ABELIAN CATEGORIES

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Contents

Introduction

This work contains a few general theorems about abelian categories and some applications of these theorems to the study of modules. We spent a lot of paper on reminders and on the statement of basic properties of abelian categories; there are two reasons for this: the first is that the subject is quite new and that we want to give the statements the right form for us.

The second reason is that we would like to convince non-specialists and offer them an overview: so let's say to this group of readers that abelian categories were introduced by Buchsbaum and Grothendieck to generalize homological methods of Cartan and Eilenberg[6]. A comparison will show immediately the interest of this notion: let $(M_i)_{i\in I}$ be a family of modules over a ring A, let T_i be the lattice of the submodules of $M_i(i \in I)$ and let $\mathrm{Hom}_A(M_i, M_j)$ be the abelian group formed of $A-$linear maps from M_i to $M_j(i, j \in I)$; likewise, the goal of homological algebra is the study of M_i by means of the groups $\mathrm{Hom}_A(M_i, M_j)$ and the composition laws that connect these groups. For a suitable choice of the family $(M_i)_{i\in I}$, the data of $\mathrm{Hom}_A(M_i, M_j)$ and the composition laws makes it possible to reconstruct the lattice $T_i(T_i$ is the lattice of 'subobjects' of $M_i)$. The data are therefore more abundant in homological algebra; it follows that the results are more accurate.

The results we get apply mainly to the following special cases:

a. Categories of right modules over a right noetherian ring.− Our statements then translate into the classical language of the theory of modules; we partially do this translation to conform to established translations. There is no doubt, however, that the study of the categories of modules involves more general abelian categories(see the concept of quotient category).
b. Category of quasi coherent sheaves over a noetherian scheme(Note by the translator : the author used the word 'prescheme')(This application is treated in Part VI).
c. Category of commutative algebraic groups(see[19], this application will be the subject of a subsequent publication).
d. Category of connected, commutative and cocommutative Hopf algebras(which are associative and coassociative) over a field (this application does not deserve a publication).

The Part I consists of reminders and complements to the published literature until this day. We insist especially on the concept of equivalence of two categories: two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if they are isomorphic when defining the morphisms in the following way: a morphism from $\mathcal{A}$ to $\mathcal{B}$ is the class of functors from $\mathcal{A}$ to $\mathcal{B}$ which are isomorphic to a given functor. If we adopt this point of view, we have, for example, the following result: Let A be a ring with identity such that any unital right $A-$module which is fintely generated projective is free; let $\mathcal{A}$(resp. $\mathcal{B}$) be the category of right unital $A-$modules(resp. $M_n(A)-$modules); the group of automorphisms of

$\mathcal{A}$(resp. $\mathcal{B}$) is the quotient of the group of automorphisms of the ring A(resp. the ring $M_n(A)$) by the subgroup of inner automorphisms. As categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent(cf.Part.V.§1), we find that the quotients considered are isomorphic.

The Part II presents the topics we are working on later: let's say roughly that an abelian category is *noetherian* if all objects of this category are noetherian, that an abelian category is *locally noetherian* if there are inductive limits, if the inductive limit functor is exact and if any object is an inductive limit of noetherian objects. We show that any noetherian category $\mathcal{A}$ is equivalent to the category of noetherian objects of a locally noetherian category $\mathcal{B}$. Reciprocally the category $\mathcal{B}$ is equivalent to the category of contravariant left exact additive functors from $\mathcal{A}$ to the category abelian groups. The category $\mathcal{B}$ is therefore equivalent to a quotient category of the category of all contravariant addtive functors from $\mathcal{A}$ to the category of abelian groups; it follows that $\mathcal{B}$ is 'almost' a category of modules(cf.Part.II, §1).

In the Part III, we are interested in the 'language modulo $\mathcal{C}$' of Serre: let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and let T and S be two additive functors, $T : \mathcal{A} \to \mathcal{B}$, $S : \mathcal{B} \to \mathcal{A}$. We suppose that T is exact, that S is (note by the translator: the author means 'right adjoint') adjoint to T and that $T \circ S$ is isomorphic to the identity functor of $\mathcal{B}$. Under these conditions, the category Ker$T$, which is formed of objects A of $\mathcal{A}$ such that TA is a zero object, is épaisse [10]; furthermore, T defines by passing to the quotient an equivalence between the quotient category of $\mathcal{A}$ by Ker$T$ [10], and the category $\mathcal{B}$.

Reciprocally, let $\mathcal{C}$ be a épaisse subcategory of $\mathcal{A}$, and let T be the canonical functor from $\mathcal{A}$ to $\mathcal{A}/\mathcal{C}$ [10]. We say that $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$ if T has a right adjoint functor.

When $\mathcal{A}$ is the category of modules over a commutative ring A, we can give the following example: let Σ be a multiplicative subset of A, let $\mathcal{B}$ be the category of modules over the ring A_Σ and let T be the functor which associates with any $A-$module M the 'localized' M_Σ. The functor T defines by passing to the quotient an equivalence between $\mathcal{A}/\text{Ker }T$ and $\mathcal{B}$.

The Part IV contains some results about locally noetherian categories: we prove that an injective object of such a category $\mathcal{A}$ is the direct sum of a family of indecomposable injective objects. We also take care of Krull dimension of $\mathcal{A}$; for this, we 'filter' $\mathcal{A}$ using localizing subcategories $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$ such that the quotients $\mathcal{A}_{i+1}/\mathcal{A}_i$ are locally finite categories (which means $\mathcal{A}_{i+1}/\mathcal{A}_i$ is locally noetherian and that any object is an inductive limit of objects of finite length); we associate with these locally finite categories complete topological rings which play the same role as complete local rings in commutative algebra.

The Part V treats the application to the theory of modules. When A contains in its center a noetherian commutative ring R, and is a finitely generated $R-$module, we explain the proposed constructions in Part V. We show, in particular, that the study of categories $\mathcal{A}_{i+1}/\mathcal{A}_i$ is equivalent to the study of the completion of A for certain topologies. This part proposes a plan to attack the non-commutative noetherian rings. It remains to be seen whether the methods considered are practicable in the study of the usual rings. Note in this regard a ring that Dieudonné introduced in the study of formal groups:

Let k be an algebraically closed field of characteristic $p > 0$, let W be the ring of Witt vectors with coefficients in k and let σ be the Frobenius automorphism of W. Denote by A the ring whose elements are the formal series of the form

$$a = w + \sum_{r=1}^{\infty} a_r.F^r + \sum_{s=1}^{\infty} b_s.V^s$$

where w, a_r and b_s run through the ring W; the multiplication is defined by the following rules:

$V.F = F.V = p$, $F.w = \sigma(w).F$ and $w.V = V.\sigma(w)$ if $w \in W$.

The ring A is noetherian (both left and right). If we take into account the category $\mathcal{A}$ of right $A-$modules, we have :

- Any simple object of $\mathcal{A}$ is isomorphic to $A/(F.A + V.A)$.
- If $\mathcal{A}_0$ is the smallest localizing subcategories of $\mathcal{A}$ which contains $A/(F.A + V.A)$, any simple object of $\mathcal{A}/\mathcal{A}_0$ is of one of the following types: $A/F.A$, $A/V.A$ or $A/(V^r - F^q).A$ with $(r, q) = 1$.
- If $\mathcal{A}_1$ is the smallest localizing subcategory of $\mathcal{A}$ which contains the above modules, any simple object of $\mathcal{A}/\mathcal{A}_1$ is isomorphic to A.

This work has already been presented in three exposes:

- Objets injectifs dans les catégories abéliennes, Séminaire Dubreil-Pisot, tome 12, 1958-1959, no.17, 32 pages.
- La localisation dans les anneaux non commutatifs, Séminarie Dubreil-Pisot, tome 13, 1959-1960, no.2, 35 pages.
- Sur les catégories localement noethériennes et leurs applications aux algèbres étudiées par Dieudonné (Groupes formels), Séminaire J.-P. Serre, 1959-1960.

Part 1. Some reminders on categories

The object of this part is to recall some definitions and notations of categories. We suppose that the reader has read the chapters I and II of [10]. However, as time has done its work since 1957, we add some complements to the article of Grothendieck; these complements are due in large part to Grothendieck himself and they will be exposed in a future book [7]. That's why we allow ourselves to omit most demonstrations.

1. The Grothendieck Universe

A set $\mathfrak{U}$ is a *universe* if the following axioms are satisfied:

U_1 : If $(X_i)_{i \in I}$ is a family of sets belonging to $\mathfrak{U}$, and if I is an element of $\mathfrak{U}$, then the union $\bigcup_i X_i$ is an element of $\mathfrak{U}$.
U_2 : If x belongs to $\mathfrak{U}$, then the set $\{x\}$ with an element belongs to $\mathfrak{U}$.
U_3 : If x belongs to X and if X belongs to $\mathfrak{U}$, then x belongs to $\mathfrak{U}$.
U_4 : If X is a set belonging to $\mathfrak{U}$, the set $\beta(X)$ of subsets of X is an element of $\mathfrak{U}$.
U_5 : The pair (x, y) is an element of $\mathfrak{U}$ if and only if x and y are elements of $\mathfrak{U}$.

It is necessary to add to the usual axioms of the theory of sets an axiom ensuring that any set belonging to universe. In this case there exists a smallest universe containing a given set. *We choose once and for all a universe* $\mathfrak{U}$ *which will not 'vary' in everything that follows.* Of course, we suppose that $\mathfrak{U}$ is large enough that the set $\mathbb{Z}$ of integers and other sets if necessary are elements of $\mathfrak{U}$. The reader will be albe to practice proving the following corollaries of the axioms:

- If Y is a subset of X, and if X belongs to $\mathfrak{U}$, then Y belongs to $\mathfrak{U}$.
- If X and Y are two sets belonging to $\mathfrak{U}$, the union $X \cup Y$ and the cartesian product $X \times Y$ are elements of $\mathfrak{U}$.
- If x and y belongs to $\mathfrak{U}$, the set $\{x, y\}$ belongs to $\mathfrak{U}$.
- If $(X_i)_{i \in I}$ is a family of sets belonging to $\mathfrak{U}$, and if I belongs to $\mathfrak{U}$, then the product $\prod_{i \in I} X_i$ belongs to $\mathfrak{U}$.
- If X is a set belonging to $\mathfrak{U}$, the cardinal of X is strictly less than the cardinal of $\mathfrak{U}$.

2. Definition of categories

A category $\mathcal{C}$ is constituted by the following data:

- A set $\mathcal{OC}$ whose elements are called *the objects* of $\mathcal{C}$.
- For each pair (M, N) of objects of $\mathcal{C}$, we give ourselves a set denoted by $\mathrm{Hom}_{\mathcal{C}}(M, N)$ whose elements are called *morphisms* from M to N [We will often write $f : M \to N$ or $M \xrightarrow{f} N$ instead of $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$].
- For each triple (M, N, P) of objects of $\mathcal{C}$, we give ourselves a map μ from the product $\mathrm{Hom}_{\mathcal{C}}(M, N) \times \mathrm{Hom}_{\mathcal{C}}(N, P)$ to $\mathrm{Hom}_{\mathcal{C}}(M, P)$ [We

will write $g \circ f$ or $g \circ_{\mathcal{C}} f$ instead of $\mu(f, g)$; we will say that μ is the composition map].

These data are subject to the axioms C_1 and C_2:

C_1 : Let f,g and h be three morphisms : $f : M \to N$, $g : N \to P$ and $h : P \to Q$. Under these conditions, we have the law $h \circ (g \circ f) = (h \circ g) \circ f$.

C_2 : For any object M of $\mathcal{C}$, there exists a morphism $1_M : M \to M$ such that we have $1_M \circ f = f$ and $g \circ 1_M = g$ whenever these equalities make sense.

If $\mathcal{C}$ is a category, there exists for each object M only one morphism 1_M satisfying C_2. It is called *the identity morphism* of M. If $f : M \to N$ is a morphism of $\mathcal{C}$, the object M is called *the source of* f, the object N is called *the target of* f. We will denote by $\mathcal{MC}$ from now on the disjoint union of the sets $\mathrm{Hom}_{\mathcal{C}}(M, N)$ and we will call it the set of morphisms of $\mathcal{C}$. In the following, $\mathfrak{U}$ **Ens** or **Ens** (resp. $\mathfrak{U}$ **Ab** or **Ab**) denotes the *category of sets* (resp. *category of abelian groups*):

- The objects of **Ens** (resp. of **Ab**) are the sets belonging to $\mathfrak{U}$ (resp. the abelian groups whose underlying sets belong to $\mathfrak{U}$).
- If M and N are two objects of **Ens** (resp. of **Ab**), a morphism from M to N is a map from M to N (resp. a linear map from M to N).
- The law $(f, g) \to g \circ f$ is the usual law of composition of maps.

The given definitions coincide with those of [10], except that the objects of a category here are the elements of a set. The reader should refer to [10] for the definition of notions such as the following : commutative diagram, monomorphism, sub-object (called sous-truc in [10]), generator, etc. We are content to specify some notations and abuses of language:

A *sub-object* of M (resp. *a quotient* of M) is a monomorphism $i_M^N : N \to M$ (resp. an epimorphism $p_Q^M : M \to Q$). We will often say that N is a sub-object of M and that i_N^M is the canonical monomorphism fromm N to M (resp. that Q is a quotient of M and that p_Q^M is the canonical epimorphism from M to Q).

If $\mathcal{C}$ is a category, the *dual category* of $\mathcal{C}$ is denoted by $\mathcal{C}^o$. If $(\mathcal{C}_i)_{i\in I}$ is a family of categories, we denote by $\prod_{i\in I} \mathcal{C}_i$ the *product category* :

- The objects of the product $\prod_{i\in I} \mathcal{C}_i$ are the elements of product of the sets $\mathcal{OC}_i$.
- If $(M_i)_{i\in I}$ and $(N_i)_{i\in I}$ are two objects of the product category, a morphism from the first to the second is an element of the product

$$\prod_{i\in I} \mathrm{Hom}_{\mathcal{C}_i}(M_i, N_i).$$

- The composition is defined by the formula

$$(f_i) \circ (g_i) = (f_i \circ_{\mathcal{C}_i} g_i).$$

Finally let's note that a *full subcategory* of $\mathcal{C}$ is a category $\mathcal{D}$ satisfying the following conditions :

- The objects of $\mathcal{D}$ are objects of $\mathcal{C}$.
- If M is an object of $\mathcal{D}$, any object of $\mathcal{C}$ isomorphic to M is an object of $\mathcal{D}$.
- If M and N are two objects of $\mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(M,N)$ is equal to $\mathrm{Hom}_{\mathcal{C}}(M,N)$.
- The composition law of $\mathcal{D}$ is induced by that of $\mathcal{C}$.

We will say from now on that a category $\mathcal{C}$ is a $\mathfrak{U}-$category if $\mathrm{Hom}_{\mathcal{C}}(M,N)$ belongs to the universe $\mathfrak{U}$ for any pair (M,N) of objects of $\mathcal{C}$. Unless stated otherwise explicitly, *all categories considered in this article are* $\mathfrak{U}-$*categories.* Whenever we talk about an *inductive system* (resp. *a projective system*) of objects of a $\mathfrak{U}-$category, we suppose implicitly that this inductive system (resp. this projective system) is indexed by a set belonging to $\mathfrak{U}$. Whenever we talk about a direct sum or direct product of a family of objects of a $\mathfrak{U}-$category, this family will be supposed to be indexed by an element of $\mathfrak{U}$. We say that $\mathcal{C}$ is a *category with inductive limits* if any inductive system (indexed by an element of $\mathfrak{U}$) of objects of $\mathcal{C}$ has an inductive limit. Similarly, we say that $\mathcal{C}$ is a *category with generators* if there exists a family of generators which is indexed by an element of $\mathfrak{U}$.

3. Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A functor F from $\mathcal{C}$ to $\mathcal{D}$ (we write $F:\mathcal{C}\to\mathcal{D}$) is constituted by the following data:

- A map $M \rightsquigarrow FM$ from $\mathcal{OC}$ to $\mathcal{OD}$.
- For any pair (M,N) of objects of $\mathcal{C}$, we are given a map $F(M,N)$ from $\mathrm{Hom}_{\mathcal{C}}(M,N)$ to $\mathrm{Hom}_{\mathcal{D}}(FM,FN)$.[We will also denote it by F instead of $F(M,N)$.]

We suppose also that $(Ff)\circ(Fg)$ is equal to $F(f\circ g)$ whenever the source of f coincides with the target of g.

For example, if $\mathcal{C}$ is a full subcategory of $\mathcal{D}$, we call canonical functor from $\mathcal{C}$ to $\mathcal{D}$ the functor F which is defined by the following equalities: $FM = M$ if $M \in \mathcal{OC}$; $Ff = f$ if $f \in \mathcal{MC}$. In the case where $\mathcal{C}$ coincides with $\mathcal{D}$, we will also say that F is the identity functor of $\mathcal{C}$. We will then denote F by $I_{\mathcal{C}}$.

The reader should note, that with the terminology of [10], we only consider covariant functors here. When no confusion is caused, it will happen to us not to explain the maps $F(M,N)$, but only the map from $\mathcal{OC}$ to $\mathcal{OD}$ which is associated with F. The symbol $M \rightsquigarrow FM$ will then denote the functor F.

The functors from $\mathcal{C}$ to $\mathcal{D}$ is the objects of a category which we denote by $\mathbf{Hom}(\mathcal{C},\mathcal{D})$ and whose morphisms are defined by the following way : let F and G be two functors from $\mathcal{C}$ to $\mathcal{D}$; a morphism (or functorial morphism) φ from F to G consists of the data, for each obect M of $\mathcal{C}$, of a morphism $\varphi(M): FM \to GM$. We further assume that for any morphism $f: M \to N$

of $\mathcal{C}$, the following diagram is commutative :

$$\begin{array}{ccc} FM & \xrightarrow{\varphi(M)} & GM \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Gf} \\ FN & \xrightarrow{\varphi(N)} & GN \end{array}$$

If $\varphi : F \to G$ and $\psi : G \to H$ are two functorial morphisms, the composition $\psi \circ \varphi$ is defined by the formula:

$$(\psi \circ \varphi)(M) = \psi(M) \circ_{\mathcal{D}} \varphi(M).$$

It follows in particular that φ is an isomorphism of $\mathbf{Hom}(\mathcal{C}, \mathcal{D})$ if and only if $\varphi(M)$ is an isomorphism for each object M of $\mathcal{C}$; we say then that φ is a *functorial isomorphism.*

Now consider two objects X and Y of $\mathcal{C}$; as we suppose that $\mathcal{C}$ is a $\mathfrak{U}-$category, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an element of $\mathfrak{U}$. We denote by $\dot{X}$, $\mathrm{Hom}_{\mathcal{C}}(X, .)$ or $\mathrm{Hom}(X, .)$ the functor $Y \rightsquigarrow \mathrm{Hom}(X, Y)$ from $\mathcal{C}$ to **Ens**. If $f : X \to X'$ is a morphism of $\mathcal{C}$, we denote by $\dot{f}$ or by $\mathrm{Hom}(f, .)$ the functorial morphism which maps an element g of $\dot{X}'Y = \mathrm{Hom}(X', Y)$ to the element $g \circ f$ of $\dot{X}Y = \mathrm{Hom}(X, Y)$.

The maps $X \rightsquigarrow \dot{X}$ and $f \rightsquigarrow \dot{f}$ obviously define a functor from $\mathcal{C}^o$ to $\mathbf{Hom}(\mathcal{C}, \mathbf{Ab})$. We will examine this functor a little more closely: for this, let F be any functor from $\mathcal{C}$ to **Ab**; each element ξ of FX defines a functorial morphism $\dot{\xi}$ from $\dot{X}$ to F: for each object Y of $\mathcal{C}$, $\dot{\xi}(Y)$ is the map $f \rightsquigarrow (Ff)(\xi)$ from $\dot{X}Y = \mathrm{Hom}(X, Y)$ to FY.

Proposition 1. *For any object X of $\mathcal{C}$, the map $\xi \rightsquigarrow \dot{\xi}$ from FX to* $\mathrm{Hom}(\dot{X}, F)$ *is bijective.*

In fact, let φ be a functorial morphism from $\dot{X}$ to F; let $\dot{\varphi}$ be the image of 1_X under the map $\varphi(X)$ from $\mathrm{Hom}(X, X)$ to FX. In the case where φ is equal to $\dot{\xi}$, $\dot{\varphi}$ is no other than ξ. It remains to prove that $\varphi = \dot{\xi}$ if $\xi = \dot{\varphi}$. In other words, we have to show that for any object Y of $\mathcal{C}$ and for any morphism $f : X \to Y$, we have the equality

$$\varphi(Y)(f) = (Ff)(\xi)$$

This results from the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(X, Y) & \xrightarrow{\varphi(Y)} & FY \\ {\scriptstyle \mathrm{Hom}(X,f)}\uparrow & & {\scriptstyle Ff}\uparrow \\ \mathrm{Hom})(X, X) & \xrightarrow{\varphi(X)} & FX \end{array}$$

and from the fact that $\dot{\xi} = \varphi(X)(1_X)$.

Corollary 1 ([11])**.** *If X and X' are two objects of $\mathcal{C}$, the map $f \rightsquigarrow$* $\mathrm{Hom}(f, .)$ *from* $\mathrm{Hom}(X', X)$ *to the set of functorial morphisms from* $\mathrm{Hom}(X, .)$ *to* $\mathrm{Hom}(X', .)$ *is bijective.*

We prove the corollary by replacing F by $\dot{X}'$ in the proposition 1.

Corollary 2. *Any isomorphism from the functor* $\mathrm{Hom}(X, .)$ *to the functor* $\mathrm{Hom}(X', .)$ *is induced by an isomorphism from* X' *to* X.

We will say in the following that a functor F from $\mathcal{C}$ to **Ens** is representable [11] if F is isomorphic to a functor $\mathrm{Hom}(X, .)$. We also say that X is a *representative of* F. One such representative is evidently defined up to an isomorphism.

In particular, if the functor F which associates with any object the set $\{\emptyset\}$ is representable, we choose once and for all a representative O of F. This representative is called an *initial object of* $\mathcal{C}$. For any object Y of $\mathcal{C}$, we then denote by η_Y the only morphism from O to Y.

By duality, the final object O' of $\mathcal{C}$, if exists, is such that $\mathrm{Hom}(Y, O')$ contains one and only one element ε_Y for any object Y.

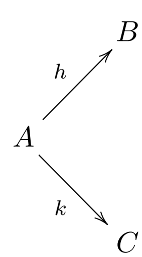
We say that O is *zero* if it is both initial and final. If M and N are then two objects of $\mathcal{C}$, we denote by $O_{M,N}$ or simply O instead of $\eta_M \circ \varepsilon_N$.

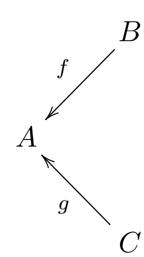
We give another example: we recall that a *direct sum* of two objects M and N is a triple (P, u, v) formed of an object P and of two morphisms, $u : M \to P$ and $v : N \to P$; furthermore, we suppose that the maps $\mathrm{Hom}(u, X)$ and $\mathrm{Hom}(v, X)$ define for any object X an isomorphism between $\mathrm{Hom}(P, X)$ and the cartesian product $\mathrm{Hom}(M, X) \times \mathrm{Hom}(N, X)$. The object P is hence a representative of functor

$$X \rightsquigarrow \mathrm{Hom}(M, X) \times \mathrm{Hom}(N, X)$$

If the direct sum of M and N exists, we choose one that we denote it by $(M\Sigma N, j_M, j'_N)$. We then say that $M\Sigma N$ is the direct sum of M and N and that j_M and j'_N are the canonical morphisms. The dual notion is the notion of *direct product* of M and N: if it exists, we denote the product by $M\Pi N$; the canonical morphisms are denoted by q_M and q'_N.

Now consider the diagrams (1) and (2):

(1) 

(2) 

A *fibre sum* of diagram (1) is a triple (P, u, v) formed of an object P and of two morphisms, $u : B \to P$, $v : C \to P$, such that we have: $u \circ h = v \circ k$; for any couple (b, c) of morphisms such that $b \circ h$ and $c \circ k$ are defined and equal, there exists one and only one morphism a whose source is P, which has the same target as b and c and satisfies the equalities $b = a \circ u$, $c = a \circ v$. If the fibre sum of diagram (1) exists, we choose one that we denote it by

$$(B\Sigma_A C, j_B, j'_C).$$

We then say that $B\Sigma_A C$ is the fibre sum of diagram (1) and that j_B and j'_C are the canonical morphisms.

The dual notion is that of *fibre product*: if it exists, we denote by $B\Pi_A C$ the fibre product of diagram (2); the canonical morphisms are denoted by q_B and q'_C.

4. Additive categories (after Grothendieck)

If $\mathcal{C}$ is a category satisfying the following axioms:

C Ad 1 There exists a zero object O.

C Ad 2 For any couple (M, N) of objects of $\mathcal{C}$, the direct product of M and N, as well as the direct sum of M and N exist.

Lemma 1. *If M and N are two objects of $\mathcal{C}$, there exists one and only one morphism $h(M, N) : M\Sigma N \to M\Pi N$ such that we have*

$$q_M \circ h(M, N) \circ j_M = 1_M, \quad q'_N \circ h(M, N) \circ j_M = 0, \quad q'_N \circ h(M, N) \circ j'_N = 1_N$$
$$q_M \circ h(M, N) \circ j'_N = 0$$

If we denote by Π(reps. Σ) the functor

$$(M, N) \rightsquigarrow M\Pi N \text{ [resp. } (M, N) \rightsquigarrow M\Sigma N \text{]}$$

from the product category $\mathcal{C}\Pi\mathcal{C}$ to $\mathcal{C}$, the lemma 1 can be completed by the

Lemma 2. *The morphisms $h(M, N)$ define a functorial morphism from the functor Σ to the functor Π.*

Now suppose that $\mathcal{C}$ satisfies one more axiom:

C Ad 3 For any couple (M, N) of objects of $\mathcal{C}$, the morphism $h(M, N)$ from $M\Sigma N$ to $M\Pi N$ is an isomorphism.

Under these conditions, we can suppose that $M\Sigma N$ is chosen to be equal to $M\Pi N$ and such that $h(M, N)$ is the identity morphism of $M\Pi N$. We then denote by $M \bigoplus N$ instead of $M\Sigma N$ or $M\Pi N$. It is easy to prove that the identification between the direct sum and the direct product is compatible with 'the exchange of factors M and N' and with the canonical isomorphisms between $(M\Sigma N)\Sigma P$ and $M\Sigma(N\Sigma P)$ and between $(M\Pi N)\Pi P$ and $M\Pi(N\Pi P)$.

If M is an object of $\mathcal{C}$, we denote by Δ_M (resp. Σ_M) the morphism, called *diagonal*, from M to $M \oplus M$ (resp. from $M \oplus M$ to M) which is defined by the equalities $q_M \circ \Delta_M = 1_M$, $q'_M \circ \Delta_M = 1_M$ (resp. $\Sigma_M \circ j_M = 1_M$, $\Sigma_M \circ j'_M = 1_M$). If f and g are two morphisms of the same source M and of the same target N, we denote by $f + g$ the composite of the following morphisms:

$$M \xrightarrow{\Delta_M} M \oplus M \xrightarrow{f \oplus g} N \oplus N \xrightarrow{\Sigma_M} N$$

Lemma 3. *The internal law $(f, g) \to f + g$ is commutative and associative. The morphism $O_{N,M}$ is an identity element for this law.*

Lemma 4. *Let $f, f' : M \to N$ and $g, g' : N \to P$ be morphisms of $\mathcal{C}$. We have the equalities $g \circ (f + f') = g \circ f + g \circ f'$ and $(g + g') \circ f = g \circ f + g' \circ f$.*

There are categories satisfying the axioms C Ad 1, 2, 3 and for which the monoids $\operatorname{Hom}(M, N)$ are not abelian groups. This cannot happen if the category $\mathcal{C}$ satisfies the axiom C Ad 4.

C Ad 4 For any object M, there is a morphism $c(M) : M \to M$ such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\ O\ } & M \\ \downarrow{\scriptstyle \Delta_M} & & \uparrow{\scriptstyle \Sigma_M} \\ M \oplus M & \xrightarrow{1_M \oplus c(M)} & M \oplus M \end{array}$$

The commutativity of the diagram implies that $c(M)$ is an inverse element of 1_M in $\operatorname{Hom}(M, N)$. It follows that a morphism $f : M \to N$ has the composite as the inverse

$$c(N) \circ f = f \circ c(M).$$

Proposition 2. *If $\mathcal{C}$ is a category, the following two assertions are equivalent:*

a. *The category $\mathcal{C}$ satisfies the axioms* C Ad 1, 2, 3 and 4.
b. *The category $\mathcal{C}$ satisfies* C Ad 1 *and one or the other assertion of* C Ad 2*; in addition, we can equip the sets* $\operatorname{Hom}(M, N)$ *with abelian group structure in such a way that the laws of composition are bilinear maps.*

The implication $(a) \Rightarrow (b)$ results from the previous lemmas. The implication $(b) \Rightarrow (a)$ is classical; It is shown in [10].

We call any category satisfying axioms C Ad 1, 2, 3 and 4 an *additive category*.

Proposition 3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two additive categories and let F be a functor from $\mathcal{C}$ to $\mathcal{D}$. The following assertions are equivalent:*

a. *For any couple (M, N) of objects of $\mathcal{C}$, the maps $F(M, N)$ from* $\operatorname{Hom}_{\mathcal{C}}(M, N)$ *to* $\operatorname{Hom}_{\mathcal{D}}(FM, FN)$ *is linear.*
b. *For any couple (M, N) of objects of $\mathcal{C}$, the triple*

$$(F(M \oplus N), Fj_M, Fj'_N)$$

is a direct sum of F_M and F_N.

If the equivalent conditions of the proposition 3 are fulfilled, we say that F is an additive functor. We express the condition (b) by saying that an *additive functor* commutes with the direct sum. Of course, an additive functor also commutes with the direct product.

Unless explicitly stated otherwise, in all following, *the functors from an additive category to another are supposed to be additive.* For example let $\mathcal{C}$ be an additive category. We have seen that for any couple (M, N) of objects of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(M,N)$ is equipped with a natural structure of abelian group. We again denote by $\mathrm{Hom}_{\mathcal{C}}(M,N)$ this abelian group. The lemma 4 then shows that the functor $Y \rightsquigarrow \mathrm{Hom}_{\mathcal{C}}(M,Y)$ from $\mathcal{C}$ to $\mathbf{Ab}$ is additive for any M.

We remark also that the dual category of an additive category $\mathcal{C}$ is additive. Thus $X \rightsquigarrow \mathrm{Hom}_{\mathcal{C}}(X,N)$ is an additive functor from $\mathcal{C}^o$ to $\mathbf{Ab}$.

5. Abelian Categories

Let $\mathcal{C}$ be an additive category and $f : M \to N$ a morphism of $\mathcal{C}$. We suppose that f has kenel, cokernel, image, coimage [10]. Let i and j (resp. p and q) be the canonical morphisms from the kernel $\mathrm{Ker}\ f$ to M and from the image $\mathrm{Im}\ f$ to N (resp. from N to the cokernel $\mathrm{Coker}\ f$ and from M to the coimage $\mathrm{Coim}\ f$). We know that there is one and only one morphism ϑ from $\mathrm{Coim}\ f$ to $\mathrm{Im}\ f$ such that we have $f = j \circ \vartheta \circ q$. We say that ϑ is the canonical morphism from $\mathrm{Coim}\ f$ to $\mathrm{Im}\ f$.

An *additive category* $\mathcal{C}$ *is abelian* if the following axioms are satisfied:

C Ab 1 For any morphism f, the kernel and cokernel of f exist.
C Ab 2 For any morphism f, the canonical morphism from $\mathrm{Coim}\ f$ to $\mathrm{Im}\ f$ is an isomorphism.

The dual category of an abelian category is abelian.

We suppose that the reader is familiar with elementary arguments of abelian categories. When only a finite number of objects and morphisms are considered, the arguments used are those of abelian groups. To increase this similarity, we usually use the following notations: if N is a sub-object of M, M/N denotes the cokernel of i_N^M; if N and P are two sub-objects of M, $N+P$ is the image of the morphism from $N \oplus P$ to M which is defined by i_N^M and i_P^M; in the same way $M \cap N$ denotes the kernel of the morphism from M to $M/N \oplus M/P$ which is defined by the canonical epimorphisms from M to M/N and M/P. If $f : M \to N$ is a morphism of an abelian category and P is a sub-object of M, $f(P)$ denotes the image of $f \circ i_N^M$; in the same way, if Q is a sub-object of N, $f^{-1}(Q)$ denotes the kernel of the composite of f with the canonical morphism from N to N/Q.

The theorems of Noether's isomorphism remain valid in an abelian category. The same is true of the results on the composition series of an object, of the Jordan-Holder composition series, of the length of an object. In the same way, we refer to [6] for the usual results on the exact sequences, the exact functors, left exactness, right exactness, the direct factors, the injective objects.

Proposition 4. *If $\mathcal{C}$ is an abelian category, there is a fiber sum (resp. fiber product) for the diagram (1)of paragraph 3 [resp. for the diagram (2) of paragraph 3].*

We recall only the construction of $B\Sigma_A C$: if j_B and j'_C are the canonical morphisms from B and C to $B \oplus C$, $B\Sigma_A C$ is the cokernel of the morphism

$j_B \circ h - j'_C \circ k$. The canonical morphism j from B to $B\Sigma_A C$ is the composite of j_B and the canonical morphism from $B \oplus C$ to $B\Sigma_A C$. The kernel of j is the image of h from Ker k; the canonical morphism from C to $B\Sigma_A C$ induces an isomorphism between Coker k and Coker j.

Recall also the construction of $B\Pi_A C$: if q_B and q'_C are the canonical morphisms from $B \oplus C$ to B and C, $B\Pi_A C$ is the kernel of the morphism $f \circ q_B - g \circ q'_C$. The canonical morphism q from $B\Pi_A C$ to B is the composite of the canonical morphism from $B\Pi_A C$ to $B \oplus C$ and q_B. The cokernel of q is the quotient of B from $f^{-1}(g(C))$; the canonical morphism from $B\Pi_A C$ to C induces an isomorphism between Ker q and Ker g.

6. Categories with generators and exact inductive limits

Proposition 5. *If $\mathcal{C}$ is an abelian category with generators and X is an object of $\mathcal{C}$, there exists a set belonging to $\mathfrak{U}$ and having the same cardinal as the set of all sub-objects of X.*

Indeed let $(X_i)i \in I$ be a family of generators of $\mathcal{C}$, the set I belonging to $\mathfrak{U}$. We know that the union of the sets $\mathrm{Hom}(X_i, X)$ is an element of $\mathfrak{U}$. We associate to every sub-object Y of X the subset EY of E which is formed of the morphisms whose images is contained in Y. If Y' is a sub-object of X which is not contained in Y, there is an element of EY not belonging to $E(Y \cap Y')$, since (X_i) is a family of generators of $\mathcal{C}$. In other words, the map $Y \to EY$ is an injective map from the set of sub-objects of X to the set of subsets of E. This proves the proposition.

Suppose now, and for the end of this paragraph, *that $\mathcal{C}$ is a category with generators and inductive limits*. According to [10], it is the same to say that $\mathcal{C}$ is a category with generators and that there is a direct sum for any family of objects indexed by a set belonging to $\mathfrak{U}$. If these conditions are satisfied, any increasing filtering family $(M_n)_{n\in E}$ of sub-objects of an object M has an upper bound: according to the previous proposition, we can suppose that E belongs to $\mathfrak{U}$; the canonical morphisms from M_n to M then define a morphism from the direct sum $\sum\limits_{n\in E} M_n$ to M. The image of this morphism is the upper bound of M_n.

Let I be a directed set, (M_i, u_{ji}) and (N_i, v_{ji}) be two inductive systems, indexed by I, of morphisms of $\mathcal{C}$ ($I \in \mathfrak{U}$). A morphism from the first to the second is, by definition, a family of morphisms from the source M_i, to the target N_i such that we have $f_u \circ u_{ji} = v_{ji} \circ f_i$ if $j > i$. Thus is defined the category of inductive systems indexed by I, and this category is abelian. Furthermore, the functor $(M_i, u_{ji}) \rightsquigarrow \varinjlim M_i$ is right exact. If this last functor is exact for any $I \in \mathfrak{U}$, we say that $\mathcal{C}$ is a *category with exact inductive limits.*

Proposition 6. *If $\mathcal{C}$ is a category with generators and inductive limits, the following assertions are equivalent:*

- *a.$\mathcal{C}$ is a category with exact inductive limits.*

- *b.If (P_i) is an increasing filtering family of sub-objects of P, the canonical morphisms from $\varinjlim P_i$ to P is an isomorphism from $\varinjlim P_i$ to the upper bound $\sup_{i \in I} P_i$.*
- *c.If (P_i) is an increasing filtering family of sub-objects of P, and if Q is a sub-object of P, we have the equality*

$$(\sup P_i) \cap Q = \sup(P_i \cap Q)$$

The inductive systems of an *abelian category with generators and exact inductive limits* $\mathcal{C}$ treat themselves in the same way as the inductive systems of abelian groups. We will use a lot later in the theory of semi-simple objects of such a category: an object S of $\mathcal{C}$ is called *simple* if it is not zero and if it does not contain any sub-object distinct from S and from O. An object is called semi-simple if it is isomorphic to a direct sum of simple objects. The theory is analogous in every respect to the theory of semi-simple modules [4]. We recall only a few results:

If $\mathcal{S}$ is a full sub-category of $\mathcal{C}$ whose objects are the semi-simple objects of $\mathcal{C}$; let $\mathscr{C}$ be the set of isomorphism classes of simple objects. If M is an object of $\mathcal{S}$ and if λ belongs to $\mathscr{C}$; we denote by M_λ the isotypic component of type λ of M: this is the sum of simple sub-objects of M belonging to λ. We denote also by $\mathcal{S}_\lambda$ the full sub-category of $\mathcal{S}$ whose objects are the objects M such that we have $M = M_\lambda$. In these conditions we have the following results:

- If $f : M \to N$ is a morphism of $\mathcal{S}$, $f(M_\lambda)$ is contained in N_λ.
- The functor $M \rightsquigarrow (M_\lambda)_{\lambda \in \mathscr{C}}$ defines an equivalence between $\mathcal{S}$ and the product category $\prod_\lambda \mathcal{S}_\lambda$ (see the paragraph 8 for the definition of the equivalences).
- If S is a simple object of $\mathcal{S}_\lambda$, the functor $M \rightsquigarrow \mathrm{Hom}(S, M)$ defines an equivalence between $\mathcal{S}_\lambda$ and the category of vector spaces [over the division ring $\mathrm{Hom}(S, S)$] whose underlying set belongs to $\mathfrak{U}$.

Note, to finish, that the theorem of Krull-Remak-Schmidt is valid in an abelian category with generators and exact inductive limits. Precisely, we say that an object M is indecomposable if it is not zero and if any direct factor of M is equal to O or M. We then show the following theorem:

Theorem 1. *Let $\mathcal{C}$ be an abelian category with generators and exact inductive limits. Let $(M_i)_{i \in I}$ and $(N_j)_{j \in J}$ be two families of indecomposable objects whose ring of endomorphisms is local ($I \in \mathfrak{U}$, $J \in \mathfrak{U}$). If the direct sums*

$$\sum_{i \in I} M_i \quad \textit{and} \quad \sum_{j \in J} N_j$$

are isomorphic, there is a bijection h from I to J such that M_i is isomorphic to $N_{h(i)}$.

Recall that a ring A, commutative or not, is called local if it has a non-zero unit element and if the quotient of A by its Jacobson radical is a division ring.

Azumaya has proved this theorem 'in the case modules' [2]. The reader is suggested to resume this demonstration using the proposition to avoid the arguments in which Azumaya talks about elements of a module.

7. Adjoint functors

The reader will find in [20] the proof of these results that is here:

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories, and let $T : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{A}$ be functors. We denote by $\mathrm{Hom}_{\mathcal{B}}(T.,\ .)$ and $\mathrm{Hom}_{\mathcal{A}}(.\ ,\ S.)$ the following functors:

$$\mathrm{Hom}_{\mathcal{B}}(T.,\ .) : \quad (A, B) \rightsquigarrow \mathrm{Hom}_{\mathcal{B}}(TA, B)$$
$$\mathrm{Hom}_{\mathcal{A}}(.\ ,\ S.) : \quad (A, B) \rightsquigarrow \mathrm{Hom}_{\mathcal{A}}(A, SB)$$

These functors are defined in the product category $\mathcal{A}^o \prod \mathcal{B}$ and they take their values in the category **Ens** of sets. A functorial morphism

$$\psi : \quad \mathrm{Hom}_{\mathcal{B}}(T.,\ .) \to \mathrm{Hom}_{\mathcal{A}}(.\ ,\ S.)$$

defines, for every couple (A, B) of objects of $\mathcal{A}$ and of $\mathcal{B}$, a map

$$\psi(A, B) : \quad \mathrm{Hom}_{\mathcal{B}}(TA, B) \to \mathrm{Hom}_{\mathcal{A}}(A, SB)$$

In particular, if B is equal to TA, the image of the identity morphism of TA under the map $\psi(A, TA)$ is a morphism $\Psi(A) : A \to STA$.

Lemma 5. *The morphisms $\Psi(A)$ define a functorial morphism Ψ from the identity functor $I_{\mathcal{A}}$ of $\mathcal{A}$ to the functor $S \circ T$. The map $\psi \rightsquigarrow \Psi$ is bijective.*

We recall only how we can rebuild ψ knowing Ψ: the functor S defines for every couple (A, B) a map

$$S(TA,\ B) : \quad \mathrm{Hom}_{\mathcal{B}}(TA, B) \to \mathrm{Hom}_{\mathcal{A}}(STA, SB)$$

The map $\psi(A, B)$ is the composite of $S(TA, B)$ and the map from $\mathrm{Hom}_{\mathcal{A}}(STA, SB)$ to $\mathrm{Hom}_{\mathcal{A}}(A, SB)$ which is induced by $\Psi(A)$.

In an analogous way, a morphism $\varphi : \ \mathrm{Hom}_{\mathcal{A}}(.\ ,\ S.) \to \mathrm{Hom}_{\mathcal{B}}(T.,\ .)$ defines, for every object B of $\mathcal{B}$, a morphism $\Phi(B) : \ TSB \to B$.

Lemma 6. *The morphisms $\Phi(B)$ define a functorial morphism Φ from the functor $T \circ S$ to the identity functor $I_{\mathcal{B}}$ of $\mathcal{B}$. The map $\varphi \rightsquigarrow \Phi$ is bijective.*

With these previous notations, suppose we are given functorial morphisms ψ and φ and let Ψ and Φ be the morphisms associated to ψ and φ by the previous lemmas:

Proposition 7. *For the functorial composed morphism $\psi \circ \varphi$ to be the identity of the functor $\mathrm{Hom}_{\mathcal{A}}(.\ ,\ S.)$, it is necessary and sufficient that the composed morphism*

$$S \xrightarrow{\Phi_S} STS \xrightarrow{S\Phi} S$$

is the identity morphism of the functor S.

Proposition 8. *For the functorial composed morphism $\varphi \circ \psi$ to be the identity of the functor $\mathrm{Hom}_{\mathcal{B}}(T.,\ .)$, it is necessary and sufficient that the composed morphism*

$$T \xrightarrow{T\Phi} TST \xrightarrow{\Phi_T} T$$

is the identity morphism of the functor T.

If there are two functorial morphisms φ and ψ such that $\psi \circ \phi$ and $\varphi \circ \psi$ are the identity morphisms, we sat that the functor S is (*right*) *adjoint to* T. In this terminology, S can be adjoint to T without T being adjoint to S. If S and S' are two functors adjoint to T, there is a functorial isomorphism between $\mathrm{Hom}_{\mathcal{A}}(.\ ,\ S.)$ and $\mathrm{Hom}_{\mathcal{A}}(.\ ,\ S'.)$. This isomorphism defines for any object B of $\mathcal{B}$ an isomorphism from the functor $\mathrm{Hom}_{\mathcal{A}}(.\ ,\ SB)$ to the functor $\mathrm{Hom}_{\mathcal{A}}(.\ ,\ S'B)$. The corollary of the proposition 1 implies the following proposition:

Proposition 9. *If S and S' are two functors adjoint to T, there is a functorial isomorphism between S and S'. Likewise, if S is a functor adjoint to two functors T and T', there is a functorial isomorphism between T and T'.*

Proposition 10. *If T is a functor from $\mathcal{A}$ to $\mathcal{B}$, the following assertions are equivalent:*

a. *There is a functor adjoint to T.*
b. *For every object B of $\mathcal{B}$, the functor $A \rightsquigarrow \mathrm{Hom}_{\mathcal{B}}(TA, B)$ is representable.*

It is clear that (a) implies (b). Conversely, choose for every object B of $\mathcal{B}$ a representative SB of the functor $A \rightsquigarrow \mathrm{Hom}_{\mathcal{B}}(TA, B)$ and a functorial isomorphism $\varphi(B):\ \ \mathrm{Hom}_{\mathcal{A}}(.\ ,\ SB) \to \mathrm{Hom}_{\mathcal{B}}(T.,\ B)$. If $f : B \to B'$ is a morphism of $\mathcal{B}$, there is one and only one morphism $Sf : SB \to SB'$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{A}}(.\ ,\ SB) & \xrightarrow{\varphi(B)} & \mathrm{Hom}_{\mathcal{B}}(T.,\ B) \\
\downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{A}}(.\ ,\ Sf)} & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{B}}(T.\ ,\ f)} \\
\mathrm{Hom}_{\mathcal{A}}(.\ ,\ SB') & \xrightarrow{\varphi(B')} & \mathrm{Hom}_{\mathcal{B}}(T.,\ B')
\end{array}$$

We see without difficulty that the map $B \rightsquigarrow SB$ and $f \rightsquigarrow Sf$ define a functor adjoint to T.

