{x,U_1})_y^a
\end{array}$$

is cocartesian. Hence we have a natural isomorphism $P_x/(P_x \setminus \varepsilon_x^{-1}(\mathfrak{p})) \xrightarrow{\sim} (P_{x,U_1})_y^a/\mathcal{O}_{X,y}^*$. The same argument gives that $P_{\overline{x}}/(P_{\overline{x}} \setminus \mathfrak{p}) \xrightarrow{\sim} (P_{\overline{x},U})_{\overline{y}}^a/\mathcal{O}_{X,\overline{y}}^*$. Since we have a commutative diagram

$$\begin{array}{ccc}
P_{\overline{x}}/(P_{\overline{x}} \setminus \mathfrak{p}) & \underset{\sim}{\xrightarrow{s_U}} & P_{\overline{y}} \\
{\scriptstyle \varepsilon_x}\uparrow\wr & & {\scriptstyle \varepsilon_y}\uparrow\wr \\
P_x/(P_x \setminus \varepsilon_x^{-1}(\mathfrak{p})) & \xrightarrow{s_{U_1}} & P_y,
\end{array}$$

and, by assumption, the vertical morphisms are isomorphisms, the map s_{U_1} is an isomorphism. Since we have a factorization $s_{U_1}: P_x/(P_x \setminus \varepsilon_x^{-1}(\mathfrak{p})) \xrightarrow{\sim} (P_{x,U_1})^a_y/\mathcal{O}^*_{X,y} \to P_y$, this yields that the map $(P_{x,U_1})^a_y/\mathcal{O}^*_{X,y} \underset{s_{U_1}}{\xrightarrow{\sim}} M^{\mathrm{Zar}}_{X,y}/\mathcal{O}^*_{X,y} = P_y$ is also an isomorphism, as wanted. □

The following proposition is the étale version of Theorem 11.6 from [13].

Proposition 2.6. *Let (X, M_X) be a log-regular scheme. Then the set $X_{\mathrm{tr}} = \{x \in X | M_{X,\overline{x}} = \mathcal{O}^*_{X,\overline{x}}\}$ is dense open in X and $M_X = \mathcal{O}_X \cap j_*\mathcal{O}^*_{X_{\mathrm{tr}}}$, where $j : X_{\mathrm{tr}} \hookrightarrow X$ is the natural inclusion.*

Proof. Choose an étale covering $\sum_{i\in I} U_i \to X$ such that every log-scheme (U_i, M_{U_i}) has a chart $P_{i,U_i} \to M_{U_i}$, where P_i is a fine and saturated monoid. ¿From [13, 11.6] we know that $U_{i,\mathrm{tr}}$ is dense and open in U_i and that $M^{\mathrm{Zar}}_{U_i} = \mathcal{O}_{U_i} \cap j_*\mathcal{O}^*_{U_{i,\mathrm{tr}}}$. We claim that $M_{U_i} = \mathcal{O}_{U_i} \cap j_*\mathcal{O}^*_{U_{i,\mathrm{tr}}}$ as well. Indeed, let $T \to U_i$ be an étale scheme over U_i. Then $P_i \to \Gamma(U_i, M_{U_i}) \to \Gamma(T, M_T)$ gives a chart of the log-scheme (T, M_T). The log-scheme (T, M_T) is log-regular (the map $T \to U_i$ being étale), hence, again by [13, 11.6], $M^{\mathrm{Zar}}_T = \mathcal{O}_T \cap j_*\mathcal{O}^*_{T_{\mathrm{tr}}}$, as wanted.

The above shows that X_{tr} is dense and open in X. Consider now the structure map $M_X \xrightarrow{\alpha_X} \mathcal{O}_X$. To show that it induces an isomorphism $M_X = \mathcal{O}_X \cap j_*\mathcal{O}^*_{X_{\mathrm{tr}}}$ it suffices to show that analogous equalities hold on the covering log-schemes (U_i, M_{U_i}), which we have done above. □

Let (X, M_X) be a log-regular scheme and let $x \in X$. Set $Y = \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}), M_Y = (M_{X,\overline{x}})^a$. We have the following

Lemma 2.7. *The map that sends a point $y \in Y$ of codimension one with $M_{Y,\overline{y}} \neq \mathcal{O}^*_{Y,\overline{y}}$ to the inverse image $\mathfrak{p}_y \in \mathrm{Spec}(P_{\overline{x}}) = \mathrm{Spec}(M_{X,\overline{x}})$ of the maximal ideal of $\mathcal{O}_{Y,\overline{y}}$ under the morphism $M_{X,\overline{x}} \xrightarrow{\alpha_{\overline{x}}} \mathcal{O}_{Y,\overline{y}}$ defines a one-to-one correspondence between points $y \in Y$ of codimension one with $M_{Y,\overline{y}} \neq \mathcal{O}^*_{Y,\overline{y}}$ and prime ideals $\mathfrak{p}_y \in \mathrm{Spec}(P_{\overline{x}})$ of height one. Moreover*

1. *$\mathfrak{p}_y\mathcal{O}_{X,\overline{x}} = J_y$, where J_y is the ideal of y in $\mathcal{O}_{X,\overline{x}}$;*
2. *the composite $P_{\overline{x}} \to \mathcal{O}_{Y,\overline{y}} \xrightarrow{v_{\overline{y}}} \mathbf{Z}$ coincides with the valuation map $P^{gp}_{\overline{x}} \xrightarrow{v_{\mathfrak{p}_y}} P^{gp}_{\mathfrak{p}_y}/P^*_{\mathfrak{p}_y} \simeq \mathbf{Z}$.*

Proof. Note that the log-scheme (Y, M_Y) is log-regular, Zariski, and if we take any section $s : P_{\overline{x}} \to M_{X,\overline{x}}$ of the projection $M_{X,\overline{x}} \to P_{\overline{x}}$, then the induced map $P_{\overline{x},Y} \to M_Y$ is a chart of M_Y.

The fact that the primes $\mathfrak{p}_y$ are of height one and have the second stated property is proved in [17, 4.4.10]. The fact that the map $y \mapsto \mathfrak{p}_y$ is a one-to-one correspondence and the first property follow from [13, 7.3] and [17, 4.4.9]. □

Set $P_{\overline{x}}(1) = \{\mathfrak{p} \in \mathrm{Spec}(P_{\overline{x}}) | \mathrm{ht}(\mathfrak{p}) = 1\}$. In the case (X, M_X) is regular, we will use the above lemma to identify this set with the corresponding subset of $\mathrm{Spec}(\mathcal{O}_{X,\overline{x}})$.

2.1.2. *Criteria for log-structure to be Zariski.* We will now state two criteria for log-structure to be Zariski. The first one is the descent criterium of Kato from [14] that applies to any fs log-scheme. The second one is a more geometric criterium that can be used in the case of log-regular schemes.

Proposition 2.8. *([14, 4.2.3]) Let (X, M_X) be any fs log-scheme. The log-structure M_X is Zariski if and only if, for every $x \in X$, the map $\varepsilon_x : P_x \to P_{\overline{x}}$ does not admit nontrivial automorphisms, i.e, the only automorphism of $P_{\overline{x}}$ over P_x is the identity.*

Proof. Let (Y, M_Y) be an étale covering of (X, M_X) such that M_Y is Zariski. Then there is an isomorphism $\theta : p_1^*M_Y \simeq p_2^*M_Y$ on $Y \times_X Y$ satisfying a cocycle condition. Take $y \in Y$ and $z \in p_1^{-1}(y) \cap p_2^{-1}(y)$, and consider the induced automorphism $t_y : P_y \xrightarrow{\sim} (p_1^*M_Y)_z/\mathcal{O}^*_{Y\times_X Y} \underset{\sim}{\xrightarrow{\theta_z}} (p_2^*M_Y)_z/\mathcal{O}^*_{Y\times_X Y} \xleftarrow{\sim} P_y$ of P_y. It is easy to see [14, 4.2.3.2] that M_X is Zariski if and only if all the automorphisms t_y are equal to the identity. Since any t_y induces (faithfully) an automorphism of $P_{\overline{x}}$ over P_x, where x is the image of y in X, our proposition follows. □

Proposition 2.9. *Let (X, M_X) be any log-regular scheme. The log-structure M_X is Zariski if and only if, for every $x \in X$, the map $\mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to \mathrm{Spec}(\mathcal{O}_{X,x})$ is injective on primes from $P_{\overline{x}}(1)$.*

Proof. Only the "if" part of the proposition is not obvious. Assume that for any $x \in X$, the morphism $\mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to \mathrm{Spec}(\mathcal{O}_{X,x})$ is injective on primes of height one. Fix $x \in X$. We will show that $\varepsilon_x : P_x \to P_{\overline{x}}$ is an isomorphism. Arguing like in Lemma 2.4, we see that it is an injection. To show that the morphism $M^{\mathrm{Zar}}_{X,x} \xrightarrow{\varepsilon_x} M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$ is surjective, take any $h \in M_{X,\overline{x}} = \mathcal{O}_{X,\overline{x}} \cap j_*\mathcal{O}^*_{X_{\mathrm{tr}},\overline{x}}$ (see Proposition 2.6), where $X_{\mathrm{tr},\overline{x}} = X_{\mathrm{tr}} \times_X \mathrm{Spec}(\mathcal{O}_{X,\overline{x}})$. Consider the Weil divisor D of h. We have $D = \mathrm{div}(h) = \sum_y a_y\{y\}^-$ for some $a_y \in \mathbf{N}$, where the sum is over points $y \in \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \setminus X_{\mathrm{tr},\overline{x}}$ of codimension 1 in $\mathrm{Spec}(\mathcal{O}_{X,\overline{x}})$. By Lemma 2.7, each $y \in P_{\overline{x}}(1)$

Let $y' \in \mathrm{Spec}(\mathcal{O}_{X,x})\setminus X_{\mathrm{tr},x}$, $X_{\mathrm{tr},x} = X_{\mathrm{tr}} \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$, be the image of y under the map $\mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to \mathrm{Spec}(\mathcal{O}_{X,x})$. Consider the Weil divisor $D' = \sum_y a_y\{y'\}^-$ on $\mathrm{Spec}(\mathcal{O}_{X,x})$. We claim that $\varepsilon_x^*(D') = D$. Since the morphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\overline{x}}$ is unramified, it suffices to show that if two points $y_1, y_2 \in \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \setminus X_{\mathrm{tr},\overline{x}}$ from $P_{\overline{x}}(1)$ have the same image y' in $\mathrm{Spec}(\mathcal{O}_{X,x})$, then $y_1 = y_2$. But that we have assumed.

On the other hand, since D is principal, by faithfully flat descent, so is D', i.e., there exists a function $h' \in K(\mathcal{O}_{X,x})$ such that $\mathrm{div}(h') = D'$. Since $\varepsilon_x^*(D') = D = \mathrm{div}(h)$, $h = \varepsilon_x(h')u$ for some $u \in \mathcal{O}^*_{X,\overline{x}}$, as wanted. □

2.2. **Cospecializations.** We will study now the behaviour of the stalks $P_{\overline{x}} = (M_X/\mathcal{O}^*_X)_{\overline{x}}$ of the sheaf $M_X/\mathcal{O}^*_X$ (and of the induced cones $\sigma_{\overline{x}}$) under cospecializations.

Let (X, M_X) be any fs log-scheme. Denote by $F(X)$ the fan of (X, M_X), $F(X) = \{x \in X | I_{\overline{x}}\mathcal{O}_{X,\overline{x}} = \mathfrak{m}_{\overline{x}}\}$, where $\mathfrak{m}_{\overline{x}}$ is the maximal ideal in $\mathcal{O}_{X,\overline{x}}$.

Let $x \in X$. Consider the lattice $L_{\overline{x}} = (P^{gp}_{\overline{x}})^\vee$. Any valuation $v_{\mathfrak{p}}$, $\mathfrak{p} \in P_{\overline{x}}(1)$, defines a map $L^\vee_{\overline{x}} \to \mathbf{Z}$, hence an element $\rho_{\mathfrak{p}} \in L_{\overline{x}}$. Denote by $\sigma_{\overline{x}} \subset L_{\overline{x},\mathbf{R}}$ the (rational convex polyhedral) cone generated by $\rho_{\mathfrak{p}}$, $\mathfrak{p} \in P_{\overline{x}}(1)$.

Lemma 2.10. *We have*

(1) $P_{\overline{x}} = \sigma^\vee_{\overline{x}} \cap L^\vee_{\overline{x}}$;
(2) *the cones $\sigma_{\overline{x}}$ and $\sigma^\vee_{\overline{x}}$ are strongly convex;*

(3) $\dim \sigma_{\overline{x}} = \dim \sigma_{\overline{x}}^{\vee} = \mathrm{rank}(P_{\overline{x}}^{gp})$;
(4) *if* (X, M_X) *is log-regular then* $\dim \sigma_{\overline{x}} \leq \dim(\mathcal{O}_{X,\overline{x}})$ *for every* $x \in X$ *and if in addition* $x \in F(X)$, *then* $\dim \sigma_{\overline{x}} = \dim(\mathcal{O}_{X,\overline{x}})$;
(5) *the elements* $\rho_{\mathfrak{p}}$ *are indivisible and the rays* $\rho_{\mathfrak{p}}$ *are exactly the one-dimensional faces of* $\sigma_{\overline{x}}$.

