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Arithmetic of values of L-functions and generalized multiple zeta values over number fields

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This thesis is dedicated to my dear father.

“In the beginning was the myth. God, in his search for self-expression, invested the souls of Hindus, Greeks, and Germans with poetic shapes and continues to invest each child’s soul with poetry every day.”

Hermann Hesse, ***Peter Camenzind***

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Résumé

L’objectif principal de cette thèse est de généraliser les multizetas au cas où le corps de base Q est remplacé par un corps de nombres quelconque. La motivation derrière cette construction vient des travaux de A. Goncharov sur les corrélateurs de Hodge et de la philosophie plectique de J. Nekovář et A. Scholl.

On commence par la construction des fonctions de Green plectiques supérieures. Hecke a prouvé que l’intégration d’une série d’Eisenstein appropriée sur le groupe de classes des idèles du corps de nombres donné, multipliée par un caractère du groupe des classes des idèles, est équale à la fonction L associée à ce caractère. Remplacant la série d’Eisenstein par les fonctions de Green plectiques supérieures, une intégration similaire donne des nouveaux résultats, qui généralisent les multizetas classiques et les multi-polylogarithmes.

D’après le principe plectique, un sous-groupe de l’anneau des entiers du corps de nombres donné joue un rôle essentiel dans ces travaux.

Mots-clefs

Corrélateurs de Hodge, formule de Hecke, multizetas, multi-polylogarithmes, principe plectique.

Abstract

The principal objective of this thesis is to generalize multiple zeta values to the case when the ground field Q is replaced by an arbitrary number field. The motivation behind the construction comes from the work of A. Goncharov on Hodge correlators and the plectic philosophy of J. Nekovář and A. Scholl.

We start by constructing the higher plectic Green functions. Hecke once proved that the integral of the restriction of a suitable Eisenstein series over $\mathbb{Q}$ to the idele class group of a given number field multipled an idele class character of finite order is equal to the L-function of this charator. By replacing Eisenstein seris with our higher plectic Green functions, a similar integration gives new results, which give the generalization of classical multiple zeta values and multiple polyloarithms.

According to the plectic principle, a non-trivial subgroup of the ring of integers of a given number field plays an essential role in this work.

Keywords

Hodge correlators, Hecke's formula, multiple zeta values, multiple polylogarithms, plectic principle.

ARITHMETIC OF VALUES OF L-FUNCTIONS AND GENERALIZED MULTIPLE ZETA VALUES OVER NUMBER FIELDS

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Chapter 1

Introduction

The principal objective of this thesis is to generalize multiple zeta values to the case when the ground field $\mathbb{Q}$ is replaced by an arbitrary number field. The motivation behind the construction comes from the work of A. Goncharov on Hodge correlators and the plectic philosophy of J. Nekovář and A. Scholl.

We start by constructing the higher plectic Green functions. Hecke once proved that the integral of the restriction of a suitable Eisenstein series over $\mathbb{Q}$ to the idele class group of a given number field multiplied an idele class character of finite order is equal to the L-functions of this character. By replacing Eisenstein series with our higher plectic Green functions, a similar integration gives new results, which give the generalization of classical multiple zeta values and multiple polylogarithms.

According to the plectic principle, a non-trivial subgroup of the ring of integers of the given number field plays an essential role in this work.

1.1 Generalizations of classical Zeta Values

Classically, the multiple zeta functions are generalizations of the Riemann zeta function, defined by

$$\zeta(s_1, \ldots, s_k) = \sum_{\substack{0<n_1<n_2<\cdots<n_k \\ n_j \in \mathbb{N}}} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}, \quad s_j \in \mathbb{C}$$

and converge when $Re(s_j) + \ldots + Re(s_k) > k - j + 1$ for all j. When $s_1, \ldots, s_k$ are all positive integers (with $s_k > 1$) these sums are often called multiple zeta values (MZVs). The k in the definition is named the length of a MZV, and the sum $m = \sum_{j=1}^{k} s_j$ is called the weight.

The MZVs are the periods of mixed Tate motives, namely the MZVs are iterated integrals on $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$. Mixed motives and their realizations related to classical modular forms were studied by various mathematicians (Beilinson, Beilinson-Levin, Manin, Goncharov, F. Brown, ...) Moreover, there exist mysterious modular phenomena in the ring of MZVs related to the depth filtration. A good geometric understanding of these phenomena seems that we should put MZVs and modular forms for $SL_2(\mathbb{Z})$ in a common framework. Recently, F. Brown proposed so-called multiple modular values and mixed modular motives.

On the other hand, A. Goncharov used ideas from quantum physics on Hodge theoretical setting to construct so-called Hodge correlators, which provide a new method to describe the corresponding variations of real mixed Hodge structures and a new way to understand periods of motives. He proved that classical polylogarithms, elliptic polylogarithms and their generalizations are all Hodge correlators.

However, all the work mentioned above is built up over the rational field $\mathbb{Q}$. What the situation will be if $\mathbb{Q}$ is replaced by an arbitrary number field K is still largely open. One natural but non-trivial question is

Question 1.1.1 *How should we define multiple zeta values over arbitrary number fields?*

What should we put in the missing place in the following diagram to complete this diagram?

$$\begin{tikzcd}
\text{Classical zeta values}/\mathbb{Q} \arrow[r] \arrow[d] & \text{(partial)Dedekind zeta functions}/K \arrow[d, dotted] \\
\text{Mutiple zeta values}/\mathbb{Q} \arrow[r, dotted] & \mathbf{?}
\end{tikzcd}$$

In this thesis, we propose a potential method to answer this question, inspired by A. Goncharov's theory on Hodge correlators [11] and by the plectic principle due to J. Nekovář and A. Scholl [15]. The starting point is to generalize the Hecke formula.

1.2 Hecke's Formula

Hecke [13] proved that the integral of the restriction of a suitable Eisenstein series over $\mathbb{Q}$ to the idele class group of a given number field multiplied by an idele class character χ of finite order is equal to the L-functions of χ, up to some Γ factors. In fact, Hecke's formula is one of the typical examples within the theory of automorphic periods.

More precisely, Let K be a number field of degree $[K : \mathbb{Q}] = r_1 + 2r_2$.

$$K_{\mathbb{R}} = K \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.$$

Define the norm map

$$N = N_{K/\mathbb{Q}} \otimes id : K_{\mathbb{R}}^{\times} \longrightarrow \mathbb{R}^{\times}.$$

In order to state Hecke's formula, we will need the following data:

(1) Let $U \subset O_{K,+}^{\times}$ be a subgroup of finite index, where

$$O_{K,+}^{\times} = O_K^{\times} \cap (K_{\mathbb{R}}^{\times})_+ , \quad (K_{\mathbb{R}}^{\times})_+ = (\mathbb{R}_+^{\times})^{r_1} \times (\mathbb{C}^{\times})^{r_2} .$$

(2) Let $I \subset K$ be a fractional O_K-ideal.

(3) $\exists\, m \in \mathbb{N} \setminus \{0\}, \quad \phi : I/mI \longrightarrow \mathbb{C}$ be a function such that

$$\forall \epsilon \in U, \quad \forall \alpha \in I \setminus \{0\}, \qquad \phi(\epsilon\alpha) = \phi(\alpha).$$

We consider the following embedding (defined up to a conjugation)

$$GL_K(1) \hookrightarrow GL_{\mathbb{Q}}([K : \mathbb{Q}]).$$

Let $E(g,s,\phi)$ be the Eisenstein series defined by

$$E(g,s,\phi) = \sum_{x\in I\smallsetminus\{0\}} \frac{\phi(x)}{\|g\cdot x\|^s},$$

where $g \in GL_{\mathbb{Z}}(I)(\mathbb{R}) \cong GL_{\mathbb{Q}}([K:\mathbb{Q}])(\mathbb{R})$ and $\|\cdot\|$ is a scalar product on $K_{\mathbb{R}}$.

Hecke proved the following formula. [13]

$$\int_{U_{\mathbb{R}}/U} E(u,s,\phi)\ \mathrm{d}^{\times}\mu(u) = C(r_1,r_2,s,\mathrm{d}^{\times}\mu(u)) \left(\sum_{x\in I\smallsetminus\{0\}} \frac{\phi(x)}{|N_{K/\mathbb{Q}}(x)|^{s/[K:\mathbb{Q}]}} \right),$$

where

$$U_{\mathbb{R}} = \mathrm{Ker}\left(N_{K/\mathbb{Q}}\otimes 1 : (K_{\mathbb{R}}^{\times})_{+} \longrightarrow \mathbb{R}_{+}^{\times}\right),$$

and $C(r_1,r_2,s,\mathrm{d}^{\times}\mu(u))$ is some Γ factor and $\mathrm{d}^{\times}\mu(u)$ is a Haar measure. In fact we can also make the formulation in an adelic setting.

In a more concrete setting, we can rewrite Hecke's formula as follows.

Theorem 1.2.1 (Hecke's Formula)

$$\int_{U_{\mathbb{R}}/U} \left(\sum_{\alpha\in I\smallsetminus\{0\}} \frac{\phi(\alpha)}{\|u\alpha\|^{[K:\mathbb{Q}]s}} \right) \mathrm{d}^{\times}\mu(u) = \frac{2\pi^{r_2}}{[K:\mathbb{Q}]2^{r_1}} \frac{\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}}{\Gamma([K:\mathbb{Q}]s/2)} \sum_{\alpha\in(I\smallsetminus\{0\})/U} \frac{\phi(\alpha)}{\mid N_{K/\mathbb{Q}}(\alpha)\mid^{s}}.$$

One of the objectives of this thesis is to try to generalize this formula in order to produce suitable "secondary"arithmetic objects. First of all we would like to precise the meaning of "secondary".

The generalization of Hecke's formula begins with replacing the Eisenstein series by some non-trivial objects, namely so-called higher plectic Green functions, which depend on one basic plectic Green function $g_{I,\nu}(x,u)$ (Definition 4.1.1) and on some combinatorial data. We will give the details about the construction of higher plectic Green functions in the fourth and fifth chapters.

1.3 Main construction and main results

In our work, we consider a generalization of $\log|1-e^{2\pi ix}|^2$ on the compact real torus

$$S^1 = \{e^{2\pi ix} | x\in\mathbb{R}/\mathbb{Z}\} \subset \mathbb{C}^{\times}.$$

This function is the restriction of the Green function $G(1,y) = \log|1-y|^2$ of the origin of $\mathbb{C}^{\times}$ to S^1. We are going to consider corresponding objects on tori with real multiplication.

Let F be a totally real field of degree $r = [F:\mathbb{Q}]$, we have the trace map $\mathrm{Tr} = \mathrm{Tr}_{F/\mathbb{Q}}$ and the norm map $N = N_{F/\mathbb{Q}}$. Let $\mathscr{D}$ be the different and let I be a fractional ideal of F and

$$I^{*} = \{a\in F \mid \mathrm{Tr}(aI)\in\mathbb{Z}\} = \mathscr{D}^{-1}I^{-1}.$$

Let $||\cdot||$: $F_{\mathbb{R}} = F\otimes\mathbb{R} \xrightarrow{\sim} \mathbb{R}^{\mathrm{Hom}(F,\mathbb{R})} \longrightarrow \mathbb{R}_{+}\cup\{0\}$ be the standard euclidean norm. As above, let

$$U_{\mathbb{R}} = \mathrm{Ker}(N:(F_{\mathbb{R}}^{\times})_{+} \longrightarrow \mathbb{R}_{+}^{\times}) = \{(u_1,\dots,u_r)\in\mathbb{R}_{+}^{r} \mid u_1\cdots u_r = 1\}.$$

Definition 1.3.1 (Plectic Green function) *The "plectic" Green function associated to the ideal I is defined as follows*

$$g_I(x,u) = \lim_{\eta\to 0^+} \sum_{n\in I^*\setminus\{0\}} \frac{e^{2\pi i \mathrm{Tr}(nx)}}{\|un\|^{r+\eta}}, \quad x \in F_{\mathbb{R}}/I,\ u \in U_{\mathbb{R}},$$

which can also be defined as $\sum_{n\in I^*\setminus\{0\}} \frac{e^{2\pi i \mathrm{Tr}(nx)}}{\|un\|^{r}}$*, viewed as a distribution on* $(F_{\mathbb{R}}/I) \times U_{\mathbb{R}}$.

In fact, for each $x \in F_{\mathbb{R}}/I$, $g_I(x,u)$ *converges by generalized alternating test or Dirichlet test.*

We can also add an additional choice of multisigns $\nu \in \{0,1\}^{\mathrm{Hom}(F,R)}$. Then the modified "plectic" Green function becomes

$$g_I^{\nu}(x,u) = \lim_{\eta\to 0^+} \sum_{n\in I^*\setminus\{0\}} \mathrm{sgn}(n)^{\mu} \frac{e^{2\pi i \mathrm{Tr}(nx)}}{\|un\|^{r+\eta}}, \qquad x \in F_{\mathbb{R}}/I,\ u \in U_{\mathbb{R}}.$$

There is also a second method to add a multisign

$$g_{I,\nu}(x,u) = \lim_{\eta\to 0^+} \sum_{\substack{n\in I^*\setminus\{0\} \\ \mathrm{sgn}(n)=(-1)^{\mu}}} \frac{e^{2\pi i \mathrm{Tr}(nx)}}{\|un\|^{r+\eta}}, \qquad x \in F_{\mathbb{R}}/I,\ u \in U_{\mathbb{R}}.$$

Similarly, we can generalize the definition of plectic Green functions to any number field.

Definition 1.3.2 ***(Plectic Green functions for arbitrary number fields)***
Let K be a number field with r_1 real places $(v_i)_{1\le i\le r_1}$ and r_2 complex places $(w_j)_{1\le j\le r_2}$ and of degree $[K:\mathbb{Q}] = r_1 + 2r_2$. We have the trace map $\mathrm{Tr} = \mathrm{Tr}_{K/\mathbb{Q}}$

$$K_{\mathbb{R}} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

and we have a Hermitian product

$$\langle\ |\ \rangle : K_{\mathbb{R}} \times K_{\mathbb{R}} \longrightarrow \mathbb{C}$$

$$\langle (x_v, z_w) | (x'_v, z'_w) \rangle = \sum_v x_v x'_v + \sum_w z_w z'_w,$$

where

$$x = (x_v, z_w).$$

Let

$$\|\cdot\| = \langle \cdot|\cdot \rangle^{1/2},$$

then the plectic Green function can be defined as

$$g_I(x,u) = \sum_{n\in I^*\setminus\{0\}} \frac{e^{2\pi i \mathrm{Tr}(nx)}}{\|un\|^{r_1+2r_2}}, \quad x \in K_{\mathbb{R}}/I,\ u \in U_{\mathbb{R}},$$

which is a distribution on $(K_{\mathbb{R}}/I) \times U_{\mathbb{R}}$.

Now it is natural to ask

Question 1.3.3 *What are we going to obtain if we integrate the basic plectic Green function over* $U_{\mathbb{R}}/U$ *?*

The answer will be that we can recover the Hecke formula. Now let us see the situation over F.

Theorem 1.3.4 (Reinterpretation of Hecke's formula) *If* $x \in F_{\mathbb{R}} = F \otimes \mathbb{R}$*, if* $U \subset O_{F,+}^{\times}$ *a subgroup of finite index and if* $\forall \epsilon \in U$*,* $(\epsilon - 1) \cdot x \in I$*, which is equivalent to*

$$xI \in (F_{\mathbb{R}}/I)^{U},$$

and if $\nu : \mathrm{Hom}(F, \mathbb{R}) \longrightarrow \{0,1\}$*, consider*

$$g_I^{\nu}(x,u) = \lim_{\eta \to 0^+} \sum_{n \in (I^* \smallsetminus \{0\})/U} \mathrm{sgn}(n)^{\nu} \frac{e^{2\pi i \mathrm{Tr}(nx)}}{\|un\|^{r+\eta}},$$

where $u \in U_{\mathbb{R}} = \mathrm{Ker}\left(N : (F_{\mathbb{R}}^{\times})_{+} \longrightarrow \mathbb{R}_{+}^{\times}\right)$ *and* $\|\cdot\|$ *is the standard scalar product on* $F_{\mathbb{R}} \cong \mathbb{R}^r$*.*

$$\forall \epsilon \in U, \quad g_I^{\nu}(x, \epsilon u) = g_I^{\nu}(x,u).$$

By integration in a similar way as Hecke did, we obtain

$$\int_{U_{\mathbb{R}}/U} g_I^{\nu}(x,u)du = \frac{2^{r-1}\Gamma(1/2)^r}{\Gamma(r/2)} \lim_{\eta \to 0^+} \sum_{n \in (I^* \smallsetminus \{0\}/U)} \mathrm{sgn}(n)^{\nu} \frac{e^{2\pi i \mathrm{Tr}(nx)}}{\prod_{j=1}^{r} |n_j|^{(r+\eta)/r}}.$$

Note that

$$N(n) = N_{F/\mathbb{Q}}(n) = \prod_{j=1}^{r} n_j,$$

hence

$$\int_{U_{\mathbb{R}}/U} g_I^{\nu}(x,u)du = \frac{2^{1-r}\Gamma(1/2)^r}{r\Gamma(r/2)} \lim_{\eta \to 0^+} \sum_{n \in (I^* \smallsetminus \{0\})/U} \mathrm{sgn}(n)^{\nu} \frac{e^{2\pi i \mathrm{Tr}(nx)}}{|N(n)|^{(r+\eta)/r}}.$$

It is easy to see that we recover the Hecke formula, the right-hand side is a linear combination of special values $L(1, \chi_F)$ *for certain Dirichlet characters* χ_F *of* F *of signature* ν*.*

By using just this ”plectic” Green function, we will construct a function (or a distribution) $G_{I,\Gamma,S}(\cdot,\cdot)$ on

$$O_{F,+}^{\times} \backslash (F_{\mathbb{R}}/I)^S \times U_{\mathbb{R}} = (F_{\mathbb{R}}/I)^S \times_{O_{F,+}^{\times}} EO_{F,+}^{\times},$$

which depends on a given graph Γ and a subset S of the set of its vertices.

Definition 1.3.5 ***(Higher plectic Green function over a totally real field)***
Let Γ *be a finite connected non-oriented graph,* $V(\Gamma)$ *the set of vertices and* $E(\Gamma)$ *the non-empty set of edges. Let* $S \subset V(\Gamma)$ *be a subset of the set of vertices. Loops are forbidden here (i.e. the endpoints of each edge are distinct), but multiple edges are allowed. For each vertex* $v \in V(\Gamma)$*, let* $x_v \in F_{\mathbb{R}}/I$ *be a variable which decorates the vertex* v*; for each edge*

$e \in E(\Gamma)$, we fix an orientation $\overrightarrow{e} = (v_0(e) \longrightarrow v_1(e))$ and we associate a variable n_e varying in $\in I^* \smallsetminus \{0\}$. Then for each edge e, we can associate a plectic Green function.

$$g_I(x_{v_1(e)} - x_{v_0(e)}, u) = \lim_{\eta\to 0^+} \sum_{n_e\in I^*\backslash\{0\}} \frac{e^{2\pi i \mathrm{Tr}(n_e(x_{v_1(e)}-x_{v_0(e)}))}}{||un_e||^{r+\eta}}.$$

*Then the **higher plectic Green function** attached to (Γ, S) is defined as*

$$G_{I,\Gamma,S}(\{x_v\}_{v\in S}, u) = \int_{(F_\mathbb{R}/I)^{V(\Gamma)\smallsetminus S}} \prod_{e\in E(\Gamma)} g_I(x_{v_1(e)} - x_{v_0(e)}, u) \prod_{v\in V(\Gamma)\smallsetminus S} dx_v,$$

where $x_v \in F_\mathbb{R}/I, u \in U_\mathbb{R}$ and dx is a fixed Haar measure en $F_\mathbb{R}$.

There are variants of these functions (and numbers) depending on an additional choice of multisigns $\nu(e) \in \{0,1\}^{Hom(F,\mathbb{R})}$ (and an orientation) for each edge e, which means that we can replace $g_I(\cdot,\cdot)$ by $g_{I,\nu}(\cdot,\cdot)$ (or $g_I^\nu(\cdot,\cdot)$).

Moreover, we can replace each $e \in E(\Gamma)$ by a chain of $k_e \geq 1$ edges (which is equivalent to that we add $k_e - 1 \geq 0$ new vertices to each edge $e \in E(\Gamma)$) to get a new graph $\Gamma(\underline{k})$ with $|V(\Gamma(\underline{k}))| = |V(\Gamma)| + \sum_e (k_e - 1)$ and the subset S is unchanged.

For example, the case of $k_e = 3$ is as follows.

Figure 1.1: The subdivision of the edge e.

Hence we have defined a subdivision map:

Definition 1.3.6 *(Subdivision map)*

$$\underline{k} : E(\Gamma) \longrightarrow \mathbb{N} \smallsetminus \{0\}$$

$$\underline{k} : e \longmapsto k_e$$

Combining a subdivision of edges and formal Fourier expansion, we get

Proposition 1.3.7 ***(Subdivision of edges)***

$$G_{I,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S}, u) = vol(F_\mathbb{R}/I)^{|V(\Gamma)\backslash S|+|k|-|E(\Gamma)|} \lim_{\eta\to 0^+} \sum\nolimits'_{\{n,c(n)\in H_1(\Gamma,S)\otimes I^*\}} \frac{e^{2\pi i \mathrm{Tr}(\sum_{v\in S}(\partial n)_v x_v)}}{\prod_{e\in E(\Gamma)} ||un_e||^{k_e(r+\eta)}},$$

where $x \in F_\mathbb{R}/\mathbb{R}, u \in U_\mathbb{R}$, $|\underline{k}| = \sum_{e\in E(\Gamma)} |k_e|$, $n = (n_e)_{e\in E(\Gamma)}$,

$$\partial n : V(\Gamma) \longrightarrow I^*,$$

$$\partial n(v) = \sum_{e\in E(\Gamma), v_1(e)=v} n_e - \sum_{e\in E(\Gamma), v_0(e)=v} n_e,$$

and $\sum'$ means that ∂n is supported at S and each n_e is nonzero.

Remark 1.3.8 *We can also add multisigns here, there are two methods. The first way is to associate each edge a multisign* $\nu_e = (\nu_e^1, \dots, \nu_e^r)$*, then the formula in 4.1.9 will then contain terms* $sgn(n_e)^{\nu(e)}$*:*

$$G^{\nu}_{I,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S}, u) = vol(F_{\mathbb{R}}/I)^{|V(\Gamma)\setminus S|+|k|-|E(\Gamma)|} \lim_{\delta\to 0^+} \sum\nolimits'_{\{n,c(n)\in H_1(\Gamma,S)\otimes I^*\}} \operatorname{sgn}(n_e)^{\nu(e)} \frac{e^{2\pi i \operatorname{Tr}(\sum_{v\in S}(\partial n)_v x_v)}}{\prod_{e\in E(\Gamma)} \|un_e\|^{k_e(r+\delta)}}.$$

The second method to add mutisigns by taking into account the $n_e \in I^* \setminus \{0\}$*, such that* $\operatorname{sgn}(n_e) = (-1)^{\nu_e}$ *for a given multisign* ν_e *to each edge* e*. Then*

$$G_{I,\nu,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S}, u) = vol(F_{\mathbb{R}}/I)^{|V(\Gamma)\setminus S|+|k|-|E(\Gamma)|} \lim_{\delta\to 0^+} \sum\nolimits'_{\substack{\{n,c(n)\in H_1(\Gamma,S)\otimes I^*\} \\ \operatorname{sgn}(n_e)=(-1)^{\nu(e)}}} \frac{e^{2\pi i \operatorname{Tr}(\sum_{v\in S}(\partial n)_v x_v)}}{\prod_{e\in E(\Gamma)} \|un_e\|^{k_e(r+\delta)}}.$$

In fact, we usually prefer the second definition $G_{I,\nu,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S}, u)$*.*

If we ask again the same question as in Question (1.3.3) for higher plectic Green functions, the results will be highly non-trivial, some non-trivial iterated integrals make their appearance in a subtle way. However, if the number field F is just the rational field $\mathbb{Q}$, the integration $\int_{U_{\mathbb{R}}/U}$ is trivial and we prove that the higher plectic Green functions deliver linear combinations of classical zeta values.

1.3.1 Generalized MZVs for totally real fields

If we suppose that there exists a subgroup $U \subset O^{\times}_{F,+}$ of finite index stabilizing $\{x_v\}$ for each $v \in S$, we can apply the Hecke transform to define a new function as follows. In fact, each x_v is taken as a torsion point, then the subgroup U is the smallest group stabilizing all x_v.

Definition 1.3.9

$$\mathscr{F}_{I,\Gamma,S}(\{x_v\}_{v\in S}) = (O^{\times}_{F,+} : U)^{-1} \int_{U_{\mathbb{R}}/U} G_{I,\Gamma,S}(\{x_v\}_{v\in S}, u) d^{\times}u,$$

where $d^{\times}u = \frac{du_1\cdots du_{r-1}}{u_1\cdots u_{r-1}}$*,* $\prod_{j=1}^r u_j = 1$ *and* $U_{\mathbb{R}}/U = BU \cong (\mathbb{S})^{r-1}$ *is the classifying space of* $U \cong \mathbb{Z}^{r-1}$.

Remark: In the same way, we can define $\mathscr{F}^{\nu}_{I,\Gamma,S}(\{x_v\}_{v\in S})$ and $\mathscr{F}_{I,\nu,\Gamma,S}(\{x_v\}_{v\in S})$.

Then the generalized multiple zeta value is defined as

Definition 1.3.10 (Generalized multiple zeta values)

$$Z_I(\Gamma, S) = \mathscr{F}_{I,\Gamma,S}(\{0\}_{v\in S}).$$

Similarly, we have

$$Z^{\nu}_I(\Gamma, S) = \mathscr{F}^{\nu}_{I,\Gamma,S}(\{0\}_{v\in S}),$$

and

$$Z_{I,\nu}(\Gamma, S) = \mathscr{F}_{I,\nu,\Gamma,S}(\{0\}_{v\in S}).$$

1.3.2 The case $F = \mathbb{Q}$.

In order to understand our generalized multiple zeta values, we need to study the relation between $Z_{I,\nu}(\cdot,\cdot)$ and the classical multiple zeta values at the first place. In this direction, we have the following two theorems.

Theorem 1.3.11 *(**Relation to multiple zeta values**) (Theorem (4.2.1))*
Let F be the rational field $\mathbb{Q}$ and Γ any tree with rank $d = \mathrm{rank}(H_1(\Gamma, S)) \geq 2$, where $S = \partial\Gamma$. Assume that we are given a "sign" map $\nu : E(\Gamma) \longrightarrow \{0,1\}$ and a subdivision map $k : E(\Gamma) \longrightarrow \mathbb{N} \setminus \{0\}$ as in Remark 4.1.10 and in Definition 4.1.8, respectively. Then the generalized multiple zeta value $Z_{I,\nu}(\Gamma(\underline{k}), \partial\Gamma(\underline{k}))$ $(I = \mathbb{Z})$ can be expressed as a finite $\mathbb{Z}$-linear combination of classical multiple zeta values (MZVs) of depth d and weight $|k| = \sum_e k_e$.

Moreover, we have an explicit relation to multiple polylogarithms.

Theorem 1.3.12 *(**Relation to multiple polylogarithms**) (Theorem (4.3.1))*
If $F = \mathbb{Q}$, $I = \mathbb{Z}$, $x_v \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$, then $G_{I,\nu,\Gamma,\partial\Gamma}(\{x_v\}_{v\in\partial\Gamma}, 1)$ is a finite $\mathbb{Z}$-linear combination of the values of multiple polylogarithms evaluated at some N-th roots of unity.

These two theorems will be proved in the third section of Chapter 4. In order to understand the statement of the theorems, let us see one simple example.

Example 1.3.13 *Let Γ_1 be the graph in Figure 1.2.*

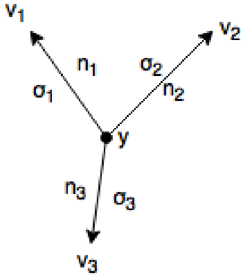

Figure 1.2: A plane trivalent tree with one internal vertex

The rank of Γ_1 is $2 = rank(H_1(\Gamma_1, \partial\Gamma_1))$. To each edge e_i $(i = 1,2,3)$ we add $\sigma_i - 1 \geq 0$ points. The only internal vertex is denoted by y, each external vertex v_i is decorated by the variable x_{v_i}. For each e_i $(i = 1,2)$, the given sign ν_i equals 0. For the edge e_3, the sign $\nu_3 = 1$. In fact, due to the formal Fourier convolution (Lemma 4.1.6), the constraint $n_1 + n_2 + n_3 = 0$ and $\nu_1 = \nu_2 = 0$ imply that $\nu_3 = 1$.

$$G_{I,\nu,\Gamma_1,\partial\Gamma_1}(\{x_v\}_{v\in\partial\Gamma_1}, 1) = \sum_{\substack{n_1+n_2+n_3=0, n_i\in\mathbb{Z}\setminus\{0\} \\ \mathrm{sgn}(n_i)=(-1)^{\nu_i}}} \frac{e^{2\pi i(n_1 x_{v_1} + n_2 x_{v_2} + n_3 x_{v_3})}}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|n_3|^{\sigma_3}},$$

where $\nu_1 = \nu_2 = 0$, $\nu_3 = 1$. Then

$$Z_{I,\nu}(\Gamma_1, \partial\Gamma_1) = \sum_{\substack{n_1+n_2+n_3=0, n_i\in\mathbb{Z}\setminus\{0\} \\ \mathrm{sgn}(n_i)=(-1)^{\nu_i}}} \frac{1}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|n_3|^{\sigma_3}} = \sum_{n_1,n_2\in\mathbb{N}\setminus\{0\}} \frac{1}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|n_1+n_2|^{\sigma_3}}$$

$$Z_{I,\nu}(\Gamma_1,\partial\Gamma_1)=\sum_{n_1,n_2\in\mathbb{N}\setminus\{0\}}\frac{1}{n_1^{\sigma_1}n_2^{\sigma_2}(n_1+n_2)^{\sigma_3}}.$$

We show in Example 4.2.7 that

$$Z_{I,\nu}(\Gamma_1,\partial\Gamma_1)=\sum_{r+s=\sigma_1+\sigma_2}\left(C_{r-1}^{\sigma_1-1}+C_{r-1}^{\sigma_2-1}\right)\zeta(s,r+\sigma_3),\qquad r\geq 1,s\geq 1,$$

where $C_b^a=\binom{b}{a}$ *the binomial coefficient.*

Here $\zeta(s,r+\sigma_3)$ is a classical double zeta value of weight $r+s+\sigma_3=\sigma_1+\sigma_2+\sigma_3$, which means that $Z_{I,\nu}(\Gamma_1,\partial\Gamma_1)$ can be expressed as a $\mathbb{Z}$-linear combination of double zeta values of this weight.

1.3.3 The case of a general totally real field

If $[F:\mathbb{Q}]=r>1$, then the generalized multiple zeta value $Z_I(\cdot,\cdot)$ is highly non-trivial, which can be written as a finite linear combination of values of generalized polylogarithms evaluated at some elements in F. We will give here a vague version of our result. For more details, we will send readers to the fifth chapter.

Theorem 1.3.14 (Main result over general totally real field-vague version) *Let F be a totally real field of degree $[F:\mathbb{Q}]=r>1$. Given a graph and a subdivision map defined as before. Moreover, we assume that if $2\nmid r$, the subdivision map is given by $k_e\in 2\mathbb{N}$ for each edge. Then we have the following expression of our generalized multiple zeta value*

$$Z_{I,\nu}(\Gamma,\partial\Gamma)=\sum_{\left\{\substack{n_e\in(I^*\setminus\{0\})/U\\ \partial n=0\\ \mathrm{sgn}(n_e)=(-1)^\nu}\right\}}\sum_{C(q,r)}\sum_{1\leq m\leq r-1}\sum_j(\alpha_j(n_e))\mathscr{L}_m(\beta_j(n_e)),$$

where $\mathscr{L}_m(\cdot)$ is a generalized m-logarithm which is a non-trivial iterated integral, and $\alpha_j(n_e)$ and $\beta_j(n_e)$ are rational functions of the conjugates of n_e with coefficients in $\mathbb{Q}$ and the sum $\sum_{C(q,r)}$ comes from some reduction results. Every sum here is finite.

The mentioned reduction results will be given in two reduction theorems Theorem 5.4.6, Theorem 5.4.12.

In fact, the contribution of 2-valency internal vertices can be covered by the subdivision map, then we will assume that the graph Γ is in a general position with no internal vertices of 2-valency and a subset of vertices S. Hence the generalized multiple zeta value associated to Γ is

$$Z_{I,\nu}(\Gamma,S)=(O_{F,+}^{\times}:U)^{-1}\sum_{\substack{(n_e)\in(I^*\setminus\{0\})^{|E(\Gamma)|}/U\\ \pi_v=0,\forall v\in V(\Gamma)\setminus S}}\int_{U_{\mathbb{R}}}\frac{\mathrm{d}^\times u}{\prod_e\|n_eu\|^{r\sigma_e}},$$

where

$$\pi_v=\sum_{e\in E(\Gamma),v_1(e)=v}n_e-\sum_{e\in E(\Gamma),v_0(e)=v}n_e.$$

We define the **basic integral** in Definition (5.0.2) associated to the generalized multiple zeta value $Z_{I,\nu}(\Gamma, S)$ as follows

$$\mathbb{I}(q, r; n_j, \sigma_j; \Gamma, S) = \int_{U_{\mathbb{R}}} \left(\prod_{j=1}^{q} \|n_j u\|^{-r\sigma_j} \right) \mathrm{d}^\times u, \tag{1.1}$$

where σ_j is a positive integer given by the subdivision map and the number q is an integer depending on (Γ, S), $q = |E(\Gamma)|$.

Therefore we can rewrite the generalized multiple zeta value

$$Z_{I,\nu}(\Gamma, S) = (O_{F,+}^\times : U)^{-1} \sum_{\substack{(n_j) \in (I^* \setminus \{0\})^q / U \\ \pi_v = 0, \forall v \in V(\Gamma) \setminus S}} \mathbb{I}(q, r; n_j, \sigma_j; \Gamma, S).$$

Again following the two reduction theorems Theorem (5.4.6) and Theorem (5.4.12) and under some minor assumptions, we have the following expression

$$\mathbb{I}(q, r; n_i, \sigma_i; \Gamma, S) = \sum_{\substack{1 \le i_1, \cdots, i_r \le q \\ i_j \ne i_h, j \ne h}} C_{i_1, \cdots, i_r} D(\underline{m'_{i_h}}) I_r(L_{i_h}),$$

where

$$I_r(L_i) = \int_{U_{\mathbb{R}}} \prod_{i=1}^{r} \frac{1}{L_i(u)} \mathrm{d}^\times u$$

is defined as the fundamental integral of the Hecke transform and $D(\overline{m'_{i_h}})$ is a differential operator with respect to the coefficients of $L_i(u) = \sum_{j=1}^{r} n_{i,j}^2 u_j$ and $\overline{C_{i_1,\cdots,i_r}}$ is certain constant obtained from the reduction.

Therefore the key calculation turns to that of $I_r(L_i)$. See Theorem (5.6.5) for precise expression.

Let us see the case of $r = 2$, we have the following theorem

Theorem 1.3.15 *(5.5.5) Given* $\underline{k} = (k_1, \ldots, k_q)$ *and* $|\underline{k}| = k_1 + \ldots + k_q$.
(1)If $2 \nmid |\underline{k}|$, *then after several differentiation with respect to the coefficients* α_j, β_j, *the basic integral associated to* $Z_{I,\nu}(\Gamma, S)$

$$\mathbb{I}(q, 2; \lambda^{(i)}; k_i; \Gamma, S) = \int_{\mathbb{R}_+} \frac{1}{\prod_{j=1}^{q} (\alpha_j^2 u + \beta_j^2 u^{-1})^{k_j}} \mathrm{d}^\times u$$

can be written as product of π *and an element of* $\mathbb{Q}(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_q)$, *where* $\lambda^{(i)} = (\alpha_i^2, \beta_i^2)$.
(2) If 2 *divides* $|\underline{k}|$, *then* $\mathbb{I}(q, 2; \lambda^{(i)}; k_i; \Gamma, S)$ *can be written as*

$$\sum_{j=1}^{q} \gamma_j \cdot \log\left(\frac{\beta_j}{\alpha_j} \right),$$

where $\gamma_j \in \mathbb{Q}(\alpha_1^2, \ldots, \alpha_q^2, \beta_1^2, \ldots, \beta_q^2)$.

Example 1.3.16 *If we are given the graph* Γ *in Figure 1.3 and* $S = \{v_1, v_2, v_3\}$. *Given the subdivision map* $\underline{k} = (k_1, k_2, k_3)$ *and* $k_j \in \mathbb{N}_+$ *and* $\lambda_j = (\alpha_j, \beta_j) \in I^*$, $1 \le j \le 3$.

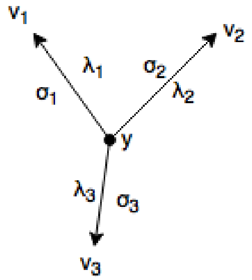

Figure 1.3: The graph Γ.

Then there exist differentials $D_{\alpha_j,\beta_j;k_j}$ such that the corresponding integral of the higher plectic Green function can be written as

$$\int_{U_{\mathbb{R}}} G_{I,\nu,\Gamma(\underline{k}),\partial\Gamma(\underline{k})}(\{0,0,0\},u)\,\mathrm{d}^{\times}u = \sum_{\alpha_j,\beta_j} D_{\alpha_j,\beta_j;k_j}\mathbb{I}(3,2;\lambda^{(i)};k_i=1;\Gamma,S),$$

where

$$\mathbb{I}(3,2;\lambda^{(i)};k_i=1;\Gamma,S) = \int_0^{\infty} \frac{1}{\prod_{j=1}^{3}(\alpha_j^2 u+\beta_j^2 u^{-1})}\,\frac{du}{u}$$
$$= \frac{\pi}{2(\alpha_1\beta_2+\alpha_2\beta_1)(\alpha_1\beta_3+\alpha_3\beta_1)(\alpha_2\beta_3+\alpha_3\beta_2)}.$$

1.4 Inspiration

For this moment, we should talk about two sources of inspiration of our construction and give some simple examples in order to illustrate our strategy.

1.4.1 Goncharov's Hodge correlators

In A. Goncharov's recent work [11], he constructed so-called Hodge correlators from just one fundamental object, namely the Green function, and certain Feynman diagrams, as in perturbative quantum field theory. In fact, the Green function contributes as the propagator which in Feynman diagrams serves to calculate the rate of collisions in quantum field theory.

Goncharov introduced such an idea to a Hodge-theoretic setting. He discovered that the so-called Hodge correlators are the coefficients of twistor connections, which describe the corresponding variations of real mixed Hodge structures. He proved that classical polylogarithms, elliptic polylogarithms and their generalizations are all Hodge correlators.

For example, over $\mathbb{C}^{\times}$, the Green function $G(x,y)$ is $\log|x-y|^2$. Applying a non-trivial integral formula of Levin, Goncharov proved that the usual polylogarithms can be written as Hodge correlators. Based on such a construction, he obtained higher multipolylogarithms, more precisely, he proved that the generating series of cycle multiple polylogarithms is a Hodge correlator, with the generating series of classical polylogarithms serving as a Green function. His construction of Hodge correlators on $\mathbb{C}^{\times}$ gives a variant of multiple polylogs, but a precise relation to the classical definition is not made explicit.

In our construction, we tried to define the plectic Green function $g_{I,\nu}(x,u)$ as an analogue of the Green function over a number field. Then Goncharov's integral for constructing Hodge correlators is replaced by the formal convolution. The new ingredient here is the non-trivial integral $\int_{U_{\mathbb{R}}}$ which delivers new results over a number field. The philosophy behind the integral $\int_{U_{\mathbb{R}}}$ is the plectic principle.

1.4.2 Plectic Philosophy

Over a totally real field F, J. Nekovář and A. Scholl [15] formulated what they call the plectic conjecture. The geometric objects in this conjecture are Shimura varieties/stacks whose definition groups are restrictions of scalars from an algebraic group over F. More concretely, they work with abelian varieties with real multiplication by O_F, where O_F is the ring of integers of F. We will try to explain the plectic principle in the following.

If B is a connected complex manifold. Let X/B be a family of abelian varieties with real multiplication, and $s : B \longrightarrow X$ a nonzero torsion section fixed by a subgroup of finite index $U \subset O_{F,+}^{\times}$. This subgroup U acts naturally on X and acts trivially on B, then we should consider the following diagram

$$\begin{tikzcd}
X \times_U EU \arrow[r] \arrow[d] & X \arrow[d] \\
B \times_U EU \arrow[r, "\pi"] & B,
\end{tikzcd}$$

where EU is the total space over the classifying space BU of the group U and $B \times_U EU = B \times EU/U$. When $B = \{pt\}$, then X is a variety, we have the following situation

$$\begin{tikzcd}
X \times_U EU \arrow[d] \\
BU.
\end{tikzcd}$$

Nekovář and Scholl [15] constructed in their work a U-equivariant current $\tilde{\theta}$ on $\tilde{X} = X \times U_{\mathbb{R}}$, where $U_{\mathbb{R}} = EU$. In fact $\tilde{\theta}$ is a plectic generalization of the (slightly modified) $\log|\theta(\tau, z)|$ of the absolute value of the standard Theta function on the elliptic curve $E = \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$, which is the Green function on E.

So $s^*(\tilde{\theta})$ is U-equivariant on $\tilde{B} = B \times U_{\mathbb{R}}$. Then $s^*(\tilde{\theta})$ can descend to a current on $B \times U_{\mathbb{R}}/U$, and we can compute the trace

$$\pi_*(s^*(\tilde{\theta})) = \int_{U_{\mathbb{R}}/U} s^*(\tilde{\theta}),$$

which gives very interesting functions, such as generalized Eisenstein-Kronecker-Lerch series.

The above integral, as well as its variants involving more complicated functions than $\tilde{\theta}$, can be computed by integrating suitable expressions depending on $\|ux\|$ over U_R. For this purpose, the Hecke transform is introduced and used in their work. Here is a typical example.

Proposition 1.4.1 ***(Hecke transform)****[16]*

Let $U_{\mathbb{R}} \subset (\mathbb{R}_+^\times)^r$ *be the subgroup*

$$U_{\mathbb{R}} = \left\{(u_j) | \prod u_j = 1\right\}$$

with the measure $\mathrm{d}^\times u = \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{r-1}}{u_{r-1}}$*. Let* $\|\cdot\|$ *be the Euclidean norm on* $\mathbb{C}^r$*, on which* U *acts by multiplication. Let* $(p_j) \in \mathbb{Z}^r$*,* $p = \sum p_j$*. Then for any* $x \in (\mathbb{C}^\times)^r$ *and* $s \in \mathbb{C}$ *with* $Re(s) > 0$,

$$\int_{U_{\mathbb{R}}} \|ux\|^{-2s} \prod_j u_j^{-2p_j} \mathrm{d}^\times u = \frac{2^{r-1}}{\Gamma(s)} \prod_j \Gamma(\frac{p+s}{r} - p_j) |x_j|^{2(p_j - (p+s)/r)}.$$

1.5 Final remarks

The classical polylogarithms have been interpreted by Deligne [12] in the language of variations of Hodge structures. It is natural to ask what kind of geometric objects or motivic avatar should correspond to our generalized polylogarithms ? Moreover, classical multiple zeta values are periods of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$, i.e. periods of mixed Tate motives. For generalized multiple zeta values there should be a relation to some plectic invariant of $\mathbb{G}_m \otimes I \setminus \{1\}$. More generally, the analogue of the values of multiple polylogarithms at roots of unity should be related to some plectic object attached to $\pi_{1,\cdots,1}^{plec}(\mathbb{G}_m \otimes I \setminus (N-torsion))$. We will continue to study these problems after this thesis.