When $\mathcal{A}$ and $\mathcal{B}$ are two *additive categories*, we suppose, in accordance with our conventions, that S and T are additive functors. What precedes then remains valuable if we want to consider that the functors $\mathrm{Hom}_{\mathcal{B}}(T.\ ,\ .)$ and $\mathrm{Hom}_{\mathcal{A}}(.\ ,\ S.)$ take their values in the category **Ab** of abelian groups. This implies that the maps $\psi(A, B)$ and $\varphi(A, B)$ are linear. Furthermore, we have the following proposition:

Proposition 11. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, let T be a (additive) functor from $\mathcal{A}$ to $\mathcal{B}$, and let S be a functor adjoint to T. Then S is left exact and T is right exact.*

Indeed let

$$0 \to B' \xrightarrow{f} B \xrightarrow{g} B'' \to 0$$

be an exact sequence in $\mathcal{B}$. We have the following commutative diagram (A is an arbitrary object of $\mathcal{A}$):

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A,\ SB') & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A,\ SB) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A,\ SB'') \\ & \downarrow{\scriptstyle \varphi(A,B')} & & \downarrow{\scriptstyle \varphi(A,B)} & & \downarrow{\scriptstyle \varphi(A,B'')} \\ 0 \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(TA,\ B') & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(TA,\ B) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(TA,\ B'') \end{array}$$

The second row is exact and the vertical arrows are bijections. It follows that the first row is exact and that S is left exact. We show in an analogous way that T is right exact.

8. Equivalence of categories

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. An equivalence between $\mathcal{A}$ to $\mathcal{B}$ consists of the data of two functors, $T : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{A}$, and of two isomorphisms of functors, $\Psi : I_{\mathcal{A}} \to S \circ T$ and $\Phi' : I_{\mathcal{B}} \to T \circ S$. Furthermore we suppose that the isomorphisms $T\Psi : T \to TST$ and $\Phi' T : T \to TST$ coincide.

It results from the conditions imposed on T, S, Ψ and Φ' that the functorial morphisms $S\Phi' : S \to STS$ and $\Psi S : S \to STS$ also coincide. If Ψ' and Φ are the inverse functorial isomorphisms of Ψ and Φ', we see consequently that Ψ and Φ make S a functor adjoint to T. In particular, the data of T determines S up to an isomorphism of functors. Similarly, the functorial morphisms Ψ' and Φ' make T a functor adjoint to S.

Let ψ, ψ', φ and φ' be the functorial morphisms associated respectively to Ψ, Ψ', Φ and Φ' as has been explained in the previous paragraph. The data of Ψ determines ψ and hence φ which is the inverse functorial morphism of ψ. We can deduce that if S and T are given, the data of Ψ determines Φ'; and conversely....

Lemma 7. *Let (T, S, Ψ, Φ') be an equivalence between $\mathcal{A}$ and $\mathcal{B}$:*

a. *For any couple (M, N) of objects of $\mathcal{A}$, the map $T(M, N)$ from $\mathrm{Hom}_{\mathcal{A}}(M, N)$ to $\mathrm{Hom}_{\mathcal{B}}(TM, TN)$ is bijective.*
b. *Any object P of $\mathcal{B}$ is isomorphic to an object of the form TM.*

Proposition 12. *If T is a functor from $\mathcal{A}$ to $\mathcal{B}$, the following assertions are equivalent:*

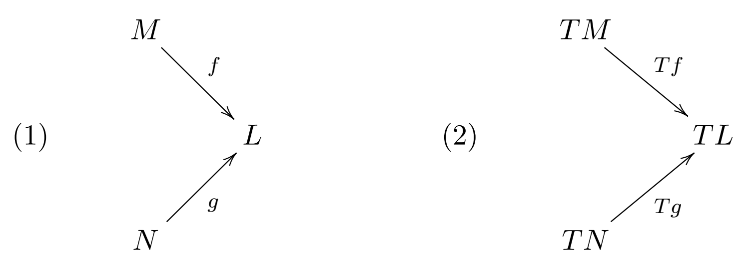
a. *There exist a functor $S : \mathcal{B} \to \mathcal{A}$ and isomorphisms $\Psi : I_{\mathcal{A}} \to S \circ T$ and $\Phi' : I_{\mathcal{B}} \to T \circ S$ such that (T, S, Ψ, Φ') is an equivalence between $\mathcal{A}$ and $\mathcal{B}$.*
b. *The functor T satisfies the conditions (a) and (b) of the above lemma.*

c. *There is a functor $S : \mathcal{B} \to \mathcal{A}$ such that $S \circ T$ is isomorphic to the functor $I_{\mathcal{A}}$ and that $T \circ S$ is isomorphic to the functor $I_{\mathcal{B}}$.*

We will say from now on that two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there is a functor T satisfying the assertions of the previous proposition. In this case we say also that T defines an equivalence from $\mathcal{A}$ to $\mathcal{B}$. We finally note that the assertion (c) fits conveniently into the following formalism:

If $\mathfrak{B}$ is a universe of which the universe $\mathfrak{U}$ is an element, we can build a new category $\mathcal{E}$: the objects of $\mathcal{E}$ are the categories whose set of morphisms belong to $\mathfrak{B}$ (we identify the objects with the identity morphisms); if $\mathcal{A}$ and $\mathcal{B}$ are two objects of $\mathcal{E}$, $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ is the set of isomorphism classes of functors from $\mathcal{A}$ to $\mathcal{B}$, the composition being done in an obvious way. We remark that $\mathcal{E}$ is not a $\mathfrak{U}-$category. The assertion (c) affirms that the class of functors isomorphic to T is an isomorphism of the category $\mathcal{E}$.

Finally, we offer the reader some easy exercises, so we will freely use the results: let (T, S, Ψ, Φ') be an equivalence between $\mathcal{A}$ and $\mathcal{B}$. If f is a monomorphism (resp. an epimorphism), Tf is a monomorphism (resp. an epimorphism). If M is a generator of $\mathcal{A}$m (resp. a projective object of $\mathcal{A}$), TM is a generator of $\mathcal{B}$ (resp. a projective object of $\mathcal{B}$). If P is the fibre product of the diagram (1), TP is the fibre product of (2); we also say that T commutes with fibre products. The functor T 'commutes' also the the fibre sums, the inductive limits.... Generally speaking, the homological properties of two equivalent categories are the 'same'.



The previous exercises and the proposition 3 have the following consequence:

Proposition 13. *Let T be a functor defining an equivalence between two additive categories $\mathcal{A}$ and $\mathcal{B}$. Then T is an additive functor. If $\mathcal{A}$ and $\mathcal{B}$ are two abelian categories, T is also an exact functor.*

The last assertion results from the proposition 11.

9. Abelian categories with enough injectives

This paragraph is devoted to some results on the injective objects of an abelian category $\mathcal{B}$. These results are related to the notion of derived category which is due to Cartier and of which we present here a 'piece':

Let $\mathcal{I}$ be an additive category and let $\mathcal{M}$ be the category of morphisms of $\mathcal{I}$: an object of $\mathcal{M}$ is a morphism of I; if $d : M \to N$ and $d' : M' \to N'$ are two objects of $\mathcal{M}$, $\mathrm{Hom}_{\mathcal{M}}(d, d')$ is the set of couples (α, β), $\alpha \in \mathrm{Hom}_{\mathcal{I}}(M, M')$, $\beta \in \mathrm{Hom}_{\mathcal{I}}(N, N')$, such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{d} & N \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ M' & \xrightarrow{d'} & N' \end{array}$$

It is well known that by defining in the obvious way the composition of the morphisms, we make $\mathcal{M}$ an additive category.

With the above notations, we say that (α, β) is *homotopic to zero* if there is an element h of $\mathrm{Hom}_{\mathcal{I}}(N, M')$ such that we have $\alpha = h \circ d$. The morphisms homotopic to zero form a sub-group of $\mathrm{Hom}_{\mathcal{M}}(d, d')$ and we denote by $\mathrm{Hom}_{\mathcal{K}}(d, d')$ the quotient of $\mathrm{Hom}_{\mathcal{M}}(d, d')$ by this sub-group. It is then clear that the bilinear maps

$$\mathrm{Hom}_{\mathcal{M}}(d, d') \times \mathrm{Hom}_{\mathcal{M}}(d', d'') \to \mathrm{Hom}_{\mathcal{M}}(d, d'')$$

defines by passing to the quotient of the bilinear maps

$$\mathrm{Hom}_{\mathcal{K}}(d, d') \times \mathrm{Hom}_{\mathcal{K}}(d', d'') \to \mathrm{Hom}_{\mathcal{K}}(d, d'')$$

So we can define a new additive category $\mathcal{KI}$: the objects of $\mathcal{KI}$ coincides with the morphisms of $\mathcal{I}$ (hence with the objects of $\mathcal{M}$); if d and d' are two such morphisms, $\mathrm{Hom}_{\mathcal{KI}}(d, d')$ is chosen equal to $\mathcal{K}(d, d')$; the law of composition, finally, is defined by passing to the law of composition of $\mathcal{M}$.

In the case which interests us, $\mathcal{I}$ is a full sub-category of $\mathcal{B}$ whose objects are the injectives of $\mathcal{B}$. We denote by Ker the functor $d \rightsquigarrow \mathrm{Ker}\ d$ from $\mathcal{M}$ to $\mathcal{B}$. If d and d' are two objects of $\mathcal{M}$, it is clear that the map $\mathrm{Ker}(d, d')$ from $\mathrm{Hom}_{\mathcal{M}}(d, d')$ to $\mathrm{Hom}_{\mathcal{B}}(\mathrm{Ker}\ d, \mathrm{Ker}\ d')$ defines, by passing to the quotient, the maps from $\mathrm{Hom}_{\mathcal{K}}(d, d')$ to $\mathrm{Hom}_{\mathcal{B}}(\mathrm{Ker}\ d, \mathrm{Ker}\ d')$. These maps define in fact a functor (still denoted by Ker!) from $\mathcal{KI}$ to $\mathcal{B}$.

Proposition 14. *Let $\mathcal{B}$ be an abelian category, and let $\mathcal{I}$ be a full sub-category of $\mathcal{B}$ whose objects are the injectives of $\mathcal{B}$. Suppose the following conditions are satisfied: for any object M of $\mathcal{B}$, there is a monomorphism from M to an object of $\mathcal{I}$. The functor* Ker *defines then an equivalence from $\mathcal{KI}$ to $\mathcal{B}$.*

Let's quickly say how to construct a functor S adjoint to Ker.

For any object M of $\mathcal{B}$, we choose an exact sequence

$$0 \to M \xrightarrow{f_M} I_0 M \xrightarrow{g_M} I_1 M$$

where $I_1 M$ and $I_0 M$ are objects belonging to $\mathcal{I}$. For any morphism $u : M \to N$, we choose the morphisms u_0 and u_1 which make the following diagram commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{f_M} & I_0M & \xrightarrow{g_M} & I_1M \\ & & \downarrow u & & \downarrow u_0 & & \downarrow u_1 \\ 0 & \longrightarrow & N & \xrightarrow{f_N} & I_0N & \xrightarrow{g_N} & I_1N \end{array}$$

We can easily see that the image of (u_0, u_1) in $\mathrm{Hom}_{\mathcal{KI}}(g_{\mathcal{M}}, g_{\mathcal{N}})$ depends only on u. If this image is denoted by Su and if SM denotes the morphism g_M, the maps $M \to SM$ and $u \to Su$ define a functor adjoint to Ker.

Corollary 3. *Suppose the hypothesis of the previous proposition is satisfied. If the category $\mathcal{I}$ is equivalent to the product of a family $(\mathcal{I}_n)_{n\in E}$ of additive categories, the category $\mathcal{B}$ is equivalent to the product of a family $(\mathcal{KI})_{n\in E}$.*

Now let $\mathcal{C}$ be another abelian category. Any additive functor F from $\mathcal{I}$ to $\mathcal{C}$ extends in an obvious way to a functor $\mathcal{K}F$ from $\mathcal{KI}$ to $\mathcal{KC}$. By composition, we obtain a left exact functor F' from $\mathcal{B}$ to $\mathcal{C}$:

$$\mathcal{B} \xrightarrow{s} \mathcal{KI} \xrightarrow{\mathcal{K}_F} \mathcal{KC} \xrightarrow{\mathrm{Ker}} \mathcal{C}$$

Conversely, if G is a functor from $\mathcal{B}$ to $\mathcal{C}$, we denote by G' the composite of the canonical functor from $\mathcal{I}$ to $\mathcal{B}$ and G.

Corollary 4. *Suppose the hypothesis of the previous proposition is satisfied. A functor G from $\mathcal{B}$ to $\mathcal{C}$ is left exact if and only if it is isomorphic to $(G')'$. Any functor F from $\mathcal{I}$ to $\mathcal{C}$ is isomorphic to $(F')'$.*

This corollary shows that the study of left exact functors from $\mathcal{B}$ to $\mathcal{C}$ is equivalent to the study of all the additive functors from $\mathcal{I}$ to $\mathcal{C}$. We will use this fact in the next corollary and at the beginning of part II.

Corollary 5. *Let $\mathcal{B}$ (resp. $\mathcal{C}$) be an abelian category, let $\mathcal{I}$ (resp. $\mathcal{J}$) be a full sub-category of $\mathcal{B}$ (resp. of $\mathcal{C}$) satisfying the conditions of the previous proposition 14. A functor G from $\mathcal{B}$ to $\mathcal{C}$ defines an equivalence between the two categories if and only if the following are satisfied: G is left exact; GI is an object of $\mathcal{J}$ for any object I of $\mathcal{I}$; the functor $I \rightsquigarrow GI$ defines an equivalence between $\mathcal{I}$ and $\mathcal{J}$.*

We will say from now on that $\mathcal{B}$ has enough injectives if the full sub-category of $\mathcal{B}$ which is formed of the injective objects satisfying the condition of the proposition 14. When this happens, the proposition 14 shows that the data of this category defines $\mathcal{B}$ up to an equivalence. The proposition 14 is so useful whenever we know injective objects of $\mathcal{B}$ (cf. part IV).

Let's go back to the hypothesis of corollary 4. With the useful notations in the proof of the proposition 14, we see that $(G')'$ is none other than the functor $M \rightsquigarrow \mathrm{Ker}\ (Gg_M)$. It follows that $(G')'$ is the zeroth derived functor of G [6]. Furthermore, the morphisms Gf_M define a functorial morphism $\rho : G \to (G')' = R^0G$ and the couple (R^0G, ρ) satisfy the conditions of the proposition 15:

Proposition 15. *Let $\mathcal{B}$ be an abelian category with enough injective objects. For any functor $G : \mathcal{B} \to \mathcal{C}$ from $\mathcal{B}$ to an abelian category $\mathcal{C}$, there is a functor R^0G and a morphism $\rho : G \to R^0G$ which satisfy the following conditions:*

The functor R^0G is left exact; in addition, if $\sigma : G \to H$ is a morphism from G to a left exact functor H, there is one morphism $\tau : R^0G \to H$ and only one such that we have $\sigma = \tau \circ \rho$.

Let $\rho' : G \to R'G$ be a functorial morphism satisfying the same universal property as $\rho : G \to R^0G$; it is clear that there is then one isomorphism u from $R'G$ to R^0G an only one such that we have $\rho = u \circ \rho'$. The proposition 15 hence determines R^0G up to a functorial isomorphism. We will try to extend these results when we do not have enough injective objects.

Part 2. Left exact functors and injective envelopes

According to a result of Grothendieck [10], any abelian category with generators and exact inductive limits have enough injective objects. We begin here the study of these injective objects. We also show that how we can reduce the study of certain categories to the study of a category with generators and exact inductive limits.

10. Categories of functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two additive categories; *we suppose that the set of objects and the set of morphisms of* $\mathcal{C}$ *are the elements of the universe* $\mathfrak{U}$. Under these conditions, we denote by $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, and we call the category of functors from $\mathcal{C}$ to $\mathcal{D}$ the full sub-category of $\mathbf{Hom}(\mathcal{C},\mathcal{D})$ whose objects are the additive functors from $\mathcal{C}$ to $\mathcal{D}$ (cf. part I, § 3).

Proposition 1. *With the hypothesis that we made above,* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ *is an additive category. If* $\mathcal{D}$ *is an abelian category, it's the same for* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$. *If* $\mathcal{D}$ *is a category with inductive limits (resp. exact inductive limits, resp. with projective limits),* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ *is a category with inductive limits (resp. exact inductive limits, resp. with projective limits).*

If E and F are two objects of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, $\mathrm{Hom}(E,F)$ is obviously a subset of the product $\prod_{M\in\mathcal{O}\mathcal{C}} \mathrm{Hom}(EM,FM)$. This shows that $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ is a $\mathfrak{U}-$category. The rest is almost clear.

The remainder of this paragraph is devoted to the study of $\mathrm{Fun}(\mathcal{C},\boldsymbol{Ab})$ *when* $\mathcal{C}$ *is an additive category whose set of objects and the set of morphisms belong to* $\mathfrak{U}$. We denote by $\mathbb{Z}[\mathcal{C}]$ the ring that follows and that does not have a unit element in general:

- The underlying abelian group of $\mathbb{Z}[\mathcal{C}]$ is the direct sum of groups $\mathrm{Hom}_{\mathcal{C}}(M,N)$ where M and N go through the objects of $\mathcal{C}$.
- Let $f\in\mathrm{Hom}_{\mathcal{C}}(M,N)$ and $g\in\mathrm{Hom}_{\mathcal{C}}(P,Q)$ be two elements of $\mathbb{Z}[\mathcal{C}]$. The product $f.g$ is the composite morphism $f\circ g$ when Q coincides with M. Otherwise $f.g$ is equal to zero.

For any object X of $\mathcal{C}$, the identity morphism 1_X is an idempotent of $\mathbb{Z}[\mathcal{C}]$. If X and Y are two distinct objects of $\mathcal{C}$, the idempotents 1_X and 1_Y are orthogonal: $1_X.1_Y=1_Y.1_X=0$. It follows that, for any left $\mathbb{Z}[\mathcal{C}]-$module M, the element 1_X defines a projection from M to a direct factor $1_X(M)$. Furthermore, the sum of abelian sub-groups $1_X(M)$ is direct and it is equal to $\mathbb{Z}[\mathcal{C}].M$.

Lemma 1. *If* N *is a sub-module of* M *(resp.* Q *a quotient of* M*), and if* $\mathbb{Z}[\mathcal{C}].M$ *is equal to* M*, then* $\mathbb{Z}[\mathcal{C}].N$ *is equal to* N *(resp.* $\mathbb{Z}[\mathcal{C}].Q$ *is equal to* Q*).*

Indeed, an element m of M belongs to $\mathbb{Z}[\mathcal{C}].M$ if and only if there is a finite number of objects of $\mathcal{C}$, say X, Y, ..., Z such that we have

$$m = 1_X.m + 1_Y.m + \cdots + 1_Z.m$$

When this condition is satisfied for any element of M, it is the same for N and Q.

The previous lemma allows us to define a new abelian category $\mathcal{A}$:

- The objects of $\mathcal{A}$ are the left modules over $\mathbb{Z}[\mathcal{C}]$ such that $\mathbb{Z}[\mathcal{C}].M$ is equal to M, and whose underlying set belongs to $\mathfrak{U}$.
- If M and N are two objects of $\mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(M, N)$ is the set of $\mathbb{Z}[\mathcal{C}]-$linear maps from M to N; the composition of morphisms is done in the usual way.

In particular, the underling left module of $\mathbb{Z}[\mathcal{C}]$ and the left ideal of $\mathbb{Z}[\mathcal{C}].1_X$ are the objects of $\mathcal{A}$. A morphism φ from $\mathbb{Z}[\mathcal{C}].1_X$ to an object M of $\mathcal{A}$ is determined by the data of the element $\varphi(1_X)$ of $1_X(M)$. The abelian group $\mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}[\mathcal{C}].1_X), M)$ is hence identified with $1_X(M)$, and $\mathbb{Z}[\mathcal{C}].1_X$ is a projective object of $\mathcal{A}$. It is the same for $\mathbb{Z}[\mathcal{C}]$ which is the direct sum of $\mathbb{Z}[\mathcal{C}].1_X$. Hence:

Lemma 2. *The underlying left module $\mathbb{Z}[\mathcal{C}]$ is a projective generator of $\mathcal{A}$.*

Proposition 2. *The categories $\mathcal{A}$ and $\mathrm{Fun}(\mathcal{C}, \boldsymbol{Ab})$ are equivalent.*

We will construct the functors

$$T : \mathcal{A} \to \mathrm{Fun}(\mathcal{C}, \mathbf{Ab}) \quad \text{and} \quad S : \mathrm{Fun}(\mathcal{C}, \mathbf{Ab}) \to \mathcal{A}$$

such that $S \circ T$ and $T \circ S$ are isomorphic respectively to the identity functors of $\mathcal{A}$ and of $\mathrm{Fun}(\mathcal{C}, \mathbf{Ab})$.

For this let M be an object of $\mathcal{A}$. For any object X of $\mathcal{C}$, we denote by $(TM)X$ the abelian group $1_X(M)$. For any morphism $f : X \to Y$, we denote by $(TM)(f)$ the $\mathbb{Z}-$linear map from $1_X(M)$ to $1_Y(M)$ which is induced by f. We have thus defined a functor TM from $\mathcal{C}$ to $\mathbf{Ab}$. Moreover, if $\varphi : M \to N$ is a $\mathbb{Z}[\mathcal{C}]-$linear map, φ maps $1_X(M)$ to $1_X(N)$. We denote by $(T\varphi)X$ the map from $1_X(M)$ to $1_X(N)$ thus defined. When X runs through the objects of $\mathcal{C}$, the maps $(T\varphi)X$ obviously define a functorial morphism from TM to TN. The verification of the following lemma is left to the reader:

Lemma 3. *The maps $M \to TM$, $\varphi \to T\varphi$ define a functor from $\mathcal{A}$ to* $\mathrm{Fun}(\mathcal{C}, \boldsymbol{Ab})$.

Conversely, let F be an object of $\mathrm{Fun}(\mathcal{C}, \mathbf{Ab})$. If $f : X \to Y$ is a morphism of $\mathcal{C}$ and $m = (m_X)$ is an element of $\sum\limits_{X \in \mathcal{OC}} FX$, we define a product $f.m$ with the help of the formula

$$\text{projection from } f.m \text{ to } FZ = \begin{cases} 0 & \text{if } \ Z \neq Y \\ (Ff)(m_X) & \text{if } \ Z = Y \end{cases}$$

The direct sum $\sum_{X\in\mathcal{OC}} FX$ is thus equipped with a structure of left module over the ring $\mathbb{Z}[\mathcal{C}]$. We denote this module by SF which is obviously an object of $\mathcal{A}$. Similarly, if $\psi : F \to G$ is a functorial morphism, we denote by $S\psi$ the linear map:

$$\sum_X \psi(X) : \sum_X FX \to \sum_X GX$$

Lemma 4. *The maps* $F \to SF$, $\psi \to S\psi$ *define a functor* S *from* $\mathrm{Fun}(\mathcal{C}, \boldsymbol{Ab})$ *to* $\mathcal{A}$.

It is clear that T and S satisfies the required conditions. This ends the proof of the proposition 2. The lemma 2 thus implies the following proposition.

Proposition 3. *The category* $\mathrm{Fun}(\mathcal{C}, \boldsymbol{Ab})$ *has a projective generator.*

Indeed, $\mathbb{Z}[\mathcal{C}]$ is a projective generator of $\mathcal{A}$. It follows that $T\mathbb{Z}[\mathcal{C}]$ is a projective generator of $\mathrm{Fun}(\mathcal{C}, \mathbf{Ab})$. This functor is none other than the functor $Y \rightsquigarrow \sum_{X\in\mathcal{OC}} \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

11. Left exact functors with values in an abelian category

We suppose now that $\mathcal{C}$ is an abelian category whose set of morphisms and the set of objects belong to $\mathfrak{U}$. If $\mathcal{D}$ is an abelian category, we want to study here the full sub-category of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are the left exact functors from $\mathcal{C}$ to $\mathcal{D}$. This full sub-category will be denoted by $\mathrm{Sex}(\mathcal{C}, \mathcal{D})$ (S means sinister, ex means exact).

Proposition 4. *Let* $\mathcal{C}$ *be an abelian* $\mathfrak{U}-$*category whose set of morphisms and the set of objects belong to* $\mathfrak{U}$. *Let* $\mathcal{D}$ *be an abelian* $\mathfrak{U}-$*category with generators and exact inductive limits. For any functor from* $\mathcal{C}$ *to* $\mathcal{D}$ *there exists a functor* R^0F *and a morphism* $\rho : F \to R^0F$ *which satisfy the following conditions:*

The functor R^0F *is left exact; furthermore, if* $\sigma : F \to G$ *is a morphism from* F *to a left exact functor* G*, there exists a morphism*

$$\tau : R^0F \to G$$

and only one such that we have $\sigma = \tau \circ \rho$.

We have already proven the proposition 4 when $\mathcal{C}$ has enough injective objects (cf. part I, proposition 15). It is well-known that on the other hand we can define the derived functors of F under the assumptions of the proposition 4. In this study, we are interested only in zeroth derived functor of F; we use a method that Buchsbaum uses in the construction of the satellite functors.

Some lemmas will precede the proof of the proposition 4: for any object A of $\mathcal{C}$, we consider the set $\mathcal{S}_A$ of the exact sequences of the form

$$S : 0 \to A \xrightarrow{u} M \xrightarrow{v} N \to 0$$

If

$$S = (0 \to A \xrightarrow{u} M \xrightarrow{v} N \to 0) \text{ and } S' = (0 \to A \xrightarrow{u'} M' \xrightarrow{v'} N' \to 0)$$

are two elements of $\mathcal{S}_A$, we will write $S \leqslant S'$ if there is a diagram of the type

$$(\bigstar) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \\ 0 & \longrightarrow & A & \xrightarrow{u'} & M' & \xrightarrow{v'} & N' & \longrightarrow & 0 \end{array}$$

Lemma 5. *The relation $S \leqslant S'$ is a filtering (right) preorder relation, that is, a binary relation which is reflexive and transitive with the addition property that every pair of elements has an upper bound.*

It is clear that we are dealing with a preorder relation. So let

$$S = (0 \to A \xrightarrow{u} M \xrightarrow{v} N \to 0) \text{ and } S' = (0 \to A \xrightarrow{u'} M' \xrightarrow{v'} N' \to 0)$$

be two elements of $\mathcal{S}_A$. We consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u'} & M' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle j'_{M'}} \\ M & \xrightarrow{j_M} & M \sum_A M' \end{array}$$

According to the end of the paragraph 7 (part I), j_M and $j'_{M'}$ are monomorphisms. We can deduce that the sequence

$$S'' = (0 \to A \xrightarrow{j_M \circ u} M \Sigma_A M' \to \text{Coker } (j_M \circ u) \to 0)$$

belongs to $\mathcal{S}_A$ and that we have $S'' \geqslant S$, $S'' \geqslant S'$.

If F is a functor from $\mathcal{C}$ to $\mathcal{D}$, and if $S \leqslant S'$, the diagram ($\bigstar$) implies the equality $(Fn) \circ (Fv) = (Fv') \circ (Fm)$. It follows that Fm induces a morphism f from Ker (Fv) to Ker (Fv').

Lemma 6. *The morphism f does not depend on the choice of m and of n.*

Indeed let $m' : M \to M'$ be another morphism such that we have $u' = m' \circ u$; let $n' : N \to N'$ be the unique morphism such that $n' \circ v$ is equal to $v' \circ m'$. Then we have $(m' - m) \circ u = 0$; whence the existence of a $w : N \to M'$ such that $m' - m = w \circ v$. It follows that Fm' is equal to $Fm + (Fw) \circ (Fv)$. The lemma results from that $(Fw) \circ (Fv)$ vanishes on Ker (Fv).

From now on we denote by FS the object Ker Fv and by $F(S', S)$ the morphism f from Ker Fv to Ker Fv'. If $S' \leqslant S$ and $S' \geqslant S$, the previous lemma shows that $F(S', S)$ is an isomorphism whose inverse is $F(S, S')$. Hence it is allowable to lay down the definition

$$(RF)A = \varinjlim_{S \in \mathcal{S}_A} FS.$$

Suppose we are in addition given a morphism $g : A \to A'$. For any sequence

$$S = (0 \to A \xrightarrow{u} M \xrightarrow{v} N \to 0)$$

we have the commutative and exact diagram which follows (end of § 6, part I):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\quad u \quad} & M & \xrightarrow{\quad v \quad} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle j'_M} & & \downarrow{\scriptstyle 1_N} & & \\
0 & \longrightarrow & A' & \xrightarrow{j'_A} & A'\Sigma_A M & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Let fS be the sequence of the second row. The morphism Fj'_M induces obviously a morphism from FS to $F(fS)$. We denote by F(f,S) the composite of this morphism and the canonical morphism from $F(fS)$ to $(RF)A'$. The reader verifies the relations $F(f,S) = F(f,S') \circ F(S',S)$ if $S' \geqslant S$. By passing to the limit, the morphisms $F(f,S)$ thus define a morphism $(RF)f$ from $(RF)A$ to $(RF)A'$.

Lemma 7. a. *The maps $A \to (RF)A$ and $f \to (RF)f$ define a functor RF from $\mathcal{C}$ to $\mathcal{D}$.*

b. *RF vanishes if and only if the following conditions are satisfied: for any object A of $\mathcal{C}$, FA is the* sup *of* Ker (Fi) *when i runs through the monomorphisms of source A.*

c. *The functor $F \rightsquigarrow RF$ is left exact.*

d. *If $\varphi : F \to G$ is a functorial morphism and if $R(\mathrm{Coker}\varphi)$ is zero, then $R(\mathrm{Coker} R\varphi)$ is zero.*

e. *The functor $F \rightsquigarrow RF$ commutes with the inductive limits.*

We leave to the reader the verification of (a), (b), (e). Prove (c): for this let $0 \to F' \xrightarrow{\varphi} F \xrightarrow{\psi} F'' \to 0$ be an exact sequence of functors from $\mathcal{C}$ to $\mathcal{D}$. For any object A of $\mathcal{C}$ and any element S of $\mathcal{S}_A$,

$$S = (0 \to A \xrightarrow{u} M \xrightarrow{v} N \to 0),$$

we have the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F'M & \xrightarrow{\varphi(M)} & FM & \xrightarrow{\psi(M)} & F''M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F'_v} & & \downarrow{\scriptstyle F_v} & & \downarrow{\scriptstyle F''_v} & & \\
0 & \longrightarrow & F'N & \xrightarrow{\varphi(N)} & FN & \xrightarrow{\psi(N)} & F''N & \longrightarrow & 0
\end{array}$$

The morphisms $\varphi(M)$ and $\psi(M)$ thus induce an exact sequence

$$0 \to F'S \to FS \to F''S$$

([4], lemma III, p. 32). The assertion (c) results from this exact sequence by passing to the inductive limit.

For the sake of clarity, we prove the assertion (d) only when $\mathcal{D}$ is the category **Ab** of abelian groups; the condition $R(\mathrm{Coker}\varphi) = 0$ can then be stated as follows: for any object A of $\mathcal{C}$ and any $a \in GA$, there is a monomorphism $i : A \to B$ and a $b :\in FB$ such that we have

$$(Gi)(a) = \varphi(B)(b)$$

We will see that this property is still true when we replace F by RF, G by RG and φ by $R\varphi$:

Indeed let a be an element of $(RG)A$. There is an exact sequence

$$S = (0 \to A \xrightarrow{u} M \xrightarrow{v} N \to 0)$$

and an element a' of GS whose image under the canonical map from GS to $(RG)A$ is a. So there is also a monomorphism $j : M \to B$ and an element b' of FB such that we have $(Gj)(a') = \varphi(B)(b')$. If b is the image of b' in $(RF)B$ and if $i = j \circ u$, we find the sought equality

$$((RG)i)(a) = ((R\varphi)(B))(b)$$

Lemma 8. *If $g : A' \to A$ is a monomorphism, so is $(RF)g$.*

Indeed associate with any sequence $S = (0 \to A \xrightarrow{u} M \xrightarrow{v} N \to 0)$ the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \\
 & & {\scriptstyle g}\uparrow & & {\scriptstyle 1_M}\uparrow & & \uparrow & & \\
0 & \longrightarrow & A' & \xrightarrow{u \circ g} & M & \xrightarrow{p^M_{N'}} & N' & \longrightarrow & 0
\end{array}$$

where N' is the cokernel of $u \circ g$. If $g^{-1}S$ denotes the sequence of the second row, it is clear that the sequences $g^{-1}S$ constitute a cofinal sub-set of $\mathcal{S}_{A'}$. We conclude that $(RF)A'$ is equal to $\varinjlim_S (g^{-1}S)$ and that $(RF)g$ is the inductive limit of the canonical monomorphisms of $F(g^{-1}S)$ in FS. This proves the lemma.

Lemma 9. *We suppose that for any monomorphism g of $\mathcal{C}$, Fg is a monomorphism. Then:*

a. *The morphisms $F(S', S)$ are monomorphisms.*
b. *The functor RF is left exact.*

To prove (a) we use the notations of lemma 1. If $S' \geqslant S$, the exact sequence

$$S'' = (0 \to A \xrightarrow{j_M \circ u} M\Sigma_A M' \to \text{Coker } j_M \circ u \to 0)$$

'dominates' both S and S' (i.e. $S'' \geqslant S$, $S'' \geqslant S'$); in addition $F(S'', S)$ is induced by the morphism Fj_M. We conclude that $F(S'', S) = F(S'', S') \circ F(S', S)$ and $F(S', S)$ are monomorphisms.

Prove (b): let $0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$ be an exact sequence of $\mathcal{C}$ and let $S = (0 \to A \xrightarrow{u} M \xrightarrow{v} N \to 0)$ be an element of $\mathcal{S}_A$. We have the following commutative and exact diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_{A'}} & & \downarrow{\scriptstyle u} & & \downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{u\circ f} & M & \xrightarrow{j_M} & M\Sigma_A A'' & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle v} & & \downarrow & & \\
 & & & & N & \xrightarrow{1_N} & N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

With the notations already used, we conclude from the exact diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & F(f^{-1}S) & \longrightarrow & FS & \longrightarrow & F(gS) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(f^{-1}S) & \longrightarrow & FM & \longrightarrow & F(M\Sigma_A A'')
\end{array}$$

By passing to the limit, it results the following exact sequence:

$$0 \to (RF)A' \to (RF)A \to \varinjlim_{S} F(gS)$$

Furthermore, the assertion (a) shows that the canonical morphism from $\varinjlim_S F(gS)$ to $(RF)A''$ is a monomorphism; this completes the proof.

We are now able to prove the proposition 4: let A be an object of $\mathcal{C}$ and σ be a functorial morphism from F to a left exact functor G. For any exact sequence

$$S = (0 \to A \xrightarrow{u} M \xrightarrow{v} N \to 0)$$

we have the commutative diagram

$$\begin{array}{ccccccc}
 & & & & FM & \xrightarrow{Fv} & FN \\
 & & & & \downarrow{\scriptstyle \sigma(M)} & & \downarrow{\scriptstyle \sigma(N)} \\
0 & \longrightarrow & GA & \xrightarrow{Gu} & GM & \xrightarrow{Gv} & GN
\end{array}$$

Let $\rho_F(A,S)$ be the morphism from FA to FS which is induced by F_u. There is one morphism $\tau_F(A,S)$ and only one whose composite with $\rho_F(A,S)$ is equal to $\sigma(A)$. By passing to the inductive limit we see that there is one morphism $\tau_F(A)$ and only one from $(RF)A$ to GA whose composite with $\rho_F(A) = \varinjlim_S \rho_F(A,S)$ is equal to $\sigma(A)$. If the functor RF is left exact, the proposition is proved. Otherwise, we repeat the operation. The lemmas 4 and 5 show that the functor $R(RF)$ is left exact. In any case we can choose R^0F equal to $R(RF)$, ρ being equal to $\rho_{RF} \circ \rho_F$.

Proposition 5. *Suppose the hypothesis of the proposition 4 are satisfied. The category* $\mathrm{Sex}(\mathcal{C},\mathcal{D})$ *of left exact functors from* $\mathcal{C}$ *to* $\mathcal{D}$ *is an abelian category with exact inductive limits. If* $\mathcal{D}$ *is a category with projective limits, so is* $\mathrm{Sex}(\mathcal{C},\mathcal{D})$*.*

It is clear that $\mathrm{Sex}(\mathcal{C},\mathcal{D})$ is an additive category. If $\varphi : F \to G$ is a morphism of left exact functors, Ker φ is the left exact functor $M \rightsquigarrow$ Ker $\varphi(M)$. Similarly, if H is the functor $M \rightsquigarrow$ Coker $\varphi(M)$, the left exact functor Coker φ is none other than R^0H. This shows that the axiom C Ab 1 is verified.

Show that the axiom C Ab 2 is verified: for this, we denote by K and L the functors $M \rightsquigarrow$ Coim $\varphi(M)$ and $M \rightsquigarrow$ Im $\varphi(M)$ respectively. The canonical morphisms $\vartheta(M)$ from Coim $\varphi(M)$ to Im $\varphi(M)$ define a functorial isomorphism ϑ from K to L. Furthermore, we have the exact sequences of functors:

$$0 \to \mathrm{Ker}\ \varphi \to F \to K \to 0, \quad 0 \to L \to G \to H \to 0.$$

Since $R^0 = R \circ R$ is a left exact functor, we have the exact sequences:

$$0 \to \mathrm{Ker}\ \varphi \to F \to R^0K \to 0, \quad 0 \to R^0L \to G \to R^0H \to 0.$$

The second exact sequence shows that R^0L is none other than Imφ; in addition, R^0K is equal to Coim φ and $R^0\vartheta$ is the canonical morphism from Coim φ to Im φ: it's an isomorphism.

The rest of the proposition is clear.

Corollary 1. *The functor* $F \rightsquigarrow R^0F$ *is an exact functor from* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ *to* $\mathrm{Sex}(\mathcal{C},\mathcal{D})$*.*

We already know that $F \rightsquigarrow R^0F$ is a left exact functor. So it suffices to prove that R^0v is an epimorphism of $\mathrm{Sex}(\mathcal{C},\mathcal{D})$ if v is an epimorphism of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$. But the equality Coker $v = 0$ [in the category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$] implies, according to the lemma 3(d), the equality $R(\mathrm{Coker}\ Rv) = 0$ and $R(\mathrm{Coker}\ R(Rv)) = R(\mathrm{Coker}\ R^0v) = 0$.

When $\mathcal{C}$ has enough injective objects, the proposition 5 results also from the proposition 1: indeed, if $\mathcal{I}$ denotes the full sub-category of $\mathcal{C}$ formed of the injective objects of $\mathcal{C}$, the categories $\mathrm{Sex}(\mathcal{C},\mathcal{D})$ and $\mathrm{Fun}(\mathcal{I},\mathcal{D})$ are equivalent (cf. part I, corollary 4).

12. Left exact functors with values in the category of abelian groups

In this paragraph, $\mathcal{C}$ is always a category whose set of morphisms and the set of objects belongs to $\mathfrak{U}$. We will complete the study of the preceding paragraph when $\mathcal{D}$ is the category **Ab** of abelian groups.

If X is an object of $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X,\ .)$ is a left exact functor from $\mathcal{C}$ to **Ab** which we denote also by $\dot{X}$. With these notations, the functor $X \rightsquigarrow \dot{X}$ from C^o to $\mathrm{Fun}(\mathcal{C},\mathbf{Ab})$ is itself left exact.

Proposition 6. *The functor* $X \rightsquigarrow \dot{X}$ *is an exact functor from* $\mathcal{C}^o$ *to* $\mathrm{Sex}(\mathcal{C}, \boldsymbol{Ab})$*. The functors* $\dot{X}$ *form a family of generators of* $\mathrm{Sex}(\mathcal{C}, \boldsymbol{Ab})$*.*

We already know that $X \rightsquigarrow \dot{X}$ is a left exact functor from C^o to $\mathrm{Fun}(\mathcal{C}, \mathbf{Ab})$. It remains to prove that any monomorphism $u : X \to X'$ induces an epimorphism $\dot{u} : \dot{X}' \to \dot{X}$. This means that $R^0(\mathrm{Coker}\ \dot{u})$ or $R(\mathrm{Coker}\ \dot{u})$ is zero, which can thus be stated as: for any object A of $\mathcal{C}$ and any $a \in \dot{X}A$, there is a monomorphism $i : A \to B$ and a $b : \dot{X}'B$ such that we have

$$(\dot{X}i)(a) = \dot{u}(B)(b)$$

This last assertion is almost clear. Finally, the second part of the proposition offers no difficulty, because $\dot{X}$ already form a family of generators of $\mathrm{Fun}(\mathcal{C}, \mathbf{Ab})$.

Corollary 2. *Any injective object* F *of* $\mathrm{Sex}(\mathcal{C}, \boldsymbol{Ab})$ *is an exact functor from* $\mathcal{C}$ *to* $\boldsymbol{Ab}$*.*

Indeed, if F is an injective object, the functor $X \rightsquigarrow FX$ is the composite of two exact functors $X \rightsquigarrow \dot{X}$ and $G \rightsquigarrow \mathrm{Hom}(G, F)$.

Corollary 3. *In the category* $\mathrm{Sex}(\mathcal{C}, \boldsymbol{Ab})$ *any sub-object of a representable functor is an upper bound of representable functors.*

Indeed let Y be an object of $\mathcal{C}$, and let $i : F \to \dot{Y}$ be a non zero monomorphism of $\mathrm{Sex}(\mathcal{C}, \mathbf{Ab})$. According to the proposition 6, there is a non zero morphism $\varphi : \dot{X} \to F$; in addition $\mathrm{Im}\ \varphi = \mathrm{Im}\ i \circ \varphi$ is representable, which proves that F contains non zero representable sub-objects. I claim that the upper bound G of these sub-objects is F: otherwise we can choose φ in such way that $\mathrm{Im}\ \varphi$ is not a sub-object of G: this is absurd.

13. Noetherian categories

Let $\mathcal{C}$ be an abelian category. An object M of $\mathcal{C}$ is said to be *noetherian* (resp. *artinian*) if any ascending sequence (resp. descending) of sub-objects of M is stationary. The following lemma is easy to prove:

Lemma 10. *Let* M *be an object of* $\mathcal{C}$ *and let* N *be a sub-object of* M*. Then* M *is noetherian (resp. artinian) if and only if* N *and* M/N *are noetherian (resp. artinian).*

Lemma 11. *Let* $\mathcal{C}$ *be a category with generators and exact inductive limits, let* M *be a noetherian object of* $\mathcal{C}$ *and let* $(P_i)_{i \in I}$ *be an ascending filtering family of sub-objects of an object* P $(I \in \mathfrak{U})$*. Then the map*

$$\varphi_M : \sup_i \mathrm{Hom}(M, P_i) \to \mathrm{Hom}(M, \sup_i P_i)$$

is bijective.

It suffices to show that φ_M is surjective. So let u be a morphism from M to $\sup_i P_i$. The equality $u^{-1}(\sup_i P_i) = \sup_i u^{-1}(P_i)$ shows that $u^{-1}(P_i)$ is equal to M for some i large enough. This proves the lemma.

We prove in the same way the dual statement:

Lemma 12. *Let $\mathcal{C}$ be a category with cogenerators and exact projective limits, let M be an artinian object of $\mathcal{C}$ and let $(P_i)_{i\in I}$ be an decreasing filtering family of sub-objects of an object P ($I \in \mathfrak{U}$). Then the map*

$$\varphi_M : \sup_i \operatorname{Hom}(P/P_i, M) \to \operatorname{Hom}(P/\inf_i P_i, M)$$

We will now say that *an abelian category $\mathcal{C}$ is noetherian* (resp. *artinian*) if the following conditions are satisfied:

- Any object of $\mathcal{C}$ is noetherian (resp. artinian).
- There is a family $(M_i)_{i\in I}$ of objects of $\mathcal{C}$ such that I is an element of $\mathfrak{U}$ and that any object of $\mathcal{C}$ is isormorphic to an object of this family.

The second condition simply ensures that the universe $\mathfrak{U}$ was chosen large enough. It implies that any noetherian category (resp. artinian) is equivalent to a noetherian category (resp. artinian) whose set of objects belong to $\mathfrak{U}$.

An abelian category which is at the same time noetherian and artinian is called *finite*: all the objects of such a category are of finite length.

We say that an abelian category $\mathcal{C}$ is *locally noetherian* if the following conditions are satisfied:

- $\mathcal{C}$ is a category with exact inductive limits.
- There is a family $(M_i)_{i\in I}$ of noetherian generators of $\mathcal{C}$ whose set of indices I is an element of $\mathfrak{U}$.

The second condition means that the noetherian objects of $\mathcal{C}$ form a noetherian category, and that any object M is the upper bound of the noetherian sub-objects. If there is a family $(M_i)_{i\in I}$ of generators of finite length, we say that $\mathcal{C}$ is *locally finite*.

Now let $\mathcal{C}$ be an artinian category and suppose, for simplicity, that the set $\mathcal{OC}$ of objects of $\mathcal{C}$ is an element of $\mathfrak{U}$. If X is an object of $\mathcal{C}$, any increasing filtering family of representable sub-objects of $\dot{X}$ contains a maximal element. The corollary 3 implies that any sub-object of $\dot{X}$ belonging to $\operatorname{Sex}(\mathcal{C}, \mathbf{Ab})$ is representable. Then any quotient of a representable functor is representable, any representable functor is noetherian, any functor is an upper bound of representable functors, any noetherian functor is representable.

Proposition 7. *If $\mathcal{C}$ is an artinian category,* $\operatorname{Sex}(\mathcal{C}, \boldsymbol{Ab})$ *is a locally noetherian category.*

We have already established the proposition when the set $\mathcal{OC}$ belongs to $\mathfrak{U}$. In the general case $\mathcal{C}$ is equivalent to an artinian category C' such that $\mathcal{OC}'$ belongs to $\mathfrak{U}$. Moreover the category $\operatorname{Sex}(\mathcal{C}, \mathbf{Ab})$ is equivalent to $\operatorname{Sex}(\mathcal{C}', \mathbf{Ab})$. The proposition results

Theorem 1 (cf. [11], [19])**.** *Let $\mathcal{C}$ be a noetherian category. There is a locally noetherian category $\mathcal{D}$ such that $\mathcal{C}$ is equivalent to the category of noetherian objects of $\mathcal{D}$. Furthermore, this condition determines $\mathcal{D}$ up to an equivalence.*

Indeed, the dual category $\mathcal{C}^o$ is artinian. It results from the preceding remark that $\mathrm{Sex}(\mathcal{C}^o, \mathbf{Ab})$ is a locally noetherian and that the functor $X \rightsquigarrow \mathrm{Hom}_{\mathcal{C}}(., X)$ defines an equivalence from $\mathcal{C}$ to the category formed of noetherian objects of $\mathrm{Sex}(\mathcal{C}^o, \mathbf{Ab})$.