Remark 2.11. *An element of a lattice is called em indivisible if it is minimal along its ray.*

Proof. To see property (1), compute $\sigma_{\overline{x}}^{\vee} \cap L_{\overline{x}}^{\vee} = \{x \in P_{\overline{x}}^{gp} | \forall \mathfrak{p} \in P_{\overline{x}}(1),\ v_{\mathfrak{p}}(x) \geq 0\}$. But, by [13, 5.8], $P_{\overline{x}} = \bigcap_{\mathfrak{p} \in P_{\overline{x}}(1)} P_{\overline{x},\mathfrak{p}}$ and, by [17, 1.1.5], $P_{\overline{x},\mathfrak{p}} = \{x \in P_{\overline{x}}^{gp} | v_{\mathfrak{p}}(x) \geq 0\}$. Hence $P_{\overline{x}} = \{x \in P_{\overline{x}}^{gp} | \forall \mathfrak{p} \in P_{\overline{x}}(1),\ v_{\mathfrak{p}}(x) \geq 0\}$, as wanted.

For properties (2) and (3), we compute from property (1) that

$$\sigma_{\overline{x}}^{\vee} \cap (-\sigma_{\overline{x}}^{\vee}) \cap L_{\overline{x}}^{\vee} = P_{\overline{x}} \cap P_{\overline{x}}^{-1} = P_{\overline{x}}^{*} = \{1\}$$

and that $\dim \sigma_{\overline{x}}^{\vee} = \mathrm{rank}\, P_{\overline{x}}^{gp}$. Hence both cones $\sigma_{\overline{x}}$ and $\sigma_{\overline{x}}^{\vee}$ are strongly convex and $\dim \sigma_{\overline{x}} = \mathrm{rank}\, P_{\overline{x}}^{gp}$.

Property (4) follows from property (3) and the fact that $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,\overline{x}})$ and $\dim(\mathcal{O}_{X,\overline{x}}) = \dim(\mathcal{O}_{X,\overline{x}}/I_{\overline{x}}\mathcal{O}_{X,\overline{x}}) + \mathrm{rank}_{\mathbf{Z}}((M_X^{gp}/\mathcal{O}_X^*)_{\overline{x}})$ (since (X, M_X) is log-regular).

Property (5) follows from the definition of $\rho_{\mathfrak{p}}$ and the fact that the cone $\sigma_{\overline{x}}$ is strongly convex and that the elements $\rho_{\mathfrak{p}}$ form a minimal set of generators of the cone $\sigma_{\overline{x}}$. To see the last fact assume that $\rho_{\mathfrak{p}_0} = \sum_{i \neq 0} a_i \rho_{\mathfrak{p}_i}$, $a_i \in \mathbf{R}_+$. Take $y \in P_{\overline{x}}$ such that $v_{\mathfrak{p}_0}(y) = 0$ and $v_{\mathfrak{p}_i}(y) > 0$, $i \neq 0$. Then

$$0 = v_{\mathfrak{p}_0}(y) = < \rho_{\mathfrak{p}_0}, y > = \sum_{i \neq 0} a_i < \rho_{\mathfrak{p}_i}, y > = \sum_{i \neq 0} a_i v_{\mathfrak{p}_i}(y).$$

Hence $a_i = 0$, $i \neq 0$, and $\rho_{\mathfrak{p}_0} = 0$, which is not true. □

Lemma 2.12. *For any points* $x, y \in X$, $x \in \{y\}^-$, *and any cospecialization map* $f : P_{\overline{x}} \to P_{\overline{y}}$

(1) *the induced map* $f : L_{\overline{x}}^{\vee} \to L_{\overline{y}}^{\vee}$ *is surjective;*
(2) *the induced map* $f^* : L_{\overline{y}} \to L_{\overline{x}}$ *is injective;*
(3) *the induced specialization map* $f^* : \sigma_{\overline{y}} \to \sigma_{\overline{x}}$ *maps* $\sigma_{\overline{y}}$ *isomorphically onto a face of* $\sigma_{\overline{x}}$;
(4) *if* (X, M_X) *is log-regular and* $y \in X$ *corresponds to the prime ideal* $I_{\overline{x}}\mathcal{O}_{X,\overline{x}}$ *(cf.* [13, 7.3]*), then the cospecialization* $f : \mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X,\overline{y}}$ *factors through the local ring of* $\mathcal{O}_{X,\overline{x}}$ *at* $I_{\overline{x}}\mathcal{O}_{X,\overline{x}}$ *and induces an isomorphism* $f : P_{\overline{x}} \xrightarrow{\sim} P_{\overline{y}}$.

Proof. Let $f : \mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X,\overline{y}}$ be the original cospecialization map. Let $z \in \mathrm{Spec}(\mathcal{O}_{X,\overline{x}})$ be the point corresponding to the preimage of the maximal ideal of $\mathcal{O}_{X,\overline{y}}$ under this map. Take any section $s_{\overline{x}} : P_{\overline{x}} \to M_{X,\overline{x}}$ of the projection $M_{X,\overline{x}} \to P_{\overline{x}}$. Let $\mathfrak{p} \in \mathrm{Spec}(P_{\overline{x}})$ be the preimage of the maximal ideal of $\mathcal{O}_{Y,\overline{z}}$ under the morphism $P_{\overline{x}} \xrightarrow{s_{\overline{x}}} M_{Y,\overline{z}} \to \mathcal{O}_{Y,\overline{z}}$ (using the notation introduced above). Then the cospecialization f factors as $f : P_{\overline{x}} \to P_{\overline{x}}/(P_{\overline{x}} \setminus \mathfrak{p}) \xrightarrow{\sim} P_{\overline{x},\mathfrak{p}}/P_{\overline{x},\mathfrak{p}}^* \xrightarrow{\sim} P_{\overline{y}}$ (see [17, 2.1.4] for the reason why we have the stated isomorphisms). This shows assertions (1) and (2).

Concerning (3), let $\rho_i, i = 1, \ldots, n$, be the rays spanning $\sigma_{\overline{y}}$ corresponding to ideals $\mathfrak{p}_i \in P_{\overline{y}}(1)$. Let $\mathfrak{q}_i = f^*(\mathfrak{p}_i) \in \mathrm{Spec}(P_{X,\overline{x}})$. By the above computations, it is clear that the prime ideals $\mathfrak{q}_i$ are in $P_{\overline{x}}(1)$ (and are contained in $\mathfrak{p}$) and that they are all distinct. The image of $\sigma_{\overline{y}}$ by f^* is equal to the face of $\sigma_{\overline{x}}$ spanned by the rays corresponding to the ideals $\mathfrak{q}_i$.

For property (4), note that there exists a cospecialization $f' : \mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X,\overline{y}}$ that factors through the local ring at $I_{\overline{x}}\mathcal{O}_{X,\overline{x}}$. Computing as above we get that $f' : P_{\overline{x}} \to P_{\overline{y}}$ is an isomorphism. In particular, $\dim(\sigma_{\overline{x}}) = \dim(\sigma_{\overline{y}})$. Property (4) follows now easily from property (3). □

Remark 2.13. (1) *If (X, M_X) is log-regular, then the points $w \in \operatorname{Spec}(\mathcal{O}_{X,\overline{y}})$ of codimension one with $M_{X,\overline{w}} \neq \mathcal{O}^*_{X,\overline{w}}$ map via the cospecialization $f : \operatorname{Spec}(\mathcal{O}_{X,\overline{y}}) \to \operatorname{Spec}(\mathcal{O}_{X,\overline{x}})$ to distinct points of codimension one in $\mathcal{O}_{X,\overline{x}}$. These are the points corresponding to the ideals $\mathfrak{q}_i$.*

(2) *We have the following*

Lemma 2.14. *If τ is a face of $\sigma_{\overline{x}}$, then $P_{\overline{x}} \cap \tau^\perp$ is a face of $P_{\overline{x}} = \sigma^\vee_{\overline{x}} \cap L^\vee_{\overline{x}}$ and $\mathfrak{p}_\tau = P_{\overline{x}} \setminus P_{\overline{x}} \cap \tau^\perp$ is a prime ideal in $P_{\overline{x}}$, with $\operatorname{ht}(\mathfrak{p}_\tau) = \dim(\tau)$. The map $\tau \mapsto \mathfrak{p}_\tau$ is a one-to-one correspondence between the faces of $\sigma_{\overline{x}}$ and the elements of $\operatorname{Spec}(P_{\overline{x}})$.*

Proof. It is basically [8, 1.2.10]. The equality $\operatorname{ht}(\mathfrak{p}_\tau) = \dim(\tau)$ follows because

$$\operatorname{rank}(P^{gp}_{\overline{x}}) = \dim(\tau) + \dim(P_{\overline{x}} \cap \tau^\perp) = \dim(\tau) + \dim(P_{\overline{x}} \setminus \mathfrak{p}_\tau),$$

and [13, 5.5] $\operatorname{ht}(\mathfrak{p}_\tau) + \dim(P_{\overline{x}} \setminus \mathfrak{p}_\tau) = \operatorname{rank}(P^{gp}_{\overline{x}})$. □

The proof of Lemma 2.12 shows that the cospecialization map $f^* : \sigma_{\overline{y}} \to \sigma_{\overline{x}}$ maps $\sigma_{\overline{y}}$ onto the face of $\sigma_{\overline{x}}$ corresponding to the ideal $\mathfrak{p}$. Indeed, the image of f^* is the face $\tau \prec \sigma_{\overline{x}}$ spanned by the elements $\rho_{\mathfrak{q}_i} \in L_{\overline{x}}$ corresponding to the valuations at the ideals $\mathfrak{q}_i$. We easily compute that

$$P_{\overline{x}} \cap \tau^\perp = \{a \in P_{\overline{x}} | v_{\mathfrak{q}_i}(a) = 0, \forall \mathfrak{q}_i\} = \{a \in P_{\overline{x}} | v_{\mathfrak{p}_i}(f(a)) = 0, \forall \mathfrak{p}_i \in P_{\overline{y}}(1)\}.$$

Hence $P_{\overline{x}} \cap \tau^\perp = \ker(f)$. On the other hand, the ideal $\mathfrak{p}$ is the preimage under f of the ideal $P_{\overline{y}} \setminus \{1\}$. Hence $\ker(f) = P_{\overline{y}} \setminus \mathfrak{p}$, as wanted.

3. Saturated fractional ideals

We will show in this section (cf. Proposition 3.14) that the classical relationship between saturated fractional ideals and certain convex functions [15] carries over to fs log-schemes.

Definition 3.1. *(*[7, 2.1]*)*

(1) *Let (X, M_X) be any fs log-scheme. A subsheaf J of M_X is called an* ideal *of M_X if the stalk $J_{\overline{x}}$ is an ideal of $M_{X,\overline{x}}$ for any $x \in X$.*

(2) *A subsheaf J of M_X (resp. M_X^{gp}) is called a* coherent ideal *(resp. a* coherent fractional ideal*) if étale locally on X there exists a chart $P \to M_X$ and an ideal J_0 of P (resp. a fractional ideal $J_0 \subset P^{gp}$* [13, 5.7]*) such that J is generated by J_0, i.e., J is the image of the map $M_X \times J_0 \to M_X$ (resp. $M_X \times J_0 \to M_X^{gp}$), $(a, b) \mapsto ab$.*

Let $J \subset M_X^{gp}$ be a coherent fractional ideal. Take $x \in X$. Let $J_{\overline{x}} \subset P^{gp}_{\overline{x}} \simeq M^{gp}_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$ denote (abusively) the image of the ideal $J_{\overline{x}}$ in $P^{gp}_{\overline{x}}$. It is a fractional ideal. Let $f_{J,\overline{x}} : \sigma_{\overline{x}} \to$

$\mathbf{R}$ be the function $f_{J,\overline{x}}(a) = \min\{\mu(a)|\mu \in J_{\overline{x}}\}$. We easily check that if $J_{\overline{x}} = \bigcup_{i=1}^{r} P_{\overline{x}}a_i$, $a_i \in P_{\overline{x}}^{gp} = L_{\overline{x}}^{\vee}$, then $f_{J,\overline{x}}(a) = \min_i\{a_i(a)\}$ and that $f_{J,\overline{x}}$ is a

(*) continuous, homogeneous function, convex and piecewise linear, integral on $L_{\overline{x}} \cap \sigma_{\overline{x}}$.