Chapter 2

Goncharov's Theory of Hodge correlators

In this chapter, we will explain the idea of A. Goncharov's theory of Hodge correlators, where we are inspired for the construction. This theory provides a new point of view of periods. Our explanation is based on several simple examples.

2.1 Introduction

In Goncharov's recent work [11], he constructed so-called Hodge correlators from just one fundamental object, namely the Green function, and certain Feynman diagrams, as in perturbative quantum field theory. In fact, the Green function contributes as the propagator which in Feynman diagrams serves to calculate the rate of collisions in quantum field theory.

Goncharov introduced such an idea to a Hodge-theoretic setting. He discovered that the so-called Hodge correlators are the coefficients of twistor connections, which describe the corresponding variations of real mixed Hodge structures. He proved that classical polylogarithms, elliptic polylogarithms and their generalizations are all Hodge correlators.

For example, over $\mathbb{C}^{\times}$, the Green function $G(x,y)$ is $\log|x-y|^2$. Applying a non-trivial integral formula of Levin, Goncharov proved that the usual polylogarithm can be writen as a Hodge correlator. Based on such a construction, he obtained higher multipolylogarithms, more precisely, he proved that the generating series of cycle multiple polylogarithms is a Hodge correlator, with the generating series of classical polylogarithms serving as a Green function. His construction of Hodge correlators on $\mathbb{C}^{\times}$ gives a variant of multiple polylogs, but a precise relation to the classical definition is not made explicit.

In addition, there are motivic correlators in the motivic Lie algebra, whose periods are given by the Hodge correlators. For a modular curve and its cusps, the Hodge correlators generalize the Rankin-Selberg integrals. In fact, the simplest case of them is the Rankin-Selberg convolution of two cuspidal Hecke eigenforms. The motivic correlators on a modular curve give Beilinson's elements in motivic cohomology, hence the Beilinson-Kato Euler system. The motivic correlators on the limit of the tower of modular curves give an automorphic adelic description of the Hodge correlators on modular curves.

2.2 Definition of Hodge correlators after Goncharov

Let X be a smooth compact complex curve and S a subset of X. There is a Green function $G(x,y)$ associated on X^2 provided by a volume form. Given a tree T, each edge e of T contributes a Green function on $X^{\{\text{vertices of } e\}}$, which we lift to a function on $X^{\{\text{vertices of } T\}}$. Moreover, there is a canonical poly differential map

$$\omega_m : \wedge^{m+1}\mathscr{A}_X^0 \longrightarrow \mathscr{A}_X^m,$$

where $\mathscr{A}_X^k$ is the space of smooth k-forms on X. Then applying this poly differential map to the Green functions assigned to the edge of T, we get a differential form of the top degree on $X^{\{\text{internal vertices of } T\}}$. Integrating such differential form, we get the integral assigned to T. If we decorate the external vertices by elements $a_0, \dots, a_n$ $(a_i \in S)$, see Figure2.1 . By taking the sum over all trees T, that is decorated by $a_0, \dots, a_n$, we obtain the Hodge correlator associated to the cyclic word $W = \mathcal{C}\left(\{a_0\} \otimes \cdots \otimes \{a_n\}\right)$.

For the rigorous definition in details, one should refer to the work of Goncharov [11].

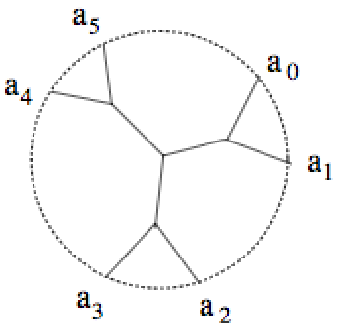

Figure 2.1: A plane trivalent tree decorated by $\mathcal{C}\left(\{a_0\} \otimes \cdots \otimes \{a_5\}\right)$.

Remark 2.2.1 *Compared to Goncharov's setting, we will work on number field, therefore we should find an analogue of Green's function over number field, which is also the starting point of our construction.*

2.3 Examples of Hodge correlators

Correlators on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus S$ and polylogarithms Let $X = \mathbb{P}^1$, $S = \{\infty\}$. Given the following cyclic word

$$W_n = \mathcal{C}\left(\{1\} \otimes \{z\} \otimes \{0\} \otimes \cdots \otimes \{0\}\right),$$

the unique W_n-decorated Feynman diagram with no internal vertices incident to two $\{0\}$'s, see Figure 2.2

We denote by $\mathbf{L}_n(z)$ the corresponding correlator [11]. Let $Li_n(z)$ be the classical n-polylogrithm on $\mathbb{CP}^1 \smallsetminus \{0,1,\infty\}$. Further there is a single-valued version

$$\text{if } n \text{ odd } \mathcal{L}_n(z) = \operatorname{Re}\left(\sum_{k=0}^{n-1} \beta_k \log^k |z| \cdot Li_{n-k}(z)\right),$$

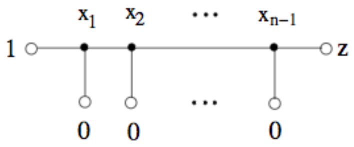

Figure 2.2: The Feynman diagram for the classical n-logarithms.

$$\text{if } n \text{ even } \mathcal{L}_n(z) = \operatorname{Im}\left(\sum_{k=0}^{n-1} \beta_k \log^k |z| \cdot Li_{n-k}(z)\right),$$

where $\beta_k = \frac{2^k B_k}{k!}$, B_k is the Bernoulli number.

Then the Levin formula gives us

$$\mathbb{L}_n^*(z) = 4^{-(n-1)} \sum_{\substack{n\in 2\mathbb{Z} \\ 1\leq k\leq n-2}} \binom{2n-k-3}{n-1} \frac{2^{k+1}}{(k+1)!} \mathcal{L}_{n-k}(z) \log^k |z|.$$

If we take

$$\mathbb{L}_n(z) = 4^{n-1} \binom{2n-2}{n-1}^{-1} \mathbb{L}_n^*(z).$$

Then Goncharov showed that the Hodge correlator can be related to the classical polylogrithms as follows

$$-\mathbf{L}_n(z) = (2\pi i)^{-n} \mathbb{L}_n(z).$$

Chapter 3

Plectic Principle

3.1 Equivariant plectic Green Currents

As we have explained in the introduction, the plectic principle, we just repeat the main idea here.

Let X/B be a family of abelian varieties with real multiplication, and $s : B \longrightarrow X$ a nonzero torsion section fixed by a subgroup of finite index $U \subset O_{F,+}^{\times}$. This subgroup U acts naturally on X and acts trivially on B, then we should consider the following diagram

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where EU is the total space over the classifying space BU of the group U and $B \times_U EU = B \times EU/U$. When $B = \{pt\}$, then X is a variety, we have the following situation

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Nekovář and Scholl [15] constructed in their work a U-equivariant current $\tilde{\theta}$ on $\tilde{X} = X \times U_{\mathbb{R}}$, where $U_{\mathbb{R}} = EU$. $\tilde{\theta}$ is a plectic generalization of the (slightly modified) $\log|\theta(\tau, z)|$ of the absolute value of the standard Theta function on the elliptic curve $E = \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$, which is the Green function on E.

So $s^*(\tilde{\theta})$ is U-equivariant on $\tilde{B} = B \times U_{\mathbb{R}}$. Then $s^*(\tilde{\theta})$ can descend to a current on $B \times U_{\mathbb{R}}/U$, and we can compute the trace

$$\pi_*(s^*(\tilde{\theta})) = \int_{U_{\mathbb{R}}/U} s^*(\tilde{\theta}),$$

which gives very interesting functions, such as generalized Eisenstein-Kronecker-Lerch series.

The above integral, as well as its variants involving more complicated functions than $\tilde{\theta}$, can be computed by integrating suitable expressions depending on $\|ux\|$ over U_R. For this purpose, the Hecke transform is introduced and used in their work. Here is a typical example.

In the speculative plectic conjecture[15], they consider a framework :

$$\begin{tikzcd}
GL_{2,F} \ltimes \mathbb{G}_{a,F}^2 \arrow[d] & Res_{L/F}(GL_{1,L}) \ltimes \mathbb{G}_{a,F}^2 \arrow[l] \arrow[d] \\
GL_{2,F} & Res_{L/F}(GL_{1,L}) \arrow[l]
\end{tikzcd}$$

where L is a totally imgainary quadratic extension of F. The diagram above gives rise to a Shimura stacks:

$$\begin{tikzcd}
\mathscr{A} \arrow[d] & \mathscr{A}_\tau \arrow[l] \arrow[d] \\
Y & \{\tau\} \arrow[l]
\end{tikzcd}$$

As in the previous remarks, Y is just an open Hilbert modular variety attached to $GL_{2,F}$, while $\mathscr{A}$ is the universal oject (in the sense of stacks) of Y. Morally $\mathscr{A}$ is the quotient $[U \backslash A]$, where A is the non-existent universal Hilbert-Blumenthal abelian scheme over Y, $U \subset O_{F,+}^\times$ is a subgroup of finite index in the group of totally positive units of F. If we consider that the group U acts trivially on Y, then we can get a stack $\mathscr{Y} = [U \backslash Y]$, which gives us a bigger diagram:

$$\begin{tikzcd}
\mathscr{A} \arrow[d] & \mathscr{A}_\tau \arrow[l] \arrow[d] \\
\mathscr{Y} \arrow[d] & {[U \backslash \{\tau\}]} \arrow[l] \arrow[d] \\
Y & \{\tau\}. \arrow[l]
\end{tikzcd}$$

3.2 Some explanation of the philosophy

In this subsection, we would like to give some geometric explanations about the reason that the subgroup $U \subset O_{F,+}^\times$ plays a non-trivial role in plectic principle.

In fact a universal Hilbert-Blumenthal abelian scheme over Y doesn't exist. First of all, we will try to explain something about the difference between two moduli problems associated to two groups. The picture is as follows:

$$\begin{tikzcd}
G^* \arrow[r, hook] \arrow[d] & G \arrow[d, "\nu"] \\
\mathbb{G}_m \arrow[r, hook] & Res_{F/\mathbb{Q}}\mathbb{G}_{m,F}
\end{tikzcd}$$

For an typical example, we take $G = Res_{F/\mathbb{Q}} GL_{2,F}$, and G^* is defined by Cartien product (fiber product).

$$G^* = \{g \in G | det(g) \in \mathbb{G}_m\}$$

Usually, we have the moduli problem $\mathscr{M}_{G^*}$(resp. $\mathscr{M}_G$) associated to G^* (resp. G). The fact is that the functor $\mathscr{M}_{G^*}$ is representable, which means that there is an universal abelian variety (scheme) $\mathscr{A}_{G^*}$ over $\mathscr{M}_{G^*}$. The group of integers of $O_{F,+}^{\times}$ acts on $\mathscr{M}_{G^*}$,

$$O_{F,+}^{\times} \circlearrowright \mathscr{M}_{G^*}, \quad \mathbb{Z}^{[F:\mathbb{Q}]-1} \cong O_{F,+}^{\times}.$$

The action of $O_{F,+}^{\times}$ on the connected components is trivial. The reason is that:

$$\mathcal{H}^d = G(\mathbb{R})/(K_\infty \backslash Z(\mathbb{R})),$$

where $d = [F : \mathbb{Q}]$ and

$$Z_{G^*}(\mathbb{R}) \circlearrowright \mathcal{H}^d/G^*(\mathbb{Z}),$$

where Z_{G^*} is the center of G^*. Therefore the center Z_{G^*} acts trivially on $\mathcal{H}^d/G^*(\mathbb{Z})$.

But the functor $\mathscr{M}_G$ is just a coarse moduli space. If we write,

$$\mathscr{M}_G^{gross} = \mathscr{M}_{G^*}/O_{F,+}^{\times}$$

(i) The neutral component of $\mathscr{M}_G^{gross}$ = the neutral component of $\mathscr{M}_{G^*}$.
(ii) But the set $\{$ the connected components $\}$ of $\mathscr{M}_G^{gross} \neq$ the set $\{$ the connected components $\}$ of $\mathscr{M}_G$.

Remark 3.2.1 *We know that the action of* $O_{F,+}^{\times}$ *permutes the connected components of* $\mathscr{M}_G^{gross}$.

$$\begin{tikzcd} \mathscr{M}_{G^*} \arrow[d, "p"] \\ \mathscr{M}_G^{gross} \end{tikzcd}$$

where p is a covering map (revetement).

Now we consider the previous moduli problems in the setting of stacks.

$$\mathscr{M}_G^{chp} = [\mathscr{M}_{G^*}/O_{F,+}^{\times}]$$

Remark 3.2.2 ***Attention***: $\mathscr{M}_G^{chp}$ *is not an algebraic stack, because the group* $O_{F,+}^{\times}$ *is not a finite group.*

But over $\mathscr{M}_G^{chp}$, we still have a stack :

$$\begin{tikzcd} {[\mathscr{A}/O_{F,+}^{\times}]} \arrow[d, "f"'] \\ \mathscr{M}_G^{chp}, \end{tikzcd}$$

where f is a representable morphism, involving a tautological abelian variety.
Now we turn to toroidal compactifications. We have a diagram of compactifications for $\mathscr{M}_{G^*}$.

$$\begin{tikzcd}
\mathscr{A}_{G^*} \arrow[r, hook] \arrow[d] & \mathscr{A}^{tor}_{G^*} \arrow[d] \\
O^{\times}_{F,+} \circlearrowright \mathscr{M}_{G^*} \arrow[r] & \mathscr{M}^{tor}_{G^*} \circlearrowleft O^{\times}_{F,+}
\end{tikzcd}$$

Similarly, there is another diagram for $\mathscr{M}^{chp}_{G}$

$$\begin{tikzcd}
{[\mathscr{A}_{G^*}/O^{\times}_{F,+}]} \arrow[r] \arrow[d, "f"'] & {[\mathscr{A}^{tor}/O^{\times}_{F,+}]} \arrow[d] \\
{[\mathscr{M}^{chp}_{G}]} \arrow[r] & {[\mathscr{M}^{tor}_{G^*}/O^{\times}_{F,+}]}
\end{tikzcd}$$

In conclusion :
$\mathscr{M}_{G^*} = G^*(\mathbb{Q})\backslash G^*(\mathbb{A})/Z_{G^*}(\mathbb{R})K_\infty$ has a universal abelian variety.

$\mathscr{M}^{gross}_{G} = G(\mathbb{Q})\backslash G(\mathbb{A})/Z_{G}(\mathbb{R})K_\infty$ does not have a universal abelian variety.

$$\mathscr{M}^{chp}_{G} = [G(\mathbb{Q})\backslash G(\mathbb{A})/Z_{G}(\mathbb{R})K_\infty].$$

Chapter 4

Generalization of Multiple Zeta Values (I): General construction and results over $\mathbb{Q}$

4.1 General construction

4.1.1 Plectic Green currents

In our work, we consider a generalization of $\log|1-e^{2\pi ix}|^2$ on the compact real torus

$$S^1 = \{e^{2\pi ix}|x \in \mathbb{R}/\mathbb{Z}\} \subset \mathbb{C}^\times.$$

This function is the restriction of the Green function $G(1,y) = \log|1-y|^2$ of the origin of $\mathbb{C}^\times$ to S^1. We are going to consider corresponding objects on tori with real multiplication.

Let F be a totally real field of degree $r = [F:\mathbb{Q}]$, we have the trace map $\mathrm{Tr} = \mathrm{Tr}_{F/\mathbb{Q}}$ and the norm map $N = N_{F/\mathbb{Q}}$. Let $\mathscr{D}$ be the different and let I be a fractional ideal of F and

$$I^* = \{a \in F \mid \mathrm{Tr}(aI) \in \mathbb{Z}\} = \mathscr{D}^{-1}I^{-1}.$$

Let $\|\cdot\|$: $F_\mathbb{R} = F \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{\mathrm{Hom}(F,\mathbb{R})} \longrightarrow \mathbb{R}_+ \cup \{0\}$ be the standard euclidean norm. As above, let

$$U_\mathbb{R} = \mathrm{Ker}(N : (F_\mathbb{R}^\times)_+ \longrightarrow \mathbb{R}_+^\times) = \{(u_1,\dots,u_r) \in \mathbb{R}_+^r \mid u_1\cdots u_r = 1\}.$$

Then we can define the "plectic" Green function associated to the ideal I

Definition 4.1.1 (Plectic Green function)

$$g_I(x,u) = \lim_{\eta\to 0^+} \sum_{n\in I^*\setminus\{0\}} \frac{e^{2\pi i\mathrm{Tr}(nx)}}{\|un\|^{r+\eta}}, \quad x \in F_\mathbb{R}/I,\ u \in U_\mathbb{R},$$

which can also be defined as $\sum\limits_{n\in I^*\smallsetminus\{0\}} \frac{e^{2\pi i\mathrm{Tr}(nx)}}{\|un\|^r}$*, viewed as a distribution on* $(F_\mathbb{R}/I) \times U_\mathbb{R}$.

In fact, for each $x \in F_\mathbb{R}/I$, $g_I(x,u)$ *converges by generalized alternating test or Dirichlet test.*

We can also add an additional choice of multisigns $\nu \in \{0,1\}^{\mathrm{Hom}(F,R)}$. Then the modified "plectic" Green function becomes

$$g_I^{\nu}(x,u) = \lim_{\eta\to 0^+} \sum_{n\in I^*\backslash\{0\}} \mathrm{sgn}(n)^{\nu} \frac{e^{2\pi i \mathrm{Tr}(nx)}}{||un||^{r+\eta}}, x \in F_{\mathbb{R}}/I,\ u \in U_{\mathbb{R}}.$$

There is also a second method to add a multisign

$$g_{I,\nu}(x,u) = \lim_{\eta\to 0^+} \sum_{\substack{n\in I^*\backslash\{0\} \\ \mathrm{sgn}(n)=(-1)^{\nu}}} \frac{e^{2\pi i \mathrm{Tr}(nx)}}{||un||^{r+\eta}}, x \in F_{\mathbb{R}}/I,\ u \in U_{\mathbb{R}}.$$

Remark 4.1.2 *($O_{F,+}^{\times}$–equivariance)*
$\forall \epsilon \in O_{F,+}^{\times}$, we have

$$g_I(\epsilon x, \epsilon u) = g_I(x,u).$$

This is also true for $g_I^{\nu}(\cdot,\cdot)$ and $g_{I,\nu}(\cdot,\cdot)$.

Remark 4.1.3 *(Dependence on I)*
1. If $\alpha \in F_+^{\times}$ and $N(\alpha) = 1$, then $(\alpha I)^* = \alpha^{-1} I^*$. Hence

$$g_{\alpha I}(\alpha x, \alpha u) = g_I(x,u).$$

2. $\forall \alpha \in F_+^{\times}$, $g_{\alpha I}(\alpha x, u) = N(\alpha) g_I(x,u)$. Therefore up to rescaling, $g_I(\cdot,\cdot)$ depends only on the class of I in the class group Cl_F^+. This is also true for $g_I^{\nu}(\cdot,\cdot)$ and $g_{I,\nu}(\cdot,\cdot)$.

4.1.2 Higher plectic Green currents

By using just this "plectic" Green function, we will construct a function (or a distribution) $G_{I,\Gamma,S}(\cdot,\cdot)$ on

$$O_{F,+}^{\times}\backslash (F_{\mathbb{R}}/I)^S \times U_{\mathbb{R}} = (F_{\mathbb{R}}/I)^S \times_{O_{F,+}^{\times}} EO_{F,+}^{\times},$$

which depends on a given graph Γ and a subset S of the set of its vertices.

Definition 4.1.4 ***(Higher plectic Green function)***
Let Γ be a finite connected non-oriented graph, $V(\Gamma)$ the set of vertices and $E(\Gamma)$ the non-empty set of edges. Let $S \subset V(\Gamma)$ be a subset of the set of vertices. Loops are forbidden here (i.e. the endpoints of each edge are distinct), but multiple edges are allowed. For each vertex $v \in V(\Gamma)$, let $x_v \in F_{\mathbb{R}}/I$ be a variable which decorates the vertex v; for each edge $e \in E(\Gamma)$, we fix an orientation $\overrightarrow{e} = (v_0(e) \longrightarrow v_1(e))$ and we associate an element $n_e \in I^* \smallsetminus \{0\}$. Then for each edge e, we can associate a plectic Green function

$$g_I(x_{v_1(e)} - x_{v_0(e)}, u) = \lim_{\eta\to 0^+} \sum_{n_e\in I^*\backslash\{0\}} \frac{e^{2\pi i \mathrm{Tr}(n_e(x_{v_1(e)} - x_{v_0(e)}))}}{||un_e||^{r+\eta}}.$$

*Then the **higher plectic Green function** attached to (Γ, S) is*

$$G_{I,\Gamma,S}(\{x_v\}_{v\in S}, u) = \int_{(F_{\mathbb{R}}/I)^{|V(\Gamma)\smallsetminus S|}} \prod_{e\in E(\Gamma)} g_I(x_{v_1(e)} - x_{v_0(e)}, u) \prod_{v\in V(\Gamma)\smallsetminus S} dx_v,$$

where $x_v \in F_{\mathbb{R}}/I, u \in U_{\mathbb{R}}$ and dx is a fixed Haar measure en $F_{\mathbb{R}}$.

Roughly speaking, the higher plectic Green function is defined by integration of the product of basic plectic Green functions associated to each edge respect to all the variables decorating the vertex $v \in V(\Gamma) \smallsetminus S$. As a result, the variables of the higher plectic Green function are the variables x_v decorating vertex $v \in S$. We should also mention that the higher plectic Green function does not depend on the orientation that we fix for each edge e.

There are variants of these functions (and numbers) depending on an additional choice of multisigns $\nu(e) \in \{0,1\}^{Hom(F,R)}$ (and an orientation) for each edge e, which means that we can replace $g_I(\cdot,\cdot)$ by $g_{I,\nu}(\cdot,\cdot)$ (or $g_I^{\nu}(\cdot,\cdot)$).

Remark 4.1.5 *By the very definition, $G_{I,\Gamma,S}(\cdot,\cdot)$ inherits a $O_{F,+}^{\times}$-invariant property. $\forall \epsilon \in O_{F,+}^{\times}$, we have*

$$G_{I,\Gamma,S}(\{\epsilon x_v\}_{v\in S}, \epsilon u) = G_{I,\Gamma,S}(\{x_v\}_{v\in S}, u).$$

Therefore, our higher plectic Green function $G_{I,\Gamma,S}(\cdot,\cdot)$ is a function (or distribution) on

$$O_{F,+}^{\times}\backslash(F_{\mathbb{R}}/I)^S \times U_{\mathbb{R}} = (F_{\mathbb{R}}/I)^S \times_{O_{F,+}^{\times}} EO_{F,+}^{\times},$$

which depends on the given graph Γ and the subset S of the set of its vertices. Here $EO_{F,+}^{\times}$ is the total space of the group $O_{F,+}^{\times}$.

Lemma 4.1.6 (Convolution on $F_{\mathbb{R}}/I$) *On $F_{\mathbb{R}}/I$ we have a formal convolution. Let $\chi_n(x) = e^{2\pi i \mathrm{Tr}(xn)}$. If $A(x) = \sum_{m\in I^*} a(m)\chi_m(x)$ and $B(y) = \sum_{n\in I^*} b(n)\chi_n(y)$, then*

$$\int_{F_{\mathbb{R}}/I} A(x-y)B(y)dy = \int_{F_{\mathbb{R}}/I} \sum_{m,n\in I^*} a(m)b(n)\chi_m(x-y)\chi_n(y)dy = vol(F_{\mathbb{R}}/I)\sum_{n\in I^*} a(n)b(n)\chi(n).$$

Proof 4.1.2.1 *The Proof of Lemma 4.1.6 is straightforward.*

$$\int_{F_{\mathbb{R}}/I} A(x-y)B(y)dy = \int_{F_{\mathbb{R}}/I} \sum_{m,n\in I^*} a(m)b(n)e^{2\pi i \mathrm{Tr}((x-y)m)}e^{2\pi i \mathrm{Tr}(yn)}dy$$

$$= \int_{F_{\mathbb{R}}/I} \sum_{m=n} a(m)b(n)e^{2\pi i \mathrm{Tr}(xm)}e^{2\pi i \mathrm{Tr}(y(n-m))}dy + \int_{F_{\mathbb{R}}/I} \sum_{m\neq n} a(m)b(n)e^{2\pi i \mathrm{Tr}(xm)}e^{2\pi i \mathrm{Tr}(y(n-m))}dy.$$

If $m \neq n$, then

$$\int_{F_{\mathbb{R}}/I} b(n)e^{2\pi i \mathrm{Tr}(y(n-m))}dy = 0.$$

Therefore we obtain

$$\int_{F_{\mathbb{R}}/I} A(x-y)B(y)dy = \int_{F_{\mathbb{R}}/I} \sum_{m=n} a(m)b(n)e^{2\pi i \mathrm{Tr}(xm)}e^{2\pi i \mathrm{Tr}(y(n-m))}dy$$

$$= \int_{F_{\mathbb{R}}/I} \sum_{n\in I^*} a(n)b(n)e^{2\pi i \mathrm{Tr}(xn)} \cdot \int_{F_{\mathbb{R}}/I} 1\, dy = vol(F_{\mathbb{R}}/I)\sum_{n\in I^*} a(n)b(n)e^{2\pi i \mathrm{Tr}(xn)}.$$

By the definition of $g_I(\cdot,\cdot)$, we can rewrite the higher plectic Green function as

$$G_{I,\Gamma,S}(\{x_v\}_{v\in S}, u) = \lim_{\eta\to 0^+} \sum_{n:E(\Gamma)\to I^*\smallsetminus\{0\}} \prod_{e\in E(\Gamma)} \|un_e\|^{-r-\eta} \int_{(F_{\mathbb{R}}/I)^{V(\Gamma)\smallsetminus S}} e^{2\pi i \mathrm{Tr}(\sum_{e\in E(\Gamma)} n_e(x_{v_1(e)}-x_{v_0(e)}))} \prod_{v\in V(\Gamma)\smallsetminus S} dx_v, \tag{4.1}$$

where we define the following map

$$n : E \longrightarrow I^* \smallsetminus \{0\}; \quad n : e \longmapsto n_e \in I^* \smallsetminus \{0\}.$$

Let us define the chain complex for the graph Γ.

$$\delta : C_1(\Gamma) = \mathbb{Z}[E(\Gamma)] \longrightarrow C_0(\Gamma) = \mathbb{Z}[V(\Gamma)],$$

where $\delta : (v_0 \to v_1) \longmapsto [v_1] - [v_0]$ is the boundary map of the chain complex and $\mathbb{Z}[X]$ denotes the free abelian group on a set X. We can also define the relative chain complex for (Γ, S), namely,

$$C_1(\Gamma) \longrightarrow C_0(\Gamma)/C_0(S).$$

We can associate to n the following element $c(n)$ in $C_1(\Gamma) \otimes_{\mathbb{Z}} I^*$ of the graph Γ.

$$c(n) = \sum_{e \in E(\Gamma)} n_e \cdot \overrightarrow{e} \in C_1(\Gamma) \otimes_{\mathbb{Z}} I^*.$$

If we let

$$\pi_v = \sum_{e \in E(\Gamma), v_1(e) = v} n_e - \sum_{e \in E(\Gamma), v_0(e) = v} n_e, \tag{4.2}$$

then

$$e^{2\pi i \mathrm{Tr}(\sum_{e \in E(\Gamma)} n_e (x_{v_1(e)} - x_{v_0(e)}))} = e^{2\pi i \mathrm{Tr}(\sum_{v \in V(\Gamma)} \pi_v x_v)}.$$

We define the boundary map

$$\partial n : V(\Gamma) \longrightarrow I^*,$$
$$\partial n(v) = \delta c(n)|_v = \pi_v,$$

where $|_v$ means taking the coefficient of the vertex v.

By using the previous convolution formula, we obtain that only the terms with $\partial n|_{V(\Gamma) \smallsetminus S} = 0$ (which means that

$$\forall v \in V(\Gamma) \smallsetminus S, \quad \pi_v = 0)$$

contribute to the integral in (4.1). Note that

$$\{c(n) \mid \partial n|_{V(\Gamma) \smallsetminus S} = 0\} = H_1(\Gamma, S) \otimes_{\mathbb{Z}} I^*.$$

Then we get a formal Fourier convolution description of $G_{I,\Gamma,S}(\cdot, \cdot)$.

Proposition 4.1.7 ***(Fourier Expansion)***

$$G_{I,\Gamma,S}(\{x_v\}_{v \in S}, u) = vol(F_{\mathbb{R}}/I)^{|V(\Gamma) \smallsetminus S|} \lim_{\eta \to 0^+} {\sum}'_{\{n, c(n) \in H_1(\Gamma,S) \otimes I^*\}} \frac{e^{2\pi i \mathrm{Tr}(\sum_{v \in S} (\partial n)_v x_v)}}{\prod_{e \in E(\Gamma)} \|u n_e\|^{r+\eta}},$$

where $\sum'$ *means that we consider only* n *such that* $c(n) \in H_1(\Gamma, S) \otimes I^*$ *and* $\forall e \in E(\Gamma), n_e \in I^* \smallsetminus \{0\}$.

The proof is straightforward by applying Lemma (4.1.6).

We can replace each $e \in E(\Gamma)$ by a chain of $k_e \geq 1$ edges, which is equivalent to that we add $k_e - 1 \geq 0$ new vertices to each edge $e \in E(\Gamma)$ to get a new graph $\Gamma(\underline{k})$ with $|V(\Gamma(\underline{k}))| = |V(\Gamma)| + \sum_e (k_e - 1)$ and the subset S is unchanged.

For example, the case of $k_e = 3$ is as follows.

Hence we have defined a subdivision map:

Figure 4.1: The subdivision of the edge e.

Definition 4.1.8 *(Subdivision map)*

$$\underline{k} : E(\Gamma) \longrightarrow \mathbb{N} \smallsetminus \{0\}$$

$$\underline{k} : e \longmapsto k_e$$

Combining a subdivision of edges and formal Fourier expansion, we get

Proposition 4.1.9 ***(Subdivision of edges)***

$$G_{I,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S},u) = vol(F_{\mathbb{R}}/I)^{|V(\Gamma)\backslash S|+|\underline{k}|-|E(\Gamma)|}\lim_{\eta\to 0^+}\sum\nolimits'_{\{n,c(n)\in H_1(\Gamma,S)\otimes I^*\}}\frac{e^{2\pi i \mathrm{Tr}(\sum_{v\in S}(\partial n)_v x_v)}}{\prod_{e\in E(\Gamma)}\|un_e\|^{k_e(r+\eta)}},$$

where $x \in F_{\mathbb{R}}/\mathbb{R}, u \in U_{\mathbb{R}}$, $|\underline{k}| = \sum_{e\in E(\Gamma)}|k_e|$, $n = (n_e)_{e\in E(\Gamma)}$,

$$\partial n : V(\Gamma) \longrightarrow I^*,$$

$$\partial n(v) = \sum_{e\in E(\Gamma),v_1(e)=v} n_e - \sum_{e\in E(\Gamma),v_0(e)=v} n_e,$$

and $\sum'$ *means that* ∂n *is supported at* S *and each* n_e *is nonzero.*

Remark 4.1.10 *We can also add multisigns here, there are two methods. The first way is to associate each edge a multisign* $\nu_e = (\nu_e^1,\dots,\nu_e^r)$*, then the formula in 4.1.9 will then contain terms* $sgn(n_e)^{\nu(e)}$*:*

$$G^{\nu}_{I,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S},u) = vol(F_{\mathbb{R}}/I)^{|V(\Gamma)\backslash S|+|\underline{k}|-|E(\Gamma)|}\lim_{\eta\to 0^+}\sum\nolimits'_{\{n,c(n)\in H_1(\Gamma,S)\otimes I^*\}}\mathrm{sgn}(n_e)^{\nu(e)}\frac{e^{2\pi i \mathrm{Tr}(\sum_{v\in S}(\partial n)_v x_v)}}{\prod_{e\in E(\Gamma)}\|un_e\|^{k_e(r+\eta)}}.$$

The second method to add mutisigns by taking into account the $n_e \in I^* \smallsetminus \{0\}$*, such that* $\mathrm{sgn}(n_e) = (-1)^{\nu_e}$ *for a given multisign* ν_e *to each edge* e*. Then*

$$G_{I,\nu,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S},u) = vol(F_{\mathbb{R}}/I)^{|V(\Gamma)\backslash S|+|\underline{k}|-|E(\Gamma)|}\lim_{\eta\to 0^+}\sum\nolimits'_{\substack{\{n,c(n)\in H_1(\Gamma,S)\otimes I^*\}\\ \mathrm{sgn}(n_e)=(-1)^{\nu(e)}}}\frac{e^{2\pi i \mathrm{Tr}(\sum_{v\in S}(\partial n)_v x_v)}}{\prod_{e\in E(\Gamma)}\|un_e\|^{k_e(r+\eta)}}.$$

In fact, we usually prefer the second definition $G_{I,\nu,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S},u)$*.*

Remark 4.1.11 *We should say some words about the convergence of* $G_{I,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S},u)$ *or* $G_{I,\nu,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S},u)$*.*

Let F *be a real quadratic field,* $F = \mathbb{Q}(\sqrt{D})$ *and* D *is positive squarefree. Given the graph* Γ *defined as follows*

In this case, the higher Green function associated is

$$G_{I,\nu,\Gamma,\partial\Gamma}(\{x_1,x_2,x_3\},u) = vol(F_{\mathbb{R}}/I)\lim_{\eta\to 0^+}\sum_{\substack{n_1,n_2\in I^*\\ n_1+n_2+n_3=0\\ \mathrm{sgn}(n_i)=(-1)^{\nu_i}}}\frac{e^{2\pi i \mathrm{Tr}(n_1x_1+n_2x_2+n_3x_3)}}{\|n_1u\|^{1+\eta}\|n_2u\|^{1+\eta}\|n_3u\|^{1+\eta}},$$

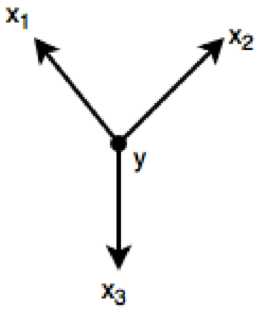


Figure 4.2: A plane trivalent tree with one internal vertex

where

$$n_i = (n_{i,1}, n_{i,2}) \in \mathbb{R}^2, \quad 1 \le i \le 3.$$

If we write

$$n_{i,1} = a_i + b_i\sqrt{D}, \quad n_{i,2} = a_i - b_i\sqrt{D},$$

and we are given the signature $\nu_1 = (\nu_1^1, \nu_1^1) = (0,0)$ *and* $\nu_2 = (\nu_2^1, \nu_2^1) = (0,0)$*, then*

$$G_{I,\nu,\Gamma,\partial\Gamma}(\{x_1,x_2,x_3\},u) = vol(F_{\mathbb{R}}/I)\times$$

$$\lim_{\eta\to 0^+} \sum_{\substack{n_1,n_2\in\mathbb{R}^2\cap I^* \\ \mathrm{sgn}(n_i)=(-1)^{\nu_i}}} \frac{e^{2\pi i \mathrm{Tr}(n_1x_1+n_2x_2+n_3x_3)}}{\prod_{i=1}^3 \left((a_i+b_i\sqrt{D})^2u_1^2 + (a_i-b_i\sqrt{D})^2u_2^2\right)^{1+\eta}},$$

where $u_1u_2 = 1$*. We can deduce that*

$$|G_{I,\nu,\Gamma,\partial\Gamma}(\{x_1,x_2,x_3\},u)| \le vol(F_{\mathbb{R}}/I)\times$$

$$\lim_{\eta\to 0^+} \sum_{n_1,n_2\in\mathbb{R}_+^2\cap I^*} \frac{1}{\left((a_1+b_1\sqrt{D})^2u_1^2\right)^{1+\eta}\left((a_2+b_2\sqrt{D})^2u_1^2\right)^{1+\eta}\left((a_1+a_2+(b_1+b_2)\sqrt{D})^2u_1^2\right)^{1+\eta}}.$$

Since $\mathbb{R}_+^2 \cap I^*$ *is a lattice, then we can transform the previous inequality as*

$$|G_{I,\nu,\Gamma,\partial\Gamma}(\{x_1,x_2,x_3\},u)| \le \mathrm{Constant} \times vol(F_{\mathbb{R}}/I)\times$$

$$\lim_{\eta\to 0^+} \sum_{m_1,m_2,k_1,k_2\in\mathbb{N}_+^2\smallsetminus\{0\}} \frac{1}{\left((m_1+k_1\sqrt{D})^2u_1^2\right)^{1+\eta}\left((m_2+k_2\sqrt{D})^2u_1^2\right)^{1+\eta}\left((m_1+m_2+(k_1+k_2)\sqrt{D})^2u_1^2\right)^{1+\eta}}.$$

$$|G_{I,\nu,\Gamma,\partial\Gamma}(\{x_1,x_2,x_3\},u)| < +\infty.$$

In conclusion, $G_{I,\nu,\Gamma,\partial\Gamma}(\{x_1,x_2,x_3\},u)$*, associated to the given graph, is absolutely convergent.*

If a subdivision map is given, then the associated higher Green function

$$G_{I,\nu,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S},u)$$

is also convergent, since the exponent of each n_e *is bigger than 1.*

For more general graphs, similar arguments for convergence are still valid. If the degree of the field F *is bigger than 2, we can still obtain the convergence of* $G_{I,\nu,\Gamma,\partial\Gamma}(\{x_1,x_2,x_3\},u)$ *in a very similar way. Since it is difficult to give a precise description of an element in* F*, we will omit the argument and leave it to the reader for this moment.*

4.1.3 Generalized MZV for totally real fields

If we suppose that there exists a subgroup $U \subset O_{F,+}^{\times}$ of finite index stabilizing $\{x_v\}$ for each $v \in S$, we can apply the Hecke transform to define a new function as follows. In fact, if each x_v is a torsion point in $F_{\mathbb{R}}/I$, then such a subgroup $U \subset O_{F,+}^{\times}$ exists. Such a subgroup U exists if and only if each x_v lies in the torsion subgroup of $F_{\mathbb{R}}/I$.

Definition 4.1.12

$$\mathscr{F}_{I,\Gamma,S}(\{x_v\}_{v\in S}) = (O_{F,+}^{\times} : U)^{-1} \int_{U_{\mathbb{R}}/U} G_{I,\Gamma,S}(\{x_v\}_{v\in S}, u) d^{\times}u,$$

where $d^{\times}u = \frac{du_1\cdots du_{r-1}}{u_1\cdots u_{r-1}}$, $\prod_{j=1}^{r} u_j = 1$ *and* $U_{\mathbb{R}}/U = BU \cong (\mathbb{S})^{r-1}$ *is the classifying space of* $U \cong \mathbb{Z}^{r-1}$*. In the same way, we can define* $\mathscr{F}_{I,\Gamma,S}^{\nu}(\{x_v\}_{v\in S})$ *and* $\mathscr{F}_{I,\nu,\Gamma,S}(\{x_v\}_{v\in S})$*.*

Then the generalized multiple zeta value is defined as

Definition 4.1.13 (Generalized Multiple Zeta Values)

$$Z_I(\Gamma, S) = \mathscr{F}_{I,\Gamma,S}(\{0\}_{v\in S}).$$

Similarily, we have

$$Z_I^{\nu}(\Gamma, S) = \mathscr{F}_{I,\Gamma,S}^{\nu}(\{0\}_{v\in S}),$$

and

$$Z_{I,\nu}(\Gamma, S) = \mathscr{F}_{I,\nu,\Gamma,S}(\{0\}_{v\in S}).$$

Remark 4.1.14 *In fact, the construction for* $G_{I,\nu,\Gamma(\underline{k}),S}(\{x_v\}_{v\in S}, u)$ *also works for arbitrary number fields. If a complex place exists, we can just replace the Euclidean norm by Hermitian norm and we can add signs just for the real places. For an imaginary quadratic field, there is no Hecke transform.*

Remark 4.1.15 *We need to say something about the convergence of* $Z_{I,\nu}(\Gamma, S)$*. Since we will give more details about the precise expression in Chapter 5, we will only mention again Example (1.3.16) in the Introduction for real quadratic field* $F = \mathbb{Q}(\sqrt{D})$*.*

$$\int_{U_{\mathbb{R}}} G_{I,\nu,\Gamma,\partial\Gamma}(\{0,0,0\}, u)\, \mathrm{d}^{\times}u = \sum_{\alpha_j,\beta_j} \mathbb{I}(3,2;\lambda^{(i)};k_i = 1;\Gamma,\partial\Gamma),$$

where $\lambda^{(i)} = (\alpha_i, \beta_i)$ *and*

$$\mathbb{I}(3,2;\lambda^{(i)};k_i = 1;\Gamma,\partial\Gamma) = \int_0^{\infty} \frac{1}{\prod_{j=1}^{3}(\alpha_j^2 u + \beta_j^2 u^{-1})} \frac{du}{u}$$

$$= \frac{\pi}{2(\alpha_1\beta_2 + \alpha_2\beta_1)(\alpha_1\beta_3 + \alpha_3\beta_1)(\alpha_2\beta_3 + \alpha_3\beta_2)}.$$

Since $\lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)} = 0$ *due to the formal convolution, then* $\alpha_3 = -\alpha_1 - \alpha_2$, $\beta_3 = -\beta_1 - \beta_2$ *and*

$$\alpha_1 = a + b\sqrt{D}, \quad \beta_1 = a - b\sqrt{D};$$

$$\alpha_2 = c + d\sqrt{D}, \quad \beta_2 = c - d\sqrt{D}.$$

Let $\lambda^{(2)'} = (\beta_2, \alpha_2)$*, then* $\alpha_1\beta_2 + \alpha_2\beta_1 = (\lambda^{(1)}|\lambda^{(2)'})$ *and* $(\cdot|\cdot)$ *is the scalar product.*

$$N(\lambda^{(1)}) = \alpha_1\beta_1 = a^2 - b^2D,$$
$$\alpha_1\beta_3 + \alpha_3\beta_1 = -2N(\lambda^{(1)}) - (\lambda^{(1)}|\lambda^{(2)'}).$$

Therefore we obtain

$$2(\alpha_1\beta_2+\alpha_2\beta_1)(\alpha_1\beta_3+\alpha_3\beta_1)(\alpha_2\beta_3+\alpha_3\beta_2) = 2(\lambda^{(1)}|\lambda^{(2)'})(2N(\lambda^{(1)})+(\lambda^{(1)}|\lambda^{(2)'}))(2N(\lambda^{(2)})+(\lambda^{(1)}|\lambda^{(2)'})).$$

Moreover we have

$$(\lambda^{(1)}|\lambda^{(2)'})) \geq 2\left(N(\lambda^{(1)})N(\lambda^{(2)'})\right)^{1/2},$$
$$(2N(\lambda^{(1)}) + (\lambda^{(1)}|\lambda^{(2)'}))(2N(\lambda^{(2)}) + (\lambda^{(1)}|\lambda^{(2)'})) \geq 12N(\lambda^{(1)})N(\lambda^{(2)'}),$$

then

$$2(\alpha_1\beta_2 + \alpha_2\beta_1)(\alpha_1\beta_3 + \alpha_3\beta_1)(\alpha_2\beta_3 + \alpha_3\beta_2) \geq Const\left(N(\lambda^{(1)})N(\lambda^{(2)'})\right)^{3/2}.$$

Hence as a result, we have

$$Z_{I,\nu}(\Gamma, \partial\Gamma) = \sum_{\lambda^{(1)},\lambda^{(2)'}\in(\mathbb{R}^2\cap I^*)/U} \mathbb{I}(3,2;\lambda^{(i)};k_i = 1;\Gamma,\partial\Gamma)$$
$$\leq \sum_{\lambda^{(1)},\lambda^{(2)'}\in(\mathbb{R}^2\cap I^*)/U} \frac{1}{\left(N(\lambda^{(1)})N(\lambda^{(2)'})\right)^{3/2}}.$$

In conclusion $Z_{I,\nu}(\Gamma, \partial\Gamma)$ *absolutely converges.*

4.1.4 Generalized MZV for arbitrary number fields

The definition of generalized MZV for arbitrary number fields is the same as we define for totally real fields. All we need to do is to define the basic plectic Green function.