It remains to show the uniqueness of $\mathcal{D}$ by showing that $\mathcal{D}$ is necessarily equivalent to the category $\mathrm{Sex}(\mathcal{C}^o, \mathbf{Ab})$. We suppose for simplicity that $\mathcal{C}$ is the category of noetherian objects of $\mathcal{D}$. We then associate to each object X of $\mathcal{D}$ the left exact functor $Y \rightsquigarrow \mathrm{Hom}_{\mathcal{D}}(Y, X)$ from $\mathcal{C}^o$ to $\mathbf{Ab}$; this defines a functor $T : X \rightsquigarrow \mathrm{Hom}_{\mathcal{D}}(., X)$ from $\mathcal{D}$ to $\mathrm{Sex}(\mathcal{C}^o, \mathbf{Ab})$. We finish the proof by showing that T defines an equivalence:

- For this, we show first that any object F of $\mathrm{Sex}(\mathcal{C}^o, \mathbf{Ab})$ is isomorphic to a functor of type $TX = \mathrm{Hom}_{\mathcal{D}}(., X)$: this is obviously true when F is noetherian; in the general case, the set of noetherian sub-objects can be indexed by a set belonging to $\mathfrak{U}$. It then results in the existence of an inductive system of noetherian objects of $\mathcal{D}$, let $(X_i, u_{ji})_{j,i \in J}$ be such that J is an element of $\mathfrak{U}$, that the u_{ji} are monomorphisms, and that F is isomorphic to the inductive limit of the functors TX_i. If X is the inductive limit of the X_i, the equality
$$T(\sup_i X_i) = \sup_i TX_i \quad \text{(lemma 11)}$$
shows that F is isomorphic to TX.
- We then show the equality $\mathrm{Hom}_{\mathcal{D}}(X, Z) = \mathrm{Hom}(TX, TZ)$: if X and Z are noetherian, this results from the corollary of the proposition 1 (part I). We go from there to the case where X is noetherian and Z is arbitrary: if Z' is a noetherian sub-object of Z, we have the diagram
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(X, Z') & \xrightarrow{\mathrm{Hom}(X, i^Z_{Z'})} & \mathrm{Hom}_{\mathcal{D}}(X, Z) \\ {\scriptstyle T(X,Z')}\downarrow & & \downarrow{\scriptstyle T(X,Z)} \\ \mathrm{Hom}(TX, TZ') & \xrightarrow{\mathrm{Hom}(TX, Ti^Z_{Z'})} & \mathrm{Hom}(TX, TZ) \end{array}$$
The map $T(X, Z')$ is an isomorphism. By passing to the inductive limit of the noetherian sub-objects of Z, we see that $T(X, Z)$ is bijective (lemma 11). The general case is then proved by a similar argument [if X and Z are arbitrary, consider the projective system formed of $\mathrm{Hom}(X', Z)$ where X' runs through the noetherian sub-objects of X].

Lemma 13. *A functor F of* $\mathrm{Sex}(\mathcal{C}^o, \boldsymbol{Ab})$ *is exact if and only if F is an injective object of* $\mathrm{Sex}(\mathcal{C}^o, \boldsymbol{Ab})$. *(We always suppose the category $\mathcal{C}$ is noetherian.)*

The corollary 2 proves half of this lemma. Conversely, suppose that F is exact. We have seen that $\dot{X}$ forms a family of generators and that any sub-object of $\dot{X}$ is representable ($\mathcal{C}$ is a noetherian category). Any monomorphism $i : G \to \dot{X}$ thus induces a surjection from $FX = \mathrm{Hom}(\dot{X}, F)$ to $\mathrm{Hom}(G, F)$. The assertion results therefore from a lemma of GROTHENDIECK ([10], lemma 1 of the theorem 1.10.1).

In the equivalence between $\mathcal{D}$ and $\mathrm{Sex}(\mathcal{C}^o, \mathbf{Ab})$, the injective objects of $\mathcal{D}$ correspond to the exact functors and the inductive limits of $\mathcal{D}$ correspond to the inductive limit of functors. We then deduce the following corollary.

Corollary 4. *Any inductive limit of injective objects of $\mathcal{D}$ is an injective object. If $(X_i, u_{ji})_{i,j\in I}$ is an inductive system of $\mathcal{D}$ and if Y is a noetherian object, we have the equality*

$$\mathrm{Hom}_{\mathcal{D}}(Y, \varinjlim_i X_i) = \varinjlim_i \mathrm{Hom}_{\mathcal{D}}(Y, X_i)$$

Corollary 5. *Any locally noetherian category is a category with projective limits.*

Because it is so for the category $\mathrm{Sex}(\mathcal{C}^o, \mathbf{Ab})$.

We leave it to the reader to formulate the dual statements of the preceding statements.

14. INJECTIVE ENVELOPES IN ABELIAN CATEGORIES

Let $\mathcal{C}$ be an abelian category and let M be an object of $\mathcal{C}$. An essential extension of M is a monomorphism $i : M \to P$ which satisfies the following condition: if f is a morphism of source P and if $f \circ i$ is a monomorphism, then f is a monomorphism.

It's the same to say that a sub-object Q of P is zero if $Q \cap i(M)$ is zero. If M is a sub-object of P, and if i is the canonical monomorphism from M to P, we say that P is an essential extension of M.

Lemma 14. *Let i and j be two monomorphisms, $i : M \to P$, $j : P \to Q$. Then $j \circ i$ is an essential extension if and only if j is an essential extension of P and i is an essential extension of M.*

Indeed, suppose that $j \circ i$ is an essential extension of M: if f is a morphism of source Q and if $f \circ j$ is a monomorphism, then $f \circ j \circ i$ is a monomorphism; we can say that f is a monomorphism and that j is an essential extension of P.

Now let R be a sub-object of P such that $R \cap i(M) = 0$; let f be the canonical morphism from Q to $Q/j(R)$. Under these conditions, $f \circ j \circ i$ is a monomorphism and R is zero. This shows that i is an essential extension of M.

The converse is clear.

Lemma 15. *Let $i : M \to P$, $j : N \to Q$ be two essential extensions. The morphism $i \oplus j : M \oplus N \to P \oplus Q$ is an essential extension of $M \oplus N$.*

We may suppose, for simplicity, that M and N are two sub-objects of P and Q, with $i = i_M^P$, $j = i_N^Q$. Let R be a non zero sub-object of $P \oplus Q$. Then one of the objects $q_P(R)$ or $q'_Q(R)$ is not zero. If $q_P(R)$ is not zero, so are $q_P(R) \cap M$ and $R_1 = q_P^{-1}(M) \cap R$. If R_1 is contained in $q_Q'^{-1}(N)$, we showed that $R \cap (M \oplus N)$ is non zero. Otherwise, $q'_Q(R_1)$ is not zero and 'intersects' with N; the sub-object $R_2 = q_Q'^{-1}(N) \cap R_1$ cannot be zero. In all the cases, a non zero sub-object of $P \oplus Q$ 'intersects' with $M \oplus N$, which was needed to prove.

A sub-object N of M is said to be *irreducible in* M if N is different from M and if, for any couple (P, Q) of sub-objects of M contained in N, the relation $P \cap Q = N$ implies $P = N$ or $Q = N$. It's the same to say that M/N is an essential extension of any non zero sub-object. If O is irreducible in M, we also say that M is *coirreducible*.

Consider an object M of $\mathcal{C}$ and r sub-objects N_1, N_2, $\cdots$, N_r. Let p_l be the canonical epimorphism from M to M/N_l and let q_l be the canonical projection from $M/N_1 \oplus M/N_2 \oplus \cdots \oplus M/N_r$ to the $l-$th factor M/N_l. There is one morphism u from M to $M/N_1 \oplus \cdots \oplus M/N_r$ and only one such that $q_l \circ u$ is equal to p_l for any l.

Lemma 16. *If the sub-objects N_l are irrducible in M, if O is the intersection of the N_l and is not the intersection of less than r of the N_l, then u is an essential extension of M.*

Indeed let P_l be the intersection of N_m for $m \neq l$. The morphisms $q_m \circ u$ cancel P_l if $m \neq l$ and $q_l \circ u$ induces a monomorphism u_l from P_l to M/N_l. The canonical monomorphisms from P_l to M induce a morphism v from the direct sum $\oplus_l P_l$ to M; moreover, it is clear that $u \circ v$ is none other than the morphism $\oplus_l u_l$ from $\oplus_l P_l$ to $\oplus_l M/N_l$. Since u_l is an essential extension of P_l for any l, the previous lemma shows that $\oplus_l u_l$, u and v are essential extensions.

If I is an injective object of $\mathcal{C}$, any monomorphism $u : I \to P$ induces an isomorphism from I to a direct factor of P. If Q is a complement of $u(I)$, the intersection $Q \cap u(I)$ is zero. It follows that u can only be an essential extension of I if u is an isomorphism. We will now say that a monomorphism $i : M \to I$ is an *injective envelope* of M if i is an essential extension of M and if I is an injective object of $\mathcal{C}$.

Proposition 8. *Let $i : M \to I$ be an injective envelope of M and let $j : M \to J$ be a monomorphism. The following conditions are equivalent:*

a. *The morphism j is an injective envelope of M*
b. *The morphism j is an essential extension of M and any essential extension of J is an isomorphism.*
c. *The object J is injective; moreover, for any monomorphism h from M to an injective H and for any morphism l from H to J such that $l \circ h = j$, l is an epimorphism.*

If these conditions are satisfied, there is an isomorphism u from J to I such that $u \circ j = i$.

The implication $(a) \Rightarrow (b)$ is clear. Prove that (b) implies (a): since I is injective, there is a morphism u from J to I such that $u \circ j = i$; since j is an essential extension, u is a monomorphism, hence an essential extension according to the lemma 14; it follows that u is an isomorphism.

Prove that (a) implies (c): since H is injective, there is a morphism v from I to H such that $v \circ i = h$. The morphism $l \circ v$ is a monomorphism from I to J, hence an essential extension of I, hence an isomorphism. It follows that l is an epimorphism.

The last assertion has been proved along the way.

If $i : M \to I$ is an injective envelope of M, the previous proposition shows that the object I is determined up to an isomorphism. Although in general this isomorphism is not unique, we will sometimes say improperly that I is the injective envelope of M. If $j : N \to J$ is another injective envelope, we say that M and N have the same envelope if I is isomorphic to J.

Proposition 9. *Let $M_1, \cdots, M_r$ be r objects of $\mathcal{C}$; for any integer i between 1 and r, let $u_i : M_i \to I_i$ be an injective envelope of M_i. Then the morphism $\oplus_i u_i$ from $\oplus_i M_i$ to $\oplus_i I_i$ is an injective envelope of $\oplus_i M_i$.*

Indeed, any direct product of injective objects is an injective object: this shows that $\oplus_i I_i$ is an injective object. Moreover, the lemma 15 shows that $\oplus_i u_i$ is an essential extension.

In the following corollary we use the notations of the lemma 16.

Corollary 6. *Suppose the hypothesis of the lemma 16 is satisfied and for any i let u_i be an injective envelope of M/N_i. Then $(\oplus_i u_i) \circ u$ is an injective envelope of M.*

The previous corollary discovers the connection between the notion of injective envelope and the decomposition of O as an intersection of irreducible sub-modules in M.

We will now say that $\mathcal{C}$ is a *category with injective envelopes* if, for any object M, there is an injective envelope of M.

Proposition 10. *The following two assertions are equivalent:*

a. *$\mathcal{C}$ is a category with injective envelopes.*
b. *For any object M of $\mathcal{C}$, there is a monomorphism from M to an injective object; furthermore, if M is a sub-object of an object N, there is a sub-object Q of N, such that $M \cap Q$ is zero, and which is maximal among the sub-objects of N satisfying this condition.*

$(a) \Rightarrow (b)$: Because if M is a sub-object of N and if $i : M \to I$ is an injective envelope of M, there is a morphism φ from N to I such that $i = \varphi \circ i_M^N$. It suffices then to put $Q = \text{Ker } \varphi$.

$(b) \Rightarrow (a)$: Indeed let j be a monomorphism from M to an injective J; let Q be a sub-object of J which is maximal for the equality $Q \cap j(M) = O$.

Similarly, let I be a sub-object of J, containing M and maximal for the equality $I \cap Q = O$ (to see that a such I exist, consider the quotient J/M). If p is the canonical epimorphism from J to J/Q, $p \circ j$ is an essential extension of M and p induces an isomorphism from I to $p(I)$; the isomorphism conversely extends to a monomorphism h from J/Q to J; then $h(J/Q)$ is an essential extension of I. Thus I is equal to $h(J/Q)$; we have $I + Q = J$, $I \cap Q = O$ and the monomorphism from M to I which is induced by j is an injective envelope of M.

Proposition 11 ([17])**.** *Let $\mathcal{C}$ be an abelian category with injective envelopes, let M be a non zero object of $\mathcal{C}$ and let $i : M \to I$ be an injective envelope of M. The following assertions are equivalent:*

a. *M is coirreducible.*
b. *I is indecomposable.*
c. *I is the injective envelope of any non zero sub-object of I.*
d. *The ring of endomorphisms of I is a local ring.*

The equivalence of the assertions (a), (b), (c) is clear. We simply prove that (b) and (c) implies (d); indeed, if $u : I \to I$ is a monomorphism, $u(I)$ is a direct factor of I; the assertion (b) thus implies that u is an automorphism. Therefore it suffices to show that Ker $u \neq O$ and Ker $v \neq O$ implies Ker $(u + v) \neq O$: but according to (c), Ker $u \cap$ Ker $v \neq O$ and Ker $(u + v)$ contains this intersection.

Corollary 7. *Let M and N be two irreducible objects of a category with injective envelopes. Then M and N have the same injective envelope if and only if M contains a sub-object M', non zero and isomorphic to a sub-object N' of N.*

If N is a sub-object of M, we sometimes call *complement* of N in M any sub-object Q of M which is maximal for the equality $M \cap Q = O$. If such a Q exist, the complements of Q which contains N are the elements maximal of the set of sub-objects of M which are essential extensions of N. This situation is often exploited. In particular the reader verifies that the direct factors of an injective object I coincide with the sub-objects of I 'which are complement of one of their complements'.

Proposition 12. *Let $\mathcal{C}$ be an abelian category with injective envelopes, let M be an object of $\mathcal{C}$ and let N be a sub-object of M. Two complements of N in M have the same injective envelope. If Q is a complement of N in M, the injective envelope of M is isomorphic to the direct sum of the injective envelope of N and the injective envelope of Q.*

Indeed let Q and Q' be two complements of N in M, and let P be a maximal element of the set of sub-objects of M which are essential extensions of N. Then P is a complement to both Q and Q'. It follows that Q and Q' have the same injective envelope as M/P, which proves the first assertion. On the other hand, $N + Q$ is isomorphic to the direct sum of $N \oplus Q$ and M

is an essential extension of $N + Q$. It follows that M has the same injective envelope as $N \oplus Q$; hence the proof thanks to proposition 9.

We leave it to the reader the research of the dual of the preceding statements. Let's just say that an *essential covering* of M is an epimorphism $p : P \to M$ which satisfies the following condition: if f is a morphism of target P, and if $p \circ f$ is an epimorphism, then f is an epimorphism. It's the same to say that a sub-object Q' of P is equal to P providing that $Q' + \operatorname{Ker} p$ is equal to P.

The dual notion of that of injective envelope is the notion of *projective cover*. Similarly, if M is the quotient of P by a sub-object M', a complement of M is a quotient P/Q' such that $Q' + M'$ is equal to P and that Q' is minimal among the sub-objects of P satisfying this condition.

15. Categories with generators and exact inductive limits

In this paragraph, we suppose that $\mathcal{C}$ is an abelian category with generators and exact inductive limits; the letter U denotes a generator of $\mathcal{C}$.

Lemma 17. *Any sub-object M of an object N has a complement Q in N.*

Indeed, let E be the set of sub-objects P of N such that $M \cap P = O$. It suffices to show that E is an inductive ordered set: for this, let F be a totally ordered sub-set of E. Since there is a set belonging to $\mathfrak{U}$ and having the same cardinal as F, the upper bound S of the elements P of F is defined. Furthermore we have the formulas

$$S \cap M = (\sup_{P \in F}) \cap M = \sup_{P \in F} (P \cap M) = O$$

This proves the lemma.

Theorem 2. *Any abelian category with generators and exact inductive limits is a category with injective envelopes.*

For any object M of $\mathcal{C}$ there is indeed a monomorphism from M to an injective object: this fact is proved in [10], theorem 1.10.1. It remains valid under our assumptions. Thus the theorem results from the proposition 10 and from the lemma 17.

Proposition 13. *Let I be an ordered set belonging to $\mathfrak{U}$. Suppose we are given two increasing maps $\alpha \to M_\alpha$, $\alpha \to N_\alpha$ from I to the set of sub-objects of an object P; suppose that M_α is a sub-object of N for each α, and that N_α is an essential extension of M_α. Then $\sup_\alpha N_\alpha$ is an essential extension of $\sup_\alpha M_\alpha$.*

Indeed let Q be a non zero sub-object of $\sup_\alpha N_\alpha$. The formula

$$Q = Q \cap \sup_\alpha N_\alpha = \sup_\alpha (Q \cap N_\alpha)$$

shows that $Q \cap N_\alpha$ is non zero for at least one α. It follows that

$$Q \cap M_\alpha = (Q \cap N_\alpha) \cap M_\alpha$$

is not zero. Thus the sub-object Q intersects $\sup_\alpha M_\alpha$ with a non zero sub-object.

Corollary 8. *If J is a set belonging to $\mathfrak{U}$, and if $(u_i)_{i \in J}$ is a family of essential extensions, then $\Sigma_i u_i$ is an essential extension.*

The corollary has already been proved when J is finite. We go from there to the general case thanks to the proposition 13.

We end this paragraph with an application of the previous theorem, or rather the theorem 1.10.1 of [10]. The statement and the proof are due to students of Eilenberg.

Theorem 3. *Let $\mathcal{C}$ be an abelian category whose set of objects is an element of $\mathfrak{U}$. There is an exact and faithful functor from $\mathcal{C}$ to the category **Ab** of abelian groups.*

Recall that an exact functor is said to be *faithful* if Ff is non zero for any non zero morphism f. It is the same to say that FX is not zero if X is a non zero object.

We have seen that $\mathrm{Sex}(\mathcal{C}, \mathbf{Ab})$ is an abelian category with exact inductive limits. Furthermore, the functor

$$Y \rightsquigarrow \sum_{X \in \mathcal{O}\mathcal{C}} \mathrm{Hom}(X, Y)$$

is a projective generator U of $\mathrm{Sex}(\mathcal{C}, \mathbf{Ab})$. If $i : U \to F$ is a monomorphism from U to an injective F, the corollary 2 shows that F is an exact functor. In addition $i(Y)$ is, for any object Y of $\mathcal{C}$, a n injective map from $U(Y)$ to $F(Y)$; since UY is not zero, neither is FY. This proves that F is faithful.

The theorem 3 is frequently used to extend to the abelian categories of the results proved for abelian groups; we give an example: consider in $\mathcal{C}$ the exact and commutative diagram (★).

$$(\bigstar) \qquad \begin{array}{ccccccc} & & A & \xrightarrow{u} & B & \xrightarrow{v} & C \\ & & \downarrow a & & \downarrow b & & \downarrow c \\ 0 & \longrightarrow & A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' \end{array}$$

The morphisms u and v induce morphisms $u'' : \mathrm{Ker}\ a \to \mathrm{Ker}\ b$ and $v'' : \mathrm{Ker}\ b \to \mathrm{Ker}\ c$. We claim that the sequence (★★) is exact:

$$(\bigstar\bigstar) \qquad \mathrm{Ker}\ a \xrightarrow{u''} \mathrm{Ker}\ b \xrightarrow{v''} \mathrm{Ker}\ c.$$

Since F is an exact and faithful functor, it suffices to verify the exactness of the sequence (★★★):

$$(\bigstar\bigstar\bigstar) \qquad F\mathrm{Ker}\ a \xrightarrow{Fu''} F\mathrm{Ker}\ b \xrightarrow{Fv''} F\mathrm{Ker}\ c.$$

However, the sequence (★★★) can be constructed from (★★★★) just like (★★) has been constructed from (★); thus it suffices to prove our assertion when (★) is a diagram of abelian groups, which is easy to do.

$$(\bigstar\bigstar\bigstar\bigstar)\qquad \begin{array}{ccccccc} & & FA & \xrightarrow{Fu} & FB & \xrightarrow{Fv} & FC \\ & & \downarrow{\scriptstyle Fa} & & \downarrow{\scriptstyle Fb} & & \downarrow{\scriptstyle Fc} \\ 0 & \longrightarrow & FA' & \xrightarrow{Fu'} & FB' & \xrightarrow{Fv'} & FC' \end{array}$$

Part 3. Localization of abelian categories

Let E be a topological space, U be an open subset of E, $\mathcal{A}$ the category of sheaves of abelian groups over E and let $\mathcal{B}$ be the category of sheaves of abelian groups over U. Let $T : \mathcal{A} \to \mathcal{B}$ be the functor $M \rightsquigarrow M|U$, where $M|U$ denotes the restriction of M to U; finally let S be the direct image functor from $\mathcal{A}$ to $\mathcal{B}$. It is well known that T is exact, that S is adjoint to T and that $T \circ S$ is isomorphic to the identity functor of $\mathcal{B}$. We study here the functors $T : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{A}$ which have the properties that we have just stated, $\mathcal{A}$ and $\mathcal{B}$ being replaced by any abelian categories. The results of this part will be used at part IV.

The categories we consider in this part are not necessarily $\mathfrak{U}-$categories. However, we will use the results of the part I by replacing, if needed, the universe $\mathfrak{U}$ with a larger universe.

16. Quotient categories

Recall that a full sub-category $\mathcal{C}$ of an abelian category $\mathcal{A}$ is called *épaisse* if the following condition is satisfied [10]: for any exact sequence of $\mathcal{A}$ of the form (★), M is an object of $\mathcal{C}$ if and only if M' and M'' are objects of $\mathcal{C}$:

$$(\bigstar) \quad 0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$$

The data of $\mathcal{A}$ and of $\mathcal{C}$ allows to construct a new abelian category that we denote by $\mathcal{A}/\mathcal{C}$ which we call the *quotient category of* $\mathcal{A}$ *by* $\mathcal{C}$ [10]:

- The objects of $\mathcal{A}/\mathcal{C}$ coincide with the objects of $\mathcal{A}$.
- If M and N are two objects of $\mathcal{A}$, M' and N' two sub-objects of M and N, the canonical morphisms from M' to M and from N' to N define a linear map

  $$\mathrm{Hom}_{\mathcal{A}}(i^M_{M'}, p^N_{N/N'}) : \mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{\mathcal{A}}(M', N/N').$$

  When M' and N' go through the sub-objects of M and N such that M/M' and N' are objects of $\mathcal{C}$, the abelian groups $\mathrm{Hom}_{\mathcal{A}}(M', N/N')$ clearly define an inductive system. We will put by definition:

  $$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) = \varinjlim_{M', N'} \mathrm{Hom}_{\mathcal{A}}(M', N/N').$$

  The set $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ is hence provided with a structure of abelian group.
- It remains to define the law of bilinear composition:

  $$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) \times \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(N, P) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, P)$$

  For this, let $\overline{f}$ be an element of $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ and let $\overline{g}$ be an element of $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(N, P)$. The element $\overline{f}$ is the image of a morphism $f : M' \to N/N'$ where M/M' and N' are objects of $\mathcal{C}$. Similarly, $\overline{g}$ is the image of a morphism $f : N'' \to P/P'$ where N/N'' and P' are objects of $\mathcal{C}$. If M'' denotes the inverse image $f^{-1}((N'' + N')/N')$, it is easy to see that M/M'' belongs to $\mathcal{C}$; we denote by f' the morphism from M'' to $N''+N'/N'$ which is induced by f. Similarly, $g(N''\cap N')$ is an object of $\mathcal{C}$; if P'' denotes the sum $P'+g(N''\cap N')$, it is easy to see that P'' belongs to $\mathcal{C}$; we denote by g' the morphism from $N''/N''\cap N'$ to P/P'' which is induced by g.

Let h be the composite of f', the canonical isomorphism from $N''+N'/N'$ to $N''/N''\cap N'$ and of g'. The image $\overline{h}$ of h in $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M,P)$ depends only on $\overline{f}$ and $\overline{g}$ and not of f and g. It is thus lawful to define the law of composition of $\mathcal{A}/\mathcal{C}$ by the equality $\overline{g}\circ\overline{f}=\overline{h}$. This law of composition is bilinear; they make $\mathcal{A}/\mathcal{C}$ a category.

In the following, the letter T denotes the functor (called *canonical*) from $\mathcal{A}$ to $\mathcal{A}/\mathcal{C}$ which is defined in the following way: $TM=M$ for any object M of $\mathcal{A}$; if $f:M\to N$ is a morphism of $\mathcal{A}$, Tf is the image of f in the inductive limit $\varinjlim \mathrm{Hom}_{\mathcal{A}}(M',N/N')$.

Lemma 1. *$\mathcal{A}/\mathcal{C}$ is an additive category and T is an additive functor from $\mathcal{A}$ to $\mathcal{A}/\mathcal{C}$.*

This lemma is a direct consequence of the propositions 2 and 3 of part I.

Lemma 2. *Let $u:M\to N$ be a morphism of $\mathcal{A}$. The morphism Tu is zero (resp. a monomorphism, resp. an epimorphism) if and only if* Im u *belongs to $\mathcal{C}$ (resp.* Ker u *belongs to $\mathcal{C}$, resp.* Coker u *belongs to $\mathcal{C}$).*

If Im u belongs to $\mathcal{C}$, the image of u in $\mathrm{Hom}_{\mathcal{A}}(M,N/\mathrm{Im}\ u)$ is indeed zero. The same goes, a fortiori, for the image of u in the inductive limit of groups $\mathrm{Hom}_{\mathcal{A}}(M',N/N')$. Conversely, if Tu is zero, we can choose the sub-objects M' and N' in such a way that the morphism from M' to N/N' which is induced by u, is zero. This means that $u(M')$ is contained in N' and belongs to $\mathcal{C}$. Since we have on the other hand an exact sequence of the form

$$0\to u(M')\to \mathrm{Im}\ u\to M/(M'+\mathrm{Ker}\ u)\to 0$$

it follows that Im u belongs to $\mathcal{C}$.

Now suppose that Tu is a monomorphism; let i be the canonical morphism from Ker u to M. Since $u\circ i$ is zero, the same goes for $Tu\circ Ti$. We can say that Ti is zero, hence that $\mathrm{Ker}\ u=\mathrm{Im}\ i\in\mathcal{C}$. Conversely, suppose that Ker u belongs to $\mathcal{C}$. Let $\overline{f}:TP\to TM$ be a non zero morphism of $\mathcal{A}/\mathcal{C}$: this morphism is the image of a morphism $f:P'\to M/M'$, where P/P' and M' belong to $\mathcal{C}$. Replacing M' by $M'+\mathrm{Ker}\ u$, we can assume in addition that M' contains Ker u. In this case, u induces a monomorphism u' from M/M' to $N/u(M')$. Since $\overline{f}$ is non zero, Im f and Im $(u'\circ f)$ do not belong to $\mathcal{C}$. This shows that $(Tu)\circ\overline{f}$ is non zero and that Tu is a monomorphism.

We prove in an analogous way the last assertion of the lemma 2.

Lemma 3. *Let $u:M\to N$ be a morphism of $\mathcal{A}$, $i:K\to M$ be the kernel of u and let $p:N\to C$ be the cokernel of u. The morphism Tu has a kernel (resp. a cokernel); furthermore, Ti (resp. Tp) induces an isomorphism from TK to the kernel of Tu (resp. from the cokernel of Tu to TC).*

We already know that Ti is a monomorphism. Thus let $\overline{f}$ be a morphism from TX to TM such that $Tu \circ \overline{f}$ is zero: we must prove the existence of a morphism $\overline{g} : TX \to TK$ such that we have $Ti \circ \overline{g} = \overline{f}$.

However, $\overline{f}$ is the image of an element f of $\mathrm{Hom}_{\mathcal{A}}(X', M/M')$ where X/X' and M belong to $\mathcal{C}$. We thus have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \xrightarrow{i} & K & \xrightarrow{u} & M & \longrightarrow & N \\
 & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle r} \\
0 & \longrightarrow & K/K \cap M' & \xrightarrow{i'} & M/M' & \xrightarrow{u'} & N/u(M')
\end{array}$$

where p, q and r are the canonical morphisms. Since $Tu \circ \overline{f}$ is zero, the image of X' under $u' \circ f$ belongs to $\mathcal{C}$. If X'' denotes the inverse image $f^{-1}(\mathrm{Im}\ i')$, it follows that X'/X'' and X/X'' belong to $\mathcal{C}$. In addition, the restriction of f to X'' is the composite of a morphism $g : X'' \to K/K \cap M'$ and i'. If $\overline{g}$ is the image of g in $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(X, K)$, we have the required equality $Ti \circ \overline{g} = \overline{f}$.

The second part of the lemma is proved in the 'dual' way.

Lemma 4. *Let $u : M \to N$ be a morphism of $\mathcal{A}$. Then Tu is an isomorphism if and only if* Ker u *and* Coker u *belong to $\mathcal{C}$.*

If Tu is an isomorphism, Tu is both a monomorphism and an epimorphism. According to the lemma 2, it follows that Ker u and Coker u are objects of $\mathcal{C}$. Conversely, suppose the latter condition is satisfied; let q be the canonical epimorphism from M to Coim u, j be the canonical morphism from Im u to N and let ϑ be the canonical isomorphism from Coim u to Im u.

The identity morphism of Coim u is an element of $\mathrm{Hom}_{\mathcal{A}}(\mathrm{Coim}\ u, M/i(K))$ (the notations are those of lemma 3). This element has an image $\overline{q}$ in $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(\mathrm{Coim}\ u, M)$ and it is clear that $\overline{q}$ is an inverse morphism of Tq. This shows that Tq is an isomorphism. In the same way, Tj is an isomorphism. It follows that $Tu = Tj \circ T\vartheta \circ Tq$ is an isomorphism.

Proposition 1. *If $\mathcal{C}$ is an épaisse sub-category of $\mathcal{A}$, the category $\mathcal{A}/\mathcal{C}$ is abelian. Furthermore, the canonical functor from $\mathcal{A}$ to $\mathcal{A}/\mathcal{C}$ is exact.*

Indeed let $\overline{f} : TM \to TN$ be a morphism of $\mathcal{A}/\mathcal{C}$. We show that $\overline{f}$ has kernel, cokernel, coimage, image and that the canonical morphism from the coimage to the image is an isomorphism.

Since $\overline{f}$ is the image of an element f of $\mathrm{Hom}_{\mathcal{A}}(M', N/N')$, where M/M' and N' are objects of $\mathcal{C}$. We deduce from the commutative diagram

$$\begin{array}{ccc}
TM & \xrightarrow{\overline{f}} & TN \\
{\scriptstyle Ti^{M}_{M'}}\big\uparrow & & \big\downarrow{\scriptstyle Tq^{N}_{N/N'}} \\
TM & \xrightarrow{Tf} & T(N/N')
\end{array}$$

where $Ti^M_{M'}$ and $Tq^N_{N/N'}$ are isomorphisms. This diagram shows that $\overline{f}$ has a kernel, cokernel, ..., if and only if so is Tf. In other words, it is permissable to assume that $\overline{f}$ is of the form Tf.

In this case the lemma 3 shows that $\overline{f}$ has kernel, cokernel, coimage, image. We denote by q, j and ϑ respectively the canonical morphisms from M to Coim f, from Im f to N and from Coim f to Im f. The lemma 3 shows that Tq induces an isomorphism q_1 from Coim Tf to T(Coim f); similarly, Tj induces an isomorphism from T(Im f) to Im Tf. Finally, it is easy to verify that the canonical morphism from Coim Tf to Im Tf is the composite $j_1 \circ T\vartheta \circ q_1$. This morphism is thus an isomorphism.

The last assertion of the proposition 5 results directly from the lemma 3.

Corollary 1. *Let $\mathcal{C}$ be an épaisse sub-category of $\mathcal{A}$ and let*

$$0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$$

be an exact sequence of $\mathcal{A}/\mathcal{C}$. Then there is a diagram of the form (1), commutative, exact, and satisfying the following conditions: u, v and w are isomorphisms of $\mathcal{A}/\mathcal{C}$; in addition, $0 \to M_1 \xrightarrow{f_1} N_1 \xrightarrow{g_1} P_1 \to 0$ is an exact sequence of $\mathcal{A}$.

$$(1)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & N & \xrightarrow{g} & P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & TM_1 & \xrightarrow{Tf_1} & TN_1 & \xrightarrow{Tg_1} & TP_1 & \longrightarrow & 0 \end{array}$$

The morphism f is indeed the image of an element $f' \in \mathrm{Hom}_{\mathcal{A}}(M', N/N')$ where M/M' and N' belong to $\mathcal{C}$. Since Tf' is a monomorphism, Ker f' belong to $\mathcal{C}$. If q denotes the canonical morphism from M' to Coim f', it is thus possible to make the following choices: $M_1 =$ Coim f'; $N_1 = N/N'$; $f_1 =$ morphism induced by f; $P_1 =$ Coker f_1; $g_1 =$ canonical morphism; $u = (Tq) \circ (Ti^M_{M'})^{-1}$; $v' = Tq^N_{N/N'}$; $w =$ morphism induced by $(Tg_1) \circ v$.

Corollary 2. *Let $\mathcal{C}$ be an épaisse sub-category of $\mathcal{A}$ and let G be an exact functor from A to an abelian category $\mathcal{D}$. If GM is zero for any object M in $\mathcal{C}$, there is one and only one functor H from $\mathcal{A}/\mathcal{C}$ to $\mathcal{D}$ such that we have $G = H \circ T$.*

Indeed let M and N be two objects of $\mathcal{A}$, M' and N' be sub-objects of M and N such that M/M' and N' belong to $\mathcal{C}$. Consider the following commutative diagram, where φ and ψ denote respectively the maps $\mathrm{Hom}_{\mathcal{A}}(i^M_{M'}, p^N_{N/N'})$ and $\mathrm{Hom}_{\mathcal{D}}(Gi^M_{M'}, Gp^N_{N/N'})$:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(M,N) & \xrightarrow{G(M,N)} & \mathrm{Hom}_{\mathcal{D}}(GM,GN) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ \mathrm{Hom}_{\mathcal{A}}(M',N/N') & \xrightarrow{G(M',N/N')} & \mathrm{Hom}_{\mathcal{D}}(GM',G(N/N')) \end{array}$$

It is clear that ψ is a bijection map; the maps $\psi^{-1} \circ G(M', N/N')$ thus define a map $H(M, N)$ from

$$\varinjlim \mathrm{Hom}_{\mathcal{A}}(M', N/N')$$

to $\mathrm{Hom}_{\mathcal{D}}(GM, GN)$. When M and N go through the objects of $\mathcal{A}$ (or of $\mathcal{A}/\mathcal{C}$), the maps $H(M, N)$ determine a functor H from $\mathcal{A}/\mathcal{C}$ to $\mathcal{D}$; this functor associate with any object M of $\mathcal{A}/\mathcal{C}$ the object GM of $\mathcal{D}$; it satisfies the equality $G = H \circ T$.

The proof of the uniqueness of H is left to the reader.

Corollary 3. *Let $\mathcal{C}$ be an épaisse sub-category of $\mathcal{A}$ and H a functor from $\mathcal{A}/\mathcal{C}$ to an abelian category $\mathcal{D}$. The functor H is exact if and only if the functor $H \circ T$ is exact.*

It is clear that $H \circ T$ is exact if H is exact. Conversely, suppose that $H \circ T$ is exact. If $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is an exact sequence of $\mathcal{A}/\mathcal{C}$, we consider a diagram of the form (1). Since the sequence

$$0 \to HTM_1 \xrightarrow{HTf_1} HTN_1 \xrightarrow{HTg_1} HTP_1 \to 0$$

is exact, so is the sequence

$$0 \to HM \xrightarrow{Hf} HN \xrightarrow{Hg} HP \to 0$$

17. Properties of the section functor

Let $\mathcal{A}$ be an abelian category, $\mathcal{C}$ be an épaisse sub-category of $\mathcal{A}$, $\mathcal{B}$ be the quotient category $\mathcal{A}/\mathcal{C}$ and let T be the canonical functor from $\mathcal{A}$ to $\mathcal{B}$.

The arguments of the previous paragraph is self-dual: it also applies both to the category $\mathcal{A}$ and to the dual category $\mathcal{A}^o$. *We will assume at the beginning of this paragraph that there is a functor S adjoint to T.* This condition is not self-dual; we leave it to the reader the task of formulating the dual of the statements which will follow.

A functor adjoint to T is defined up to an isomorphism. We choose once and for all such a functor S and we call it *section functor*. We use also the notations of part I, § 6; in particular, the letters φ, ψ, Φ, Ψ will have the following meanings: φ is a functorial isomorphism from $\mathrm{Hom}_{\mathcal{A}}(.,\ S.)$ to $\mathrm{Hom}_{\mathcal{A}}(T., .)$ and ψ is the inverse isomorphism of φ; the letter Φ (resp. the letter Ψ) denotes the functorial morphism from $T \circ S$ to $1_{\mathcal{B}}$ (resp. from $1_{\mathcal{A}}$ to $S \circ T$) which is associated to φ (resp. to ψ).

Proposition 2. *The section functor is left exact.*

The assertion is indeed verified for any (right) adjoint functor (proposition 11, part I).

Lemma 5. *If M is an object of $\mathcal{A}$, the following assertions are equivalent:*

a. *For any morphism $u : P \to Q$ such that* $\mathrm{Ker}\, u$ *and* $\mathrm{Coker}\, u$ *belong to* $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{A}}(u, M)$ *is a bijection from* $\mathrm{Hom}_{\mathcal{A}}(Q, M)$ *to* $\mathrm{Hom}_{\mathcal{A}}(P, M)$.

b. *Any sub-object M of M belong to $\mathcal{C}$ is zero; furthermore, any exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ such that P belongs to $\mathcal{C}$, splits (i.e. f induces an isomorphism from M to a direct factor of N).*
c. *For any object P of $\mathcal{A}$, $T(P, M)$ is a bijection from $\mathrm{Hom}_{\mathcal{A}}(P, M)$ to $\mathrm{Hom}_{\mathcal{B}}(TP, TM)$.*

$(a) \Rightarrow (b)$: indeed let L be a sub-object of M belong to $\mathcal{C}$. The identity morphism 1_M is the image of an element v of $\mathrm{Hom}_{\mathcal{A}}(M/L, M)$ under the map $\mathrm{Hom}_{\mathcal{A}}(p^M_{M/L}, M)$. In other words, we have an equality of the type $1_M = v \circ p^M_{M/L}$; thus $p^M_{M/L}$ is a monomorphism and L is zero.

Similarly, there is an element g of $\mathrm{Hom}_{\mathcal{A}}(N, M)$ such that we have $1_M = g \circ f$. This proves that f is an isomorphism from M to a direct factor of N.

$(b) \Rightarrow (c)$: Since any sub-object of M belonging to $\mathcal{C}$ is zero, $\mathrm{Hom}_{\mathcal{B}}(TP, TM)$ is indeed the inductive limit of the abelian groups $\mathrm{Hom}_{\mathcal{A}}(P', M)$, when P' runs through the sub-objects of P such that P/P' belong to $\mathcal{C}$. In particular, any element of $\mathrm{Hom}_{\mathcal{B}}(TP, TM)$ is the image of a morphism $f : P' \to M$.

Denote by $M \sum_{P'} P$ the fiber sum of the diagram defined by the morphisms f and $i^P_{P'}$. The canonical morphism j_M is a monomorphism and Coker j_M is a isomorphic to P/P' (part I, end of § 5). According to the assertion (b), there is a morphism p from $M \sum_{P'} P$ to M such that we have $p \circ j_M = 1_M$; we deduce that f is equal to $(p \circ j'_P) \circ i^P_{P'}$. This proves that $T(P, M)$ is surjective.

Now let g be a non zero morphism from P to M. The image of g is non zero and does not belong to $\mathcal{C}$. It follows that Tg is non zero (lemma 2, § 1). This proves that $T(P, M)$ is injective.

$(c) \Rightarrow (a)$: indeed we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{A}}(P, M) & \xrightarrow{T(P,M)} & \mathrm{Hom}_{\mathcal{B}}(TP, TM) \\
{\scriptstyle \mathrm{Hom}(u,M)} \uparrow & & \uparrow {\scriptstyle \mathrm{Hom}(Tu,TM)} \\
\mathrm{Hom}_{\mathcal{A}}(Q, M) & \xrightarrow{T(Q,M)} & \mathrm{Hom}_{\mathcal{B}}(TQ, TM)
\end{array}$$

The maps $T(P, M)$ and $T(Q, M)$ are bijective. If Ker u and Coker u belong to $\mathcal{C}$, Tu is an isomorphism (lemma 4, § 1). Thus the same is true for $\mathrm{Hom}(u, M)$.

We will say from now on that the object M is $\mathcal{C}-$closed if it satisfies the equivalent conditions of the previous lemma. Our purpose is to prove that M is $\mathcal{C}-$ closed if and only if the morphism $\Psi(M)$ from M to STM is an isomorphism. For this, we will need the

Lemma 6. *For any object N of $\mathcal{A}/\mathcal{C}$, SN is $\mathcal{C}-$ closed.*

Indeed let's check the assertion (a) of the lemma 5; we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(P, SN) & \xrightarrow{\varphi(P,N)} & \mathrm{Hom}_{\mathcal{B}}(TP, N) \\ {\scriptstyle \mathrm{Hom}(u,SN)}\uparrow & & \uparrow{\scriptstyle \mathrm{Hom}(Tu,N)} \\ \mathrm{Hom}_{\mathcal{A}}(Q, SN) & \xrightarrow{\varphi(Q,M)} & \mathrm{Hom}_{\mathcal{B}}(TQ, N) \end{array}$$

The maps $\varphi(P, N)$ and $\varphi(Q, N)$ are bijective. It follows that $\mathrm{Hom}(u, SN)$ is a bijection if Tu is an isomorphism.

Proposition 3. a. *The functorial morphism Φ is an isomorphism from $T \circ S$ to $1_{\mathcal{B}}$.*

b. *For any object M of $\mathcal{A}$,* Ker $\Psi(M)$ *and* Coker $\Psi(M)$ *belong to $\mathcal{C}$.*

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{A}}(M, SN) & & \xrightarrow{\quad\varphi(M,N)\quad} & & \mathrm{Hom}_{\mathcal{B}}(TM, N) \\ & {\scriptstyle T(M,SN)}\searrow & & \nearrow{\scriptstyle \mathrm{Hom}(TM,\Phi(N))} & \\ & & \mathrm{Hom}_{\mathcal{B}}(TM, TSN) & & \end{array}$$

Prove (a): indeed let M be an object of $\mathcal{A}$ and let N be an object of $\mathcal{A}/\mathcal{C}$. We have the above commutative diagram.

The map $\varphi(M, N)$ is a bijection as well as $T(M, SN)$ (lemma 5 and 6). It follows that $\mathrm{Hom}(TM, \Phi(M))$ is a bijection for any M. The assertion (a) results from that any object of $\mathcal{A}/\mathcal{C}$ is of the form TM (corollary 2 of the proposition 1, part I).

Now let's prove (b) showing that $T\Psi(M)$ is an isomorphism for any M. However the composite of $T\Psi(M)$ and $\Phi(TM)$ are the identity morphism of TM (proposition 8, part I). Since Φ is a functorial isomorphism, the result is demonstrated.

Corollary 4. *An object M of $\mathcal{A}$ is $\mathcal{C}-$ closed if and only if $\Psi(M)$ is an isomorphism from M to STM.*

We already know that STM is $\mathcal{C}-$closed. So suppose that M is $\mathcal{C}-$closed: since any object of M belonging to $\mathcal{C}$ is zero, Ker $\Psi(M)$ is zero; this proves that $\Psi(M)$ is a monomorphism. The assertion (b) of the lemma 5 then shows that Coker $\Psi(M)$ is isomorphic to a direct factor of STM. Since any sub-object of STM belonging to $\mathcal{C}$ is zero, Coker $\Psi(M)$ is zero; this proves the corollary.

We will say from now on that an épaisse sub-category $\mathcal{C}$ of $\mathcal{A}$ is a *localizing sub-category* of $\mathcal{A}$ if there is a functor S adjoint to T. In this case, the left exact functor $S \circ T$ is called *localizing functor*. This localizing functor is exact if and only if S is exact (corollary 3 of the proposition 1).

Proposition 4. *If $\mathcal{C}$ is an épaisse sub-category of $\mathcal{A}$, the following assertions are equivalent:*

a. *$\mathcal{C}$ is a localizing sub-category of $\mathcal{A}$.*
b. *Any object M of $\mathcal{A}$ contains a sub-object which is maximal among the sub-objects of M belonging to $\mathcal{C}$; furthermore, if any sub-object of M belonging to $\mathcal{C}$ is zero, there is a monomorphism from M to a $\mathcal{C}-$ closed object.*

$(a) \Rightarrow (b)$:With the usual notations it is indeed clear that Ker $\Psi(M)$ is maximal among the sub-objects of M belonging to $\mathcal{C}$. If Ker $\Psi(M)$ is zero, $\Psi(M)$ is a monomorphism from M to a $\mathcal{C} -$ closed object. $(b) \Rightarrow (a)$:Let $u : M \to N$ be a morphism of $\mathcal{A}$. We say that u is a $\mathcal{C}-$*envelope* of M if N is $\mathcal{C} -$ closed and if Ker u and Coker u are the objects of $\mathcal{C}$. We will first show that any object M has a $\mathcal{C} -$ envelope:

Indeed let M' be the largest sub-object of M belonging to $\mathcal{C}$ and let i be a monomorphism from M/M' to a $\mathcal{C} -$ closed object R. On the other hand let N be the inverse image in R of the largest sub-object of Coker i belonging to $\mathcal{C}$. The condition (a) of the lemma 5 shows that N is $\mathcal{C} -$ closed. If j is the morphism from M/M' to N which is induced by i, then $j \circ p^M_{M/M'}$ is a $\mathcal{C} -$ envelope of M.