Lemma 3.2. *The maps $f_{J,\overline{x}}$, $x \in X$, are compatible with cospecializations $g : P_{\overline{x}} \to P_{\overline{y}}$, $x, y \in X$.*

Proof. By taking a chart, it is easy to see that the map $g : J_{\overline{x}} \to J_{\overline{y}}$ is surjective. Let $g^* : \sigma_{\overline{y}} \to \sigma_{\overline{x}}$ be the induced map and take any $a \in \sigma_{\overline{y}}$. We have

$$\begin{aligned} f_{J,\overline{y}}(a) &= \min\{\mu(a)|\mu \in J_{\overline{y}}\} = \min\{g(\mu)(a)|\mu \in J_{\overline{x}}\} \\ &= \min\{\mu(g^*(a))|\mu \in J_{\overline{x}}\} = f_{J,\overline{x}}(g^*(a)), \end{aligned}$$

as wanted. □

Lemma 3.3. *Let (X, M_X) be a log-regular scheme. The natural restriction map $\{f_{\overline{x}} : \sigma_{\overline{x}} \to \mathbf{R}|x \in X\} \to \{f_{\overline{x}} : \sigma_{\overline{x}} \to \mathbf{R}|x \in F(X)\}$ induces a one-to-one correspondence between the set of functions $\{f_{\overline{x}} : \sigma_{\overline{x}} \to \mathbf{R}|x \in X\}$ satisfying (*) and compatible with cospecializations, and its subset indexed by the set $\{x \in F(X)\}$.*

Proof. We will construct the inverse to the restriction. Consider functions $\{f_{\overline{x}} : \sigma_{\overline{x}} \to \mathbf{R}|x \in F(X)\}$ satisfying (*) and compatible with cospecializations. We want to extend them to all $x \in X$. Take any point $x \in X$. Let $y \in F(X)$ be the point corresponding to the prime ideal $I_{\overline{x}}\mathcal{O}_{X,\overline{x}}$. Take any cospecialization $g : \mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X,\overline{y}}$. Recall (Lemma 2.12.4) that $g^* : \sigma_{\overline{y}} \to \sigma_{\overline{x}}$ is an isomorphism. Set $f_{g,\overline{x}} = f_{\overline{y}}(g^*)^{-1}$. This definition does not depend on the cospecialization chosen. Indeed, let $h : \mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X,\overline{y}}$ be also a cospecialization. Since the cospecializations $g, h : P_{\overline{x}} \to P_{\overline{y}}$ are isomorphisms, it is easy to check that both h and g factor through the local ring at $I_{\overline{x}}\mathcal{O}_{X,\overline{x}}$. Then there exists a cospecialization $t : \mathcal{O}_{X,\overline{y}} \to \mathcal{O}_{X,\overline{y}}$ such that $th = g$. Compatibility with cospecializations gives

$$f_{g,\overline{x}} = f_{\overline{y}}(g^*)^{-1} = f_{\overline{y}}(t^*)^{-1}(h^*)^{-1} = f_{\overline{y}}(h^*)^{-1} = f_{h,\overline{x}},$$

as wanted.

The new functions $\{f_{\overline{x}} : \sigma_{\overline{x}} \to \mathbf{R}|x \in X\}$ satisfy (*) and it is not difficult to check that they are compatible with cospecializations. Since it is clear that the above construction yields the inverse to the restriction map, we are done. □

We will now "glue" the functions from the set $\{f_{\overline{x}}|x \in X\}$ into a global section of the sheaf of sets $\mathcal{F}_X$ associated to the presheaf $\mathcal{P}_X$:

$$(U \to X) \mapsto \mathrm{Map}(\mathrm{Hom}((M_X/\mathcal{O}_X^*)(U), \mathbf{R}^+), \mathbf{R}).$$

To understand the sheaf $\mathcal{F}_X$ we need to digress a bit. Let $\mathcal{L}$ be a sheaf on a locally noetherian topological space Y. The sheaf $\mathcal{L}$ is constructible (cf., [16, 2.3]) if there exists a locally finite partition Σ of Y into locally closed subsets such that each $\mathcal{L}_S$ is constant, for $S \in \Sigma$. Here we say that $\mathcal{L}$ is constant on S if the natural map $\mathcal{L}(S) \to \mathcal{L}_y$ is an isomorphism, for each $y \in S$. In particular, S is connected. The partition above is called a trivializing partition for $\mathcal{L}$.

It is easy to check [11, 1.11] that if (X, M_X) is a Zariski fs log-scheme then the sheaf $M_X/\mathcal{O}_X^*$ is constructible. Hence every point $x \in X$ has an open neighbourhood such that

x is central for $M_X/\mathcal{O}_X^*$ in the terminology of Ogus [16], i.e., there exists a trivializing partition Σ of U such that x belongs to the closure of every $S \in \Sigma$. In particular, if $y \in S \subset U$ then there exists a point $w \in S$ such that $x \in \{w\}^-$. This gives a "path" between x and y in U. Indeed, for any two points y and w belonging to the same stratum, there exist points $y = a_0, \dots, a_n$ and $b_0, \dots, b_n = w$ in the same stratum such that $a_i \in \{b_i\}^-$ and $a_{i+1} \in \{b_i\}^-$. Any such path $g : y \mapsto w$ induces a morphism $g : P_{\overline{y}} \mapsto P_{\overline{w}}$ that is clearly an isomorphism.

Lemma 3.4. *For any $x \in X$ we can find a neighbourhood $\overline{x} \to U \to X$ with a section $s_U : P_{\overline{x}} \to M_X(U)$ of the projection $M_X(U) \to P_{\overline{x}}$ such that the map $s'_U : P_{\overline{x}} \to M_X/\mathcal{O}_X^*(U)$ is an isomorphism.*

Proof. Take a neighbourhood $\overline{x} \to U \to X$ with a section $s_U : P_{\overline{x}} \to M_X(U)$ of the projection $M_X(U) \to P_{\overline{x}}$ such that the image x_1 of $\overline{x} \to U$ is central in U. Clearly s'_U is injective. To show surjectivity, take any $a \in M_X/\mathcal{O}_X^*(U)$ and consider $b = a/s'_U(a_{\overline{x}}) \in M_X^{gp}/\mathcal{O}_X^*(U)$. We claim that $b_{\overline{y}} = 1$, $y \in U$. Take a point $w \in U$ such that $x_1 \in \{w\}^-$ and y and w are in the same stratum S. Since any path in S $g : P_{\overline{y}}^{gp} \mapsto P_{\overline{w}}^{gp}$ is an isomorphism, we may assume that $y = w$. Now for any cospecialization $g : P_{\overline{x_1}}^{gp} \to P_{\overline{y}}^{gp}$, $b_{\overline{y}} = g(b_{\overline{x}}) = 1$, and we are done. This gives $b = 1$ and the desired surjectivity. □

Corollary 3.5. *For any $x \in X$, the natural morphism $\mathcal{F}_{X,\overline{x}} \to \mathrm{Map}(\sigma_{\overline{x}}, \mathbf{R})$ is an isomorphism.*

Proof. Since $\mathrm{Hom}(P_{\overline{x}}, \mathbf{R}^+) = \sigma_{\overline{x}}$, this is clear from the above lemma. □

Corollary 3.6. *Let $P \to M_X(X)$ be a chart with an fs monoid P. Let $f : (X, M_X) \to T = \mathrm{Spec}(\mathbf{Z}[P])$ be the induced map. Then the natural morphism $f^*\mathcal{F}_T \to \mathcal{F}_X$ is an isomorphism.*

Proof. Let $x \in X$. It suffices to show that the map on stalks $\mathcal{F}_{X,x} \leftarrow (f^*\mathcal{F}_T)_x \simeq (\mathcal{F}_T)_{f(x)}$ is an isomorphism. Via the previous corollary, this is equivalent to showing that the map $\mathrm{Map}(\sigma_x, \mathbf{R}) \leftarrow \mathrm{Map}(\sigma_{f(x)}, \mathbf{R})$ induced by the map $P_{f(x)} \to P_x$ is an isomorphism. Since both monoids $P_{f(x)}$ and P_x are isomorphic to $P/(P \setminus \mathfrak{p})$, where $\mathfrak{p}$ is the preimage of the maximal ideal at x in P, this is clear. □

Let f be a global section of the sheaf $\mathcal{F}_X$. Since the above corollary yields that $\mathcal{F}_{X,\overline{x}} \xrightarrow{\sim} \mathrm{Map}(\sigma_{\overline{x}}, \mathbf{R})$, $x \in X$, the function f defines functions $f_{\overline{x}} : \sigma_{\overline{x}} \to \mathbf{R}$, $x \in X$, which are clearly compatible with cospecializations.

Proposition 3.7. *The above map $\phi : f \mapsto \{f_{\overline{x}} | x \in X\}$ defines a one-to-one correspondence between the global sections f of the sheaf $\mathcal{F}_X$ (such that every stalk $f_{\overline{x}}$ satisfies (*)) and the set of functions $\{f_{\overline{x}} \in \mathrm{Map}(\sigma_{\overline{x}}, \mathbf{R}) | x \in X\}$ compatible with cospecializations (and satysfying (*)).*

Proof. We will construct the inverse ϕ^{-1} to ϕ. Since the map ϕ is injective and functorial with respect to étale morphisms, it suffices to construct ϕ^{-1} étale locally.

Let $x \in X$ and assume that we are given a chart $s_X : P_{\overline{x},X} \to M_X$ compatible with the projection $M_{X,\overline{x}} \to P_{\overline{x}}$ and such that x is central in X. Set $\tilde{f} \in \mathcal{F}_X(X)$ to be the section given by the map

$$[M_X/\mathcal{O}_X^*(X) \to \mathbf{R}^+] \xrightarrow{s_X} [P_{\overline{x}} \to \mathbf{R}^+] \xrightarrow{f_{\overline{x}}} \mathbf{R}.$$

Using the fact that x is central in X and the compatibility of the functions $\{f_{\overline{y}}|y \in X\}$ and $\phi(\tilde{f})$ with cospecializations, it is now easy to check that $\phi(\tilde{f}) = \{f_{\overline{y}}|y \in X\}$, as wanted. □

We will now establish a correspondence between global sections of $\mathcal{F}_X$ whose stalks satisfy (*) and certain fractional ideals.

Definition 3.8. *(*[13, 11.1]*) Let P be a fine and saturated monoid. A fractional ideal $I \subset P^{gp}$ is called* saturated *if it is equal to its saturation, i.e., to the ideal $I^c := \{a \in P^{gp}|\exists q \in \mathbf{N} \setminus \{0\},\ a^q \in I^q\}$.*

It is easy to check that the fractional ideal I^c is itself saturated, i.e., that $(I^c)^c = I^c$.

Definition 3.9. *A coherent fractional ideal $J \subset M_X^{gp}$ is called* saturated *if étale locally there exists a chart (P, J_0) as in Definition 3.1 with $J_0 \subset P^{gp}$ saturated.*

We have the obvious

Lemma 3.10. *A coherent fractional ideal $J \subset M_X^{gp}$ is saturated if and only if all the ideals $J_{\overline{x}} \subset P_{\overline{x}}^{gp}$, $x \in X$, are saturated.*

Definition 3.11. *Let $J \subset M_X^{gp}$ be a coherent fractional ideal. Its saturation $J^c \subset M_X^{gp}$ is a fractional ideal defined by (étale locally) saturating the ideals generating J. We have $(J^c)_{\overline{x}} = (J_{\overline{x}})^c \subset P_{\overline{x}}^{gp}$ for $x \in X$.*

Let now $f \in \mathcal{F}_X(X)$ be a section satisfying (*). For every $x \in X$, consider the fractional ideal $J_{\overline{x}} = \{a \in P_{\overline{x}}^{gp}|a(z) \geq f_{\overline{x}}(z), \forall z \in \sigma_{\overline{x}}\}$ in $P_{\overline{x}}^{gp}$. For every étale open $U \to X$, let $J_f(U)$ be the set of all $a \in M_X^{gp}(U)$ such that for any étale neighbourhood $\overline{x} \xrightarrow{h_{\overline{x}}} U$ of $\overline{x} \to X$, $x \in X$, $h_{\overline{x}}^*(a) \in J_{\overline{x}}$. We easily check that $J_f \subset M_X^{gp}$ is an ideal.

Lemma 3.12. *The sheaf J_f is a saturated fractional ideal.*

Proof. First, take any cospecialization $g : P_{\overline{x}} \to P_{\overline{y}}$, $x, y \in X$. We claim that g maps $J_{\overline{x}}$ onto $J_{\overline{y}}$. Indeed, if $a \in J_{\overline{x}}$ and $z \in \sigma_{\overline{y}}$, then $g(a)(z) = a(g^*(z)) \geq f_{\overline{x}}(g^*(z)) = f_{\overline{y}}(z)$. Hence $g(a) \in J_{\overline{y}}$ and g maps $J_{\overline{x}}$ to $J_{\overline{y}}$. For surjectivity, take any $a \in J_{\overline{y}}$ and any $y \in P_{\overline{x}}^{gp}$ in the preimage of a. We have $y(g^*(z)) = a(z) \geq f_{\overline{y}}(z) = f_{\overline{x}}(g^*(z))$. Write $g^*(\sigma_{\overline{y}}) = \sigma_{\overline{x}} \cap u^{\perp}$, for some $u \in P_{\overline{x}}$. Since $y \geq f_{\overline{x}}$ on $g^*(\sigma_{\overline{y}})$ if and only if $y + Nu \geq f_{\overline{x}}$ on $\sigma_{\overline{x}}$ for some N, $y + Nu \in J_{\overline{x}}$. Since $y + Nu$ also maps to a, we are done.