Definition 4.1.16 ***(Plectic Green functions for arbitrary number fields)***
Let K *be a number field with* r_1 *real places* $(v_i)_{1\leq i\leq r_1}$ *and* r_2 *complex places* $(w_j)_{1\leq j\leq r_2}$ *and of degree* $[K:\mathbb{Q}] = r_1 + 2r_2$*. We have the trace map* $\mathrm{Tr} = \mathrm{Tr}_{K/\mathbb{Q}}$

$$K_\mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

and we have a Hermitian product

$$\langle\ |\ \rangle : K_\mathbb{R} \times K_\mathbb{R} \longrightarrow \mathbb{C}$$
$$\langle(x_v, z_w)|(x'_v, z'_w)\rangle = \sum_v x_v x'_v + \sum_w z_w z'_w,$$

where

$$x = (x_v, z_w).$$

Let

$$\|\cdot\| = \langle\cdot|\cdot\rangle^{1/2},$$

then the plectic Green function can be defined as

$$g_I(x,u) = \sum_{n\in I^*\setminus\{0\}} \frac{e^{2\pi i\mathrm{Tr}(nx)}}{\|un\|^{r_1+2r_2}}, \quad x \in K_\mathbb{R}/I,\ u \in U_\mathbb{R},$$

which is a distribution on $(K_\mathbb{R}/I) \times U_\mathbb{R}$*.*

4.2 Relation to classical MZVs

4.2.1 Relation to MZVs and examples

In order to give an explicit relation to the classical definition, we obtain the following result.

Theorem 4.2.1 (Relation to multiple zeta values) *Let F be the rational field $\mathbb{Q}$. Let Γ be any tree with rank $d = \operatorname{rank}(H_1(\Gamma, S)) \geq 2$, where $S = \partial\Gamma$. Assume that we are given a "sign" map $\nu : E(\Gamma) \longrightarrow \{0, 1\}$ and a subdivision map $\underline{k} : E(\Gamma) \longrightarrow \mathbb{N} \smallsetminus \{0\}$ as in Remark 4.1.10 and in Definition 4.1.8, respectively. Then the generalized multiple zeta value $Z_{I,\nu}(\Gamma(\underline{k}), \partial\Gamma(\underline{k}))$ $(I = \mathbb{Z})$ can be expressed as a finite $\mathbb{Z}$-linear combination of classical multiple zeta values (MZVs) of depth d and weight $|\underline{k}| = \sum\limits_{e} k_e$.*

Remark 4.2.2 *Before proving this result in general we consider first the case when Γ is a plane trivalent tree, i.e., a tree whose internal vertices are of valency 3. A plane trivalent tree Γ with $m + 1$ external vertices has $2m - 1$ edges and $m - 1$ internal vertices. The rank of Γ is $d = \operatorname{rank}(H_1(\Gamma, \partial\Gamma)) = m$. A tree is a connected graph with no loops, no multiple edges and $H_1(\Gamma, \mathbb{Z}) = 0$.*

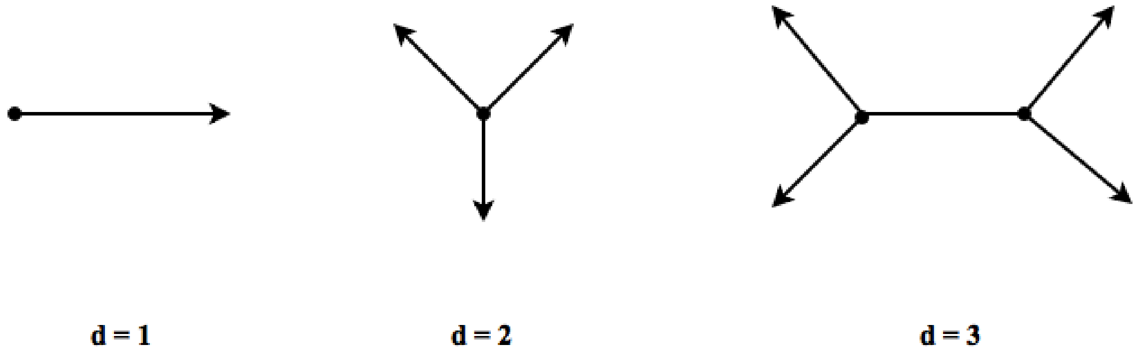


Figure 4.3: The simplest examples of plane trivalent trees of small ranks.

Remark 4.2.3 *In this theorem we only consider the situation that the subset $S = \partial\Gamma$. We would like to explain that such choice is quite general. In fact, $Z_{I,\nu}(\Gamma, S) = 0$ if $\partial\Gamma \not\subset S$, therefore we must have $\partial\Gamma \subset S$. If one internal vertex is contained in S, namely $\partial\Gamma \subsetneq S$, the situation can be reduced to a new tree with fewer external vertices due to the formal convolution Lemma (4.1.6)*

When $F = \mathbb{Q}$ and the ideal $I = \mathbb{Z}$, the Hecke transform is trivial. The generalized multiple zeta values are given by

$$Z_{I,\nu}(\Gamma, S) = G_{I,\nu,\Gamma,S}(\{0\}_{v \in S}, 1).$$

Before giving the proof in full generality, we will illustrate the statement of Theorem 4.2.1 by several examples.

Some Examples.

Remark 4.2.4 *In Theorem 4.2.1 and 4.3.1, we will take into account all arbitary trees, and we prove the theorems when the valency of any internal vertex* $val(v_{internal}) \geq 3$. *We will now explain the case:* $val(v_{internal}) = 2$.

Let $\widetilde{\Gamma}$ *be one of the trees in Figure 4.4.*

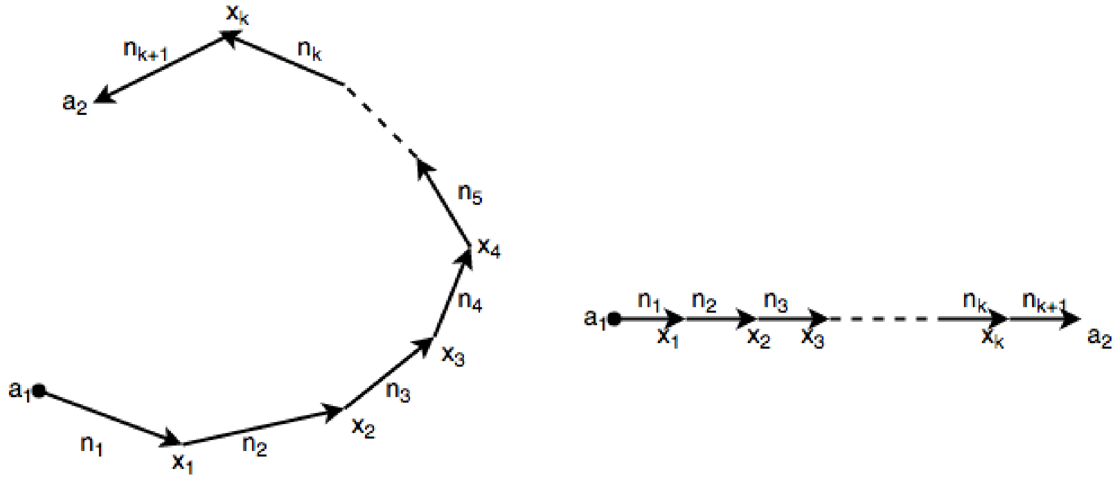


Figure 4.4: All internal vertices have valency 2

We fix the signature $\nu_1 = 0$ *for* n_1, *then implicitly all other signatures are also determined. Because of the formal convolution, we have* $n_{i+1} - n_i = 0$ *for each internal vertex* x_i *(*$1 \leq i \leq k$*).*

$$G_{\mathbb{Z},\nu,\widetilde{\Gamma},\partial\widetilde{\Gamma}}(\{a_1,a_2\},1) = \int_{(\mathbb{R}/\mathbb{Z})^k} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}\smallsetminus\{0\} \\ \mathrm{sgn}(n_j)=(-1)^{\nu_j} \\ n_j=n_{j+1},1\leq j\leq k}} \frac{e^{2\pi i\left(\sum_{2\leq j\leq k} n_j(x_j-x_{j-1}))+n_1(x_1-a_1)+n_{k+1}(a_2-x_k)\right)}}{\prod_{j=1}^{k+1}|n_j|} dx_1\cdots dx_k$$

$$= \sum_{n_1\in\mathbb{N}\smallsetminus\{0\}} \frac{e^{2\pi i n_1(a_2-a_1)}}{n_1^{k+1}}.$$

$$Z_{\mathbb{Z},\nu}(\widetilde{\Gamma},\partial\widetilde{\Gamma}) = G_{\mathbb{Z},\nu,\widetilde{\Gamma},\partial\widetilde{\Gamma}}(\{0,0\},1) = \sum_{n_1\in\mathbb{N}\smallsetminus\{0\}} \frac{1}{n_1^{k+1}} = \zeta(k+1).$$

This result tells us that such a graph delivers the same result as the subdivision map of adding k *points for the tree with one edge and two external vertices* a_1, a_2.

Non-tree case. We will show some examples for graphs which are not trees.

Example 4.2.5 *Let* $\widehat{\Gamma}$ *be the graph in Figure 4.5.*

We fix the signature $\nu_1 = 0$ *for* n_1 *and the signature* $\mu_1 = 0$ *for* m_1, *then other signatures are implicitly determined by the constraint at each internal vertex because of the formal*

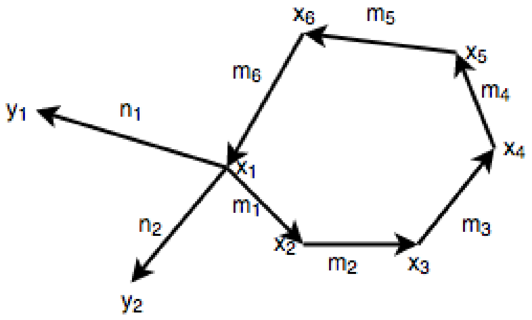


Figure 4.5: The graph $\widehat{\Gamma}$.

Fourier convolution, namely $n_1+n_2-m_6+m_1=0$ *for* x_1 *and* $m_{k+1}-m_k=0$ *for each* x_k *(*$2\le k\le 5$*).*

$$G_{\mathbb{Z},\nu,\widehat{\Gamma},\partial\widehat{\Gamma}}(\{y_1,y_2\},1) = \int_{(\mathbb{R}/\mathbb{Z})^6} \sum_{\substack{n_k,m_l\in\mathbb{Z}\setminus\{0\}\\ \mathrm{sgn}(n_k)=(-1)^{\nu_k}\\ \mathrm{sgn}(m_l)=(-1)^{\mu_l}\\ n_1+n_2+m_1-m_6=0\\ m_l=m_{l+1},1\le l\le 5}} \frac{e^{2\pi i(\sum_{l=1}^5 m_l(x_{l+1}-x_l)+m_6(x_1-x_6)+n_1(y_1-x_1)+n_2(y_2-x_1))}}{\prod_{l=1}^6 |m_l|\cdot|n_1|\cdot|n_2|} dx_1\cdots dx_6.$$

By formal Fourier convolution,

$$G_{\mathbb{Z},\nu,\widehat{\Gamma},\partial\widehat{\Gamma}}(\{y_1,y_2\},1) = \sum_{n_1,m_1\in\mathbb{N}\setminus\{0\}} \frac{e^{2\pi i(n_1(y_1-y_2))}}{m_1^6 n_1^2} = \zeta(6)\sum_{n=0}^{\infty}\frac{e^{2\pi i n(y_1-y_2)}}{n^2},$$

and

$$Z_{\mathbb{Z},\nu}(\widehat{\Gamma},\partial\widehat{\Gamma}) = \sum_{n_1,m_1\in\mathbb{N}\setminus\{0\}} \frac{1}{m_1^6 n_1^2} = \zeta(6)\zeta(2).$$

By the shuffle relation, we get

$$Z_{\mathbb{Z},\nu}(\widehat{\Gamma},\partial\widehat{\Gamma}) = \sum_{r+s=8}(C_{r-1}^5 + C_{r-1}^1)\zeta(s,r).$$

The result is equal to the value of

$$Z_{\mathbb{Z},\nu}(\Gamma_b,\{s_1,s_2\}) = \sum_{\substack{n_1,m_1\in\mathbb{N}\setminus\{0\}\\ n_2=-n_1}} \frac{e^{2\pi i(n_1y_1+n_2y_2)}}{m_1^6n_1^2} = \sum_{n_1,m_1\in\mathbb{N}\setminus\{0\}} \frac{e^{2\pi i(n_1(y_1-y_2))}}{m_1^6n_1^2},$$

given by Figure 4.6.

From the discussion above, we can see that different graphs can deliver the same value.

Example 4.2.6 *Let* Γ *be the graph in Figure 4.7.*

The graph $\Gamma(\underline{k})$ *is just a chain obtained after adding* $k-1$ *points to* Γ*. The given sign is* $\nu_i=\nu$*,* $1\le i\le k$*. Then*

$$G_{I,\nu,\Gamma(k),\partial\Gamma(\underline{k})}(\{x_v\}_{v\in\partial\Gamma(\underline{k})},1) = \sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ \mathrm{sgn}(n)=(-1)^{\nu}}} \frac{e^{2\pi i n(x_2-x_1)}}{|n|^k},$$

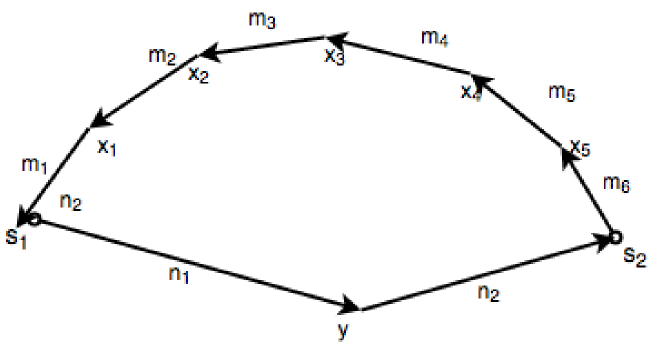


Figure 4.6: The graph Γ_b.

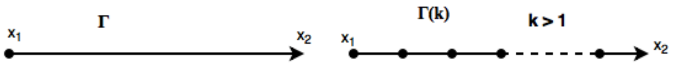


Figure 4.7: A plane tree of rank 1.

and

$$Z_{I,\nu}(\Gamma(k),\partial\Gamma(\underline{k})) = \sum_{\substack{n\in\mathbb{Z}\smallsetminus\{0\}\\ \mathrm{sgn}(n)=(-1)^{\nu}}} \frac{1}{|n|^k} = \zeta(k).$$

Moreover, we even do not need sign ν, then one obtain

$$Z_I(\Gamma(k),\partial\Gamma(\underline{k})) = \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \frac{1}{|n|^k} = 2\zeta(k).$$

Case of trees of internal valency ≥ 3.

Example 4.2.7 *Let Γ_1 be the graph in Figure 4.8.*

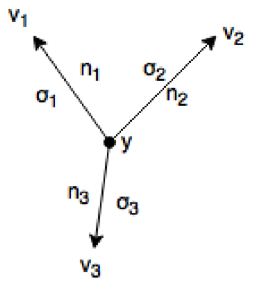


Figure 4.8: A plane trivalent tree with one internal vertex

The rank of Γ_1 is $2 = rank(H_1(\Gamma_1,\partial\Gamma_1))$. To each edge e_i $(i = 1,2,3)$ we add $\sigma_i - 1 \geq 0$ points. The only internal vertex is denoted by y, each external vertex v_i is decorated by the variable x_{v_i}. For each e_i $(i = 1,2)$, the given sign ν_i equals 0. For the edge e_3, the sign $\nu_3 = 1$. In fact, the constraint $n_1 + n_2 + n_3 = 0$ and $\nu_1 = \nu_2 = 0$ imply that $\nu_3 = 1$.

$$G_{\mathbb{Z},\nu,\Gamma_1,\partial\Gamma_1}(\{x_v\}_{v\in\partial\Gamma_1},1) = \sum_{\substack{n_1+n_2+n_3=0,n_i\in\mathbb{Z}\smallsetminus\{0\}\\ \mathrm{sgn}(n_i)=(-1)^{\nu_i}}} \frac{e^{2\pi i(n_1x_{v_1}+n_2x_{v_2}+n_3x_{v_3})}}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|n_3|^{\sigma_3}},$$

where $\nu_1 = \nu_2 = 0$, $\nu_3 = 1$. *Then*

$$Z_{\mathbb{Z},\nu}(\Gamma_1, \partial\Gamma_1) = \sum_{\substack{n_1+n_2+n_3=0, n_i \in \mathbb{Z}\setminus\{0\} \\ \mathrm{sgn}(n_i)=(-1)^{\mu_i}}} \frac{1}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|n_3|^{\sigma_3}} = \sum_{n_1,n_2\in\mathbb{N}\setminus\{0\}} \frac{1}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|n_1+n_2|^{\sigma_3}}$$

$$Z_{\mathbb{Z},\nu}(\Gamma_1, \partial\Gamma_1) = \sum_{n_1,n_2\in\mathbb{N}\setminus\{0\}} \frac{1}{n_1^{\sigma_1}n_2^{\sigma_2}(n_1+n_2)^{\sigma_3}}.$$

Recall the following Eisenstein's trick

Formula 4.2.8

$$\frac{1}{m^i n^j} = \sum_{r+s=i+j}\left(\frac{C_{r-1}^{i-1}}{(m+n)^r n^s} + \frac{C_{r-1}^{j-1}}{(m+n)^r m^s}\right).$$

then $\frac{1}{n_1^{\sigma_1}n_2^{\sigma_2}} = \sum_{r+s=\sigma_1+\sigma_2}\left(\frac{C_{r-1}^{\sigma_1-1}}{(n_1+n_2)^r n_2^s} + \frac{C_{r-1}^{\sigma_2-1}}{(n_1+n_2)^r n_1^s}\right)$, $r, s \geq 1$,

where $C_b^a = \binom{b}{a}$. *Hence*

$$Z_{\mathbb{Z},\nu}(\Gamma_1, \partial\Gamma_1) = \sum_{n_1,n_2\in\mathbb{N}\setminus\{0\}} \sum_{r+s=\sigma_1+\sigma_2}\left(\frac{C_{r-1}^{\sigma_1-1}}{(n_1+n_2)^{r+\sigma_3}n_2^s} + \frac{C_{r-1}^{\sigma_2-1}}{(n_1+n_2)^{r+\sigma_3}n_1^s}\right),$$

$$= \sum_{r+s=\sigma_1+\sigma_2}\left\{C_{r-1}^{\sigma_1-1}\left(\sum_{n_1,n_2\in\mathbb{N}\setminus\{0\}}\frac{1}{(n_1+n_2)^{r+\sigma_3}n_2^s}\right) + C_{r-1}^{\sigma_2-1}\left(\sum_{n_1,n_2\in\mathbb{N}\setminus\{0\}}\frac{1}{(n_1+n_2)^{r+\sigma_3}n_1^s}\right)\right\}.$$

Since

$$\sum_{n_1,n_2\in\mathbb{N}\setminus\{0\}}\frac{1}{(n_1+n_2)^{r+\sigma_3}n_2^s} = \sum_{0<n_2<n_1'}\frac{1}{(n_1')^{r+\sigma_3}n_2^s} = \zeta(s, r+\sigma_3),$$

where $n_1' = n_1 + n_2$. *We can express the generalized multiple zeta value for* Γ_1 *as follows:*

$$Z_{\mathbb{Z},\nu}(\Gamma_1, \partial\Gamma_1) = \sum_{r+s=\sigma_1+\sigma_2}\left(C_{r-1}^{\sigma_1-1}\zeta(s, r+\sigma_3) + C_{r-1}^{\sigma_2-1}\zeta(s, r+\sigma_3)\right).$$

$$= \sum_{r+s=\sigma_1+\sigma_2}\left(C_{r-1}^{\sigma_1-1} + C_{r-1}^{\sigma_2-1}\right)\zeta(s, r+\sigma_3).$$

Here $\zeta(s, r+\sigma_3)$ is a classical double zeta value of weight $r+s+\sigma_3 = \sigma_1+\sigma_2+\sigma_3$, which means that $Z_{\mathbb{Z},\nu}(\Gamma_1, \partial\Gamma_1)$ can be expressed as a $\mathbb{Z}$-linear combination of double zeta values of this weight.

Example 4.2.9 *Let* Γ_1' *be the diagram as in Figure 2. The rank of* Γ_1' *is* $rank(H_1(\Gamma_1', \partial\Gamma_1')) = 3$. *In fact, the* Γ_1' *is no longer a plane trivalent tree. For each edge* $e_i (1 \leq i \leq 4)$, *we add* $\sigma_i - 1$ $(\sigma_i \geq 1)$ *points. The only internal vertex is denoted by* y, *each external vertex* v_i *is decorated by* x_{v_i}. *For each* e_i $(i = 1, 2, 3)$, *the given sign* ν_i *equals* 0; *for* e_4 *the sign* $\nu_4 = 1$.

$$G_{\mathbb{Z},\nu,\Gamma_1',\partial\Gamma_1'}(\{x_v\}_{v\in\partial\Gamma_1'}, 1) = \sum_{\substack{n_1+n_2+n_3+n_4=0, n_i\in\mathbb{Z}\setminus\{0\}; \\ \mathrm{sgn}(n_i)=(-1)^{\nu_i}, 1\leq i\leq 4}} \frac{e^{2\pi i(n_1x_{v_1}+n_2x_{v_2}+n_3x_{v_3}+n_4x_{v_4})}}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|n_3|^{\sigma_3}|n_4|^{\sigma_4}},$$

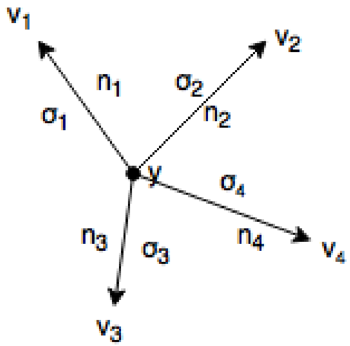


Figure 4.9: A plane trivalent tree with one internal vertex and 4 edges

where $\nu_1 = \nu_2 = \nu_3 = 0$, $\nu_4 = 1$. *Then*

$$Z_{\mathbb{Z},\nu}(\Gamma_1', \partial\Gamma_1') = G_{I,\nu,\Gamma_1',\partial\Gamma_1'}(\{0\}_{v\in\partial\Gamma_1'}, 1)$$

$$= \sum_{\substack{n_1+n_2+n_3+n_4=0, n_i\in\mathbb{Z}\\ \mathrm{sgn}(n_j)=(-1)^{v_j}, 1\leq j\leq 4}} \frac{1}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|n_3|^{\sigma_3}|n_4|^{\sigma_4}}$$

$$Z_{\mathbb{Z},\nu}(\Gamma_1', \partial\Gamma_1') = \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{1}{n_1^{\sigma_1}n_2^{\sigma_2}n_3^{\sigma_3}(n_1+n_2+n_3)^{\sigma_4}}.$$

Firstly, we use Eisenstein's trick for

$$\frac{1}{n_2^{\sigma_2}n_3^{\sigma_3}} = \sum_{r_1+s_1=\sigma_2+\sigma_3} \left(\frac{C^{\sigma_2-1}_{r_1-1}}{(n_2+n_3)^{r_1}n_3^{s_1}} + \frac{C^{\sigma_3-1}_{r_1-1}}{(n_2+n_3)^{r_1}n_2^{s_1}} \right)$$

Then

$$Z_{\mathbb{Z},\nu}(\Gamma_1', \partial\Gamma_1') =$$

$$\sum_{r_1+s_1=\sigma_2+\sigma_3} \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \left(\frac{C^{\sigma_2-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4}(n_2+n_3)^{r_1}n_3^{s_1}n_1^{\sigma_1}} + \frac{C^{\sigma_3-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4}(n_2+n_3)^{r_1}n_2^{s_1}n_1^{\sigma_1}} \right)$$

Secondly, we use twice Eisenstein's trick for the terms involving n_1 *and* (n_2+n_3). *Then we obtain*

$$Z_{\mathbb{Z},\nu}(\Gamma_1', \partial\Gamma_1') = \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=r_1+\sigma_1}} \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{C^{r_1-1}_{r_2-1}C^{\sigma_2-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4+r_2}n_3^{s_1}n_1^{s_2}}$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=r_1+\sigma_1}} \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{C^{\sigma_1-1}_{r_2-1}C^{\sigma_2-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4+r_2}(n_2+n_3)^{s_2}n_3^{s_1}}$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2'+s_2'=r_1+\sigma_1}} \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{C^{r_1-1}_{r_2'-1}C^{\sigma_3-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4+r_2'}n_2^{s_1}n_1^{s_2'}}$$

$$+\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r'_2+s'_2=r_1+\sigma_1}}\ \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{C^{\sigma_1-1}_{r'_2-1}C^{\sigma_3-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4+r'_2}(n_2+n_3)^{s'_2}n_2^{s_1}}$$

Finally, we use Eisenstein's trick for the terms involving n_1 and n_2(respectively, n_1 and n_3).

$$Z_{\mathbb{Z},\nu}(\Gamma'_1,\partial\Gamma'_1)=\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=r_1+\sigma_1\\ r_3+s_3=s_1+s_2}}\ \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{C^{s_1-1}_{r_3-1}C^{r_1-1}_{r_2-1}C^{\sigma_2-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4+r_2}(n_1+n_3)^{r_3}n_1^{s_3}}$$

$$+\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=r_1+\sigma_1\\ r_3+s_3=s_1+s_2}}\ \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{C^{s_2-1}_{r_3-1}C^{r_1-1}_{r_2-1}C^{\sigma_2-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4+r_2}(n_1+n_3)^{r_3}n_3^{s_3}}$$

$$+\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=r_1+\sigma_1}}\ \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{C^{\sigma_1-1}_{r_2-1}C^{\sigma_2-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4+r_2}(n_2+n_3)^{s_2}n_3^{s_1}}$$

$$+\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r'_2+s'_2=r_1+\sigma_1\\ r'_3+s'_3=s_1+s'_2}}\ \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{C^{s_1-1}_{r'_3-1}C^{r_1-1}_{r'_2-1}C^{\sigma_3-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4+r'_2}(n_1+n_2)^{r'_3}n_1^{s'_3}}$$

$$+\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r'_2+s'_2=r_1+\sigma_1\\ r'_3+s'_3=s_1+s'_2}}\ \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{C^{s'_2-1}_{r'_3-1}C^{r_1-1}_{r'_2-1}C^{\sigma_3-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4+r'_2}(n_1+n_2)^{r'_3}n_2^{s'_3}}$$

$$+\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r'_2+s'_2=r_1+\sigma_1}}\ \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{C^{\sigma_1-1}_{r'_2-1}C^{\sigma_3-1}_{r_1-1}}{(n_1+n_2+n_3)^{\sigma_4+r'_2}(n_2+n_3)^{s'_2}n_2^{s_1}}$$

Then

$$Z_{\mathbb{Z},\nu}(\Gamma'_1,\partial\Gamma'_1)=\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=r_1+\sigma_1\\ r_3+s_3=s_1+s_2}}\left(C^{s_1-1}_{r_3-1}C^{r_1-1}_{r_2-1}C^{\sigma_2-1}_{r_1-1}+C^{s_2-1}_{r_3-1}C^{r_1-1}_{r_2-1}C^{\sigma_2-1}_{r_1-1}\right)\zeta(\sigma_4+r_2,r_3,s_3)$$

$$+\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r'_2+s'_2=r_1+\sigma_1\\ r'_3+s'_3=s_1+s'_2}}\left(C^{s_1-1}_{r'_3-1}C^{r_1-1}_{r'_2-1}C^{\sigma_3-1}_{r_1-1}+C^{s'_2-1}_{r'_3-1}C^{r_1-1}_{r'_2-1}C^{\sigma_3-1}_{r_1-1}\right)\zeta(\sigma_4+r'_2,r'_3,s'_3)$$

$$+\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=r_1+\sigma_1}} C^{\sigma_1-1}_{r_2-1}C^{\sigma_2-1}_{r_1-1}\zeta(\sigma_4+r_2,s_2,s_1)+\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r'_2+s'_2=r_1+\sigma_1}} C^{\sigma_1-1}_{r'_2-1}C^{\sigma_3-1}_{r_1-1}\zeta(\sigma_4+r'_2,s'_2,s_1),$$

where $\sigma_4+r_2+r_3+s_3=\sigma_4+r_2+s_2+s_1=\sigma_1+\sigma_2+\sigma_3+\sigma_4$*.*
We can see that $Z_{\mathbb{Z},\nu}(\Gamma'_1,\partial\Gamma'_1)$ *is a* $\mathbb{Z}$*-linear combination of triple-zeta values of weight* $\sigma_1+\sigma_2+\sigma_3+\sigma_4$*.*

Example 4.2.10 *Let* Γ_2 *be the diagram as in Figure 4.10.*

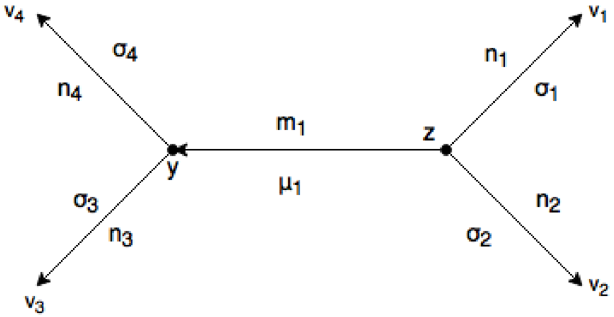


Figure 4.10: A plane trivalent tree with two internal vertices

The rank is equal to $rank(H_1(\Gamma_2,\partial\Gamma_2))=3$*. For each external edge* e_i $(1\leq i\leq 3)$*, the given sign* ν_i *equals* 0*, therefore* $\nu_4=1$*.*

$$G_{I,\nu,\Gamma_2,\partial\Gamma_2}(\{x_v\}_{v\in S},1)=\int_{(\mathbb{R}/\mathbb{Z})^2}\sum_{\substack{n_i,m_i\in\mathbb{Z}\smallsetminus\{0\}\\ \mathrm{sgn}(n_i)=(-1)^{\nu_i}}}\frac{e^{2\pi i((x_{v_3}-y)n_3+(x_{v_4}-y)n_4)}}{|n_3|^{\sigma_3}|n_4|^{\sigma_4}}\frac{e^{2\pi i((x_{v_1}-z)n_1+(x_{v_2}-z)n_2+(y-z)m_1)}}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|m_1|^{\mu_1}}dxdy$$

By the formal Fourier convolution, we get

$$G_{I,\nu,\Gamma_2,\partial\Gamma_2}(\{x_v\}_{v\in S},1)=\sum_{\substack{n_1+n_2+m_1=0,n_3+n_4-m_1=0;n_i,m_i\in\mathbb{Z}\smallsetminus\{0\}\\ \mathrm{sgn}(n_i)=(-1)^{\nu_i}}}\frac{e^{2\pi i(n_1x_{v_1}+n_2x_{v_2}+n_3x_{v_3}+n_4x_{v_4})}}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|n_3|^{\sigma_3}|n_4|^{\sigma_4}|m_1|^{\mu_1}}.$$

We can see that for the first internal vertex z, we have a constraint condition $n_1+n_2+m_1=0$ *and for the internal vertex* y*, we have* $n_3+n_4-m_1=0$, $n_1+n_2+n_3+n_4=0$*.*

$$Z_{I,\nu}(\Gamma_2,\partial\Gamma_2)=G_{I,\nu,\Gamma_2,\partial\Gamma_2}(\{0\}_{v\in S},1)=\sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}}\frac{1}{n_3^{\sigma_3}(n_1+n_2+n_3)^{\sigma_4}}\frac{1}{n_1^{\sigma_1}n_2^{\sigma_2}(n_1+n_2)^{\mu_1}}$$

Applying Eisenstein's trick 4.2.8 again, we get

$$Z_{I,\nu}(\Gamma_2,\partial\Gamma_2)=\sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}}\frac{1}{n_3^{\sigma_3}(n_1+n_2+n_3)^{\sigma_4}}\sum_{s_1+t_1=\sigma_1+\sigma_2}\left(\frac{C^{\sigma_1-1}_{s_1-1}}{(n_1+n_2)^{s_1+\mu_1}n_2^{t_1}}+\frac{C^{\sigma_2-1}_{s_1-1}}{(n_1+n_2)^{s_1+\mu_1}n_1^{t_1}}\right)$$

$$=\sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}}\sum_{s_1+t_1=\sigma_1+\sigma_2}\left(\underbrace{\frac{C^{\sigma_1-1}_{s_1-1}}{n_3^{\sigma_3}(n_1+n_2+n_3)^{\sigma_4}(n_1+n_2)^{s_1+\mu_1}n_2^{t_1}}}_{①}+\underbrace{\frac{C^{\sigma_2-1}_{s_1-1}}{n_3^{\sigma_3}(n_1+n_2+n_3)^{\sigma_4}(n_1+n_2)^{s_1+\mu_1}n_1^{t_1}}}_{②}\right).$$

Since

$$\frac{1}{n_3^{\sigma_3}(n_1+n_2)^{s_1+\mu_1}} = \sum_{s_2+t_2=\sigma_3+s_1+\mu_1}\left(\frac{C_{s_2-1}^{\sigma_3-1}}{(n_1+n_2+n_3)^{s_2}(n_1+n_2)^{t_2}} + \frac{C_{s_2-1}^{s_1+\mu_1-1}}{(n_1+n_2+n_3)^{s_2}n_3^{t_2}}\right),$$

$$\text{①} = \sum_{s_2+t_2=\sigma_3+s_1+\mu_1}\left(\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_1+n_2)^{t_2}n_2^{t_1}} + \frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{s_1+\mu_1-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}n_3^{t_2}n_2^{t_1}}\right),$$

$$\text{②} = \sum_{s_2+t_2=\sigma_3+s_1+\mu_1}\left(\frac{C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{\sigma_3-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_1+n_2)^{t_2}n_1^{t_1}} + \frac{C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{s_1+\mu_1-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}n_3^{t_2}n_1^{t_1}}\right).$$

Since

$$\frac{1}{n_3^{t_2}n_2^{t_1}} = \sum_{s_3+t_3=t_1+t_2}\left(\frac{C_{s_3-1}^{t_2-1}}{(n_3+n_2)^{s_3}n_2^{t_3}} + \frac{C_{s_3-1}^{t_1-1}}{(n_3+n_2)^{s_3}n_3^{t_3}}\right),$$

$$\frac{1}{n_3^{t_2}n_1^{t_1}} = \sum_{s'_3+t'_3=t_1+t_2}\left(\frac{C_{s'_3-1}^{t_2-1}}{(n_3+n_1)^{s'_3}n_1^{t'_3}} + \frac{C_{s'_3-1}^{t_1-1}}{(n_3+n_1)^{s'_3}n_3^{t'_3}}\right),$$

we can rewrite ① and ② as follows.

$$\text{①} = \sum_{s_2+t_2=\sigma_3+s_1+\mu_1}\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_1+n_2)^{t_2}n_2^{t_1}} + \sum_{\substack{s_2+t_2=\sigma_3+s_1+\mu_1\\ s_3+t_3=t_1+t_2}}\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s_3-1}^{t_2-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_3+n_2)^{s_3}n_2^{t_3}}$$
$$+\sum_{\substack{s_2+t_2=\sigma_3+s_1+\mu_1\\ s_3+t_3=t_1+t_2}}\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s_3-1}^{t_1-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_3+n_2)^{s_3}n_3^{t_3}}$$

$$\text{②} = \sum_{s_2+t_2=\sigma_3+s_1+\mu_1}\frac{C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{\sigma_3-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_1+n_2)^{t_2}n_1^{t_1}} + \sum_{\substack{s_2+t_2=\sigma_3+s_1+\mu_1\\ s'_3+t'_3=t_1+t_2}}\frac{C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s'_3-1}^{t_2-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_3+n_1)^{s'_3}n_1^{t'_3}}$$
$$+\sum_{\substack{s_2+t_2=\sigma_3+s_1+\mu_1\\ s'_3+t'_3=t_1+t_2}}\frac{C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s'_3-1}^{t_1-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_3+n_1)^{s'_3}n_3^{t'_3}}$$

Finally we get:

Formula 4.2.11

$$Z_{I,\nu}(\Gamma_2,\partial\Gamma_2) = \sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1}}\ \sum_{n_1,n_2,n_3\in\mathbb{N}\setminus\{0\}}\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_1+n_2)^{t_2}n_2^{t_1}}$$

$$+\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_3+t_3=t_1+t_2}}\ \sum_{n_1,n_2,n_3\in\mathbb{N}\setminus\{0\}}\left(\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s_3-1}^{t_2-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_3+n_2)^{s_3}n_2^{t_3}} + \frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s_3-1}^{t_1-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_3+n_2)^{s_3}n_3^{t_3}}\right)$$

$$+\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1}}\ \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{\sigma_3-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_1+n_2)^{t_2}n_1^{t_1}}$$

$$+\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s'_3+t'_3=t_1+t_2}}\ \sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \left(\frac{C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s'_3-1}^{t_2-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_3+n_1)^{s'_3}n_1^{t'_3}}+\frac{C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s'_3-1}^{t_1-1}}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_3+n_1)^{s'_3}n_3^{t'_3}}\right).$$

However,

$$\sum_{n_1,n_2,n_3\in\mathbb{N}\smallsetminus\{0\}} \frac{1}{(n_1+n_2+n_3)^{s_2+\sigma_4}(n_1+n_2)^{t_2}n_2^{t_1}} = \sum_{0<k_1<k_2<k_3} \frac{1}{k_3^{s_2+\sigma_4}k_2^{t_2}k_1^{t_1}} = \zeta(t_1,t_2,s_2+\sigma_4),$$

where $k_1=n_1, k_2=n_1+n_2, k_3=n_1+n_2+n_3$. *Therefore*

$$Z_{I,\nu}(\Gamma_2,\partial\Gamma_2) = \sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1}} \left(C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}+C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{\sigma_3-1}\right)\zeta(t_1,t_2,s_2+\sigma_4)$$

$$+\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_3+t_3=t_1+t_2}} \left(C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s_3-1}^{t_2-1}+C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s_3-1}^{t_1-1}\right)\zeta(t_3,s_3,s_2+\sigma_4)$$

$$+\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s'_3+t'_3=t_1+t_2}} \left(C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s'_3-1}^{t_2-1}+C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s'_3-1}^{t_1-1}\right)\zeta(t'_3,s'_3,s_2+\sigma_4).$$

We have expressed our $Z_{I,\nu}(\Gamma_2,\partial\Gamma_2)$ *as a* $\mathbb{Z}$*-linear combination of triple zeta values* $\zeta(t_1,t_2,s_2+\sigma_4)$, *whose weight is* $t_1+t_2+s_2+\sigma_4=\sigma_1+\sigma_2+\sigma_3+\mu_1$, *and* $\zeta(t_3,s_3,s_2+\sigma_4)$ *and* $\zeta(t'_3,s'_3,s_2+\sigma_4)$, *whose weights are also* $\sigma_1+\sigma_2+\sigma_3+\mu_1$.