Now we are able to construct a functor adjoint to T: any object N of $\mathcal{A}/\mathcal{C}$ can be considered as an object of $\mathcal{A}$; we choose a $\mathcal{C} -$ envelope of this object and $u(N) : N \to SN$. Any object M of $\mathcal{A}$ then gives rise to the following maps:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{B}}(TM, N) = \mathrm{Hom}_{\mathcal{B}}(TM, TN) & \xrightarrow{\mathrm{Hom}(TM, Tu(N))} & \mathrm{Hom}_{\mathcal{B}}(TM, TSN) \\
 & & \uparrow {\scriptstyle T(M,SN)} \\
 & & \mathrm{Hom}_{\mathcal{A}}(M, SN)
\end{array}$$

These maps are bijective and define a functorial isomorphism from $\mathrm{Hom}_{\mathcal{B}}(T., N)$ to $\mathrm{Hom}_{\mathcal{A}}(., SN)$. In other words, $\mathrm{Hom}_{\mathcal{B}}(T., N)$ is a representable functor and the proposition results from the proposition 10 (part I).

Now suppose that $\mathcal{C}$ is a localizing sub-category of $\mathcal{A}$ and consider a morphism $u : M \to N$ of $\mathcal{A}$. Let M' (resp. N') be the largest sub-object of M (resp. of N) belonging to $\mathcal{C}$. Since $u(M')$ is contained in N', u induces a morphism u' from M/M' to N/N'. We then have the following assertion which will be very useful in the paragraph 3.

Lemma 7. *With the above notations and hypothesis, Tu is an essential extension of TM if and only if u' is an essential extension of M/M'.*

Since $Tp^M_{M/M'}$ and $Tp^N_{N/N'}$ are isomorphisms, Tu is an essential extension if and only if so is Tu'. In other words, if suffices to establish the proof when M' and N' are zero. We make this hypothesis in the following, and we choose the sub-objects of TN (resp. of TM) among the TP, where P is a sub-object of N (resp. of M).

Suppose that u is an essential extension of M, and let TP be a sub-object of N. If TP is non zero, it is the same for P and for $u^{-1}(P)$. This last object does not belong to $\mathcal{C}$. Since $(Tu)^{-1}(TP)$ is isomorphic to $T(u^{-1}(P))$, it follows that $(Tu)^{-1}(TP)$ is non zero and that Tu is an essential extension of TM.

Conversely, suppose that Tu is an essential extension and let P be a non zero sub-object of N. Since any sub-object of N belonging to $\mathcal{C}$ is zero, TP can not be zero. It follows that $u^{-1}(P)$ is non zero.

Lemma 8. *Let $\mathcal{C}$ be a localizing sub-category of $\mathcal{A}$ and let $(U_i)_{i\in I}$ be a family of generators of $\mathcal{A}$. Then $(TU_i)_{i\in I}$ is a family of generators of $\mathcal{A}/\mathcal{C}$.*

Indeed let $u : M \to N$ be a monomorphism of $\mathcal{A}/\mathcal{C}$, which is not an isomorphism. We must prove that the map $\prod_i \mathrm{Hom}(TU_i, u)$ from $\prod_i \mathrm{Hom}(TU_i, M)$ to $\prod_i \mathrm{Hom}(TU_i, N)$ is not surjective. However Su is not an isomorphism (prop. 3 (a)). It then follows that the map $\prod_i \mathrm{Hom}(U_i, Su)$ from $\prod_i \mathrm{Hom}(U_i, SM)$ to $\prod_i \mathrm{Hom}(U_i, SN)$ is not surjective. The lemma is a consequence of this fact and of the properties of the adjoint functors.

Lemma 9. *Let $\mathcal{C}$ be a localizing sub-category of a $\mathfrak{U}-$category with generators $\mathcal{A}$. Then $\mathcal{A}/\mathcal{C}$ is a $\mathfrak{U}-$category with generators.*

Indeed let M and N be two objects of $\mathcal{A}$. According to the proposition 5 (part I), the set of sub-objects M' of M such that M/M' belong to $\mathcal{C}$, can be indexed by a set belongs to $\mathfrak{U}$. It is the same for the set of sub-objects N' of N which belong to $\mathcal{C}$. It follows that the inductive limit $\varinjlim \mathrm{Hom}_{\mathcal{A}}(M', N/N')$ is an object of the category $\mathfrak{U}\mathbf{Ab}$. This proves that $\mathcal{A}/\mathcal{C}$ is a $\mathfrak{U}-$category.

The existence of generators follow from the lemma 8.

In practice there are a lot of localizing sub-categories. The proposition which follows can sometimes detect them: let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, T_1 be a functor from $\mathcal{A}$ to $\mathcal{B}$ and let S_1 be a functor adjoint to T_1. We keep the letters Φ_1, Ψ_1, φ_1, ψ_1 their usual meaning (cf. part I, § 7).

Proposition 5. *Suppose, with the above notations, that T_1 is an exact functor and that Φ_1 is an isomorphism from $T_1 \circ S_1$ to $1_{\mathcal{B}}$. Then* Ker T_1 *is a localizing sub-category of $\mathcal{A}$ and T_1 induces an equivalence between* $\mathcal{A}/$Ker T_1 *and $\mathcal{B}$.*

Indeed let T be the canonical functor from $\mathcal{A}$ to $\mathcal{A}/\mathrm{Ker}\ T_1$; (Ker T_1 denotes the épaisse sub-category of $\mathcal{A}$ whose objects are vanished under T_1); let R be the unique functor from $\mathcal{A}/\mathrm{Ker}\ T_1$ to $\mathcal{B}$ such that we have $T_1 = R\circ T$. If f is a morphism of $\mathcal{A}$, $T_1 f$ is an isomorphism if and only if $T_1\mathrm{Ker}\ f$ and $T_1\mathrm{Coker}\ f$ are zero, that is to say if and only if Tf is an isomorphism. We apply this remark to the particular case where f is the morphism $\Psi_1(M)$ from M to $(S_1 \circ T_1)M$. Indeed we know that the composite of $T_1\Psi_1(M)$ and $\Phi_1(T_1 M)$

is the identity morphism of T_1M. Since $\Phi_1(T_1M)$ is an isomorphism, it follows that $T_1\Psi_1(M)$, hence $T\Psi_1(M)$, are isomorphisms. On the other hand, since the objects M of $\mathcal{A}$ coincide with the objects TM of $\mathcal{A}/\text{Ker } T_1$, the morphisms $T\Psi_1(M)$ define an isomorphism from the identity functor of $\mathcal{A}/\text{Ker } T_1$ to the functor $(TS_1)R$. We conclude that $(TS_1)R$ and $R(TS_1)$ are isomorphisms from the identity functors to $\mathcal{A}/\text{Ker } T_1$ and to $\mathcal{B}$. This means that R and TS_1 define an equivalence between $\mathcal{A}/\text{Ker } T_1$ and $\mathcal{B}$.

We leave it to the reader to prove that $S_1 \circ R$ is a functor adjoint to T, which shows that Ker T_1 is a localizing sub-category of $\mathcal{A}$.

18. Categories with injective envelopes

We use the notations of the previous paragraph.

Proposition 6. *Let $\mathcal{C}$ be an épaisse sub-category of an abelian category $\mathcal{A}$. Let $i : M \to I$ be an injective envelope of an object M of $\mathcal{A}$ which does not contain any non zero sub-object in $\mathcal{C}$. Then I is $\mathcal{C}-$closed and the morphism Ti is an injective envelope of TM.*

Indeed let N be a sub-object of I belonging to $\mathcal{C}$; then $i^{-1}(N)$ belongs to $\mathcal{C}$ and is thus zero; since i is an essential extension, it follows that N is zero. This proves that I is $\mathcal{C} -$ closed [lemma 5(b)].

An argument already used in the proof of lemma 7 shows that Ti is an essential extension. It thus remains to show that TI is injective: for this suppose we are given a monomorphism $f' : TI \to TM$; this monomorphism is the image of a morphism $f : I' \to M/M'$, where I' and M' are the sub-objects of I and of M such that I/I' and M' belong to $\mathcal{C}$. Since Tf is a monomorphism, Ker f belongs to $\mathcal{C}$ and is thus zero. In other words, f is a monomorphism and the canonical morphism i from I' to I is the composite of f and a morphism $g : M/M' \to I$. This leads to the following commutative diagram:

$$\begin{array}{ccc} TI' & \xrightarrow{Tf} & T(M/M') \\ {\scriptstyle Ti}\downarrow & \overset{Tg}{\swarrow} & \uparrow{\scriptstyle Tp^M_{M/M'}} \\ TI & \xrightarrow{f'} & TM \end{array}$$

Since Ti and $Tp^M_{M/M'}$ are isomorphisms, f' is an isomorphism from TI to a direct factor of TM.

Corollary 5. *If $\mathcal{C}$ is an épaisse sub-category of a category $\mathcal{A}$ with injective envelopes, the following assertions are equivalent:*

a. *$\mathcal{C}$ is a localizing sub-category of $\mathcal{A}$.*
b. *Any object M of $\mathcal{A}$ contains a sub-object which is maximal among the sub-objects of M belonging to $\mathcal{C}$.*

The corollary results from the propositions 6 and 4.

Corollary 6. *Let $\mathcal{C}$ be a localizing sub-category of a category $\mathcal{A}$ with injective envelopes. Then $\mathcal{A}/\mathcal{C}$ is a category with injective envelopesl any injective object of $\mathcal{A}/\mathcal{C}$ is isomorphic to an object TI where I is an injective object of $\mathcal{A}$ not containing any non zero sub-object of $\mathcal{C}$. Conversely, any injective object J of $\mathcal{A}$ is isomorphic to a direct sum $J_1 \oplus SJ_2$ where J_1 is the injective envelope of an object of $\mathcal{C}$ and J_2 is an injective envelope of $\mathcal{A}/\mathcal{C}$. Furthermore, these conditions determine J_1 and J_2 up to isomorphisms.*

According to the corollary 5, any object of $\mathcal{A}/\mathcal{C}$ is indeed isomorphic to an object TM, where M does not contain any non zero sub-object of $\mathcal{C}$. The first two assertions thus result directly from the proposition 6. To prove the rest, we choose for J_1 the injective envelope of the largest sub-object of J belonging to $\mathcal{C}$.

Corollary 7. *The hypothesis are those of corollary 6. We suppose in addition that the injective envelope (in $\mathcal{A}$) of an object of $\mathcal{C}$ belongs to $\mathcal{C}$. Then TJ is an injective object of $\mathcal{A}/\mathcal{C}$ if J is an injective object of $\mathcal{A}$.*

With the notations of the corollary 6, TJ is actually reduced to TSJ_2.

The corollary 7 is often used in the following way: let $F : \mathcal{A}/\mathcal{C} \to \mathcal{D}$ be a functor from $\mathcal{A}/\mathcal{C}$ to an abelian category $\mathcal{D}$ with $\mathcal{C}$ satisfying the conditions of the corollary 7. Since the functor T is exact and transforms injectives to injectives, we know [10] that the derived functors $R^n(F \circ T)$ are identified with the functors $(R^nF) \circ T$. This is true in particular if F is the functor $N \rightsquigarrow \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(TM, N)$. We deduce the corollary 8 and 9:

Corollary 8. *The hypothesis are those of the corollary 7. The functor $N \rightsquigarrow \mathrm{Ext}^n_{\mathcal{A}/\mathcal{C}}(TM, TN)$ is isomorphic to the $n-$th derived functor of hte functor $N \rightsquigarrow \mathrm{Hom}_{\mathcal{A}}(M, STN)$.*

Corollary 9. *The hypothesis are those of the corollary 7. The homological dimension of $\mathcal{A}/\mathcal{C}$ is smaller than or equal to the homological dimension of $\mathcal{A}$.*

We recall that the homological dimension of a category $\mathcal{A}$ is the smallest integer n such that $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$ is zero for any couple (M, N) of objects of $\mathcal{A}$.

The calculus of groups $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$ can be done in another way when the section functor is exact: indeed, S transforms the injective objects of $\mathcal{A}/\mathcal{C}$ to injective objects of $\mathcal{A}$. Therefore, if G is a functor from $\mathcal{A}$ to an abelian category $\mathcal{D}$, the functors $(R^nG) \circ S$ and $R^n(G \circ S)$ are still isomorphic. This is true in particular if G is the functor $X \rightsquigarrow \mathrm{Hom}_{\mathcal{A}}(M, X)$.

Corollary 10. *The hypothesis are those of the corollary 6. We suppose in addition that the section functor is exact. The functors $N \rightsquigarrow \mathrm{Ext}^n_{\mathcal{A}/\mathcal{C}}(TM, N)$ and $N \rightsquigarrow \mathrm{Ext}^n_{\mathcal{A}}(M, SN)$ are then isomorphic.*

The following proposition examines the case when the section functor is exact:

Proposition 7. *Let $\mathcal{C}$ be a localizing sub-category of a category $\mathcal{A}$ with injective envelopes. The following assertions are equivalent:*

a. *The localization functor (or the section functor) is exact.*
b. *If N is a sub-object of M and if N and M are $\mathcal{C}-$ closed, then M/N is $\mathcal{C}-$ closed.*
c. *If I is an injective object not containing any non zero sub-object of $\mathcal{C}$, and if N is a $\mathcal{C}-$ closed sub-object of I, then I/N is $\mathcal{C}-$ closed.*

$(a) \Rightarrow (b) \Rightarrow (c)$: clear.

$(c) \Rightarrow (a)$: If L is the localization functor $S \circ T$, we will show that the first derived functor $R^1 L$ of L is zero.

First suppose that M is an object of $\mathcal{C}$; let $i : M \to I$ be an injective envelope of M and $p : I \to N$ be the cokernel of i. Since Lp is an isomorphism, the exact sequence $0 \to 0 = LM \xrightarrow{Li} LI \xrightarrow{Lp} LN \to R^1LM \to 0$ shows that R^1LM is zero.

Now suppose that M does not contain any non zero sub-object of $\mathcal{C}$ and keep the previous notations. Since Lp is an epimorphism, R^1LM is zero.

In the general case, let M' be the largest sub-object of M belonging to $\mathcal{C}$ and let $M'' = M/M'$. The exact sequence $R^1LM' \to R^1LM \to R^1LM''$ proves the nullity of R^1LM.

Corollary 11. *Let $\mathcal{C}$ be a localizing sub-category of a category $\mathcal{A}$ with injective envelopes. If the homological dimension of $\mathcal{A}$ is 0 or 1, the localization functor is exact.*

19. Categories with generators and exact inductive limits

We always use the notations of the previous paragraphs.

Proposition 8. *If $\mathcal{C}$ is an épaisse sub-category of an abelian $\mathfrak{U}-$category $\mathcal{A}$ with generators and exact inductive limits, the following assertions are equivalent:*

a. *$\mathcal{C}$ is a localizing sub-category of $\mathcal{A}$.*
b. *The inductive limit (in $\mathcal{A}$) of an inductive system of objects of $\mathcal{C}$ belongs to $\mathcal{C}$.*

$(a) \Rightarrow (b)$: Indeed let (M_i, φ_{ji}) be an inductive system of objects of $\mathcal{C}$ with the indexing set belonging to the universe $\mathfrak{U}$. Let M be the inductive limit of this system and let φ_i be the canonical morphism from M_i to M. According the corollary 5, M contains a sub-object M' which belongs to $\mathcal{C}$ and which contains all of the sub-objects of M belonging to $\mathcal{C}$; in particular, M' contains $\operatorname{Im} \varphi_i$ for any i and is thus equal to M.

$(b) \Rightarrow (a)$: Indeed let M be an object of $\mathcal{A}$. If P and N are two sub-objects of M belonging to $\mathcal{C}$, $P + N$ is isomorphic to a quotient of $P \oplus N$. It follows that $P + N$ belongs to $\mathcal{C}$ and the sub-objects of M belonging to $\mathcal{C}$ form an increasing filtering set of sub-objects of M. According to (b), the upper bound of these sub-objects belong to $\mathcal{C}$ and it is the largest sub-object of M belonging to $\mathcal{C}$; hence the result, thanks to the corollary 5.

Corollary 12. *Let $\mathcal{A}$ be an abelian $\mathfrak{U}-$category with generators and exact inductive limits and let (I_j) be a family of injective objects of $\mathcal{A}$. The objects M of $\mathcal{A}$ such that* $\mathrm{Hom}(M, I_j)$ *is zero for any j are the objects of a localizing sub-category of $\mathcal{A}$. Conversely, any localizing sub-category of $\mathcal{A}$ can be defined in this way.*

Indeed let $\mathcal{C}$ be the sub-category of $\mathcal{A}$ whose objects M are such that $\mathrm{Hom}(M, I_j)$ is zero for any j. If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of $\mathcal{A}$, we have also the exact sequences

$$0 \to \mathrm{Hom}(M'', I_j) \to \mathrm{Hom}(M, I_j) \to \mathrm{Hom}(M', I_j) \to 0$$

These exact sequences show that M belong to $\mathcal{C}$ if and only if M' and M'' belong to $\mathcal{C}$. In other words, $\mathcal{C}$ is an épaisse sub-category of $\mathcal{A}$. On the other hand, any inductive system (M_i, φ_{hi}) of $\mathcal{A}$ gives rise to the 'equalities'

$$\mathrm{Hom}_{\mathcal{A}}(\varinjlim_i M_i, I_j) = \varprojlim_i \mathrm{Hom}_{\mathcal{A}}(M_i, I_j).$$

It follows that $\varinjlim_i M_i$ belongs to $\mathcal{C}$ if the M_i belong to $\mathcal{C}$.

Conversely, if $\mathcal{C}$ is a localizing sub-category of $\mathcal{A}$, $\mathcal{A}/\mathcal{C}$ is a category with injective envelopes; thus an object M of $\mathcal{A}$ belongs to $\mathcal{C}$ if and only if $\mathrm{Hom}_{\mathcal{A}}(M, I)$ is zero for any $\mathcal{C}-$closed injective I.

Proposition 9. *Let $\mathcal{C}$ be a localizing sub-category of an abelian $\mathfrak{U}-$category $\mathcal{A}$ with generators and exact inductive limits. The categories $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are $\mathfrak{U}-$categories with generators and exact inductive limits. In addition, the canonical functor T from $\mathcal{A}$ to $\mathcal{A}/\mathcal{C}$ commutes with the inductive limits.*

Let $(U_\lambda)_{\lambda \in \Lambda}$ be a family of generators of $\mathcal{A}$. The quotients, belonging to $\mathcal{C}$, of the objects U_λ form a family of generators of $\mathcal{C}$. Furthermore, any inductive system of objects of $\mathcal{C}$ has an inductive limit in $\mathcal{A}$. The proposition 8 shows that this inductive limit even belongs to $\mathcal{C}$. It follows that $\mathcal{C}$ is a $\mathfrak{U}-$category with generators and exact inductive limits.

The lemma 9 proves that $\mathcal{A}/\mathcal{C}$ is a $\mathfrak{U}-$category with generators. Now consider an inductive system of objects of $\mathcal{A}$ $(M_i, \varphi_{ji})_{i,j \in I}$. Let φ_i be the canonical morphisms from M_i to the inductive limit $\varinjlim M_i$. We then have the 'equalities':

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(T \varinjlim M_i, N) &= \mathrm{Hom}_{\mathcal{A}}(\varinjlim M_i, SN) \\ &= \varprojlim \mathrm{Hom}_{\mathcal{A}}(M_i, SN) = \varprojlim \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(TM_i, N) \end{aligned}$$

These equalities show that $(T \varinjlim M_i, T\varphi_i)$ is an inductive limit of the inductive system $(TM_i, T\varphi_{ji})$. Thus the functor T commutes with the inductive limits.

Now let $(N_i, \psi_{ji})_{i,j \in I}$ be an inductive system of objects of $\mathcal{A}/\mathcal{C}$ and ψ_i be the canonical morphism from SN_i to $\varinjlim SN_i$. The above proves that

$$(T \varinjlim SN_i, (T\psi_i) \circ \Phi(N_i)^{-1})$$

is an inductive limit of (N_i, φ_{ji}). We will write briefly

$$\varinjlim N_i = T \varinjlim SN_i.$$

The last 'formula' shows that in particular that an inductive limit of monomorphisms is a monomorphism; this completes the proof.

Corollary 13. *Let $\mathcal{C}$ be a localizing sub-category of a locally noetherian $\mathfrak{U}-$category $\mathcal{A}$. Then $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are locally noetherian $\mathfrak{U}-$categories. Furthermore, the section functor commutes with the inductive limits.*

It remains to prove that S commutes with the inductive limits. If (N_i, ψ_{ij}) is an inductive system of objects of $\mathcal{A}/\mathcal{C}$, $\varinjlim N_i$ is equal to $T \varinjlim SN_i$. Thus it suffices to prove that the functorial morphism Ψ defines an isomorphism from $\varinjlim SN_i$ to $ST \varinjlim SN_i$: that is to say that $\varinjlim SN_i$ is $\mathcal{C}-$ closed.

This last assertion results from the lemma 5 (a): with the notations of this lemma, it indeed suffices to verify the condition (a) when P and Q are noetherian objects. The groups $\mathrm{Hom}_{\mathcal{A}}(P, \varinjlim SN_i)$ and $\mathrm{Hom}_{\mathcal{A}}(A, \varinjlim SN_i)$ then are identified respectively with $\varinjlim \mathrm{Hom}_{\mathcal{A}}(P, SN_i)$ and $Hom_{\mathcal{A}}(Q, \varinjlim SN_i)$ (corollary 4, part II); the proof follows directly from these facts (lemma 6).

Corollary 14. *Let $\mathcal{C}$ be a localizing sub-category of a locally finite $\mathfrak{U}-$category $\mathcal{A}$. Then $\mathcal{A}/\mathcal{C}$ and $\mathcal{C}$ are locally finite $\mathfrak{U}-$categories.*

Now suppose that $\mathcal{A}$ is a locally noetherian $\mathfrak{U}-$category. Let $\mathcal{A}'$ be the full sub-category of $\mathcal{A}$ which is defined by the noetherian objects of $\mathcal{A}$. If $\mathcal{C}$ is a localizing sub-category of $\mathcal{A}$, we denote by $\mathcal{C}'$ the full sub-category of $\mathcal{A}$ which is defined by the noetherian objects of $\mathcal{C}$. We leave it to the reader to show what follows:

Proposition 10. *The map $\mathcal{C} \to \mathcal{C}'$ is a bijective map from the set of localizing sub-categories of $\mathcal{A}$ to the set of épaisse sub-categories of $\mathcal{A}'$.*

In particular, if $\mathcal{A}$ is a locally finite $\mathfrak{U}-$category, any localizing sub-category $\mathcal{C}$ is characterized by the simple objects that it contains. We can say that $\mathcal{A}/\mathcal{C}$ is obtained from $\mathcal{A}$ by neglecting a number of simple objects.

20. Some examples of localizing subcategories

20.1. **Sheaves of modules**

Let E be a topological space, A be a sheaf of rings over E and let $\mathcal{A}$ be the category of sheaves of $A-$modules (or of $\mathcal{A}-$modules). If F is a closed subset of E, U the open complement of F in E, we use the following notations:

- If M is an $A-$module, $M|U$ denotes the restriction to U.
- T_1 denotes the functor $M \rightsquigarrow M|U$.
- $\mathcal{B}$ denotes the category of $A|U-$modules.
- S_1 denotes the direct image functor: if N is an $A|U-$module, $S_1 N$ is the direct image of N in E.

It is clear that T_1 is an exact functor and that Ker T_1 is formed of $A-$modules whose restriction to U is zero. Since S_1 is a functor adjoint to T_1 and that $T_1 \circ S_1$ is isomorphic to $1_{\mathcal{B}}$, the proposition 5 shows that the categories $\mathcal{B}$ and $\mathcal{A}/\mathrm{Ker}\ T_1$ are equivalent.

20.2. **Left exact functors**

The hypothesis and the notations are that of the proposition 4 (part II). Let $\mathcal{A}$ be the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, $\mathcal{B}$ be the category $\mathrm{Sex}(\mathcal{C}, \mathcal{D})$, T_1 be the functor R^0 and S_1 be the canonical functor from $\mathrm{Sex}(\mathcal{C}, \mathcal{D})$ to $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The proposition 4 (part II) says that S_1 is adjoint to T_1, thus any injective object of $\mathrm{Sex}(\mathcal{C}, \mathcal{D})$ is injective in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

20.3. **Satellite Functors** [6].

Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories. We suppose, for simplicity, that the category $\mathcal{C}$ 'has enough projective objects'. Then let $\mathcal{A}$ be the following category:

- an object of $\mathcal{A}$ is a 'connected sequence of functors' (F_n) from $\mathcal{C}$ to $\mathcal{D}$, $n \geqslant 0$;
- if (F_n) and G_n are two objects of $\mathcal{A}$, a morphism from the first to the second is a sequence of functorial morphisms $f_n : F_n \to G_n$; we suppose in addition that the morphisms f_n commute with the connecting homomorphisms;
- the composition of morphisms is defined in the obvious way.

The objects (F_n) whose first component F_0 is zero form an épaisse subcategory $\mathcal{B}$ of the abelian category $\mathcal{A}$. Furthermore, the reader verifies easily that the functor $(F_n) \rightsquigarrow F_0$ defines an equivalence between $\mathcal{A}/\mathcal{B}$ and $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Conversely, let F be a functor from $\mathcal{C}$ to $\mathcal{D}$ and denote by S_nF the $n-$th satellite functor of F: the functor $F \rightsquigarrow (S_nF)$ is then adjoint to the functor $(F_n) \rightsquigarrow F_0$.

The reader will find an analogous interpretation of the right satellite functors S^nF when he states the dual proposition of the propositions of this part.

20.4. **Localization in relation to the center**

If $\mathcal{A}$ is an additive category, we call *center of* $\mathcal{A}$ the set of functorial morphisms from the identity functor $1_{\mathcal{A}}$ of $\mathcal{A}$ to itself. The addition and the composition of functorial morphisms define on this set a structure of ring. This ring will be denoted by $Z(\mathcal{A})$; the reader verifies that $Z(\mathcal{A})$ is a commutative ring.

Now we suppose that $\mathcal{A}$ is a $\mathfrak{U}-$category with generators and exact inductive limits. If Σ is a multiplicative subset of $Z(\mathcal{A})$, we denote by $\mathcal{A}_{\Sigma}$ the full sub-category of $\mathcal{A}$ which follows: an object M belongs to $\mathcal{A}_\Sigma$ if M is the upper bound of the sub-objects N such that $s(N)$ is zero for at least one element s of Σ. It is clear that $\mathcal{A}_\Sigma$ is an épaisse sub-category and closed in $\mathcal{A}$. For any object M of $\mathcal{A}$ we can thus choose an $\mathcal{A}_\Sigma -$envelope $j_M : M \to M_\Sigma$.

Proposition 11. *With the above hypothesis, the following assertions are equivalent:*

a. *M is an $\mathcal{A}_\Sigma -$ closed object.*
b. *For any element s of Σ, $s(M)$ is an automorphism of M.*

$(a) \Rightarrow (b)$: Indeed it is clear that $s(M)$ is a monomorphism from M to M. It follows that Im $s(M)$ is $\mathcal{A}_\Sigma -$closed. According to the lemma 5, Im $s(M)$ is thus a direct factor of M and any complement of Im $s(M)$ belongs to $\mathcal{A}_\Sigma$. This is possible only if Im $s(M) = M$.

$(b) \Rightarrow (a)$: Indeed it is clear that M does not contain any non zero sub-object belonging to $\mathcal{A}_\Sigma$. We suppose, for simplicity, that M is a sub-object of M_Σ and that j_M is the canonical morphism from M to M_Σ. Then let N be a sub-object of M_Σ, containing M and such that Im $s(N)$ is contained in M for at least one element s of Σ. If i denotes the canonical morphism from M to N, we have the equality $1_M = s(M)^{-1} \circ s(N) \circ i$. This equality shows that M is a direct factor of N; it follows that M is equal to N, thus to M_Σ.

Corollary 15. *The localization functor $M \rightsquigarrow M_\Sigma$ is exact.*

This results from the proposition 11 and 7 (b).

Proposition 12. *The hypothesis and the notations are that of the proposition 11. We suppose in addition that $\mathcal{A}$ is a locally noetherian category. The injective envelope of an object of $\mathcal{A}_\Sigma$ belongs to $\mathcal{A}_\Sigma$.*

Indeed suppose that N is an essential extension of a sub-object M belonging to $\mathcal{A}_\Sigma$. We want to show that N belongs to $\mathcal{A}_\Sigma$; for this it suffices to prove that any noetherian sub-object N' of N belongs to $\mathcal{A}_\Sigma$; since N' is essential extension of $M \cap N'$, we can thus bring us back to the case where N is noetherian.

In this case, there is an element $s$of Σ such that $s(M)$ is zero. Since N is noetherian, Ker $(s(N))^n$ is equal to Ker $(s(N))^{n+1}$ for n large enough. We conclude from the equality Ker $(s(N))^n \cap$Im $(s(N))^n = 0$. Since Ker $(s(N))^n$ contains M and that N is an essential extension of M, Im $(s(N))^n$ is zero. In other words, $s^n(N)$ is zero and N belongs to $\mathcal{A}_\Sigma$.

Part 4. Locally noetherian categories

Unless explicitly stated otherwise, all categories considered in this chapter are abelian $\mathfrak{U}-$categories with generators and exact inductive limits.

This chapter is dedicated to the study of injective objects in a locally noetherian $\mathfrak{U}-$category. We derive from this study the following philosophy: a locally noetherian category is an 'extension' of a locally finite category; a locally finite category compares itself to a category of modules.

21. The Krull dimension of an abelian category

Let $\mathcal{A}$ be a category(abelian...). We will associate with any ordinal α a localizing subcategory $\mathcal{A}_\alpha$ of $\mathcal{A}$. The ordinal having no element is denoted by -1 for historical reasons; for the same reason, the ordinal having a finite number of n elements will be denoted by $n-1$. If α and β are two ordinals, the symbol $\alpha \top \beta$ will denote the ordinal obtained by 'first taking α, then β'. If α and β are two finite ordinals, we have for example the formula $\alpha \top \beta = \alpha + \beta + 1$. Denote by T_α the canonical functor from $\mathcal{A}$ to $\mathcal{A}/\mathcal{A}_\alpha$.

The construction of subcategories $\mathcal{A}_\alpha$ is done by transfinite induction:

- $\mathcal{A}_{-1}$ is the localizing subcategoy of $\mathcal{A}$ whose objects are the zero objects of $\mathcal{A}$. The functor T_{-1} is then the identity functor of $\mathcal{A}$.
- If the ordinal α has a predecessor β, $\mathcal{A}_\alpha$ is the smallest localizing sub-category containing all objects M such that $T_\beta M$ is of finite length.
- If α is a limit ordinal, $\mathcal{A}_\alpha$ is the smallest localizing sub-category containing all the sub-categories $\mathcal{A}_\beta$ for $\beta < \alpha$.

We denote by $\mathcal{A}_\omega$ the smallest localizing sub-category containing all the sub-categories $\mathcal{A}_\alpha$; the quotient category $\mathcal{A}/\mathcal{A}_\omega$ does not have any simple object. When M is an object of A_ω, *the Krull dimension of* M is the smallest ordinal α such that M belongs to $\mathcal{A}_\alpha$ (notation: Kdim $M = \alpha$). If $\mathcal{A}_\omega$ coincides with $\mathcal{A}$, the smallest ordinal α such that $\mathcal{A}_\alpha$ is equal to $\mathcal{A}$ is called *the Krull dimension of* $\mathcal{A}$ (notation: Kdim $\mathcal{A} = \alpha$). The following proposition results from the definitions.

Proposition 1. *Let $\mathcal{C}$ be a localizing sub-category of $\mathcal{A}$. Then $\mathcal{A}_\omega$ coincides with $\mathcal{A}$ if and only if $\mathcal{C}_\omega$ and $(\mathcal{A}/\mathcal{C})_\omega$ is $\mathcal{A}/\mathcal{C}$. In this case we have the inequalities*

$$\sup(\text{Kdim } \mathcal{C}, \text{Kdim } \mathcal{A}/\mathcal{C}) \leqslant \text{Kdim } \mathcal{A} \leqslant (\text{Kdim } \mathcal{C}) \top (\text{Kdim } \mathcal{A}/\mathcal{C})$$

We assume in the rest of the paragraph that *the Krull dimension of $\mathcal{A}$ is defined*, which means $\mathcal{A}_\omega$ coincide with $\mathcal{A}$. For any ordinal $\alpha <$ Kdim $\mathcal{A}$, the category $\mathcal{A}_{\alpha\top 0}/\mathcal{A}_\alpha$ is then of zero dimension.

If M is an object of $\mathcal{A}$, *the type* of this object will be the set of objects of $\mathcal{A}$ isomorphic to M. We denote by $\text{Sp}(\mathcal{A})$ (*spectrum of* $\mathcal{A}$) the set of types of indecomposable injective objects of $\mathcal{A}$. For any ordinal $\alpha <$ Kdim $\mathcal{A}$, we denote by $\text{Sp}_\alpha(\mathcal{A})$ the set of types which are formed of objects which are

$\mathcal{A}_\alpha -$ closed indecomposable injective and containing non-zero sub-objects of $\mathcal{A}_{\alpha\top 0}$. It is clear that the sets $\mathrm{Sp}_\alpha(\mathcal{A})$ are pairwise disjoint and that their union is equal to $\mathrm{Sp}(\mathcal{A})$.

If I is an indecomposable injective object whose type belongs to $\mathrm{Sp}_\alpha(\mathcal{A})$, $T_\alpha I$ contains a sub-object of finite length of $\mathcal{A}/\mathcal{A}_\alpha$. It results that the socle of $T_\alpha I$ is not zero and is a simple object of $\mathcal{A}_{\alpha\top 0}/\mathcal{A}_\alpha$. We denote by $s(I)$ the smallest nonzero $\mathcal{A}_\alpha -$ closed sub-object of I. The canonical morphism from $s(I)$ to I then induces an isomorphism from $T_\alpha s(I)$ to the socle of $T_\alpha I$. By abuse of language, we say that $s(I)$ is the socle of I. We will treat $s(I)$ indifferently as an object of $\mathcal{A}$ or as an object of $\mathcal{A}/\mathcal{A}_\alpha$.

Proposition 2. *Let $\mathcal{A}$ be a category whose Krull dimension is defined. For any ordinal $\alpha <$ Kdim $\mathcal{A}$, the map $I \mapsto s(I)$ induces a bijection between $\mathrm{Sp}_\alpha(A)$ and the set of types of simple objects of $\mathcal{A}/\mathcal{A}_\alpha$.*

Let I and J be two indecomposable injectives whose types belong to $\mathrm{Sp}_\alpha\mathcal{A}$. The corollary of the proposition 11 (part.II), implies that I and J are isomorphic if and only if $s(I)$ and $s(J)$ are isomorphic. On the other hand let M be a simple object of $\mathcal{A}_{\alpha\top 0}/\mathcal{A}_\alpha$ and let $\mu : M \to EM$ be an injective envelope of M in $\mathcal{A}/\mathcal{A}_\alpha$. If S_α is a right adjoint functor of T_α, $S_\alpha EM$ is an indecomposable injective whose socle is isomorphic to M (as an object of $\mathcal{A}/\mathcal{A}_\alpha$. This completes the proof.

If we compare the proposition 2 and the proposition 11 (part.II) we see how the notions of spectrum, of simple object and of coirreducible object are related.

Now let $\mathcal{C}$ be a localizing sub-category of $\mathcal{A}$, let T be the canonical functor from $\mathcal{A}$ to $\mathcal{A}/\mathcal{C}$ and let S be a right adjoint functor of T. If I is an indecomposable injective of $\mathcal{C}$ and if EI is injective envelope of I in $\mathcal{A}$, it is immediate that EI is again indecomposable. The map $I \to EI$ induces an injection from $\mathrm{Sp}(\mathcal{C})$ to $\mathrm{Sp}(\mathcal{A})$; we always identify $\mathrm{Sp}(\mathcal{C})$ with the image of this injection. Likewise, if J is an indecomposable injective of $\mathcal{A}/\mathcal{C}$, SJ is an indecomposable injective of $\mathcal{A}$. The map $J \to SJ$ induces an injection from $\mathrm{Sp}(\mathcal{C}/\mathcal{A})$ to $\mathrm{Sp}(\mathcal{A})$; we always identify $\mathrm{Sp}(\mathcal{A}/\mathcal{C})$ with the image of this injection. The corollary 2 of the proposition 6 (part. III), then shows that $\mathrm{Sp}(\mathcal{A}/\mathcal{C})$ is the complement of $\mathrm{Sp}(\mathcal{C})$ in $\mathrm{Sp}(\mathcal{A})$. In addition the proposition 2 implies the

Corollary 1. *Let $\mathcal{C}$ be a localizing sub-category of a category $\mathcal{A}$ whose Krull dimension is defined. An object A of $\mathcal{A}$ belongs to $\mathcal{C}$ if and only if* $\mathrm{Hom}_\mathcal{A}(M, I)$ *is zero for any indecomposable injective of $\mathcal{A}$ whose type belongs to* $\mathrm{Sp}(\mathcal{A}/\mathcal{C})$.

Let $\mathcal{D}$ be a localizing sub-category of $\mathcal{A}$ whose objects M are such that $\mathrm{Hom}_\mathcal{A}(M, I)$ is zero for any indecomposable injective whose type belongs to $\mathrm{Sp}(\mathcal{A}/\mathcal{C})$ (corollary of the proposition 8, part.III). It is clear that $\mathcal{D}$ contains $\mathcal{C}$.If $\mathcal{D}$ does not coincide with $\mathcal{C}$, there is an object M of $\mathcal{D}$ which is not zero and which does not contain any non zero sub-object in $\mathcal{C}$. Therefore let β

be the smallest ordinal such that M contains a non zero sub-object in $\mathcal{A}_\beta$. It is clear that β has a predecessor. Thus there is a sub-object N of M such that $T_\alpha N$ is simple. If I is the injective envelope of N, the type of I belongs to $\mathrm{Sp}(\mathcal{A}/\mathcal{C})$ and $\mathrm{Hom}_\mathcal{A}(M, I)$ is not zero: a contradiction.

Corollary 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two sub-categories of a category $\mathcal{A}$ whose Krull dimension is defined. Then $\mathcal{C}$ coincides with $\mathcal{D}$ if and only if $\mathrm{Sp}(\mathcal{C})$ coincides with $\mathrm{Sp}(\mathcal{D})$.*

This results from corollary 1.

If I is an indecomposable injective of $\mathcal{A}$, we remark that the ring of endomorphisms of $s(I)$ (in $\mathcal{A}$ or in $\mathcal{A}/\mathcal{A}_\alpha$) is a division ring. If A is the local ring of endomorphisms of I, this division ring is the quotien of A by its Jacobson radical: let f be an endomorphism of I (in $\mathcal{A}$ or in $\mathcal{A}/\mathcal{A}_\alpha$); the image of $s(I)$ under f is zero or is a simple object (in $\mathcal{A}/\mathcal{A}_\alpha$); thus the morphism f induces an endomorphism f' of $s(I)$; since I is injective, any endomorphism of $s(I)$ extends reciprocally to an endomorphism of I. This shows that the map $f \rightsquigarrow f'$ is surjective. On the other hand f' is non zero if and only if f is an automorphism. We have thus proved the

Proposition 3. *Let $\mathcal{A}$ be a category whose Krull dimension is defined, and let I be an indecomposable injective object of $\mathcal{A}$. The division ring of endomorphisms of the socle $s(I)$ is the quotient of the ring of endomorphisms of I by its Jacobson radical.*

The following proposition will give us some information on the injective envelopes in the category $\mathcal{A}$:

Proposition 4. *Let M be an object of a category $\mathcal{A}$ whose Krull dimension is defined. There is a family (f_α) of morphisms of $\mathcal{A}$, which is indexed by the ordinals $\alpha < \mathrm{Kdim}\ \mathcal{A}$ and which satisfies the conditions* (*a*), (*b*), (*c*) *and* (*d*)*:*

a. *f_α is a morphism from an object N_α to M.*
b. *Any sub-object of N_α belongs to $\mathcal{A}_\alpha$ is zero.*
c. *$T_\alpha N_\alpha$ is a semi-simple object of $\mathcal{A}/\mathcal{A}_\alpha$.*
d. *The morphism f from $\Sigma_\alpha N_\alpha$ to M which is defined by the family (f_α) is an essential extension of $\Sigma_\alpha N_\alpha$.*

For any family (f_α) satisfying (*a*), (*b*), (*c*) *and* (*d*), *$T_\alpha f_\alpha$ is an isomorphism from $T_\alpha N_\alpha$ to the socle of $T_\alpha M$.*

Indeed denote by M_α the largest sub-object of M belonging to $\mathcal{A}_\alpha$ ($\alpha <$ Kdim $- \mathcal{A}$). Let Q_α be a complement of M_α in $M_{\alpha \top 0}$ (part II, § 5); let i_α be the canonical monomorphism from Q_α to M.

Lemma 1. *The morphism j from $\sum_\alpha Q_\alpha$ to M which is defined by the family (i_α) is an essential extension of $\sum_\alpha Q_\alpha$.*

This is clear if Kdim $\mathcal{A}$ is equal to -1; thus we suppose that the property is demonstrated for all the categories $\mathcal{B}$ such that Kdim $\mathcal{B} <$ Kdim $\mathcal{A}$; then we prove that the property is true for $\mathcal{A}$. Indeed two cases are possible:

- The ordinal Kdim $\mathcal{A}$ has a predecessor β: we know that then the morphism h from $M_\beta \oplus Q_\beta$ to M which is defined by the canonical morphism from M_β and Q_β to M, is an essential extension. On the other hand, j is the composite of a morphism i from $\sum_\alpha Q_\alpha$ to $M_\beta \oplus Q_\beta$ and h; since i is an essential extension according to the hypothesis of the induction and the lemma 15 (part II), it follows that j is an essential extension (lemma 14, part II).
- Kdim $\mathcal{A}$ is a limit ordinal: then M is the upper bound of the M_β, for $\beta <$ Kdim $\mathcal{A}$. Let j_β be the morphism from $\sum_{\gamma<\beta} Q_\gamma$ to M_β which is defined by the morphisms i_γ. Since j_β is an essential extension according to the hypothesis of the induction, the assertion results from the proposition 13 (part II).

The lemma being shown, the construction of a family (f_α) is simple: indeed let N_α be a sub-object of Q_α and let g_α be the canonical morphism from N_α to Q_α; we suppose that $T_\alpha g_\alpha$ is an isomorphism from $T_\alpha N_\alpha$ to the socle of $T_\alpha Q_\alpha$. If P is a non zero sub-object of Q_α, $T_\alpha P$ contains a simple sub-object and $T_\alpha P \cap T_\alpha N_\alpha$ is non zero. It follows that $P \cap N_\alpha$ is non zero. This proves that g_α is an essential extension. If we put $f_\alpha = i_\alpha \circ g_\alpha$, the conditions (a), (b), (c) and (d) are verified.

Conversely, let (f_α) be a family of morphisms satisfying the conditions (a), (b), (c) and (d). It is clear that Im $T_\alpha f_\alpha$ is the largest sub-object of Im $T_\alpha f$ which belongs to $\mathcal{A}_{\alpha\intercal 0}$. If S is a simple sub-object of $T_\alpha M$, $S \cap$ Im $T_\alpha f$ is a non zero sub-object of $T_\alpha M$ and belongs to $\mathcal{A}_{\alpha\intercal 0}$. It follows that $S \cap$ Im $T_\alpha f$ is contained in Im $T_\alpha f_\alpha$; thus any simple sub-object of $T_\alpha M$ 'intersects' with Im $T_\alpha f_\alpha$, which proves the last assertion.

Theorem 1. *Let I be an injective object of a category $\mathcal{A}$ whose Krull dimension is defined. There is a family $(I_i)_{i\in \boldsymbol{L}}$ of indecomposable injectives of $\mathcal{A}$, such that I is isomorphic to the injective envelope of the direct sum $\sum_{i\in \boldsymbol{L}}$.*

If $(J_n)_{n\in \boldsymbol{N}}$ is a second family of indecomposable injectives, and if $\sum_l I_l$ and $\sum_n J_n$ have the same injective envelope, there is a bijection h from L to N such that I_l is isomorphic to $J_{h(l)}$.

First we prove the existence of a family $(I_l)_{l\in \mathbf{L}}$: we use the previous proposition by choosing M equal to I. If EN_α is 'the' injective envelope of N_α, it is clear that I is isomorphic to the injective envelope of the direct sum $\sum_\alpha EN_\alpha$. It is thus sufficient to prove that EN_α is the injective envelope of a direct sum of indecomposable injectives. However N_α does not contain any non zero sub-object belonging to $\mathcal{A}_\alpha$; in addition, $T_\alpha N_\alpha$ is a semi-simple object of $\mathcal{A}/\mathcal{A}_\alpha$. We deduce the existence of a family $(N_\alpha^\sigma)_{\alpha\in \mathbf{S}}$ of sub-objects of N_α, which satisfies the following conditions: $T_\alpha N_\alpha^\sigma$ is a simple object; the sum of N'_α and N_α^σ is direct; the canonical monomorphism from N'_α to N_α induces an isomorphism from $T_\alpha N'_\alpha$ to $T_\alpha N_\alpha$. If EN_α^σ is the injective envelope of N_α^σ, it follows that N_α and $\sum_\alpha EN_\alpha^\sigma$ have the same injective envelope. Since EN_α^σ is indecomposable, it first part of the theorem is proved.