Next, let's check that, for any $x \in X$, the natural morphism $\pi_{\overline{x}} : J'_{f,\overline{x}} \to J_{\overline{x}}$ is an isomorphism, where $J'_{f,\overline{x}}$ is the image of $J_{f,\overline{x}}$ in $P_{\overline{x}}^{gp}$. Consider a neighbourhood $\overline{x} \xrightarrow{h_{\overline{x}}} U$ of $\overline{x} \to X$ such that a section $s_{\overline{x}} : P_{\overline{x}} \to M_{X,\overline{x}}$ extends to a chart $s_U : P_{\overline{x},U} \to M_X(U)$ and the image x_1 of $h_{\overline{x}}$ is central in U. We will check that $s_U(J_{\overline{x}}) \subset J_f(U)$. For that, take a point $\overline{z} \xrightarrow{h_{\overline{z}}} U$, $z \in X$, over $z_1 \in U$. Let $w_1 \in U$ over $w \in X$, be such that $x_1 \in \{w_1\}^-$ and w_1, z_1 are in the same stratum S. Let $a \in J_{\overline{x}}$. We want $h_{\overline{z}}^*(s_U(a)) \in J_{\overline{z}}$. Take a point $\overline{w} \xrightarrow{h_{\overline{w}}} U$ over w_1, and let $g : P_{\overline{z}} \mapsto P_{\overline{w}}$ be a path in S compatible with $h_{\overline{z}}$ and $h_{\overline{w}}$, i.e., $gh_{\overline{z}}^* = h_{\overline{w}}^*$. Then $h_{\overline{w}}^*(s_U(a)) = gh_{\overline{z}}^*(s_U(a))$. Since g maps $J_{\overline{z}}$ isomorphically onto $J_{\overline{w}}$, we may assume that $z_1 = w_1$. Let $g : P_{\overline{x}} \to P_{\overline{z}}$ be a cospecialization compatible with $h_{\overline{x}}$ and $h_{\overline{z}}$, i.e., $gh_{\overline{x}}^* = h_{\overline{z}}^*$. Then $h_{\overline{z}}^*(s_U(a)) = gh_{\overline{x}}^*(s_U(a)) = g(a)$. Since, by the computation at the begining of this proof, $g(a) \in J_{\overline{z}}$, we are done.

This yields that the map $J_f(U) \xrightarrow{h^*_{\overline{x}}} J_{\overline{x}}$ is a surjection. Hence $\pi_{\overline{x}}$ is surjective as well. To see that it is injective, take any $a \in J'_{f,\overline{x}}$. By shrinking U if necessary, we may assume that $a \in J'_f(U)$, where $J'_f(U)$ is the image of $J_f(U)$ in $M_X^{gp}/\mathcal{O}_X^*(U)$. Arguing like in the proof of Lemma 3.4 we see that $\pi_{\overline{x}}(a) = 1$ implies $a = 1$ for small enough U, as wanted.

The above yields, that J_f is starry in the terminology of [16, 2.4], i.e., that for any cospecialization $g : P_{\overline{x}} \to P_{\overline{y}}$, $x, y \in X$, the induced map $g' : J'_{f,\overline{x}} \to J'_{f,\overline{y}}$ is surjective. This, by [16, 2.6], implies that J_f is coherent.

To check that J_f is saturated it suffices now to check that so is every $J'_{f,\overline{x}} \simeq J_{\overline{x}}$, $x \in X$. That can be shown exactly as in the proof of Theorem 9 in [15]. □

Remark 3.13. *When (X, M_X) is log-regular, in view of Lemma 3.3, in the definition of the ideal J_f we could have just used the points $x \in F(X)$.*

Proposition 3.14. *We have the following*

(1) *for any $f \in \mathcal{F}_X(X)$ with stalks satisfying (*), $f_{J_f} = f$;*
(2) *for any coherent fractional ideal J, $J_{f_J} = J^c$;*
(3) *the maps $J \mapsto f_J$ and $f \mapsto J_f$ define a one-to-one correspondence between saturated fractional ideals of M_X^{gp} and the the global sections of $\mathcal{F}_X$ satisfying (*).*

Proof. The first statement is clear and the third one follows from the first two.

For the second one note that there is a natural inclusion $J \hookrightarrow J_{f_J}$. Since the ideal J_{f_J} is saturated, this inclusion induces a map $J^c \hookrightarrow J_{f_J}$. To check that this is an isomorphism we pass to stalks and argue like in the proof of Theorem 9 in [15]. □

4. Logarithmic blow-ups

We will now recall the definition of log-blow-ups (see [7, 2.2], [9, 1.6], [10] for details). Let (X, M_X) be an fs log-scheme equipped with a chart $P_X \to M_X$ for a fine and saturated monoid P. Let $J \subset P$ be an ideal. The log-blow-up of (X, M_X) at JM_X is defined by blowing-up the ideal generated by J in $\mathbf{Z}[P]$, saturating the blow-up and pulling it back to X via the map $X \to \mathrm{Spec}(\mathbf{Z}[P])$. The resulting scheme can be equipped with an fs log-structure. The construction can also be globalized, letting us blow-up coherent ideals J on any fs log-scheme (X, M_X). A version of log-blow-up, where we skip the saturation, we will call an unsaturated log-blow-up.

Definition 4.1. *A coherent ideal $J \subset M_X$ is called* invertible *if étale locally it can be generated by a single element or, equivalently, the pair $((X, M_X), J)$ admits a chart* [16, 2.2] *with principal ideals.*

A trivial but a crucial observation here is that a pullback of an invertible ideal is always invertible.

Proposition 4.2. *(Universal Property of Logarithmic Blowing-Up) Let (X, M_X) be an fs log-scheme and let $J \subset M_X$ be a coherent ideal. Let $(\widetilde{X}, M_{\widetilde{X}})$ be the log-blow-up of (X, M_X) at J. If $f : (Z, M_Z) \to (X, M_X)$ is any morphism of fs log schemes such that $f^{-1}J$ (the ideal generated by the image of $f^{-1}J$ in M_Z) is an invertible ideal, then there exists a unique morphism $g : (Z, M_Z) \to (\widetilde{X}, M_{\widetilde{X}})$ factoring f.*

Proof. We will treat first the case of unsaturated log-blow-ups. By uniqueness it suffices to argue étale locally on (Z, M_Z). We may thus assume an existence of the following fs-chart

$$\begin{array}{ccc} (Q, f(J_0)Q) & \longrightarrow & (M_Z, f^{-1}J) \\ {\scriptstyle f}\uparrow & & {\scriptstyle f}\uparrow \\ (P, J_0) & \longrightarrow & (M_X, J), \end{array}$$

where $f(J_0)Q = cQ$ is principal. Take $a \in J_0$ and consider the map

$$g'_a : \mathbf{Z}[P_a] \to \mathbf{Z}[Q] \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P_a] \xrightarrow{f} \mathbf{Z}[Q_{f(a)}],$$

where P_a is a submonoid of P^{gp} generated by P and elements of the form ba^{-1}, $b \in J_0$. Write $c = f(b)q$, $b \in J_0$, $q \in Q$. Then $Q_{f(b)} = Q$. In particular, the map g'_b induces a map $g' : \mathrm{Spec}(\mathbf{Z}[Q]) \to \mathrm{Proj}(\oplus_{n\geq 0}(J_0)^n)$, where (J_0) is the ideal generated by J_0 in $\mathbf{Z}[P]$. Since

$$(\widetilde{X}, M_{\widetilde{X}}) = \mathrm{Proj}(\mathcal{O}_X \otimes (\oplus_{n\geq 0}(J_0)^n)) = (X, M_X) \times_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Proj}(\oplus_{n\geq 0}(J_0)^n),$$

g' yields the wanted map $g : (Z, M_Z) \to (\widetilde{X}, M_{\widetilde{X}})$.

For the uniqueness, let's assume that we have two good maps: g_1 and g_2. We may argue locally and hence assume that we have a chart as above and that g_1 and g_2 are induced by maps $g'_1, g'_2 : P_a \to Q$, $a \in J$, that composed with the natural map $P \to P_a$ yield the map $P \to Q$. Since Q is integral, it follows that $g'_1 = g'_2$, and consequently $g_1 = g_2$.

We get the statement of the theorem for the saturated log-blow-ups by the universal property of saturation. □

Log-blow-ups have a particularly nice description in the case of log-regular schemes.

Proposition 4.3. *Let (X, M_X) be a log-regular scheme and let $J \subset M_X$ be a coherent ideal. Let $\pi : (\widetilde{X}, M_{\widetilde{X}}) \to (X, M_X)$ be the log-blow-up of (X, M_X) at J. If $f : Z \to X$ is the blow-up of the coherent sheaf of ideals $J\mathcal{O}_X$, then there exists a unique morphism $g : \widetilde{X} \to \widetilde{Z}$, where $\widetilde{Z}$ is the normalization of Z, factoring f. Moreover, the morphism g is an isomorphism.*

Remark 4.4. *The reader will notice that $J\mathcal{O}_X$ is indeed a Zariski sheaf, so the above statement makes sense.*

Proof. Again, let's treat first the case of unsaturated log-blow-up. Assume that we have a chart $\beta : (P, J_0) \to (M_X, J)$ and that X is affine. Then $\widetilde{X} = \mathrm{Proj}(\mathcal{O}_X \otimes_{\mathbf{Z}[P]} (\oplus_{n\geq 0}(J_0)^n))$. Since $Z = \mathrm{Proj}(\oplus_{n\geq 0}(J\mathcal{O}_X)^n)$, we have a natural map $g_1 : Z \to \widetilde{X}$ induced by the morphism $P \xrightarrow{\beta} \Gamma(X, M_X) \xrightarrow{\alpha} \mathcal{O}_X$ [6, 3.6.2] (notice that the morphism $\mathcal{O}_X \otimes_{\mathbf{Z}[P]} (J_0)^n \to (J\mathcal{O}_X)^n$ is surjective). We claim that g_1 is an isomorphism. Indeed, it suffices to show that the map $\mathcal{O}_X \otimes_{\mathbf{Z}[P]} (J_0)^n \to (J\mathcal{O}_X)^n = \alpha\beta((J_0)^n)\mathcal{O}_X$ is an isomorphism, or by localizing, that so is the map $\mathcal{O}_{X,x} \otimes_{\mathbf{Z}[P]} (J_0)^n \to \alpha\beta((J_0)^n)\mathcal{O}_{X,x}$, $x \in X$. There is an exact sequence

$$0 \to (J_0)^n \to \mathbf{Z}[P] \to \mathbf{Z}[P]/(J_0)^n \to 0.$$

After tensoring it with $\mathcal{O}_{X,x}$ over $\mathbf{Z}[P]$ we get the following exact sequence

$$\mathrm{Tor}_1^{\mathbf{Z}[P]}(\mathcal{O}_{X,x}, \mathbf{Z}[P]/(J_0)^n) \to \mathcal{O}_{X,x} \otimes_{\mathbf{Z}[P]} (J_0)^n \xrightarrow{\alpha\beta} \mathcal{O}_{X,x}.$$

Since (X, M_X) is log-regular, $\mathrm{Tor}_1^{\mathbf{Z}[P]}(\mathcal{O}_{X,x}, \mathbf{Z}[P]/(J_0)^n) = 0$ [13, 6.1.iii], and we are done. Set $g = g_1^{-1}$.