Example 4.2.12 *Let* Γ_3 *be the diagram shown in Figure 4.11.*

The rank of Γ_3 *is* $rank(H_1(\Gamma_3,\partial\Gamma_3))=4$, *then the rank of* Γ_3 *is* 4. *For each external edge* e_i $(1\le i\le 4)$, *the given sign* ν_i *equals* 0, *therefore* $\nu_5=1$ *by the constraint* $n_1+\cdots+n_5=0$.

$$G_{I,\nu,\Gamma_3,\partial\Gamma_3}(\{x_v\}_{v\in\partial\Gamma_3})$$

$$=\int\sum_{n_i\in\mathbb{N}\smallsetminus\{0\}}\frac{e^{2\pi i((x_{v_4}-y_3)n_4+(x_{v_5}-y_3)n_5)}}{|n_4|^{\sigma_4}|n_5|^{\sigma_5}}\frac{e^{2\pi i((x_{v_3}-y_2)n_3+(y_3-y_2)m_2)}}{|n_3|^{\sigma_3}|m_2|^{\mu_2}}\frac{e^{2\pi i((x_{v_1}-y_1)n_1+(x_{v_2}-y_1)n_2+(y_2-y_1)m_1)}}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|m_1|^{\mu_1}}\underline{dx},$$

where $\underline{dx}=dx_1dx_2dx_3$. *By applying the formal Fourier convolution,*

$$Z_{I,\nu}(\Gamma_3,\partial\Gamma_3)=G_{I,\nu,\Gamma_3,\partial\Gamma_3}(\{0\}_{v\in\partial\Gamma_3},1)=\sum_{\substack{n_4+n_5-m_2=0\\ n_3+m_2-m_1=0\\ n_1+n_2+m_1=0\\ n_1,n_2,n_3,n_4\in\mathbb{N}\smallsetminus\{0\}}}\frac{1}{|n_4|^{\sigma_4}|n_5|^{\sigma_5}}\frac{1}{|n_3|^{\sigma_3}|m_2|^{\mu_2}}\frac{1}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|m_1|^{\mu_1}}$$

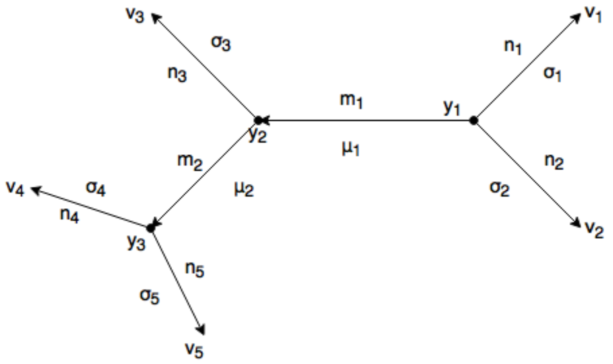


Figure 4.11: A plane trivalent tree with three internal vertices

$$= \sum_{\substack{n_1+n_2+n_3+n_4+n_5=0\\ n_1+n_2+n_3+m_2=0\\ n_1+n_2+m_1=0\\ n_1,n_2,n_3,n_4\in\mathbb{N}\smallsetminus\{0\}}} \frac{1}{|n_4|^{\sigma_4}|n_5|^{\sigma_5}} \frac{1}{|n_3|^{\sigma_3}|m_2|^{\mu_2}} \frac{1}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|m_1|^{\mu_1}}$$

$$= \sum_{n_1,n_2,n_3,n_4\in\mathbb{N}\smallsetminus\{0\}} \frac{1}{n_4^{\sigma_4}(n_1+n_2+n_3+n_4)^{\sigma_5}} \underbrace{\frac{1}{n_3^{\sigma_3}(n_1+n_2+n_3)^{\mu_2}} \frac{1}{n_1^{\sigma_1}n_2^{\sigma_2}(n_1+n_2)^{\mu_1}}}_{③}.$$

We can use the result of ③ in the formula 4.2.11 to get:

$$Z_{I,\nu}(\Gamma_3,\partial\Gamma_3) = \sum_{n_1,n_2,n_3,n_4\in\mathbb{N}\smallsetminus\{0\}} \left(\frac{1}{|n_4|^{\sigma_4}|n_1+n_2+n_3+n_4|^{\sigma_5}}\right)\times$$

$$\left\{ \sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1}} \underbrace{\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}}{(n_1+n_2+n_3)^{s_2+\mu_2}(n_1+n_2)^{t_2}n_2^{t_1}}}_{ⓐ} \right.$$

$$+ \sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_3+t_3=t_1+t_2}} \left(\underbrace{\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s_3-1}^{t_2-1}}{(n_1+n_2+n_3)^{s_2+\mu_2}(n_3+n_2)^{s_3}n_2^{t_3}}}_{ⓑ} + \underbrace{\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{s_1+\mu_1-1}C_{s_3-1}^{t_1-1}}{(n_1+n_2+n_3)^{s_2+\mu_2}(n_3+n_2)^{s_3}n_3^{t_3}}}_{ⓒ} \right)$$

$$+ \sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1}} \underbrace{\frac{C_{s_1-1}^{\sigma_2-1}C_{s_2-1}^{\sigma_3-1}}{(n_1+n_2+n_3)^{s_2+\mu_2}(n_1+n_2)^{t_2}n_1^{t_1}}}_{ⓓ}$$

$$+\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s'_3+t'_3=t_1+t_2}}\left(\underbrace{\frac{C^{\sigma_2-1}_{s_1-1}C^{s_1+\mu_1-1}_{s_2-1}C^{t_2-1}_{s'_3-1}}{(n_1+n_2+n_3)^{s_2+\mu_2}(n_3+n_1)^{s'_3}n_1^{t'_3}}}_{\textcircled{e}}+\underbrace{\frac{C^{\sigma_2-1}_{s_1-1}C^{s_1+\mu_1-1}_{s_2-1}C^{t_1-1}_{s'_3-1}}{(n_1+n_2+n_3)^{s_2+\mu_2}(n_3+n_1)^{s'_3}n_3^{t'_3}}}_{\textcircled{f}}\right)\Bigg\}$$

Remark: By careful observation, we can see that on the factorisation factors (without the exponent) of the denominators of these six terms ⓐ . . . ⓕ *, there is an action of the symmetric group S_3. For example, to pass from the factors of the denominator of* ⓐ *(without considering the power) to that of* ⓒ*, we let $\sigma = \sigma(123)$ acting on $(n_1+n_2+n_3)(n_1+n_2)n_2$ by $(n_{\sigma\cdot 1}+n_{\sigma\cdot 2}+n_{\sigma\cdot 3})(n_{\sigma\cdot 1}+n_{\sigma\cdot 2})n_{\sigma\cdot 2}$, which also explains why there are $6 = |S_3|$ terms.*

Thanks to the remark, in order to reexpress $Z_I(\Gamma_3, \partial\Gamma_3)$, we only need to understand:

$$Sum_1 = \sum_{n_1,n_2,n_3,n_4\in\mathbb{N}\smallsetminus\{0\}}\left(\frac{1}{n_4^{\sigma_4}(n_1+n_2+n_3+n_4)^{\sigma_5}}\right)\times\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1}}\underbrace{\frac{C^{\sigma_1-1}_{s_1-1}C^{\sigma_3-1}_{s_2-1}}{(n_1+n_2+n_3)^{s_2+\mu_2}(n_1+n_2)^{t_2}n_2^{t_1}}}_{\textcircled{a}}. \tag{4.3}$$

Apply formula 4.2.8 again:

$$\frac{1}{n_4^{\sigma_4}(n_1+n_2+n_3)^{s_2+\mu_2}} = \sum_{s_4+t_4=\sigma_4+s_2+\mu_2}\left(\frac{C^{\sigma_4-1}_{s_4-1}}{(n_1+n_2+n_3+n_4)^{s_4}(n_1+n_2+n_3)^{t_4}}+\frac{C^{s_2+\mu_2-1}_{s_4-1}}{(n_1+n_2+n_3+n_4)^{s_4}n_4^{t_4}}\right).$$

Then

$$Sum_1 = \sum_{n_1,n_2,n_3,n_4\in\mathbb{N}\smallsetminus\{0\}}\ \sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_4+t_4=\sigma_4+s_2+\mu_2}}\frac{C^{\sigma_1-1}_{s_1-1}C^{\sigma_3-1}_{s_2-1}C^{\sigma_4-1}_{s_4-1}}{(n_1+n_2+n_3+n_4)^{s_4+\sigma_5}(n_1+n_2+n_3)^{t_4}(n_1+n_2)^{t_2}n_2^{t_1}}$$

$$+\sum_{n_1,n_2,n_3,n_4\in\mathbb{N}\smallsetminus\{0\}}\ \sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_4+t_4=\sigma_4+s_2+\mu_2}}\frac{C^{\sigma_1-1}_{s_1-1}C^{\sigma_3-1}_{s_2-1}C^{s_2+\mu_2-1}_{s_4-1}}{(n_1+n_2+n_3+n_4)^{s_4+\sigma_5}(n_4)^{t_4}(n_1+n_2)^{t_2}n_2^{t_1}}.$$

Again,

$$\frac{1}{(n_4)^{t_4}(n_1+n_2)^{t_2}} = \sum_{s_5+t_5=t_4+t_2}\left(\frac{C^{t_4-1}_{s_5-1}}{(n_1+n_2+n_4)^{s_5}(n_1+n_2)^{t_5}}+\frac{C^{t_2-1}_{s_5-1}}{(n_1+n_2+n_4)^{s_5}n_4^{t_5}}\right).$$

Then

$$\frac{C^{\sigma_1-1}_{s_1-1}C^{\sigma_3-1}_{s_2-1}C^{s_2+\mu_2-1}_{s_4-1}}{(n_1+n_2+n_3+n_4)^{s_4+\sigma_5}(n_4)^{t_4}(n_1+n_2)^{t_2}n_2^{t_1}} =$$

$$\sum_{s_5+t_5=t_4+t_2}\frac{C^{\sigma_1-1}_{s_1-1}C^{\sigma_3-1}_{s_2-1}C^{s_2+\mu_2-1}_{s_4-1}C^{t_4-1}_{s_5-1}}{(n_1+n_2+n_3+n_4)^{s_4+\sigma_5}(n_1+n_2+n_4)^{s_5}(n_1+n_2)^{t_5}n_2^{t_1}}$$

$$+\sum_{s_5+t_5=t_4+t_2}\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}C_{s_4-1}^{s_2+\mu_2-1}C_{s_5-1}^{t_2-1}}{(n_1+n_2+n_3+n_4)^{s_4+\sigma_5}(n_1+n_2+n_4)^{s_5}n_4^{t_5}n_2^{t_1}}.$$

Once again

$$\frac{1}{n_4^{t_5}n_2^{t_1}}=\sum_{s_6+t_6=t_5+t_1}\left(\frac{C_{s_6-1}^{t_5-1}}{(n_4+n_2)^{s_6}n_2^{t_6}}+\frac{C_{s_6-1}^{t_1-1}}{(n_4+n_2)^{s_6}n_4^{t_6}}\right)$$

Finally, we get

$$Sum_1=\sum_{n_1,n_2,n_3,n_4\in\mathbb{N}\setminus\{0\}}\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_4+t_4=\sigma_4+s_2+\mu_2}}\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}C_{s_4-1}^{\sigma_4-1}}{(n_1+n_2+n_3+n_4)^{s_4+\sigma_5}(n_1+n_2+n_3)^{t_4}(n_1+n_2)^{t_2}n_2^{t_1}}$$

$$+\sum_{n_1,n_2,n_3,n_4\in\mathbb{N}\setminus\{0\}}\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_4+t_4=\sigma_4+s_2+\mu_2\\ s_5+t_5=t_4+t_2}}\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}C_{s_4-1}^{s_2+\mu_2-1}C_{s_5-1}^{t_4-1}}{(n_1+n_2+n_3+n_4)^{s_4+\sigma_5}(n_1+n_2+n_4)^{s_5}(n_1+n_2)^{t_5}n_2^{t_1}}$$

$$+\sum_{n_1,n_2,n_3,n_4\in\mathbb{N}\setminus\{0\}}\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_4+t_4=\sigma_4+s_2+\mu_2\\ s_5+t_5=t_4+t_2\\ s_6+t_6=t_5+t_1}}\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}C_{s_4-1}^{s_2+\mu_2-1}C_{s_5-1}^{t_2-1}C_{s_6}^{t_5-1}}{(n_1+n_2+n_3+n_4)^{s_4+\sigma_5}(n_1+n_2+n_4)^{s_5}(n_4+n_2)^{s_6}n_2^{t_6}}$$

$$+\sum_{n_1,n_2,n_3,n_4\in\mathbb{N}\setminus\{0\}}\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_4+t_4=\sigma_4+s_2+\mu_2\\ s_5+t_5=t_4+t_2\\ s_6+t_6=t_5+t_1}}\frac{C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}C_{s_4-1}^{s_2+\mu_2-1}C_{s_5-1}^{t_2-1}C_{s_6}^{t_1-1}}{(n_1+n_2+n_3+n_4)^{s_4+\sigma_5}(n_1+n_2+n_4)^{s_5}(n_4+n_2)^{s_6}n_4^{t_6}}$$

By exchanging the sum symbols, we get

Formula 4.2.13

$$Sum_1=\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_4+t_4=\sigma_4+s_2+\mu_2}}C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}C_{s_4-1}^{\sigma_4-1}\zeta(t_1,t_2,t_4,s_4+\sigma_5)$$

$$+\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_4+t_4=\sigma_4+s_2+\mu_2\\ s_5+t_5=t_4+t_2}}C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}C_{s_4-1}^{s_2+\mu_2-1}C_{s_5-1}^{t_4-1}\zeta(t_1,t_5,s_5,s_4+\sigma_5)$$

$$+\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_4+t_4=\sigma_4+s_2+\mu_2\\ s_5+t_5=t_4+t_2\\ s_6+t_6=t_5+t_1}}C_{s_1-1}^{\sigma_1-1}C_{s_2-1}^{\sigma_3-1}C_{s_4-1}^{s_2+\mu_2-1}C_{s_5-1}^{t_2-1}C_{s_6}^{t_5-1}\zeta(t_6,s_6,s_5,s_4+\sigma_5)$$

$$+\sum_{\substack{s_1+t_1=\sigma_1+\sigma_2\\ s_2+t_2=\sigma_3+s_1+\mu_1\\ s_4+t_4=\sigma_4+s_2+\mu_2\\ s_5+t_5=t_4+t_2\\ s_6+t_6=t_5+t_1}} C^{\sigma_1-1}_{s_1-1}C^{\sigma_3-1}_{s_2-1}C^{s_2+\mu_2-1}_{s_4-1}C^{t_2-1}_{s_5-1}C^{t_1-1}_{s_6}\zeta(t_6,s_6,s_5,s_4+\sigma_5),$$

where

$$t_1+t_2+t_4+s_4+\sigma_5 = t_1+t_5+s_5+s_4+\sigma_5 = t_6+s_6+s_5+s_4+\sigma_5 = \sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\mu_1+\mu_2.$$

In conclusion, since $Z_{I,\nu}(\Gamma_3,\partial\Gamma_3)$ is just the summation of the 6 sums as Sum_1, therefore $Z_{I,\nu}(\Gamma_3,\partial\Gamma_3)$ is indeed a $\mathbb{Z}$-linear combination of quadruple zeta values of weight $\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\mu_1+\mu_2$.

Remark 4.2.14 *We should mention that Example (4.2.7) and Example (4.2.9) are typical examples of Mordell-Tornheim zeta values ([BZ2010]), which are defined as*

$$T(s_1,\ldots,s_r;s) := \sum_{m_1=1}^{\infty}\ldots\sum_{m_r=1}^{\infty}\frac{1}{m_1^{s_1}\ldots m_r^{s_r}(m_1+\ldots+m_r)^s},$$

where $s_1,\ldots,s_r$ and s are complex numbers with $s_1+\ldots+s_r+s=w$ and r is the depth and w is the weight.

For given subdivision map and sign respectively for Γ_1 and Γ_1', it is not difficult to see that

$$Z_{\mathbb{Z},\nu}(\Gamma_1,\partial\Gamma_1) = T(\sigma_1,\sigma_2;\sigma_3);\quad Z_{\mathbb{Z},\nu}(\Gamma_1',\partial\Gamma_1') = T(\sigma_1,\sigma_2,\sigma_3;\sigma_4).$$

We should point out that Bradley and Zhou ([BZ2010]) have proven that any Mordell-Tornheim sum with positive integer arguments can be expressed as a rational linear combination of multiple zeta values of the same weight and depth. In fact given a plan tree T with only one internal vertex and m edges, given a subdivision map $\underline{k}=(\sigma_i)_{1\le i\le m}$ and an appropriate sign ν, we will always have

$$Z_{\mathbb{Z},\nu}(\mathrm{T},\partial\mathrm{T}) = T(\sigma_1,\sigma_2,\ldots,\sigma_{m-1};\sigma_m).$$

In this sense, Mordell-Tornheim zeta values are special case of our generalized multiple zeta values associated to special graphs. Besides, we give details of proof for Example (4.2.7) and Example (4.2.9) and keep a uniform way of demonstration as also shown in Example (4.2.10) and Example (4.2.12), in order to make readers pay attention to the appearance of a permutation group on the index.

However when the given graph has more internal vertices, our generalized multiple zeta value is no longer a Mordell-Tornheim zeta value, neither a generalized Witten zeta value or their generalization- generalized zeta value associated to root systems, defined and studied by Komori, Matsumoto, and Tsumura. For example, the generalized zeta value associated to $sl(l+1)$ ([MT06]) is

$$\zeta_{sl(l+1)}(\underline{s}) = \sum_{m_1,\ldots,m_r=1}^{\infty}\prod_{j=1}^{l}\prod_{k=1}^{l-j+1}\left(\sum_{t=k}^{j+k-1}m_t\right)^{s_{jk}}.$$

$$\zeta_{sl(4)}(s_1,s_2,s_3,s_4,s_5,s_6) = \sum_{m_1,m_2,m_3=1}^{\infty}\frac{1}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(m_1+m_2+m_3)^{s_6}}.$$

They define (section 5 of ([MT06]))

$$\mathcal{T}(s_1, s_2, s_3, s_4, s_5) = \zeta_{sl(4)}(s_1, s_2, s_3, s_4, 0, s_5).$$

Then give some evaluation of $\mathcal{T}(s_1, s_2, s_3, s_4, s_5)$ for special $s_1, \ldots, s_5$, which are indeed a linear combination of 3-depth MZVs. Example (4.2.10) shows that

$$Z_{I,\nu}(\Gamma_2, \partial\Gamma_2) = G_{I,\nu,\Gamma_2,\partial\Gamma_2}(\{0\}_{v\in S}, 1) = \sum_{n_1,n_2,n_3\in\mathbb{N}\setminus\{0\}} \frac{1}{n_3^{\sigma_3}(n_1+n_2+n_3)^{\sigma_4}} \frac{1}{n_1^{\sigma_1} n_2^{\sigma_2}(n_1+n_2)^{\mu_1}}$$

is a finite $\mathbb{Z}$-linear combination of triple zeta values for general σ_i and μ_1. And

$$Z_{I,\nu}(\Gamma_2, \partial\Gamma_2) = \mathcal{T}(\sigma_1, \sigma_2, \sigma_3, \mu_1, \sigma_4).$$

Moreover when the given graph is no longer a plane trivalent tree, our generalized multiple zeta value is not contained in work of Komori, Matsumoto, and Tsumura. Besides our higher plectic Green functions can be related to multiple polylogarithms (see more details in next section). Therefore our construction is quite new.

4.2.2 Proof of the theorem

Proof of Theorem 4.2.1

Proof 4.2.2.1 ***(I). The graph Γ is a plane trivalent tree.***
Inspired by the previous examples, we will prove the theorem 4.2.1 by induction on the number of internal vertices of a given tree. For simplicity, we will first consider only plane trivalent trees. Later we will prove this theorem for any tree.

Step 1 *: Let Γ be a given plane trivalent tree with N internal vertices w_j $(1 \leq j \leq N)$, $N+2$ external vertices v_i $(1 \leq i \leq N+2)$, $N-1$ internal edges and $N+2$ external edges. The subdivision map k is given by $k_{e_i} = \sigma_{e_i}$ $(\sigma_{e_i} \geq 1)$ if e_i $(1 \leq i \leq N+2)$ is an external edge with endpoint v_i, and $k_{f_j} = \mu_{f_j}$ $(\mu_{f_j} \geq 1)$ if f_j $(1 \leq j \leq N-1)$ is an internal edge. Recall that for each edge, we give a sign $\nu_e \in \{0, 1\}$.*

Remark: In fact, we can see that the orientation of each edge has no importance by changing the sign for each edge. Moreover, we can also assume that for each external edge e_i with the sign $\nu_{e_i} = 0,\ 1 \leq i \leq N+1$ (we shall see that this forces $\nu_{e_{N+2}} = 1$). It is easy to see that we will lose no generality.

By the results of the above examples, we know that Therem 4.2.1 holds when $N = 1, 2, 3$. Now our inductive hypothesis is that if $N = n$ $(n \geq 1)$, the theorem is true. Moreover, we assume that:

Let Γ be a plane trivalent tree with N $(N \leq n)$ internal vertices and rank $d = rank(H_1(\Gamma, \partial\Gamma))$. Then we have a relation $d = N+1$. The generalized multiple zeta value associated $Z_{I,\nu}(\Gamma, \partial\Gamma)$ can be written as follows

$$Z_{I,\nu}(\Gamma, \partial\Gamma) = \sum_{n_1,\ldots,n_{N+1}\in\mathbb{N}\setminus\{0\}} \sum_{\gamma\in S_d} \sum_{\substack{t_i^\gamma \\ 1\leq i\leq d}} \frac{C_{\gamma,t_i^\gamma}}{n_{\gamma\cdot 1}^{t_1^\gamma}(n_{\gamma\cdot 1}+n_{\gamma\cdot 2})^{t_2^\gamma}\cdots(n_{\gamma\cdot 1}+\ldots+n_{\gamma\cdot d})^{t_d^\gamma}},$$

which implies that

$$Z_{I,\nu}(\Gamma, \partial\Gamma) = \sum_{\gamma\in S_d} \sum_{\substack{t_i^\gamma \\ 1\leq i\leq d}} C_{\gamma,t_i^\gamma}\zeta(t_1^\gamma, \cdots, t_d^\gamma),$$

where S_d is the symmetric group, $C_{\gamma,t_i^\gamma} \in \mathbb{Z}$ is a constant depending on γ and t_i^γ. The upper mute symbol γ of t_j^γ implies the dependence of γ, and

$$t_1^\gamma + \ldots + t_d^\gamma = \sum_e \sigma_e + \sum_f \mu_f, \quad \forall \gamma \in S_d,$$

and

$$t_i^\gamma \geq 1, \quad \forall \gamma \in S_d, \quad 1 \leq i \leq d.$$

Therefore the first sum

$$\sum_{\substack{t_i^\gamma \in \mathbb{N} \setminus \{0\} \\ 1 \leq i \leq d}}$$

is a finite sum.

For convenience, we will define a new quantity

$$\mathcal{O}_{I,\nu}(\Gamma, \partial\Gamma, (n_i)_i, (m_j)_j) = \prod_{1 \leq i \leq N+2} \frac{1}{|n_i|^{\sigma_i}} \prod_{1 \leq j \leq N-1} \frac{1}{|m_j|^{\mu_j}}.$$

In fact for each internal vertex we have $\pi_v = 0$, hence each m_j can be written as a linear combination of the n_i, thus

$$\mathcal{O}_{I,\nu}(\Gamma, \partial\Gamma, (n_i)_i, (m_j)_j) = \mathcal{O}_{I,\nu}(\Gamma, \partial\Gamma, (n_i)_i).$$

If there is no ambiguity, we will for simplicity write:

$$\mathcal{O}_{I,\nu}(\Gamma, \partial\Gamma) = \mathcal{O}_{I,\nu}(\Gamma, \partial\Gamma, (n_i)_i).$$

The fact that Theorem 4.2.1 holds means that we have a new expression for $\mathcal{O}_{I,\nu}(\Gamma, \partial\Gamma)$, namely

$$\mathcal{O}_{I,\nu}(\Gamma, \partial\Gamma) = \sum_{\gamma \in S_d} \sum_{\substack{t_i^\gamma \\ 1 \leq i \leq d}} \frac{C_{\gamma,t_i^\gamma}}{n_{\gamma\cdot 1}^{t_1^\gamma} (n_{\gamma\cdot 1} + n_{\gamma\cdot 2})^{t_2^\gamma} \cdots (n_{\gamma\cdot 1} + \ldots + n_{\gamma\cdot d})^{t_d^\gamma}},$$

$$t_1^\gamma + \ldots + t_d^\gamma = \sum_e \sigma_e + \sum_f \mu_f, \quad \forall \gamma \in S_d,$$

then

$$Z_{I,\nu}(\Gamma, \partial\Gamma) = \sum_{\substack{n_i \in \mathbb{N} \setminus \{0\}, \\ 1 \leq i \leq d}} \mathcal{O}_{I,\nu}(\Gamma, \partial\Gamma).$$

***Step 2**: Now we will prove **the case of** $N = n + 1$.*

For a plane trivalent tree, if the number of internal vertices is increased by 1, then the rank of the tree is increased by 1, too.

Now we give a clockwise order for all external vertices v_i. We will also give an order for all internal vertices, such that the internal vertex w_N, decorated by the variable x_N, is connected with the two external vertices v_{N+1} and v_{N+2} by external edges $\overrightarrow{e_{N+1}} = (w_N \longrightarrow v_{N+1})$ and $\overrightarrow{e_{N+2}} = (w_N \longrightarrow v_{N+2})$.

In order to deduce the case $N = n + 1$ from the case $N - 1 = n$, we do an operation: we cut down the internal edge f_{N-1}, one of whose ends is the internal vertex w_N and associate

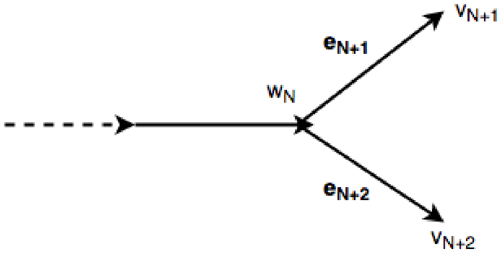


Figure 4.12: the internal vertex w_N is connected with the two external vertices v_{N+1} and v_{N+2}.

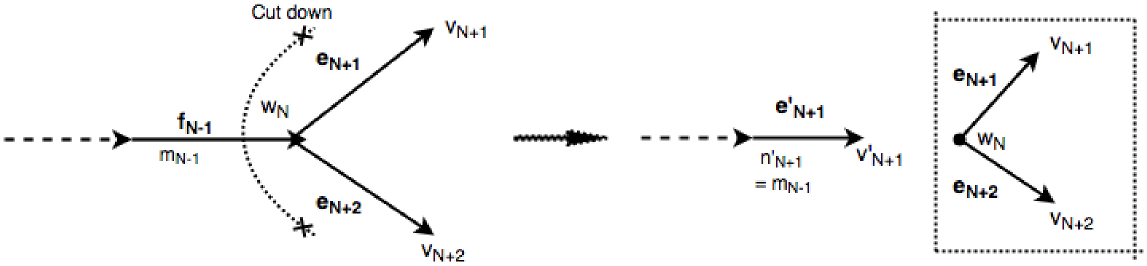


Figure 4.13: cutting down the internal edge f_{N-1} to get a new tree Γ'.

a new external vertex denoted as v'_{N+1} and denote the new external edge as e'_{N+1} to which we associate $n_{e'_{N+1}} = m_{N-1}$ and the subdivision μ_{N-1}, then we build a new plane trivalent tree Γ' with $N - 1 = n$ internal vertices and whose rank is $d - 1$, where d is the rank of Γ.

In the definition of

$$G_{I,\Gamma,S}(\{x_v\}_{v\in S}, 1) = \int_{(F_{\mathbb{R}}/I)^{V(\Gamma)\smallsetminus S}} \prod_{e\in E(\Gamma)} g_I(x_{v_0(e)} - x_{v_1(e)}, 1) \prod_{v\in V(\Gamma)\smallsetminus S} dx_v,$$

where

$$g_I(x_{v_0(e)} - x_{v_1(e)}, 1) = \lim_{\delta\to 0^+} \sum_{h_e\in I^*\backslash\{0\}} \frac{e^{2\pi i(h_e(x_{v_0(e)} - x_{v_1(e)}))}}{||uh_e||^{r+\delta}},$$

and $F = \mathbb{Q}$ $(r = 1)$, $I^* = \mathbb{Z}$. We denote h_e by n_e if e is external and denote h_e by m_e if e is internal. And

$$Z_{I,\nu}(\Gamma, \partial\Gamma) = G_{I,\nu,\Gamma,\partial\Gamma}(\{0\}_{v\in\partial\Gamma}, 1).$$

Then by formal Fourier convolution, we have

$$G_{I,\nu,\Gamma,\partial\Gamma}(\{0\}_{v\in\partial\Gamma}, 1) = \sum_{\substack{n_i\in\mathbb{Z}\smallsetminus\{0\}\\ \mathrm{sgn}(n_i)=(-1)^{\nu_i}, 1\le i\le d}} \frac{1}{|n_{N+1}|^{\sigma_{N+1}}|n_{N+2}|^{\sigma_{N+2}}|m_{N-1}|^{\mu_{N-1}}} \prod_{1\le i\le N} \frac{1}{|n_i|^{\sigma_i}} \prod_{1\le j\le N-2} \frac{1}{|m_j|^{\mu_j}},$$

where $d = N + 1$ and on each internal vertex, we have a constraint $\pi_v = 0$ as in (4.2).

In fact, it is not difficult to see that

$$\frac{1}{|m_{N-1}|^{\mu_{N-1}}} \prod_{1\le i\le N} \frac{1}{|n_i|^{\sigma_i}} \prod_{1\le j\le N-2} \frac{1}{|m_j|^{\mu_j}} = \mathscr{O}_{I,\nu}(\Gamma', \partial\Gamma'),$$

and

$$Z_{I,\nu}(\Gamma', \partial\Gamma') = \sum_{n_i\in\mathbb{N}\smallsetminus\{0\}, 1\le i\le N} \mathscr{O}_{I,\nu}(\Gamma', \partial\Gamma').$$

Then

$$G_{I,\nu,\Gamma,\partial\Gamma}(\{0\}_{v\in\partial\Gamma},1)=\sum_{n_i\in\mathbb{N}\smallsetminus\{0\},1\le i\le N+1}\frac{1}{|n_{N+1}|^{\sigma_{N+1}}|n_{N+2}|^{\sigma_{N+2}}}\cdot\mathscr{O}_{I,\nu}(\Gamma',\partial\Gamma').$$

Since the number of the internal vertices of Γ' is n, then the theorem for Γ' holds by the inductive hypothesis.

Hence we have the following equality:

$$\mathscr{O}_{I,\nu}(\Gamma',\partial\Gamma')=\sum_{\gamma\in S_{d-1}}\sum_{t_1^\gamma,\dots,t_{d-1}^\gamma}\frac{\tilde{C}_{\gamma,t_i^\gamma}}{n_{\gamma\cdot 1}^{t_1^\gamma}(n_{\gamma\cdot 1}+n_{\gamma\cdot 2})^{t_2^\gamma}\cdots(n_{\gamma\cdot 1}+\ldots+n_{\gamma\cdot(d-1)})^{t_{d-1}^\gamma}},$$

where

$$t_1^\gamma+\ldots+t_{d-1}^\gamma=\sum_{1\le i\le N}\sigma_i+\sum_{1\le j\le N-1}\mu_j,\qquad \forall\gamma\in S_{d-1},(d=N+1).$$

Now we need to calculate

$$P_\gamma=\frac{1}{|n_{N+1}|^{\sigma_{N+1}}|n_{N+2}|^{\sigma_{N+2}}}\times\frac{\tilde{C}_{\gamma,t_i^\gamma}}{n_{\gamma\cdot 1}^{t_1^\gamma}(n_{\gamma\cdot 1}+n_{\gamma\cdot 2})^{t_2^\gamma}\cdots(n_{\gamma\cdot 1}+\ldots+n_{\gamma\cdot(d-1)})^{t_{d-1}^\gamma}},$$

from which we will deduce

$$Z_{I,\nu}(\Gamma,\partial\Gamma)=\sum_{\substack{n_i\in\mathbb{N}\smallsetminus\{0\},\\ 1\le i\le N+1}}\sum_{\gamma\in S_{d-1}}P_\gamma.$$

Recall: the rank d of a plane trivalent tree with N internal vertices equals $N+1$. Then

$$Z_{I,\nu}(\Gamma,\partial\Gamma)=\sum_{\substack{n_i\in\mathbb{N}\smallsetminus\{0\},\\ 1\le i\le N+1}}\sum_{\gamma\in S_N}P_\gamma.$$

Step 3: Calculation of P_γ.

For simplicity, we assume that $\gamma=1$. In fact, this assumption will be no loss of generality.

$$P_1=\frac{1}{n_{N+1}^{\sigma_{N+1}}|n_{N+2}|^{\sigma_{N+2}}}\times\frac{\tilde{C}_{1,t_i^1}}{n_1^{t_1^1}(n_1+n_2)^{t_2^1}\cdots(n_1+\ldots+n_N)^{t_N^1}}.$$

For any plane tree Γ, we have

$$n_1+n_2+\ldots+n_{N+1}+n_{N+2}=0;\quad n_i>0;\quad 1\le i\le N+1.$$

Then

$$P_1=\frac{1}{n_{N+1}^{\sigma_{N+1}}(n_1+\ldots+n_{N+1})^{\sigma_{N+2}}}\times\frac{\tilde{C}_{1,t_i^1}}{n_1^{t_1^1}(n_1+n_2)^{t_2^1}\cdots(n_1+n_2+\ldots+n_N)^{t_N^1}}.$$

Now for the final result, we will apply N times the Eisenstein trick 4.2.8.

1st time:

$$\frac{1}{n_{N+1}^{\sigma_{N+1}}(n_1+n_2+\ldots+n_N)^{t_N^1}}=\sum_{r_1+s_1=\sigma_{N+1}+t_N^1}\frac{C_{r_1-1}^{\sigma_{N+1}-1}}{(n_1+n_2+\ldots+n_{N+1})^{r_1}(n_1+n_2+\ldots+n_N)^{s_1}}$$

$$+\sum_{r_1+s_1=\sigma_{N+1}+t_N^1} \frac{C_{r_1-1}^{t_N^1-1}}{(n_1+n_2+\ldots+n_{N+1})^{r_1} n_{N+1}^{s_1}}.$$

Then

$$P_1=\underbrace{\sum_{r_1+s_1=\sigma_{N+1}+t_N^1} \frac{C_{r_1-1}^{\sigma_{N+1}-1}\tilde{C}_{1,t_i^1}}{n_1^{t_1^1}(n_1+n_2)^{t_2^1}\cdots(n_1+n_2+\ldots+n_N)^{s_1}(n_1+n_2+\ldots+n_{N+1})^{r_1+\sigma_{N+2}}}}_{a_1^1}$$

$$+\underbrace{\sum_{r_1+s_1=\sigma_{N+1}+t_N^1} \frac{C_{r_1-1}^{t_N^1-1}\tilde{C}_{1,t_i^1}}{n_1^{t_1^1}n_{N+1}^{s_1}(n_1+n_2)^{t_2^1}\cdots(n_1+n_2+\ldots+n_{N-1})^{t_{N-1}^1}(n_1+n_2+\ldots+n_{N+1})^{r_1+\sigma_{N+2}}}}_{b_1^1},$$

where the lower index 1 *for* a_1^1 *refers to* P_1 *and the upper index* 1 *indicates the 1st time use of Eisenstein's trick.*

Denote $C_{1,t_i^1}=C_{r_1-1}^{\sigma_{N+1}-1}\tilde{C}_{1,t_i^1}$. *Now will see that*

$$\sum_{\substack{n_i\in\mathbb{N}\setminus\{0\}\\ 1\le i\le N+1}} a_1^1=\sum_{r_1+s_1=\sigma_{N+1}+t_N^1}\ \sum_{\substack{n_i\in\mathbb{N}\setminus\{0\}\\ 1\le i\le N+1}} \frac{C_{1,t_i^1}}{n_1^{t_1^1}(n_1+n_2)^{t_2^1}\cdots(n_1+n_2+\ldots+n_N)^{s_1}(n_1+n_2+\ldots+n_{N+1})^{r_1+\sigma_{N+2}}}$$

$$=\sum_{r_1+s_1=\sigma_{N+1}+t_N^1} C_{1,t_i^1}\zeta(t_1^1,\cdots,t_N^1,s_1,r_1+\sigma_{N+2}),$$

where

$$t_1^1+\cdots+t_N^1+s_1+r_1+\sigma_{N+2}=\sum_{1\le i\le N+2}\sigma_i+\sum_{1\le j\le N-1}\mu_j,$$

since

$$t_1^\gamma+\ldots+t_{d-1}^\gamma=\sum_{1\le i\le N}\sigma_i+\sum_{1\le j\le N-1}\mu_j,\qquad \forall\gamma\in S_{d-1},\quad (d=N+1).$$

$\zeta(t_1^1,\cdots,t_N^1,s_1,r_1+\sigma_{N+2})$ *is a* $(N+1)$*-tuple zeta value of weight* $\sum_{1\le i\le N+2}\sigma_i+\sum_{1\le j\le N-1}\mu_j$.

Let us continue this procedure. We apply for the second time Eisenstein's trick for b_1^1 *by considering*

$$\frac{1}{n_{N+1}^{s_1}(n_1+n_2+\ldots+n_{N-1})^{t_{N-1}^1}}=\sum_{r_2+s_2=s_1+t_{N-1}^1}\frac{C_{r_2-1}^{s_1-1}}{(n_1+n_2+\ldots+n_{N-1}+n_{N+1})^{r_2}(n_1+n_2+\ldots+n_{N-1})^{s_2}}$$

$$+\sum_{r_2+s_2=s_1+t_{N-1}^1}\frac{C_{r_2-1}^{t_{N-1}^1-1}}{(n_1+n_2+\ldots+n_{N-1}+n_{N+1})^{r_2}n_{N+1}^{s_2}}.$$

Then

$$b_1^1=\underbrace{\sum_{\substack{r_1+s_1=\sigma_{N+1}+t_N^1\\ r_2+s_2=s_1+t_{N-1}^1}}\frac{C_{r_2-1}^{s_1-1}C_{r_1-1}^{t_N^1-1}\tilde{C}_{1,t_i^1}}{n_1^{t_1^1}\cdots(n_1+\ldots+n_{N-1})^{s_2}(n_1+\ldots+n_{N-1}+n_{N+1})^{r_2}(n_1+\ldots+n_N+n_{N+1})^{r_1+\sigma_{N+2}}}}_{a_1^2}$$

$$+\underbrace{\sum_{\substack{r_1+s_1=\sigma_{N+1}+t_N^1\\ r_2+s_2=s_1+t_{N-1}^1}} \frac{C_{r_2-1}^{t_{N-1}^1-1} C_{r_1-1}^{t_N^1-1} \tilde{C}_{1,t_i^1}}{n_1^{t_1^1} n_{N+1}^{s_2} (n_1+n_2)^{t_2^1} \cdots (n_1+\ldots+n_{N-2})^{t_{N-2}^1} (n_1+\ldots+n_{N-1}+n_{N+1})^{r_2} (n_1+\ldots+n_{N+1})^{r_1+\sigma_{N+2}}}}_{b_1^2},$$

We denote $C_{\sigma(N,N+1)} = C_{r_2-1}^{s_1-1} C_{r_1-1}^{t_N^1-1} \tilde{C}_{1,t_i^1}$. *Then*

$$\sum_{\substack{n_i\in\mathbb{N}\smallsetminus\{0\}\\ 1\le i\le N+1}} a_1^2 = \sum_{\substack{r_1+s_1=\sigma_{N+1}+t_N^1\\ r_2+s_2=s_1+t_{N-1}^1\\ n_i\in\mathbb{N}\smallsetminus\{0\}\\ 1\le i\le N+1\\ g=\sigma(N,N+1)}} \frac{C_{\sigma(N,N+1)}}{n_1^{t_1^1} \cdots (n_1+\ldots+n_{N-1})^{s_2} (n_1+\ldots+n_{N-1}+n_{N+1})^{r_2} (n_1+\ldots+n_{N+1})^{r_1+\sigma_{N+2}}}$$

$$= \sum_{\substack{r_1+s_1=\sigma_{N+1}+t_N^1\\ r_2+s_2=s_1+t_{N-1}^1}} C_{\sigma(N,N+1)} \zeta(t_1^1, \cdots, t_{N-2}^1, s_2, r_2, r_1+\sigma_{N+2}),$$

where $t_1^1 + \cdots + t_{N-2}^1 + s_2 + r_2 + r_1 + \sigma_{N+2} = \sum_{1\le i\le N+2} \sigma_i + \sum_{1\le j\le N-1} \mu_j$.

We will continue the procedure by considering b_1^2. *In fact, after the* k*-th time use of Eisenstein's trick* ($k \ge 1$)*,*

$$\frac{1}{n_{N+1}^{s_{k-1}} (n_1+n_2+\ldots+n_{N-(K-1)})^{t_{N-(k-1)}^1}} = \sum_{\substack{r_k+s_k\\ =s_{k-1}+t_{N-(k-1)}^1}} \frac{C_{r_k-1}^{s_k-1}}{(n_1+\ldots+n_{N-(k-1)}+n_{N+1})^{r_k} (n_1+\ldots+n_{N-(k-1)})^{s_k}}$$

$$+ \sum_{r_k+s_k=s_{k-1}+t_{N-(k-1)}^1} \frac{C_{r_k-1}^{t_{N-(k-1)}^1-1}}{(n_1+n_2+\ldots+n_{N-(k-1)}+n_{N+1})^{r_k} n_{N+1}^{s_k}},$$

then we will get two sums a_1^k *and* b_1^k*, where* $1 \le k \le N-1$ *and*

$$a_1^k(n_1, \cdots, n_{N+1}) = \sum_{\substack{r_1+s_1=\sigma_{N+1}+t_N^1\\ r_2+s_2=s_1+t_{N-1}^1\\ \vdots\\ r_k+s_k=s_{k-1}+t_{N-(k-1)}^1}} C_{r_k-1}^{s_{k-1}-1} C_{r_{k-1}-1}^{t_{N-(k-2)}^1} \cdots C_{r_1-1}^{t_{N-1}^1} \tilde{C}_{1,t_i^1} \times M,$$

where

$$M = \frac{1}{\left[\prod_{i=1}^{N-k} (\sum_{p=1}^{i} n_p)^{t_i^1}\right] (n_1+\cdots+n_{N-(k-1)})^{s_k}}$$

$$\times \frac{1}{\left[\prod_{j=N-(k-1)}^{N-1} ((\sum_{q=1}^{j} n_q) + n_{N+1})^{r_{N-j+1}}\right] (n_1+\cdots+n_{N+1})^{r_1+\sigma_{N+2}}}.$$

$$b_1^k = \sum_{\substack{r_1+s_1 \\ =\sigma_{N+1}+t_N^1; \\ r_2+s_2 \\ =s_1+t_{N-1}^1; \\ \cdots \\ r_k+s_k \\ =s_{k-1}+t_{N-k+1}^1}} \frac{C_{r_k-1}^{t_{N-k+1}^1} \cdots C_{r_1-1}^{t_N^1-1} \tilde{C}_{1,t_i^1}}{n_{N+1}^{s_k} \prod_{j=1}^{N-k} (n_1 + \cdots + n_j)^{t_j^1} \prod_{j=N-k+1}^{N-1} (n_1 + \cdots n_j + n_{N+1})^{r_{N-j+1}} (n_1 + \cdots + n_{N+1})^{r_1+\sigma_{N+2}}}.$$

Similarly, after the $(k+1)$-th time use of Eisenstein's trick for b_1^k, we obtain

$$b_1^k = a_1^{k+1} + b_1^{k+1}.$$

After N steps, we have $b_1^{N-1} = a_1^N + b_1^N$, where

$$b_1^{N-1} = \sum_{\substack{r_1+s_1=\sigma_{N+1}+t_N^1 \\ r_2+s_2=s_1+t_{N-1}^1 \\ \vdots \\ r_{N-1}+s_{N-1}=s_{N-2}+t_1^1}} \frac{C_{r_{N-1}-1}^{t_2^1} \cdots C_{r_1-1}^{t_N^1-1} \tilde{C}_{1,t_i^1}}{n_1^{t_1^1} n_{N+1}^{s_N} \prod_{j=2}^{N-1} (n_1 + \cdots n_j + n_{N+1})^{r_{N-j+1}} (n_1 + \cdots + n_{N+1})^{r_1+\sigma_{N+2}}},$$

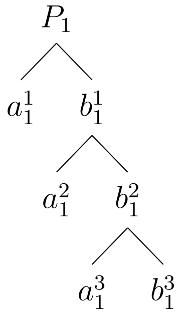
$$a_1^N = \sum_{\substack{r_1+s_1=\sigma_{N+1}+t_N^1 \\ r_2+s_2=s_1+t_{N-1}^1 \\ \vdots \\ r_N+s_N=s_{N-1}+t_1^1}} \frac{C_{r_N-1}^{s_{N-1}-1} C_{r_{N-1}-1}^{t_2^1} \cdots C_{r_1-1}^{t_{N-1}^1} \tilde{C}_{1,t_i^1}}{n_1^{s_N} \left[\prod_{j=1}^{N-1} ((\sum_{q=1}^{j} n_q) + n_{N+1})^{r_{N-j+1}}\right] (n_1 + \cdots + n_{N+1})^{r_1+\sigma_{N+2}}},$$

and

$$b_1^N = \sum_{\substack{r_1+s_1=\sigma_{N+1}+t_N^1 \\ r_2+s_2=s_1+t_{N-1}^1 \\ \vdots \\ r_N+s_N=s_{N-1}+t_1^1}} \frac{C_{r_N-1}^{t_1^1-1} \cdots C_{r_1-1}^{t_N^1-1} \tilde{C}_{1,t_i^1}}{n_{N+1}^{s_{N+1}} (n_1 + n_{N+1})^{r_N+r_{N+1}} \prod_{j=2}^{N-1} (n_1 + \cdots n_j + n_{N+1})^{r_{N-j+1}} (n_1 + \cdots + n_{N+1})^{r_1+\sigma_{N+2}}}.$$

The sums $\sum_{n_i \in \mathbb{N} \smallsetminus \{0\}} a_1^k$ and $\sum_{n_i \in \mathbb{N} \smallsetminus \{0\}} b_1^N$ (for each $1 \le k \le N$) are $\mathbb{Z}$-linear combinations of $(N+1)$-tuple zeta values.

For example, for a plane trivalent tree with one internal vertex, the corresponding rooted tree illustrating the procedure of calculations is as follows.