Prove the second assertion: denote by $\mathbf{L}_\alpha$ (resp. by $\mathbf{N}_\alpha$) the set of l (resp. of n) such that the socle of I_l (resp. of J_n) 'belongs' to $\mathrm{Sp}_\alpha(\mathcal{A})$. With the notations of the previous proposition, it is then clear that N_α, $\sum_{l\in\mathbf{L}_\alpha} I_l$ and $\sum_{n\in\mathbf{N}_\alpha} J_n$ have the same injective envelope. If we consider these objects as the objects of $\mathcal{A}/\mathcal{A}_\alpha$, it follows that N_α, $\sum_{l\in\mathbf{L}_\alpha} I_l$ and $\sum_{n\in\mathbf{N}_\alpha} J_n$ have the same socle; in other words, the direct sums $\sum_{l\in\mathbf{L}_\alpha} s(I_l)$ and $\sum_{n\in\mathbf{N}_\alpha} s(J_n)$ are isomorphic in $\mathcal{A}/\mathcal{A}_\alpha$. The properties of semi-simple objects imply the existence of a bijection h_α from $\mathbf{L}_\alpha$ to $\mathbf{N}_\alpha$ such that $s(I_l)$ is isomorphic to $s(J_{h_\alpha(l)})$. Then I_l is isomorphic to $J_{h_\alpha(l)}$.

Proposition 5 (exchange theorem)**.** *Suppose the hypothesis of theorem 1 are satisfied. Let $(I_l)_{l\in\boldsymbol{L}}$ be a family of indecomposable injectives and $u : \sum_{l\in\boldsymbol{L}} I_l \to I$ be an injective envelope. If J is a direct factor of I, there is a sub-set $\boldsymbol{N}$ of $\boldsymbol{L}$ which satisfies the following condition: any maximal essential extension of $u(\sum_{l\in\boldsymbol{N}} I_l)$ in I is a complement of J.*

Indeed let $\mathbf{N}$ be the maximal element of the set of sub-sets $\mathbf{M}$ of $\mathbf{L}$ which satisfies the following condition: J and $u(\sum_{l\in\mathbf{M}} I_l)$ have zero intersection. We will prove that I is an essential extension of the sum $J + u(\sum_{l\in\mathbf{N}} I_l)$: indeed let L be a maximal essential extension of this sum in I. Since $\mathbf{N}$ is maximal, L 'intersects' all $u(I_l)$; in other words, $u(I_l)$ is an essential extension of $L \cap u(I_l)$ for any $l \in \mathbf{L}$. It follows that I is an essential extension of L. Whence the equality $L = I$. This completes the proof.

22. The structure of injective objects in a locally noetherian category

The statements of the previous paragraphs can be simplified when $\mathcal{A}$ is a locally noetherian category. Indeed the corollary 4 (part II) has the following consequence:

Proposition 6. *Let $\mathcal{A}$ be a locally noetherian category and $(I_l)_{l\in\boldsymbol{L}}$ be a family of injective objects of $\mathcal{A}$. The direct sum $\sum_{l\in\boldsymbol{L}} I_l$ is then an injective object of $\mathcal{A}$.*

Proposition 7. *For any locally noetherian category $\mathcal{A}$, the localizing sub-category $\mathcal{A}_\omega$ coincides with $\mathcal{A}$; furthermore, the categories $\mathcal{A}_{\alpha\intercal 0}/\mathcal{A}_\alpha$ are locally finite.*

Indeed we know that $\mathcal{A}/\mathcal{A}_\omega$ does not have any simple object. Since $\mathcal{A}$ is a locally noetherian category, it is the same for $\mathcal{A}/\mathcal{A}_\omega$. If $\mathcal{A}_\omega$ does not coincide with $\mathcal{A}$, the category $\mathcal{A}/\mathcal{A}_\omega$ thus contains a non zero noetherian object M. For any maximal proper sub-object N of M, the quotient M/N is simple, a contradiction.

The last assertion comes from the fact that any locally noetherian category whose Krull dimension is zero, is locally finite.

Modulo the propositions 6 and 7, the theorem 1 and the proposition 5 are stated in the following way:

Theorem 2 (Matlis [17]). *Let $\mathcal{A}$ be a locally noetherian category. Any injective I is isomorphic to the direct sum $\sum_{l\in\boldsymbol{L}} I_l$ of a family $(I_l)_{l\in\boldsymbol{L}}$ of indecomposable injective objects.*

If $(J_n)_{n\in\boldsymbol{N}}$ is a second family of indecomposable objects, and if the direct sums $\sum_{l\in\boldsymbol{L}} I_l$ and $\sum_{n\in\boldsymbol{N}} J_n$ are isomorphic, there is a bijection h from $\boldsymbol{L}$ to $\boldsymbol{N}$ such that I_l is isomorphic to $J_{h(l)}$.

The second assertion of the theorem 2 results also from theorem 1 (part I).

Proposition 8 (exchange theorem). *Let $\mathcal{A}$ be a locally noetherian category and $(I_l)_{l\in\boldsymbol{L}}$ be a family of indecomposable injectives of $\mathcal{A}$. If J is a direct factor of $\sum_{l\in\boldsymbol{L}} I_l$, there is a sub-set $\boldsymbol{N}$ of $\boldsymbol{L}$ such that $\sum_{l\in\boldsymbol{N}} I_l$ is a complement of J in $\sum_{l\in\boldsymbol{L}} I_l$.*

We suppose in the following of the this paragraph that $\mathcal{A}$ is a locally noetherian category. We say that the category $\mathcal{A}$ is *connected* if it is not equivalent to the product of two non null categories (note by translator: here a null category should be the category with one object and one morphism).

Corollary 3. *Any locally noetherian category $\mathcal{A}$ is equivalent to the product of a family $(\mathcal{A}_N)_{n\in E}$ of connected categories.*

Indeed let $\mathcal{I}$ be the full sub-category of $\mathcal{A}$ whose objects are the injective objects of $\mathcal{A}$. According to the corollary 3 (part I), it suffices to show that $\mathcal{I}$ is the product of a family $(\mathcal{I}_n)_{n\in E}$ of additive categories such that $\mathcal{K}\mathcal{I}_\backslash$ is a connected category for each n. For this we agree to say that two indecomposable injectives I and J are equivalent if there is a finite sequence (I_l), $1 \leqslant l \leqslant k$, of indecomposable injectives satisfying the following conditions: $I_1 = I$, $I_k = J$; if $1 \leqslant l \leqslant k-1$, the abelian groups $\mathrm{Hom}(I_l, I_{l+1})$ and $\mathrm{Hom}(I_{l+1}, I_l)$ are not both zero. We denote by E the set of equivalence classes thus obtained (Brauer call such a class a *block* ; to be precise let's say that Brauer consider the indecomposable projective modules over an artinian ring A; he says that two indecomposable projectives P and Q belong to the same block if there is a sequence $P_1 = P$, P_2, $\cdots$, $P_k = Q$ formed of indecomposable projectives such that the groups $\mathrm{Hom}_A(P_l, P_{l+1})$ and $\mathrm{Hom}_A(P_{l+1}, P_l)$ are never both zero; we will see in paragraph 3 that the theory of Brauer is a particular case of the one we are developing here). If n is such a class, let $\mathcal{I}_n$ be the full sub-category of $\mathcal{I}$ which is defined by the direct sums of indecomposable injectives of n. It is clear that the category $\mathcal{I}$ is equivalent to the product of the categories $\mathcal{I}_n$; furthermore, $\mathcal{I}_n$ is not equivalent to the product of two non null additive categories; whence the assertion.

Let $\mathcal{A}_n$ be the localizing sub-category of $\mathcal{A}$ which is defined by the objects M such that $\mathrm{Hom}(M, I)$ is zero for any indecomposable injectives not belonging to n. It is clear that the categories $\mathcal{A}_n$ and $\mathcal{K}\mathcal{I}_\backslash$ are equivalent. The category $\mathcal{A}$ is thus equivalent to the product of the connected categories $\mathcal{A}_n$. We say that $\mathcal{A}_n$ is a *connected component* of $\mathcal{A}$.

We end this paragraph with a generalization of the primary decomposition of Lasker-Noether. For any element s of the spectrum of $\mathcal{A}$, we choose an indecomposable injective I_s of type s. If M is an object of $\mathcal{A}$, we say that *s is associated with M* if the injective envelope of M contains a sub-object isomorphic to I_s. The set of elements s of $\mathrm{Sp}(\mathcal{A})$ which are associated with M is denoted by $\mathrm{ass}(M)$. We say that M is *isotypic of type s* if $\mathrm{ass}(M)$ has s as the only element.

Proposition 9. *Let M be a noetherian object of a locally noetherian category $\mathcal{A}$. There is a map $s \rightsquigarrow N_s$ from a finite subset $\boldsymbol{L}$ of $\mathrm{Sp}(\mathcal{A})$ to the set of sub-objects of M which fulfills the following conditions: M/N_s is isotypic of type s; the intersection of N_s is zero; if $\boldsymbol{P}$ is a subset of $\boldsymbol{L}$ which is distinct from $\boldsymbol{L}$, the intersection $\bigcap_{s\in\boldsymbol{P}} N_s$ is not zero. If these conditions are fulfilled, $\boldsymbol{L}$ is equal to $\mathrm{ass}(M)$.*

Indeed let $i : M \to I$ be an injective envelope of M. We suppose for simplicity that M is a sub-object of I and that i is the canonical morphism from M to I. We suppose also that I is the direct sum of a family $(I_l)_{l\in\mathbf{L}}$ of indecomposable injectives. The sub-objects $M \cap I_l$ of M are then different from O for any l and their sum is direct. Since M is noetherian, the set $\mathbf{L}$ is finite. If s is an element of $\mathrm{ass}(M)$, denote by J_s the direct sum of I_l isomorphic to I_s. In these conditions, it suffices to choose $\mathbf{L}$ equal to $\mathrm{ass}(M)$ and N_s equal to $M \cap (\sum_{t\neq s} J_t)$.

Conversely, suppose we are given $\mathbf{L}$ and the map $s \rightsquigarrow N_s$. The canonical morphisms $q_s : M \to M/N_s$ defines a monomorphism q from M to the direct sum of N and M/N_s. We conclude the formula $\mathrm{ass}(M) \subset \mathrm{ass}(N) = \mathbf{L}$. On the other hand let M_s be the intersection of N_t for t different from s. The morphism q_s induces a monomorphism from M_s to M/N_s. Since $\mathrm{ass}(M_s)$ is not zero and that $\mathrm{ass}(M/N_s) = \{s\}$, we have the following formulas: $s \in \mathrm{ass}(M_s) \subset \mathrm{ass}(M)$. This proves that $\mathbf{L}$ is contained in $\mathrm{ass}(M)$, which remained to be demonstrated.

We leave it to the reader to write in the language of this article the many propositions of the primary decompositions. We also leave him the task of formulating the dual of the preceding statements.

23. PSEUDO-COMPACT MODULES

We saw in the previous paragraphs the locally finite categories are introduced to the study of locally noetherian categories. The end of this part will be devoted to a more in-depth study of locally finite categories. We start by giving an example:

Call *left pseudo-compact ring* a ring A with unit element, topological, separated, complete, whose underlying set belong to $\mathfrak{U}$ and which satisfies the axiom APC:

APC: The ring A has a base of neighbourhood of O formed of left ideals l of finite colength [i.e. $\text{long}_A(A/l) < +\infty$].

Unless explicitly stated otherwise, we consider in the following of this part that the left unitary $A-$modules whose underlying set belong to $\mathfrak{U}$. If M is such an $A-$module, the discrete topology makes M a topological module over the topological ring A if and only if the annihilator of any element m of M is an open left ideal (thus of finite colength). The *discrete topological $A-$modules* form a locally finite category that we denote it by dis(A) (for the definition of topological modules, cf. Bourbaki, *Topologie*, III, §6, no.6, 3rd.Edition).

The objects of finite length of dis(A) are the discrete topological modules of finite length. These modules define a finite abelian category $\mathcal{T}(A)$ and the theorem 1 (part II) shows that the data of $\mathcal{T}(A)$ determine dis(A) up to an equivalence. Since the dual category $\mathcal{T}(A)^o$ is also a finite abelian category, the same theorem proves the existence of a locally finite category $\mathcal{D}$ such that $\mathcal{T}(A)^o$ is equivalent to the categories of objects of finite length of $\mathcal{D}$. We will show that the dual category $\mathcal{D}^o$ is equivalent to a category of $A-$modules:

Call *pseudo-compact $A-$module* any left $A-$module, topological, separate, complete M which satisfies the axiom MPC:

MPC: The module M has a base of neighbourhood of O formed of sub-modules N such that M/N is of finite length.

If M and N are two pseudo-compact $A-$modules, a morphism from M to N will be a continuous $A-$linear map from M to N; the composition of morphisms is the usual composition of maps. We have thus defined a new category that we denote by $PC(A)$. It is clear $PC(A)$ is an additive category. We will see that $PC(A)$ is also an abelian category. The proof of this fact relies on a well-known algebraic lemma (Bourbaki, *Topologie*, I Appendice, 3rd.Edition):

Lemma 2. *Let B be a ring, I be a directed set, (M_i, f_{ji}) and (N_i, g_{ji}) be two projective systems, indexed by I and formed of left $B-$modules. Let (h_i) be a morphism from the first to the second, and suppose h_i is surjective and has artinian kernel for each i. Then $\varprojlim h_i$ is a surjective map from $\varprojlim M_i$ to $\varprojlim N_i$.*

This lemma results in the following two propositions:

Proposition 10. *Let M be a pseudo-compact $A-$module, (M_i) be a decreasing filtering family of closed sub-modules of M. The canonical map from M to the projective limit $\varprojlim M/M_i$ is surjective and has $\inf_i M_i$ as the kernel.*

Indeed let (N_j) be the decreasing filtering family of open sub-modules of M. In the following diagram, all the arrows represent the 'obvious' maps:

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & M & \xrightarrow{p} & \varprojlim_j M/N_j \\
 & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle s} \\
0 & \longrightarrow & \varprojlim_{i,j}(M_i+N_j)/M_i & \longrightarrow & \varprojlim_i M/M_i & \xrightarrow{r} & \varprojlim_{i,j} M/(M_i+N_j)
\end{array}$$

Since M is complete, p is an isomorphism; according to lemma 2, s is surjective. It follows that r is surjective. On the other hand we have the equality:

$$\varprojlim_{i,j}(M_i+N_j)/M_i = \varprojlim_i \varprojlim_j (M_i+N_j)/M_i = \varprojlim_i M_i/M_i = 0,$$

because M_i is closed and is the intersection of $M_i + N_j$. It follows that r is an isomorphism and that q is surjective. Finally, the kernel of q is the projective limit of M_i, that is to say $\inf_i M_i$.

Proposition 11. *Let M be a pseudo-compact $A-$module, N be a closed sub-module of M and M_i a decreasing filtering family of closed sub-modules of M. Then the sub-modules $N + \inf_i M_i$ and $\inf(N + M_i)$ coincide.*

Indeed we have the following exact and commutative diagram (the arrows represent the 'obvious' arrows):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & \varprojlim_i N/(N\cap M_i) & \longrightarrow & \varprojlim_i M/M_i & \longrightarrow & \varprojlim_i M/(M_i+N) & &
\end{array}$$

N is obviously a pseudo-compact module for the topology induced by M. The previous proposition thus shows that l and j are surjections. It follows that the kernel of h is the image of the kernel of j ([6], lemma III, 3.3). Since Ker h is equal to $\inf_i(M_i + N)/N$ and that Ker j is equal to $\inf_i M_i$, we have 'the equality'

$$\inf_i(M_i + N)/N = ((\inf_i M_i) + N)/N.$$

Theorem 3. *The pseudo-compact modules over the pseudo-compact ring A form an abelian category with cogenerators and exact projective limits. The dual category is locally finite.*

Indeed let M be a pseudo-compact module and let $\mathrm{Hom}_{PC(A)}(M, \ .)$ be the functor $N \rightsquigarrow \mathrm{Hom}_{PC(A)}(M,N)$ from $\mathcal{T}(A)$ to $\mathbf{Ab}$. We can verify directly that the functor $M \rightsquigarrow \mathrm{Hom}_{PC(A)}(M, \ .)$ defines an equivalence between $PC(A)$ and the category $\mathrm{Sex}(\mathcal{T}(A), \mathbf{Ab})$ (cf. part II). However we prefer a demonstration that explains the construction of the kernels and of the cokernels.

If $f : M \to N$ is a morphism of pseudo-compact modules, the underlying module of Ker f is formed of elements m of M such that $f(m)$ is zero. This sub-module is closed and it is equipped with the topology induced by M.

The coimage of f is the quotient module $M/\text{Ker } f$ which is equipped with the quotient topology. Show that $M/\text{Ker } f$ is a pseudo-compact module: the verification of axiom MPC is immediate; it remains to be seen that $M/\text{Ker } f$ is complete, or that the canonical map from $M/\text{Ker } f$ to $\varprojlim M/(\text{Ker } f + N_j)$ is an isomorphism where N_j runs through the projective system of open sub-modules of M. This results from the proposition 10.

Let N' be the set-theoretic image of M under the map f and let us confuse N' with $M/\text{Ker } f$ for the convenience of reasoning. The topology induced from N in N' is separated, linear (i.e. define by the sub-modules) and it is coarser than the quotient topology. Let (N'_j) be a decreasing filtering family towards O of open sub-modules of N' for the induced topology. If P is a sub-module of N' which is open for the quotient topology, the family $(P + N'_j)$ is decreasing filtering. Since N'/P is artinian, this family has a smallest elelment. Since finally $\inf_j (P + N'_j)$ is equal to $P + \inf_j N'_j$ according to the proposition 11, we see that this smallest element is equal to P. In other words, P contains N'_j when N'_j is small enough: the induced topology coincides with the quotient topology.

In particular, N' is complete for the topology induced from N; so N' is closed in N. It follows that Im f and Coim f are isomorphic and that Coker f is the quotient N/N' equipped with the quotient topology. This proves that $PC(A)$ is an abelian category.

If (M_i, f_{ji}) is a projective system of pseudo-compact modules, the projective limit of this system is defined in the following way: the underlying module of $\varprojlim M_i$ is the projective limit of the underlying modules; the topology of $\varprojlim M_i$ is the topology of the projective. The exactness of projective limits equals one or the other of propositions 10 and 11 (dual of the proposition 6, part I).

Finally, when n runs through the positive integers and M runs through the pseudo-compact sub-modules of finite colength of A^n, the quotients A^n/M define a family of cogenerators of $PC(A)$. This completes the proof of the theorem.

Corollary 4. *The pseudo-compact ring A is the topological direct product of indecomposable and closed left ideals. Any projective pseudo-compact module is the topological direct product of indecomposable projective pseudo-compact modules. Any indecomposable projective pseudo-compact module is isomorphic to a left ideal $A.e$, where e is a primitive idempotent of A.*

Recall that an idempotent e is said to be primitive if e is not the sum of two idempotents e' and e'' such that $e'.e'' = e''.e' = 0$.

Any morphism from A to a pseudo-compact module M is indeed of the form $a \to a.m$, where $m \in M$. It follows that A is a projective pseudo-compact module. The first two assertions of the corollary thus results by duality from theorem 2. On the other hand, any indecomposable projective P is 'the' projective envelope of a simple pseudo-compact module S. Since S is isomorphic to a quotient of A, P is isomorphic to a direct factor of A.

Corollary 5. *Let A be a pseudo-compact ring, $(e_i)_{i \in I}$ be a summable family of pairwise orthogonal idempotents. If e is the sum of this family, the canonical injections from $A.e_l$ to $A.e$ extends to an isomorphism from the direct product $\prod_i A.e_i$ to $A.e$.*

Recall that two idempotents e and f are said to be orthogonal if $e.f = f.e = 0$.

Indeed it is clear that e is an idempotent of A. Denote by f_i the map $a.e \to a.e_i$ from $A.e$ to $A.e_i$. These maps define a morphism f from $A.e$ to the direct product $\prod_{i \in I} A.e_i$. If J is a finite sub-set of I, f induces a surjection from $A.e$ to $\prod_{i \in J} A.e_i$. It follows that f is a surjection (exactness of projective limits)and that the kernel Ker f is a direct factor of $A.e$. Suppose Ker f is non zero and let l be the left ideal of A, open and not containing Ker f. For any finite sub-set J of I, $A.e$ is the direct product of $A.e_i$, $i \in J$, Ker f and a third factor. Thus we have a decomposition of e into a finite sum of pairwise orthogonal idempotents, let

$$e = \sum_{i \in J} e_i + e' + e''$$

where e' gives rise to Ker f.

For J large enough, let $e - \sum_{i \in J} e_i$ must belong to l; thus $e' + e''$ and $e' = e'.(e' + e'')$ belong to l: this is absurd. In other words, Ker f is zero and the corollary is proved.

Corollary 6. *Let A be a pseudo-compact ring, $\mathfrak{a}$ be a closed two-sided ideal of A and $(e_i)_{i \in I}$ be a summable family of pairwise orthogonal idempotents of $A/\mathfrak{a}$. There is a summable family $(f_i)_{i \in I}$ of pairwise orthogonal idempotents of A such that e_i is the image of f_i by the canonical map from A to $A/\mathfrak{a}$.*

The ring $A/\mathfrak{a}$ is indeed pseudo-compact for the quotient topology. It follows that the map $f : a \to (a.e_i)$ from $A/\mathfrak{a}$ to the product $\prod_i A.e_i$ is surjective (corollary 5). Let p be the canonical map from A to $A/\mathfrak{a}$, $u_i : P_i \to A.e_i$ be a projective cover of $A.e_i$ and u be the product $\prod_i u_i$.

Since A is projective, there is a morphism g from A to $\prod_i P_i$ such that $u \circ g$ is equal to $f \circ p$. Since u is a projective cover and that $u \circ g$ is surjective, there is a morphism h from $\prod_i P_i$ to A such that $h \circ g$ is the identity map of

$\prod_i P_i$. Then it suffices to choose for f_i the image of 1 in the projection from A to $h(P_i)$ which annihilate Ker $g$and the modules $h(P_j)$, $j \neq i$.

$$\begin{array}{ccc} A & \xrightarrow{p} & A/\mathfrak{a} \\ {\scriptstyle h}\uparrow\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \prod_i P_i & \xrightarrow{u} & \prod_i A.e_i \end{array}$$

Remark 1. *The demonstrations of this paragraph remain valid when we replace APC and MPC by the weaker conditions here:*

APC': *The ring A has a base of neighborhoods of O formed of left ideals $\mathfrak{l}$ such that $A/\mathfrak{l}$ are artinian.*

MPC': *The module M has a base of neighborhoods of O formed of sub-modules N such that M/N are artinian.*

The only modifications to be made in the preceding statements are the following: the left $A-$modules satisfying MPC' form an abelian category whose dual category is locally noetherian. Furthermore, the last assertion of corollary 4 is no longer true.

The topological rings satisfying the condition APC' have been studied by H. Leptin *using different methods (*[13], [14]*). We will only need the pseudo-compact rings.*

24. The duality between locally finite categories and pseudo-compact modules

We propose to show that any locally finite category is equivalent to the dual category of a category of pseudo-compact modules; we start by some definitions:

A full sub-category of a category $\mathcal{A}$ is called *closed* if the following conditions are satisfied: any sub-object and any quotient of an object of $\mathcal{C}$ belong to $\mathcal{C}$; the direct sum (in $\mathcal{A}$) of two objects of $\mathcal{C}$ belong to $\mathcal{C}$; the inductive limit (in $\mathcal{A}$) of an inductive system of objects of $\mathcal{C}$ belongs to $\mathcal{C}$.

If M is an object of $\mathcal{A}$, M contains a sub-object $\mathcal{C}M$ which belongs to $\mathcal{C}$ and which contains all the other sub-objects of M belonging to $\mathcal{C}$. If $f : M \to N$ is a morphism of $\mathcal{A}$, $f(\mathcal{C}M)$ is contained in $\mathcal{C}N$; furthermore, the functor $M \rightsquigarrow \mathcal{C}M$ is adjoint to the canonical functor from $\mathcal{C}$ to $\mathcal{A}$; in particular, the functor $M \rightsquigarrow \mathcal{C}M$ is left exact (proposition 11, part I). For example, the semi-simple objects of $\mathcal{A}$ are the objects of a closed subcategory $\mathcal{S}$. If M is an object of $\mathcal{A}$, the sub-object $\mathcal{S}M$ is called the socle of M.

If $\mathcal{C}$ and $\mathcal{D}$ are two closed sub-categories of $\mathcal{A}$, we denote by $\mathcal{C}.\mathcal{D}$ the following closed sub-category: an object M of $\mathcal{A}$ belongs to $\mathcal{C}.\mathcal{D}$ if $M/\mathcal{D}M$ is an object of $\mathcal{C}$. The product $(\mathcal{C}, \mathcal{D}) \rightsquigarrow \mathcal{C}.\mathcal{D}$ is obviously associative. Thus we can say the power $\mathcal{C}^n$ of a closed sub-category $\mathcal{C}$ of $\mathcal{A}$.

Now let $\mathcal{A}$ be a locally finite category and $\mathcal{S}$ be the closed sub-category of $\mathcal{A}$ which is formed of the semi-simple objects of $\mathcal{A}$. If M is an object of $\mathcal{A}$, we denote by M_n the largest sub-object of M which belongs to the closed sub-category $\mathcal{S}^{n+1}$ (for example, M_0 denotes the socle of M). Since $\mathcal{A}$ is a locally finite category, it is clear that M is the upper bound of the sub-objects M_n. We will examine the case where M is an injective object of $\mathcal{A}$.

Proposition 12. *Let I be an injective object of a locally finite category $\mathcal{A}$, A be the ring of endomorphisms of I and $\mathfrak{r}$ be the set of endomorphisms of I whose kernel contain the socle I_0. Then $\mathfrak{r}$ is the Jacobson radical of A; the intersection of the powers $\mathfrak{r}^n$ is zero; the quotient ring $A/\mathfrak{r}$ is isomorphic to the product of rings of endomorphisms of a family of vector spaces.*

Indeed let $\mathfrak{r}^{(n)}$ be the two-sided ideal formed of the endomorphisms of I whose kernel contain I_{n-1}. If f is an element of $\mathfrak{r}^{(n)}$, it is clear that $f(I_{n+m})$ is contained in I_m. It follows that $\mathfrak{r}^{(m)}$, $\mathfrak{r}^{(n)}$ is contained in $\mathfrak{r}^{(n+m)}$. In particular $\mathfrak{r}^n$ is contained in $\mathfrak{r}^{(n)}$.

On the other hand, the canonical morphisms from I_n to I and from I to I/I_n give rise to the exact sequences

$$0 \to \mathrm{Hom}(I/I_n, I) \xrightarrow{i_n} \mathrm{Hom}(I, I) \xrightarrow{p_n} \mathrm{Hom}(I_n, I) \to 0$$

The ideal $\mathfrak{r}^{(n+1)}$ is nothing else than the image of i_n. The map thus p_n defines an isomorphism from $A/\mathfrak{r}^{(n+1)}$ to $\mathrm{Hom}(I_n, I)$. The formulas

$$A = \mathrm{Hom}(I, I) = \mathrm{Hom}(\varinjlim I_n, I) = \varprojlim \mathrm{Hom}(I_n, I) = \varprojlim A/\mathfrak{r}^{(n+1)}$$

then show that A is separated and complete for the filtration defined by the ideals $\mathfrak{r}^{(n)}$. The formula $(1-x)^{-1} = 1 + x + x^2 + \cdots$ shows furthermore that the inverse of $1 - x$ exists if x belongs to $\mathfrak{r}$; in other words, $\mathfrak{r}$ is contained in the Jacobson radical of A.

Finally remark that for any morphism f from I_0 to I, $f(I_0)$ is contained in I_0. The quotient $A/\mathfrak{r}$ is thus equal to $\mathrm{Hom}(I_0, I_0)$: this is the ring of endomorphisms of a semi-simple object. The last assertion of the proposition results from there (cf. part I, § 6). It follows also that $\mathfrak{r}$ is an intersection of maximal ideals and contains the Jacobson radical; this completes the proof.

When M runs through the sub-objects of finite length of I, the canonical morphism from I to I/M defines an injection from $\mathrm{Hom}(I/M, I)$ to A. The image of this map is a left ideal which we denote by $\mathfrak{l}(M)$. If M and N are two sub-objects of finite length, the exactness of the functor $X \rightsquigarrow \mathrm{Hom}(X, I)$ implies the equality $\mathfrak{l}(M + N) = \mathfrak{l}(M) \cap \mathfrak{l}(N)$. This equality shows that the left ideals $\mathfrak{l}(M)$ form a base of neighbourhoods of O for the topology which makes A a topological group. We always equip A with this topology that we will call *natural.*

Proposition 13. a. *Let $\mathcal{A}$ be a locally finite category, I be an injective object of $\mathcal{A}$ and let A be the ring of endomorphisms of I. Equipped with the natural topology, A is a pseudo-compact ring.*

b. *Conversely, any pseudo-compact ring A is isomorphic to the ring of endomorphisms of an injective of a locally finite category. In particular, the Jacobson radical $\mathfrak{r}$ of A is the intersection of the maximal open left ideals; the intersection of $\mathfrak{r}^n$ is zero and $A/\mathfrak{r}$ is isomorphic to the product of rings of endomorphisms of a family of vector spaces.*

a. Show first that A is a topological ring. Since there is a base of neighbourhoods of O formed of left ideals, it suffices to prove the following thing: for any sub-object M of finite length of I and for any element a of A, there is a sub-object N of finite length such that $\mathfrak{l}(N).a$ is contained in $\mathfrak{l}(M)$. It suffices to choose N equal to $a(M)$.

   On the other hand we have the 'equalities'

$$A = \mathrm{Hom}(I, I) = \mathrm{Hom}(\varinjlim M, I) = \varprojlim \mathrm{Hom}(M, I) = \varprojlim A/\mathfrak{l}(M).$$

   These equalities show both that A is separated and that A is complete. IT remains to prove that the $A-$modules $A/\mathfrak{l}(M)$ are of finite length: for this, we study first the $A-$module $\mathrm{Hom}(S, I)$ when S is a simple object of $\mathcal{A}$; we denote by I_S the isotypic component of type S of I_0 and by B the ring of endomorphisms of I_S.

   For any endomorphism f of I, $f(I_S)$ is contained in I_S; it follows that f induces an endomorphism g of I_S and that the map $p : f \rightsquigarrow g$ from A to B is an epimorphism of rings. This epimorphism is compatible with the canonical bijection between the $A-$module $\mathrm{Hom}(S, I)$ and the $B-$module $\mathrm{Hom}(S, I_S)$. It follows that $\mathrm{Hom}(S, I)$ is zero or is a simple $A-$module following that I_S is zero or non zero.

   Now let M be a sub-object of finite length of I and let

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset M$$

   be a Jordan-Holder composition series of M. The $A-$module $A/\mathfrak{l}(M)$ is isomorphic to $\mathrm{Hom}(M, I)$ and the modules $\mathrm{Hom}(M/M_i, I)$ define a composition series of $\mathrm{Hom}(M, I)$. The quotient of $\mathrm{Hom}(M/M_i, I)$ by $\mathrm{Hom}(M/M_{i+1}, I)$ is isomorphic to $\mathrm{Hom}(M_{i+1}/M_i, I)$. It follows that the length of $A/\mathfrak{l}(M)$ is lower or equal to the length of M. This proves (a).

b. Let $\mathcal{A}$ be the dual category of $PC(A)$. The underlying left $A-$module A is projective; thus it is an injective object of $\mathcal{A}$. The assertion (b) results from the fact that A is the ring of endomorphisms of this injective object.

The proposition 13 being proved, let us take our interest on the functor $M \rightsquigarrow \mathrm{Hom}(M, I)$: the abelian group $\mathrm{Hom}(M, I)$ is equipped with a structure of left module over the ring A of endomorphisms of I. On the other hand, when N runs through the sub-objects of finite length of M, the canonical epimorphisms from M to M/N define an injection from $\mathrm{Hom}(M/N, I)$ to $\mathrm{Hom}(M, I)$. The image of this map is a sub-module that we denote by $\mathfrak{l}(N)$. In a general way, the arguments we used when M was equal to I remain valid here and they have the following consequences: the sub-modules $\mathfrak{l}(N)$ of $\mathrm{Hom}(M, I)$ form a base of neighbourhoods of O for a topology which makes $\mathrm{Hom}(M, I)$ a pseudo-compact $A-$module. We always equip $\mathrm{Hom}(M, I)$ with this topology that we will call *natural*.

If $f : M \to M'$ is a morphism of $\mathcal{A}$, $\mathrm{Hom}(f, I)$ is a continuous map from $\mathrm{Hom}(M', I)$ to $\mathrm{Hom}(M, I)$: since $\mathrm{Hom}(f, I)$ is a linear map, it indeed suffices to prove that, for any sub-object N of finite length of M, there is a sub-object N' of finite length of M' such that $\mathrm{Hom}(f, I)$ maps $\mathfrak{l}(N')$ to $\mathfrak{l}(N)$. Now it suffices to choose N' equal to $f(N)$. This shows that $\mathrm{Hom}(f, I)$ is a morphism of pseudo-compact modules. We thus have defined a functor from $\mathcal{A}$ to the dual category of $PC(A)$.

Theorem 4. *Let I be an injective object of a locally finite category $\mathcal{A}$, A be the ring of endomorphisms of I and let F be the functor $M \rightsquigarrow \mathrm{Hom}(M, I)$. The functor F defines by passing to the quotient an equivalence between $\mathcal{A}/\mathrm{Ker}\ F$ and the dual category of the category $PC(A)$ of pseudo-compact modules over A.*

The sub-category Ker F is formed of M such that $\mathrm{Hom}(M, I) = 0$ and it is localizing (corollary 3, part III). According to the proposition 10 (part III), Ker F is the smallest localizing sub-category containing the simple objects S such that $\mathrm{Hom}(S, I)$ is zero. We can also say that the localizing sub-category Ker F is generated by the simple objects which 'do not involve' in the decomposition of the socle of I into isotypic components.

Let T be the canonical functor from $\mathcal{A}$ to $\mathcal{A}/\mathrm{Ker}\ F$. Then TI is an injective object of $\mathcal{A}/\mathrm{Ker}\ F$ and the socle of TI contains the simple objects of $\mathcal{A}/\mathrm{Ker}\ F$ of all types. Furthermore, since I is $\mathrm{Ker}\ F-$ closed, the map $T(M, I)$ from $\mathrm{Hom}(M, I)$ to $\mathrm{Hom}(TM, TI)$ is a bijection. We thus can replace $\mathcal{A}$ by the locally finite category $\mathcal{A}/\mathrm{Ker}\ F$ and replace I by TI. In other words, we can go back to the case where the category Ker F is zero: That's what we suppose in the following of this proof.

Lemma 3. *For any object M of $\mathcal{A}$, the map $F(M, I)$ from $\mathrm{Hom}_{\mathcal{A}}(M, I)$ to $\mathrm{Hom}_{PC(A)}(FI, FM)$ is a bijection.*

Indeed let f be a morphism from M to I. The image of f by the map $F(M, I)$ is nothing else than $\mathrm{Hom}_{\mathcal{A}}(f, I)$: this last map associates with any element a of $\mathrm{Hom}_{\mathcal{A}}(I, I)$ the element $a \circ f$ of $\mathrm{Hom}_{\mathcal{A}}(M, I)$. Now the morphisms from $A = FI$ to FM are of the form $a \rightsquigarrow a \circ f$. This proves the lemma.

Lemma 4. *Let M be an object of $\mathcal{A}$, $(I_j)_{j \in E}$ be a family of objects isomorphic to I and let J be the direct sum of the family $(I_j)_{j \in E}$. The map $F(M, J)$ from $\mathrm{Hom}_{\mathcal{A}}(M, J)$ to $\mathrm{Hom}_{PC(A)}(FJ, FM)$ is bijective.*

We consider first the case where the object M is of finite length. We know that then the canonical map u from the direct sum of groups $\mathrm{Hom}_{\mathcal{A}}(M, I_j)$ to $\mathrm{Hom}_{\mathcal{A}}(M, J)$ is a bijection (corollary 4, part II); it is the same for the canonical morphism v from the direct sum of $\mathrm{Hom}_{PC(A)}(FI_j, FM)$ to $\mathrm{Hom}_{PC(A)}(FJ, FM)$. In addition, $F(M, I_j)$ is a bijection for any j. The assertion thus results from the equality

$$v \circ (\sum F(M, I_j)) = F(M, J) \circ u$$

In the general case, $\mathrm{Hom}_{\mathcal{A}}(M, J)$ is the projective limit of groups $\mathrm{Hom}_{\mathcal{A}}(N, J)$ when N runs through the sub-objects of finite length of M. Similarly, $\mathrm{Hom}_{PC(A)}(FJ, FM)$ is the projective limit of groups $\mathrm{Hom}_{PC(A)}(FJ, FN)$. We furthermore have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(M, J) & \xrightarrow{F(M,J)} & \mathrm{Hom}_{PC(A)}(FJ, FM) \\ \downarrow \wr & & \downarrow \wr \\ \varprojlim \mathrm{Hom}_{\mathcal{A}}(M, J) & \xrightarrow{\varprojlim F(N,J)} & \varprojlim \mathrm{Hom}_{PC(A)}(FJ, FN) \end{array}$$

Since $F(N, J)$ is an isomorphism for any N, it is the same for $F(M, J)$, which demonstrates the lemma 4.

If the category Ker F is zero, denote by $\mathcal{I}$ the full sub-category of $\mathcal{A}$ whose objects are isomorphic to the direct sum of a family of objects equal to I. Similarly, let $\mathcal{L}$ the full sub-category of $PC(A)$ whose objects are isomorphic to the direct product of a family of pseudo-compact modules equal to A. The lemma 4 shows that the functor F induces an equivalence between $\mathcal{I}$ and the category $\mathcal{L}^o$ of $\mathcal{L}$. The corollary 3 (part I) shows that F defines an equivalence between $\mathcal{A}$ and the category $PC(A)^o$.

Corollary 7. *The hypothesis and the notations are those of theorem 4. If I contains an indecomposable injective of each type, the functor F defines an equivalence between $\mathcal{A}$ and the dual category of $PC(A)$.*

We suppose from now on that I contains an indecomposable injective of each type. If M is any sub-object of I, we denote by $\mathfrak{l}(M)$ the left ideal of A formed of endomorphisms of I whose kernel contains M.

Corollary 8. *The hypothesis and the notations are those of corollary 7. The map $M \rightsquigarrow \mathfrak{l}(M)$ is a bijection from the set of sub-objects of I to the set of closed left ideals of A. The ideal $\mathfrak{l}(M)$ is two-sided if and only if the sub-object M of I is characteristic[i.e. $f(M)$ is contained in M for any endomorphism f of I].*

The demonstration of corollary 8 is left to the reader. We associate with any closed sub-category $\mathcal{C}$ of $\mathcal{A}$ the largest sub-object $\mathcal{C}I$ of I which belongs to $\mathcal{C}$. If f is an endomorphism of I, $f(\mathcal{C}I)$ is contained in $\mathcal{C}I$. In other words, $\mathcal{C}I$ is a characteristic sub-object of I.

Conversely, let J be a characteristic sub-object of I. The objects M for which the canonical map from $\mathrm{Hom}(M, J)$ to $\mathrm{Hom}(M, I)$ is bijective, form a closed sub-category $J\mathcal{A}$ to $\mathcal{A}$. Furthermore, if I contains an indecomposable injective of each type, the maps $\mathcal{C} \to \mathcal{C}I$ and $J \to J\mathcal{A}$ define a one to one correspondence between the set of closed sub-categories from $\mathcal{A}$ to the set of characteristic sub-objects of I. In other words, the map $\mathcal{C} \to \mathfrak{l}(\mathcal{C}I)$ is a bijection from the set of closed sub-categories of $\mathcal{A}$ to the set of closed two-sided ideals of A. In the corollary which follows, we denote by $\mathfrak{l}(\mathcal{C})$ instead of $\mathfrak{l}(\mathcal{C}I)$.

Corollary 9. *The hypothesis and the notations are those of corollary 7. If* $\mathcal{C}$ *and* $\mathcal{D}$ *are two closed sub-categories of* $\mathcal{A}$*, the two-sided ideal* $\mathfrak{l}(\mathcal{C}.\mathcal{D})$ *is the closure of the product* $\mathfrak{l}(\mathcal{C}).\mathfrak{l}(\mathcal{D})$*.*

The proof of the corollary 9 is left to the reader. It follows that with the notations of the proposition 12, $\mathfrak{r}^{(n)}$ is the closure of the product $\mathfrak{r}^n$.

If the injective I contains one and only one indecomposable injective of each type, the socle I_0 of I contains one and only one simple object of each type. We then say that the ring A of endomorphisms of I is the *the pseudo-compact ring associated with* $\mathcal{A}$. This ring is determined up to an isomorphism by the data of the category $\mathcal{A}$. Two equivalent locally finite categories have the associated pseudo-compact rings which are isomorphic.

We will say that a pseudo-compact ring A is sober if the quotient of $\mathcal{A}$ by the Jacobson radical is a product of division rings. We will say that two pseudo-compact rings A and B are equivalent if the categories $PC(A)$ and $PC(B)$ are equivalent. We then have the following results: the pseudo-compact ring associated with a locally finite category is sober. Any pseudo-compact ring is equivalent to a sober pseudo-compact ring. Two sober pseudo-compact rings are equivalent if and only if they are isomorphic.

If A is a pseudo-compact ring, the theorem 4 applies in particular to the category $\mathrm{dis}(A)$ of the discrete topological $A-$modules. If A^* is the pseudo-compact ring associated to $\mathrm{dis}(A)$, the categories $\mathrm{dis}(A)$ and $PC(A^*)$ are dual [i.e. the categories $\mathrm{dis}(A)$ and $PC(A^*)^o$ are equivalent]. It follows that the categories $\mathcal{T}(A)$ and $\mathcal{T}(A^*)$, formed of discrete modules of finite length over A and A^*, are dual. According to the theorem 1 (part II) finally, any duality between $\mathcal{T}(A)$ and $\mathcal{T}(A^*)$ extends to a duality between $PC(A)$ and $\mathrm{dis}(A^*)$. If A is sober, A is thus isomorphic to $(A^*)^*$ and we can say that A^* is the dual pseudo-compact ring of A.

The corollary 7 has the following consequence:

Corollary 10. *If* $\mathcal{A}$ *is a locally finite category and if* A *is the pseudo-compact ring associated with* $\mathcal{A}$*, then* $\mathcal{A}$ *is equivalent to the category of discrete topological modules over the dual ring* A^* *of* A*.*

Corollary 11. *Let* I *be an injective object of a locally finite category* $\mathcal{A}$ *and let* A *be the ring of endomorphisms of* I*. If* I *contains an indecomposable injective of each type, the center* $Z[\mathcal{A}]$ *of the category* $\mathcal{A}$ *(part III, §5) is isomorphic to the center of the ring* A*.*

An element z of $Z[\mathcal{A}]$ is indeed a morphism from the identity functor $I_{\mathcal{A}}$ to itself. In particular, z defines an endomorphism $z(I) : I \to I$. Since $z(I)$ commutes with all the endomorphisms of I, $z(I)$ belongs to the center of A.

We leave it to the reader to verify that the map $z \to z(I)$ is a bijection from $Z[\mathcal{A}]$ to the center of A.

Corollary 12. *Any commutative pseudo-compact ring A is isomorphic to its dual A^*. In particular, the category $\mathcal{T}(A)$ of discrete topological $A-$modules of finite length is equivalent to the dual category $\mathcal{T}(A)^o$.*

Since A is a sober pseudo-compact ring, it indeed suffices to show that the pseudo-compact rings A and A^* are equivalent, that is to say that the categories $\mathcal{T}(A)$ and $\mathcal{T}(A)^o$ are equivalent. We will exhibit a contravariant functor from $\mathcal{T}(A)$ to $\mathcal{T}(A)^o$ which defines an equivalence between $\mathcal{T}(A)$ and $\mathcal{T}(A)^o$.

When $\mathfrak{m}$ runs through the maximal open ideals of A, the quotients $A/\mathfrak{m}$ run through the simple objects of each type of the category $\mathrm{dis}(A)$. We denote by E an injective envelope [in $\mathrm{dis}(A)$] of the direct sum of these quotients.

For any $A-$module M, the abelian group $\mathrm{Hom}_A(M,E)$ is then equipped with a structure of $A-$module: if φ is an element of $\mathrm{Hom}_A(M,E)$ and if a is an element of A, $a.\varphi$ is the map $x \to a.\varphi(x) = \varphi(a.x)$ from M to E. In particular, the annihilator of $\mathrm{Hom}_A(M,E)$ contains the annihilator of M. It follows that $\mathrm{Hom}_A(M,E)$ is a discrete topological $A-$module if M is an object of $\mathcal{T}(A)$. Furthermore, $\mathrm{Hom}_A(M,E)$ is isomorphic to $A/\mathfrak{m}$ if M is isomorphic to $A/\mathfrak{m}$. The exactness of the functor $M \rightsquigarrow \mathrm{Hom}_A(M,E)$ thus implies that $\mathrm{Hom}_A(M,E)$ is a discrete topological $A-$module of the same length as M when M runs through the objects of $\mathcal{T}(A)$. We denote by D the contravariant functor from $\mathcal{T}(A)$ to $\mathcal{T}(A)$ which we have just defined: $DM = \mathrm{Hom}_A(M,E)$.

It suffices to show that the functor $D \circ D$ is isomorphic to the identity functor of $\mathcal{T}(A)$ (part I, proposition 12): if M is a discrete topological $A-$module of finite length and if m is an element of M, we denote by m' the $A-$linear map from $\mathrm{Hom}_A(M,E)$ to E which is defined by the formula $m'(f) = f(m)$. When M varies, the maps $m \to m'$ define a morphism from the identity functor of $\mathcal{T}(A)$ to $D \circ D$. In addition, the map $m \to m'$ is bijective when M is of the form $A/\mathfrak{m}$. The exactness of the functor $D \circ D$ implies that it is still bijective when M is of finite length. This completes the proof.

Part 5. Applications to the study of modules

Here we pick up some properties of commutative rings. We try to generalize to not necessarily commutative rings.