Let's return now to the case when $(\widetilde{X}, M_{\widetilde{X}})$ is saturated. The existence of the map $g : \widetilde{X} \to \tilde{Z}$ follows from the previous paragraph, together with the fact that $\widetilde{X}$ is normal since $\widetilde{X}, M_{\widetilde{X}}$ is log-regular. It remains to prove that g is unique. From the universal property of (classical) blowing-up, it suffices to show that the sheaf of ideals $\pi^{-1}(J\mathcal{O}_X)$ is invertible. We can argue locally on $\widetilde{X}$. Assume that we are in the local setting described above and consider the open $\mathrm{Spec}(\mathbf{Z}[P_a^{\mathrm{sat}}] \otimes_{\mathbf{Z}[P]} \mathcal{O}_X)$, $a \in J_0$, of $\widetilde{X}$, where the superscript "sat" refers to saturating the monoid. It suffices to show that the morphism $\mathbf{Z}[P_a^{\mathrm{sat}}] \otimes_{\mathbf{Z}[P]} \mathcal{O}_X \xrightarrow{a} \mathbf{Z}[P_a^{\mathrm{sat}}] \otimes_{\mathbf{Z}[P]} \mathcal{O}_X$ is injective. Since the morphism $\mathbf{Z}[P_a^{\mathrm{sat}}] \xrightarrow{a} \mathbf{Z}[P_a^{\mathrm{sat}}]$ is injective, that follows (just like above) from the fact that $\mathrm{Tor}_1^{\mathbf{Z}[P]}(\mathcal{O}_{X,x}, \mathbf{Z}[P_a^{\mathrm{sat}}]/a\mathbf{Z}[P_a^{\mathrm{sat}}]) = 0$ [13, 6.1.ii]. □

Everything we have done in this and the previous section could be done with the logarithmic structures in the Zariski instead of the étale topology. To relate the Zariski and the étale log-blow-ups we have the following

Proposition 4.5. *Let (X, M_X) be a Zariski fs log-scheme and let $J \subset M_X$ be a Zariski coherent ideal. Let (Y, M_Y) be the log-blow-up of the coherent ideal JM_X. Then the log-scheme (Y, M_Y) is Zariski and the corresponding Zariski log-scheme (Y, M_Y^{Zar}) is uniquely isomorphic over (X, M_X^{Zar}) to the log-blow-up of J.*

Remark 4.6. *By Proposition 2.5, the log-structures M_Y^{Zar} and M_X^{Zar} are fs, so the above statement makes sense.*

Proof. The structure morphism $\pi : (Y, M_Y) \to (X, M_X)$ induces a morphism $\pi_1 : (Y, M_Y^{\mathrm{Zar}}) \to (X, M_X^{\mathrm{Zar}})$. Since the ideal $\pi^{-1}(JM_X)$ is invertible, using the local description of log-blow-ups, we easily check that so is the ideal $\pi_1^{-1}J$. Hence, by Proposition 4.2, there is a unique (X, M_X^{Zar})-morphism $\phi : (Y, M_Y^{\mathrm{Zar}}) \to (Z, M_Z)$, where (Z, M_Z) is the Zariski log-blow-up of J. Similarly, there is a unique (X, M_X)-morphism $\phi' : (Y, \varepsilon^* M_Y^{\mathrm{Zar}}) \to (Y, M_Y)$.To check that ϕ and ϕ' are isomorphisms we may argue locally, where this fact is clear from the description of log-blow-ups. □

We will now relate Zariski log-blow-ups to subdivisions in the sense of Kato [13, 9.9]. Let (X, M_X) be a Zariski log-regular scheme. Recall [13, 10.1] that the fan $F(X)$ has a structure of a monoidal space [13, 9.1], i.e., a topological space T endowed with a sheaf of monoids M_T such that $M_{T,t}^* = \{1\}$ for any $t \in T$. Assume that the fan $F(X)$ comes from a classical fan, i.e., that the monoidal space corresponding to the fan $F(X)$ is equipped with a surjective morphism $h : L^\vee \to M_{F(X)}^{gp}$, where L is a finitely generated free abelian group, such that the induced map $\mathrm{Mor}(\mathrm{Spec}(\mathbf{N}), F(X)) \to L$ is injective. Let $\Sigma(X)$ denote the associated (classical) fan (embedded in $L_{\mathbf{R}}$).

Assume that we are given functions $f = \{f_x\}, x \in F(X)$, satisfying (*) and compatible with cospecializations. Since there is a one-to-one correspondence between the cones σ_x associated to (X, M_X) and the cones σ'_x of the fan $\Sigma(X)$:

$$\sigma'_x \cap L = \mathrm{Im}(\sigma_x \cap L_x \xrightarrow{h^*} L),$$

which is compatible with cospecializations and inclusions of faces, the map f induces a map $f : \cup\sigma'_x \to \mathbf{R}$ which is continuous, piecewise-linear, homogeneous, integral and convex on every cone σ'_x. Let $\Sigma(X)_f$ denote the fan obtained by subdividing the fan $\Sigma(X)$ using the function f and let $F(X)_f$ denote the corresponding fan in the sense of Kato [13, 9.5].

Define a map $\pi : F(X)_f \to F(X)$ by sending every cone $\tau \in \Sigma(X)_f$ to the smallest cone τ' in $\Sigma(X)$ containing it. Define a homomorphism $t : \pi^{-1}M_{F(X)} \to M_{F(X)_f}$ by sending $M_{F(X)}(U(\tau')) = \mathrm{Hom}(\tau' \cap L, \mathbf{N})$ to $M_{F(X)_f}(U(\tau)) = \mathrm{Hom}(\tau \cap L, \mathbf{N})$ via the map $\tau \cap L \to \tau' \cap L$ (here, for a point y of a fan, $U(y)$ denotes the smallest open set containing y). We easily check that the pair (π, t) defines a morphism of monoidal spaces $F(X)_f \to F(X)$ [13, 9.1], which is a (proper) subdivision in the sense of Kato [13, 9.6], i.e., the morphisms $M^{gp}_{F(X),\pi(\tau)} \to M^{gp}_{F(X)_f,\tau}$ are surjective and the induced map $\mathrm{Mor}(\mathrm{Spec}(\mathbf{N}), F(X)_f) \to \mathrm{Mor}(\mathrm{Spec}(\mathbf{N}), F(X))$ is a bijection.

Theorem 4.7. *Let (X, M_X) be a Zariski log-regular scheme. Let $(Y, M_Y) \to (X, M_X)$ be the log-blow-up of the coherent ideal $J_f \subset M_X$. There is a unique morphism $g : (X, M_X) \times_{F(X)} F(X)_f \to (Y, M_Y)$ over (X, M_X) yielding an isomorphism of (Y, M_Y^{Zar}) with the base-change $(X, M_X^{\mathrm{Zar}}) \times_{F(X)} F(X)_f$ of (X, M_X) by the subdivision $F(X)_f$.*

Proof. Set $\pi : (X, M_X) \times_{F(X)} F(X)_f \to (X, M_X)$ to be equal to the natural projection. Assume for the moment that the coherent ideal $\pi^{-1}(J_f)$ is invertible. Proposition 4.2 then yields the existence and uniqueness of the map g in the statement of the theorem. To prove that g is an isomorphism we may argue locally. Let $x \in F(X)$ and assume that we are given a chart $P_{x,X} \to M_X$ compatible with the projection $M_{X,x} \to P_x$ and such that $F(X) \simeq \mathrm{Spec}(P_x)$. Let $J_0 \subset P_x$ be an ideal. Since the change of lattice via $h : L_x = P_x^{gp} \hookrightarrow L$ changes isomorphically the relevant fans, we may assume that $L = L_x$. We want to show that g is an isomorpism of the log-blow-up (Y, M_Y) of (X, M_X) at J_0M_X with the base-change of (X, M_X) by the subdivision of σ_x into the biggest possible cones on which $f = f_{J_0M_X}$ is linear and that the pullback of the ideal J_0M_X to $(X, M_X) \times_{F(X)} F(X)_f$ is invertible.

Recall that the monoidal space $F(X)_f$ is defined by taking as the set of points the elements of $\Sigma(X)_f$ and as a basis of open sets the sets $U(\tau) = \{\tau' \in \Sigma(X)_f | \tau' \prec \tau\}$, $\tau \in \Sigma(X)_f$. The sheaf $M_{F(X)_f}$ is defined by setting $M_{F(X)_f}(U(\tau)) = \mathrm{Hom}(\tau \cap L, \mathbf{N})$. Consider the monoid $P_\tau = \tau^\vee \cap L^\vee$. We easily check that the natural morphism of monoids $P_\tau \to \mathrm{Hom}(\tau \cap L, \mathbf{N})$ induces an isomorphism $P_\tau/P_\tau^* \xrightarrow{\sim} \mathrm{Hom}(\tau \cap L, \mathbf{N}) = M_{F(X)_f}(U(\tau))$. Thus $U(\tau) \simeq \mathrm{Spec}(P_\tau) \simeq \mathrm{Spec}(P_\tau/P_\tau^*)$.

We claim that if $J_0 = \bigcup_i P_{\overline{x}} r_i$ then the monoidal space $F(X)_f$ is covered by open sets isomorphic to $\mathrm{Spec}(P_{r_i}^{\mathrm{sat}})$. Indeed, let τ_i be the cone in σ_x cut out by the half-spaces $H_j^+ = \{y | (r_j - r_i)(y) \geq 0\}$. Clearly the open sets $U(\tau_i) \simeq \mathrm{Spec}(P_{\tau_i})$ cover $F(X)_f$. Using the correspondence between saturated monoids and cones, it is easy to check that the natural inclusion $P_{r_i}^{\mathrm{sat}} \subset P_{\tau_i}$ is actually an isomorphism.

The log-blow-up (Y, M_Y) is covered by the open subschemes $Y_i = X \times_{\mathbf{Z}[P_x]} \mathbf{Z}[P_{r_i}^{\mathrm{sat}}]$. The natural maps $Y_i \to \mathrm{Spec}(P_{r_i}^{\mathrm{sat}}/P_{r_i}^*)$ glue to a map of monoidal spaces $\beta : (Y, M_Y/\mathcal{O}_Y^*) \to (F(X)_f, M_{F(X)_f})$ compatible with the projections $(Y, M_Y/\mathcal{O}_Y^*) \to (X, M_X/\mathcal{O}_X^*)$ and $F(X)_f \to F(X)$. Hence β fuctors uniquely through the base change $X' = (X, M_X/\mathcal{O}_X^*) \times_{F(X)} F(X)_f$

[13, 9.9]

$$\beta : (Y, M_Y/\mathcal{O}_Y^*) \xrightarrow{\beta'} X' \to (F(X)_f, M_{F(X)_f}).$$

To show that β' is an isomorphism it suffices now to argue locally on the open sets Y_i. By the universal property of base-change, the map $\beta'_{|Y_i}$ factors through the log-scheme $X'_i = (X, M_X/\mathcal{O}_X^*) \times_{F(X)} \mathrm{Spec}(P_{r_i}^{\mathrm{sat}}/P_{r_i}^*) = X \times_{\mathbf{Z}[P_x]} \mathbf{Z}[P_{r_i}^{\mathrm{sat}}]$ [13, 9.9]. Since it commutes with the projections to X and to $\mathrm{Spec}(\mathbf{Z}[P_{r_i}^{\mathrm{sat}}])$, it has to be the identity morphism, as wanted.

By uniqueness $g = \beta^{-1}$, hence g is itself an isomorphism. Since the ideal $J_0 \subset P_x$ becomes principal in every $P_{r_i}^{\mathrm{sat}}$, the above computation also shows that $\pi^{-1}(J_0 M_X)$ is invertible, as wanted. ☐

The next two corollaries show that log-blow-ups are stable under base changes and compositions.

Corollary 4.8. *Let $f : (Y, M_Y) \to (X, M_X)$ be a morphism of fs log-schemes and let $J \subset M_X$ be a coherent ideal. Let $\pi : (\widetilde{X}, M_{\widetilde{X}}) \to (X, M_X)$ be the log-blow-up of J and let $g : (\widetilde{Y}, M_{\widetilde{Y}}) \to (Y, M_Y)$ be the log-blow-up of the inverse image ideal $f^{-1}(J)M_Y$. Then there exists a unique morphism $\psi : (Y, M_Y) \times_{(X,M_X)} (\widetilde{X}, M_{\widetilde{X}}) \to (\widetilde{Y}, M_{\widetilde{Y}})$ compatible with the maps f and g. Moreover, the morphism ψ is an isomorphism.*

Proof. We will give two proof. For the first one, notice that $g^{-1}f^{-1}(J)$ is invertible. Hence Proposition 4.2 gives an existence of a unique (X, M_X)-morphism $(\widetilde{Y}, M_{\widetilde{Y}}) \to (\widetilde{X}, M_{\widetilde{X}})$, hence a (Y, M_Y)-morphism $\phi : (\widetilde{Y}, M_{\widetilde{Y}}) \to (Y, M_Y) \times_{(X,M_X)} (\widetilde{X}, M_{\widetilde{X}})$.

On the other hand, the ideal $\pi^{-1}f^{-1}(J)$ is invertible on $(Y, M_Y) \times_{(X,M_X)} (\widetilde{X}, M_{\widetilde{X}})$, hence, again by Proposition 4.2, there exists a unique (Y, M_Y)-morphism $\psi : (Y, M_Y) \times_{(X,M_X)} (\widetilde{X}, M_{\widetilde{X}}) \to (\widetilde{Y}, M_{\widetilde{Y}})$. The composition $\psi\phi$ is by Proposition 4.2 the identity. The composition $\phi\psi$ composed with the projection $(Y, M_Y) \times_{(X,M_X)} (\widetilde{X}, M_{\widetilde{X}}) \to (\widetilde{X}, M_{\widetilde{X}})$ is equal to this projection (by Proposition 4.2), and hence $\phi\psi$ is the identity as well.