Finally, we have

$$P_1 = a_1^1 + b_1^1 = a_1^1 + a_1^2 + b_1^2 = \cdots = \left(\sum_{1\leq j\leq k} a_1^j\right) + b_1^k = \left(\sum_{1\leq l\leq N} a_1^l\right) + b_1^N.$$

$$\sum_{n_i\in\mathbb{N}\setminus\{0\}} a^k_{1,(n_1,\cdots,n_{N+1})} = \sum_{\substack{r_1+s_1=\sigma_{N+1}+t^1_N \\ r_2+s_2=s_1+t^1_{N-1} \\ \vdots \\ r_k+s_k=s_{k-1}+t^1_{N-(k-1)}}} C_{\tau_k,t_i^1}\zeta(t_1^1,\cdots,t^1_{N-k},s_k,r_k,\cdots,r_2,r_1+\sigma_{N+2}),$$

where $C_{\tau_k,t_i^1} = C_{r_k-1}^{t^1_{N-(k-1)}}\cdots C_{r_1-1}^{t^1_{N-1}}\tilde{C}_1$, $\tau_1 = Id$ *and* $\tau_k(k\geq 2)$ *is the permutation*

$$\tau_k = \left(\begin{smallmatrix} N-k+2 & N-k+3 & N-k+4 & \cdots & N & N+1 \\ N+1 & N-k+2 & N-k+3 & \cdots & N-1 & N \end{smallmatrix}\right)$$

which signifies that through the action of this permutation on the indices (without considering the powers) of denominators of a_1^1*, we will recover the factors of* a_1^k*.*

$$t_1^1+\cdots+t^1_{N-k}+s_k+r_k+\cdots+r_2+r_1+\sigma_{N+2} = \sum_{1\leq i\leq N+2}\sigma_i + \sum_{1\leq j\leq N-1}\mu_j.$$

It is easy to see that $\zeta(t_1^1,\cdots,t^1_{N-k},s_k,r_k,\cdots,r_2,r_1+\sigma_{N+2})$ *is a* $(N+1)$*-tuple zeta value. Therefore* P_1 *is indeed a* $\mathbb{Z}$*-linear combination of* $(N+1)$*-tuple zeta values of weight* $\sum_{1\leq i\leq N+2}\sigma_i + \sum_{1\leq j\leq N-1}\mu_j$*.*

We can obtain similar results for other P_γ *and get* $a^i_\gamma(1\leq i\leq N+1)$ *with numerator* $C_{\tau_i\cdot\gamma}$*. Therefore we get*

$$Z_{I,\nu}(\Gamma,\partial\Gamma) = \sum_{n_1,\ldots,n_{N+1}\in\mathbb{N}\setminus\{0\}}\ \sum_{1\leq i\leq N+1}\ \sum_{\gamma\in S_{d-1}} a^i_\gamma$$

$$= \sum_{n_1,\ldots,n_{N+1}\in\mathbb{N}\setminus\{0\}}\ \sum_{\alpha\in S_d}\sum_{\tilde{t}_i^\alpha}\frac{C_{\alpha,\tilde{t}_i^\alpha}}{n_{\alpha\cdot1}^{\tilde{t}_1^\alpha}(n_{\gamma\cdot1}+n_{\alpha\cdot2})^{\tilde{t}_2^\alpha}\cdots(n_{\alpha\cdot1}+\ldots+n_{\alpha\cdot d})^{\tilde{t}_d^\alpha}},$$

$$= \sum_{\alpha\in S_d}\sum_{\tilde{t}_i^\alpha} C_{\alpha,\tilde{t}_i^\alpha}\zeta(\tilde{t}_1^\alpha,\cdots,\tilde{t}_d^\alpha),$$

where $\alpha = \tau_i\cdot\gamma$ $(1\leq i\leq N+1$ *and* $\gamma\in S_{d-1})$ *is an element of* S_d*. Note that* τ_i $(1\leq i\leq N+1)$ *and* $\gamma\in S_{d-1}$ *generate the symmetric group* S_d *in the sense that*

$$S_d = S_{N+1} = \coprod_{i=1}^{N+1}\tau_i S_N = \coprod_{i=1}^{N+1}\tau_i S_{d-1}.$$

So we have finished the proof of the case $N = n+1$, $Z_{I,\nu}(\Gamma,\partial\Gamma)$ *is indeed a finite* $\mathbb{Z}$*-linear combination of* $(N+1)$*-tuple zeta values of weight* $\sum_{1\leq i\leq N+2}\sigma_i + \sum_{1\leq j\leq N-1}\mu_j$*.*

(II). Γ ***is an arbitary tree.***

The demonstration is quite similar to the previnous proof for any plane trivalent tree. Let Γ *be an arbitrary plane tree with* N *internal vertices. For each internal vertex* $w_j(1\leq j\leq N)$*, the valency* $val(w_j) = 3+\beta_j$, $\beta_j\geq 0$*. Then* Γ *has* $N-1$ *internal edges,* $N+2+$

$\sum_{1\le j\le N}\beta_j$ external edges and $N+2+\sum_{1\le j\le N}\beta_j$ external vertices v_i. Therefore the rank $d=\mathrm{rank}(H_1(\Gamma,\partial\Gamma))$ is not equal $N+1$ any more, but we have

$$d=N+1+\sum_{1\le j\le N}\beta_j.$$

Again we give a clockwise order to the set of all external vertices v_i. We will also give an order for all internal vertices, such that the internal vertex w_N, decorated by the variable x_N, is connected with the external vertices

$$v_{N+1+\sum_{1\le j\le N-1}\beta_j},\dots,v_{N+2+\sum_{1\le j\le N}\beta_j}$$

by external edges $\overrightarrow{e_{N+1+\sum_{1\le j\le N-1}\beta_j}}=(w_N\longrightarrow v_{N+1+\sum_{1\le j\le N-1}\beta_j})$, ... ,$\overrightarrow{e_{N+2+\sum_{1\le j\le N}\beta_j}}=(w_N\longrightarrow v_{N+2+\sum_{1\le j\le N}\beta_j})$.

***The initial step**: the case $N=1$. Let $\beta=\beta_1$ (≥ 0) be the valency of the internal vertex of Γ'_1. When $\beta=1$, the theorem is correct due to example 4.2.9 for Γ'_1. Now we will prove the case for a general β (≥ 1), where the rank d of the tree is $2+\beta$.*

For each edge $e_i(1\le i\le 3+\beta)$, we add $\sigma_i-1(\sigma_i\ge 1)$ points. The only internal vertex is denoted by x, each external vertex v_i is decorated by x_{v_i}. For each $e_i(i=1,\dots,2+\beta)$, the given sign ν_i equals 0. Then the sign $\nu_{3+\beta}$ is forced to be 1.

$$G_{I,\nu,\Gamma'_1,\partial\Gamma'_1}(\{x_v\}_{v\in\partial\Gamma'_1},1)=\sum_{\substack{n_1+\cdots+n_{3+\beta}=0,n_i\in\mathbb{Z}\smallsetminus\{0\};\\ \mathrm{sgn}(n_i)=(-1)^{\nu_i},1\le i\le 2+\beta}}\frac{e^{2\pi i(n_1x_{v_1}+\cdots+n_{3+\beta}x_{v_{3+\beta}})}}{\prod_{i=1}^{3+\beta}|n_i|^{\sigma_i}},$$

where $\nu_1=\cdots=\nu_{2+\beta}=0$. Then

$$Z_{I,\nu}(\Gamma'_1,\partial\Gamma'_1)=G_{I,\nu,\Gamma'_1,\partial\Gamma'_1}(\{0\}_{v\in\partial\Gamma'_1},1)=\sum_{\substack{n_1+\cdots+n_{3+\beta}=0,n_i\in\mathbb{Z}\smallsetminus\{0\};\\ \mathrm{sgn}(n_i)=(-1)^{\nu_i},1\le i\le 2+\beta}}\frac{1}{\prod_{i=1}^{3+\beta}|n_i|^{\sigma_i}}$$

$$Z_{I,\nu}(\Gamma'_1,\partial\Gamma'_1)=\sum_{\substack{n_i\in\mathbb{N}\smallsetminus\{0\}\\ 1\le i\le 2+\beta}}\frac{1}{\prod_{i=1}^{2+\beta}n_i^{\sigma_i}(n_1+\cdots+n_{2+\beta})^{\sigma_{3+\beta}}}.$$

We define

$$Q=\frac{1}{\prod_{i=1}^{2+\beta}n_i^{\sigma_i}(n_1+\cdots+n_{2+\beta})^{\sigma_{3+\beta}}}.$$

Then

$$Z_{I,\nu}(\Gamma'_1,\partial\Gamma'_1)=\sum_{\substack{n_i\in\mathbb{N}\smallsetminus\{0\}\\ 1\le i\le 2+\beta}}Q.$$

Definition 4.2.15 *If $0\le k\le\beta$,*

$$Q^k\left(\begin{smallmatrix} n_1, & \cdots, & n_{k+1}; & n_{k+2}, & \cdots, & n_{2+\beta}\\ {}_kt_1, & \cdots, & {}_kt_{k+1}; & \sigma_{k+2}, & \cdots, & \sigma_{2+\beta}\end{smallmatrix}\right)$$

$$=\frac{1}{n_{k+2}^{\sigma_{k+2}}\cdots n_{2+\beta}^{\sigma_{2+\beta}}(n_1+\cdots+n_{2+\beta})^{\sigma_{3+\beta}}(n_1+\ldots+n_{k+1})^{{}_kt_{k+1}}\cdots(n_1+n_2)^{{}_kt_2}n_1^{{}_kt_1}}.$$

And $Q^0\left(\begin{smallmatrix} n_1; & n_2, & \cdots, & n_{2+\beta} \\ \sigma_1; & \sigma_2, & \cdots, & \sigma_{2+\beta}\end{smallmatrix}\right) = Q$.

When $k = \beta$,

$$Q^\beta\left(\begin{smallmatrix} n_1, & \cdots, & n_{\beta+1}; & n_{2+\beta} \\ {}_\beta t_1, & \cdots, & {}_\beta t_{\beta+1}; & \sigma_{2+\beta}\end{smallmatrix}\right)$$

$$= \frac{1}{n_{\beta+2}^{\sigma_{\beta+2}}(n_1 + \cdots + n_{2+\beta})^{\sigma_{3+\beta}}(n_1 + \ldots + n_{\beta+1})^{{}_\beta t_{\beta+1}} \cdots (n_1 + n_2)^{{}_\beta t_2} n_1^{{}_\beta t_1}}.$$

By the calculation of P_1 *for any plane trivalent tree, we know that* $Q^\beta\left(\begin{smallmatrix} n_1, & \cdots, & n_{\beta+1}; & n_{2+\beta} \\ {}_\beta t_1, & \cdots, & {}_\beta t_{\beta+1}; & \sigma_{2+\beta}\end{smallmatrix}\right)$ *can be written as*

$$= \sum_{\tau_j(\beta)} \sum_{\tilde{t}_i^{\tau_j}} \frac{C_{\tau_j(\beta), \tilde{t}_i^{\tau_j}}}{n_{\tau_j \cdot 1}^{\tilde{t}_1^{\tau_j}} (n_{\tau_j \cdot 1} + n_{\tau_j \cdot 2})^{\tilde{t}_2^{\tau_j}} \cdots (n_{\tau_j \cdot 1} + \ldots + n_{\tau_j \cdot (\beta+2)})^{\tilde{t}_{\beta+2}^{\tau_j}}},$$

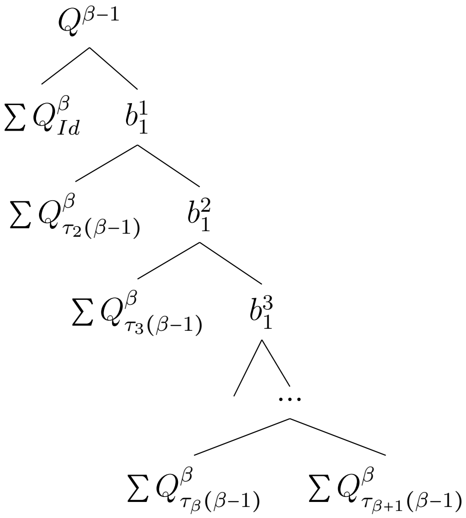
$$\tilde{t}_1^{\tau_j} + \cdots \tilde{t}_{\beta+2}^{\tau_j} = \sum_{i=1}^{\beta+3} \sigma_i, \quad \forall j.$$

where the permutation $\tau_1(\beta) = Id$ *and if* $2 \le j \le \beta + 2$

$$\tau_j(\beta) = \left(\begin{smallmatrix} 2+\beta-(j-1) & 2+\beta-(j-2) & \cdots & 1+\beta & 2+\beta \\ 2+\beta & 2+\beta-(j-1) & \cdots & \beta & 1+\beta \end{smallmatrix}\right)$$

and $C_{\tau_j(\beta), \tilde{t}_i^{\tau_j}} \in \mathbb{Z}$.

If $k = \beta - 1$. *We have one rooted tree illustrating the procedure of reducing to the case* $k = \beta$ *by a repeated times of Eisenstein's trick.*



$$Q^{\beta-1} = Q^{\beta-1}\left(\begin{smallmatrix} n_1, & \cdots, & n_\beta; & n_{\beta+1}, & n_{\beta+2} \\ {}_{\beta-1}t_1, & \cdots, & {}_{\beta-1}t_\beta^1; & \sigma_{\beta+1}, & \sigma_{\beta+2}\end{smallmatrix}\right)$$

Denote

$$Q^\beta_{\tau_j(\beta-1)} = Q^\beta\left(\tau_j(\beta - 1) \cdot \left(\begin{smallmatrix} n_1, & \cdots, & n_{\beta+1}; & n_{\beta+2} \\ {}_\beta t_1, & \cdots, & {}_\beta t_{\beta+1}; & \sigma_{\beta+2}\end{smallmatrix}\right)\right),$$

where the permutation $\tau_1(\beta - 1) = Id$, *and if* $2 \le j \le \beta + 1$,

$$\tau_j(\beta - 1) = \left(\begin{smallmatrix} 1+\beta-(j-1) & 2+\beta-(j-2) & \cdots & \beta+2 \\ 1+\beta & 2+\beta-(j-1) & \cdots & \beta+1 \end{smallmatrix}\right)$$

acts on the indices i, l *of* n_i *and* ${}_\beta t_l$.

Define

$$Q^{\beta}_{\tau_j(\beta-1)} = \sum_{\substack{r_1+s_1=\sigma_{1+\beta}+{}_{\beta-1}t_{\beta} \\ r_2+s_2=s_1+{}_{\beta-1}t^1_{\beta-1} \\ \cdots \\ r_j+s_j=s_{j-1}+{}_{\beta-1}t_{1+\beta-j}}} Q^{\beta}\left(\tau_j(\beta-1)\cdot\left(\begin{smallmatrix} n_1, & \cdots, & n_{\beta+1}; & n_{\beta+2} \\ {}_{\beta}t_1, & \cdots, & {}_{\beta}t_{\beta+1}; & \sigma_{\beta+2}\end{smallmatrix}\right)\right),$$

where

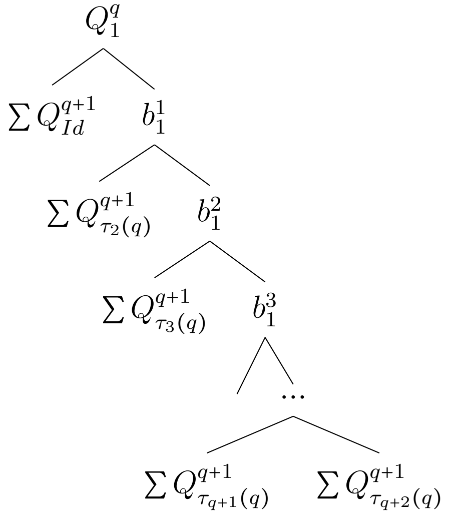
$$\tau_j(\beta-1)\cdot\left(\begin{smallmatrix} n_1, & \cdots, & n_{\beta+1}; & n_{\beta+2} \\ {}_{\beta}t_1, & \cdots, & {}_{\beta}t_{\beta+1}; & \sigma_{\beta+2}\end{smallmatrix}\right) =$$

$$\left(\begin{smallmatrix} n_{\tau_j(\beta-1)\cdot 1}, & \cdots, & n_{\tau_j(\beta-1)\cdot({}_{\beta-1}t^1_{\beta-j})}, & n_{\tau_j(\beta-1)\cdot({}_{\beta-1}t_{\beta+1-j})}, & n_{\tau_j(\beta-1)\cdot({}_{\beta-1}t_{\beta_N+2-j})}, & \cdots & n_{\tau_j(\beta-1)\cdot(\beta+1)}; & n_{\beta+2} \\ {}_{\beta-1}t_1, & \cdots, & {}_{\beta-1}t_{\beta-j}, & s_j, & r_j, & \cdots, & r_1; & \sigma_{\beta+2}\end{smallmatrix}\right).$$

Then

$$Q^{\beta-1} = \sum Q^{\beta}_{Id} + \sum Q^{\beta}_{\tau_2(\beta-1)} + \cdots + \sum Q^{\beta}_{\tau_\beta(\beta-1)} + \sum Q^{\beta}_{\tau_{\beta+1}(\beta-1)}.$$

Continuing this procedure we can write Q^q *as a sum of* $Q^{q+1}_{\tau_j(q)}$ *for* $0 \le q \le \beta$. *We can get the same result for* Q^q_σ *and* $Q^{q+1}_{\tau_j(q)\sigma}$ *and* $\sigma = \tau_l(q-1)$ *for some* $1 \le l \le q+1$.



Then

$$Q^q = \sum Q^{q+1}_{Id} + \sum Q^{q+1}_{\tau_2(q)} + \cdots + \sum Q^{q+1}_{\tau_{q+1}(q)} + \sum Q^{q+1}_{1,\tau_{q+2}(q)}.$$

In conclusion, $Q^0\left(\begin{smallmatrix} n_1; & n_2, & \cdots, & n_{\beta+2} \\ \sigma_1; & \sigma_2, & \cdots, & \sigma_{\beta+2}\end{smallmatrix}\right) = Q$ *can be written as*

$$= \sum_{\widehat{\tau}_k}\sum_{\widetilde{t}^{\widehat{\tau}_k}_i} \frac{C_{\widehat{\tau}_k,\widetilde{t}^{\widehat{\tau}_k}_i}}{n^{\widetilde{t}^{\widehat{\tau}_k}_1}_{\widehat{\tau}_k\cdot 1}(n_{\widehat{\tau}_k\cdot 1}+n_{\widehat{\tau}_k\cdot 2})^{\widetilde{t}^{\widehat{\tau}_k}_2}\cdots(n_{\widehat{\tau}_k\cdot 1}+\ldots+n_{\widehat{\tau}_k\cdot(\beta+2)})^{\widetilde{t}^{\widehat{\tau}_k}_{\beta+2}}}.$$

Therefore

$$Z_{I,\nu}(\Gamma',\partial\Gamma') = \sum_{\widehat{\tau}_k}\sum_{\widetilde{t}^{\widehat{\tau}_k}_i} C_{\widehat{\tau}_k,\widetilde{t}^{\widehat{\tau}_k}_i}\zeta(\widetilde{t}^{\widehat{\tau}_k}_1, \widetilde{t}^{\widehat{\tau}_k}_2, \cdots, \widetilde{t}^{\widehat{\tau}_k}_{\beta+2}),$$

$$\widetilde{t}^{\widehat{\tau}_k}_1 + \widetilde{t}^{\widehat{\tau}_k}_2 + \cdots + \widetilde{t}^{\widehat{\tau}_k}_{\beta+2} = \sum_{i=1}^{\beta+3}\sigma_i,$$

where

$$\widehat{\tau}_k = \tau_{l_0}(0)\cdots\tau_{l_{\beta-1}}(\beta-1)\tau_{l_\beta}(\beta),$$

where $\tau_1(m) = Id$ *and, if* $2 \leq l \leq d' + 1 + m$, $0 \leq m \leq \beta$,

$$\tau_l(m) = \left(\begin{smallmatrix} 2+m-(l-1) & 2+m-(k-2) & \cdots & 1+m & 2+m \\ 2+m & 2+m-(k-1) & \cdots & m & 1+m \end{smallmatrix}\right).$$

$$|\{\tau_l(m); \quad 2 \leq l \leq m+2, \quad 0 \leq m \leq \beta\}| = (\beta+2)(\beta+1)\cdots 2 = (\beta+2)! = |S_d|.$$

Hence, theorem 4.2.1 holds for a tree with one internal vertex of valency $3+\beta$ $(\beta \geq 1)$*.*

The inductive step:

What we will change for induction in the proof is the following :
Recall that $d = N + 1 + \sum_{1\leq j\leq N}\beta_j$*, then let* $d' = N + \sum_{1\leq j\leq N-1}\beta_j$*.*

In order to deduce the case $N = n+1$ *to the case* $N-1 = n$*, we apply the following operation: we cut down the internal edge* f_{N-1}*, one of whose ends is the internal vertex* w_N *and associate the new external vertex denoted as* $v'_{N+1+\sum_{1\leq j\leq N-1}\beta_j} = v'_{d'+1}$ *and denote the new external edge as* $e'_{N+1+\sum_{1\leq j\leq N-1}\beta_j} = e'_{d'+1}$ *to which we associate* $n_{e'_{N+1+\sum_{1\leq j\leq N-1}\beta_j}} = m_{N-1}$ *and the subdivision* μ_{N-1}*, then we build up a new plane tree* Γ' *with* $N-1 = n$ *internal vertices and whose rank is* $d' = d - 1 - \beta_N$*, where* d *is the rank of* Γ*.*

$$Z_{I,\nu}(\Gamma', \partial\Gamma') = \sum_{n_i\in\mathbb{N}\smallsetminus\{0\}, 1\leq i\leq d'} \mathscr{O}_{I,\nu}(\Gamma', \partial\Gamma').$$

Then

$$Z_{I,\nu}(\Gamma, \partial\Gamma) = G_{I,\nu,\Gamma,\partial\Gamma}(\{0\}_{v\in\partial\Gamma}, 1) = \sum_{\substack{n_i\in\mathbb{N}\smallsetminus\{0\},\\ 1\leq i\leq d}} \frac{1}{\prod_{k=d'-N}^{d-N} |n_{N+1+k}|^{\sigma_{N+1+k}}} \cdot \mathscr{O}_{I,\nu}(\Gamma', \partial\Gamma').$$

Since the number of the internal vertices of Γ' *is n, then the theorem for* Γ' *holds by the inductive hypothesis.*

Hence we have the following equality:

$$\mathscr{O}_{I,\nu}(\Gamma', \partial\Gamma') = \sum_{\gamma\in S_{d'}} \sum_{t_1^\gamma,\ldots,t_{d'}^\gamma} \frac{\tilde{C}_{\gamma,t_i^\gamma}}{n_{\gamma\cdot 1}^{t_1^\gamma}(n_{\gamma\cdot 1}+n_{\gamma\cdot 2})^{t_2^\gamma}\cdots(n_{\gamma\cdot 1}+\ldots+n_{\gamma\cdot(d')})^{t_{d'}^\gamma}},$$

where

$$t_1^\gamma + \ldots + t_{d'}^\gamma = \sum_{1\leq i\leq d'} \sigma_i + \sum_{1\leq j\leq N-1} \mu_j, \quad \forall \gamma \in S_{d'}.$$

Now

$$P_\gamma = \frac{1}{\prod_{k=d'-N}^{d-N} |n_{N+1+k}|^{\sigma_{N+1+k}}} \frac{\tilde{C}_{\gamma,t_i^\gamma}}{n_{\gamma\cdot 1}^{t_1^\gamma}(n_{\gamma\cdot 1}+n_{\gamma\cdot 2})^{t_2^\gamma}\cdots(n_{\gamma\cdot 1}+\ldots+n_{\gamma\cdot(d')})^{t_{d'}^\gamma}},$$

where $d' = N + \sum_{1\leq j\leq N-1}\beta_j$ *and*

$$\sum_{i=1}^{d+1} n_i = 0.$$

Then

$$P_\gamma = \frac{1}{n_{d'+1}^{\sigma_{d'+1}} \cdots n_{d'+1+\beta_N}^{\sigma_{d'+1+\beta_N}} (n_1 + \cdots + n_{d'+1+\beta_N})^{\sigma_{d'+2+\beta_N}}} \frac{\tilde{C}_{\gamma, t_i^\gamma}}{n_{\gamma\cdot 1}^{t_1^\gamma} (n_{\gamma\cdot 1} + n_{\gamma\cdot 2})^{t_2^\gamma} \cdots (n_{\gamma\cdot 1} + \ldots + n_{\gamma\cdot(d')})^{t_{d'}^\gamma}},$$

and

$$Z_{I,\nu}(\Gamma, \partial\Gamma) = \sum_{\substack{n_i \in \mathbb{N}\setminus\{0\},\\ 1\le i\le d}} \sum_{\gamma\in S_{d'}} \sum_{t_1^\gamma,\ldots,t_{d'}^\gamma} P_\gamma.$$

Without losing generality, we can focus on P_1, where 1 is the identity permutation.

***Calculation of** P_1.*

$$P_1 = \frac{1}{n_{d'+1}^{\sigma_{d'+1}} \cdots n_{d'+1+\beta_N}^{\sigma_{d'+1+\beta_N}} (n_1 + \cdots + n_{d'+1+\beta_N})^{\sigma_{d'+2+\beta_N}}} \frac{\tilde{C}_{1, t_i^1}}{(n_1 + \ldots + n_{d'})^{t_{d'}^1} \cdots (n_1 + n_2)^{t_2^1} n_1^{t_1^1}}.$$

If $\beta_N = 0$, then we return to the calculation of P_1 for a plane trivalent tree. If $\beta_N > 0$, the number of monomials before the polynomial $(n_1 + \cdots + n_{d'+1+\beta_N})^{\sigma_{d'+2+\beta_N}}$ is no more 1.

Therefore, the calculation of P_1 turns to reducing the number of monomials $n_{d'+k}^{\sigma_{d'+k}}$. We have to show that this number indeed can be reduced, then finally conclude the calculation by the case where the number is one.

As before, we will heavily use Eisenstein's trick. We will introduce several pieces of notation in order to simplify the demonstration.

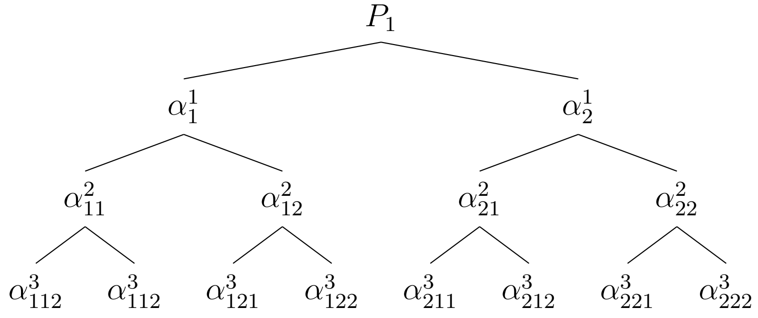
Notation 4.2.16 *(1) We will write $Eis(a^{s_1}, b^{s_2})$ for the operation*

$$\frac{1}{a^{s_1} b^{s_2}} = \sum_{r_1+r_2=s_1+s_2} \frac{C_{r_1-1}^{s_1-1}}{(a+b)^{r_1} b^{r_2}} + \frac{C_{r_1-1}^{s_2-1}}{(a+b)^{r_1} a^{r_2}}.$$

(2) Let $n(k_1, k_2)$ be the sum:

$$\sum_{j=k_1}^{k_2} n_j = n_{k_1} + \cdots + n_{k_2}.$$

In every use of $Eis(\cdot,\cdot)$, there are two kinds of fractions $\frac{C_{r_1-1}^{s_1-1}}{(a+b)^{r_1} b^{r_2}}$ and $\frac{C_{r_1-1}^{s_2-1}}{(a+b)^{r_1} a^{r_2}}$. In order to reduce the number of terms $n_{d'+k}^{\sigma_{d'+k}}$, we draw a rooted tree to visualize the results after a repeated use of $Eis(\cdot,\cdot)$.



For example, the operation $Eis(n_{d'+1}^{\sigma^{d'+1}}, n(1,d')^{\sigma_{d'}})$ yields

$$P_1 = \alpha_1^1 + \alpha_2^1,$$

$$\alpha_1^1 = \sum_{\substack{r_1+s_1 \\ =\sigma_{d'+1}+t^1_{d'}}} \frac{C_{r_1-1}^{\sigma_{d'+1}-1}\tilde{C}_{1,t_i^1}}{n_{d'+2}^{\sigma_{d'+2}}\cdots n_{d'+1+\beta_N}^{\sigma_{d'+1+\beta_N}} n(1,d'+1+\beta_N)^{\sigma_{d'+2+\beta_N}} n(1,d'+1)^{r_1}(n(1,d')^{s_1}\cdots(n_1+n_2)^{t_2^1} n_1^{t_1^1}},$$

$$\alpha_2^1 = \sum_{\substack{r_1+s_1 \\ =\sigma_{d'+1}+t^1_{d'}}} \frac{C_{r_1-1}^{t^1_{d'}-1}\tilde{C}_{1,t_i^1}}{n_{d'+1}^{s_1}\cdots n_{d'+1+\beta_N}^{\sigma_{d'+1+\beta_N}} n(1,d'+1+\beta_N)^{\sigma_{d'+2+\beta_N}} n(1,d'+1)^{r_1} n(1,d'-1)^{t^1_{d'-1}}\cdots(n_1+n_2)^{t_2^1} n_1^{t_1^1}}.$$

Definition 4.2.17

$$\alpha(k:i_1,j_1;\cdots;i_k,j_k) = \alpha^k_{\underbrace{1\cdots1}_{i_1}\underbrace{2\cdots2}_{j_1}\underbrace{1\cdots1}_{i_2}\underbrace{2\cdots2}_{j_2}\underbrace{\cdots\cdots}_{\cdots}\underbrace{1\cdots1}_{i_k}\underbrace{2\cdots2}_{j_k}},$$

where

$$\sum_{l=1}^{k}(i_l+j_l) = k; \quad i_l \geq 0; \quad j_l \geq 0.$$

The upper symbol k means that we have used Eisenstein's trick k-times. In the first i_1 applications of $Eis(\cdot,\cdot)$ we consider only the term on the left, in the next j_1 applications only the term on the right, etc. If we descend p times along the left branch, the number of monomials $n^?_{d'+k}$ decreases by p. However, on the right branch, the number of monomials does not decrease. Moreover, the number of factors of the denominator of P_1 is equal to

$$d'+2+\beta_N = N+2+\sum_{1\leq j\leq N}\beta_j = d+1.$$

During the repeated applications of $Eis(\cdot,\cdot)$, the number of factors of denominators of $\alpha(k:i_1,j_1;\cdots;i_k,j_k)$ remains unchanged.

Note that

$$\alpha(k:i_1,j_1;\cdots;i_k,j_k) = \alpha(k+1:i_1,j_1;\cdots;i_k,j_k;1,0)+\alpha(k+1:i_1,j_1;\cdots;i_k,j_k;0,1);$$

$$P_1 = \sum_{\substack{i_1,\cdots,i_k\geq 0 \\ j_1,\cdots,j_k\geq 0 \\ \sum_{l=1}^{k}(i_l+j_l)=k}} \alpha(k:i_1,j_1;\cdots;i_k,j_k).$$

For example, $\alpha_1^1 = \alpha(1:1,0)$, $\alpha_2^1 = \alpha(1:0,1)$.

We will consider first two special cases.

Proposition 4.2.18 *If $i_1 = k$ $(1 \leq k \leq \beta_N)$, then we get*

$$\alpha^k_{\underbrace{1\cdots1}_{k}} = \sum_{\substack{r_1+s_1 \\ =t^1_{d'}+\sigma_{d'+1} \\ r_2+s_2 \\ =r_1+\sigma_{d'+2} \\ \cdots \\ \cdots \\ \cdots \\ r_k+s_k \\ =r_{k-1}+\sigma_{d'+k}}} \frac{C_{r_k-1}^{\sigma_{d'+k}-1}\cdots C_{r_1-1}^{\sigma_{d'+1}}\tilde{C}_1}{\prod_{j=k}^{\beta_N} n_{d'+j+1}^{\sigma_{d'+j+1}} n(1,d'+1+\beta_N)^{\sigma_{2+d'+\beta_N}} n(1,d'+k)^{r_k}\prod_{j=1}^{k} n(1,d'+j-1)^{s_j}\prod_{j=1}^{d'-1} n(1,j)^{t_j^1}}.$$

When $k = \beta_N$, then

$$\alpha^{\beta_N}_{\underbrace{1\cdots1}_{\beta_N}} = \sum_{\substack{r_1+s_1 \\ =t^1_{d'}+\sigma_{d'+1} \\ r_2+s_2 \\ =r_1+\sigma_{d'+2} \\ \cdots \\ \cdots \\ \cdots \\ r_{\beta_N}+s_{\beta_N} \\ =r_{\beta_N-1}+\sigma_{d'+\beta_N}}} \frac{C^{\sigma_{d'+\beta_N}-1}_{r_{\beta_N}-1}\cdots C^{\sigma_{d'+1}}_{r_1-1}\tilde{C}_1}{n^{\sigma_{d'+\beta_N+1}}_{d'+\beta_N+1} n(1,d'+1+\beta_N)^{\sigma_{2+d'+\beta_N}} n(1,d'+\beta_N)^{r_{\beta_N}} \prod_{j=1}^{\beta_N} n(1,d'+j-1)^{s_j} \prod_{j=1}^{d'-1} n(1,j)^{t^1_j}}.$$

From the calculation of P_1 for any plane trivalent tree, it is easy to see that $\alpha^{\beta_N}_{\underbrace{1\cdots1}_{\beta_N}}$ can be written as a $\mathbb{Z}$-linear combination of $d = d' + 1 + \beta_N$-tuple zeta values.

Remark 4.2.19 *The example shows that if some $i_l > 0$, then the number of monomials $n^?_{d'+k}$ of $\alpha(k : i_1, j_1; \cdots; i_k, j_k)$ decrease.*

Proposition 4.2.20 *If $j_k = k$, then*

$$\alpha^{k}_{\underbrace{2\cdots2}_{k}} = \sum_{\substack{r_1+s_1 \\ =\sigma_{d'+1}+t^1_{d'} \\ r_2+s_2 \\ =s_1+t^1_{d'-1} \\ \cdots \\ \cdots \\ \cdots \\ r_k+s_k \\ =s_{k-1}+t^1_{d'-k+1}}} \frac{C^{t^1_{d'-k+1}-1}_{r_k-1}\cdots C^{t^1_{d'}-1}_{r_1-1}\tilde{C}_1}{n^{s_k}_{d'+1} \prod_{j=d'+2}^{d'+1+\beta_N} n^{\sigma_j}_j n(1,d'+1+\beta_N)^{\sigma_{d'+2+\beta_N}} \prod_{j=0}^{k-1}(n(1,d'-j)+n_{d'+1})^{r_{j+1}} \prod_{j=1}^{d'-k} n(1,j)^{t^1_j}}.$$

Then

$$\alpha^{d'-1}_{\underbrace{2\cdots2}_{d'-1}} = \sum_{\substack{r_1+s_1 \\ =\sigma_{d'+1}+t^1_{d'} \\ r_2+s_2 \\ =s_1+t^1_{d'-1} \\ \cdots \\ \cdots \\ \cdots \\ r_{d'-1}+s_{d'-1} \\ =s_{d'-2}+t^1_2}} \frac{C^{t^1_2-1}_{r_{d'-1}-1}\cdots C^{t^1_{d'}-1}_{r_1-1}\tilde{C}_1}{n^{s_{d'-1}}_{d'+1} \prod_{j=d'+2}^{d'+1+\beta_N} n^{\sigma_j}_j n(1,d'+1+\beta_N)^{\sigma_{d'+2+\beta_N}} \prod_{j=0}^{d'-2}(n(1,d'-j)+n_{d'+1})^{r_{j+1}} n^{t^1_1}_1}.$$

Now we will apply once again $Eis(n^{s_{d'-1}}_{d'+1}, n^{t^1_1}_1)$ and get:

$$\alpha^{d'}_{\underbrace{2\cdots2}_{d'-1}\underbrace{1}_{1}} = \sum_{\substack{r_1+s_1 \\ =\sigma_{d'+1}+t^1_{d'} \\ r_2+s_2 \\ =s_1+t^1_{d'-1} \\ \cdots \\ \cdots \\ \cdots \\ r_{d'}+s_{d'} \\ =s_{d'-1}+t^1_1}} \frac{C^{s_{d'-1}-1}_{r_{d'}-1} C^{t^1_2-1}_{r_{d'-1}-1}\cdots C^{t^1_{d'}-1}_{r_1-1}\tilde{C}_1}{\prod_{j=d'+2}^{d'+1+\beta_N} n^{\sigma_j}_j n(1,d'+1+\beta_N)^{\sigma_{d'+2+\beta_N}} \prod_{j=0}^{d'-2}(n(1,d'-j)+n_{d'+1})^{r_{j+1}} (n_1+n_{d'+1})^{r_{d'}} n^{s_{d'}}_1};$$

$$\alpha^{d'}_{\underbrace{2\cdots 2}_{d'}} = \sum_{\substack{r_1+s_1\\ =\sigma_{d'+1}+t^1_{d'}\\ r_2+s_2\\ =s_1+t^1_{d'-1}\\ \cdots\\ \cdots\\ \cdots\\ r_{d'}+s_{d'}\\ =s_{d'-1}+t^1_1}} \frac{C^{t^1_1-1}_{r_{d'}-1} C^{t^1_2-1}_{r_2-1} \cdots C^{t^1_{d'}-1}_{r_1-1} \tilde{C}_1}{\prod_{j=d'+2}^{d'+1+\beta_N} n_j^{\sigma_j} n(1, d'+1+\beta_N)^{\sigma_{d'+2+\beta_N}} \prod_{j=0}^{d'-2} (n(1, d'-j) + n_{d'+1})^{r_{j+1}} (n_1 + n_{d'+1})^{r_{d'}} n_{d'+1}^{s_{d'}}}.$$

Remark 4.2.21 *Even in the extreme case, when $i_1 + \cdots i_k = 0$, $\alpha(k : i_1, j_1; \cdots; i_k, j_k)$ has no index 1, we can still reduce the number of monomials $n^{?}_{d'+k}$ before the polynomial $n(1, d' + 1 + \beta_N)$.*

Therefore we have shown that the number of monomials $n^{?}_{d'+k}$ before $n(1, d' + 1 + \beta_N)$ indeed can be reduced.

However, in order to demontrate by the mathematical induction that $\mathscr{O}_{I,\nu}(\Gamma, \partial\Gamma)$ can be written in the form

$$\sum_{\gamma \in S_d} \sum_{\substack{t^\gamma_i \\ 1 \le i \le d}} \frac{C_\gamma}{n^{t^\gamma_1}_{\gamma \cdot 1} (n_{\gamma\cdot 1} + n_{\gamma \cdot 2})^{t^\gamma_2} \cdots (n_{\gamma \cdot 1} + \ldots + n_{\gamma \cdot d})^{t^\gamma_d}}$$

with $d = d' + 1 + \beta_N$, and therefore conclude that

$$Z_{I,\nu}(\Gamma, \partial\Gamma) = \sum_{\substack{n_i \in \mathbb{N} \setminus \{0\},\\ 1 \le i \le d}} \mathscr{O}_{I,\nu}(\Gamma, \partial\Gamma),$$

we will use, in the body of the first mathematical induction, another induction on $\sharp$, the number of monomials $n^{\sigma_{d'+k}}_{d'+k}$ $(1 \le k \le 1 + \beta_N)$ before

$$(n_1 + n_2 + \cdots + n_{d'+1+\beta_N})^{\sigma_{d'+2+\beta_N}}$$

in the denominator of

$$P_1 = \frac{1}{n^{\sigma_{d'+1}}_{d'+1} \cdots n^{\sigma_{d'+1+\beta_N}}_{d'+1+\beta_N} (n_1 + \cdots + n_{d'+1+\beta_N})^{\sigma_{d'+2+\beta_N}}} \frac{\tilde{C}_1}{(n_1 + \ldots + n_{d'})^{t^1_{d'}} \cdots (n_1 + n_2)^{t^1_2} n_1^{t^1_1}}.$$

Definition 4.2.22 *If $0 \le k \le \beta_N$,*

$$P^k_1 \begin{pmatrix} n_1, \; \cdots, \; n_{d'+k}; \; n_{d'+k+1}, \; \cdots, \; n_{d'+1+\beta_N} \\ {}_k t^1_1, \; \cdots, \; {}_k t^1_{d'+k}; \; \sigma_{d'+k+1}, \; \cdots, \; \sigma_{d'+1+\beta_N} \end{pmatrix}$$

$$= \frac{1}{n^{\sigma_{d'+1+k}}_{d'+1+k} \cdots n^{\sigma_{d'+1+\beta_N}}_{d'+1+\beta_N} (n_1 + \cdots + n_{d'+1+\beta_N})^{\sigma_{d'+2+\beta_N}}} \frac{\tilde{C}_1}{(n_1 + \ldots + n_{d'+k})^{{}_k t^1_{d'+k}} \cdots (n_1 + n_2)^{{}_k t^1_2} n_1^{{}_k t^1_1}}.$$

And $P^0_1 \begin{pmatrix} n_1, \; \cdots, \; n_{d'}; \; n_{d'+1}, \; \cdots, \; n_{d'+1+\beta_N} \\ {}_0 t^1_1, \; \cdots, \; {}_0 t^1_{d'}; \; \sigma_{d'+1}, \; \cdots, \; \sigma_{d'+1+\beta_N} \end{pmatrix} = P_1$.

Fix $\beta_N > 0$. We make the inductive step on $\sharp$.

$$\sharp = 1 + \beta_N - k$$

①*If* $\sharp = 1$, *then* $k = \beta_N$.

$$P_1^{\beta_N}\left(\begin{smallmatrix} n_1, & \cdots, & n_{d'+\beta_N}; & n_{d'+1+\beta_N} \\ {}_{\beta_N}t_1^1, & \cdots, & {}_{\beta_N}t_{d'+\beta_N}^1; & \sigma_{d'+1+\beta_N} \end{smallmatrix}\right)$$

$$= \frac{1}{n_{d'+1+\beta_N}^{\sigma_{d'+1+\beta_N}}(n_1+\cdots+n_{d'+1+\beta_N})^{\sigma_{d'+2+\beta_N}}} \frac{\tilde{C}_1}{(n_1+\ldots+n_{d'+\beta_N})^{{}_{\beta_N}t_{d'+\beta_N}^1}\cdots(n_1+n_2)^{{}_{\beta_N}t_2^1} n_1^{{}_{\beta_N}t_1^1}}.$$

By the demonstration for any plane trivalent tree, $P_1^{\beta_N}(n_1,\cdots,n_{d'+1+\beta_N})$ *can be written as*

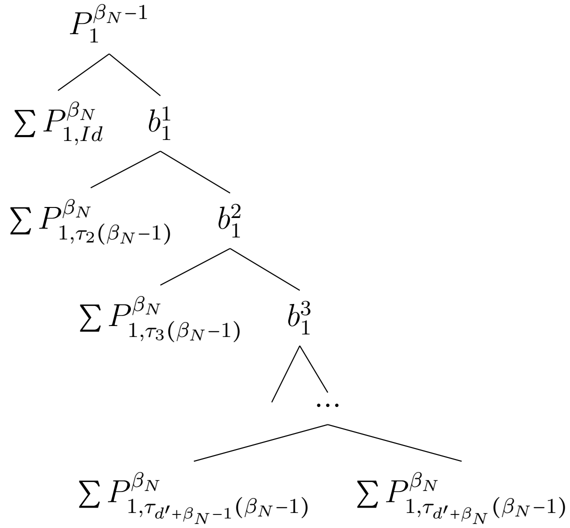
$$= \sum_{\tau_j(\beta_N)} \sum_{\tilde{t}_i^{\tau_j}} \frac{C_{\tau_j(\beta_N),\tilde{t}_i^{\tau_j}}}{n_{\tau_j\cdot 1}^{\tilde{t}_1^{\tau_j}}(n_{\tau_j\cdot 1}+n_{\tau_j\cdot 2})^{\tilde{t}_2^{\tau_j}}\cdots(n_{\tau_j\cdot 1}+\ldots+n_{\tau_j\cdot(d'+1+\beta_N)})^{\tilde{t}_{d'+1+\beta_N}^{\tau_j}}},$$

where the permutation $\tau_1(\beta_N) = Id$ *and if* $2 \le j \le d'+\beta_N+1$

$$\tau_j(\beta_N) = \left(\begin{smallmatrix} d'+1+\beta_N-(j-1) & d'+1+\beta_N-(j-2) & \cdots & d'+\beta_N & d'+1+\beta_N \\ d'+1+\beta_N & d'+1+\beta_N-(j-1) & \cdots & d'+\beta_N-1 & d'+\beta_N \end{smallmatrix}\right)$$

and $C_{\tau_j(\beta_N),\tilde{t}_i^{\tau_j}} = C_{r_j-1}^{t_{N-(j-1)}^1}\cdots C_{r_1-1}^{t_{N-1}^1}\tilde{C}_1$.