Unless explicitly stated otherwise, all the rings considered in this part admit a unit element. All the modules considered are right unitary modules. We suppose in addition that the underlying set of a ring or of a module is an element of the universe $\mathfrak{U}$.

25. Categories of modules

Let A be a ring. We call *category of (right) modules* over A and we denote by mod$A$ the following category:

- an object of mod$A$ is a module over A;
- if M and N are two modules over A, a morphism from M to N is an $A-$linear map from M to N;
- the composition of morphisms coincide with the composition of maps.

It is clear that mod$A$ is an abelian $\mathfrak{U}-$category with exact inductive limits. Furthermore, the underlying right $A-$module A_d of the ring A is a projective generator of mod$A$. The category mod$A$ is locally noetherian if and only if the ring A is right noetherian.

We define in an analogous way the category of left modules over A.

Now consider an object U of an abelian category $\mathcal{B}$ and let $\chi : A \to \mathrm{Hom}_{\mathcal{B}}(U, U)$ be a homomorphism of rings with unit element. When M runs through the objects of $\mathcal{B}$, the composition of morphisms define a bilinear map from $\mathrm{Hom}_{\mathcal{B}}(U, M) \times A$ to $\mathrm{Hom}_{\mathcal{B}}(U, M) : (f, a) \to f \circ \chi(a)$. This map makes $\mathrm{Hom}_{\mathcal{B}}(U, M)$ an $A-$module. This functor will be denote by $\mathrm{Hom}_{\mathcal{B}}({}_{\chi}U,\ .)$, or $\mathrm{Hom}(U,\ .)$ when no confusion is caused.

Proposition 1. *Let A be a ring, $\mathcal{B}$ be an abelian $\mathfrak{U}-$category and let S be a functor from $\mathcal{B}$ to* mod$A$. *The following assertions are equivalent:*

a. *S is adjoint to a functor $T : \mathrm{mod}A \to \mathcal{B}$.*
b. *There is a homomorphism χ from A to the ring of endomorphisms of an object U of $\mathcal{B}$ which satisfies the following conditions: S is isomorphic to the functor $\mathrm{Hom}_{\mathcal{B}}({}_{\chi}U,\ .)$; furthermore, any family of objects isomorphic to U has a direct sum.*

$(a) \Rightarrow (b)$: Indeed let ψ be a functorial isomorphism from $\mathrm{Hom}_{\mathcal{B}}(T.,\ .)$ to $\mathrm{Hom}_A(., S.)$. Identify A with the ring of endomorphisms of A_d and put $U = TA_d$, $\chi = T(A_d, A_d)$. For any object N of $\mathcal{B}$, $\psi(A_d, N)$ is a bijective map from $\mathrm{Hom}_{\mathcal{B}}(TA_d, N)$ to $\mathrm{Hom}_A(A_d, SN)$. It is easy to see that this map is $A-$linear. Since $\mathrm{Hom}_A(A_d, SN)$ is isomorphic to SN, it results in a functorial isomorphism from $\mathrm{Hom}_{\mathcal{B}}(U,\ .)$ to S.

On the other hand, let $(U_i)_{i \in I}$ be a family formed of objects isomorphic to U and indexed by a set belonging to $\mathfrak{U}$. Let u_i be an isomorphism from U_i to U and put $A_i = A_d$ for any i. If v_i is the canonical map from A_i to the direct sum $\sum_i A_i$, the following equalities show that the morphisms $(Tv_i) \circ u_i$ make $T(\sum_i A_i)$ a direct sum of the family $(U_i)_{i \in I}$:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{B}}(T(\sum_i A_i), N) &= \mathrm{Hom}_A(\sum_i A_i, SN) \\ &= \prod \mathrm{Hom}_A(A_i, SN) = \prod \mathrm{Hom}_{\mathcal{B}}(TA_i, N)\end{aligned}$$

$(b) \Rightarrow (a)$: Indeed choose for any $A-$module M an exact sequence

$$\sum A_j \xrightarrow{q} \sum A_i \xrightarrow{p} M \to 0$$

where $(A_i)_{i \in I}$ and $(A_j)_{j \in J}$are two families of modules equal to A_d. Put $U_j = U$ for any j and $U_i = U$ for any i. Any isomorphism from $\mathrm{Hom}_{\mathcal{B}}({}_\chi U, \ .)$ to S induces functorial isomorphisms:

$$\psi_0 : \mathrm{Hom}_{\mathcal{B}}(\sum_{i \in I} U_i, \ .) \to \mathrm{Hom}_A(\sum_{i \in I} A_i, S.),$$

$$\psi_1 : \mathrm{Hom}_{\mathcal{B}}(\sum_{j \in J} U_j, \ .) \to \mathrm{Hom}_A(\sum_{j \in J} A_j, S.).$$

Furthermore, there is a morphism q' from $\sum_j U_j$ to $\sum_i U_i$ such that we have

$$\mathrm{Hom}_A(q, \ .).\psi_0 = \psi_1 \circ \mathrm{Hom}_{\mathcal{B}}(q', \ .).$$

If we put $TM = \mathrm{Coker}\ q'$, ψ_0 induces an isomorphism $\psi(M, \ .)$ from $\mathrm{Hom}_{\mathcal{B}}(TM, \ .)$ to $\mathrm{Hom}_A(M, S.)$. This shows that the functor

$$N \rightsquigarrow \mathrm{Hom}_A(M, SN)$$

is representable for any M; the sought implication thus results from the dual proposition of the proposition 10 (part I).

When $\mathcal{B}$ is the category of modules over a ring B, the homomorphism χ makes U an $A - B-$bimodule (left $A-$module, right $B-$module). In this case, we can choose for T the functor $M \rightsquigarrow M \otimes_A U$. The functorial isomorphism from $\mathrm{Hom}_B(T., \ .)$ to $\mathrm{Hom}_A(., S.)$ which is described in [6] will be said to be *canonical*.

Proposition 1 (repeated)**.** *Let A be a ring, $\mathcal{B}$ be an abelian category and let T be a functor from* mod$A$ *to $\mathcal{B}$. The following assertions are equivalent:*

a. *There is a functor S adjoint to T.*
b. *The functor T is right exact and it commutes with direct sums.*

$(a) \Rightarrow (b)$: The proposition 11 (part I) shows that T is right exact. An argument analogous to the one which has been used for proving the implication $(a) \Rightarrow (b)$ of the proposition 1, shows that T commutes with the direct sums.

$(b) \Rightarrow (a)$: Indeed put $TA_d = U$ and $\chi = T(A_d, A_d)$. We are going to define a functorial morphism from $\mathrm{Hom}_{\mathcal{B}}(T., \ .)$ to $\mathrm{Hom}_A(., S.)$, where S is the functor $\mathrm{Hom}_{\mathcal{B}}({}_\chi U, \ .)$; for this, let M be an $A-$module, N be an object of $\mathcal{B}$ and let f be a morphism from TM to N. If m is an element of M, we denote by g_m the map $a \rightsquigarrow m.a$ from A to M. We easily verify that the map $m \rightsquigarrow f \circ T g_m$ is an $A-$linear map f' from M to SN. When M and N vary, the maps $f \rightsquigarrow f'$ define a functorial morphism $\psi : \mathrm{Hom}_{\mathcal{B}}(T., \ .) \to \mathrm{Hom}_A(., S.)$.

It remains to demonstrate that the maps $\psi(M, N) : f \rightsquigarrow f'$ are bijective. This is obvious true when M is equal to A_d, N being arbitrary. Since the contravariant functors $\mathrm{Hom}_{\mathcal{B}}(T., N)$ and $\mathrm{Hom}_A(., SN)$ transform the direct sums to direct products, $\psi(M, N)$ is again bijective when M is free. In the general case, we choose an exact sequence

$$L_1 \xrightarrow{q} L_0 \xrightarrow{p} M \to 0$$

where L_0 and L_1 are free modules. The contravariant functors $\mathrm{Hom}_{\mathcal{B}}(T., N)$ and $\mathrm{Hom}_A(., SN)$ are left exact and the maps $\psi(L_1, N)$ and $\psi(L_0, N)$ are bijective. It follows from a classical argument that $\psi(M, N)$ is bijective.

Corollary 1. *Let S be a functor from an abelian category $\mathcal{B}$ to* mod$A$. *The following assertions are equivalent:*

a. *S defines an equivalence between $\mathcal{B}$ and* mod$A$.
b. *There is an isomorphism χ from A to the ring of endomorphisms of an object U of $\mathcal{B}$, which satisfies the following conditions: S is isomorphic to the functor $\mathrm{Hom}_{\mathcal{B}}({}_{\chi}U, \ .)$; the object U is a projective generator of $\mathcal{B}$; any family $(U_i)_{i \in I}$ of objects isomorphic to U has a direct sum; furthermore, the canonical map from $\sum_{i \in I} \mathrm{Hom}_{\mathcal{B}}(U, U_i)$ to $\mathrm{Hom}_{\mathcal{B}}(U, \sum_{i \in I} U_i)$ is bijective.*

$(a) \Rightarrow (b)$: Indeed, S is adjoint to a functor $T : \mathrm{mod}A \to \mathcal{B}$. If we put $U = TA_d$ and $\chi = T(A_d, A_d)$, we saw that S is isomorphic to the functor $\mathrm{Hom}_{\mathcal{B}}({}_{\chi}U, \ .)$. Since T defines an equivalence, $T(A_d, A_d)$ is a bijective map. Since A_d is a projective generator of mod$A$, TA_d is a projective generator of $\mathcal{B}$. Finally, since S defines an equivalence, S commutes with the direct sums. The same is true for the functor $\mathrm{Hom}_{\mathcal{B}}({}_{\chi}U, \ .)$ which proves the last assertion of (b).

$(b) \Rightarrow (a)$: We indeed show that the functor $S' = \mathrm{Hom}_{\mathcal{B}}({}_{\chi}U, \ .)$ defines an equivalence between $\mathcal{B}$ and mod$A$: since U is projective, S' is an exact functor. Since χ is an isomorphism, $S'U$ is identified with A_d. If $(U_i)_{i \in I}$ is a family of objects equal to U, the last assertion of (b) shows that $S'(\sum_{i \in I} U_i)$ is a free $A-$module. If $(U_j)_{j \in J}$ is another family of objects equal to U, the same condition provides the following 'equalities':

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{B}}(\sum_i U_i, \sum_j U_j)\\&=\prod_i\sum_j \mathrm{Hom}_{\mathcal{B}}(U_i,U_j)=\prod_i\sum_j \mathrm{Hom}_A(S'U_i,S'U_j)\\&=\mathrm{Hom}_A(\sum_i S'U_i,\sum_j S'U_j)=\mathrm{Hom}_A(S'\sum_i U_i, S'\sum_j U_j).\end{aligned}$$

Denote by $\mathcal{P}$ (resp. $\mathcal{Q}$) the full sub-categories of mod$A$ (resp. of $\mathcal{B}$) formed of the free $A-$modules (resp. of the objects isomorphic to the direct sum of a family of objects equal to U). What precedes proves that S' induces an equivalent between $\mathcal{Q}$ and $\mathcal{P}$ [proposition 12(b), part I]. The implication thus results from the dual of the proposition 14, corollary 5, (part I).

We remark that we can replace the last two conditions of the assertion (b) by the stronger conditions here: $\mathcal{B}$ is a category with generators and exact inductive limits and U is an object of finite type of $\mathcal{B}$ (i.e. any increasing filtering family of sub-objects of U whose upper bound is equal to U, is stationary).

Corollary 2. *Let A and B be two rings and S be a functor from* mod$B$ *to* mod$A$*. The following conditions are equivalent:*

a. *S defines an equivalence between* mod$B$ *and* mod$A$*.*
b. *There is an isomorphism χ from A to the ring of endomorphisms of a $B-$module U, which satisfies the following conditions: S is isomorphic to the functor* $\mathrm{Hom}_B({}_\chi U,\ .)$*; U is a projective $B-$module of finite type; furthermore, U is a projective generator of* mod$B$*.*

It is obviously the same to give a homomorphism χ from A to the ring of endomorphisms of an $B-$module U and to give U a structure of $A-B-$bimodule (left $A-$module, right $B-$module). In the corollary 3, U is such a bimodule; we denote by ${}_AU$ (resp. by U_B) the underlying left $A-$module (resp. the underlying right $B-$module) of U. If a is an element of A (resp. b an element of B), a_U (resp. b_U) denotes the endomorphism $x \rightsquigarrow a.x$ (resp. $x \rightsquigarrow x.b$) of the underling abelian group of U.

Corollary 3 (Morita[18])**.** *Let A and B be two rings and U be an $A-B-$bimodule (left $A-$module, right $B-$module). The following assertions are equivalent:*

a. *The functor $M \rightsquigarrow M\otimes_A U$ defines an equivalence between* mod$A$ *and* mod$B$*.*
b. *The functor $N \rightsquigarrow \mathrm{Hom}_B(U,N)$ defines an equivalence between* mod$A$ *and* mod$B$*.*
c. *U_B (resp. ${}_AU$) is a projective $B-$module (resp. a projective left $A-$module) of finite type; The map $a \rightsquigarrow a_U$ (resp. $b \rightsquigarrow b_U$) is an isomorphism from A (resp. from the opposite ring of B) to the ring of endomorphisms of U_B (resp. of ${}_AU$).*

The equivalence of (a) and (b) results from the fact that the functor

$$S : N \rightsquigarrow \mathrm{Hom}_B(U, N)$$

is adjoint to the functor $T : M \rightsquigarrow M \otimes_A U$.

$(a) \Rightarrow (c)$: Indeed we have $TA_d = U$ and the map $a \rightsquigarrow a_U$ coincides with $T(A_d, A_d)$. It follows from the demonstration of the corollary 1 that U_B is a $B-$module of finite type and that $a \rightsquigarrow a_U$ is a bijection from A to the ring of endomorphisms of U_B.

According to the corollary 2, there is a $B - A-$bimodule V such that T is isomorphic to the functor $M \rightsquigarrow \mathrm{Hom}_A(V, M)$. Furthermore, V_A is projective, of finite type and the map $b \rightsquigarrow b_V$ is an isomorphism from B to the ring of endomorphisms of V_A.

The ring B operates on the right on the abelian group $\mathrm{Hom}_A(V, A_d)$; the ring of endomorphisms of A_d operates on the left on $\mathrm{Hom}_A(V, A_d)$; this group is thus equipped with a structure of $A - B-$bimodule. Since U is identified with TA_d, the $A - B-$bimodule U and $\mathrm{Hom}_A(V, A_d)$ are isomorphic. The properties of V_A and the following lemma thus imply that ${}_AU$ is projective, of finite type and that the map $b \rightsquigarrow b_U$ is an isomorphism from the ring B^o, opposite to B, to the ring of endomorphisms of ${}_AU$.

Lemma 1. *If M is a left projective $A-$module and of finite type, the left $A-$module $\mathrm{Hom}_A(M, A_d)$ is projective and of finite type. The functor $M \rightsquigarrow \mathrm{Hom}_A(M, A_d)$ defines a duality between the left $A-$modules, projective and of finite type, and the left $A-$modules, projective and of finite type.*

The structure of left $A-$module of $\mathrm{Hom}_A(M, A_d)$ is obviously defined in the following way: if a belongs to A and f belongs to $\mathrm{Hom}_A(M, A_d)$, $a.f$ is such that we have $(a.f)(x) = a.f(x)$ if $x \in M$. The lemma 1 is well known and we will ignore the demonstration.

$(c) \Rightarrow (b)$: According to the corollary 2, if suffices to prove that U_B is a generator of the category mod$B$. For this we will show that B_d is a direct factor of the product U_B^n:

Consider first two left $A-$modules P and Q and let ${}_sA$ be the underlying left $A-$module of the ring A. The abelian group $\mathrm{Hom}_A(P,_s A)$ is equipped in a natural way a structure of a right $A-$module, so that it is legal to talk about the tensor product $\mathrm{Hom}_A(P,_s A) \otimes_A Q$. If q is an element of Q and f an element of $\mathrm{Hom}_A(P,_s A)$, we denote by $v_{q,f}$ the $A-$linear map $p \rightsquigarrow f(p).q$. When q and f vary, the function $(q, f) \rightsquigarrow v_{q,f}$ defines a linear map $v(P, Q)$ from $\mathrm{Hom}_A(P,_s A) \otimes_A Q$ to $\mathrm{Hom}_A(P, Q)$. When P and Q vary, the maps $v(P, Q)$ define a morphism from the functor $(P, Q) \rightsquigarrow \mathrm{Hom}_A(P,_s A) \otimes_A Q$ to the functor $(P, Q) \rightsquigarrow \mathrm{Hom}_A(P, Q)$. It follows in particular that $v(P, Q)$ is a morphism of left modules over the ring C of endomorphisms of Q.

When P is equal to ${}_sA$, $v(P, Q)$ is a bijection. We deduce easily that $v(P, Q)$ is a bijection whenever P is a left projective $A-$module of finite type. If this is true, the left $C-$module $\mathrm{Hom}_A(P, Q)$ is isomorphic to the left $C-$module $\text{Hom}_A(P,_s A) \otimes_A Q$ which is a direct factor of the product Q^n.

We obtain the required result by choosing P and Q equal to ${}_AU$. The ring C is then identified with the opposite ring of B.

Corollary 4. *Let A and B be two rings and U be an $A-B-$bimodule (left $A-$module, right $B-$module). The following assertions are equivalent:*

a. *U_B is a projective $B-$module of finite type; it is a generator of the category* mod$B$*; furthermore, the map $a \rightsquigarrow a_U$ is an isomorphism from A to the ring of endomorphisms of U_B.*
b. *U satisfies the assertion (b) of the corollary 3.*
c. *${}_AU$ is a left projective $A-$module and of finite type; it is a generator of the category of left $A-$modules; furthermore, the map $b \to b_U$ is an isomorphism from the opposite ring of B to the ring of endomorphisms of ${}_AU$.*

The equivalence between (a) and (b) results from the corollaries 2 and 3. It is the same for the equivalence of (b) and (c) provided we replace A and B respectively by B^o and A^o.

Now we give some applications of what precedes:

a. First of all let $\mathfrak{V}$ be a universe such that $\mathfrak{U} \in \mathfrak{V}$, $\mathfrak{U}$ being the universe that we have chosen once and for all at the beginning of this work. On the other hand let $\mathcal{E}$ be the category whose objects are the categories $\mathcal{C}$ such that $\mathcal{MC} \in \mathfrak{V}$ and $\mathcal{OC} \in \mathfrak{V}$, the morphisms of $\mathcal{E}$ being the isomorphism classes of functors (cf. part I, § 8). If A is a ring whose underlying set belongs to $\mathfrak{U}$, the category mod$A$ is an object of $\mathcal{E}$; what precedes allow to determine the group of automorphisms of this object:

   According to the corollaries 2 and 3, a functor $T : \text{mod}A \to \text{mod}A$ defines an equivalence if it is isomorphic to a functor $M \rightsquigarrow M \otimes_A U$, where U is an $A-A-$bimodule satisfying the assertion (c) of the corollary 3. We can thus identify the elements of $G(A)$ with the types of $A-A-$bimodules satisfying the assertion (c) of the corollary 3. Modulo this identification, the law of $G(A)$ is defined by the map $(U, V) \rightsquigarrow V \otimes_A U$. In addition, if U satisfies the assertion (c) of the corollary 3, it is the same for the $A-A-$bimodule $\text{Hom}_A(U, A_d)$; the types of U and $\text{Hom}_A(U, A_d)$ are then the inverse elements of the other.
b. Let B be a ring and U be a free $B-$module B_d^n. The ring of endomorphisms of B_d^n is the ring $M_n(B)$ formed of $n \times n$ matrices with coefficients in B. The corollary 2 shows that the functor $N \rightsquigarrow \text{Hom}_B(B_d^n, M)$ defines an equivalence between B and $M_n(B)$.

   In particular, if M is a $B-$sub-module of B_d^n, $\text{Hom}_B(B_d^n, M)$ is identified with the right ideal $\mathfrak{v}(M)$ of $M_n(B)$ which is formed of the endomorphisms f of B_d^n whose image is contained in M. We find that the map $M \rightsquigarrow \mathfrak{v}(M)$ is a bijection from the set of $B-$sub-modules of B_d^n to the set of right ideals of $M_n(B)$.

c. The example which follows is due to Azumaya [3]: let R be a commutative ring and A be a finite $R-$algebra (i.e. the underlying $R-$module of A is of finite type) which is faithful (i.e. the map $r \rightsquigarrow r.1_A$ is an injection from R to A). We denote by A^o the opposite algebra of A and by A^e the algebra $A \otimes_R A^o$ [6]. The map $(x, a \otimes b) \rightsquigarrow b.x.a$ defines on A a structure of right A^e-module; the map $(r, x) \rightsquigarrow r.x$ defines a structure of left $R-$module on A: the algebra A is thus equipped with a structure of $R-A^e-$bimodule (left R, right A^e). It is this structure that we consider in the following of this paragraph.

Lemma 2. *Let R be a commutative ring, A be an $R-$algebra, S be a commutative $R-$algebra, B be the $S-$algebra deduced from A by extension of scalars and B^e be the tensor product $B \otimes_S B^o$. We suppose that A_{A^e} is a projective A^e-module; B_{B^e} is then a projective B^e-module.*

Indeed let $\mu(A|R)$ be the map $a \otimes b \rightsquigarrow b.a$ from A^e to A. The A^e module A_{A^e} is projective if and only if $\mu(A|R)$ induces an isomorphism from a direct factor of A^e to A. In this case $\mu(A|R) \otimes_R S$ induces an isomorphism from a direct factor of $A^e \otimes_R S$ to $A \otimes_R S$. The lemma results from that $\mu(A|R) \otimes_R S$, $A^e \otimes_R S$ and $A \otimes_R S$ are 'identified' respectively with $\mu(B|S)$, B^e and B.

Lemma 3. *The hypothesis are those of the lemma 2; for any prime ideal $\mathfrak{p}$ of R, $(A/\mathfrak{p}A)_\mathfrak{p}$ is then a semi-simple separable algebra over $(R/\mathfrak{p}R)_\mathfrak{p}$.*

We apply the lemma 2 by choosing for S the algebra $(R/\mathfrak{p}R)_\mathfrak{p}$. In this case, B coincides with $(A/\mathfrak{p}A)_\mathfrak{p}$. Since S is a division ring, we know on the other hand that B_{B^e} is a projective B^e-module if and only if B is a separable $S-$algebra (cf. [6], chap. VI).

Lemma 4. *The hypothesis are those of the lemma 2. If i is the canonical map from the center $Z(A)$ of A to A, then $i \otimes_R S$ is an isomorphism from $Z(A) \otimes_R S$ to the center of $A \otimes -RS$.*

The centers of A and of B are indeed identified with $\mathrm{Hom}_{A^e}(A, A)$ and $\mathrm{Hom}_{B^e}(B, B)$. The lemma results from this and from the fact that the canonical map from $\mathrm{Hom}_{A^e}(M, N) \otimes_R S$ to $\mathrm{Hom}_{B^e}(M \otimes_R S, N \otimes_R S)$ is a bijection when M is a projective A^e-module of finite type.

Lemma 5. *If R is a commutative ring and M is a projective $R-$module of finite type, the following assertions are equivalent:*

a. *M is a generator of* $\mathrm{mod}R$.
b. *M is a faithful $R-$module (i.e. the annihilator of M is zero).*

The implication $(a) \Rightarrow (b)$ is clear. Prove the converse: it suffices to show that $\mathrm{Hom}_R(M, N)$ is non zero if N is non zero; so let $\mathfrak{m}$ be a maximal ideal of R such that $N_\mathfrak{m}$ is non zero. It is clear that $M_\mathfrak{m}$ is a projective $R_\mathfrak{m}-$module (thus free) and that the annihilator of $M_\mathfrak{m}$ is zero. Since M is of finite presentation, $(\mathrm{Hom}_R(M,N))_\mathfrak{m}$ is identified with $\mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m},N_\mathfrak{m})$ [5]. Since this last module is non zero, it is the same for $(\mathrm{Hom}_R(M,N))_\mathfrak{m}$, thus for $\mathrm{Hom}_R(M,N)$.

Proposition 2. *Let R be a commutative ring and A be a finite and faithful $R-$algebra. The following assertions are equivalent:*

a. *The functor $N \rightsquigarrow N\otimes_R A$ defines an equivalence between* mod$R$ *and* modA^e.
b. *The functor $M \rightsquigarrow \mathrm{Hom}_{A^e}(A,M)$ defines an equivalence between* modA^e *and* mod$R$.
c. *The right A^e-module A_{A^e} is projective; furthermore, the map $r \rightsquigarrow r.1_A$ is an isomorphism from R to the center of A.*
d. *The left $R-$module $_RA$ is projective; furthermore, the map $b \rightsquigarrow b_A$ is an isomorphism from the opposite ring of A^e to the ring of endomorphisms of $_RA$.*
e. *The left $R-$module is projective; for any maximal ideal $\mathfrak{m}$ of R, $A/\mathfrak{m}A$ is in addition a simple and central algebra over A.*

The equivalence between (a) and (b) results from the corollary 3. The equivalence between (a) and (d) results from the lemma 5 and the corollaries 2 and 4. The implication $(b)\Rightarrow(c)$ results from the corollary 3 and from the fact that $\mathrm{Hom}_{A^e}(A,A)$ is the center of A.

$(c)\Rightarrow(b)$: According to the corollary 3, it suffices to prove that A_{A^e} is a generator of modA^e. It is equivalent to say that, for any maximal right ideal $\mathfrak{m}$ of A^e, there is a A^e-linear map $f: A\to A^e$ such that Im f is not contained in $\mathfrak{m}$. Now $\mathfrak{m}\cap R$ is a prime ideal $\mathfrak{p}$ of R and $A^e/\mathfrak{m}$ is a simple module over the ring $B^e = B\otimes_S B^o$, where $S=(R/\mathfrak{p}R)_\mathfrak{p}$ and $B=(A/\mathfrak{p}A)_\mathfrak{p}$. Since B is a simple and central algebra over S according to the lemma 3 and 4, B^e is a simple ring and the abelian group $\mathrm{Hom}_{B^e}(B,A^e/\mathfrak{m})$ is non zero. Since $\mathrm{Hom}_{B^e}(B,A^e/\mathfrak{m})$ is identified with $(\mathrm{Hom}_{A^e}(A,A^e/\mathfrak{m}))_\mathfrak{p}$, we conclude that $\mathrm{Hom}_{A^e}(A^e/\mathfrak{m})$ is non zero. Finally, since A is projective, any non zero morphism from A to $A^e/\mathfrak{m}$ factors through A^e by a morphism $f: A\to A^e$ such that Im $f \nsubseteq \mathfrak{m}$.

(c), $(d)\Rightarrow(e)$: Results from the lemmas 3 and 4.

$(e)\Rightarrow(d)$: Indeed put $L_R(A)=\mathrm{Hom}_R(A,A)$. On the other hand let $\varphi(A|R)$ be the map $b\rightsquigarrow b_A$ from the ring $A^o\otimes_R A$, the opposite ring of A^e, to $L_R(A)$.

Since $_RA$ is projective, $L_R(A)$ and $A^o\otimes_R A$ are projective $R-$modules of finite type. To show that $\varphi(A|R)$ is a bijection, it thus suffices to prove that $\varphi(A|R)\otimes_R R/\mathfrak{m}$ is a bijection for any maximal ideal $\mathfrak{m}$ of R. Now $(A^o\otimes_R A)\otimes_R R/\mathfrak{m}$, $L_R(A)\otimes_R R/\mathfrak{m}$ and $\varphi(A|R)\otimes_R R/\mathfrak{m}$ are identified respectively with $B^o\otimes_S B$, $L_S(B)$ and $\varphi(B|S)$, where we have put $B=A/\mathfrak{m}$ and $S=R/\mathfrak{m}$. Since B is a simple and central algebra over $R/\mathfrak{m}$, $\varphi(B|S)$ is a bijection for any $\mathfrak{m}$; the assertion (c) then follows.

Corollary 5. *Suppose the equivalent conditions of the proposition 2 are satisfied. The map* $\mathfrak{a} \to \mathfrak{a}.A$ *is a bijection from the set of ideals of* R *to the set of two-sided ideals of* A.

Indeed the canonical injection from $\mathfrak{a}$ to R defines an injection $\mathfrak{a} \otimes_R A$ to $R \otimes_R A = A$. The image of this injection is nothing else than $\mathfrak{a}.A$ and we confuse $\mathfrak{a} \otimes_R A$ with this image. Since the functor $N \rightsquigarrow N \otimes_R A$ defines an equivalence between mod$R$ and nidA^e, the map $\mathfrak{a} \to \mathfrak{a} \otimes_R A$ is a bijection from the set of $R-$sub-modules of R to the set of A^e-sub-modules of A; these coincide with the two-sided ideals of A. We refer the reader to [1] for the other applications of the proposition 2.

26. The localization

Let A be a ring. A topology on A is said to be *(right) linear* if it makes A a topological ring and if there is a base of neighbourhoods of O formed of right ideals. In this case, the set F of open right ideals satisfies the following conditions:

a. If $\mathfrak{m}$ is a right ideal containing an element $\mathfrak{l}$ of F, then $\mathfrak{m}$ belongs to F.
b. If $\mathfrak{m} \in F$ and if $\mathfrak{l} \in F$, then $\mathfrak{m} \cap \mathfrak{l} \in F$.
c. If $\mathfrak{l} \in F$ and if $a \in A$, then $(\mathfrak{l} : a) = \{x | a.x \in \mathfrak{l}\}$ belongs to F.

We say that a set F, formed of right ideals of A, is *topologizing* if the conditions (a), (b) and (c) are satisfied. We say also that an $A-$module M is $F-$negligible if the annihilator of any element of M is a right ideal belonging to F. If $\mathcal{F}$ is the full sub-category of mod$A$ whose objects are the $F-$negligible $A-$modules, $\mathcal{F}$ is a closed sub-category (cf. part IV, § 4).

Lemma 6. *The map* $F \to \mathcal{F}$ *is a bijection from the set of topologizing sets of right ideals to the set of closed sub-categories of* mod$A$.

If we know $\mathcal{F}$, we can indeed find F: a right ideal $\mathfrak{l}$ belongs to F if and only if $A/\mathfrak{l}$ is an object of $\mathcal{F}$.

If F and G are two topologizing sets of right ideals, $F.G$ denotes the topologizing set associated to the product $\mathcal{F}.\mathcal{G}$ (part IV, § 4). It is the same to say that an ideal $\mathfrak{m}$ belongs to $F.G$ if for any $a \in A$, there is an element $\mathfrak{l}$ of F such that $(a.\mathfrak{l} + \mathfrak{m})/\mathfrak{m}$ is $G-$negligible. In particular, if $\mathfrak{p}$ is an element of F and $\mathfrak{q}$ is an element of G, then $\mathfrak{p}.\mathfrak{q}$ belongs to $F.G$. With these conventions, the closed sub-category $\mathcal{F}$ is localizing if and only if $F.F$ is equal to F. The localizing sub-categories of mod$A$ therefore correspond bijectively to the idempotent topologizing sets of right ideals.

Now let F be an idempotent topologizing set of right ideals. For any $A-$module M, we choose an $\mathcal{F}-$envelope that we denote by $u_M : M \to M_F$ (cf. part III). Since M_F is an $A-$module, any element m from M_F defines an $A-$linear map from A_d to $M_F : a \rightsquigarrow m.a$. This map extends in one and only one way to an $A-$linear map from $A_F = (A_d)_F$ to M_F; we thus define a bilinear map:

$$M_F \times A_F \to M_F.$$

We leave it to the reader to verify that, if M is equal to A_d, this bilinear map makes A_F a ring. If M is arbitrary, this map makes M_F an A_F-module. In particular, if $\mathfrak{l}$ is a right ideal of A, we can suppose that we chose for $\mathfrak{l}_F$ a right ideal of A_F.

In the following, we always equip A_F and M_F with the structures that we have just made explicit. We also use the following notations: T is the canonical functor from mod$A$ to mod$A/\mathcal{F}$; S' is the functor from mod$A/\mathcal{F}$ to modA_F which is induced by the localizing functor; ρ is the functor which associates with any A_F-module N the underlying $A-$module (for the structure of $A-$module induced by u_A). With these notations, $\rho \circ S'$ is a functor adjoint to T.

Lemma 7. *Let M be an $A-$module, N be an A_F-module. The map $\rho(N, M_F)$ from $\mathrm{Hom}_{A_F}(N, M_F)$ to $\mathrm{Hom}_A(\rho N, \rho M_F)$ is bijective.*

Indeed it is clear that $\rho(N, M_F)$ is an injection. So let f be an $A-$linear map from N to M_F. If n is an element of N, we want to show that we have $f(n.a) = f(n).a$ for any a of A_F. Now the maps $g : a \rightsquigarrow f(n.a)$ and $h : a \rightsquigarrow f(n).a$ are $A-$linear, and we have the equality $g \circ u_A = h \circ u_A$. We conclude the equality of g and h (definition of $\mathcal{F}-$closed objects, lemma 5, § 2, part III).

Let F' be the set of right ideals $\mathfrak{l}$ of A_F such that $\rho(A_F/\mathfrak{l})$ is $F-$negligible. It is clear that F' is idempotent topologizing, and that an A_F-module N is $F'-$negligible if and only if ${}_\rho N$ is $F-$negligible. We can thus talk about the canonical functor T' from modA_F to the quotient category mod$A_F/\mathcal{F}'$.

Proposition 3. *Let A be a ring and F be an idempotent topologizing set of right ideals. With the above notations, the functor $T \circ \rho$ defines by passing to the quotient an equivalence between* mod$A_F/\mathcal{F}'$ *and* mod$A/\mathcal{F}$.

Indeed let $\mathcal{B}$ be the quotient category mod$A/\mathcal{F}$; show that the functor S' is adjoint to $T \circ \rho$: if M is an object of $\mathcal{B}$, N an object of modA_F, the abelian group $\mathrm{Hom}_{\mathcal{B}}(T\rho N, M)$ is identified with $\mathrm{Hom}_A(\rho N, \rho S'M)$, because $\rho \circ S'$ is adjoint to T. According to the lemma 7, $\mathrm{Hom}_A(\rho N, \rho S'M)$ 'is identified' with $\mathrm{Hom}_{A_F}(N, S'M)$. This proves that S' is adjoint to $T \circ \rho$. The proposition thus follows from the proposition 5 (part III).

Corollary 6. *An A_F-module N is $\mathcal{F}'-$closed (resp. $\mathcal{F}'-$closed and injective) if and only if ρN is $\mathcal{F}-$closed (resp. $\mathcal{F}-$closed and injective).*

This corollary results from the previous proposition and from the corollary 4 (part III).

If M is an $A-$module, we denote by FM the largest $F-$negligible submodule of M . The underlying abelian group of M_F is identified with $\mathrm{Hom}_A(A_d, M_F)$; this last group is itself isomorphic to $\mathrm{Hom}_{\mathrm{mod}A/\mathcal{F}}(A_d, M)$, that is to say, the inductive limit $\varinjlim_{\mathfrak{l} \in F} \mathrm{Hom}_A(\mathfrak{l}, M/FM)$.

It is comfortable to equip this inductive limit with a structure of $A-$module such that the previous result from M_F to $\varinjlim \mathrm{Hom}_A(\mathfrak{l}, M/FM)$ is an isomorphism of $A-$modules. This results in the following proposition:

Proposition 4. *Let A be a ring and F be an idempotent topologizing set of right ideals. The localization functor defined by F is isomorphic to the functor $M \rightsquigarrow \varinjlim_{\mathfrak{l}\in F} \mathrm{Hom}_A(\mathfrak{l}, M/FM)$.*

Corollary 7. *The hypothesis are those of the proposition. We suppose in addition that the injective envelope of an object of $\mathcal{F}$ belongs to $\mathcal{F}$. The localization functor defined by F is then isomorphic to the functor $M \rightsquigarrow \varinjlim_{\mathfrak{l}\in F} \mathrm{Hom}_A(\mathfrak{l}, M)$.*

Indeed consider the following exact sequence:

$$0 \to \varinjlim \mathrm{Hom}_A(\mathfrak{l}, FM) \xrightarrow{i} \varinjlim \mathrm{Hom}_A(\mathfrak{l}, M) \xrightarrow{p} \varinjlim \mathrm{Hom}_A(\mathfrak{l}, M/FM)$$
$$\xrightarrow{d} \varinjlim \mathrm{Ext}^1_A(\mathfrak{l}, FM)$$

We check immediately that $\mathrm{Hom}_A(\mathfrak{l}, N)$ is zero if N is $F-$negligible. It results that p is an injection. On the other hand, the functor $N \rightsquigarrow \mathrm{Ext}^1_A(\mathfrak{l}, N)$ is the satellite of the functor $N \rightsquigarrow \mathrm{Hom}_A(\mathfrak{l}, N)$. If N is $F-$negligible, there is an exact sequence

$$0 \to N \xrightarrow{u} I \xrightarrow{v} N' \to 0$$

where I is an injective $F-$negligible $A-$module. We conclude that $\mathrm{Ext}^1_A(\mathfrak{l}, N)$ is zero if N is $F-$negligible. In particular, the map d is surjective.

Corollary 8. *Let A be a ring and let F be an idempotent topologizing set of right ideals. Supppose that there is a sub-set G of F, cofinal and formed of right ideals of finite type. If the localization functor defined by F is exact, this functor is isomorphic to the functor $M \rightsquigarrow M \otimes_A A_F$; furthermore, the categories* $\mathrm{mod}A/\mathcal{F}$ *and* $\mathrm{mod}A_F$ *are then equivalent.*

Show first that the functor $M \rightsquigarrow \varinjlim_{\mathfrak{l}\in G} \mathrm{Hom}_A(\mathfrak{l}, M)$ commutes with direct sums. This is obviously true for the functors $M \rightsquigarrow FM$ and $M \rightsquigarrow M/FM$; if N is an $A-$module of finite type, the functor $\mathrm{Hom}_A(N, \ .)$ commutes with direct sums. It follows that the functor $M \to \mathrm{Hom}_A(\mathfrak{l}, M)$ commutes with direct sums if $\mathfrak{l}$ is an element of G. The first assertion of the corollary then results [see the proposition 1 (repeated) and the remark that precedes it].

On the other hand let $\mathcal{B}$ be the quotient category $\mathrm{mod}A/\mathcal{F}$, let T be the canonical functor from $\mathrm{mod}A$ to $\mathcal{B}$ and let S be a functor adjoint to T. The functor $\mathrm{Hom}_{\mathcal{B}}(A_F, \ .)$ is isomorphic to the functor $\mathrm{Hom}_A(A, S.)$; since S is exact, A_F is a projective object of $\mathcal{B}$; since S commutes with direct sums, it is the same for $\mathrm{Hom}_{\mathcal{B}}(A_F, \ .)$. The last assertion thus results from the corollary 1.

Here is an application of the corollary 6: let S be a multiplicative subset of A (sub-set of A such that $a.b \in S$ if $a \in S$ and $b \in S$). The data of

S admit to define an idempotent and topologizing set F_S of right ideals: a right ideal $\mathfrak{l}$ belongs to F_S if, for any $a \in A$, there is an $s \in S$ such that $a.s$ belongs to $\mathfrak{l}$. It is the same to say that an $A-$module M is F_S-negligible if any element m of M is annihilated by an element s of S.

Any ideal belonging to F_S obviously meet S; the converse is true if we have $\forall s \in S$, $\forall a \in A$, $\exists t \in S$, $\exists b \in A$ such that we have $a.t = s.b$. We will find this condition again; for this, we denote by M_S instead of M_{F_S}:

Definition 1. *Let $\varphi : A \to B$ be a homomorphism of rings with unit element and S be a multiplicative sub-set of A. We say that (B, φ) is a ring of right fractions of A for S if we have:*

a. Ker φ *is F_S-negligible; in other words, if $\varphi(a)$ is zero, there is $s \in S$ such that we have $a.s = 0$.*
b. *The image under φ of any element of S is invertible (left and right).*
c. *Any element b of B is of the form $b = (\varphi a).\varphi(s)^{-1}$, $a \in A$, $s \in S$.*

Proposition 5. *Let A be a ring and S be a multiplicative sub-set of A. The following assertions are equivalent:*

a. *There is a ring of right fractions of A for S.*
b. *S satisfies the following conditions:*

($\bigstar$) $\forall s \in S, \quad \forall a \in A, \quad \exists t \in S, \quad \exists b \in A,$ *such that we have* $a.t = s.b.$

($\bigstar\bigstar$) *if* $a \in A$,*if* $s \in S$ *and if* $s.a = 0$,
there is $t \in S$ *such that* $a.t = 0$

c. *The image of an element of S under the map $u_A : A \to A_S$ is invertible.*

If these conditions are fulfilled and if (B, φ) is a ring of right fractions of A for S, there is one and only one isomorphism ψ from B to A_F such that we have $\psi \circ \varphi = u_A$. The functor $M \rightsquigarrow M_S$ is then exact if it is isomorphic to the functor $M \rightsquigarrow M \otimes_A A_S$.

$(a) \Rightarrow (b)$: The assertion ($\bigstar\bigstar$) is clear because $\varphi(a)$ is zero if a is zero. On the other hand, $\varphi(s)^{-1}.\varphi(a)$ is an element of B, and is thus of the form

$$\varphi(s)^{-1}.\varphi(a) = \varphi(c).\varphi(r)^{-1}, \quad r \in S, \quad c \in A.$$

We conclude the equality $\varphi(a).\varphi(r) = \varphi(s).\varphi(c)$. In other words, $a.r - s.c$ belongs to Ker φ; consequently there is a $u \in S$ such that we have $a.r.u - s.c.u = 0$. We put $t = r.u$ and $b = c.u$.

$(b) \Rightarrow (c)$: We show first that the conditions ($\bigstar$) and ($\bigstar\bigstar$) imply the exactness of the functor $M \rightsquigarrow M_S$. Suppose we are given a diagram

$$\begin{array}{ccccc} & & \mathfrak{l} & & \\ & & \downarrow{\scriptstyle u} & & \\ M & \xrightarrow{p} & N & \longrightarrow & 0 \end{array}$$

where $\mathfrak{l}$ belongs to F_S and where M and Ker p are F_S-closed. It suffices to show that $\mathfrak{l}$ contains an ideal $\mathfrak{m} \in F_S$ such that the restriction of u to $\mathfrak{m}$ is of the form $p \circ v$, where $v : \mathfrak{m} \to M$ [proposition 7(b), part III].

If s belongs to $\mathfrak{l} \cap S$, $u(s)$ is of the form $p(m)$, where $m \in M$; furthermore, $m.a$ is zero if $s.a$ is zero. Thus we put $\mathfrak{m} = s.A$ and $v(s.a) = m.a$.

The exactness of the functor $M \rightsquigarrow M_S$ implies that $u_A(s)$ is right invertible for any $s \in S$ (indeed let g_s be the map $a \rightsquigarrow s.a$; since Coker g_s is F_S-negligible, g_s induces an epimorphism from A_S to A_S). It follows that $u_A(s).(s'.u_A(s) - 1)$ is zero if s' is such that we have $u_A(s).s' = 1$. Since $u_A(s)$is a left and right regular element [condition (★★)], we have $s'.u_A(s) = 1$.

(c) ⇒ (a): This is clear, similar to the uniqueness of the ring of fractions.

The last assertion results from the corollary 8.

Remark 1. *The conditions* (★) *and* (★★) *are always satisfied when S is contained in the center of A. We then find the localization in relation to the center (part III,* § 5*).*

Remark 2. *If the conditions* (★) *and* (★★) *of the proposition 5 are satisfied, we can exhibit a construction of M_S closer to the established traditions: for this, we should say that two elements (m, s) and (n, t) of $M \times S$ are equivalent if there are two elements u and v of A such that we have $s.u = t.v \in S$ and $m.u = n.v$. The underlying set of M_S is then chosen equal to the quotient of $M \times S$ by the equivalence relation that we have just defined. The definition of laws of addition and multiplication is easy.*

Remark 3. *We leave it to the reader to prove that, if A is commutative, A_F is commutative for any idempotent topologizing set F of ideals; furthermore, the map $\mathfrak{p} \rightsquigarrow \mathfrak{p}_F$ is a bijection from the set of prime ideals of A not belonging to F to the set of prime ideals of A_F not belonging to F'.*

27. The theorem of Goldie

This paragraph is devoted a theorem of Goldie that we will prove by our methods. Let A be a ring and let F the set of right ideals of A whose canonical morphism to A_d is an essential extension. For a right ideal $\mathfrak{l}$ belongs to F, it is necessary and sufficient that the following condition is satisfied: for any non zero element a of A, there is $b \in A$ such that $a.b$ is non zero and belongs to $\mathfrak{l}$. It follows easily from this condition that the set F is topologizing. We can thus talk about the largest $F-$negligible right ideal; this ideal is a characteristic sub-module of A_d, that is to say, a two-sided ideal of A. In this paragraph, we suppose that this two-sided ideal is zero; in other words, we suppose that the following assertion is true:

(★) If A is an essential extension of the right ideal $\mathfrak{l}$ and if $x.\mathfrak{l}$ is zero, $x \in A$,

then x is zero.

Lemma 8. *If the condition* (★) *is satisfied, then the set F of right ideals whose canonical morphism to A_d is an essential extension, is idempotent topologizing.*

Show that F is idempotent: Let $\mathfrak{m}$ be an element of $F.F$ and let a be a non zero element of A; there is an element $\mathfrak{l}$ of F such that $(a.\mathfrak{l}+\mathfrak{m})/\mathfrak{m}$ is $F-$negligible (cf. § 2). According to (★), the product $a.\mathfrak{l}$ is non zero. Thus there is a $l \in \mathfrak{l}$ such that $a.l$ is non zero and such that $a.l.\mathfrak{n}$ is contained in $\mathfrak{m}$ for at least one $\mathfrak{n} \in F$. This shows that $\mathfrak{m}$ belongs to F.