For the second one, the reader will notice that the only reason the statement of the last corollary holds is the fact that we take the products in the category of fine log-schemes (before saturating). It is the process of making the log-structure integral that kills the exceptional divisor. This can be seen more explicitly in the following way. Argue (étale) locally. By taking a chart of the map f, we may assume that, for some fine and saturated monoids P and Q, $X = \mathrm{Spec}(\mathbf{Z}[P])$, $Y = \mathrm{Spec}(\mathbf{Z}[Q])$, the map f is induced by a map $f : P \to Q$, and that the ideal J comes from an ideal $\Sigma \subset P$. It suffices now to show that the following diagram is cartesian

$$\begin{array}{ccc} \mathrm{Bl}_{(f(\Sigma))}(Y) & \xrightarrow{\tilde{f}} & \mathrm{Bl}_{(\Sigma)}(X) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{Z}[Q]) & \xrightarrow{f} & \mathrm{Spec}(\mathbf{Z}[P]). \end{array}$$

Here the map $\tilde{f}$ on the level of the classical blow-ups exists and is unique by the universal property of the blow-up.

It remains to show that $\tilde{f}$ is an isomorphism. Consider the covering of $\mathrm{Bl}_{(f(\Sigma))}(Y)$ by the opens $\mathrm{Spec}(\mathbf{Z}[Q_{f(a)}])$, $a \in \Sigma$. It suffices to show that the natural morphism $\mathbf{Z}[Q_{f(a)}] \leftarrow (\mathbf{Z}[(Q \oplus_P P_a)^{\mathrm{int}}]$, where the superscript "int" refers to making a monoid integral, is an isomorphism. That follows from the easy to check fact that the monoid $Q_{f(a)}$ is the pushout of the diagram $Q \stackrel{f}{\leftarrow} P \to P_a$ in the category of integral monoids. □

To prove that log-blow-ups are stable under composition we will need the following fact.

Lemma 4.9. *Let $f : (Y, M_Y) \to (X, M_X)$ be a log-blow-up of an fs log-scheme. Then the pullback morphism $f^* : \mathcal{F}_X \to f_*\mathcal{F}_Y$ is an isomorphism.*

Proof. It suffices to argue locally. Assume that $x \in X$ and we have a chart $P_X \to M_X$ with an fs monoid P and an ideal $J \subset P$ generating the ideal of our log-blow-up. We claim that we may assume (X, M_X) to be log-regular. Indeed, from the definition of the log-blow-up we have the following cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{h_1} & \mathrm{Bl}_{(J)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ X & \xrightarrow{h} & \mathrm{Spec}(\mathbf{Z}[P]) = T. \end{array} \qquad (4.1)$$

Since Corollary 3.6 yields the isomorphisms $h^*\mathcal{F}_T \stackrel{\sim}{\to} \mathcal{F}_X$ and $h_1^*\mathcal{F}_{\mathrm{Bl}_{(J)}} \stackrel{\sim}{\to} \mathcal{F}_Y$, an application of the proper base change theorem allows us to pass from the pullback f^* to the pullback $f_1^* : \mathcal{F}_T \to f_{1,*}\mathcal{F}_{\mathrm{Bl}_{(J)}}$. It suffices now to remark that the log-scheme T is log-regular (being smooth over $\mathbf{Z}$).

Let then (X, M_X) be log-regular. Assume that $x \in F(X)$ and we have a chart $P_{\overline{x},X} \to M_X$ compatible with the projection $M_{X,\overline{x}} \to P_{\overline{x}}$ and such that $F(X) \simeq \mathrm{Spec}(P_{\overline{x}})$. We have the following

Lemma 4.10. *The natural map* $\mathrm{Map}(\sigma_{\overline{x}}, \mathbf{R}) \stackrel{\sim}{\to} \mathcal{P}_X(X) \to \mathcal{F}_X(X)$ *is an isomorphism.*

Proof. For the first map in the lemma, use the isomorphism $P_{\overline{x}} \stackrel{\sim}{\to} M_X/\mathcal{O}_X^*(X)$ (Lemma 3.4). For the second map, note that, since, for any $f \in \mathcal{P}_X(X)$, $f_{\overline{x}} = f$, the map is injective. Surjectivity follows now from this and the proof of the correspondence $f \in \mathcal{F}_X(X) \leftrightarrow \{f_{\overline{y}}\}$ (cf. Proposition 3.7). □

Hence, by Theorem 4.7, it suffices to prove the exactness of the sequence

$$0 \to \mathrm{Map}(\sigma_{\overline{x}}, \mathbf{R}) \to \coprod_i \mathrm{Map}(\tau_i, \mathbf{R}) \to \coprod_{ij} \mathrm{Map}(\tau_i \cap \tau_j, \mathbf{R}),$$

where τ_i are cones of (Y, M_Y). This follows from the exact sequence

$$\coprod_{ij} \tau_i \cap \tau_j \to \coprod_i \tau_i \to \sigma_{\overline{x}} \to 0.$$

□

Corollary 4.11. *Let (X, M_X) be an fs log-scheme. Let $f : (Y, M_Y) \to (X, M_X)$ and $g : (Z, M_Z) \to (Y, M_Y)$ be log-blow-ups. Then the composition $fg : (Z, M_Z) \to (X, M_X)$ is a log-blow-up as well.*

Proof. Take the "functions" $f' \in \mathcal{F}_X(X)$ and $g' \in \mathcal{F}_Y(Y)$ corresponding to the log-blow-ups f and g respectively. By Lemma 4.9, we may think about g' as an element of $\mathcal{F}_X(X)$. Take $\varepsilon \in \mathbf{Q}^+$ and the element $h' = f' + \varepsilon g' \in \mathcal{F}_X(X)$. Consider the stalks $h'_{\overline{x}}$, $x \in X$, of h'. They are clearly compatible with cospecializations. Notice also that if we shrink X in the diagram 4.1 to a small enough neighbourhood of $\overline{x}$, then we get the following commutative diagram

$$\begin{array}{ccc} \mathcal{F}_X(X) & \xrightarrow{\sim} & \mathcal{F}_Y(Y) \\ \wr\uparrow & & \wr\uparrow \\ \mathcal{F}_T(T) \simeq \mathrm{Map}(\sigma_{\overline{x}}, \mathbf{R}) & \xrightarrow{\sim} & \mathcal{F}_{\mathrm{Bl}_{(J)}}(\mathrm{Bl}_{(J)}). \end{array}$$

This yields the following description of the function $g'_{\overline{x}}$. Let Σ be the (classical) fan obtained by subdividing $\sigma_{\overline{x}}$ according to the function $f'_{\overline{x}}$. Let $\{\tau_1, \dots, \tau_n\}$ be the cones in Σ of the same dimension as $\sigma_{\overline{x}}$. There exist points $\{y_1, \dots, y_n\}$ in Y such that $\sigma_{\overline{y_i}} = \tau_i$. The function $g'_{\overline{x}} : \sigma_{\overline{x}} \to \mathbf{R}$ is obtained by glueing the functions $g'_{\overline{y_i}}$. In particular, it is continuous, homogeneous, and satisfies (*) on every cone of Σ.

Let Σ_{fg} be the subdivision of the cone $\sigma_{\overline{x}}$ induced by first using the function $f'_{\overline{x}}$ and then the functions $g'_{\overline{y_i}}$. By [15, III.1.12], for small enough $\varepsilon_{\overline{x}} \in \mathbf{Q}^+$, the function $h'_{\overline{x}} = f'_{\overline{x}} + \varepsilon_{\overline{x}} g'_{\overline{x}}$ induces the same subdivision Σ_{fg} (notice that the proof in [15, III.1.12] is for compact polyhedral complexes; we can pass here to that case by choosing a slicing function). Set $\varepsilon = \min_{x \in F(X)}\{\varepsilon_{\overline{x}}\}$.

The functions $h'_{\overline{x}}, x \in X$, are now compatible with cospecialization and (perhaps after multiplication by the same integer) satisfy (*). Let $J_{h'} \subset M_X$ be the ideal corresponding to h' and let (W, M_W) be the log-blow-up of $J_{h'}$. We claim that there exists a unique (X, M_X)-morphism $\phi : (Z, M_Z) \to (W, M_W)$. Indeed, by Proposition 4.2, it suffices to make sure that the pullback $(fg)^{-1}J_{h'}$ is invertible. That is now a local statement. Localizing as usual around a point $\overline{x} \to X$, we see (cf. Theorem 4.7 and Corollary 4.8) that the composition fg corresponds to the subdivision Σ_{fg}. It is now clear that $(fg)^{-1}J_{h'}$ is invertible, as wanted.

It remains to show that ϕ is an isomorphism. To do that we can localize and then it suffices to evoke the characterization of the log-blow-ups as base changes by corresponding subdivisions (Theorem 4.7). □

5. Resolution of singularities

5.1. **Description of the regular locus.** We will see now that Kato's results from [13] suffice to show that, just as in the classical case, the regular locus of any log-regular scheme has a combinatorial description.

Definition 5.1. *Let (X, M_X) be a log-regular scheme. Let X_{reg} be the set of points $x \in X$ such that the cone $\sigma_{\overline{x}} \cap L_{\overline{x}}$ is* regular*, i.e., it is generated by a $\mathbf{Z}$-basis of $L_{\overline{x}}$.*

Lemma 5.2. *Let (X, M_X) be a log-regular scheme. The scheme X is regular at $x \in X$ if and only if $x \in X_{\mathrm{reg}}$.*

Proof. The statement is étale local. By Lemma 2.3 we may thus assume that (X, M_X) is a log-regular scheme in the Zariski topology.

Assume first that $x \in X_{\mathrm{reg}}$. Then $P_x = \mathbf{N}\rho_1^* \oplus \ldots \oplus \mathbf{N}\rho_d^* \simeq \mathbf{N}^d$, where $\{\rho_i^*\}$, $i = 1, \ldots, d$, is the basis of the lattice $L_x^\vee$ dual to the basis $\{\rho_i\}$, $i = 1, \ldots, d$, of L_x. Hence [13, 3.1] yields that

$$\mathcal{O}_{X,x}^\wedge \simeq k[[X_1, \ldots, X_n]] \quad \text{or} \quad \mathcal{O}_{X,x}^\wedge \simeq R[[X_1, \ldots, X_n]]/\theta,$$

depending on whether $\mathcal{O}_{X,x}^\wedge$ contains a field or not. In the first case k is a subfield of $\mathcal{O}_{X,x}^\wedge$ isomorphic to the residue field of $\mathcal{O}_{X,x}$; in the second, R is a complete discrete valuation ring in which the residue characteristic p is a prime element and such that R/pR is isomorphic to the residue field. We also have $\theta \equiv p \mod (X_1, \ldots, X_n)$. It is clear that the local ring $\mathcal{O}_{X,x}$ is regular, as wanted

Assume now that X is regular at x. We may take $x \in F(X)$. It suffices to show that P_x is generated by a basis of $L_x^\vee = P_x^{gp}$. Let $\{r_i\}$, $i = 1, \ldots, l$, be the indivisible rays of $\sigma_x^\vee$. They generate $\sigma_x^\vee$ and $L_{x,\mathbf{R}}^\vee$. Hence $l \geq \mathrm{rank}(L_x^\vee)$. Recall also that we have an isomorphism

$$\mathrm{gr}_I(\mathbf{Z}[P_x]) \otimes_{\mathbf{Z}} \mathcal{O}_{X,x}/\mathfrak{m}_x \xrightarrow{\sim} \mathrm{gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \quad [13, 6.1.v],$$

where I is the ideal of $\mathbf{Z}[P_x]$ generated by $P_x \setminus \{1\}$. In particular, $I/I^2 \otimes_{\mathbf{Z}} \mathcal{O}_{X,x}/\mathfrak{m}_x \xrightarrow{\sim} \mathfrak{m}_x/\mathfrak{m}_x^2$. Since $\mathbf{Z}\chi^{r_1} \oplus \ldots \oplus \mathbf{Z}\chi^{r_l} \subset I/I^2$ and, by assumption $\dim_{\mathcal{O}_{X,x}/\mathfrak{m}_x}(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim(\mathcal{O}_{X,x}) = \mathrm{rank}(L_x^\vee)$, this yields that $l \leq \mathrm{rank}(L_x^\vee)$. Hence $\{r_i\}$ is a basis of $L_x^\vee$, as wanted. □

Corollary 5.3. *The set $X_{\mathrm{reg}} \subset X$ is exactly the regular locus of X. It is open in X.*

Proof. Argue étale locally. Let $x \in X_{\mathrm{reg}}$ and choose an open neighbourhood U of x such that $F(U) \simeq \mathrm{Spec}(P_x)$. Then, for any $y \in U$, σ_y is a face of σ_x. Since by assumption σ_x is regular, so is σ_y, as wanted. □

5.2. **Weak resolutions of singularities.** We will show in this section (cf. Corollary 5.7) that any log-regular scheme (X, M_X) admits a weak resolution of singularities, i.e., a log-blow-up map $\pi : (Y, M_Y) \to (X, M_X)$ from a (classically) regular log-scheme (Y, M_Y). We will do it in two steps: in the first one we will pass to Zariski log-structure (this can be done for any fs log-scheme), in the second one we will resolve it following the classical algorithm for resolution of toroidal singularities [15].