② *Next, if* $k = \beta_N - 1$. *We have one rooted tree illustrating the procedure of reducing to the case* $k = \beta_N$ *by Eisenstein's trick.*



$$P_1^{\beta_N-1} = P_1^{\beta_N-1}\left(\begin{smallmatrix} n_1, & \cdots, & n_{d'+\beta_N-1}; & n_{d'+\beta_N}, n_{d'+1+\beta_N} \\ {}_{\beta_N-1}t_1^1, & \cdots, & {}_{\beta_N-1}t_{d'+\beta_N-1}^1; & \sigma_{d'+\beta_N}, \sigma_{d'+1+\beta_N} \end{smallmatrix}\right)$$

Denote

$$P_{1,\tau_j(\beta_N-1)}^{\beta_N} = P_1^{\beta_N}\left(\tau_j(\beta_N-1)\cdot\left(\begin{smallmatrix} n_1, & \cdots, & n_{d'+\beta_N}; & n_{d'+1+\beta_N} \\ {}_{\beta_N}t_1^1, & \cdots, & {}_{\beta_N}t_{d'+\beta_N}^1; & \sigma_{d'+1+\beta_N} \end{smallmatrix}\right)\right),$$

where the permutation $\tau_1(\beta_N-1) = Id$, *and if* $2 \le j \le d'+\beta_N$,

$$\tau_j(\beta_N-1) = \left(\begin{smallmatrix} d'+\beta_N-(j-1) & d'+1+\beta_N-(k-2) & \cdots & d'+\beta_N \\ d'+\beta_N & d'+1+\beta_N-(j-1) & \cdots & d'+\beta_N-1 \end{smallmatrix}\right)$$

acts on the indices i, l *of* n_i *and* ${}_{\beta_N}t_l^1$.

Define

$$\sum P^{\beta_N}_{1,\tau_j(\beta_N-1)} = \sum_{\substack{r_1+s_1=\sigma_{d'+\beta_N}+{}_{\beta_N-1}t^1_{d'+\beta_N-1}\\ r_2+s_2=s_1+{}_{\beta_N-1}t^1_{d'+\beta_N-2}\\ \cdots\\ r_j+s_j=s_{j-1}+{}_{\beta_N-1}t^1_{d'+\beta_N-j}}} P_1^{\beta_N}\left(\tau_j(\beta_N-1)\cdot\left(\begin{smallmatrix} n_1, & \cdots, & n_{d'+\beta_N}; & n_{d'+1+\beta_N}\\ {}_{\beta_N}t^1_1, & \cdots, & {}_{\beta_N}t^1_{d'+\beta_N}; & \sigma_{d'+1+\beta_N}\end{smallmatrix}\right)\right),$$

where

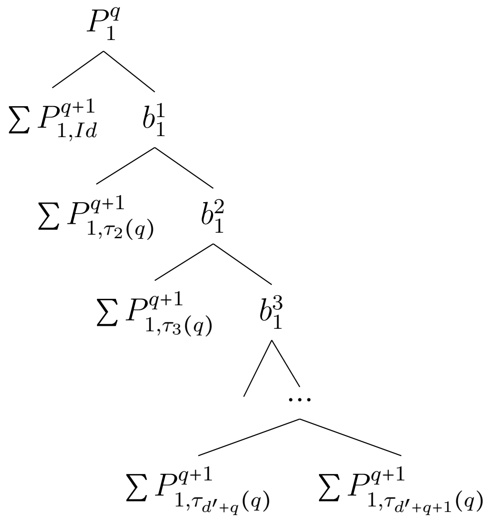
$$\tau_j(\beta_N-1)\cdot\left(\begin{smallmatrix} n_1, & \cdots, & n_{d'+\beta_N}; & n_{d'+1+\beta_N}\\ {}_{\beta_N}t^1_1, & \cdots, & {}_{\beta_N}t^1_{d'+\beta_N}; & \sigma_{d'+1+\beta_N}\end{smallmatrix}\right) =$$

$$\left(\begin{smallmatrix} n_{\tau_j(\beta_N-1)\cdot 1}, & \cdots, & n_{\tau_j(\beta_N-1)\cdot({}_{\beta_N-1}t^1_{d'+\beta_N-j-1})}, & n_{\tau_j(\beta_N-1)\cdot({}_{\beta_N-1}t^1_{d'+\beta_N-j})}, & n_{\tau_j(\beta_N-1)\cdot({}_{\beta_N-1}t^1_{d'+\beta_N-j+1})}, & \cdots & n_{\tau_j(\beta_N-1)\cdot(d'+\beta_N)}; & n_{d'+1+\beta_N}\\ {}_{\beta_N-1}t^1_1, & \cdots, & {}_{\beta_N-1}t^1_{d'+\beta_N-j-1}, & s_j, & r_j, & \cdots, & r_1; & \sigma_{d'+1+\beta_N}\end{smallmatrix}\right)$$

Then

$$P_1^{\beta_N-1} = \sum P^{\beta_N}_{1,Id} + \sum P^{\beta_N}_{1,\tau_2(\beta_N-1)} + \cdots + \sum P^{\beta_N}_{1,\tau_{d'+\beta_N-1}(\beta_N-1)} + \sum P^{\beta_N}_{1,\tau_{d'+\beta_N}(\beta_N-1)}.$$

Continuing this procedure we can write P_1^q *as a sum of* $P^{q+1}_{1,\tau_j(q)}$ *for* $0 \le q \le \beta_N$. *We can get the same result for* $P^q_{1,\sigma}$ *and* $P^{q+1}_{1,\tau_j(q)\sigma}$ *and* $\sigma = \tau_l(q-1)$ *for some* $1 \le l \le d'+q$.



Then

$$P_1^q = \sum P^{q+1}_{1,Id} + \sum P^{q+1}_{1,\tau_2(q)} + \cdots + \sum P^{q+1}_{1,\tau_{d'+q}(q)} + \sum P^{q+1}_{1,\tau_{d'+q+1}(q)}.$$

In conclusion, $P_1^0\left(\begin{smallmatrix} n_1, & \cdots, & n_{d'}; & n_{d'+1}, & \cdots, & n_{d'+1+\beta_N}\\ {}_0t^1_1, & \cdots, & {}_0t^1_{d'}; & \sigma_{d'+1}, & \cdots, & \sigma_{d'+1+\beta_N}\end{smallmatrix}\right) = P_1$ *can be written as*

$$= \sum_{\widehat{\tau}}\sum_{\widetilde{t_i^{\tau}}} \frac{C_{\widehat{\tau},\widetilde{t_i^{\tau}}}}{n_{\widehat{\tau}_k\cdot 1}^{\widetilde{t_1^{\tau}}}(n_{\widehat{\tau}\cdot 1}+n_{\widehat{\tau}\cdot 2})^{\widetilde{t_2^{\tau}}}\cdots(n_{\widehat{\tau}\cdot 1}+\ldots+n_{\widehat{\tau}\cdot(d'+1+\beta_N)})^{\widetilde{t^{\tau}_{d'+1+\beta_N}}}}$$

$$= \sum_{\widehat{\tau}}\sum_{\widetilde{t_i^{\tau}}} C_{\widehat{\tau},\widetilde{t_i^{\tau}}}\zeta(\widetilde{t_1^{\tau}}, \widetilde{t_2^{\tau}}, \cdots, \widetilde{t^{\tau}_{d'+1+\beta_N}}),$$

where

$$\widehat{\tau} = \tau_{l_0}(0)\cdots\tau_{l_{\beta_{N-1}}}(\beta_N-1)\tau_{l_{\beta_N}}(\beta_N),$$

where $\tau_1(m) = Id$ *and, if* $2 \le l \le d' + 1 + m$, $0 \le m \le \beta_N$,

$$\tau_l(m) = \begin{pmatrix} d'+1+m-(l-1) & d'+1+m-(k-2) & \cdots & d'+m & d'+1+m \\ d'+1+m & d'+1+m-(k-1) & \cdots & d'+m-1 & d'+m \end{pmatrix}.$$

Let the set

$$D = \{\widehat{\tau} = \prod_{0 \le m \le \beta_N} \tau_{l_m}(m); \quad 2 \le l_m \le d' + 1 + m\},$$

$$|D| = (d' + 1 + \beta_N)(d' + \beta_N) \cdots (d' + 1).$$

For other $\gamma \in S_{d'}$*, we can obtain the same result of* P_γ*, and*

$$S_d = \coprod_{\widehat{\tau} \in D} \widehat{\tau} \cdot S_{d'}.$$

Remark 4.2.23 *Eisenstein's trick has been formalized by Sczech in his theory of Eisenstein cocycles. It also appears in one of the proofs of the shuffle relations for MZV's. This is probably no coincidence.*

4.3 Relation to multiple polylogarithms

In fact, the result of theorem 4.2.1 can be generalized to any higher Green function

$$G_{I,\nu,\Gamma,\partial\Gamma}(\{x_v\}_{v \in \partial\Gamma}, 1)$$

for any arbitary tree Γ.

Theorem 4.3.1 ***(Relation to multiple polylogarithms)***
If $F = \mathbb{Q}, I^* = \mathbb{Z}$, $x_v \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$***, then*** $G_{I,\nu,\Gamma,\partial\Gamma}(\{x_v\}_{v\in\partial\Gamma}, 1)$ ***is a finite*** $\mathbb{Z}$***-linear combination of the values of multiple polylogarithms evaluated at some*** N***-th roots of unity.***

Let us first review Example 4.2.7.

$$G_{I,\nu,\Gamma_1,\partial\Gamma_1}(\{x_v\}_{v\in\partial\Gamma_1}, 1) = \sum_{\substack{n_1+n_2+n_3=0, n_j \in \mathbb{Z}\setminus\{0\} \\ \mathrm{sgn}(n_j)=(-1)^{\nu_j}}} \frac{e^{2\pi i(n_1 x_{v_1} + n_2 x_{v_2} + n_3 x_{v_3})}}{|n_1|^{\sigma_1} |n_2|^{\sigma_2} |n_3|^{\sigma_3}},$$

where $\nu_1 = \nu_2 = 0$, $\nu_3 = 1$ and $x_{v_1}, x_{v_2}, x_{v_3} \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$.

$$G_{I,\nu,\Gamma_1,\partial\Gamma_1}(\{x_v\}_{v\in\partial\Gamma_1}, 1) = \sum_{n_1,n_2 \in \mathbb{N}\setminus\{0\}} \frac{e^{2\pi i(n_1(x_{v_1}-x_{v_3})+n_2(x_{v_2}-x_{v_3}))}}{n_1^{\sigma_1} n_2^{\sigma_2} (n_1+n_2)^{\sigma_3}}$$

$$G_{I,\nu,\Gamma_1,\partial\Gamma_1}(\{x_v\}_{v\in\partial\Gamma_1}, 1)$$

$$= \sum_{n_1,n_2 \in \mathbb{N}\setminus\{0\}} \sum_{r+s=\sigma_1+\sigma_2} e^{2\pi i(n_1(x_{v_1}-x_{v_3})+n_2(x_{v_2}-x_{v_3}))} \cdot \left(\frac{C_{r-1}^{\sigma_1-1}}{(n_1+n_2)^{r+\sigma_3} n_2^s} + \frac{C_{r-1}^{\sigma_2-1}}{(n_1+n_2)^{r+\sigma_3} n_1^s} \right),$$

$$= \sum_{r+s=\sigma_1+\sigma_2} \left\{ C_{r-1}^{\sigma_1-1} \left(\sum_{n_1,n_2 \in \mathbb{N}\setminus\{0\}} \frac{e^{2\pi i(n_1(x_{v_1}-x_{v_3})+n_2(x_{v_2}-x_{v_3}))}}{(n_1+n_2)^{r+\sigma_3} n_2^s} \right) + C_{r-1}^{\sigma_2-1} \left(\sum_{n_1,n_2 \in \mathbb{N}\setminus\{0\}} \frac{e^{2\pi i(n_1(x_{v_1}-x_{v_3})+n_2(x_{v_2}-x_{v_3}))}}{(n_1+n_2)^{r+\sigma_3} n_1^s} \right) \right\}.$$

Note that

$$e^{2\pi i(n_1(x_{v_1}-x_{v_3})+n_2(x-v_2-x_{v_3}))} = e^{2\pi i((n_1+n_2)(x_{v_1}-x_{v_3})+n_2(x_{v_2}-x_{v_1}))},$$

and

$$e^{2\pi i(n_1(x_{v_1}-x_{v_3})+n_2(x_{v_2}-x_{v_3}))} = e^{2\pi i(n_1(x_{v_1}-x_{v_2})+(n_1+n_2)(x_{v_2}-x_{v_3}))}.$$

Let $z_1 = e^{2\pi i(x_{v_2}-x_{v_1})}$ and $z_2 = e^{2\pi i(x_{v_1}-x_{v_3})}$, let $y_1 = e^{2\pi i(x_{v_1}-x_{v_2})}$ and $y_2 = e^{2\pi i(x_{v_2}-x_{v_3})}$. In fact, let the permutation $\sigma(12)$ acts on the indices of $v_j (1 \le j \le 3)$, then

$$y_{\sigma(12)\cdot j} = z_j, \quad j = 1, 2.$$

Then we obtain

$$G_{I,\nu,\Gamma_1,\partial\Gamma_1}(\{x_v\}_{v\in\partial\Gamma_1}, 1) =$$

$$\sum_{r+s=\sigma_1+\sigma_2} \left\{ C_{r-1}^{\sigma_1-1} \left(\sum_{n_1,n_2\in\mathbb{N}\setminus\{0\}} \frac{z_1^{n_2} z_2^{n_1+n_2}}{(n_1+n_2)^{r+\sigma_3} n_2^s} \right) + C_{r-1}^{\sigma_2-1} \left(\sum_{n_1,n_2\in\mathbb{N}\setminus\{0\}} \frac{y_1^{n_1} y_2^{n_1+n_2}}{(n_1+n_2)^{r+\sigma_3} n_1^s} \right) \right\}.$$

In conclusion,

$$G_{I,\nu,\Gamma_1,\partial\Gamma_1}(\{x_v\}_{v\in\partial\Gamma_1}, 1) = \sum_{r+s=\sigma_1+\sigma_2} \left(C_{r-1}^{\sigma_1-1} Li_{s,r+\sigma_3}(z_1, z_2) + C_{r-1}^{\sigma_2-1} Li_{s,r+\sigma_3}(y_1, y_2) \right).$$

Proof 4.3.0.2 (Proof of Theorem 4.3.1) *For a general tree Γ, by the proof of Theorem 4.2.1, $\mathscr{O}_{I,\nu}(\Gamma, \partial\Gamma)$ can be written as*

$$\sum_{\gamma\in S_d} \sum_{\substack{t_j^\gamma \\ 1\le j\le d}} \frac{C_{\gamma,t_j^\gamma}}{n_{\gamma\cdot 1}^{t_1^\gamma}(n_{\gamma\cdot 1}+n_{\gamma\cdot 2})^{t_2^\gamma}\cdots(n_{\gamma\cdot 1}+\ldots+n_{\gamma\cdot d})^{t_d^\gamma}}$$

with d the rank of Γ, and therefore conclude that

$$Z_{I,\nu}(\Gamma, \partial\Gamma) = \sum_{\substack{n_j\in\mathbb{N}\setminus\{0\}, \\ 1\le j\le d}} \mathscr{O}_{I,\nu}(\Gamma, \partial\Gamma).$$

By careful observation, we get an expression

$$G_{I,\nu,\Gamma,\partial\Gamma}(\{x_v\}_{v\in\partial\Gamma}, 1) = \sum_{\substack{n_j\in\mathbb{N}\setminus\{0\}, \\ 1\le i\le d}} e^{2\pi i(\sum_{j=1}^{d+1} n_j x_{v_j})} \left(\sum_{\gamma\in S_d} \sum_{\substack{t_j^\gamma \\ 1\le i\le d}} \frac{C_{\gamma,t_j^\gamma}}{n_{\gamma\cdot 1}^{t_1^\gamma}(n_{\gamma\cdot 1}+n_{\gamma\cdot 2})^{t_2^\gamma}\cdots(n_{\gamma\cdot 1}+\ldots+n_{\gamma\cdot d})^{t_d^\gamma}} \right),$$

where $C_{\gamma,t_j^\gamma} \in \mathbb{Z}$. In fact, the Eisenstein trick is applied for the denominators, during these repeated operations the numerator remains unchanged. A simple calculation gives

$$\sum_{j=1}^{d+1} n_j x_{v_j} = \sum_{l=1}^{d} (\sum_{j=1}^{l} n_j)(x_{v_l} - x_{v_{l+1}}).$$

Let

$$z_j = e^{2\pi i(x_{v_j}-x_{v_{j+1}})}, \quad 1 \le j \le d,$$

and

$$z_{\gamma\cdot j} = e^{2\pi i(x_{v_{\gamma\cdot j}} - x_{v_{\gamma\cdot(j+1)}})}, \quad \gamma \in S_d,$$

then

$$G_{I,\nu,\Gamma,\partial\Gamma}(\{x_v\}_{v\in\partial\Gamma}, 1) = \sum_{\substack{n_j\in\mathbb{N}\setminus\{0\},\\ 1\leq j\leq d}} \sum_{\gamma\in S_d} \sum_{\substack{t_i^\gamma \\ 1\leq j\leq d}} \frac{C_{\gamma,t_j^\gamma} \prod_{i=1}^{d} z_{\gamma\cdot j}^{\sum_{l=1}^{j} n_{\gamma\cdot l}}}{n_{\gamma\cdot 1}^{t_1^\gamma}(n_{\gamma\cdot 1}+n_{\gamma\cdot 2})^{t_2^\gamma}\cdots(n_{\gamma\cdot 1}+\ldots+n_{\gamma\cdot d})^{t_d^\gamma}},$$

$$= \sum_{\gamma\in S_d}\left(\sum_{\substack{t_j^\gamma \\ 1\leq j\leq d}} \sum_{\substack{n_j\in\mathbb{N}\setminus\{0\},\\ 1\leq j\leq d}} C_{\gamma,t_j^\gamma} \frac{\prod_{j=1}^{d} z_{\gamma\cdot j}^{\sum_{l=1}^{j} n_{\gamma\cdot l}}}{n_{\gamma\cdot 1}^{t_1^\gamma}(n_{\gamma\cdot 1}+n_{\gamma\cdot 2})^{t_2^\gamma}\cdots(n_{\gamma\cdot 1}+\ldots+n_{\gamma\cdot d})^{t_d^\gamma}} \right),$$

finally we obtain

$$G_{I,\nu,\Gamma,\partial\Gamma}(\{x_v\}_{v\in\partial\Gamma}, 1) = \sum_{\gamma\in S_d}\left(\sum_{\substack{t_j^\gamma \\ 1\leq j\leq d}} C_{\gamma,t_j^\gamma} Li_{t_1^\gamma,\cdots,t_d^\gamma}(z_{\gamma\cdot 1},\cdots,z_{\gamma\cdot d}) \right).$$

In conclusion, $G_{I,\nu,\Gamma,\partial\Gamma}(\{x_v\}_{v\in\partial\Gamma}, 1)$ *is indeed a finite* $\mathbb{Z}$*-linear combination of multiple polylogarithms evaluated at some* N*-th roots of unity.*

Chapter 5

Generalization of Multiple Zeta Values (II): Results for general totally real fields

The general construction of the generalized multiple zeta value $Z_I(\Gamma, S)$ defined over $\mathbb{Q}$ has a natural relation to classical MZVs. In this chapter, we will turn to the situation where $\mathbb{Q}$ is replaced by a totally real field.

Throughout this chapter, we will fix an arbitrary-chosen totally real field F with the degree $[F:\mathbb{Q}] = r$ bigger than 1. We will see that $Z_I(\Gamma, S)$ is highly non-trivial, which means that certain non-trivial iterated integrals will make their appearance.

At the first place, let us compare the concrete expression of $Z_I(\Gamma, S)$ associated to the same graph in case of $\mathbb{Q}$ and of F. Let the graph Γ_1 given as in Figure 5.1. Moreover, the subdivision map k and the sign ν are also fixed.

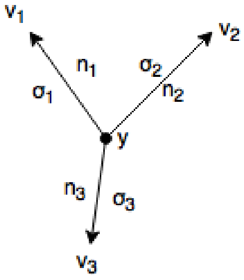


Figure 5.1: A plane trivalent tree with one internal vertex

Recall the definition

$$\mathscr{F}_{I,\nu,\Gamma,S}(\{x_v\}_{v\in S}) = (O_{F,+}^{\times} : U)^{-1} \int_{U_{\mathbb{R}}/U} G_{I,\nu,\Gamma,S}(\{x_v\}_{v\in S}, u) d^{\times}u,$$

and the generalized multiple zeta value associated to (Γ, S) is

$$Z_{I,\nu}(\Gamma, S) = \mathscr{F}_{I,\nu,\Gamma,S}(\{0\}_{v\in S}),$$

where $d^{\times}u = \frac{du_1\cdots du_{r-1}}{u_1\cdots u_{r-1}}$, $\prod_{j=1}^{r} u_j = 1$ and $U_{\mathbb{R}}/U (\cong BU)$ is the classifying space of the group $U = \mathbb{Z}^{r-1}$.

$$BU \cong (S^1)^{r-1},$$

the isomorphism is independent of the choice of U.

The calculation of the integral $\int_{U_{\mathbb{R}}/U}$ plays an important role to get the precise expression of our $Z_{I,\nu}(\Gamma, S)$.

The ground field is $\mathbb{Q}$. Then we obtain that

$$Z_{I,\nu}(\Gamma_1, \partial\Gamma_1) = \sum_{\substack{n_1+n_2+n_3=0\\ n_i\in\mathbb{Z}\\ n_1,n_2>0}} \frac{1}{|n_1|^{\sigma_1}|n_2|^{\sigma_2}|n_3|^{\sigma_3}} = \sum_{n_1,n_2\in\mathbb{N}\smallsetminus\{0\}} \frac{1}{n_1^{\sigma_1}n_2^{\sigma_2}(n_1+n_2)^{\sigma_3}}.$$

In fact, the integral $\int_{U_{\mathbb{R}}/U}$ is trivial in this case.

The ground field is a general totally real field F with $[F:\mathbb{Q}] = r > 1$. Then

$$\mathscr{F}_{I,\nu,\Gamma_1,\partial\Gamma_1}(\{x_v\}_{v\in\partial\Gamma_1}) = (O_{F,+}^{\times}:U)^{-1}\int_{U_{\mathbb{R}}/U} G_{I,\nu,\Gamma_1,\partial\Gamma_1}(\{x_v\}_{v\in\partial\Gamma_1}, u)d^{\times}u,$$

$$\mathscr{F}_{I,\nu,\Gamma_1,\partial\Gamma_1}(\{0\}_{v\in\partial\Gamma_1}) = (O_{F,+}^{\times}:U)^{-1}\sum_{\substack{(n_1,n_2)\in(I^*\smallsetminus\{0\})^2/U\\ n_1+n_2+n_3=0}}\int_{U_{\mathbb{R}}}\frac{\mathrm{d}^{\times}u}{\|n_1u\|^{r\sigma_1}\|n_2u\|^{r\sigma_2}\|n_3u\|^{r\sigma_3}}.$$

The generalized multiple zeta value associated to $(\Gamma_1, \partial\Gamma_1)$ is

$$Z_{I,\nu}(\Gamma_1,\partial\Gamma_1) = (O_{F,+}^{\times}:U)^{-1}\sum_{(n_1,n_2)\in(I^*\smallsetminus\{0\})^2/U}\int_{U_{\mathbb{R}}}\frac{\mathrm{d}^{\times}u}{\|n_1u\|^{r\sigma_1}\|n_2u\|^{r\sigma_2}\|(n_1+n_2)u\|^{r\sigma_3}}. \tag{5.1}$$

For studying the generalized MZV $Z_{I,\nu}(\Gamma_1, \partial\Gamma_1)$, we need to understand integral

$$\int_{U_{\mathbb{R}}}\frac{\mathrm{d}^{\times}u}{\|n_1u\|^{r\sigma_1}\|n_2u\|^{r\sigma_2}\|(n_1+n_2)u\|^{r\sigma_3}}.$$

In the following paragraph, we will fix a graph Γ in a general position with no internal vertices of 2-valency and a subset of vertices S. In fact, the contribution of 2-valency internal vertices can be covered by the subdivision map. Then the generalized multiple zeta value associated to Γ is

$$Z_{I,\nu}(\Gamma,S) = (O_{F,+}^{\times}:U)^{-1}\sum_{\substack{(n_e)\in(I^*\smallsetminus\{0\})^{|E(\Gamma)|}/U\\ \pi_v=0,\forall v\in V(\Gamma)\smallsetminus S}}\int_{U_{\mathbb{R}}}\frac{\mathrm{d}^{\times}u}{\prod_e\|n_eu\|^{r\sigma_e}},$$

where

$$\pi_v = \sum_{e\in E(\Gamma),v_1(e)=v} n_e - \sum_{e\in E(\Gamma),v_0(e)=v} n_e.$$

Definition 5.0.2 *We define the* ***basic integral*** *associated to the generalized multiple zeta value $Z_{I,\nu}(\Gamma, S)$ as follows*

$$\mathbb{I}(q,r;n_j,\sigma_j;\Gamma,S) = \int_{U_{\mathbb{R}}}\left(\prod_{j=1}^{q}\|n_ju\|^{-r\sigma_j}\right)\mathrm{d}^{\times}u, \tag{5.2}$$

where σ_j is a positive integer given by the subdivision map and the number q is an integer depending on (Γ, S),

$$q = |E(\Gamma)|.$$

Therefore we can rewrite the generalized multiple zeta value

$$Z_{I,\nu}(\Gamma,S)=(O_{F,+}^{\times}:U)^{-1}\sum_{\substack{(n_j)\in(I^*\smallsetminus\{0\})^q/U\\ \pi_v=0,\forall v\in V(\Gamma)\smallsetminus S}}\mathbb{I}(q,r;n_j,\sigma_j;\Gamma,S).$$

Remark 5.0.3 *Let us see one example of a graph with* 2*-valency internal vertices. We reconsider Example (4.2.5) for graph* $\widehat{\Gamma}$ *(4.5) when* $\mathbb{Q}$ *is replaced by* F *.*

$$Z_{I,\nu}(\widehat{\Gamma},\partial\widehat{\Gamma})=(O_{F,+}^{\times}:U)^{-1}\int_{U_{\mathbb{R}}/U}\sum_{\substack{n_k,m_l\in(I^*\smallsetminus\{0\})/U\\ \mathrm{sgn}(n_k)=(-1)^{\nu_k}\\ \mathrm{sgn}(m_l)=(-1)^{\mu_l}\\ n_1+n_2+m_1-m_6=0\\ m_l=m_{l+1},1\le l\le 5}}\frac{1}{\prod_{l=1}^{6}\|m_lu\|^r\cdot\|n_1u\|^r\cdot\|n_2u\|^r}\mathrm{d}^{\times}u.$$

From $m_l=m_{l+1},1\le l\le 5$*, we can actually rewrite*

$$Z_{I,\nu}(\widehat{\Gamma},\partial\widehat{\Gamma})=(O_{F,+}^{\times}:U)^{-1}\sum_{\substack{n_1,m_1\in(I^*\smallsetminus\{0\})/U\\ \mathrm{sgn}(n_1)=(-1)^{\nu_1}\\ \mathrm{sgn}(m_1)=(-1)^{\mu_1}}}\int_{U_{\mathbb{R}}/U}\frac{1}{\|m_1u\|^{6r}\cdot\|n_1u\|^{2r}}\mathrm{d}^{\times}u.$$

Remark 5.0.4 *The work to understand integrals mentioned above will be done for totally real field* F*. For a general number field* K *of degree* r_1+2r_2*,*

$$K_{\mathbb{R}}\cong\mathbb{R}^{r_1}\times\mathbb{C}^{r_2},$$

one can use polar coordinates on each factor of $\mathbb{C}^{r_2}$*. This reduces the integral (5.2) to an analogous integral for totally real fields of degree* r_1+r_2*.*

5.1 The Hecke transform

As we have seen in the previous introduction, the integral $\int_{U_{\mathbb{R}}/U}$ contributes to the expression of our generalized multiple zeta values. Moreover, the U-invariance of $G_{I,\nu,\Gamma,S}(\{x_v\}_{v\in S},u)$ enables us to firstly consider the integral $\int_{U_{\mathbb{R}}}$, more precisely the integral as (5.2). The calculation of the basic integral necessitates the Hecke transform.

Definition-Proposition 5.1.1 ***(The Hecke transform)[16]***
Let $U_{\mathbb{R}}\subset(\mathbb{R}_+^{\times})^r$ *be the subgroup*

$$U_{\mathbb{R}}=\{u=(u_1,\dots,u_r)\in(\mathbb{R}_+^{\times})^r|\prod_{j=1}^{r}u_j=1\}$$

with the measure $d^{\times}u=\frac{du_1}{u_1}\wedge\cdots\wedge\frac{du_{r-1}}{u_{r-1}}$*. Let* $\|\cdot\|$ *be the Euclidean norm on* $\mathbb{C}^r$*, on which* $U_{\mathbb{R}}$ *acts by multiplication. Let* $(p)_j\in\mathbb{Z}^r$*,* $p=\sum p_j$*. Then for any* $x=(x_1,\dots,x_r)\in(\mathbb{C}^{\times})^r$ *and* $s\in\mathbb{C}$*,* $Re(s)>0$*,* ***the Hecke transform***

$$\int_{U_{\mathbb{R}}}\|ux\|^{-2s}\prod_j u_j^{-2p_j}d^{\times}u=\frac{2^{1-r}}{r\Gamma(s)}\prod_j\Gamma(\frac{p+s}{r}-p_j)|x_j|^{2(p_j-(p+s)/r)}.$$

Proof 5.1.0.3 *Firstly we will give a more general set-up. Change of variable:*

$$H_+^{n-1} = \{u = (u_1, \cdots, u_n) | u_j > 0, u_1 \cdots, u_n = 1\}$$

$$H_+^{n-1} \times \mathbb{R}_+ \simeq \mathbb{R}_+^{n-1}; \big((u_1, \cdots, u_n), t\big) \longrightarrow x = (x_1, \cdots, x_n) = (tu_1, \cdots, tu_n).$$

$$x_1 \cdots x_n = t^n, \quad t = (x_1 \cdots x_2)^{1/n}, \quad u_i = \frac{x_i}{t} = \frac{x_i}{(x_1 \cdots x_n)^{1/n}},$$

$$\sum_1^n \frac{du_i}{u_i} = 0,$$

then,

$$dx_1 \wedge \cdots \wedge dx_n = \frac{dx_1 \wedge \cdots \wedge dx_n}{u_1 \cdots u_n} = \frac{dx_1}{u_1} \wedge \cdots \wedge \frac{dx_n}{u_n} = \big(t\frac{du_1}{u_1} + dt\big) \wedge \cdots \big(t\frac{du_n}{u_n} + dt\big),$$

$$dx_1 \wedge \cdots \wedge dx_n = t^n \wedge_{i=1}^n \frac{du_i}{u_i} + t^{n-1} \sum_{i=1}^n \big(du_1/u_1 \wedge \cdots \wedge \frac{du_{i-1}}{u_{i-1}} \wedge dt \wedge \frac{du_{i+1}}{u_{i+1}} \wedge \cdots \wedge \frac{du_n}{u_n}\big) = \omega \wedge nt^{n-1} dt.$$

where

$$\omega = (-1)^{n-i} \frac{du_1 \wedge \cdots \wedge \widehat{\mathrm{u}_\mathrm{i}} \wedge \cdots \wedge du_n}{u_1 \cdots \widehat{u_i} \cdots u_n}$$

ω is independent of i, so we fix a standard form

$$\mathrm{d}^\times u = \omega = \frac{du_1 \wedge \cdots \wedge du_{n-1}}{u_1 \cdots u_{n-1}}$$

then for every function with n-variables, we have

$$\int_{\mathbb{R}_+^n} g(x_1, \cdots, x_n) dx_1 \cdots dx_n = \int_{\mathbb{R}_+} \big(\int_{H_+^{n-1}} g(tu_1, \cdots, tu_{n-1}) \frac{du_1 \cdots du_{n-1}}{u_1 \cdots u_{n-1}} \big) nt^{n-1} dt.$$

Recall the following Γ-integrals: $\lambda, a, b > 0$, and $Re(s) > 0$ and $d^\times \lambda = \mathrm{d}\lambda/\lambda$,

$$\Gamma(s) a^{-s} = \int_{\mathbb{R}_+} \mathrm{e}^{-a\lambda} \lambda^s d^\times \lambda = 2 \int_{\mathbb{R}_+} \mathrm{e}^{-ax^2} x^{2s} \mathrm{d}^\times x,$$

where $\lambda = x^2$. Moreover $a = b^2$, we get

$$\frac{\Gamma(s/2)}{2b^s} = \int_{\mathbb{R}_+} \mathrm{e}^{-(bx)^2} x^s \mathrm{d}^\times x.$$

Now if we take $z = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n$, $\|z\|^2 = \sum_{j=1}^n |z_j|^2$, $u = (u_1, \cdots, u_n) \in H_+^{n-1}, uz = (u_1 z_1, \cdots, u_n z_n)$,

$$\Gamma(s) \|uz\|^{-2s} = \int_{\mathbb{R}_+} \mathrm{e}^{-\lambda \|uz\|^2} \lambda^s \mathrm{d}^\times \lambda = 2 \int_{\mathbb{R}_+} \mathrm{e}^{-\|tuz\|^2} t^{2s} \mathrm{d}^\times t$$

where we take $\lambda = t^2$ for the last equality. if $(a_1, \cdots, a_n) \in \mathbb{Z}^n$, $a = \frac{a_1 + \cdots + a_n}{n}$,

$$I(s) = \frac{\Gamma(s)}{2} \int_{H_+^{n-1}} \frac{u_1^{a_1} \cdots u_n^{a_n}}{\|uz\|^{2s}} \frac{du_1 \cdots du_{n-1}}{u_1 \cdots u_{n-1}} = \int_{H_+^{n-1} \times \mathbb{R}_+} u_1^{a_1} \cdots u_n^{a_n} \mathrm{e}^{-\|tuz\|^2} t^{2s} \frac{du_1 \cdots du_{n-1}}{u_1 \cdots u_{n-1}} \frac{dt}{t}.$$

$$I(s) = \int_{\mathbb{R}_+^n} \frac{x_1^{a_1}\cdots x_n^{a_n}}{(x_1\cdots x_n)^{(a_1+\cdots+a_n)/n}} \mathrm{e}^{-\|xz\|^2} (x_1\cdots x_n)^{2s/n} \frac{dx_1\cdots dx_n}{nx_1\cdots x_n}$$

$$I(s) = \frac{1}{n}\prod_{j=1}^{n} \int_{\mathbb{R}_+} x_j^{a_j-\frac{a_1+\cdots+a_n}{n}+\frac{2s}{n}} \mathrm{e}^{-|z_j|^2 x_j^2} \mathrm{d}^\times x_j = \frac{1}{n}\prod_{j=1}^{n}\frac{1}{2}\frac{\Gamma(\frac{s}{n}+\frac{a_j-a}{2})}{|z_j|^{a_j-a+\frac{2s}{n}}}.$$

In conclusion: $(a_1,\ldots,a_n) \in \mathbb{Z}^n, z \in \mathbb{C}, a = (a_1+\cdots+a_n)/n,\ \forall j, a_j - a + Re(2s/n) > 0,$ *we have the Hecke transform*

$$\int_{H_+^{n-1}} \frac{u_1^{a_1}\cdots u_n^{a_n}}{\|uz\|^{2s}} \frac{du_1\cdots du_{n-1}}{u_1\cdots u_{n-1}} = \frac{2^{1-n}}{n\Gamma(s)} \prod_{j=1}^{n} \frac{\Gamma(\frac{s}{n}+\frac{a_j-a}{2})}{|z_j|^{a_j-a+\frac{2s}{n}}}.$$

Then we get what we want to prove by taking $a_i = -2p_i$.

Remark 5.1.2 *This formula works also for any* $(a_1,\cdots,a_n) \in \mathbb{C}^n$ *and* $s \in \mathbb{C}$ *such that* $\forall j = 1,\cdots,n,\ 2Re(s) + na_j > a_1 + \cdots + a_n$.

Example 5.1.3 *First of all, we will see one special example, namely the Hecke transform of our plectic Green current*

$$g_I^\nu(x,u) = \lim_{\delta\to 0^+} \sum_{n\in I^*\setminus\{0\}} \mathrm{sgn}(n)^\nu \frac{e^{2\pi i \mathrm{Tr}(nx)}}{\|un\|^{r+\delta}}, \quad x \in F_{\mathbb{R}}/I, u \in U_{\mathbb{R}}.$$

Theorem 5.1.4 (Hecke's formula) *If* $U \subset O_{F,+}^\times$ *is a subgroup of finite index and if* $\forall \epsilon \in U$, $(\epsilon - 1)\cdot x \in I$, *which is equivalent to*

$$xI \in (F_{\mathbb{R}}/I)^U,$$

then

$$\forall \epsilon \in U, \quad g_I^\nu(x,\epsilon u) = g_I^\nu(x,u).$$

Therefore we get

$$\int_{U_{\mathbb{R}}/U} g_I^\nu(x,u)du = \lim_{\delta\to 0^+} \sum_{n\in(I^*\setminus\{0\})/U} e^{2\pi i\mathrm{Tr}(nx)} \int_{U_{\mathbb{R}}} \frac{1}{\|un\|^{r+\delta}} du.$$

By the Hecke transform, we obtain

$$\int_{U_{\mathbb{R}}/U} g_I^\nu(x,u)du = \frac{2^{r-1}\Gamma(1/2)^r}{\Gamma(r/2)} \lim_{\delta\to 0^+} \sum_{n\in(I^*\setminus\{0\}/U)} \mathrm{sgn}(n)^\nu \frac{e^{2\pi i\mathrm{Tr}(nx)}}{\prod_{j=1}^r |n_j|^{(r+\delta)/r}}.$$

Note that

$$N(n) = N_{F/\mathbb{Q}}(n) = \prod_{j=1}^{r} n_j,$$

hence

$$\int_{U_{\mathbb{R}}/U} g_I^\nu(x,u)du = \frac{2^{1-r}\Gamma(1/2)^r}{r\Gamma(r/2)} \lim_{\delta\to 0^+} \sum_{n\in(I^*\setminus\{0\})/U} \mathrm{sgn}(n)^\nu \frac{e^{2\pi i\mathrm{Tr}(nx)}}{|N(n)|^{(r+\delta)/r}}.$$

It is esay to see that the Hecke transform of the basic plectic Green function with signature delivers a linear combination of special values $L(1,\chi_F)$ *for certain Dirichlet characters* χ_F *of* F *of signature* ν.

Remark 5.1.5 *From the above theorem, we can see that the Hecke transform can give a natural reinterpretation of Hecke's formula. That is why such a formula is called by Nekovář and Scholl the Hecke transform.*

For any given graph Γ and given multiple-signs ν, the Hecke transform of the higher plectic Green function $G_{I,\nu,\Gamma,S}(\{x_v\}_{v\in S}, u)$ is quite interesting. More revelent details will be presented in the next section.

5.2 Reciprocity formula for the Hecke transform

From the expression of the Hecke transform, we observe that when the degree of F is bigger, the higher plectic Green function(current) associated to a given graph becomes more complicated. However we can always reduce to the case that the rank of the given graph is smaller than the degree of F, due to the duality given by the following formula.

Notation 5.2.1 *Let $L_1(u), \dots, L_q(u)$ be linear forms, $u = (u_1, \dots, u_r)$*

$$L_i(u) = \sum_{j=1}^{r} L_{ij}u_j, \quad L_{ij} > 0, \quad 1 \le i \le q.$$

The dual linear forms

$$L_j^*(t) = \sum_{i=1}^{q} L_{ij}t_i.$$

such that

$$\sum_{i=1}^{q} L_i(u)t_i = \sum_{j=1}^{r} L_j^*(t)u_j.$$

Definition 5.2.2 *Recall that $U_{\mathbb{R}} = \{u = (u_1, \dots, u_r) \in (\mathbb{R}_+^\times)^r | u_1 \cdots u_r = 1, u_j > 0\}$, we define*

$${}_qI_r(\{L_i\}, \{\alpha_i\}, \{\beta_j\}) = \int_{U_{\mathbb{R}}} \prod_{i=1}^{q} L_i(u)^{-\alpha_i} \prod_{j=1}^{r} u_j^{\beta_j} d^\times u.$$

Proposition 5.2.3 (Reciprocity formula for the Hecke transform) *[16]*

$$\frac{1}{q}\left(\prod_{i=1}^{q}\Gamma(\alpha_i)\right)\left({}_qI_r(\{L_i\}, \{\alpha_i\}, \{\beta_j\})\right) = \frac{1}{r}\left(\prod_{j=1}^{r}\Gamma(\beta_j^*)\right)\left({}_rI_q(\{L_j^*\}, \{\beta_j^*\}, \{\alpha_i^*\})\right),$$

where

$$\alpha_i^* = \alpha_i, \quad \beta_j^* = \beta_j + \frac{|\alpha| - |\beta|}{r}, \quad |\alpha| = \sum_{i=1}^{q}\alpha_i, \quad |\beta| = \sum_{j=1}^{r}\beta_j.$$

Example 5.2.4 *If $q = 1$, $L_1(u) = \sum_{j=1}^{r} l_j u_j$, $l_j > 0$. $\beta_j^* = \beta_j + \frac{\alpha - |\beta|}{r}$; $L_j^*(t) = l_j t$. Then by Reciprocity Formula (5.2.3), we obtain*

$$\int_{U_{\mathbb{R}}} L_1(u)^{-\alpha} \prod_{j=1}^{r} u_j^{\beta_j} d^\times u = \frac{1}{r\Gamma(\alpha)} \prod_{j=1}^{r} \frac{\Gamma(\beta_j^*)}{l_j^{\beta_j^*}}.$$

If $\beta_j = 0, \forall 1 \le j \le r$, then

$$\int_{U_{\mathbb{R}}} L_1(u)^{-\alpha} d^\times u = \frac{\Gamma(\alpha/r)^r}{r\Gamma(\alpha)} (l_1 \cdots l_r)^{-\alpha/r}.$$

Example 5.2.5 *If $q = 2$ and $\beta_j = 0$ for all j, then $\beta_j^* = \frac{\alpha_1+\alpha_2}{r}$. The reciprocity formula gives us*

$$\int_{U_{\mathbb{R}}} L_1(u)^{-\alpha_1} L_2(u)^{-\alpha_2} d^\times u = \frac{2\Gamma(\frac{\alpha_1+\alpha_2}{2})^r}{r\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{\mathbb{R}_+} \left(\prod_{j=1}^{r} (L_{1j}s + L_{2j}s^{-1})^{-\frac{\alpha_1+\alpha_2}{r}}\right) s^{\alpha_1-\alpha_2} d^\times s.$$

5.2.1 Reciprocity and Higher Plectic Green functions.

Now we will fix a graph Γ in a general position. We will explain how to apply the reciprocity formula to study higher plectic Green functions $G_{I,\nu,\Gamma,S}(\cdot,\cdot)$ and the generalized multiple zeta value $Z_{I,\nu}(\Gamma,S)$.