The set F thus defines a localization. The condition (★) implies that the canonical map from A to A_F is injective. Furthermore, we have the lemma:

Lemma 9. *A_F is an injective $A-$module.*

Indeed let $f : \mathfrak{l} \to A_F$ be an $A-$linear map from a right ideal of A to A_F. If $\mathfrak{l}$ is an element of F, f extends obviously to A (part III, § 2). If $\mathfrak{l}$ is an arbitrary right ideal, let $\mathfrak{m}$ be a complement of $\mathfrak{l}$ in A_d (part II, § 5); then $\mathfrak{l}+\mathfrak{m}$ is isomorphic to the direct sum $\mathfrak{l} \oplus \mathfrak{m}$ and f extends to an $A-$linear map $g : \mathfrak{l}+\mathfrak{m} \to A_F$. Since A is an essential extension of $\mathfrak{l}+\mathfrak{m}$, $\mathfrak{l}+\mathfrak{m}$ belongs to F and g extends to A.

Lemma 10. *Any $F-$closed $A-$module M contained in A_F is a direct factor of A_F.*

Identify A with its image in A_F and let N be a complement of $M \cap A$ in A. The module A_d is then an essential extension of $M \cap A + N$, and the quotient $A_F/(M+N)$ is $F-$negligible. It follows that the canonical injection from $M+N$ to A_F is an $F-$envelope. The lemma thus results from the formula

$$A_F = (M \oplus N)_F = M_F \oplus N_F = M \oplus N_F.$$

Lemma 11. *The ring A_F is regular in the sense of Von* Neumann *(*[4]*, § 6, exercise 15) (we also say 'absolutely flat' instead of regular,* [5]*, chap. I, § 2, exercises).*

We want to show that any cyclic ideal $a.A_F$ is given by an idempotent. For this, we denote by f_a the endomorphism $b \rightsquigarrow a.b$ of A_F. The map $a \rightsquigarrow f_a$ is an isomorphism from the ring A_F to the ring of endomorphisms of the $A-$module A_F. Since A_F is $F-$closed, Ker f_a is $F-$closed and is a direct factor of A_F (lemma 10). If M is a complement of Ker f_a, M is injective (lemma 9) and f_a induces an isomorphism from M to a direct factor of A_F. The lemma 11 thus results from the lemma 12:

Lemma 12. *Let A be a ring, M be an $A-$module, B be the ring of endomorphisms of M. If b is an element of B, the following assertions are equivalent.*

a. *The right ideal $b.B$ is a direct factor of B_d.*
b. *The left ideal $B.b$ is a direct factor of ${}_sB$.*
c. Ker b *and* Im b *are direct factors of M.*

The proof of this lemma is obvious and is left to the reader.

The corollary of the proposition 3 shows that A_F is also an injective A_F-module. We also remark that with the notations of paragraph 2, F' is none other than the set of right ideals of A_F whose canonical morphism to A_F is an essential extension. Furthermore, the ring A_F also satisfies the condition (★).

We summarize the previous lemmas in the

Theorem 1. *If the ring A satisfies the condition (★), the set of right ideals whose canonical morphism to A_d is an essential extension, is topologizing and idempotent. The ring A_F is regular and the underlying right A_F-module of A_F is injective. As an $A-$module, A_F is the injective envelope of A_d.*

Let's mention two applications of the theorem 1.

a. Let A be a ring that any non zero right ideal contains a non zero idempotent. It is the same to say that for any non zero element a of A, there is a non zero element $x \in A$ such that we have $x.a.x = x$ (this happens if A is a regular ring in the sense of Von NEUMANN). If $\mathfrak{l}$ is a right ideal of A and if $x.\mathfrak{l}$ is zero, $\mathfrak{l}$ annihilates any left ideal of the form $A.e$, where e runs through the idempotents contained in $A.x$. If x is non zero, we can choose e different from O. Then $\mathfrak{l}$ is contained in $(1-e).A$ and A_d cannot be an essential extension of $\mathfrak{l}$. This proves the condition (★).
b. Any integral ring and any quasi simple ring (there is no two-sided non zero proper ideal) can be embedded into a regular ring. We will see that, under the noetherian conditions, this regular ring is a simple ring.

Lemma 13. *If the ring A satisfies the condition (★), the following assertions are equivalent:*

a. *The ring A_F is semi-simple.*
b. *There does not exist an infinite family formed of right ideals of A whose sum is direct.*

$(a) \Rightarrow (b)$: Indeed let n be the length of the underlying A_F-module of A_F. Let $(\mathfrak{l}_i)_{i\in I}$ be a family of right ideals of A whose sum is direct. The formula

$$(\sum_{i\in I} \mathfrak{l}_i)_F = \sum_{i\in I} (\mathfrak{l}_i)_F$$

shows that the number of elements of I is smaller than n.

$(b) \Rightarrow (a)$: If the assertion (b) is true, a classical argument proves the existence of a finite family $(\mathfrak{l}_i)_{i\in I}$, formed of right ideals of A satisfying the following conditions: the sum of $\mathfrak{l}_i$ is direct and A_d is an essential extension of this direct sum; furthermore, $\mathfrak{l}_i$ is a coirreducible $A-$module (cf. part II, § 5; in a general way, it is equivalent to say that the injective envelope of an $A-$module M is the direct sum of a finite family of indecomposable injectives or to say that there does not exist an infinite family of sub-modules of M

whose sum is direct). It follows that the right ideals $(\mathfrak{l}_i)_F$ are indecomposable and that their sum is direct and equal to A_F; the regular ring A_F is thus the direct sum of a finite family of indecomposable rings. This proves (a).

Lemma 14. *Let A be a ring satisfying the following conditions:*

a. *Any nilpotent ideal is zero.*
b. *The right ideals of the form $(0 : a) = \{x|x \in A, a.x = 0\}$ satisfies the ascending chain condition.*

The ring A then satisfies the condition $(\bigstar)$.

Let a be a non zero element of A and we prove that A_d is not an essential extension of $(0 : a)$. For this, we choose an element $c \in a.A$ such that $(o : c)$ is maximal among the ideals of the form $(0 : x)$, where $x \in a.A$ and $x \neq 0$. Since $c.A$ is not nilpotent, there is an element $d \in A$ such that $a.d.c$ is different from 0. It then follows from the inclusions:

$$(0 : c) \subset (0 : d.c) \subset (0 : a.d.c)$$

and from the maximality of $(0 : c)$ that $(0 : d.c)$ is equal to $(o : a.d.c)$. In other words, the equality $a.d.c.x = 0$ implies the equality $d.c.x = 0$. This shows that the intersection $d.c.A \cap (0 : a)$ is zero and that A_d is non an essential extension of $(0 : a)$.

Theorem 2 (Goldie)**.** *Let A be a ring satisfying the following conditions:*

a. *Any nilpotent ideal is zero.*
b. *The right ideals of the form $(0 : a) = \{x|x \in A, a.x = 0\}$ satisfies the ascending chain condition.*
c. *There does not exist an infinite family formed of right ideals of A whose sum is direct.*

The ring A satisfies the condition $(\bigstar)$ and the localized A_F of A is a semi-simple ring. If S is the multiplicative sub-set formed of the regular elements of A, the couple (A_F, u_A) is a ring of right fractions of A for S.

It remains to show the last assertion: if $s \in A$ is left regular (i.e. $s.x = 0$ implies $x = 0$), the morphism $x \rightsquigarrow s.x$ defines injective endomorphisms of A and of A_F. This morphism is thus an automorphism of A_F and s is invertible in A_F; in particular, s is right regular in A. The multiplicative sub-set S is thus formed of left regular elements of A. To complete the demonstration, it suffices to show that with the notations of the paragraph 2, F is equal to F_S [proposition 5, (c)]. For this, we prove that a right ideal $\mathfrak{l}$ belongs to F if and only if $\mathfrak{l}$ contains an $s \in S$.

If s is a regular element of A, the ideal $s.A_F$ is equal to A_F. It results in that $s.A$ belongs to F.

Conversely, let $\mathfrak{l}$ be an element of F. We will construct two sequences $(e_1, \cdots, e_n)$ and $(a_1, \cdots, a_n)$, formed of elements of A_F and satisfying the following conditions: for any i, e_i is a primitive idempotent of A_F; A_F is the direct sum of indecomposable ideals $e_i.A_F$; for any i, $e_i.a_i$ is non zero and belongs to $\mathfrak{l}$; if f_i denotes the morphism $x \rightsquigarrow e_i.a_i.x$, we have the equality

$\text{Ker } f_1 \cap \cdots \cap \text{Ker } f_n = 0$. If we put $s = e_1.a_1 + \cdots + e_n.a_n$, it follows from these conditions that the image of A_F under the morphism $x \rightsquigarrow s.x$ meets any of the ideals $e_i.A_F$ (because $s.x$ belongs to $e_i.A_F$ if x belongs to the intersection of Ker f_j for $j \neq i$); this image being $F-$closed, it coincides with A_F and the morphism is bijective. The element s is thus regular and belongs to $\mathfrak{l}$.

Take for e_1 an arbitrary primitive idempotent of A_F. Let $\mathfrak{m}_1$ be an element of F such that $e_1.\mathfrak{m}_1$ is contained in $\mathfrak{l}$. Since $e_1.\mathfrak{m}_1$ is not nilpotent, there is $a_1 \in \mathfrak{m}_1$ such that $e_1.a_1.e_1$ is non zero. We conclude from there the formula

$$A_F = e_1.A_F \oplus \text{Ker } f_1.$$

Take for e_2 a primitive idempotent belonging to Ker f_1. Let $\mathfrak{m}_2$ be an element of F such that $e_2.\mathfrak{m}_2$ is contained in $\mathfrak{l}$. Since $e_2.\mathfrak{m}_2$ is not nilpotent, there is $a_2 \in \mathfrak{m}_2$ such that $e_2.a_2.e_2$ is non zero. We conclude from there the formula

$$A_F = e_1.A_F \oplus e_2.A_F \oplus \text{Ker } f_1 \cap \text{Ker } f_2.$$

Take for e_2 a primitive idempotent belonging to $\text{Ker } f_1 \cap \text{Ker } f_2$, ... The construction stops when n is equal to the length of the underlying right A_F-module of A_F. We then have the formulas

$$A_F = e_1.A_F + \cdots + e_n.A_F \quad \text{and} \quad \text{Ker } f_1 \cap \cdots \cap \text{Ker } f_n = 0.$$

Corollary 9. *Let A be a ring satisfying the conditions (b) and (c) of the theorem 2 and whose O is a prime two-sided ideal (i.e. if $\mathfrak{a}$ and $\mathfrak{b}$ are two non zero two-sided ideals, the product $\mathfrak{a}.\mathfrak{b}$ is non zero). If S is the multiplicative sub-set formed of regular elements of A, there is a ring of right fractions of A for S. This ring of fractions is simple.*

The condition (a) of the theorem 2 is indeed satisfied. If A_F was not simple, A_F would contain non zero two-sided ideals $\mathfrak{a}$ and $\mathfrak{b}$ whose product would be zero. The intersections $\mathfrak{a} \cap A$ and $\mathfrak{b} \cap A$ would then be two-sided ideals of A and the product of these ideals would be zero. This is absurd.

Corollary 10. *Let A be an right noetherian integral ring. If S is the multiplicative sub-set formed of non zero elements of A, there is a ring of right fractions of A for S. This ring is a division ring.*

The existence of the ring of fractions A_S results from the theorem 2. If A_S is not a division ring, A_S contains an idempotent e distinct from 1. Thus there is an $s \in S$ such that $e.s$ belongs to A. Since $e.s$ is not invertible in A_S, the $x \in A_S$ such that $e.s.x$ is zero form a right ideal $\mathfrak{l}$. It follows that $A \cap \mathfrak{l}$ is non zero and is annihilated by $e.s$: this is absurd.

28. Indecomposable injectives and two-sided ideals

Let A be a *right noetherian* ring. If M is an $A-$module, Ann M denotes the two-sided ideal formed of a such that we have $M.a = 0$. Similarly, if m is an element of M, Ann m denotes the right ideal formed of a such that we have $m.a = 0$.

Let I be an indecomposable injective $A-$module. If M and N are two non zero sub-modules of I, $M \cap N$ is non zero and we have

$$\text{Ann } (M \cap N) \supset \text{Ann } M + \text{Ann } N.$$

The annihilators of non zero sub-modules of I thus form an increasing filtering family of two-sided ideals of A. This family has a maximal element which we denote by $A(I)$.

We first see that $A(I)$ is a prime two-sided ideal; in other words, $A(I)$ is not equal to A and we have $a.A.b \subset A(I) \Rightarrow a \in A(I)$ or $b \in A(I)$. This results from the following lemma whose proof is left to the reader.

Lemma 15. *If $\mathfrak{a}$ is a two-sided ideal of a ring A, the following assertions are equivalent:*

a. *$\mathfrak{a}$ is a prime two-sided ideal.*
b. *There is a non zero $A-$module M such that we have* Ann $N = \mathfrak{a}$ *for any non zero sub-module N of M.*

I also claim that any prime two-sided ideal $\mathfrak{p}$ is of the type $A(I)$, where I is an indecomposable injective $A-$module: indeed, let $\oplus_k I_k$ be a decomposition of the injective envelope of $A/\mathfrak{p}$ into indecomposable injectives; let M_k be a sub-module of I_k whose annihilator is $A(I_k)$. Then $M_k \cap A/\mathfrak{p}$ is different from O and has both $\mathfrak{p}$ and $A(I_k)$ as annihilators; whence $\mathfrak{p} = A(I_k)$ for any k, and the result.

It is easy to see that $A/\mathfrak{p}$ is in fact an isotypic $A-$module (part IV, § 2), that is to say that the injectives I_k are all isomorphic. Indeed, the corollary 9 applies to the ring $A/\mathfrak{p}$. This ring is thus contained in a simple ring B which is the injective envelope of $A/\mathfrak{p}$ as right $(A/\mathfrak{p})-$module. The injective envelope I of $A/\mathfrak{p}$ considered as $A-$module contains B, and B coincides with the set of elements of I which are annihilated by $\mathfrak{p}$. The assertion thus results from the fact that B is the direct sum of right simple ideals which are isomorphic coirreducible $A-$modules.

Now let M be an arbitrary $A-$module, let I be the injective envelope of M and let $I = \sum\limits_k I_k$ be a decomposition of I into a direct sum of indecomposable injective modules. We say that the *prime two-sided ideal* $\mathfrak{p}$ *of A is associated with M* if there is a k such that we have $\mathfrak{p} = A(I_k)$. We can then group in packets $J_\mathfrak{p} = \sum\limits_{A(I_k)=\mathfrak{p}} I_k$ the indecomposable injectives associated with the same prime two-sided ideal. We thus obtain a decomposition of I, unique up to an automorphism, of the type $I = \sum\limits_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through the prime two-sided ideals associated with M.

In particular, if $M_\mathfrak{p}$ is equal to the intersection $M \cap (\sum\limits_{\mathfrak{q}\mathfrak{p}} J_\mathfrak{q})$, we have the equality $0 = \bigcap\limits_{\mathfrak{p}} M_\mathfrak{p}$, where $\mathfrak{p}$ runs through the prime ideals associated with M. In addition, the decomposition is irredundant and $M/M_\mathfrak{p}$ has $\mathfrak{p}$ as the only associated prime two-sided ideal. The following proposition results from the lemma 15:

Proposition 6. *The prime two-sided ideal* $\mathfrak{p}$ *is associated with* M *if and only if* M *has a sub-module* N *such that* $\mathfrak{p}$ *is the annihilator of* N *and of any non zero sub-modules of* N.

We thus find by a different process the results of LESIEUR-CROISOT on the 'tertiary' decomposition [16].

We now propose to see under which conditions the theory obtained is not finer than the theory of LESIEUR-CROISOT, that is to say when the established correspondence between indecomposable injectives and prime two-sided ideals is bijective. For this, we suppose the hypothesis (H) is satisfied:

(H) If $mathfrakl$ is a right ideal of A and if $\mathfrak{p}$ is the annihilator of $A/\mathfrak{l}$, there is a finite number of elements $x_1, \cdots, x_r$ of $A/\mathfrak{l}$ such that we have

$$p = \bigcap_{1 \leqslant i \leqslant r} \text{Ann } x_i$$

If the condition (H) is satisfied, we claim that, for any indecomposable injective I, the injective envelope of $A/A(I)$ is the direct sum of a finite number of modules isomorphic to I: Indeed let M be a sub-module of I whose annihilator is $A(I)$. We clearly have

$$A(I) = \bigcap_{\substack{x \neq 0 \\ x \in M}} \text{Ann } x.$$

But it results from (H) that $A(I)$ is the intersection of a finite number of $\text{Ann } x$, say $\text{Ann } x_1, \cdots, \text{Ann } x_r$; so there is a monomorphism from $A/A(I)$ to the finite direct sum $\bigoplus_{i=1}^{i=r} x_i.A$. The injective envelop of this sum is isotypic; whence the result.

Lemma 16. *If the condition* (H) *is satisfied, the correspondence* $I \rightsquigarrow A(I)$ *between types of indecomposable injectives and prime two-sided ideals is bijective.*

It remains to give examples where the condition (H) is verified:

a. All the right ideals of A are two-sided.
b. The ring A is right artinian. The radical $\mathfrak{r}(A)$ of A is then nilpotent and any prime two-sided ideal contains $\mathfrak{r}(A)$. The prime two-sided ideals of A thus correspond bijectively to those of semi-simple ring $A/\mathfrak{r}(A)$, that is to say, to the irreducible representations of A.
c. The center $Z(A)$ of A is a noetherian ring and A is a $Z(A)-$module of finite type: Let M be an $A-$module of finite type, x and y be two elements of M. If x' and y' denote the sets formed of the elements of M which are annihilated by $\text{Ann } x$ and $\text{Ann } y$, we see that x' and

y' are modules of finite type over $Z(A)$. Furthermore, we have the equality

$$\text{Ann }(x' + y') = \text{Ann } x \cap \text{Ann } y.$$

Since M is a noetherian $Z(A)-$module, there is a finite number of elements $x_1, \cdots, x_r$ of M such that we have $x'_1 + \cdots + x'_r = M$.

The condition (H) is thus verified.

d. A counter-example: Let k be a field of characteristic 0 and $k(X)$ be the field of rational fractions of an indeterminate X over k. Let d be the $k-$derivation $P(X) \to P'(X)$ of $k(X)$ and let A be the ring of operators given by d and the homotheties of $k(X)$(note by the translator: homothety means the multiplication map by a given element). The ring A is integral, any right ideal is cyclic and there is no prime two-sided ideal distinct from O or A ([4], § 5, exerc. 13). In particular, the prime two-sided ideals associated with a simple $A-$module and with A_d are zero. Since these modules do not have the same injective envelope, we see that the correspondence $I \rightsquigarrow A(I)$ is not bijective.

The set of prime two-sided ideals of A will be called from now on the *prime spectrum of* A [notation: Spec (A)]. We know that it is possible to associate with any prime two-sided ideal $\mathfrak{p}$ an indecomposable injective $A-$module $I_{\mathfrak{p}}$ which satisfies the following conditions: the injective envelope of $A/\mathfrak{p}$ is the finite direct sum of modules isomorphic to $I_{\mathfrak{p}}$. The map $\mathfrak{p} \rightsquigarrow I_{\mathfrak{p}}$ defines an injection from Spec (A) to the spectrum of the category mod$A$. We always identify Spec (A) with the image of this injection.

Proposition 7. *Let A be a right noetherian ring and $\mathcal{C}$ be the localizing sub-category of* mod$A$. *A prime two-sided ideal $\mathfrak{p}$ belongs to* $\text{Sp}(\mathcal{C})$ *(part IV, § 1) if and only if $A/\mathfrak{p}$ is an object of $\mathcal{C}$.*

It is clear that $\mathfrak{p}$ belongs to $\text{Sp}(\mathcal{C})$ if $A/\mathfrak{p}$ is an object of $\mathcal{C}$. Conversely suppose that $\mathfrak{p}$ belongs to $\text{Sp}(\mathcal{C})$. This means that $I_{\mathfrak{p}}$ contains a non zero sub-module belonging to $\mathcal{C}$. It follows that $A/\mathfrak{p}$ contains a non zero sub-module N belonging to $\mathcal{C}$. We denote by S the multiplicative subset formed of the regular elements of $A/\mathfrak{p}$ (cf. § 3). If P is a sub-group of N, the right ideal Ann P of $A/\mathfrak{p}$ which is formed of a such that $P.a$ is zero, is the intersection of $A/\mathfrak{p}$ with a right ideal of the simple ring $(A/\mathfrak{p})_S$. It follows that the right ideals of A of the form Ann P satisfy the descending chain condition. In particular, there is a finite number of elements $x_1, \cdots, x_r$ of N such that we have

$$0 = \bigcap_{i=1}^{i=r} \text{Ann } x_i.$$

Thus there is an $A-$linear map from $A/\mathfrak{p}$ to the direct sum of the modules $x_i.A$. It follows that $A/\mathfrak{p}$ belongs to $\mathcal{C}$.

If s is an element of Sp(mod$A$), and if I_s is an indecomposable injective $A-$module of type s, the two-sided ideal $A(I_s)$ belongs to Spec (A). We will say that the map $s \rightsquigarrow A(I_s)$ is the canonical projection from Sp(mod$A$) to Spec (A). This projection is identifies with Spec (A); it is thus surjective. When it is bijective, we will say that *the ring A has enough two-sided ideals.*

Let A be a ring, right noetherian and having enough two-sided ideals. If F is an idempotent topologizing set of right ideals, we saw that the product $\mathfrak{l}.\mathfrak{m}$ of two right ideals $\mathfrak{l}$ and $\mathfrak{m}$ belongs to F provided that $\mathfrak{l}$ and $\mathfrak{m}$ belong to F. The multiplicative law of A thus equip the set F with a structure of monoid. We claim that the sub-monoid of F which is generated by the prime two-sided ideals belonging to F, is cofinal: the sub-category $\mathcal{F}$ is indeed generated by the modules $A/\mathfrak{p}$, where $\mathfrak{p}$ runs through the prime two-sided ideals belonging to F (proposition 7 and corollary 2, part IV). Any noetherian module M belonging to $\mathcal{F}$ consequently has a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r \subset M_{r+1} = M,$$

whose quotients M_{i+1}/M_i are annihilated by a prime two-sided ideal $\mathfrak{p}_i \in F$. We conclude from there that the product $\mathfrak{p}_r.\mathfrak{p}_{r-1}...\mathfrak{p}_0$ annihilates M. If M is the quotient of A by an element $\mathfrak{l}$ of F, it follows that $\mathfrak{l}$ contains the product $\mathfrak{p}_r.\mathfrak{p}_{r-1}...\mathfrak{p}_0$. The proposition 7 thus implies the

Corollary 11. *Let A be a right noetherian ring with enough two-sided ideals and let E be a set of prime two-sided ideals of A. The set F of the right ideals which contains a product of the form* $\mathfrak{p}_\mathfrak{r}.\mathfrak{p}_{\mathfrak{r}-1}...\mathfrak{p}_\mathfrak{o}$ *where* $\mathfrak{p}_\mathfrak{i} \in E$*, is topologizing and idempotent. Conversely, any idempotent topologizing set of right ideals is of this form.*

If the ring A has enough two-sided ideals, the prime spectrum of A coincides with the spectrum of the category mod$A$. The latter is therefore equipped with a structure of noetherian ordered set. Furthermore, for any localizing sub-category $\mathcal{C}$ of mod$A$, Sp$(\mathcal{C})$ coincides with the set of prime two-sided ideals $\mathfrak{p}$ such that $A/\mathfrak{p}$ belongs to $\mathcal{C}$. If α is an ordinal, we denote in particular by E_α the set of prime ideals such that we have Kdim $A/\mathfrak{p} \leqslant \alpha$ (see part IV, § 1). We then have the

Corollary 12. *Let A be a right noetherian ring with enough two-sided ideals. With the above notations, the sets* E_α *can be defined in the following way:*

- E_{-1} *is empty.*
- *If the ordinal* α *has a predecessor* β*,* E_α *is formed of the prime two-sided ideals* $\mathfrak{p}$ *such that any prime two-sided ideal containing* $\mathfrak{p}$ *and distinct from* $\mathfrak{p}$ *belongs to* E_β*.*
- *If* α *is a limit ordinal,* E_α *is the union of the* E_β *for* $\beta < \alpha$*.*

It is clear that E_{-1} is empty. So let γ be any ordinal and suppose that the statements of the corollary 12 are satisfied for any ordinal $\alpha < \gamma$. If γ is a limit ordinal, these statements obviously remain satisfied when α is equal to γ. Examine the case where γ has a predecessor β and let T_β be the canonical functor from mod$A = \mathcal{A}$ to $\mathcal{A}/\mathcal{A}_\beta$ (the notations are those of the part IV).

Let $\mathfrak{p}$ be a prime two-sided ideal such that we have Kdim $A/\mathfrak{p} = \gamma$. On the other hand, let $\mathfrak{q}$ be a prime two-sided ideal containing $\mathfrak{p}$ and distinct from $\mathfrak{p}$. It is clear that Kdim $A/\mathfrak{q}$ is smaller or equal to Kdim $A/\mathfrak{p}$; if the equality holds, the quotients of Jordan-Hölder of $T_\beta(A/\mathfrak{p})$ would not be all isomorphic. It would result in the existence of a non zero sub-module N of $A/\mathfrak{p}$ which would not generate the same localizing sub-category of mod$A$ as $A/\mathfrak{p}$. Since there is an $A-$linera injection from $A/\mathfrak{p}$ to a direct sum of modules isomorphic to N (cf. the proof of the proposition 7), that is impossible. We thus have the inequality Kdim $A/\mathfrak{q} <$ Kdim $A/\mathfrak{p}$.

Conversely, suppose that $A/\mathfrak{q}$ belongs to $\mathcal{A}_\beta$ for any prime ideal $\mathfrak{q}$ containing $\mathfrak{p}$ and distinct from $\mathfrak{p}$. Let $\mathfrak{l}$ be a right ideal of $A/\mathfrak{p}$ which is maximal among the right ideals $\mathfrak{m}$ such that $A/\mathfrak{m}$ does not belong to $\mathcal{A}_\beta$. Then $T_\beta(A/\mathfrak{l})$ is a simple object; it follows that there is a prime two-sided ideal associated with $A/\mathfrak{l}$. This prime ideal contains $\mathfrak{p}$, and the hypothesis that we made shows that it coincides with $\mathfrak{p}$. Since $A/\mathfrak{l}$ belongs to $\mathcal{A}_\gamma$, Sp$(\mathcal{A}_\gamma)$ contains $\mathfrak{p}$. It follows that $A/\mathfrak{p}$ belongs to $\mathcal{A}_\gamma$ (proposition 7).

Corollary 13. *Let A be a right noetherian ring with enough two-sided ideals, $\mathfrak{p}$ be a prime two-sided ideal of A and η be a finite ordinal. The following assertions are equivalent:*

a. Kdim $A/\mathfrak{p} \leqslant \eta$.
b. *Any chain of prime two-sided ideals containing $\mathfrak{p}$ has at most $\eta + 1$ elements.*

The corollary 13 results directly from the corollary 12. If A is a noetherian commutative ring, it shows that the notion of Krull dimension that we have introduced for mod$A$ coincided with the classical notion.

To complete this paragraph, we recall the examples that we have given:

Proposition 8. *A right noetherian ring A has enough two-sided ideals if one of the following conditions is satisfied:*

a. *A is right artinian.*
b. *Any right ideal is two-sided.*
c. *The center $Z(A)$ of A is a noetherian ring and A is a $Z(A)-$module of finite type.*

29. Stability of injective envelopes

Let A be a ring and $\mathcal{C}$ be a full sub-category of mod$A$ satisfying the following conditions:

(★) If M is an $A-$module belonging to $\mathcal{C}$, any sub-module of M belongs to $\mathcal{C}$.

(★★) If M and N are two $A-$modules belonging to $\mathcal{C}$, the direct sum $M \oplus N$ belongs to $\mathcal{C}$.

Now we will denote by M an arbitrary $A-$module. If M' and M'' are two sub-modules of M such that M/M' and M/M'' belong to $\mathcal{C}$, then $M/M' \cap M''$ belongs to $\mathcal{C}$; this module is indeed isomorphic to a sub-module of $M/M' \oplus M/M''$. The sub-modules M' of M such that M/M' belongs to $\mathcal{C}$, thus form a base of neighbourhoods for O for a topology $\mathcal{T}_{\mathcal{C}}M$ which makes M a topological group. The reader verifies that the topology $\mathcal{T}_{\mathcal{C}}A_d$ makes A a topological ring. If we equip A with this topology, the topology $\mathcal{T}_{\mathcal{C}}M$ makes M a topological $A-$module. Furthermore, any $A-$linear map $f : M \to N$ is continuous when we equip M and N with topologies $\mathcal{T}_{\mathcal{C}}M$ and $\mathcal{T}_{\mathcal{C}}N$. We can summarize the situation by saying that the data of $\mathcal{C}$ defines on the one hand a structure of topological ring on A; it defines on the other hand a functor from mod$A$ to the category of topological $A-$modules.

Proposition 9. *Let A be a ring and $\mathcal{C}$ be a full sub-category of* mod$A$ *satisfying the conditions* ($\bigstar$) *and* ($\bigstar\bigstar$). *The following assertions are equivalent:*

a. *For any $A-$module M and for any sub-module N of M, the topology $\mathcal{T}_{\mathcal{C}}N$ coincides with the restriction to N of the topology of $\mathcal{T}_{\mathcal{C}}M$.*
b. *For any $A-$module M belongs to $\mathcal{C}$, the injective envelope of M belongs to $\mathcal{C}$.*

$(a) \Rightarrow (b)$: Indeed it is the same to say that M belongs to $\mathcal{C}$ or to say that the topology $\mathcal{T}_{\mathcal{C}}M$ is discrete. If M belongs to $\mathcal{C}$ and is contained in an $A-$module I, the assertion (a) implies that there is a sub-module Q of I such that I/Q belongs to $\mathcal{C}$ and that we have $Q \cap M = O$. If I is the injective envelope of M, Q is necessarily zero. Whence the assertion (b).

$(b) \Rightarrow (a)$: Indeed let N' be a sub-module of N such that N/N' belongs to $\mathcal{C}$. We must show that there is a sub-module M' of M such that M/M' belongs to $\mathcal{C}$ and that $N \cap M'$ is contained in N'. We will choose for M' a sub-module of M which is maximal for the equality $N \cap M' = N'$. Then M/M' is an essential extension of N/N' and (a) results from (b).

Corollary 14. *Let A be a ring and $\mathcal{C}$ be a full sub-category of* mod$A$ *satisfying the conditions* ($\bigstar$), ($\bigstar\bigstar$) *and the conditions* (a) *and* (b) *of the proposition 9. Let M be an $A-$module and let N be the intersection of the sub-modules M' of M such that M/M' belongs to $\mathcal{C}$. Then there is no sub-module N' of N, distinct from N and such that N/N' belongs to $\mathcal{C}$.*

Indeed the topology $\mathcal{T}_{\mathcal{C}}N$ coincides with the induced topology induced by $\mathcal{T}_{\mathcal{C}}M$. The latter topology is coarse.

We will assume from now on that A is right noetherian. The proposition 9 and the corollary 14 are then often used in the following way: let $\mathfrak{i}$ be a two-sided ideal of A and let $\mathcal{C}$ be a localizing sub-category of mod$A$ whose noetherian objects are the $A-$modules annihilated by a power of $\mathfrak{i}$. If $\mathcal{C}$ is stable under injective envelopes [that is to say if $\mathcal{C}$ satisfies the assertion (b) of the proposition 9]; the proposition 11 and the corollary 14 can be formulated as follows: if N is a sub-module of a noetherian $A-$module M, the $\mathfrak{i}-$adic topology of N (i.e. $\mathcal{T}_{\mathcal{C}}N$) is the restriction to N of the $\mathfrak{i}-$adic topology of M; in addition, if R is the intersection of the sub-modules $M.\mathfrak{i}^n$ of M, $R.\mathfrak{i}$ is equal to R. These propositions evoke the well-known results of Artin-Rees and of Krull.

In the paragraph 6, we will use the proposition 9 in a little different way: if A is a right noetherian ring, we take for $\mathcal{C}$ the localizing sub-category $(\mathrm{mod}A)_0$ whose objects are the $A-$modules of Krull dimension $\leqslant 0$. If M is an $A-$module, then we will denote by $\widehat{M}$ the completion of M for the topology $\mathcal{T}_{\mathcal{C}}M$; this completion $\widehat{M}$ is the projective limit of the modules M/M', where M' runs through the sub-modules of M such that we have Kdim $M/M' \leqslant 0$. If M is noetherian, the quotients M/M' are of finite length. In particular, $\widehat{A}$ is a right pseudo-compact ring; if M is noetherian, $\widehat{M}$ is a right pseudo-compact module over $\widehat{A}$.

It is clear that $(\mathrm{mod}A)_0$ is identified with the category of discrete topological right $\widehat{A}-$modules. Furthermore, we have the

Corollary 15. *Let A be a right noetherian ring. The functor that associates with any noetherian right $A-$module M the right pseudo-compact $A-$module $\widehat{M}$, is right exact. This functor is exact if the injective envelope of an $A-$module of zero Krull dimension has zero Krull dimension.*

This corollary results easily from the proposition 11 and of the lemma 2, part IV.

Proposition 10. *If A is a commutative noetherian ring, any localizing sub-category of* mod$A$ *is stable under injective envelopes.*

Indeed let $\mathcal{C}$ be such a localizing sub-category and let M be an $A-$module belonging to $\mathcal{C}$. If the prime ideal $\mathfrak{p}$ is associated with M, the module $A/\mathfrak{p}$ belongs to $\mathcal{C}$ (proposition 7). We will show that, for any prime ideal $\mathfrak{p}$, the injective envelope of $A/\mathfrak{p}$ belongs to the localizing sub-category generated by $A/\mathfrak{p}$; if M is an $A-$module and if a is an element of A, we denote by a_M the homothety $x \to x.a$ of M. We will say that a_M is *almost-nilpotent* if any noetherian sub-module of M is annihilated by a power of a_M.

Lemma 17. *Let $\mathfrak{p}$ be a prime ideal of a noetherian commutative ring A and let M be an $A-$module. The following assertions are equivalent:*

a. *M belongs to the localizing sub-category of* mod$A$ *which is generated by $A/\mathfrak{p}$.*
b. *For any element a of $\mathfrak{p}$, a_M is almost-nilpotent.*

If $\mathfrak{l}$ is an ideal of A, the quotient $A/\mathfrak{l}$ indeed belongs to the localizing sub-category generated by $A/\mathfrak{p}$ if and only if $\mathfrak{l}$ contains a power of $\mathfrak{p}$ (corollary 11). It follows that M belongs to the localizing sub-category generated by $A/\mathfrak{p}$ if and only if any element of M is annihilated by a power of $\mathfrak{p}$.

Lemma 18. *Let A be a noetherian commutative ring and I be an indecomposable injective $A-$module. For any element a of A, the homothety a_I is either bijective or almost-nilpotent.*

Indeed let M be a non zero noetherian sub-module of I. Since M is noetherian, the following equality is true for n large enough:

$$\mathrm{Ker}\ a_M^n \cap \mathrm{Im} a_M^n = 0.$$

Since the intersection of two non zero sub-modules of I is non zero, we have either $a_M^n = 0$ or Ker $a_M^n = 0$. In the latter case, a_M is a monomorphism from I to I, thus an automorphism, because I does not contain any injective distinct from I or from O. In the first case, any noetherian sub-module of I is annihilated by a power of a.

We are now able to prove the proposition: if I is the injective envelope of $A/\mathfrak{p}$, Ker a_I is different from O for any element a of $\mathfrak{p}$. It follows that a_i is almost-nilpotent (lemma 18) and that I belongs to the localizing subcategory of mod$A$ which is generated by $A/\mathfrak{p}$ (lemma 17).

The corollary 15 thus applies to any commutative noetherian ring. The same is true of remarks following the corollary 14 (lemmas of ARTIN-REES and of KRULL).

30. FINITE EXTENSION OF A NOETHERIAN COMMUTATIVE RING

In this paragraph, R is a commutative noetherian ring. We denote by A a unitary $R-$algebra, finite and faithful (cf. § 1). Such an algebra has enough two-sided ideals. We will be interested on the one hand in the relations between the prime spectrum of A and the spectrum of R; we will study on the other hand the structure of indecomposable injective $A-$modules. The results which we end up are those which everyone expects; the methods used are those which everyone uses.

It is permissible to suppose that R is contained in the center $Z(A)$ of A. We will then say that A is a *finite extension* of R. For any prime two-sided ideal $\mathfrak{P}$ of A, $R \cap \mathfrak{P}$ is then a prime ideal of R.

Proposition 11. *Let A be a finite extension of a ring R, commutative and noetherian. The following assertions are true:*

a. *The map $\mathfrak{P} \rightsquigarrow \mathfrak{P} \cap R$ is a surjection from* $\mathrm{Spec}(A)$ *to* $\mathrm{Spec}(R)$.
b. *If $\mathfrak{P}$ and $\mathfrak{Q}$ are two prime two-sided ideals of A, the conditions $\mathfrak{P} \supset \mathfrak{Q}$ and $\mathfrak{P} \neq \mathfrak{Q}$ imply $\mathfrak{P} \cap R \neq \mathfrak{Q} \cap R$*
c. *If $\mathfrak{p}$ is a prime ideal of R and if the $R-$module A can be generated by n elements, the inverse image of $\{\mathfrak{p}\}$ in* $\mathrm{Spec}(A)$ *contains at most n elements.*

The proposition results from the following lemmas:

Lemma 19. *Let A be a ring, S be a multiplicative subset contained in the center $Z(A)$ of A and let φ be the canonical map from A to A_S. The maps $\mathfrak{P} \rightsquigarrow \mathfrak{P}_S$ and $\mathfrak{Q} \rightsquigarrow \varphi^{-1}(\mathfrak{Q})$ define a bijective correspondence between the prime two-sided ideals of A which does not meet S and the prime two-sided ideals of A_S.*

The proof of the lemma 19 is left to the reader.

Lemma 20. *The hypothesis and the notations are those of the proposition 11. For any prime ideal $\mathfrak{p}$ of R, $A_\mathfrak{p}/\mathfrak{p}.A_\mathfrak{p}$ is a non zero finite algebra over the filed $R_\mathfrak{p}/\mathfrak{p}.R_\mathfrak{p}$. Furthermore, the map $\mathfrak{P} \rightsquigarrow \mathfrak{P}_\mathfrak{p}/\mathfrak{p}.\mathfrak{P}_\mathfrak{p}$ defines a bijection from* *the set of prime two-sided ideals* $\mathfrak{P}$ *of* A *satisfying* $\mathfrak{P} \cap R = \mathfrak{p}$*, to the set of prime two-sided ideals of* $A_\mathfrak{p}/\mathfrak{p}.A_\mathfrak{p}$*.*

It is clear that $A_\mathfrak{p}/\mathfrak{p}.A_\mathfrak{p}$ is a finite algebra over $R_\mathfrak{p}/\mathfrak{p}.R_\mathfrak{p}$. Since $\mathfrak{p}.R_\mathfrak{p}$ is the Jacobson radical of $R_\mathfrak{p}$ and that $A_\mathfrak{p}$ is an $R_\mathfrak{p}-$module of finite type, the quotient $A_\mathfrak{p}/\mathfrak{p}.A_\mathfrak{p}$ is non zero (Nakayama's lemma).

On the other hand, the prime two-sided ideals of $A_\mathfrak{p}/\mathfrak{p}.A_\mathfrak{p}$ are in bijective correspondence with the the prime two-sided ideals of $A_\mathfrak{p}$ which contains $\mathfrak{p}.A_\mathfrak{p}$, the lemma 19 shows that the latter bijectively corresponds to the prime two-sided ideals $\mathfrak{P}$ such that we have $\mathfrak{P} \cap R = \mathfrak{p}$. This proves the lemma 20.

The assertion (a) of the proposition results from the fact that $A_\mathfrak{p}/\mathfrak{p}.A_\mathfrak{p}$ is a non zero artinian ring; the prime two-sided ideals of this ring are the maximal two-sided ideals. The assertion (b) of the proposition results from the fact that $A_\mathfrak{p}/\mathfrak{p}.A_\mathfrak{p}$ does not contain nested and distinct prime two-sided ideals. Finally let $x_1, \cdots, x_n$ be generators of the underlying $R-$module of A; the images of these generators in $A_\mathfrak{p}/\mathfrak{p}.A_\mathfrak{p}$ generate the underlying $(R_\mathfrak{p}/\mathfrak{p}.R_\mathfrak{p})-$module of $A_\mathfrak{p}/\mathfrak{p}.A_\mathfrak{p}$; it results in the formula $[A_\mathfrak{p}/\mathfrak{p}.A_\mathfrak{p} : R_\mathfrak{p}/\mathfrak{p}.R_\mathfrak{p}] \leqslant n$; this proves (c).

Corollary 16. *The notations and the hypothesis are those of the proposition 11. Let* $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ *be a chain of prime ideals of* R *and* $\mathfrak{P}_0 \subset \cdots \subset \mathfrak{P}_i$ *be a chain of prime two-sided ideals of* A *such that we have* $\mathfrak{P}_j \cap R = \mathfrak{p}_j$*,* $0 \leqslant j \leqslant i < n$*. There is a chain* $\mathfrak{P}_i \subset \mathfrak{P}_{i+1} \subset \cdots \subset \mathfrak{P}_n$ *formed of prime two-sided ideals of* A *such that we have* $\mathfrak{P}_k \cap R = \mathfrak{p}_k$ *for* $i \leqslant k \leqslant n$*.*

Indeed it results from the equality $\mathfrak{P}_i \cap R = \mathfrak{p}_i$ that $A/\mathfrak{P}_i$ contains $R/\mathfrak{p}_i$ and is a finite extension. According to the assertion (a) of the proposition 11, there is thus a prime two-sided ideal $\mathfrak{Q}_{i+1}$ of $A/\mathfrak{P}_i$ such that we have

$$\mathfrak{Q}_{i+1} \cap (R/\mathfrak{p}_i) = \mathfrak{p}_{i+1}/\mathfrak{p}_i.$$

The inverse image $\mathfrak{P}_{i+1}$ of $\mathfrak{Q}_{i+1}$ in A is a prime two-sided ideal of A and satisfies the equality $\mathfrak{P}_{i+1} \cap \mathfrak{p}_{i+1}$.

We construct $\mathfrak{P}_{i+2}$ from $\mathfrak{P}_{i+1}$ as we built $\mathfrak{P}_{i+1}$ from $\mathfrak{P}_i$; the construction continues by recurrence.

Corollary 17. *The notations and the hypothesis are those of the proposition 11. If* M *is an* $A-$*module, we denote by* ${}_\rho M$ *the underlying* $R-$*module of* M*. The Krull dimension of* M *is finite if and only if the Krull dimension of* ${}_\rho M$ *is finite. In the latter case, these two dimensions are equal.*

Indeed let $\mathcal{C}$ (resp. $\mathcal{D}$) be the localizing sub-category of mod$A$ (resp. of mod$R$) which is generated by M (resp. by ${}_\rho M$). We know that $\mathcal{C}$ can be generated by the modules $A/\mathfrak{P}$, where $\mathfrak{P}$ is a prime two-sided ideal of A. Similarly, $\mathcal{D}$ can be generated by the modules ${}_\rho(A/\mathfrak{P})$, where $\mathfrak{P}$ runs through the prime two-sided ideals of A such that we have $A/\mathfrak{P} \in \mathcal{C}$. We conclude the following formulas:

$$\text{Kdim } M = \sup_{A/\mathfrak{P}\in\mathcal{C}} \text{Kdim } A/\mathfrak{P}, \quad \text{Kdim }{}_{\rho}M = \sup_{A/\mathfrak{P}\in\mathcal{C}} \text{Kdim }{}_{\rho}(A/\mathfrak{P}).$$

In other words, it suffices to prove the corollary 17 when M is of the form $A/\mathfrak{P}$, where $\mathfrak{P}$ is a prime two-sided ideal of A. In this case, the annihilator of the noetherian $R-$module ${}_{\rho}(A/\mathfrak{P})$ is $\mathfrak{p} = \mathfrak{P} \cap R$. We know that it results in the equality Kdim ${}_{\rho}(A/\mathfrak{P}) =$ Kdim $R/\mathfrak{p}$.

According to the corollary 13, it thus remains to prove the equivalence of the following assertions:

- Any chain of prime two-sided ideals of A containing $\mathfrak{P}$ has at most $n+1$ elements.
- Any chain of prime ideals of R containing $\mathfrak{p}$ has at most $n+1$ elements.

The equivalence of these assertions results from the corollary 16 and from the assertion (b) of the proposition 11.

We will now say that the map $\mathfrak{P} \rightsquigarrow \mathfrak{P} \cap R$ is the canonical map from $\text{Spec}(A) = \text{Sp}(\text{mod}A)$ to $\text{Spec}(R) = \text{Sp}(\text{mod}R)$. If $\mathcal{D}$ is a localizing sub-category of mod$R$, we denote by $\rho^{-1}(\mathcal{D})$ the localizing sub-category of mod$A$ which follows: an $A-$module M belongs to $\rho^{-1}(\mathcal{D})$ if and only if ${}_{\rho}M$ belongs to $\mathcal{D}$. The spectrum of the category $\rho^{-1}(\mathcal{D})$ is obviously the inverse image of $\text{Sp}(\mathcal{D})$ under the canonical map from Sp(mod$A$) to Sp(mod$R$). The reader verifies the converse: a localizing sub-category $\mathcal{C}$ of mod$A$ is of the form $\rho^{-1}(\mathcal{D})$ if and only if $\text{Sp}(\mathcal{C})$ is the inverse image of a subset of Sp(mod$R$).

Proposition 12. *Let A be a finite extension of a ring R, commutative and noetherian. Let $\mathcal{D}$ be a localizing sub-category of* mod$R$ *and ρ be the functor which associates with any $A-$module M the underlying $R-$module of M. The localizing sub-category $\rho^{-1}(\mathcal{D})$ of* mod$A$ *is stable under injective envelopes.*

The proof is analogous to that of the proposition 10; we will only give a sketch: if $\mathfrak{P}$ is a prime two-sided ideal of A, we show that the injective envelope of $A/\mathfrak{P}$ belongs to the localizing sub-category generated by the modules $A/\mathfrak{Q}$, where $\mathfrak{Q}$ runs through the prime two-sided ideals such that we have $\mathfrak{Q} \cap R = \mathfrak{P} \cap R$. An $A-$module M belongs to this sub-category if and only if a_M is almost-nilpotent for any element a of $\mathfrak{P} \cap R$. Finally we show that if I is an indecomposable injective and a an element of R, the homothety a_I is either bijective or almost nilpotent (cf. lemma 18).