First we will show that we can pass to Zariski log-structure. The following theorem is based on Kato's "descent theory" [14, 4.2.3].

Theorem 5.4. *For any fs log-scheme (X, M_X), there exists a coherent ideal J of M_X such that the log-blowing-up (Y, M_Y) of (X, M_X) with center J has a Zariski log-structure.*

Proof. For every $x \in X$, consider the ideal $J_{\overline{x}}$ in $P_{\overline{x}}$ equal to the product of all ideals of the form (a, b), where a, b are some irreducible elements in $P_{\overline{x}}$ (i.e. elements $c \in P_{\overline{x}}$ such that $c = xy$ implies $x = 1$ or $y = 1$). For every open in the étale topology $U \to X$, let $I(U)$ be the set of all $a \in M_X(U)$ such that for any étale neighbourhood $\overline{x} \xrightarrow{h_{\overline{x}}} U$ of $\overline{x} \to X$, $x \in X$, $h_{\overline{x}}^*(a) \in J_{\overline{x}}$.

Clearly $I \subset M_X$ is an ideal. To see that it is a coherent ideal, (by Proposition 2.6 from [16]) it suffices to show that it is starry, i.e., that for any cospecialization $g : P_{\overline{x}} \to P_{\overline{y}}$, $x, y \in X$, the induced map $g' : I'_{\overline{x}} \to I'_{\overline{y}}$ is surjective, where $I'_{\overline{x}}$ is the image of $I_{\overline{x}}$ in $P_{\overline{x}}$. For that first notice that any cospecialization $g : P_{\overline{x}} \to P_{\overline{y}}$, $x, y \in X$, maps $J_{\overline{x}}$ onto $J_{\overline{y}}$.

Indeed, this follows from the fact that the set of irreducible elements in $P_{\overline{y}}$ is the minimal set of generators and the cospecialization map g is a surjection. Then argue as in the proof of Lemma 3.12 that the stalks $I'_{\overline{x}}$ are isomorphic to $J_{\overline{x}}$.

Let now (Y, M_Y) be the log-blow-up of (X, M_X) at I. We claim that M_Y is Zariski. Our argument follows that of Kato [14, 4.2.9]. Let $y \in Y$ and $x = \pi(y)$. Let t_y be an automorphism of $P_{\overline{y}}$ over P_y. By Proposition 2.8, it suffices to show that t_y is the identity. Since t_y is always of the form described in Proposition 2.8, we can find a commutative diagram

$$\begin{array}{ccccc} P_y & \xrightarrow{\varepsilon_y} & P_{\overline{y}} & \xrightarrow{t_y} & P_{\overline{y}} \\ \pi\uparrow & & \pi\uparrow & & \pi\uparrow \\ P_x & \xrightarrow{\varepsilon_x} & P_{\overline{x}} & \xrightarrow{t_x} & P_{\overline{x}} \end{array}$$

with t_x an P_x-automorphism of $P_{\overline{x}}$. Let $E \subset P_{\overline{x}}$ be the set of irreducible elements. Since the map $\pi : P_{\overline{x}}^{gp} \to P_{\overline{y}}^{gp}$ is a surjection, it suffices to show that $t_y(\pi(a)) = \pi(a)$ for any $a \in E$. Consider the element $d = \pi(a)/\pi(t_x(a)) \in P_{\overline{y}}^{gp}$. Since, $t_x(a) \in E$, by construction we have $d \in P_{\overline{y}}$ or $d^{-1} \in P_{\overline{y}}$. Assume that $d \in P_{\overline{y}}$. Since t_y is of finite order (say n), we have $dt_y(d) \dots t_y^{n-1}(d) = 1$. Hence $d^{-1} = t_y(d) \dots t_y^{n-1}(d) \in P_{\overline{y}}$ and $d \in P_{\overline{y}}^{*}$ yielding $d = 1$, as wanted. □

In the case of log-regular schemes we can use barycentric subdivision to pass to Zariski log-structures.

Definition 5.5. *A log-regular scheme (X, M_X) is called simplicial if all the cones $\sigma_{\overline{x}}$, $x \in F(X)$, are simplicial (i.e., generated by independent vectors).*

Theorem 5.6. *For any log-regular scheme (X, M_X), there exists a coherent ideal J of M_X such that the log-blowing-up (Y, M_Y) of (X, M_X) with center J has a simplicial Zariski log-structure.*

Proof. First, we will define a set of continuous, homogeneous functions $f_{\overline{x}} : \sigma_{\overline{x}} \to \mathbf{R}^+$, for every $x \in F(X)$, convex and piecewise linear, rational on $L_{\overline{x}} \cap \sigma_{\overline{x}}$, and compatible with all the cospecialization maps.

If $\dim \sigma_{\overline{x}} = 1$, then define $f_{\overline{x}}$ by setting $f_{\overline{x}}(\rho) = 1$ for the indivisible ray ρ of $\sigma_{\overline{x}}$. Assume now that we have defined functions $f_{\overline{x}}$ for every cone $\sigma_{\overline{x}}$ of dimension at most k that are compatible with cospecializations, i.e.,

(**) for any cospecialization $g : P_{\overline{x}} \to P_{\overline{y}}$, $y \in F(X)$, the maps $f_{\overline{x}} : \sigma_{\overline{x}} \to \mathbf{R}$ and $f_{\overline{y}} : \sigma_{\overline{y}} \to \mathbf{R}$ are compatible with the map $g^* : \sigma_{\overline{y}} \to \sigma_{\overline{x}}$.

Let now $x \in F(X)$ be such that the cone $\sigma_{\overline{x}}$ has dimension $k+1$. Let τ be a proper face of $\sigma_{\overline{x}}$. Take the corresponding prime $\mathfrak{p}_\tau \in \operatorname{Spec}(P_{\overline{x}})$ and the point $z \in \operatorname{Spec}(\mathcal{O}_{X,\overline{x}})$ corresponding to the ideal $\mathfrak{q}_\tau = \mathfrak{p}_\tau \mathcal{O}_{X,\overline{x}}$ (we hope that the reader will forgive us the slightly abusive notation here). Let z_0 be its image in X. Clearly $x \in \{z_0\}^-$. Consider any cospecialization $g : P_{\overline{x}} \to P_{\overline{z}_0}$ factoring through the local ring at $\mathfrak{q}_\tau$ (this can be done because the residue field $k(z)$ is algebraic over $k(z_0)$). Define $f_{0,g}(\tau) = f_{\overline{z}_0}(g^*)^{-1}$, where $g^* : \sigma_{\overline{z}_0} \xrightarrow{\sim} \tau \subset \sigma_{\overline{x}}$ is the specialization map induced by g (see Lemma 2.12 and note [17, 4.4.9] that $\mathfrak{p}$ is the preimage of the maximal ideal of $(\mathcal{O}_{X,\overline{x}})_{\overline{z}}$ under the map $M_{X,\overline{x}} \to \mathcal{O}_{X,\overline{x}} \to (\mathcal{O}_{X,\overline{x}})_{\overline{z}}$).

We claim that this construction gives a well-defined function f_0 on the boundary $\partial\sigma_{\overline{x}}$ of $\sigma_{\overline{x}}$. First, we have to check that for any face τ, $f_{0,g}(\tau)$ does not depend on the chosen cospecialization g. Let $g_1, g_2 : \mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X,\overline{z}_0}$ be two cospecializations factoring through $(\mathcal{O}_{X,\overline{x}})_{\mathfrak{q}_\tau}$. Then there exists a cospecialization $h : \mathcal{O}_{X,\overline{z}_0} \to \mathcal{O}_{X,\overline{z}_0}$ such that $hg_2 = g_1$. Hence, by the assumption $(**)$,

$$f_{0,g_1}(\tau) = f_{\overline{z}_0}(g_1^*)^{-1} = f_{\overline{z}_0}(h^*)^{-1}(g_2^*)^{-1} = f_{\overline{z}_0}(g_2^*)^{-1} = f_{0,g_2}(\tau),$$

as wanted.

Next, we need to know that if $\tau_1 \prec \tau_2$, then $f_0(\tau_1) = f_0(\tau_2)|_{\tau_1}$. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}(P_{\overline{x}})$ be the primes corresponding to the faces τ_1, τ_2, and let $z_1, z_2 \in \mathrm{Spec}(\mathcal{O}_{X,\overline{x}})$ be the corresponding points. Clearly $z_{2,0} \in \{z_{1,0}\}^-$. Choose cospecializations

$$g_1 : \mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X,\overline{z}_{1,0}},\; g_2 : \mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X,\overline{z}_{2,0}},\; h : \mathcal{O}_{X,\overline{z}_{2,0}} \to \mathcal{O}_{X,\overline{z}_{1,0}},$$

such that g_1, g_2 factor through the local rings at $\mathfrak{p}_1\mathcal{O}_{X,\overline{x}}$ and $\mathfrak{p}_2\mathcal{O}_{X,\overline{x}}$, respectively, and $hg_2 = g_1$. Then, again by the assumption $(**)$ and the fact that $(g_2^*)^{-1}(\tau_1) = h^*(\sigma_{\overline{z}_{1,0}})$,

$$f_0(\tau_1) = f_{\overline{z}_{1,0}}(g_1^*)^{-1} = f_{\overline{z}_{1,0}}(h^*)^{-1}(g_2^*)^{-1}|\tau_1 = f_{\overline{z}_{2,0}}(g_2^*)^{-1}|\tau_1 = f_0(\tau_2)|\tau_1,$$

as wanted.

Let $\rho_0 = \rho_1 + \ldots + \rho_n$, where $\rho_1, \ldots, \rho_n$ are the indivisible rays of $\sigma_{\overline{x}}$. Then $\rho_0 \in L_{\overline{x}} \cap \mathrm{Int}(\sigma_{\overline{x}})$. Take a positive rational number C_x and set $f_{\overline{x}}(a\rho_0 + by) = aC_x + bf_0(y)$ for $a, b \geq 0$ and $y \in \partial\sigma_{\overline{x}}$. Since, by construction, the function f_0 is continuous, homogeneous, piecewise linear and convex rational on every face of the boundary $\partial\sigma_{\overline{x}}$, if C_x is large enough, the function $f_{\overline{x}}$ is convex and its associated cones are all of the form $< \rho_0, \tau >$, where τ is a cone associated to f_0 [3, Lemma 1 in the proof of Theorem 11]. Take any such C_x.

We claim that if we construct functions $f_{\overline{x}}$ as above for all $x \in F(X)$ such that the dimension of $\sigma_{\overline{x}}$ is $k+1$, then the assertion (**) holds for all the cones $\sigma_{\overline{x}}$, $x \in F(X)$, of dimension at most $k+1$. Indeed, let $g : P_{\overline{x}} \to P_{\overline{y}}$ be any cospecialization. If $\dim \sigma_{\overline{y}} \leq k$ then the functions $f_{\overline{x}}$ and $f_{\overline{y}}$ are compatible by construction. If $\dim \sigma_{\overline{y}} = k+1$, then, by Lemma 2.10, $x = y$ and, by the proof of Lemma 2.12, g is an automorphism of $P_{\overline{x}}$ (hence of $\sigma_{\overline{x}}$). In particular $g^*(\rho_0) = \rho_0$. Hence, for $a, b \geq 0$ and $t \in \partial\sigma_{\overline{x}}$,

$$\begin{aligned} f_{\overline{x}}(g^*(a\rho_0 + bt)) &= f_{\overline{x}}(a\rho_0 + bg^*(t)) = aC_x + bf_{\overline{x}}(g^*(t)) = aC_x + bf_0(g^*(t)) \\ &= aC_x + bf_0(t) = f_{\overline{x}}(a\rho_0 + bt), \end{aligned}$$

where the last equality follows from our construction. Indeed, $t = h^*(w)$ for some cospecialization $h : P_{\overline{x}} \to P_{\overline{z}}$, where $\dim \sigma_{\overline{z}} \leq k$. The composition $hg : P_{\overline{x}} \to P_{\overline{z}}$ is a cospecialization as well. By definition,

$$\begin{aligned} &f_0(g^*(t)) = f_{0,hg}(g^*(t)) = f_{\overline{z}}((hg)^*)^{-1}(g^*(t)) = f_{\overline{z}}((h^*)^{-1}(t)) = f_{\overline{z}}(w), \text{ and} \\ &f_0(t) = f_{\overline{z}}((h^*)^{-1}(t)) = f_{\overline{z}}(w), \end{aligned}$$

as wanted.