Notation 5.2.6 *Let* $\lambda^{(i)}=(\lambda_1^{(i)},\cdots,\lambda_r^{(i)})\in\mathbb{R}_+^r$, $1\le i\le q$. *let us define* d *linear forms*

$$L_i(u)=\sum_{j=1}^{r}(\lambda_j^{(i)})^2u_j,\quad 1\le i\le q,$$

and r *dual linear forms of* L_i

$$L_j^*(s)=\sum_{i=1}^{q}(\lambda_j^{(i)})^2s_i,\quad 1\le j\le r.$$

$$L_i(u^2)=\sum_{j=1}^{r}(\lambda_j^{(i)})^2u_j^2=\|\lambda^{(i)}u\|^2,$$

where $\|\cdot\|$ *is the standard Euclidean norm.*

Then the basic integral (5.0.2) associated to $Z_{I,\nu}(\Gamma,S)$

$$\mathbb{I}(q,r;\lambda^{(i)},\sigma_i;\Gamma,S)=\int_{U_{\mathbb{R}}}\prod_{i=1}^{q}\|\lambda^{(i)}u\|^{-r\sigma_i}d^\times u=\int_{U_{\mathbb{R}}}\prod_{i=1}^{q}L_i(u^2)^{-r\sigma_i/2}d^\times u.$$

The generalized MZV (5.1) is just one typical example with $q=3$.

Formula 5.2.7 *Since* $d^\times(u^2)=2^{r-1}d^\times u$*, then*

$$\mathbb{I}(q,r;\lambda^{(i)},\sigma_i;\Gamma,S)=2^{1-r}\int_{U_{\mathbb{R}}}\prod_{i=1}^{q}L_i(u)^{-r\sigma_i/2}d^\times u.$$

Again by Reciprocity Formula (5.2.3), we obtain

$$\int_{U_{\mathbb{R}}}\prod_{i=1}^{q}\|\lambda^{(i)}u\|^{-r\sigma_i}d^\times u=\frac{q}{2^{r-1}r}\frac{\left(\Gamma(\frac{\sigma_1+\cdots+\sigma_q}{2})\right)^r}{\prod_{i=1}^{d}\Gamma(r\sigma_i/2)}\int_{\substack{s_1\cdots s_q=1\\ s_i>0}}\left(\prod_{j=1}^{r}L_j^*(s)\right)^{-\frac{\sigma_1+\cdots+\sigma_q}{2}}\prod_{i=1}^{q}s_i^{r\sigma_i/2}d^\times s.$$

Example 5.2.8 $q=1$. *Let* $\lambda=(\lambda_1,\cdots,\lambda_r)$*, then*

$$\int_{U_{\mathbb{R}}}\|\lambda u\|^{-r\sigma}d^\times u=\frac{\Gamma(\sigma/2)^r}{2^{r-1}r\Gamma(r\sigma/2)}\frac{1}{(\lambda_1\cdots\lambda_r)^\sigma}=\frac{\Gamma(\sigma/2)^r}{2^{r-1}r\Gamma(r\sigma/2)}N(\lambda)^{-\sigma}.$$

This result has already been seen in Theoem 5.1.4 of the Hecke formula.

Example 5.2.9 *(1)* $q=2$, $L_j^*(s)=(\lambda_j^{(1)})^2s+(\lambda_j^{(2)})^2s^{-1}$.

$$\int_{U_{\mathbb{R}}}\|\lambda^{(1)}u\|^{-r\sigma_1}\|\lambda^{(2)}u\|^{-r\sigma_2}d^\times u=$$

$$\frac{\Gamma(\frac{\sigma_1+\sigma_2}{2})}{2^{r-2}r\Gamma(r\sigma_1/2)\Gamma(r\sigma_2/2)}\int_{\mathbb{R}_+}\left(\prod_{j=1}^{r}((\lambda_j^{(1)})^2s+(\lambda_j^{(2)})^2s^{-1})\right)^{-\frac{\sigma_1+\sigma_2}{2}}s^{r(\sigma_1-\sigma_2)/2}d^\times s.$$

(2) If $r = 2$, F *is a real quadratic field.*

$$\int_{U_{\mathbb{R}}} \prod_{i=1}^{q} \|\lambda^{(i)} u\|^{-2\sigma_i} d^\times u = \frac{1}{2} \int_{U_{\mathbb{R}}} \prod_{i=1}^{q} L_i(u)^{-\sigma_i} d^\times u = \int_{\mathbb{R}_+} \prod_{i=1}^{q} \left((\lambda_1^{(i)})^2 u + (\lambda_2^{(i)})^2 u^{-1} \right)^{-\sigma_i} d^\times u.$$

From the two cases above, it is easy to observe the duality between $(q = 2, r)$ *and* $(q, r = 2)$*, by the explicit expression of the two integrals. Therefore we can easily interchange* q *and* r *for getting the case* (q, r) *with* $q \geq r$*.*

Moreover, we will explain in the next paragraph that we only finally reduce to the case of $q = r$, thanks to a formula of Sczech which generalizes Eisenstein's trick for GL_n.

5.3 Sczech's rational functions and Eisenstein cocycles

5.3.1 Introduction

R. Sczech [18] constructed a group cocycle Ψ on the unimodular group $GL_n\mathbb{Z}$, called the Eisenstein cocycle, which represents a nontrivial cohomology class in $H^{n-1}(GL_n\mathbb{Z}, M)$ with values in a function space M. Restricting Ψ on some subgroup W of totally positive units and evaluating the elements of M on W-invariant points, one obtain a sequence of rational cohomology classes $\eta(b, f; s) \in H^{n-1}(W, \mathbb{Q}), s = 1, 2, \ldots$ By evaluation on some fundamental cycle in $H_{n-1}(W, \mathbb{Z})$, these rational classes $\eta(b, f; s)$ give rise to the numbers of partial zeta function associated with the ray class b mod f

$$\zeta(b, f; 1 - s) = \sum_{a \equiv b(f)} N(a)^{-(1-s)}, \quad Re(s) > 1,$$

where a runs over all integral ideals in the ring of integers O_F of a totally real field F, such that ab^{-1} is a principal ideal generated by a totally positive number in the coset $1 + fb^{-1}$. The Eisenstein cocycle Ψ is universal in the sense that it parametrizes all the special values of Hecke L-functions in every totally real number field of degree r, which are known to be either an algebraic number or an algebraic number times a power of π.

Sczech's construction begins with the rational function

$$f(\xi)(x) = \frac{det(\xi^{(1)}, \cdots, \xi^{(r)})}{(x, \xi^{(1)}) \cdots (x, \xi^{(r)})},$$

where $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$ is a row vector and $\xi^{(i)} \in \mathbb{R}^r$ are r nonzero column vectors and $(x, \xi^{(i)}) = \sum x_j \xi_j^{(i)}$. This function is well-defined outside the hyperplanes $(x, \xi^{(i)}) = 0$. Given a homogenous polynomial $P(X_1, \cdots, X_r)$, we apply the differential operator $P(-\partial_{x_1}, \cdots, -\partial_{x_r})$ to $f(\xi)(x)$.

Definition 5.3.1 *We define the general rational function*

$$f(\xi)(P, x) = P(-\partial_{x_1}, \cdots, -\partial_{x_r}) f(\xi)(x),$$

where ∂_{x_j} *denotes the partial derivative with respect to the variable* x_j*.*

Sczech gave several elementary observations about this rational function.

Lemma 5.3.2 *[18]*

$$\sum_{i=0}^{r}(-1)^i f(\xi^{(0)},\cdots,\widehat{\xi^{(i)}},\cdots,\xi^{(r)}) = 0.$$

For $A \in GL_r(\mathbb{R})$,

$$f(A\xi^{(1)},\cdots,A\xi^{(r)})(x) = det(A)f(\xi^{(1)},\cdots,\xi^{(r)})(xA),$$

$$Af(\xi) = f(A\xi).$$

Formula 5.3.3

$$P(-\partial_{x_1},\cdots,-\partial_{x_r})\sum_{i=0}^{r}(-1)^i f(\xi^{(0)},\cdots,\widehat{\xi^{(i)}},\cdots,\xi^{r})(x) = 0,$$

therefore

$$\sum_{i=0}^{r}(-1)^i f(\xi^{(0)},\cdots,\widehat{\xi^{(i)}},\cdots,\xi^{(r)})(P,\cdot) = 0.$$

In fact, we will recover the Eisenstein trick from the above formulas in the case $r = 2$. Now we will explain the relation between Sczech's rational function $f(\xi)(x)$ and our higher plectic Green function $G_{I,\nu,\Gamma,S}(\{x_v\}_{v\in S},u)$, and will explain how to reduce the following integral to the case of $q = r$

$$\int_{U_{\mathbb{R}}}\prod_{i=1}^{q}\|\lambda^{(i)}u\|^{-r\sigma_i}d^{\times}u = 2^{1-r}\int_{U_{\mathbb{R}}}\prod_{i=1}^{q}L_i(u)^{-r\sigma_i/2}d^{\times}u, \tag{5.3}$$

as we have mentioned at the end of previous section.

We always assume that $r\sigma_i/2 \in \mathbb{N}$ **for all** $1 \leq i \leq d$**.** So if $r \in 2\mathbb{N}$, then $\sigma_i \in \mathbb{N}$; if $r \in 2\mathbb{N}+1$, then $\sigma_i \in 2\mathbb{N}\setminus\{0\}$.

Before that, we will give a more general definition of Sczech's rational function.

Definition 5.3.4 *Let* $\underline{a} = (a_1,\cdots,a_r) \in \mathbb{N}_+^r$, $b \in \mathbb{N}_+$. *We define*

$$f(\xi)(x;\underline{a},b) = \frac{(det(\xi^1,\cdots,\xi^r))^b}{(x,\xi^1)^{a_1}\cdots(x,\xi^r)^{a_r}}.$$

In fact, we can show that $f(\xi)(x;\underline{a},b)$ can be realized as $f(\xi)(P_l,x)$ for a carefully chosen homogenous polynomial P_l.

First of all, we will consider the simple case when $r = 2$.

Example 5.3.5 *We define two homogenous polynomials of degree 2.*

$$P_{1;\xi}(X_1,X_2) = \xi_2^{(2)}X_1 - \xi_1^{(2)}X_2,$$

$$P_{2;\xi}(X_1,X_2) = \xi_1^{(1)}X_2 - \xi_2^{(1)}X_1.$$

Then

$$f(\xi)(P_{1;\xi},x) = P_{1;\xi}(-\partial_{x_1},-\partial_{x_2})f(\xi)(x) = \frac{(det(\xi^{(1)},\xi^{(2)}))^2}{(x,\xi^{(1)})^2(x,\xi^{(2)})} = f(\xi)(x;(2,1),2),$$

$$f(\xi)(P_{2;\xi},x)=P_{2;\xi}(-\partial_{x_1},-\partial_{x_2})f(\xi)(x)=\frac{\left(det(\xi^{(1)},\xi^{(2)})\right)^2}{(x,\xi^{(1)})(x,\xi^{(2)})^2}=f(\xi)(x;(1,2),2).$$

Therefore we obtain that

$$\frac{P_{1;\xi}^{a_1}(-\partial_{x_1},-\partial_{x_2})}{a_1!}\frac{P_{2;\xi}^{a_2}(-\partial_{x_1},-\partial_{x_2})}{a_2!}f(\xi)(x)=\frac{(det(\xi^1,\xi^2))^{a_1+a_2+1}}{(x,\xi^1)^{a_1+1}(x,\xi^2)^{a_2+1}}=f(\xi)(x;(a_1{+}1,a_2{+}1),a_1{+}a_2{+}1).$$

If we denote

$$P_{a_1-1,a_2-1;\xi}(X_1,X_2)=\frac{P_1^{a_1-1}}{(a_1-1)!}\frac{P_2^{a_2-1}}{(a_2-1)!},$$

then

$$f(\xi)(x;(a_1,a_2),a_1+a_2-1)=P_{a_1-1,a_2-1;\xi}(-\partial_{x_1},\partial_{x_2})f(\xi)(x)=f(\xi)(P_{a_1-1,a_2-1;\xi},x).$$

For general r. Given

$$\xi=(\xi^{(1)},\xi^{(2)},\cdots,\xi^{(r)})$$

the invertible matrix with columns $\xi^{(i)}$, then denote a new column vector $(y_1,\cdots,y_r)^\top$, where

$$y_l=\frac{det(\xi)}{(x,\xi^{(1)})\cdots(x,\xi^{(l)})^2\cdots(x,\xi^{(r)})}.$$

We have a linear equation

$$\xi\cdot(y_1,\cdots,y_r)^\top=(-\partial_{x_1}f(\xi)(x),\cdots,-\partial_{x_r}f(\xi)(x))^\top.$$

This equation has a unique solution, then we obtain a unique expression of each y_l, namely

$$(y_1,\cdots,y_r)^\top=\xi^{-1}\cdot(-\partial_{x_1}f(\xi)(x),\cdots,-\partial_{x_r}f(\xi)(x))^\top.$$

$$\xi^{-1}=\frac{1}{det(\xi)}C_\xi^\top,$$

where $C_\xi=\left(C_\xi(ij)\right)_{1\le i,j\le r}$ is the adjoint matrix of ξ.

Therefore there exist r homogenous polynomials of degree 1

$$P_{1;\xi}(X_1,\cdots,X_r),\ldots,P_{r;\xi}(X_1,\cdots,X_r),$$

where

$$P_{l;\xi}(X_1,\cdots,X_r)=\frac{C_\xi(1l)X_1+\cdots+C_\xi(rl)X_r}{det(\xi)},\quad 1\le l\le r.$$

which depend on the matrix ξ and such that

$$y_l=\frac{det(\xi)}{(x,\xi^1)\cdots(x,\xi^l)^2\cdots(x,\xi^r)}=P_{l;\xi}(-\partial_{x_1},\cdots,-\partial_{x_r})f(\xi)(x)=f(\xi)(P_{l;\xi},x),\quad\forall 1\le l\le r.$$

Continuing the calculation, denote

$$P(\underline{X};\underline{a};\xi)=\frac{P_{1;\xi}^{a_1-1}(\underline{X})}{(a_1-1)!}\frac{P_{2;\xi}^{a_2-1}(\underline{X})}{(a_2-1)!}\cdots\frac{P_{r;\xi}^{a_r-1}(\underline{X})}{(a_r-1)!},$$

where $\underline{X}=(X_1,\cdots,X_r)$. Hence we obtain

Formula 5.3.6

$$f(\xi)(x;\underline{a},1)=\frac{det(\xi)}{(x,\xi^{(1)})^{a_1}(x,\xi^{(2)})^{a_2}\cdots(x,\xi^{(r)})^{a_r}}=P(-\partial_{x_1},\cdots,-\partial_{x_r};\underline{a};\xi)f(\xi)(x).$$

Now we consider $r+1$ column vectors $\xi^{(0)},\ldots,\xi^{(r)}$. By Lemma 5.3.2,

$$(-1)^i f(\xi^{(0)},\cdots,\widehat{\xi^{(i)}},\cdots,\xi^{(r)})(x)=0.$$

Denote the new matrix

$$\xi^{\widehat{i}}=\left(\xi^{(0)},\cdots,\widehat{\xi^{(i)}},\cdots\xi^{(r+1)}\right)$$

and apply the differential operator

$$P(-\partial_{x_1},\cdots,-\partial_{x_r};\underline{a};\xi^{\hat{0}})$$

on

$$f(\xi^{(1)},\cdots,\xi^{(r)})(x)=\sum_{i=1}^{r}(-1)^{i+1}f(\xi^{(0)},\cdots,\widehat{\xi^{(i)}},\cdots,\xi^{(r)})(x).$$

then

$$\frac{det(\xi^{\hat{0}})}{(x,\xi^{(1)})^{a_1}(x,\xi^{(2)})^{a_2}\cdots(x,\xi^{(r)})^{a_r}}=f(\xi^{\hat{0}})(x;\underline{a},1)=P(-\partial_{x_1},\cdots,-\partial_{x_r};\underline{a};\xi^{\hat{0}})f(\xi^{(1)},\cdots,\xi^{(r)})(x),$$

hence we obtain

Lemma 5.3.7

$$\frac{det(\xi^{\hat{0}})}{(x,\xi^{(1)})^{a_1}(x,\xi^{(2)})^{a_2}\cdots(x,\xi^{(r)})^{a_r}}=\sum_{i=1}^{r}(-1)^{i+1}P(-\partial_{x_1},\cdots,-\partial_{x_r};\underline{a};\xi^{(\hat{0})})f(\xi^{(0)},\cdots,\widehat{\xi^{(i)}},\cdots,\xi^{(r)})(x),$$

where $\underline{a}=(a_1,\ldots,a_r)\in\mathbb{N}_+^r$.

The right hand side of the formula in Lemma 5.3.7 can be written as a finite linear combination of

$$(Constant)\times\frac{det(\xi^{(\hat{i})})}{(x,\xi^{(0)})^{b_1}\cdots(x,\widehat{\xi^{(i)}})^{b_i}\cdots(x,\xi^{(r)})^{b_r}},\quad i\geq 1,\quad b_i\in\mathbb{N}_+.$$

5.3.2 Sczech's rational function and higher plectic Green functions.

Now we denote $(\lambda^{(i)})^2=((\lambda_1^{(i)})^2,\cdots,(\lambda_r^{(i)})^2)^\top\in\mathbb{R}_+^r$ as a column vector and denote

$$\underline{\lambda}^2=((\lambda^{(1)})^2,\cdots,(\lambda^{(q)})^2)\in Mat_{r\times q}(\mathbb{R}_+)$$

and $u=(u_1,\cdots,u_r)\in\mathbb{R}_+^r$ with $u_1\cdots u_r=1$ as a row vector. Hence

$$L_i(u)=\sum_{j=1}^{r}u_j(\lambda_j^{(i)})^2=(u,(\lambda^{(i)})^2).$$

In general for the definition for our higher plectic Green function, we have

$$\prod_{i=1}^{q} L_i(u)^{-r\sigma_i/2} = \frac{1}{\prod_{i=1}^{q}(u,(\lambda^{(i)})^2)^{r\sigma_i/2}}.$$

If $q = r$,

$$\prod_{i=1}^{r} L_i(u)^{-r\sigma_i/2} = \frac{1}{\prod_{i=1}^{r}(u,(\lambda^{(i)})^2)^{r\sigma_i/2}} = det((\lambda^{(1)})^2,\cdots,(\lambda^{(r)})^2))^{-1}\left(f(\underline{\lambda}^2)(u;(r/2)\underline{\sigma},1)\right),$$

where $\underline{\sigma} = (\sigma_1,\cdots,\sigma_r)$.

Moreover if $\sigma_i = \sigma \quad (\forall 1 \le i \le q = r)$, then

$$\prod_{i=1}^{r} L_i(u)^{-r\sigma/2} = det((\lambda^{(1)})^2,\cdots,(\lambda^{(r)})^2))^{-r\sigma/2}\left(f(\underline{\lambda}^2)(u)\right)^r.$$

5.4 Two reduction theorems via Sczech's cycles

In this section we will explain two reduction theorems. By the reciprocity of q and r, we can assume that $q \ge r$. The first reduction theorem assures that we can reduce to the case $q = r$. Moreover the second reduction theorem tells us that we can only do the calculation for smallest exponents.

We will present two methods for the first reduction. We can do the reduction in a general setting, for arbitray linear forms $L_i(u) = \sum_{j=1}^{r} L_{ij}u_j, \quad 1 \le i \le q$ and for arbitary exponents $\underline{m} = (m_1,\ldots,m_q)$. However for now we will only focus on the reduction for $L_i(u) = (u,(\lambda^{(i)})^2)$ and for $\underline{m} = (r/2)\underline{\sigma}$ as in the begining of this section.

5.4.1 First Reduction

(1) Method I

ⓐ If $q = r+1$**.**

$$\prod_{i=1}^{r+1} L_i(u)^{-r\sigma_i/2} = \frac{1}{\prod_{i=1}^{r+1}(u,(\lambda^{(i)})^2)^{r\sigma_i/2}}.$$

By Lemma 5.3.7, we have the following relation

$$\frac{det(\xi^{\hat{1}})}{(x,\xi^2)^{a_2}(x,\xi^3)^{a_3}\cdots(x,\xi^{r+1})^{a_{r+1}}} = \sum_{i=2}^{r+1}(-1)^i P(-\partial_{x_1},\cdots,-\partial_{x_r};\underline{a};\xi^{\hat{1}})f(\xi^1,\cdots,\widehat{\xi^i},\cdots,\xi^{r+1})(x).$$

Take

$$\xi = \underline{\lambda}^2, \quad x = u = (u_1,\cdots,u_r), \quad \underline{a} = (r/2)\underline{\sigma^0} = (r/2)(\sigma_2,\cdots,\sigma_{r+1}).$$

then we obtain

$$\frac{det((\lambda^2)^2,\cdots,(\lambda^{r+1})^2))}{\prod_{i=2}^{r+1}(u,(\lambda^i)^2)^{r\sigma_i/2}} = \sum_{i=2}^{r+1}(-1)^i P(-\partial_{u_1},\cdots,-\partial_{u_r};(r/2)\underline{\sigma}^0;(\underline{\lambda}^2)^{\hat{1}})f((\lambda^1)^2,\cdots,\widehat{(\lambda^i)^2},\cdots,(\lambda^{r+1})^2)(u).$$

Therefore

$$\prod_{i=1}^{r+1} L_i(u)^{\frac{-r\sigma_i}{2}}$$

$$= \sum_{i=2}^{r+1}(-1)^i \frac{1}{det((\lambda^2)^2,\cdots,(\lambda^{r+1})^2))}\frac{P(-\partial_{u_1},\cdots,-\partial_{u_r};\frac{r}{2}\underline{\sigma}^0;(\underline{\lambda}^2)^{\hat{1}})f((\lambda^1)^2,\cdots,\widehat{(\lambda^i)^2},\cdots,(\lambda^{r+1})^2)(u)}{(u,(\lambda^1)^2)^{\frac{r\sigma_1}{2}}}.$$

Now combining with Formula 5.2.7 and 5.3, we have

$$\int_{U_{\mathbb{R}}}\prod_{i=1}^{r+1}\|\lambda^{(i)}u\|^{-r\sigma_i}d^\times u =$$

$$2^{1-r}\sum_{i=2}^{r+1}(-1)^i\int_{U_{\mathbb{R}}}\frac{P(-\partial_{u_1},\cdots,-\partial_{u_r};\frac{r}{2}\underline{\sigma}^0;(\underline{\lambda}^2)^{\hat{1}})f((\lambda^1)^2,\cdots,\widehat{(\lambda^i)^2},\cdots,(\lambda^{r+1})^2)(u)}{(u,(\lambda^1)^2)^{\frac{r\sigma_1}{2}}det((\lambda^2)^2,\cdots,(\lambda^{r+1})^2))}d^\times u.$$

It is easy to see that RHS is a finite linear combination of case $d = r$.

ⓑ **If** $q > r + 1$**.**

It is not difficult to obtain the same result by induction using $(q-r)$ times Lemma 5.3.7.

More precisely,

$$\frac{1}{\underbrace{(u,(\lambda^{(1)})^2)^{r\sigma_1}\cdots\underbrace{\underbrace{(u,(\lambda^{(q-r-1)})^2)^{r\sigma_{q-r-1}}\underbrace{(u,(\lambda^{(q-r)})^2)^{r\sigma_{q-r}}(u,(\lambda^{(q-r+1)})^2)^{r\sigma_{q-r+1}}\cdots(u,(\lambda^{(q)})^2)^{r\sigma_q}}_{\textcircled{1}}}_{\textcircled{2}}}_{\cdots}}_{\textcircled{p}}},$$

where $p = q-r$. The symbol ⓘ means the i-th use of Sczech's formula in Lemma 5.3.7.

For example ①:

$$\frac{1}{(u,(\lambda^{(q-r)})^2)^{r\sigma_{q-r}}\cdots(u,(\lambda^{(q)})^2)^{r\sigma_q}}$$

$$= \sum_{i_1=q-r+1}^{q}(-1)^{i_1+q-r+1}\frac{P(-\partial_{u_1},\cdots,-\partial_{u_r};r\underline{\sigma}^{(q-r)};(\underline{\lambda}^2)^{[q-r,q]})f((\lambda^{(q-r)})^2,\cdots,\widehat{(\lambda^{(i_1)})^2},\cdots,(\lambda^{(q)})^2)(u)}{(u,(\lambda^{(q-r)})^2)^{r\sigma_{q-r}}det((\lambda^{(q-r)})^2,\cdots,(\lambda^{(q)})^2)}$$

where

$$(\underline{\lambda}^2)^{[q-r,q]} = \Bigg((\lambda^{(q-r)})^2,\ldots,(\lambda^{(q)})^2\Bigg),$$

$$\underline{\sigma}^{(q-r)} = \big(\sigma_{q-r+1},\sigma_{q-r+2},\ldots,\sigma_q\big) \in \mathbb{N}_+^r.$$

It is easy to see that we can continue this process and obtain the conclusion, we have reduced the case from $q > r$ to $q = r$.

Remark 5.4.1 *Even though, this method can give a resolution for the reduction, it increases the complexity of calculation of the Hecke transform. Now we will introduce the second method, which simply bases on Sczech's cocycle relation.*

(2) Method II

Definition 5.4.2 *Let* $L_i(u) = \sum_j^r L_{ij} u_j, 1 \le i \le d$ *be linear forms defined as before. We define*

$$\underline{k} = (k_1, \cdots, k_q) \in \mathbb{N}^r.$$

Then we define

$$L(\underline{k}; q) = \frac{1}{\prod_{i=1}^d L_i(u)^{k_i}}.$$

If we denote $L_i = (L_{i1}, \cdots, L_{ir})^\top$ a column vector and denote $\underline{L} = (L_1, \cdots, L_q) \in Mat_{r \times q}(\mathbb{R})$ as a matrix. By Lemma 5.3.2, we have

Formula 5.4.3

$$L(\underline{1}^{\widehat{r+1}}(r+1); r+1) = \sum_{i=1}^r (-1)^{i-1} \frac{det(\underline{L}^{\hat{i}})}{det(\underline{L}^{\widehat{r+1}})} L(\underline{1}^{\hat{i}}(r+1); r+1),$$

where

$$\underline{k}^{\hat{i}}(r+1) = (k_1, \cdots, \widehat{k_i}, \cdots, k_{r+1}),$$

and

$$\underline{1} = (\underbrace{1, \cdots, 1}_{r+1}), \quad \underline{1}^{\hat{i}} = (1, \cdots, \underbrace{0}_{i^{th}}, \cdots, 1),$$

and

$$\underline{L}^{\hat{i}} = \left(L_1, \cdots, \widehat{L_i}, \cdots, L_{r+1}\right) \in Mat_{r \times r}(\mathbb{R}), \quad 1 \le i \le r+1.$$

Definition 5.4.4 (Coefficient Field) *We define the field*

$$K(L) = \mathbb{Q}(L_{ij}), 1 \le i \le r+1, 1 \le j \le r.$$

Then it is easy to see that

$$\frac{det(\underline{L}^{\hat{i}})}{det(\underline{L}^{\widehat{r+1}})} \in K(L).$$

If $q > r$.

$$L(\underline{m}; q) L(\underline{n}; q) = L(\underline{m} + \underline{n}; q), \quad \underline{m}, \underline{n} \ge \underline{0}.$$

Define

$$\underline{1}_{(r,q)} = (\underbrace{1, \cdots, 1}_{r}, \underbrace{0, \cdots, 0}_{q-r}),$$

then

$$L(\underline{k} + \underline{1}_{(r,q)}; q) = L(\underline{1}_{(r,q)}; q) L(\underline{k}; q),$$

namely

$$L(k_1 + 1, \cdots, k_r + 1, k_{r+1}, \cdots, k_q; q) = L(\underbrace{1, \cdots, 1}_{r}, 0, \cdots, 0; q) L(k_1, \cdots, k_q; q).$$

By Formula 5.4.3, we obtain

$$L(k_1{+}1,\cdots,k_r{+}1,k_{r+1},\cdots,k_q;q) = \sum_{i=1}^{r}(-1)^{i-1}\frac{det(\underline{L}^{\hat{i}})}{det(\underline{L}^{\widehat{r+1}})}L(k_1{+}1,\cdots,k_{i-1}{+}1,k_i,k_{i+1}{+}1,\cdots,k_{r+1}{+}1,k_{r+2},\cdots,k_q),$$

namely

$$L(\underline{k}+\underline{1}_{r,q};q) = \sum_{i=1}^{r}(-1)^{i-1}\frac{det(\underline{L}^{\hat{i}})}{det(\underline{L}^{\widehat{r+1}})}L(\underline{k}+\underline{1}^{\hat{i}}_{(r+1,q)};q),$$

with

$$\underline{L}^{\hat{i}} = \left(L_1,\cdots,\widehat{L_i},\cdots,L_{r+1}\right) \in Mat_{r\times r}(\mathbb{R}), \quad 1 \le i \le r+1,$$

and

$$\underline{1}^{\hat{i}}_{(r+1,q)} = (\underbrace{1,\cdots,1,\overbrace{0}^{i},1,\cdots,1}_{r+1},0,\cdots,0).$$

If we denote

$$\underline{m} = \underline{k}+\underline{1}_{(r,q)}, \quad \delta_i = (0,\cdots 0,\underbrace{1}_{i},0,\cdots 0),$$

then

$$\underline{m}+\delta_{r+1}-\delta_i = \underline{k}+\underline{1}^{\hat{i}}_{r+1,q}.$$

If $\underline{m} \ge \underline{1}_{r,q}$, then we obtain

$$L(\underline{m};q) = \sum_{i=1}^{r}(-1)^{i-1}\frac{det(\underline{L}^{\hat{i}})}{det(\underline{L}^{\widehat{r+1}})}L(\underline{m}+\delta_{r+1}-\delta_i;q).$$

Continuously, if $\underline{m}^{(1),i} = \underline{m}+\delta_{r+1}-\delta_i \ge \underline{1}_{r,q}$ $(1 \le i \le r)$, then

$$L(\underline{m}^{(1),i};q) = \sum_{j=1}^{r}(-1)^{i-1}\frac{det(\underline{L}^{\hat{j}})}{det(\underline{L}^{\widehat{r+1}})}L(\underline{m}^{(1),i}+\delta_{r+1}-\delta_j;q).$$

Define

$$\mathrm{Supp}(\underline{m}) = \{i|m_i \ne 0\}.$$

It is easy to observe that during the procedure of applying the cocycle relation from Lemma 5.3.2, the norm of $\underline{m}$ remains unchanged, which means that

$$|\underline{m}| = \sum_{i=1}^{q}m_i = |\underline{m}+\delta_r-\delta_i| = Constant,$$

however, we have

$$\mathrm{Supp}(\underline{m}+\delta_{r+1}-\delta_i) \subseteq \mathrm{Supp}(\underline{m}).$$

Proposition 5.4.5 *$q > r$ and all notations defined as before.*
If $\underline{m} = (m_1,\cdots,m_q) \ge \underline{1}_{r,q}$ then the cocycle relation

$$L(\underline{m};q) = \sum_{i=1}^{r}(-1)^{i-1}\frac{det(\underline{L}^{\hat{i}})}{det(\underline{L}^{\widehat{r+1}})}L(\underline{m}^{(1),i};q), \tag{5.4}$$

implies

$$|\underline{m}| = |\underline{m}^{(1),i}|,$$
$$m_i^{(1),i} = m_i - 1; \quad m_j^{(1),i} = m_j, j \neq i,$$
$$\sum_{j=1}^{r} m_j^{(1),i} < \sum_{j=1}^{r} m_j.$$

And

$$\sharp(\mathrm{Supp}(\underline{m}^{(1),i})) \leq \sharp(\mathrm{Supp}(\underline{m})), \quad \forall 1 \leq i \leq r.$$

After repeating l-times the cocycle relation, we have

$$L(\underline{m}^{(l),i_l}; q) = \sum_{i_{l+1}=1}^{r} C_{i_{l+1}} L(\underline{m}^{(l),i_l} + \delta_{r+1} - \delta_{i_{l+1}}; q), \tag{5.5}$$

where the constant

$$C_{i_{l+1}} = (-1)^{i_{l+1}-1} \frac{det(L_1, \cdots, \widehat{L}_{i_{l+1}}, \cdots, L_{r+1})}{det(L_1, \cdots, \widehat{L}_{i_l}, \cdots, L_{r+1})}$$

depends only on the coefficients of linear forms L_i and

$$\underline{m}^{(l+1),i_{l+1}} = \underline{m}^{(l),i_l} + \delta_{r+1} - \delta_{i_{l+1}},$$
$$|\underline{m}^{(l+1),i_{l+1}}| = |\underline{m}^{(l),i_l}|$$

There exists a finite l, such that

$$\sharp(\mathrm{Supp}(\underline{m}^{(l),i_l})) = \sharp(\mathrm{Supp}(\underline{m})) - 1.$$

Proof 5.4.1.1 *The proof is straightforward from repeating Sczech's cocylce relation. Now we will try to explain the procedure. If we only focus on the changement of powers of L_i, then the cocylc relation (5.4) can be translated into*

$$\underline{m} = (m_1, \cdots, m_r, m_{r+1}, m_{r+2}, \cdots, m_q) \longrightarrow \begin{cases} \underline{m}^{(1),1} = (m_1 - 1, m_2, \cdots, m_r, m_{r+1} + 1, m_{r+2}, \cdots, m_q) \\ \underline{m}^{(1),2} = (m_1, m_2 - 1, m_3, \cdots, m_r, m_{r+1} + 1, m_{r+2}, \cdots, m_q) \\ \cdots \\ \underline{m}^{(1),i} = (m_1, \cdots, m_{i-1}, m_i - 1, m_{i+1}, \cdots, m_r, m_{r+1}, m_{r+2}, \cdots, m_q) \\ \cdots \\ \underline{m}^{(1),r} = (m_1, \cdots, m_{r-1}, m_r - 1, m_{r+1} + 1, m_{r+2}, \cdots, m_q) \end{cases}.$$

The cocyle relation 5.5 can also be written as

$$\underline{m}^{(l),i_l} = (m_1^{(l),i_l}, \cdots, m_r^{(l),i_l}, m_{r+1} + l, \cdots, m_q) \longrightarrow$$

$$\longrightarrow \begin{cases} \underline{m}^{(l+1),1} = (m_1^{(l),i_l} - 1, m_2^{(l),i_l}, \cdots, m_r^{(l),i_l}, m_{r+1} + l + 1, m_{r+2}, \cdots, m_q) \\ \underline{m}^{(l+1),2} = (m_1^{(l),i_l}, m_2^{(l),i_l} - 1, m_3^{(l),i_l}, \cdots, m_r^{(l),i_l}, m_{r+1} + l + 1, m_{r+2}, \cdots, m_q) \\ \cdots \\ \underline{m}^{(l+1),i_{l+1}} = (m_1^{(l),i_l}, \cdots, m_{i_{l+1}-1}^{(l),i_l}, m_{i_{l+1}}^{(l),i_l} - 1, m_{i_{l+1}+1}^{(l),i_l}, \cdots, m_r^{(l),i_l}, m_{r+1} + l + 1, m_{r+2}, \cdots, m_q) \\ \cdots \\ \underline{m}^{(l),r} = (m_1^{(l),i_l}, \cdots, m_{r-1}^{(l),i_l}, m_r^{(l),i_l} - 1, m_{r+1} + l + 1, m_{r+2}, \cdots, m_q) \end{cases}.$$

$$m_j^{(l+1),i_{l+1}} = m_j^{(l),i_l}, \quad 1 \le j \le r, j \ne i_{l+1},$$

and

$$m_{i_{l+1}}^{(l+1),i_{l+1}} = m_{i_{l+1}}^{(l),i_l} - 1, \quad 1 \le i_l, i_{l+1} \le r,$$

and

$$|\underline{m}^{(l),i_l}| = |\underline{m}^{(l+1),i_{l+1}}|, \quad m_{r+1}^{(l),i_l} = m_{r+1} + l.$$

Denote

$$m_{i_0} = Min\{m_i, 1 \le i \le q\}.$$

$$m_{i_0}^{(l),i_0} = m_{i_0}^{(l-1),i_0} - 1 = m_{i_0}^{(l-2),i_0} - 2 = \cdots = m_{i_0} - l.$$

Take $l_0 = m_{i_0}$, then

$$m_{i_0}^{(l_0),i_0} = 0,$$

$$\sharp(\mathrm{Supp}(\underline{m}^{(l_0),i_0})) = \sharp(\mathrm{Supp}(\underline{m})) - 1.$$

Based on Proposition 5.4.5, it is not difficult to obtain the following theorem

Theorem 5.4.6 (Reduction Theorem I) *If $q > r$. There exist constants $C_{i_1,\cdots,i_{q-1}} \in K(L)$, such that*

$$L(\underline{m}; q) = \frac{1}{\prod_{i=1}^{q} L_i(u)^{m_i}} = \sum_{\substack{1 \le i_1,\cdots,i_{q-1} \le q \\ i_j \ne i_h, j \ne h}} \frac{C_{i_1,\cdots,i_{q-1}}}{\prod_{h=1}^{q-1} L_{i_h}(u)^{m'_{i_h}}},$$

where $i_1, \ldots, i_{q-1}$ runs through some possible choices of $(q-1)$ different elements from the set $\{1, \ldots, q\}$ and

$$\sum_{i=1}^{q} m_i = \sum_{h=1}^{q-1} m_{i_h}.$$

We can then deduce to obtain that there exist $C_{i_1,\cdots,i_r} \in K(L)$, such that

$$L(\underline{m}; q) = \frac{1}{\prod_{i=1}^{q} L_i(u)^{m_i}} = \sum_{\substack{1 \le i_1,\cdots,i_r \le q \\ i_j \ne i_h, j \ne h}} \frac{C_{i_1,\cdots,i_r}}{\prod_{h=1}^{r} L_{i_h}(u)^{m'_{i_h}}}.$$

where $i_1, \ldots, i_r$ runs through some possible choices of (r) different elements from the set $\{1, \ldots, q\}$ and

$$\sum_{i=1}^{q} m_i = \sum_{h=1}^{r} m'_{i_h}.$$

Proof 5.4.1.2 *We can obtain this theorem by mathematical induction on the sum $\sum_{j=1}^{r} m_j$, using Proposition 5.4.5.*

We can even give the exact expression of the coefficients $C_{i_1,\cdots,i_r} \in K(L)$ if in need. For this moment, we just write $C_{i_1,\cdots,i_r} \in K(L)$ as symbols. Let us see the following example.

Example 5.4.7 *If $r = 2$ and $q = 3$, we consider*

$$\frac{1}{L_1^2(u)L_2^3(u)L_3^3(u)}.$$

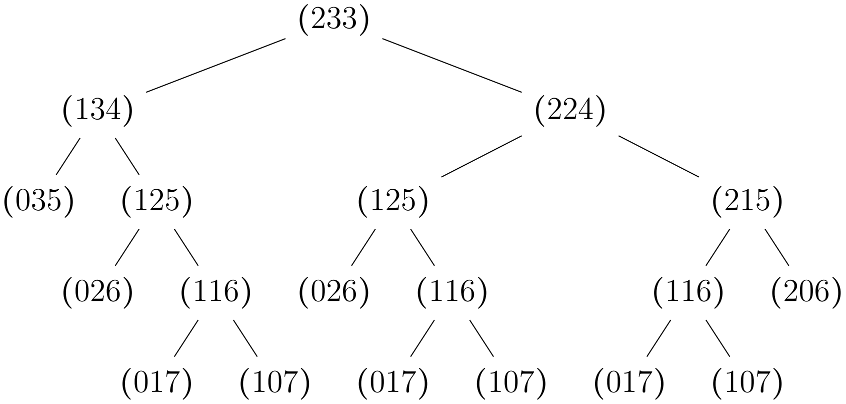
Then Sczech's cocycle relation implies

$$\frac{1}{L_1(u)L_2(u)} = \frac{C_1}{L_1(u)L_3(u)} - \frac{C_2}{L_2(u)L_3(u)},$$

where

$$C_1 = \frac{det(L_1, L_3)}{det(L_1, L_2)}, \quad C_2 = \frac{det(L_2, L_3)}{det(L_1, L_2)}.$$

We have the following tree to illustrate the procedure of applying the cocycle relation,



on the left branch of the tree, (a, b, c) (which signifies $\frac{1}{L_1^a(u)L_2^b(u)L_3^c(u)}$) $\longrightarrow (a-1, b, c+1)$ (which denotes $\frac{1}{L_1^{a-1}(u)L_2^b(u)L_3^{c+1}(u)}$) and the final fraction should be multiplied by $(-1)C_2$; on the right branch, $(a, b, c) \longrightarrow (a, b-1, c+1)$ and the final fraction should be multiplied by C_1. The $i-$depth of the tree describes the situation after $i-$times appliction of the cocyle relation. For example, the first depth of the tree gives us

$$\frac{1}{L_1^2(u)L_2^3(u)L_3^3(u)} = C_1\frac{1}{L_1^2(u)L_2^2(u)L_3^4(u)} - C_2\frac{1}{L_1^1(u)L_2^3(u)L_3^4(u)}.$$

In conclusion, we obtain

$$\frac{1}{L_1^2(u)L_2^3(u)L_3^3(u)} =$$

$$C_1^3\frac{1}{L_1^2(u)L_3^6(u)}+C_2^2\frac{1}{L_2^3(u)L_3^5(u)}-3C_1^3C_2\frac{1}{L_1(u)L_3^7(u)}+3C_1^2C_2^2\frac{1}{L_2(u)L_3^7(u)}+2C_1C_2^2\frac{1}{L_2^2(u)L_3^6(u)}.$$

It is not difficult to see that this example is one of the simplest cases of Theorem 5.4.6.

5.4.2 Further reduction.

It turns out that the essential part in the definition of Higer plectic Green function

$$\int_{U_{\mathbb{R}}} \prod_{i=1}^{q} \|\lambda^{(i)}u\|^{-r\sigma_i} \mathrm{d}^\times u = 2^{1-r} \int_{U_{\mathbb{R}}} \prod_{i=1}^{q} L_i(u)^{-r\sigma_i/2} \mathrm{d}^\times u$$

can be written as a $K(L)$-valued finite linear combination of the following intergral

$$\int_{U_{\mathbb{R}}} \prod_{i=1}^{r} \frac{1}{L_i(u)^{r\sigma_i/2}} \mathrm{d}^{\times} u,$$

where we consider only the case $2|r\sigma_i, 1 \leq i \leq r$ and we always assume that $q \geq r$.

Now we will show that we can do further reduction, which means that we only need to do the calculation of

$$\int_{U_{\mathbb{R}}} \prod_{i=1}^{r} \frac{1}{L_i(u)} \mathrm{d}^{\times} u,$$

and its variant, where $L_i(u) = \sum_{j=1}^{r} L_{ij} u_j$. In order to obtain the result of abitary exponents of $L_i(u)$, we can simply apply some differential operators on the last integral.