Corollary 18. *Let A be a ring satisfying the hypothesis of the proposition 12. If the Krull dimension of an $A-$module M is zero, the Krull dimension of the injective envelope is zero.*

The corollary 15 thus applies to the ring A. Let's take a look at the notations of this corollary. For example, if $\mathfrak{p}$ is a prime ideal of R, $\widehat{R_{\mathfrak{p}}}$ is the completion of $R_{\mathfrak{p}}$ for the $\mathfrak{p}.R_{\mathfrak{p}}-$adic topology.

In the following proposition, ΩR denotes the set of maximal ideals of R.

Proposition 13. *Let A be a finite extension of a ring R, commutative and noetherian. When M runs through the noetherian $A-$modules, the functor $M \rightsquigarrow \widehat{M}$ is isomorphic to the functor $M \rightsquigarrow \prod_{\mathfrak{m}\in\Omega R} M \otimes_R \widehat{R}_{\mathfrak{m}}$.*

Indeed let M' be a sub-module of M such that M/M' is of finite length. Then $\rho(M/M')$ is a noetherian $R-$module whose Krull dimension is zero. It follows that $\rho(M/M')$ is of finite length. On the other hand let $\mathfrak{a}$ be an ideal of R such that $R/\mathfrak{a}$ is of finite length. then $M/M.\mathfrak{a}$ is an $A-$module of finite length.

It results from these remarks that the $A-$sub-modules of M of finite colength define the same topology as the $R-$sub-modules of finite colength. Thus it suffices to prove that the completion of M for the latter topology is the product $\prod_{\mathfrak{m}\in\Omega R} M\otimes_R\widehat{R}_{\mathfrak{m}}$. This is a well-known proposition in commutative algebra.

Corollary 19. *The notations and the hypothesis are those of the proposition 13. If $Z(A)$ is the center of A, the center of $\widehat{A}$ is the ring $\widehat{Z(A)}$.*

The ring $Z(A)$ is indeed an $R-$module of finite type. According to the previous proposition, $\widehat{Z(A)}$ is thus equal to the product $\prod_{\mathfrak{m}\in\Omega R} Z(A) \otimes_R \widehat{R}_{\mathfrak{m}}$. It remains to show that $Z(A) \otimes_R \widehat{R}_{\mathfrak{m}}$ is the center of $A \otimes_R \widehat{R}_{\mathfrak{m}}$:

Lemma 21. *Let $u : R \to S$ be a homomorphism of commutative rings with unit elements. If S is $R-$flat* [15], *$Z(A) \otimes_R S$ is the center of $A \otimes_R S$.*

Indeed let $a_1, \cdots, a_r$ be generators of the $R-$algebra A; let $v_i : A \to A$ be the map defined by the formula $v_i(a) = a_i.a - a.a_i$, $1 \leqslant i \leqslant r$. The center $Z(A)$ of A is the intersection of the kernel Ker v_i. Since S is $R-$flat, it follows that $Z(A)\otimes_R S$ is the intersection of the kernels of the maps $v_i \otimes_R S$. The formulas $(v_i \otimes_R S)(x) = (a_i \otimes_R 1).x - x.(a_i \otimes_R 1)$ show that the latter intersection is the center of $A \otimes_R S$.

Now let $\mathfrak{p}$ be a prime ideal of $Z(A)$. We denote by $(\text{mod}A)_{\mathfrak{p}}$ the localizing sub-category of mod$A$ which is defined in the following way: an $A-$module M belongs to $(\text{mod}A)_{\mathfrak{p}}$ if and only if a_M is almost-nilpotent for any element a of $\mathfrak{p}$. Similarly, we denote by $(\text{mod}A)_0$ the localizing sub-category formed of the $A-$modules whose Krull dimension is zero.

Corollary 20. *The notations and the hypothesis are those of the proposition 13. The map $\mathfrak{m} \rightsquigarrow (\text{mod}A)_{\mathfrak{m}}$ is a bijection from the set of maximal ideals of $Z(A)$ to the set of connected components of the locally finite category $(\text{mod}A)_0$.*

Indeed let Z be the center of the category $\mathcal{A} = (\text{mod}A)_0$. Suppose the ring Z is isomorphic to the product of a family of rings $(Z_i)_{i\in I}$. We then identify Z with this product. If e_i is the unit element of Z_i, we denote by $\mathcal{A}_i$ the localizing sub-category of $\mathcal{A}$ whose objects are the $A-$modules $M \in \mathcal{O}\mathcal{A}$

such that we have $e_j(M) = 0$ for $j \neq i$. It is clear that the category $\mathcal{A}$ is equivalent to the product of the categories $\mathcal{A}_i$. In other words, there is a bijective correspondence between the decompositions of Z into products of rings and the decompositions of $\mathcal{A}$ into products of categories.

Now we saw at the previous paragraph that the category $(\mathrm{mod}A)_0$ is identified with the category of discrete topological right modules over the topological ring $\widehat{A}$. If M is an $A-$moduel of zero dimension, we define in particular a bilinear map $M \times \widehat{A} \to M$ which extends the bilinear map defining the structure of $A-$module of M; we denote by $m.a$ again the image of $(m.a) \in M \times \widehat{A}$ under this map. If a is an element of the center of $\widehat{A}$, we denote by a_M the $A-$linear map $m \rightsquigarrow m.a$ from M to M. When M varies, the maps $M \rightsquigarrow a_M$ defines an isomorphism from the center $\widehat{Z(A)}$ of $\widehat{A}$ to Z. The corollary 20 results from there and the previous remarks.

Corollary 21. *The notations and the hypothesis are those of the proposition 13. The following assertions are equivalent:*

a. *The map* $\mathfrak{P} \rightsquigarrow \mathfrak{P} \cap Z(A)$ *from* $\mathrm{Spec}(A)$ *to* $\mathrm{Spec}(Z(A))$ *is bijective.*
b. *Any localizing sub-category of* $\mathrm{mod}A$ *is stable under injective envelopes.*

$(a) \Rightarrow (b)$: This results from the proposition 12.

$(b) \Rightarrow (a)$: If any localizing sub-category of $\mathrm{mod}A$ is stable under injective envelopes, it is the same, *a fortiori*, for any quotient category of $\mathrm{mod}A$. In particular, if $\mathfrak{p}$ is a prime ideal of $Z(A)$, any localizing sub-category of $\mathrm{mod}A_{\mathfrak{p}}$ is stable under injective envelopes.

Thus let $\mathfrak{P}$ be a prime two-sided ideal of A such that we have $\mathfrak{P} \cap Z(A) = \mathfrak{p}$; let $\mathcal{C}$ be the localizing sub-category of $\mathrm{mod}A_{\mathfrak{p}}$ which is generated by $A_{\mathfrak{p}}/\mathfrak{P}.A_{\mathfrak{p}}$; let $\mathcal{D}$ be the localizing sub-category of $\mathrm{mod}A_{\mathfrak{p}}$ which is generated by the simple $A_{\mathfrak{p}}-$modules not annihilated by $\mathfrak{P}$. Since $\mathcal{C}$ and $\mathcal{D}$ are stable under injective envelopes, the category $(\mathrm{mod}A_{\mathfrak{p}})_0$ is equivalent to the product $\mathcal{C} \prod \mathcal{D}$ (cf. the demonstration of the corollary of the theorem 2, part IV). According to the lemma 21, $(Z(A))_{\mathfrak{p}}$ is the center of $A_{\mathfrak{p}}$. It follows that the category $\mathcal{D}$ is zero (corollary 20); this completes the proof.

The assertion (a) of the corollary 21 is for example satisfied if the $Z(A)-$algebra A satisfies the equivalent conditions of the proposition 2. It is also satisfied for the maximal orders of arithmetic.

We end this paragraph by the study of the pseudo-compact ring associated with the locally finite category $(\mathrm{mod}A)_0$. If $\mathfrak{m}$ is a maximal ideal of the commutative ring R, we denote by $E_{\mathfrak{m}}$ the injective envelope of the $R-$module $R/\mathfrak{m}$. We denote by E the direct sum of the $R-$modules $E_{\mathfrak{m}}$ when $\mathfrak{m}$ runs through the maximal ideals of R.

Let M be a right $A-$module of finite length, a be an element of A and let f be an $R-$linear map from M to E. We denote by $a.f$ the $R-$linear map from M to E which is defined by the following formula:

$$(a.f)(m) = f(m.a).$$

The map $(a, f) \rightsquigarrow a.f$ defines a structure of left $A-$module of the abelian group $\mathrm{Hom}_R(M, E)$. In the following, $\mathrm{Hom}_R(M, E)$ will always be equipped with this structure:

Proposition 14. *Let A be a finite extension of a ring R, commutative and noetherian. Let E be the direct sum of the injective envelopes of the $R-$modules $R/\mathfrak{m}$, where $\mathfrak{m}$ runs through the maximal ideals of R. The functor $M \rightsquigarrow \mathrm{Hom}_R(M, E)$ defines a duality between the right $A-$modules of finite length and the left $A-$modules of finite length.*

Let $\mathcal{A}$ be the category of right $A-$modules of finite length, $\mathcal{B}$ be the category of left $A-$modules of finite length. We see as in the demonstration of the corollary 12 (part IV) that $\mathrm{Hom}_R(M, E)$ is an $R-$module of finite length, *a fortiori* thus an $A-$module of finite length. This shows that $M \rightsquigarrow \mathrm{Hom}_R(M, E)$ is a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$; this functor will be denoted by T in the following of this proof.

Now consider a left $A-$module N of finite length. We then equip the abelian group $\mathrm{Hom}_R(N, E)$ with the structure of right $A-$module: if a is an element of A and g is an element of $\mathrm{Hom}_R(N, E)$, $g.a$ is defined by the formula $(g.a)(n) = g(a.n)$, we show as before that $\mathrm{Hom}_R(N, E)$ is a right $A-$module of finite length; the functor $N \rightsquigarrow \mathrm{Hom}_R N, E$ from $\mathcal{B}$ to $\mathcal{A}$ will be denoted by S.

According to the proposition 12 (part I), it suffices to prove that the functors $S \circ T$ and $T \circ S$ are isomorphic respectively to the functors $I_{\mathcal{A}}$ and $I_{\mathcal{D}}$: if M is an object of $\mathcal{A}$ and if m is an element of M, we denote by m' the $A-$linear map from $\mathrm{Hom}_R(M, E)$ to E which is induced by the formula

$$m'(f) = f(m).$$

When M varies, the maps $m \rightsquigarrow m'$ define a morphism from the identity functor $I_{\mathcal{A}}$ to $S \circ T$. We see as in the demonstration of the corollary 12 (part IV), that the map $m \rightsquigarrow m'$ is an isomorphism of $R-$modules; since this map is $A-$linear, it is also an isomorphism of $A-$modules. It follows that $I_{\mathcal{A}}$ is isomorphic to $S \circ T$. We prove in the same way that $I_{\mathcal{B}}$ is isomorphic to $T \circ S$.

It follows from the proposition 13 that the topology of $\widehat{A}$ can be defined by the two-sided ideals $\mathfrak{a}$ of finite colength (i.e. the underlying right $\widehat{A}-$module and left $\widehat{A}-$module of $\widehat{A}/\mathfrak{a}$ are of finite length). The topological ring $\widehat{A}$ is thus both left and right pseudo-compact. Furthermore, it is clear that the duality that we have exhibited between $\mathcal{A}$ and $\mathcal{B}$ extends to a duality between $(\mathrm{mod} A)_0$ and $PC(\widehat{A})$ formed of left pseudo-compact $\widehat{A}-$modules. Whence the

Corollary 22. *The notations and the hypothesis are those of the proposition 14. The pseudo-compact ring associated with the category of right $A-$modules of zero Krull dimension, is equivalent to the ring $\widehat{A}$. The dual ring of $\widehat{A}$ is thus equivalent to the opposite ring of $\widehat{A}$.*

Corollary 23. *The notations and the hypothesis are those of the proposition 14. Let I be an indecomposable injective right $A-$module, $\mathfrak{P}$ be the two-sided ideal associated to I and let $\mathfrak{p}$ be the intersection $\mathfrak{P} \cap R$. Then I is an artinian right module over $A_{\mathfrak{p}}$.*

Indeed let $\mathcal{C}$ be the localizing sub-category of mod$A$ whose noetherian obejcts are the right $A-$modules annihilated by an element of $R-\mathfrak{p}$. Since I is $\mathcal{C}-$closed, the structure of $A-$module of I extends in one and only one way to a structure of right $A_{\mathfrak{p}}-$module. Since the quotient category mod$A/\mathcal{C}$ is identified with mod$A_{\mathfrak{p}}$, I is an indecomposable injective $A_{\mathfrak{p}}-$module (corollary of the proposition 3). The prime two-sided ideal associated with this injective $A_{\mathfrak{p}}-$module is $\mathfrak{P}$. It follows that I is the injective envelope of a simple $A_{\mathfrak{p}}-$module; the corollary of the proposition 12 thus shows that I is an $A_{\mathfrak{p}}-$module whose Krull dimension is zero. Now the category $(\text{mod}A_{\mathfrak{p}})_0$ is dual to the category of pseudo-compact left $\widehat{A}_{\mathfrak{p}}-$modules (corollary 22). Since $\widehat{A}_{\mathfrak{p}}$ is a noetherian ring, the injective envelope of a simple object of $PC(\widehat{A}_{\mathfrak{p}})$ is noetherian. The corollary 23 is then deduced by duality.

We leave it to the reader to continue this investigation. It may in particular search for the pseudo-compact rings associated with the categories $(\text{mod}A)_{n+1}/(\text{mod}A)_n$. It may thus look for the connected components.

31. The Krull dimension of some rings

If A is a right noetherian ring, we will call *right Krull dimension of A* the Krull dimension of the category mod$A$. If G is a graded ring, we will say that G is a right noetherian graded ring if any ascending sequence of right homogeneous ideals is stationary; we will call *right Krull dimension of the graded ring G* the Krull dimension of the abelian category of graded modules of the graded ring G:

- An object of this category is a graded $G-$module whose underlying set belongs to $\mathfrak{U}$.
- If M and N are two objects of the category, a morphism from M to N is a $G-$linear map, homogeneous, of degree 0 from M to N.
- The composition of morphisms coincides with the usual composition of maps.

Proposition 15. *Let A be a ring filtered by a decreasing sequence of abelian sub-groups $A_n (n \geqslant 0)$ such that we have $A_m.A_n \subset A_{m+n}$. We suppose that A is the union of the A_n and that A is separated and complete for the topology defined by this A_n.*

a. *If the associated graded ring*

$$G(A) = \cdots \oplus A_{-1}/A_{-2} \oplus A_0/A_{-1} \oplus A_1/A_0 \oplus \cdots$$

is right noetherian, the ring A is right noetherian. If any right homogeneous ideal of $G(A)$ is generated by less than r homogeneous elements, any right ideal of A is generated by less than r elements.

b. *If the graded ring $G(A)$ is right noetherian and has a right Krull dimension smaller than n, then A is right noetherian and has a right Krull dimension smaller than n.*

To demonstrate (a) and (b), we rely on the following lemmas which are well-known:

Lemma 22. *Let M and N be two abelian groups filtered by the decreasing sequences of sub-groups M_n and N_n ($n \geqslant 0$). We suppose that the union of M_n (resp. N_n) is equal to M (resp. to N), that the intersection of M_n (resp. of N_n) is zero and that M_n is complete for the topology defined by the M_n. Let $f : M \to N$ be a homomorphism of filtered groups. If f induces a surjection of the associated graded groups, then f is a surjection and N is complete for the topology defined by the N_n. If f induces an injection of the associated graded groups, then f is an injection.*

Lemma 23. *Suppose the hypothesis of the proposition 15 are satisfied. Let M be an $A-$module equipped with a filtration by the abelian sub-groups M_n ($n \geqslant 0$), such that we have $M_m.A_n \subset M_{m+n}$. We suppose that the union of the M_n is equal to M, that their intersection is zero and that $G(M)$ is a graded $G(A)-$module generated by r homogeneous elements. Then M is generated by the representatives of the homogeneous generators of $G(M)$ and M is complete.*

Now we show (a): if $\mathfrak{l}$ is a right ideal of A, we will equip $\mathfrak{l}$ with the filtration defined by the $\mathfrak{l} \cap A_n$. Then $G(\mathfrak{l})$ is a homogeneous right ideal of $G(A)$ and it suffices to apply the previous lemmas. We also see that $\mathfrak{l}$ is complete for the filtration induced by that of A.

To show (b), it suffices to prove that we have Kdim $M \leqslant n$ when M is a right $A-$module of the form $A/\mathfrak{l}$, where $\mathfrak{l}$ is a right ideal. It is therefore clearly suffices to demonstrate the following lemma:

Lemma 24. *Let M be a filtered right $A-$module satisfying the conditions of the lemma 23. We then have* Kdim $M \leqslant$ Kdim $G(M)$.

We proceed by induction on Kdim $G(M)$: the assertion is true if Kdim $G(M) = -1$. Suppose the assertion is true if Kdim $G(M) < m$ and we show it is true if Kdim $G(M) = m$.

If not, there is an infinite sequence of sub-modules $M \supset M^1 \supset M^2 \supset \cdots$ such that we have Kdim $(M^i/M^{i+1}) \geqslant m$. Equipping the M^i with the induced filtration by that of M and M^i/M^{i+1} with the quotient filtration of that of M^i, we deduce that the Krull dimension of $G(M^i/M^{i+1}) = G(M^i)/G(M^{i+1})$ is greater or equal to m. The graded module $G(M)$ thus would have an infinite sequence of graded sub-modules whose successive quotients have Krull dimension greater or equal to m; this is contrary to the hypothesis of the induction and to the noetherian property of $G(M)$.

Now let A be a ring, σ be an automorphism of A and $A_\sigma[T]$ be the ring of Hilbert polynomials in T relatively to σ: this ring is formed of polynomials

$a_0 + T.a_1 + T^2.a_2 + \cdots + T_r.a_r$, having coefficients in A, with usual addition; on the other hand we impose the relations $a.T = T.\sigma(a)$ if $a \in A$.

Corollary 24. *Let σ be an automorphism of a ring A. If A is right noetherian ring, then $A_\sigma[T]$ is right noetherian. If A is right noetherian and has a finite right Krull dimension equal to n, then the right Krull dimension of $A_\sigma[T]$ is equal to $n+1$.*

We will equip the ring $B = A_\sigma[T]$ with the following filtration: B_n is zero if $n > 0$; if n is a positive integer, B_{-n} is formed of the Hilbert polynomials of degree $\leqslant n$. In this case, the underlying ring of the graded ring $G(B)$ can be identified with B, $G_{-n}(B)$ being identified with the set of monomials of degree n ($n \geqslant 0$). Modulo this identification, the right homogeneous ideals $\mathfrak{a}$ of $G(B)$ are of the form

$$\mathfrak{a}_0 \oplus T.\mathfrak{a}_1 \oplus T^2.\mathfrak{a}_2 \oplus T^3.\mathfrak{a}_3 \oplus \cdots \oplus T^i.\mathfrak{a}_i,$$

where $\mathfrak{a}_i$ is a right ideal of A such that we have

$$\cdots \supset \sigma^{-i}(\mathfrak{a}_i) \supset \cdots \supset \sigma^{-1}(\mathfrak{a}_i) \supset \mathfrak{a}_0$$

If A is right noetherian, it follows that $\sigma^{-i}(\mathfrak{a}_i)$ is equal to $\sigma^{-i-1}(\mathfrak{a}_{i+1})$ for i large enough. Thus there is an integer n such that we have $\sigma(\mathfrak{a}_i) = \sigma(\mathfrak{a}_{i+1})$ for $i \geqslant n$; it follows that $\mathfrak{a}$ is generated by a finite number of homogeneous elements of degree smaller or equal to n. Consequently, the graded ring $G(B)$ is right noetherian and B is right noetherian.

Now suppose that we have Kdim $A = n$. If M is an $A-$module, we equip the $G(B)-$module $M \otimes_A G(B)$ the obvious grading: the elements of degree r of $M \otimes_A G(B)$ are of the form $m \otimes_A T^r$, $m \in M$. We first prove the

Lemma 25. *Let M be a non zero $A-$module of finite Krull dimension n. The Krull dimension of the graded $G(B)-$module $M \otimes_A G(B)$ is then equal to $n+1$.*

We will demonstrate this lemma by induction on n. Since A is supposed to be right noetherian, it suffices to establish the proof when M is noetherian. We denote by $\mathcal{A}$ the category of graded $G(B)-$modules and we will use the notations of the part IV (§ 1):

If M is simple and if x is a non zero element of M, the only graded submodules of $M \otimes_A G(B)$ are generated by the $x \otimes_A T^r$, $r \geqslant 0$. It follows that the image of $M \otimes_A G(B)$ in the quotient category $\mathcal{A}/\mathcal{A}_0$ is a simple object. If M is of finite length, let $0 = M_0 \subset M_1 \subset \cdots \subset M_s = M$ be a Jordan-Hölder sequence of M. The graded modules $M_i \otimes_A G(B)$ then define a composition series of $M \otimes_A G(B)$ whose quotients have 1 as Krull dimension. It follows that $M \otimes_A G(B)$ has 1 as Krull dimension.

Now suppose the lemma is demonstrated when we have $n < m$ and we prove the lemma when n is equal to m: for this, we will consider $M \otimes_A G(B)$ as an object of the quotient object $\mathcal{A}/\mathcal{A}_{m-1}$. If M is a simple object of the quotient category mod$A$/(mod$A)_{m-1}$, the only sub-objects of $M \otimes_A G(B)$

in $\mathcal{A}/\mathcal{A}_{m-1}$ are the graded sub-modules which are generated by $M \otimes_A T^r$, $r \geqslant 0$. It follows that the image of $M \otimes_A G(B)$ in $\mathcal{A}/\mathcal{A}_m$ is simple. Finally, if M is a noetherian $A-$module such that we have Kdim $M = n$, the image of M in $\text{mod}A/(\text{mod}A)_{n-1}$ is of finite length. By using the composition series of M, we show as above that the Krull dimension of $M \otimes_A G(B)$ is $n+1$.

The lemma 25 shows in particular that the Krull dimension of the graded ring $G(B)$ is $n+1$. It follows that the Krull dimension of B is smaller or equal to $n+1$ (proposition 15). Since the descending sequence of the right ideals $T^r B$ is infinite and that the Krull dimension of $T^r.B/T^{r+1}.B$ is obviously equal to n, we also have the inequality Kdim $B \geqslant n+1$. Whence the corollary.

Corollary 25. *Let k be a filed and $\mathfrak{g}$ be a $k-$Lie-algebra of finite dimension: $[\mathfrak{g} : k] < +\infty$. The enveloping algebra of $\mathfrak{g}$ is a noetherian ring (both right and left) whose Krull dimension (both right and left) is smaller or equal to $[\mathfrak{g} : k]$.*

Indeed let U be the enveloping algebra and let U_{-n} be the vector subspace of U generated by 1 and the products of the form $g_1.g_2....g_m$, $o \leqslant n$, $m \leqslant n$, $g_i \in \mathfrak{g}$. We know that the graded ring associated with U is a ring of polynomial in $[\mathfrak{g} : k]$ indeterminates. The corollary results from this fact and of the proposition 15.

Part 6. Applications to the study of quasi coherent sheaves

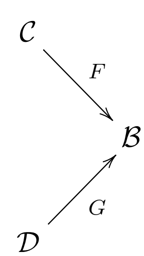
We want to study here the injective quasi coherent sheaves over a noetherian scheme [12]. Every time that we talk about a scheme $(X, \mathcal{O}_X)$, it will be implied that the underlying set of X and the étalé space associated to $\mathcal{O}_X$ are the elements of the universe $\mathfrak{U}$. Every time that we talk about an $\mathcal{O}_X-$module M, it will be implied that M is quasi coherent and that the étalé space associated to M is an element of $\mathfrak{U}$. We denote by $\mathcal{F}_X$ the category of $\mathcal{O}_X-$modules; with our conventions, $\mathcal{F}_X$ is an abelian $\mathfrak{U}-$category with exact inductive limits.

We show in the first paragraph that $\mathcal{F}_X$ can be obtained by 'recollement' of categories of modules. The following is devoted to the properties of categories of modules which are preserved by 'recollement'.

32. Recollement of abelian categories

Consider the diagram

(★)



where $\mathcal{C}$,$\mathcal{D}$ and $\mathcal{B}$ are abelian categories and where F and G are exact functors. We call *recollement of* $\mathcal{C}$ *and* $\mathcal{D}$ *along* $\mathcal{B}$ and denote it by $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ the following category:

- An object of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is a triple (C, D, σ) such that C is an object of $\mathcal{C}$, D is an object of $\mathcal{D}$ and σ an isomorphism from FC to GD. We sometimes say that (C, D, σ) is a datum of recollement.
- If (C', D', σ') and (C, D, σ) are two objects, a morphism from the first to the second is a couple (u, v) such that u is a morphism from C' to C, v a morphism from D' to D; we also impose the equality

$$\sigma \circ (Fu) = (Gv) \circ \sigma'$$

- The composition of the morphisms is defined by the formula

$$(u', v') \circ (u, v) = (u' \circ u, v' \circ v)$$

The hypothesis that we have made cause that $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is an abelian category and that the functors $(C, D, \sigma) \rightsquigarrow C$ and $(C, D, \sigma) \rightsquigarrow D$ are exact. We say that these functors are the canonical projections from $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ to $\mathcal{C}$ and $\mathcal{D}$. The notation $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is justified by the following statement:

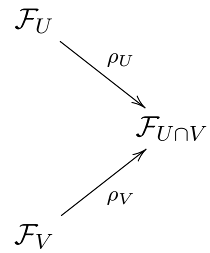
Proposition 1. *Let $\mathcal{A}$ be a category and let $S : \mathcal{A} \to \mathcal{C}$ and $T : \mathcal{A} \to \mathcal{D}$ be two functors such that $F \circ S$ is isomorphic to $G \circ T$. There is then a functor $R : \mathcal{A} \to \mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$, unique up to an isomorphism, such that S and T are isomorphic to the composite of R and the canonical projections from $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ to $\mathcal{C}$ and $\mathcal{D}$.*

Let τ be a functorial isomorphism from $F \circ S$ to $G \circ T$. We leave it to the reader to verify that the proposition 1 is satisfied if we take for R the functor $A \rightsquigarrow (FA, GA, \tau(A))$. If we consider $\mathcal{C}$, $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ as the objects of the category $\mathcal{E}$ of part I (§ 8),and if we replace F and G by the isomorphism classes of these functors, the proposition 1 implies that $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is the fibre product of the diagram (★). This shows in particular that the proposition 1 determines the category $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ up to an equivalence.

Now consider a scheme $(X, \mathcal{O}_X)$ and let U and V be two open subsets covering X. If M (resp. N) is an $(\mathcal{O}_X|U)-$module [resp. an $(\mathcal{O}_X|V)$-]module, then $F|U \cap V$ (resp. $G|U \cap V$) is an $(\mathcal{O}_X|U \cap V)-$module; we denote by ρ_U (resp. by ρ_V) the restriction functor $M \rightsquigarrow M|U \cap V$ (resp.

$N \rightsquigarrow N|U \cap V)$. Hence we have the diagram (★★)

(★★)



If P is an $\mathcal{O}_X-$module, the sheaf $(P|U)|U \cap V$, $P|U \cap V$ and $(P|V)|U \cap V$ obviously coincide; if σ_P is the identity morphism of $P|U \cap V$, the triple $(P|U, P|V, \sigma_P)$ is hence a datum of recollement.

Proposition 2. *Let U and V be two open subsets of a scheme X such that we have $U \cup V = X$. The functor $P \rightsquigarrow (P|U, P|V, \sigma_P)$ defines an equivalence between $\mathcal{F}_X$ and the recollement of $\mathcal{F}_U$ and $\mathcal{F}_V$ along $\mathcal{F}_{U \cap V}$.*

The proposition 2 results directly from the proposition 12 (part I) (cf. also [12], prop. 0.3.3).

Proposition 3. *Let $(X, \mathcal{O}_X)$ be a scheme and let U be an open subset of X such that the canonical injection $j : U \to X$ is a quasi compact morphism. Let T be the functor which associates to any $\mathcal{O}_X-$module M the restriction $M|U$ of M over the scheme $(U, \mathcal{O}_X|U)$. The functor T defines by passing to the quotient an equivalence between the categories $\mathcal{F}_X/\text{Ker } T$ and $\mathcal{F}_U$.*

We denote by $i_*(N)$ the direct image in X of the $(\mathcal{O}_X|U)-$ module N. According to the proposition 9.4.2 of [12], $j_*(N)$ is a quasi coherent $\mathcal{O}_X-$module; it follows that $S : G \rightsquigarrow j_*(G)$ is a functor adjoint to T. Hence the assertion results from the proposition 5 (part III).

The proposition 2 and 3 will be used in the following way: if X is a noetherian scheme, X is the union of a finite sequence of affine open subsets $X_1, X_2, \cdots, X_n$. The category $\mathcal{F}_{X_1 \cup X_2}$ is then equivalent to the recollement of $\mathcal{F}_{X_1}$ and $\mathcal{F}_{X_2}$ along the same quotient category $\mathcal{F}_{X_1 \cap X_2}$. That is to say, $\mathcal{F}_{X_1 \cup X_2}$ (note by the translator: the author uses '$\mathcal{F}_{X_1 \cap X_2}$') is equivalent to the recollement of two categories of modules along the common quotient category. Similarly, $\mathcal{F}_{X_1 \cup X_2 \cup X_3}$ is equivalent to the recollement of $\mathcal{F}_{X_1 \cup X_2}$ and the category of modules $\mathcal{F}_{X_3}$ along the common quotient category $\mathcal{F}_{(X_1 \cup X_2) \cap X_3}$. We continue so on until we get $\mathcal{F}_X$; this category is hence obtained by 'successive recollements' of categories of modules.

33. Properties of a recollement of abelian categories

We keep the notations of paragraph 1.

Lemma 1. *Let T be the functor $(C, D, \sigma) \rightsquigarrow D$. We suppose that* Ker F *is a localizing sub-category of $\mathcal{C}$ and that F defines by passing to the quotient an equivalence between $\mathcal{C}/\text{Ker } F$ and $\mathcal{B}$. The functor $M \rightsquigarrow (M, 0, 0)$ is then* *an isomorphism from* $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ *and* T *defines by passing to the quotient an equivalence between* $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}/\mathrm{Ker}\ T$ *and* $\mathcal{D}$*.*

Indeed let H be a functor adjoint to F, let χ be an isomorphism from $\mathrm{Hom}_{\mathcal{C}}(.,\ H.)$ to $\mathrm{Hom}_{\mathcal{B}}(F.,\ .)$ and let X be a morphism from $F\circ H$ to $I_{\mathcal{B}}$ which is associated to χ (cf. part I, § 7); we know that X is a functorial isomorphism. If S denotes the functor $D \rightsquigarrow (HGD, D, X(GD))$, $T\circ S$ is the identity functor of $\mathcal{D}$. Hence the identity morphisms 1_D define a functorial isomorphism Φ from $T\circ S$ to $I_{\mathcal{D}}$. This isomorphism Φ makes S a functor adjoint to T and the lemma results from the proposition 5 (part III).

Lemma 2. *We suppose that* Ker F *(resp.* Ker G*) is a localizing sub-category of* $\mathcal{C}$ *(resp. of* $\mathcal{D}$*) and that* F *(resp.* G*) defines by passing to the quotient an equivalence between* $\mathcal{C}/\mathrm{Ker}\ F$ *and* $\mathcal{B}$ *(resp. between* $\mathcal{D}/\mathrm{Ker}\ G$ *and* $\mathcal{B}$*). If the categories* $\mathcal{C}$ *and* $\mathcal{D}$ *are locally noetherian , it is the same for* $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$*.*

Indeed it is clear that $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is a $\mathfrak{U}-$category with exact inductive limits. If (C', D', σ') is a proper sub-object of (C, D, σ), we show that there is a noetherian sub-object of (C, D, σ) which is not contained in (C', D', σ'). For this we can suppose C' is different from C. Then there is a noetherian sub-object C'' of C which is not contained in C'. It thus remains to 'raise' FC' to a noetherian sub-object of $\mathcal{D}$.

It remains to prove that there is a set belonging to $\mathfrak{U}$ and having the same cardinal as the set of types of noetherian objects of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$; this results from the properties corresponding to $\mathcal{C}$ and $\mathcal{D}$; we leave the proof to the reader.

We always assume that the hypothesis of the lemma 2 are verified and use the notations introduced in the proof of the lemma 1. If I is an indecomposable injective object of $\mathcal{D}$, SI is an indecomposable injective object of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$. The map $I \rightsquigarrow SI$ induces an injection from $\mathrm{Sp}(\mathcal{D})$ to $\mathrm{Sp}(\mathcal{C}\prod_{\mathcal{B}}\mathcal{D})$. In accordance with the conventions of the part IV, we identify $\mathrm{Sp}(\mathcal{D})$ with the image of this injection. Similarly we identify $\mathrm{Sp}(\mathcal{B})$ and $\mathrm{Sp}(\mathcal{C})$ with the subsets of $\mathrm{Sp}(\mathcal{C}\prod_{\mathcal{B}}\mathcal{D})$. The lemma 1 and the remarks of part IV (§ 1) show that $\mathrm{Sp}(\mathcal{C}\prod_{\mathcal{B}}\mathcal{D})$ is the union of the disjoint subsets $\mathrm{Sp}(\mathcal{D})$ and $\mathrm{Sp}(\mathrm{Ker}\ T)$. Since $\mathrm{Sp}(\mathrm{Ker}\ T)$ coincides with $\mathrm{Sp}(\mathrm{Ker}\ F)$ and is contained in $\mathrm{Sp}(\mathcal{C})$, we see that $\mathrm{Sp}(\mathcal{C}\prod_{\mathcal{B}}\mathcal{D})$ is the union of $\mathrm{Sp}(\mathcal{C})$ and $\mathrm{Sp}(\mathcal{D})$. Since $\mathrm{Sp}(\mathcal{D})$ is the union of $\mathrm{Sp}(\mathrm{Ker}\ G)$ and $\mathrm{Sp}(\mathcal{B})$, we see that $\mathrm{Sp}(\mathcal{B})$ is the intersection of $\mathrm{Sp}(\mathcal{C})$ and $\mathrm{Sp}(\mathcal{D})$; thus we have the formulas

$$(\bigstar\bigstar\bigstar)\mathrm{Sp}(\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}) = \mathrm{Sp}(\mathcal{C})\cup\mathrm{Sp}(\mathcal{D}), \quad \mathrm{Sp}(\mathcal{B}) = \mathrm{Sp}(\mathcal{C})\cap\mathrm{Sp}(\mathcal{D}).$$

Theorem 1. *Let* $(X, \mathcal{O}_X)$ *be a noetherian scheme. We have the following assertions:*

a. *The category* $\mathcal{F}_X$ *is locally noetherian.*
b. *The support* Supp (I) *of an indecomposable injective* $\mathcal{O}_X-$*module is a irreducible closed sub-set of* X. *Any non zero* $\mathcal{O}_X-$*sub-module of* I *have the same support as* I.
c. *The map* $I \rightsquigarrow$ Supp (I) *induces a bijection from the spectrum of the category* $\mathcal{F}_X$ *to the set of irreducible closed subsets of* X.

When X is the affine scheme associated to a noetherian ring, the theorem results from the proposition 8 and from the second corollary of the proposition 14 of part V. According to what was said at the end of the paragraph 1, it thus suffices to prove that the theorem is true for X if it is verified for two open sub-sets U and V of X such that $X = U \cup V$. In this case we identify $\mathcal{F}_X$, $\mathcal{F}_U$, $\mathcal{F}_V$, $\mathcal{F}_{U\cap V}$, ρ_U and ρ_V respectively with $\mathcal{C} \prod\limits_{\mathcal{B}} \mathcal{D}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{B}$, F and G (cf. proposition 2). The assertion (a) is then an immediate consequence of the lemma 2.

Prove assertion (b): If I is an indecomposable injective $\mathcal{O}_X-$module, three cases are possible:

- I belongs to Sp(Ker T); in other words, I contains a non zero $\mathcal{O}_X-$sub-module whose support is contained in $V' = X - V$. It follows that I belongs to Sp($\mathcal{C}$), in other words, is isomorphic to the direct image of an indecomposable injective $(\mathcal{O}_X|U)-$module J. Since U satisfies the theorem, the support of J is an irreducible closed subset contained in V'. The support of I therefore coincides with the support of J; the assertion (b) follows.
- I contains a non zero $\mathcal{O}_X-$sub-module whose support is contained in $U' = X - U$. An argument analogous to the previous argument then shows that the support of I is an irreducible closed sub-set contained in U'.
- I belongs to Sp($\mathcal{B}$); in other words, the support of any non zero $\mathcal{O}_X-$sub-module of I meets $U \cap V$. If j is the canonical injection from $U \cap V$ to X, I is then isomorphic to the direct image $j_*(K)$ of an indecomposable injective $(\mathcal{O}_X|U \cap V)-$module K. Since the direct image of K in V coincides with $j_*(K)|V$ and is an indecomposable injective $(\mathcal{O}_X|V)-$module, the support of $I|V$ is an irreducible closed sub-set of V; furthermore, any non zero sub-module of $I|V$ has the same support as $I|V$. For the same reason, the support of $I|U$ is closed and irreducible and any non zero sub-module of $I|U$ has the same support as $I|U$. The assertion (b) follows.

The assertion (c) finally results from the classification that we have just done and from the fact that any irreducible closed sub-set of X is the closure of an irreducible closed sub-set of U or of V.

Now let us return to the diagram (★), and let R and T be the canonical projections from $\mathcal{C} \prod\limits_{\mathcal{B}} \mathcal{D}$ to $\mathcal{C}$ and $\mathcal{D}$. We suppose that the categories $\mathcal{C}$ and $\mathcal{D}$ are locally noetherian, that the hypothesis of the lemma 2 are verified and that the localizing sub-categories Ker F and Ker G are stable under injective envelopes. It follows that the sub-categories Ker R and Ker R of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ are stable under injective envelopes and that a datum of recollement (I,J,σ) is an injective object of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ if and only if I and J are injective objects of $\mathcal{C}$ and $\mathcal{D}$. We intend to seek the localizing sub-categories of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$; if $\mathcal{C}'$ (resp. $\mathcal{D}'$) is a localizing sub-category of $\mathcal{C}$ (resp. of $\mathcal{D}$), we denote for this by $F\mathcal{C}'$ (resp. $G\mathcal{D}'$) the smallest localizing sub-category of $\mathcal{B}$ which contains the objects FC (resp. GD), when C (resp. D) runs through the objects of $\mathcal{C}'$ (resp. of $\mathcal{D}'$).

Lemma 3. *Let $\mathcal{C}'$ and $\mathcal{D}'$ be localizing sub-categories of $\mathcal{C}$ and $\mathcal{D}$ such that $F\mathcal{C}'=G\mathcal{D}'$. With the above hypothesis, $\mathcal{C}'\prod_{\mathcal{B}}\mathcal{D}'$ is a localizing sub-category of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ and any localizing sub-category is of this type. If $\mathcal{C}'$ and $\mathcal{D}'$ are stable under injective envelopes, it is the same for $\mathcal{C}'\prod_{\mathcal{B}}\mathcal{D}'$.*

It is clear that the category $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is 'contained' in $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$. We will just show that any localizing sub-category $\mathcal{A}$ of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is of the form $\mathcal{C}'\prod_{\mathcal{B}}\mathcal{D}'$.: for this, let $K=(I,J,\tau)$ be an injective not containing any non zero sub-object of $\mathcal{A}$; let $E=(C,D,\sigma)$ be an arbitrary object of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$. We will first show that the canonical map from $\mathrm{Hom}(E,K)$ to $\mathrm{Hom}(C,I)$ is surjective: indeed let J' be the largest sub-object of J annihilated by G; since Ker G is stable under injective envelopes, J is the direct sum of J' and of an injective object J'' such that $GJ''=GJ$; it follows that, for any $f:C\to I$, $\tau\circ Ff\circ\sigma^{-1}$ lifts to a morphism from D to J'', thus also to a morphism g from D to J; in other words, f is the image of a morphism $(f,g):E\to K$.

Thus let $\mathcal{C}'$ and $\mathcal{D}'$ be the localizing sub-categories of $\mathcal{C}$ and $\mathcal{D}$ formed of objects C and D such that $\mathrm{Hom}_{\mathcal{C}}(C,I)=0$ and $\mathrm{Hom}_{\mathcal{D}}(D,J)=0$ when (I,J,τ) runs through the injective objects not containing any non zero sub-object of $\mathcal{A}$. What precedes implies that $\mathrm{Hom}(E,K)$ is zero if and only if $\mathrm{Hom}_{\mathcal{C}}(C,I)$ and $\mathrm{Hom}_{\mathcal{D}}(D,J)$ are zero.

In other words, (C,D,σ) belongs to $\mathcal{A}$ if and only if C and D belong to $\mathcal{C}'$ and $\mathcal{D}'$, i.e. if and only if (C,D,σ) belongs to $\mathcal{C}'\prod_{\mathcal{B}}\mathcal{D}'$. This completes the proof of the lemma.

Let $(X,\mathcal{O}_X)$ again be a noetherian scheme. If R is the union of a family of closed sub-sets of X, we denote by Cat R the full sub-category of $\mathcal{F}_X$ formed of $\mathcal{O}_X-$modules whose support is contained in R.

Proposition 4. a. *Any localizing sub-category of $\mathcal{F}_X$ is stable under injective envelopes.*

b. *The map $R \rightsquigarrow$ Cat R is a bijection from the set of sub-sets of X which is the union of closed sub-sets to the set of localizing sub-categories of $\mathcal{F}_X$.*

When X is the affine scheme associated with a noetherian ring, the proposition follows from the proposition 10 and form the first corollary of the proposition 7 of part V. According to what we have said at the end of paragraph 1, it thus suffices to prove that the proposition is true for X if it is verified for two opens U and V such that $U \cup V = X$. This last point results from the lemma 3.

Let us return to the diagram ($\bigstar$) for the last time, and suppose the hypothesis of the lemma 2 are verified. If φ is an element of the center $Z[\mathcal{C}]$ of $\mathcal{C}$, it is clear that there is one and only one element $F\varphi$ of $Z[\mathcal{B}]$ such that we have $(F\varphi)(FC) = F(\varphi(C))$ for any object C of $\mathcal{C}$. The map $\varphi \rightsquigarrow F\varphi$ is a homomorphism of rings from $Z[\mathcal{C}]$ to $Z[\mathcal{B}]$. We define in an analogous way a homomorphism $\psi \rightsquigarrow G\psi$ from $Z[\mathcal{D}]$ to $Z[\mathcal{B}]$.

If φ and ψ are elements of $Z[\mathcal{C}]$ and $Z[\mathcal{D}]$ such that $F\varphi = G\psi$, the morphisms $(\varphi(C), \psi(D)) : (C, D, \sigma) \to (C, D, \sigma)$ define obviously an element $\varphi \prod \psi$ of the center of $\mathcal{C} \prod_{\mathcal{B}} \mathcal{D}$. Furthermore it is clear that the map $(\varphi, \psi) \rightsquigarrow \varphi \prod \psi$ is a bijection from the fiber product $Z[\mathcal{C}] \prod_{Z[\mathcal{B}]} Z[\mathcal{D}]$ to $Z[\mathcal{C} \prod_{\mathcal{B}} \mathcal{D}]$.

What precedes and the fact that the center of a ring A is identified with the center of the category mod$A$ imply 'by recollement' the following result: let $(X, \mathcal{O}_X)$ be a noetherian scheme, z be a section of $\mathcal{O}_X$ on X, M an $\mathcal{O}_X-$module and z_M the endomorphism of M defined by z; the maps $z \rightsquigarrow z_M$ then define an isomorphism from $\Gamma(X, \mathcal{O}_X)$ to the center of the category $\mathcal{F}_X$.

34. Schemes and abelian categories

Let $(X, \mathcal{O}_X)$ be a noetherian scheme. We will see that the data, up to an equivalence, of the category $\mathcal{F}_X$ allows us to reconstruct the scheme $(X, \mathcal{O}_X)$; for this, we will say that a localizing sub-category $\mathcal{A}$ of $\mathcal{F}_X$ is *finite* if there is a noetherian object M such that $\mathcal{A}$ is the smallest localizing sub-category containing M; it is the same to say that $\mathcal{A}$ is of the form Cat R, where R is a closed sub-set of X.

The theorem 1 establish a bijective correspondence between X and the spectrum $\mathrm{Sp}(\mathcal{F}_X)$ of $\mathcal{F}_X$. In this correspondence, the open sub-sets correspond to the spectrums $\mathrm{Sp}(\mathcal{F}_X/\mathcal{A})$, where $\mathcal{A}$ runs through the *finite* localizing sub-categories of $\mathcal{F}_X$. More precisely, the open sub-set U corresponds to $\mathrm{Sp}(\mathcal{F}_X/\mathcal{A}(U))$ if $\mathcal{A}(U)$ denotes the closed sub-category formed of $\mathcal{O}_X-$modules whose support does not meet U.

The sets $\mathrm{Sp}(\mathcal{F}_X/\mathcal{A})$ define a structure of topological space on $\mathrm{Sp}(\mathcal{F}_X)$, it remains to equip this topological space with a sheaf of rings $\mathcal{O}$: according to the end of paragraph 2, we can take $Z[\mathcal{F}_X/\mathcal{A}]$ as rings of sections of $\mathcal{O}$ on $\mathrm{Sp}(\mathcal{F}_X/\mathcal{A})$. If $\mathcal{A}$ is contained in $\mathcal{B}$, the canonical functor $\mathcal{F}_X/\mathcal{A}$ to $\mathcal{F}_X/\mathcal{B}$

induces a homomorphism from $Z[\mathcal{F}_X/\mathcal{A}]$ to $Z[\mathcal{F}_X/\mathcal{B}]$ (cf. § 2); this is the homomorphism which is chosen as the restriction homomorphism.

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