By Lemma 3.12, the functions $f_{\overline{x}}$ (scaled perhaps by the same integer) yield ideals $J_{\overline{x}}$ that lift to a coherent ideal $J_f \subset M_X$. Let $(Y, M_Y) \xrightarrow{\pi} (X, M_X)$ be the log-blowing-up of (X, M_X) with center $J = J_f$. We claim that M_Y is Zariski. Let $y \in Y$. By Proposition 2.9 it suffices to show that the map $\varepsilon_y^* : \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ is injective on primes from $P_{\overline{y}}(1)$. Let $x = \pi(y)$. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(\mathcal{O}_{Y,\overline{y}})$ be two different prime ideals of height one from $P_{\overline{y}}(1)$. Take the corresponding (distinct) one dimensional faces τ_1 and τ_2 of $\sigma_{\overline{y}}$ (cf. Lemma 2.7). By Theorem 4.7 the map $\pi : \sigma_{\overline{y}} \to \sigma_{\overline{x}}$ embeds $\sigma_{\overline{y}}$ as a cone τ obtained by subdividing $\sigma_{\overline{x}}$ according to function $f_{\overline{x}}$. The faces τ_1, τ_2 are then two different faces of dimension one in τ. Since we used barycentric subdivision and $\pi(\tau_i)$ is the smallest face of $\sigma_{\overline{x}}$ containg τ_i, we have that $\dim(\pi(\tau_1)) \neq \dim(\pi(\tau_2))$. Hence $\operatorname{ht}(\pi(\mathfrak{p}_1)) \neq \operatorname{ht}(\pi(\mathfrak{p}_2))$ and $\pi(\mathfrak{p}_1) \neq \pi(\mathfrak{p}_2)$, as wanted.

The last statement of the theorem is obvious from the construction of the ideal J and the description of log-blow-up given in Theorem 4.7. □

Corollary 5.7. *Any log-regular scheme (X, M_X) can be weakly desingularized by a log-blow-up.*

Proof. Follows from Theorem 5.6, Theorem 5.8 and Proposition 4.5, and Corollary 4.11. □

5.3. **Resolutions of singularities.** We will show now that any log-regular scheme (X, M_X) admits a resolution of singularities, i.e., a log-blow-up map $\pi : (Y, M_Y) \to (X, M_X)$ from a (classically) regular log-scheme (Y, M_Y) that induces an isomorphism $\pi^{-1}(X_{\mathrm{reg}}) \xrightarrow{\sim} X_{\mathrm{reg}}$.

In the case of Zariski log-structures we follow the classical constructions. Unfortunately we were not able to find equally elementary method to treat the general case. Hence we employ rather brutal methods: for log-schemes without monodromy we show that the classical algorithm gives local functions f_* that (because there is no monodromy) can actually be glued to give a global desingularizing log-blow-up; for totally general log-schemes – to deal with the possible monodromy – we use locally the canonical resolution of Bierstone and Milman.

Theorem 5.8. *Any log-regular Zariski scheme (X, M_X) can be desingularized by a log-blow-up.*

Proof. Recall [15, Theorem 11] that we can find a proper subdivision $F(X)_f \to F(X)$ (for a "good" function f) of the fan $F(X)$ such that $M_{F(X)_f,x} \simeq \mathbf{N}^{r(x)}$ for some $r(x) \geq 0$. The base change $(Y, M_Y) = (X, M_X) \times_{F(X)} F(X)_f$ is then regular. Since, by Theorem 4.7, (Y, M_Y) is uniquely over (X, M_X) isomorphic to the log-blow-up of the coherent ideal $J_f \subset M_X$, we are done. □

Paraphrasing [15, Theorem 11], [3, Theorem 11], and [4, 5.1], the construction of the log-blow-ups in the above theorem can be made very explicit.

First, to make the cones of (X, M_X) simplicial, we proceed exactly as in the proof of Theorem 5.6 with the exception of defining the functions f_x on regular cones σ_x by linearly extending the boundary functions f_0. This will assure that we do not modify the regular cones.

Assume now that all the cones of (X, M_X) are simplicial. For any $x \in X$, let the multiplicity $\mu(x)$ of x be the index of $\mathbf{Z}\rho_1 + \ldots + \mathbf{Z}\rho_d]$ in L_x, where $\rho_1, \ldots, \rho_d$ are the indivisible rays of σ_x. If $\mu(x) = 1$ then, by Lemma 5.2, the scheme X is regular at x. Set $\mu(X) = \max_{x \in F(X)} \mu(x)$. Assume that $\mu(X) > 1$. We will construct a coherent ideal

$J \subset M_X$ such that the log-blow-up (Y, M_Y) of (X, M_X) at J is again simplicial and has

$(***)$ $\mu(Y) < \mu(X)$, or
$\mu(Y) = \mu(X)$ and $F(Y)$ has fewer points with multiplicity $\mu(X)$ than $F(X)$.

Let $x_0 \in F(X)$ be a point with maximal multiplicity $\mu(x_0) = \mu(X)$. Let $x \in F(X)$, $x_0 \in \{x\}^-$, be a point corresponding to a non-regular face σ_x of σ_{x_0} with minimal dimension. Take an indivisible element $u \in L_x$ of the form

$$u = \alpha_1\rho_1 + \ldots + \alpha_d\rho_d, \qquad 0 < \alpha_i < 1,\ \alpha_i \in \mathbf{Q},$$

where $\rho_1, \ldots, \rho_d$ are the indivisible rays of σ_x. Such an element exists because σ_x is simplicial, $\mu(x) > 1$, and σ_x has minimal dimension. Let $y \in \{x\}^-$, $y \in F(X)$. Define the function f_y as the convex interpolation of the map assigning 1 to u and 0 to all the indivisible rays of σ_y. Set $f_y \equiv 0$ for $y \in F(X)$, $y \in X \setminus \{x\}^-$. Since the functions f_y are compatible with cospecializations, by Lemma 3.12, they give rise to ideals J_y, $y \in F(X)$, that in turn can be glued to a coherent ideal $J \subset M_X$.

Let (Y, M_Y) be the log-blow-up of (X, M_X) at J. It remains to check that (***) holds. If $y \in \{x\}^-$, $y \in F(X)$, then we can write uniquely (the log-structure is Zariski and the cones are simplicial) $\sigma_y = \sigma_x + \tau'$, $\tau' = \sigma_z$, $z \in F(X)$, $\tau' \cap \sigma_x = \{0\}$. The cones in σ_y associated to the functions f_y are all of the form

$$\tau_{y,i} =< \rho_1, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_d, \tau', u >, \qquad i = 1, \ldots, d.$$

In particular they are simplicial. Since the set of cones of (Y, M_Y) is equal to the set of cones of (X, M_X) minus the set of cones $\{\sigma_y\}$, $y \in \{x\}^-$, $y \in F(X)$, enlarged by the set of cones $\tau_{y,i}$, $y \in \{x\}^-$, $y \in F(X)$, (***) clearly holds.

Let (X, M_X) be an fs log-scheme. We say that (X, M_X) is *without monodromy* if every cospecialization $g : P_{\overline{x}} \to P_{\overline{x}}$ is the identity.

Theorem 5.9. *Any log-regular scheme (X, M_X) without monodromy can be desingularized by a sequence of log-blow-ups.*

Proof. We will define a set of continuous, homogeneous functions $f_{\overline{x}} : \sigma_{\overline{x}} \to \mathbf{R}^+$, for every $x \in F(X)$, convex and piecewise linear, integral on $L_{\overline{x}} \cap \sigma_{\overline{x}}$, and compatible with all the cospecialization maps.

If $\dim \sigma_{\overline{x}} = 1$, then define $f_{\overline{x}}$ by setting $f_{\overline{x}}(\rho) = 1$ for the indivisible ray ρ of $\sigma_{\overline{x}}$. Assume now that we have defined functions $f_{\overline{x}}$ for every cone $\sigma_{\overline{x}}$ of dimension at most k that are compatible with cospecializations and induce a subdivision of $\sigma_{\overline{x}}$ into regular cones. Let now $x \in F(X)$ be such that the cone $\sigma_{\overline{x}}$ has dimension $k+1$. Proceeding as in the proof of Theorem 5.6 define a function f_0 on the boundary $\partial\sigma_{\overline{x}}$ of $\sigma_{\overline{x}}$ and, using compatibility with cospecializations, show that it is a well-defined function.

Now, if $\sigma_{\overline{x}}$ is regular, then define $f_{\overline{x}}$ linearly extending the function f_0 to the whole of $\sigma_{\overline{x}}$. If $\sigma_{\overline{x}}$ is not regular, pick $\rho_0 \in L_{\overline{x}}$ in the relative interior of $\sigma_{\overline{x}}$. Set $f_{\overline{x}}(a\rho_0 + by) = aC_x + bf_0(y)$ for $a, b \geq 0$, $y \in \partial\sigma_{\overline{x}}$, and for a large positive rational number C_x such that the function $f_{\overline{x}}$ is convex and its associated cones are all of the form $< \rho_0, \tau >$, where τ is a cone associated to f_0 [3, Theorem 11]. This yields a fan that can be farther (projectively) subdivided so that all the cones are regular [15, Theorem 11]. Moreover, we may take the function $g_{\overline{x}}$ yielding this subdivision to be equal to zero on all the regular cones. In particular, we may assume it to be equal to zero on the boundary of $\sigma_{\overline{x}}$. By transitivity of projective subdivisions [15, Corollary 1.12], we can find a rational positive number ε such that the sum $f_{\overline{x}} := f_{\overline{x}} + \varepsilon g_{\overline{x}}$ is a function yielding for our subdivision. Notice that, since $g_{\overline{x}}$ is equal to zero on the boundary of $\sigma_{\overline{x}}$, our new $f_{\overline{x}}$ extends the function f_0.

Since any automorphism of $\sigma_{\overline{x}}$ induced by a cospecialization is trivial, arguing as in the proof of Theorem 5.6, we see that if we construct functions $f_{\overline{x}}$ as above for all $x \in F(X)$ such that the dimension of $\sigma_{\overline{x}}$ is $k+1$, then the assertion (**) holds for all the cones $\sigma_{\overline{x}}$, $x \in F(X)$, of dimension at most $k+1$.

By Lemma 3.12, the functions $f_{\overline{x}}$ (scaled perhaps by the same integer) yield ideals $J_{\overline{x}}$ that lift to a coherent ideal $J_f \subset M_X$. Let $(Y, M_Y) \xrightarrow{\pi} (X, M_X)$ be the log-blowing-up of (X, M_X) with center $J = J_f$. Theorem 4.7 and Corollary 5.3 yield that the scheme X is regular. □

Theorem 5.10. *Any log-regular scheme (X, M_X) can be desingularized by a log-blow-up.*

Proof. We proceed here almost exactly as in the proof of Theorem 5.9, the only difference being in the treatment of cones $\sigma_{\overline{x}}$ that are not regular. For these cones, first, we pick not just any ray $\rho_0 \in L_{\overline{x}}$ in the interior of $\sigma_{\overline{x}}$ but the barycentric one, i.e., $\rho_0 = \rho_1 + \dots \rho_n$, where $\rho_1, \dots, \rho_n$ are the indivisible rays of $\sigma_{\overline{x}}$. Then we proceed as in the proof of Theorem 5.9 extending the function $f_{\overline{x}}$ to the whole of $\sigma_{\overline{x}}$ and taking the associated fan. Second, we claim that the functions $g_{\overline{x}}$ in that proof (equal to zero on the boundary of $\sigma_{\overline{x}}$) can be taken to be $G_{\overline{x}}$-invariant, where $G_{\overline{x}}$ is the finite group of automorphisms of $P_{\overline{x}}$ induced by cospecializations. Notice that if we can do that then the rest of the proof of Theorem 5.9 goes through and we are done.

The existence of the function $g_{\overline{x}}$, which is $G_{\overline{x}}$-invariant follows from the canonical desingularization of Bierstone and Milman [1]. Specifically, take the toric variety X over the complex numbers associated to the above fan. By [1, 13.2], there exists a sequence

$$X_n \to \dots \to X_2 \to X_1 \to X$$

of blow-ups such that X_n is regular. By construction [1, 13], every blow-up $\sigma_{j+1} : X_{j+1} \to X_j$ blows-up a sheaf of ideals, which is not only torus-invariant but also $G_{\overline{x}}$-invariant.

Take now the normalization of the above tower of blow-ups

$$X_n \to \dots \to X_2^{norm} \to X_1^{norm} \to X$$

Every $\sigma_{j+1} : X_{j+1}^{norm} \to X_j^{norm}$ is now a normalization of a blow-up of a torus-invariant and $G_{\overline{x}}$-invariant sheaf of ideals J_{j+1}. Clearly the associated function $\operatorname{ord} J_{j+1}$ is also $G_{\overline{x}}$-invariant. It follows that the composition $\sigma : X_n \to X$ is a normalized blow-up given by a projective subdivision of the fan of X associated to a $G_{\overline{x}}$-invariant function $g_{\overline{x}}$. Since all the ideals J_{j+1} are isomorphic to $\mathcal{O}_{U_j}$ on the nonsingular locus U_j of X_j, $g_{\overline{x}}$ is zero on the boundary of $\sigma_{\overline{x}}$, as wanted.

□

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