Remark 5.4.8 *For this further reduction, we will begin with a general setting, which means we can begin with $q \geq r$ linear forms with arbitary exponents and reduce to the case of trivial exponents. Our first reduction theorem built upon cocycle relations can give another point of view, and also make a connection with our treatment for $F = \mathbb{Q}$.*

Definition 5.4.9 *Let $\rho : \{1, \ldots, r\} \longrightarrow \{1, \ldots, q\}$ be a map. We define*

$$m_{\rho} = \prod_{i=1}^{q} (\sharp(\rho^{-1}(i)))!,$$

where $\sharp(\rho^{-1}(i)$ is the multiplicity of i. Then we define differential operator

$$D_{\rho} = \prod_{j=1}^{r} \left(-\frac{\partial}{\partial L_{\rho(j)j}} \right) m_{\rho}^{-1}.$$

Let us see some examples.

Example 5.4.10 *(1) If $q = r$ and*

$$\rho_1 : \{1, \ldots, r\} \longrightarrow \{1, \ldots, r\},$$

$$\rho_1(j) = j,$$

then the diagonal operator

$$D_{\rho_1} = \prod_{j=1}^{r} \left(-\frac{\partial}{\partial L_{jj}} \right).$$

(2) If $\rho_2 : \{1, \ldots, r\} \longrightarrow \{1, \ldots, q\}$,

$$\rho_2(j) = 1, \quad \forall j \in \{1, \cdots, r\},$$

then

$$D_{\rho_2} = \frac{1}{(r)!} \prod_{j=1}^{r} \left(-\frac{\partial}{\partial L_{1j}} \right).$$

We see investigate the acition of differential operator D_ρ on $L(\underline{m}; q)$.

Fix a map $\rho : \{1, \dots, r\} \longrightarrow \{1, \dots, q\} \quad (q \geq r)$, and define the following set

$$S_\rho = \rho\left(\{1, \dots, r\}\right) \subseteq \{1, \dots, q\}.$$

Given the fraction

$$L(\underline{m}; q) = \prod_{i=1}^{q} \frac{1}{L_i^{m_i}(u)},$$

where $\underline{m} = (m_1, \dots, m_q)$. Then a direct calculation gives

Lemma 5.4.11

$$D_\rho(L(\underline{m}; q)) = \left(\prod_{i \notin S_\rho} \frac{1}{L_i^{m_i}(u)}\right)\left(\prod_{i \in S_\rho} \frac{(m_i + \sharp(\rho^{-1}(i)) - 1)!}{(m_i - 1)!\, \sharp(\rho^{-1}(i))} \frac{1}{(L_i(u))^{m_i + \sharp(\rho^{-1}(i))}}\right).$$

Let us review Example 5.4.10.

$$D_{\rho_1}\left(\prod_{i=1}^{r} \frac{1}{L_i(u)}\right) = \left(\prod_{i=1}^{r} \frac{1}{L_i^2(u)}\right),$$

and

$$D_{\rho_2}\left(\prod_{i=1}^{d} \frac{1}{L_i(u)}\right) = \frac{1}{L_1^{r+1}(u)}\left(\prod_{i=2}^{d} \frac{1}{L_i(u)}\right).$$

Theorem 5.4.6 guaranties Now the following theorem tells us only

$$\int_{U_\mathbb{R}} \prod_{i=1}^{r} \frac{1}{L_i(u)} \mathrm{d}^\times u$$

with trivial exponents is sufficient for our calculation of higher plectic Green function.

Theorem 5.4.12 ***(Reduction Theorem II)***

Fixe

$$L(\underline{m}; r) = \prod_{i=1}^{r} \frac{1}{L_i^{m_i}(u)},$$

where $\underline{m} = (rk_1/2, \cdots, rk_r/2)$, $\underline{k} = (k_1, \cdots, k_r)$ *and* $rk_i/2 \in \mathbb{N}_+$.
(1) If $2 \nmid r$ *or If* $2|r$ *and* $|\underline{k}|$ *is even. There exists a differential operator* $D(\underline{m})$*, composed by some* D_ρ*, such that up to some rational number*

$$D(\underline{m})L(\underline{1}_{(r,r)}; r) = L(\underline{m}; r).$$

(2) If $2|r$ *and* $|\underline{k}|$ *is odd. Then exists a differential operator* $D(\underline{m})$ *composed by some* D_ρ*, such that up to some rational number*

$$D(\underline{m})L(\underline{1}_{(r,r)} + \frac{r}{2}\delta_{i_0}; r) = L(\underline{m}; r).$$

Proof 5.4.2.1 *If there exists* $l \in \mathbb{N}_+$ *such that*

$$\sum_{i=1}^{r}(m_i - 1) = |\underline{m}| - r = rl,$$

then we can find l *differential operators* $D^{j}_{j_1,\cdots,j_r}, 1 \le j \le l$*,*

$$\frac{\prod(n_{j_i})!}{\prod(n(j_i)-1)!} D^1 \cdots D^l (L(\underline{1}_{(r,r)}; r)) = L(\underline{m}; r),$$

where $n(j_i)$ *is the multiplicity of* j_i *appearing in the* l *operators. We have the following equivalence*

$$|\underline{m}| - r = rl \iff |\underline{m}| > \underline{1}_{(r,r)}, \quad r|(|\underline{m}|).$$

(1)If $2 \nmid r$ *and we request that* $rk_i/2 \in \mathbb{N}_+$*, then there exist* $l_i \in \mathbb{N}_+$ *such that* $k_i/2 = l_i$ *and*

$$|\underline{m}| = \sum_{i=1}^{r} rk_i/2 = \sum_{i=1}^{r} rl_i = r(\sum_{i=1}^{r} l_i), \quad \Longrightarrow \quad r|(|\underline{m}|).$$

Therefore in this case, the theorem is proved.

If $2|r$ *and* $|\underline{k}|$ *is even. Let define*

$$S = \{i | k_i \in 2\mathbb{N}_+ + 1\},$$

then $|S| = s$ *is even.*

①. *If* $i \notin S$*, then* k_i *is even, then* $k_i = 2l_i$*, then*

$$\sum_{i \notin S} m_i = \sum_{i \notin S} rk_i/2 = \sum_{i \notin S} rl_i = r(\sum_{i \notin S} l_i),$$

②. *If* $i \in S$*, then* k_i *is odd.*

$$m_i = rk_i/2 = r\left(\frac{k_i+1}{2}\right) - \frac{r}{2},$$

$$\sum_{i \in S} m_i = \sum_{i \in S} rk_i/2 = \sum_{i \in S} r\left(\frac{k_i+1}{2}\right) - \frac{r}{2}s = r\left(\sum_{i \in S}\left(\frac{k_i+1}{2}\right) - s/2\right).$$

Finally, we have

$$|\underline{m}| = \sum_{i=1}^{r} m_i = r(\sum_{i \notin S} l_i) + r\left(\sum_{i \in S}\left(\frac{k_i+1}{2}\right) - s/2\right),$$

which implies that

$$r|(|\underline{m}|).$$

(2) If $2|r$ *and* $|\underline{k}|$ *is odd. In this case,* $|S| = s = t+1 (t \in 2\mathbb{N}_+)$ *is odd and* $\forall i \in S, \; k_i \in 2\mathbb{N}_+ + 1$.

$$m_i = rk_i/2 = r\left(\frac{k_i-1}{2}\right) + \frac{r}{2},$$

$$\sum_{i \in S} m_i = \sum_{i \in S} rk_i/2 = \sum_{i \in S} r\left(\frac{k_i-1}{2}\right) + \frac{r}{2}s = r\left(\sum_{i \in S}\left(\frac{k_i-1}{2}\right) + t/2\right) + \frac{r}{2}$$

and

$$\sum_{i\notin S} m_i = \sum_{i\notin S} rk_i/2 = \sum_{i\notin S} rl_i = r(\sum_{i\notin S} l_i),$$

so finally

$$|\underline{m}| = r\left(\sum_{i\notin S} l_i + \sum_{i\in S}\left(\frac{k_i-1}{2}\right) + t/2\right) + \frac{r}{2}.$$

Let

$$p = \left(\sum_{i\notin S} l_i + \sum_{i\in S}\left(\frac{k_i-1}{2}\right) + t/2\right) - 1,$$

then there exist p *differential operators* $D^j_{j_1,\cdots,j_r}$, $1 \le j \le p$, *such that*

$$\prod(n_{j_i})!\prod\frac{(r/2)!}{(r/2+n(j_i))!}D^1\cdots D^p(L(\underline{1}_{(r,r)}+\frac{r}{2}\delta_{i_0};q)) = L(\underline{m};r),$$

where $i_0 \in S$.

Remark 5.4.13 *In fact, by applying more differential operators, we can deduce to* $i_0 = r$. *Moreover let define the differential operator*

$$D = \frac{1}{(r/2)!}\left(-\frac{\partial}{\partial L_{rr}}\right)^{r/2},$$

then we have

$$D\left(\frac{u_r^{r/2}}{L_1(u)\cdots L_r(u)}\right) = L(\underline{1}_{(r,r)}+\frac{r}{2}\delta_r;r).$$

Example 5.4.14 $r = 2$.

$$\frac{1}{3}D_{2,2}D_{1,2}(\frac{1}{L_1(u)L_2(u)}) = \frac{1}{L_1^2(u)L_2^4(u)}.$$

$$\frac{2!}{4!}D_{2,2}D_{1,2}(\frac{1}{L_1(u)L_2^2(u)}) = \frac{1}{L_1^2(u)L_2^5(u)}.$$

In the next section, we will give the details of the calculation, and therefore give an expression for higher plectic Green functions and generalized multiple zeta values for general totally real fields.

5.5 Generalized multiple zeta values for general F (I): special cases

For a general F, already the Hecke transform in Definition 4.1.12 leads to non-trivial integral formulas. The resulting numbers are given by the usual L-values of F in the simplest possible case when Γ is a chain (thanks to Proposition 4.1.9). After that, everything turns out to be much more complicated than any naive attempt at defining MDZV would produce. For example, if F is a real quadratic field ($r = 2$), then the terms of the infinite series defining $Z_I(\Gamma, S)$ do not necessarily involve only rational functions of the components of the $n_e \in I^*$, but sometimes also their logarithms. If F is a cubic field, there are also terms involving dilogarithm, and for higer rank field, some non-trivial iterated integrals make their appearance.

Before illustrating how the Hecke transform produces these new functions, we will give a very vague version of our main theorem.

Theorem 5.5.1 (Main result over general totally real field-vague version) *Let F be a totally real field of degree $[F : \mathbb{Q}] = r > 1$. Given a graph and a subdivision map defined as before. Moreover, we assume that if $2 \nmid r$, the subdivision map is given by $k_e \in 2\mathbb{N}$ for each edge. Then we have the following expression of our generalized multiple zeta value*

$$Z_{I,\nu}(\Gamma, \partial\Gamma) = \sum_{\left\{\substack{n_e \in (I^* \setminus \{0\})/U \\ \partial n = 0 \\ \mathrm{sgn}(n_e) = (-1)^\nu}\right\}} \sum_{C(q,r)} \sum_{1 \le m \le r-1} \sum_{j} (\alpha_j(n_e)) \mathscr{L}_m(\beta_j(n_e)),$$

where $\mathscr{L}_m(\cdot)$ is a generalized m-logarithm which is a non-trivial iterated integral, and $\alpha_j(n_e)$ and $\beta_j(n_e)$ are rational functions of the conjugates of n_e with coefficients in $\mathbb{Q}$ and the sum $\sum_{C(q,r)}$ comes from the reduction theorem. Every sum here is finite.

5.5.1 Introduction.

Definition 5.5.2 *Let us define the* ***fundamental integral of the Hecke transform*** *as follows*

$$I_r(L_i) = \int_{U_\mathbb{R}} \prod_{i=1}^{r} \frac{1}{L_i(u)} \mathrm{d}^\times u,$$

where $U_\mathbb{R} = \{u = (u_1, \cdots, u_r) \in \mathbb{R}_+^r | u_1 \cdots u_r = 1\}$ and $\mathrm{d}^\times u = \frac{du_1 \cdots du_{r-1}}{u_1 \cdots u_{r-1}}$.

Since $u_r = \frac{1}{u_1 \cdots u_{r-1}}$, then

$$\int_{U_\mathbb{R}} \prod_{i=1}^{r} \frac{1}{L_i(u)} \mathrm{d}^\times u = \int_{\mathbb{R}_+^{r-1}} \prod_{i=1}^{r} \left(\frac{u_1 \cdots u_{r-1}}{\sum_{j=1}^{r-1} (L_{ij}(u_1 \cdots u_{j-1} u_j^2 u_{j+1} \cdots u_{r-1})) + L_{ir}} \right) \mathrm{d}^\times u$$

If we define

$$y_j = u_1 \cdots u_{j-1} u_j^2 u_{j+1} \cdots u_{r-1}, \quad 1 \le j \le r-1,$$

then

$$y_1 \cdots y_{r-1} = (u_1 \cdots u_{r-1})^r = u_r^{-r}.$$

By a direct calculation, we obtain

$$\mathrm{d}^{\times}y = \frac{dy_1 \cdots dy_{r-1}}{y_1 \cdots y_{r-1}} = r\ \mathrm{d}^{\times}u.$$

$$\mathrm{d}^{\times}u = \frac{u_r^r}{r}\ dy,$$

where $dy = dy_1 \cdots dy_{r-1}$. Then we obtain that

$$I_r(L_i) = \int_{U_{\mathbb{R}}} \prod_{i=1}^{r} \frac{1}{L_i(u)} \mathrm{d}^{\times}u = \frac{1}{r} \int_{\mathbb{R}_+^{r-1}} \prod_{i=1}^{r} \frac{dy}{\sum_{j=1}^{r-1}(L_{ij}y_j) + L_{ir}}.$$

We will explain how the fundamental integral is related to the basic integral associated to the generalized multiple zeta value $Z_{I,\nu}(\Gamma, S)$. Recall that

$$Z_{I,\nu}(\Gamma, S) = (O_{F,+}^{\times} : U)^{-1} \sum_{\substack{(n_j)\in(I^*\smallsetminus\{0\})^q/U \\ \pi_v=0, \forall v\in V(\Gamma)\smallsetminus S}} \mathbb{I}(q, r; n_j, \sigma_j; \Gamma, S).$$

By the reciprocity between q and r, we can always assume $q > r$. From the first reduction theorem Theorem (5.4.6), we have

$$\frac{1}{\prod_{i=1}^{q} L_i(u)^{m_i}} = \sum_{\substack{1\le i_1,\cdots,i_r\le q \\ i_j\ne i_h, j\ne h}} \frac{C_{i_1,\cdots,i_r}}{\prod_{h=1}^{r} L_{i_h}(u)^{m_{i_h}}}.$$

where $i_1, \ldots, i_r$ runs through some possible choices of (r) different elements from the set $\{1, \ldots, q\}$ and

$$\sum_{i=1}^{q} m_i = \sum_{h=1}^{r} m'_{i_h}.$$

Then the second reduction theorem Theorem (5.4.12) tells us if $2 \nmid r$ or if $2|r$ and $|\underline{k}|$ is even. There exists a differential operator $D(\underline{m})$ (5.4.11), composed by some D_ρ, such that up to some rational number

$$D(\underline{m})L(\underline{1}_{(r,r)}; r) = L(\underline{m}; r).$$

Combining the two reduction theorems, we can say that if $2 \nmid r$ or if $2|r$ and $|\underline{k}|$ with $\underline{m} = (r/2)\underline{k}$, then there exits a finite family of differential operators $D(\underline{m'_{i_h}})$ such that

$$\frac{1}{\prod_{i=1}^{q} L_i(u)^{m_i}} = \sum_{\substack{1\le i_1,\cdots,i_r\le q \\ i_j\ne i_h, j\ne h}} C_{i_1,\cdots,i_r} D(\underline{m'_{i_h}}) \left(\frac{1}{\prod_{h=1}^{r} L_{i_h}(u)} \right),$$

where the differentials act with respect to L_i. And we have

$$\int_{U_{\mathbb{R}}} \frac{1}{\prod_{i=1}^{q} L_i(u)^{m_i}} \mathrm{d}^{\times}u = \sum_{\substack{1\le i_1,\cdots,i_r\le q \\ i_j\ne i_h, j\ne h}} C_{i_1,\cdots,i_r} D(\underline{m'_{i_h}}) \left(\int_{U_{\mathbb{R}}} \frac{\mathrm{d}^{\times}u}{\prod_{h=1}^{r} L_{i_h}(u)} \right).$$

If we take $\underline{k} = (\sigma_1, \dots, \sigma_q)$ and take $L_{ij} = \lambda_j^{(i)}$, then we obtain

$$\mathbb{I}(q, r; \lambda^{(i)}, \sigma_i; \Gamma, S) = \sum_{\substack{1 \le i_1, \cdots, i_r \le q \\ i_j \ne i_h, j \ne h}} C_{i_1, \cdots, i_r} D(\underline{m'_{i_h}}) I_r(L_{i_h}).$$

Hence we can finally get the expression of generalized multiple zeta value define over the totally real field F. The concrete calculation of $I_r(L_i)$ involves the so-called generalized polylogarithms. Therefore in the following paragraph of this section, we will give us three simplest examples to find out how non-trivial this integral is. And we will give a general form of $I_r(L_i)$ in the next section.

5.5.2 The case $r = 2$.

$$I_2(L_i) = \int_{U_{\mathbb{R}}} \frac{1}{L_1(u) L_2(u)} \, \mathrm{d}^\times u = \frac{1}{2} \int_{\mathbb{R}_+} \frac{dy}{(L_{11} y + L_{12})(L_{21} y + L_{22})}.$$

Recall the Eisenstein trick

Lemma 5.5.3 *Let V be a n-dimensional vector space, let $l_0, \dots, l_n \in V^*$ be $n+1$ linear forms. Then $\forall x = (x_1, \dots, x_n) \in V$, we have*

$$\sum_{i=0}^{n} (-1)^i \frac{det(l_0, \dots, \widehat{l_i}, \dots, l_n)}{\prod_{j \ne i} l_j(x)} = 0.$$

Theorem 5.5.4

$$I_2(L_i) = \frac{1}{2(L_{11} L_{22} - L_{21} L_{12})} \left(\log\left(\frac{L_{11}}{L_{21}}\right) - \log\left(\frac{L_{12}}{L_{22}}\right) \right).$$

Proof 5.5.2.1 *We will use the Eisenstein trick for the case*

$$l_0 = 0 \cdot y_1 + y_2, \quad l_1 = L_{11} y_1 + L_{12} y_2, \quad l_2 = L_{21} y_1 + L_{22} y_2,$$

and take $y_1 = y$ and $y_2 = 1$, then we obtain

$$\frac{det(L_1, L_2)}{(L_{11} y + L_{12})(L_{21} y + L_{22})} = \frac{L_{11}}{L_{11} y + L_{12}} - \frac{L_{21}}{L_{21} y + L_{22}},$$

where $L_1 = (L_{11}, L_{12})$ and $L_2 = (L_{21}, L_{22})$. Then

$$\begin{aligned} det(L_1, L_2) \int_{\mathbb{R}_+} \frac{dy}{(L_{11} y + L_{12})(L_{21} y + L_{22})} &= \lim_{T \to \infty} \int_0^T \left(\frac{L_{11}}{L_{11} y + L_{12}} - \frac{L_{21}}{L_{21} y + L_{22}} \right) dy \\ &= \lim_{T \to \infty} \Big[\log(L_{11} y + L_{12}) - \log(L_{21} y + L_{22}) \Big]_0^T \\ &= \lim_{T \to \infty} \log\left(\frac{L_{11} T + L_{12}}{L_{21} T + L_{22}}\right) - \log\left(\frac{L_{12}}{L_{22}}\right) = \log\left(\frac{L_{11}}{L_{21}}\right) - \log\left(\frac{L_{12}}{L_{22}}\right). \end{aligned}$$

Finally,

$$I_2(L_i) = \frac{1}{2(L_{11} L_{22} - L_{21} L_{12})} \left(\log\left(\frac{L_{11}}{L_{21}}\right) - \log\left(\frac{L_{12}}{L_{22}}\right) \right).$$

Now we turn to the definition of the higher plectic Green fuction, according to Theorem 5.4.12 we have two cases, depending on that if the sum of all exponents are even or not.

Theorem 5.5.5 *Given $\underline{k} = (k_1, \dots, k_q)$ and $|\underline{k}| = k_1 + \dots + k_q$.*
(1)If $2 \nmid |\underline{k}|$, then after several differentiation with respect to the coefficients α_j, β_j, the basic integral associated to $Z_{I,\nu}(\Gamma, S)$

$$\mathbb{I}(q, 2; \lambda^{(i)}; k_i; \Gamma, S) = \int_{\mathbb{R}_+} \frac{1}{\prod_{j=1}^{q} (\alpha_j^2 u + \beta_j^2 u^{-1})^{k_j}} \, \mathrm{d}^{\times} u$$

can be written as product of π and an element of $\mathbb{Q}(\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_q)$, where $\lambda^{(i)} = (\alpha_i^2, \beta_i^2)$.
(2) If 2 divides $|\underline{k}|$, then $\mathbb{I}(q, 2; \lambda^{(i)}; k_i; \Gamma, S)$ can be written as

$$\sum_{j=1}^{q} \gamma_j \cdot \log\left(\frac{\beta_j}{\alpha_j}\right),$$

where $\gamma_j \in \mathbb{Q}(\alpha_1^2, \dots, \alpha_q^2, \beta_1^2, \dots, \beta_q^2)$.

Remark 5.5.6 *Even though Theorem 5.4.6 brings us to the situation $q = r$, however in this lemma above we give a general setting, $q \geq r$.*

Proof 5.5.2.2 (Proof of Theorem 5.5.5)

$$\mathbb{I}(q, 2; \lambda^{(i)}; k_i; \Gamma, S) = \int_0^{\infty} \frac{u^{\left(\sum_{j=1}^{q} k_j\right)-1}}{\prod_{j=1}^{q} (\alpha_j^2 u^2 + \beta_j^2)^{k_j}} \, du.$$

Now we will begin with something more general.

Lemma 5.5.7 *If $f, g \in \mathbb{C}[X]$. $g(X) = \prod_{j=1}^{q} (X - \tau_j)$, $\tau_1, \dots, \tau_q \in \mathbb{C}$ are distinct and $deg(f) \leq q$. Then*

$$\frac{f(X)}{g(X)} = \sum_{j=1}^{q} \frac{f(\tau_j)}{g'(\tau_j)} \frac{1}{X - \tau_j},$$

where $\frac{f(\tau_j)}{g'(\tau_j)} = Res_{X=\tau_j} \frac{f(X)dX}{g(X)}$ and

$$\frac{f(\tau_j)}{g'(\tau_j)} = \frac{f(\tau_j)}{\prod_{\substack{k=1 \\ k \neq j}}^{q} (\tau_k - \tau_j)}.$$

By this lemma, we have

$$\frac{X^n}{\prod_{j=1}^{q} (X - \tau_j)} = \sum_{j=1}^{d} \frac{\tau_j^n}{\prod_{\substack{k=1 \\ k \neq j}}^{q} (\tau_k - \tau_j)} \frac{1}{X - \tau_j}, \quad n \in \mathbb{N}.$$

Now let us define

$$c_j^2 = \frac{\beta_j^2}{\alpha_j^2}, \quad 1 \leq j \leq q.$$

Then we have

$$\frac{X^n}{\prod_{j=1}^q (X+c_j^2)} = \sum_{j=1}^q \frac{(-c_j^2)^n}{\prod_{\substack{k=1\\k\neq j}}^q (c_j^2-c_k^2)} \frac{1}{X+c_j^2}, \quad \forall n \in \mathbb{N}.$$

Then

$$\frac{u^{2n}}{\prod_{j=1}^q (u^2+c_j^2)} = \sum_{j=1}^q \frac{(-c_j^2)^n}{\prod_{\substack{k=1\\k\neq j}}^q (c_j^2-c_k^2)} \frac{1}{u^2+c_j^2}.$$

Therefore

$$\int_0^\infty \frac{u^m}{\prod_{j=1}^q (\alpha_j^2 u + \beta_j^2 u^{-1})} \frac{du}{u} = \frac{1}{\prod_{j=1}^q \alpha_j^2} \int_0^\infty \frac{u^{m+q-1}}{\prod_{j=1}^q (u^2+c_j^2)}\, du.$$

Let define $m+q-1 = 2n+\varepsilon$, $\varepsilon \in \{0,1\}$. *Then*

$$\int_0^\infty \frac{u^m}{\prod_{j=1}^q (\alpha_j^2 u + \beta_j^2 u^{-1})} \frac{du}{u} = \frac{1}{\prod_{j=1}^q \alpha_j^2} \int_0^\infty \frac{u^{2n+\varepsilon}}{\prod_{j=1}^q (u^2+c_j^2)}\, du = \frac{1}{\prod_{j=1}^d \alpha_j^2} \sum_{j=1}^q \frac{(-c_j^2)^n}{\prod_{\substack{k=1\\k\neq j}}^q (c_j^2-c_k^2)} \int_0^\infty \frac{u^\varepsilon}{u^2+c_j^2}\, du.$$

①*: If* $\varepsilon = 0$, *then* $2 \nmid m+q$. *Then*

$$\begin{aligned}\int_0^\infty \frac{u^m}{\prod_{j=1}^q (\alpha_j^2 u + \beta_j^2 u^{-1})} \frac{du}{u} &= \frac{\pi}{2\prod_{j=1}^q \alpha_j^2} \sum_{j=1}^q \frac{(-c_j^2)^n (c_j)^{-1}}{\prod_{\substack{k=1\\k\neq j}}^q (c_j^2-c_k^2)} \\ &= (-1)^{(m+q-1)/2} \frac{\pi}{2} \sum_{j=1}^q \frac{\alpha_j^{q-m-2} \beta_j^{q+m-2}}{\prod_{\substack{k=1\\k\neq j}}^q \Delta_{kj}},\end{aligned}$$

where

$$\Delta_{kj} = \begin{vmatrix} \alpha_k^2 & \beta_k^2 \\ \alpha_j^2 & \beta_j^2 \end{vmatrix}$$

②*: If* $\varepsilon = 1$, *then* $2 | m+q$.

$$\int_0^\infty \frac{u^m}{\prod_{j=1}^q (\alpha_j^2 u + \beta_j^2 u^{-1})} \frac{du}{u} = \frac{(-1)}{2\prod_{j=1}^q \alpha_j^2} \sum_{j=1}^q \frac{(-c_j^2)^n}{\prod_{\substack{k=1\\k\neq j}}^q (c_j^2-c_k^2)} \log(c_j^2).$$

(Attention: one maybe worry about the convegence of $\int_0^\infty \frac{u}{u^2+c_j^2}\, du$, *however the alternating sum of such integral indeed cancels the infinite part, the similar calculation has been shown for* I_2.*)*

When we take $m = |\underline{k}|$, *we will get the result of* $I_2(q, \underline{k})$.

Now let us give one simplest example of $2 \nmid |\underline{k}|$.

Example 5.5.8 $q = 3$, $m = 0$ *and* $n = 1$, *that is*

$$\mathbb{I}(3, 2; \lambda^{(i)}; k_i = 1; \Gamma, S) = \int_0^\infty \frac{1}{\prod_{j=1}^3 (\alpha_j^2 u + \beta_j^2 u^{-1})} \frac{du}{u}$$

$$= -\frac{\pi}{2}\sum_{j=1}^{3}\frac{\alpha_j\beta_j}{\prod_{k=1,k\neq j}^{3}\Delta_{kj}} = -\frac{\pi}{2}\left(\frac{\alpha_1\beta_1}{\Delta_{12}\Delta_{13}} - \frac{\alpha_2\beta_2}{\Delta_{21}\Delta_{23}} + \frac{\alpha_3\beta_3}{\Delta_{12}\Delta_{23}}\right)$$
$$= \frac{\pi}{2(\alpha_1\beta_2+\alpha_2\beta_1)(\alpha_1\beta_3+\alpha_3\beta_1)(\alpha_2\beta_3+\alpha_3\beta_2)}.$$

Given the following graph Γ'_a and $S = \{v_1, v_2, v_3\}$, there are $k_j \in \mathbb{N}_+$ edges with given $\lambda_j = (\alpha_j, \beta_j) \in I^*$, $1 \le j \le 3$.

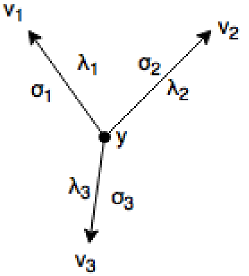


Figure 5.2: The graph Γ'_a.

Then there exist differetials $D_{\alpha_j,\beta_j;k_j}$ such that the corresponding higher plectic Green function can be written as

$$\int_{U_{\mathbb{R}}} G_{I,\nu,\Gamma,\partial\Gamma}(\{0,0,0\},u)\,\mathrm{d}^{\times}u = \sum_{\alpha_j,\beta_j} D_{\alpha_j,\beta_j;k_j}\mathbb{I}(3,2;\lambda^{(i)};k_i=1;\Gamma,S).$$

At last, we will show one simplest example when $q = r = 2$ and all exponents are even.

$$\int_{U_{\mathbb{R}}}\frac{1}{L_1(u)^{2\sigma_1}L_2(u)^{2\sigma_2}}\,\mathrm{d}^{\times}u = \int_{\mathbb{R}_+}\frac{1}{(u_1(\lambda_1^{(1)})^2+u_1^{-1}(\lambda_2^{(1)})^2)^{2\sigma_1}(u_1(\lambda_1^{(2)})^2+u_1^{-1}(\lambda_2^{(2)})^2)^{2\sigma_2}}\frac{du_1}{u_1}$$
$$= \int_{\mathbb{R}_+}\frac{u_1^{2\sigma_1+2\sigma_2-1}}{(u_1^2(\lambda_1^{(1)})^2+(\lambda_2^{(1)})^2)^{2\sigma_1}(u_1^2(\lambda_1^{(2)})^2+(\lambda_2^{(2)})^2)^{2\sigma_2}}\,du_1,$$

where $\sigma_i \in \mathbb{N}_+$. And

$$\frac{u_1^{2\sigma_1+2\sigma_2-1}}{(u_1^2(\lambda_1^{(1)})^2+(\lambda_2^{(1)})^2)^{2\sigma_1}(u_1^2(\lambda_1^{(2)})^2+(\lambda_2^{(2)})^2)^{2\sigma_2}} = D_{\underline{\lambda}^2,\sigma_1,\sigma_2}\left(\frac{u_1}{(u_1^2(\lambda_1^{(1)})^2+(\lambda_2^{(1)})^2)(u_1^2(\lambda_1^{(2)})^2+(\lambda_2^{(2)})^2)}\right),$$

where the differential operator

$$D_{\underline{\lambda}^2,\sigma_1,\sigma_2} = (-1)^{\sigma_1-2}\frac{(\partial_{\lambda_1^{(1)}})^{\sigma_1-2}}{(\sigma_1-2)!(2\lambda_1^{(1)})^{\sigma_1-2}}(-1)^{\sigma_1}\frac{(\partial_{\lambda_2^{(1)}})^{\sigma_1}}{(\sigma_1)!(2\lambda_2^{(1)})^{\sigma_1}}\frac{(\partial_{\lambda_1^{(2)}})^{\sigma_2-1}}{(\sigma_2-1)!(2\lambda_1^{(2)})^{\sigma_2-1}}\frac{(\partial_{\lambda_2^{(2)}})^{\sigma_2-1}}{(\sigma_2-1)!(2\lambda_2^{(2)})^{\sigma_2-1}}.$$

Therefore

$$\int_{U_{\mathbb{R}}}\frac{1}{L_1(u)^{2\sigma_1}L_2(u)^{2\sigma_2}}\mathrm{d}^{\times}u = \int_{\mathbb{R}_+} D_{\underline{\lambda}^2,\sigma_1,\sigma_2}\left(\frac{u_1}{(u_1^2(\lambda_1^{(1)})^2+(\lambda_2^{(1)})^2)(u_1^2(\lambda_1^{(2)})^2+(\lambda_2^{(2)})^2)}\right)du_1$$

$$= D_{\underline{\lambda}^2,\sigma_1,\sigma_2}\int_{\mathbb{R}_+}\frac{u_1}{(u_1^2(\lambda_1^{(1)})^2+(\lambda_2^{(1)})^2)(u_1^2(\lambda_1^{(2)})^2+(\lambda_2^{(2)})^2)}du_1$$

$$= D_{\underline{\lambda}^2,\sigma_1,\sigma_2}\left(det(\underline{\lambda}^2)^{-1}\int_{U_{\mathbb{R}}} f(\underline{\lambda}^2)(u)d^{\times}u\right) = D_{\underline{\lambda}^2,\sigma_1,\sigma_2}\left(det(\underline{\lambda}^2)^{-1}\left(\log\left(\frac{|\lambda_1^{(1)}|}{|\lambda_2^{(1)}|}\right)-\log\left(\frac{|\lambda_1^{(2)}|}{|\lambda_2^{(2)}|}\right)\right)\right).$$

According to Formula 5.2.7

$$\int_{U_{\mathbb{R}}}\prod_{i=1}^{2}\|\lambda^i u\|^{-2\tau_i}\,\mathrm{d}^{\times}u = \frac{1}{2}\int_{U_{\mathbb{R}}}\prod_{i=1}^{2}L_i(u)^{-\tau_i}d^{\times}u.$$

Given the following graph Γ_b' and $S=\{s_1,s_2\}$, there are $2\sigma_1\in\mathbb{N}_+$ edges with given $\lambda^1\in I^*$ and $2\sigma_2\in\mathbb{N}_+$ edges with given $\lambda^2\in I^*$.

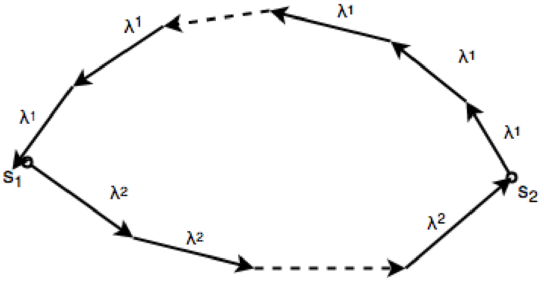


Figure 5.3: The graph Γ_b'.

The corresponding integral of higher plectic Green function can be written as

$$\int_{U_{\mathbb{R}}} G_{I,\nu,\Gamma,\partial\Gamma}(\{0,0\},u)\,\mathrm{d}^{\times}u = \sum_{\lambda^1,\lambda^2}\int_{U_{\mathbb{R}}}\prod_{i=1}^{2}\|\lambda^i u\|^{-4\sigma_i}\,d^{\times}u = \sum_{\lambda^1,\lambda^2}\frac{1}{2}\int_{U_{\mathbb{R}}}\prod_{i=1}^{2}L_i(u)^{-2\sigma_i}d^{\times}u$$

$$= \sum_{\lambda^1,\lambda^2\in I^*\smallsetminus\{0\}}\frac{1}{2}D_{\underline{\lambda}^2,\sigma_1,\sigma_2}\left(det(\underline{\lambda}^2)^{-1}\left(\log\left(\frac{|\lambda_1^{(1)}|}{|\lambda_2^{(1)}|}\right)-\log\left(\frac{|\lambda_1^{(2)}|}{|\lambda_2^{(2)}|}\right)\right)\right).$$

5.5.3 The case $r=3$.

By Thereom 5.4.12 (1), we need only calculate

$$I_3(L_i) = \int_{U_{\mathbb{R}}}\frac{1}{L_1(u)L_2(u)L_3(u)}\,\mathrm{d}^{\times}u = \frac{1}{3}\int_{\mathbb{R}_+^2}\frac{dy_1dy_2}{\prod_{i=1}^{3}(L_{i1}y_1+L_{i2}y_2+L_{i3})}.$$

Again by Eisenstein's trick, we have

$$\frac{det(L_1,L_2,L_3)}{\prod_{i=1}^{3}(L_{i1}y_1+L_{i2}y_2+L_{i3})} = \sum_{1\le i<j\le 3}(-1)^{i+j-1}\frac{det(L_i',L_j')}{(L_{i1}y_1+L_{i2}y_2+L_{i3})(L_{j1}y_1+L_{j2}y_2+L_{j3})},$$

where

$$det(L_1, L_2, L_3) = \begin{vmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{31} & L_{33} \end{vmatrix}$$

$$det(L_i', L_j') = \begin{vmatrix} L_{i1} & L_{i2} \\ L_{j1} & L_{j2} \end{vmatrix}.$$

Now let us define some new quantities.

$$\Delta = det(L_1, L_2, L_3); \quad R_{ij} = det(L_i', L_j'); \quad S_{ij} = \begin{vmatrix} L_{i1} & L_{i3} \\ L_{j1} & L_{j3} \end{vmatrix}$$

and

$$m_{ij}(x) = \begin{vmatrix} L_{i1} & L_{i2}x + L_{i3} \\ L_{j1} & L_{j2}x + L_{j3} \end{vmatrix} = R_{ij}x + S_{ij},$$

and

$$l_i(y_1, y_2) = L_{i1}y_1 + L_{i2}y_2 + L_{i3}.$$

Then

$$\frac{det(L_1, L_2, L_3)}{\prod_{i=1}^3 (L_{i1}y_1 + L_{i2}y_2 + L_{i3})} = \sum_{1 \le i < j \le 3} (-1)^{i+j-1} \frac{R_{ij}}{l_i(y_1, y_2) l_j(y_1, y_2)}$$

$$= \sum_{1 \le i < j \le 3} (-1)^{i+j-1} \frac{m_{ij}(y_2)}{l_i(y_1, y_2) l_j(y_1, y_2)} \frac{R_{ij}}{m_{ij}(y_2)}.$$

Therefore

$$\Delta \cdot I_3 = \frac{1}{3} \sum_{1 \le i < j \le 3} (-1)^{i+j-1} \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \frac{m_{ij}(y_2)}{l_i(y_1, y_2) l_j(y_1, y_2)} dy_1 \right) \frac{R_{ij}}{m_{ij}(y_2)} dy_2.$$

Replacing L_{i2} by $L_{i2}y_2 + L_{i3}$ in the result of Example 5.5.2 we obtain

$$\int_{\mathbb{R}_+} \frac{m_{ij}(y_2)}{l_i(y_1, y_2) l_j(y_1, y_2)} dy_1 = \log\left(\frac{L_{i1}}{L_{j1}}\right) - \log\left(\frac{L_{i2}y_2 + L_{i3}}{L_{j2}y_2 + L_{j3}}\right).$$

We need to calculate

$$\Delta \cdot I_3 = \frac{1}{3} \sum_{1 \le i < j \le 3} (-1)^{i+j-1} \int_{\mathbb{R}_+} \left(\log\left(\frac{L_{i1}}{L_{j1}}\right) - \log\left(\frac{L_{i2}y_2 + L_{i3}}{L_{j2}y_2 + L_{j3}}\right) \right) \frac{R_{ij}}{m_{ij}(y_2)} dy_2.$$

In conclusion, we obtain the following result

Theorem 5.5.9 (The case $r = 3$)

$$3\Delta \cdot I_3 = J_0^{(3)} - J_1^{(3)},$$

where

$$J_0^{(3)} = \sum_{1 \le i < j \le 3} (-1)^{i+j-1} \left(\log\left(\frac{L_{i1}}{L_{i2}}\right) - \log\left(\frac{L_{j1}}{L_{j2}}\right) \right) \left(\log\left(\frac{R_{ij}}{S_{ij}}\right) \right);$$

$$J_1^{(3)} = \sum_{1\le i<j\le 3} (-1)^{i+j-1} I(\frac{L_{i3}}{L_{i2}}, \frac{L_{j3}}{L_{j2}}; \frac{S_{ij}}{R_{ij}}),$$

where

$$I(\alpha_1, \alpha_2; \beta) = \int_{\mathbb{R}_+} \log\left(\frac{y+\alpha_1}{y+\alpha_2}\right) d\log(y+\beta).$$

We will give the proof of the theorem above in several steps. Moreover we prove that if $r = 3$, the term $J_1^{(3)}$ involves dilogarithms.

①:**contribution of** $\log\left(\frac{L_{i1}}{L_{j1}}\right)$

First of all, it is easy to observe that

$$\frac{R_{ij}}{m_{ij}(y_2)} dy_2 = d\log(m_{ij}(y_2)).$$

Then let us define

$$J_0(1) = \sum_{1\le i<j\le 3} (-1)^{i+j-1} \int_{\mathbb{R}_+} \log\left(\frac{L_{i1}}{L_{j1}}\right) \frac{R_{ij}}{m_{ij}(y_2)} dy_2$$

$$= \sum_{1\le i<j\le 3} (-1)^{i+j-1} \int_{\mathbb{R}_+} (\log(L_{i1}) - \log(L_{j1})) d\log(m_{ij}(y_2))$$

$$= \lim_{T\to\infty} \sum_{1\le i<j\le 3} (-1)^{i+j-1} (\log(L_{i1}) - \log(L_{j1})) \Bigg[\log(m_{ij}(y_2)) \Bigg]_0^T.$$

Then

$$J_0(1) = \lim_{T\to\infty} \sum_{1\le i<j\le 3} (-1)^{i+j-1} (\log(L_{i1}) - \log(L_{j1})) (\log(R_{ij}T + S_{ij}) - \log(S_{ij}))$$

$$= \lim_{T\to\infty} \left(\log(L_{11}) \log\left(\frac{R_{12}T + S_{12}}{R_{13}T + S_{13}}\right) - \log(L_{21}) \log\left(\frac{R_{12}T + S_{12}}{R_{23}T + S_{23}}\right) + \log(L_{31}) \log\left(\frac{R_{13}T + S_{13}}{R_{23}T + S_{23}}\right) \right)$$
$$- \sum_{1\le i<j\le 3} (-1)^{i+j-1} (\log(L_{i1}) - \log(L_{j1})) \log(S_{ij}).$$

Thus

$$J_0(1) = \log(L_{11}) \log\left(\frac{R_{12}}{R_{13}}\right) - \log(L_{21}) \log\left(\frac{R_{12}}{R_{23}}\right) + \log(L_{31}) \log\left(\frac{R_{13}}{R_{23}}\right)$$
$$- \sum_{1\le i<j\le 3} (-1)^{i+j-1} (\log(L_{i1}) - \log(L_{j1})) \log(S_{ij}).$$

Notice that

$$\log(L_{11}) \log\left(\frac{R_{12}}{R_{13}}\right) - \log(L_{21}) \log\left(\frac{R_{12}}{R_{23}}\right) + \log(L_{31}) \log\left(\frac{R_{13}}{R_{23}}\right)$$
$$= \sum_{1\le i<j\le 3} (-1)^{i+j-1} (\log(L_{i1}) - \log(L_{j1})) \log(R_{ij}),$$

In conclusion, we have

Formula 5.5.10

$$J_0(1) = \sum_{1\le i<j\le 3} (-1)^{i+j-1} \left(\log(L_{i1}) - \log(L_{j1})\right)(\log(R_{ij}) - \log(S_{ij}))$$

$$= \sum_{1\le i<j\le 3} (-1)^{i+j-1} \log\left(\frac{L_{i1}}{L_{j1}}\right)\left(\log\left(\frac{R_{ij}}{S_{ij}}\right)\right).$$

②: **contribution of** $\log\left(\frac{L_{i2}y_2+L_{i3}}{L_{j2}y_2+L_{j3}}\right)$

$$J_1 = \sum_{1\le i<j\le 3} (-1)^{i+j-1} \int_{\mathbb{R}_+} \log\left(\frac{L_{i2}y_2 + L_{i3}}{L_{j2}y_2 + L_{j3}}\right) \frac{R_{ij}}{m_{ij}(y_2)}\, dy_2.$$

Again, we have $\frac{R_{ij}}{m_{ij}(y_2)}dy_2 = d\log(m_{ij}(y_2))$, hence

$$J_1 = J_1(0) + J_1(1),$$

where

$$J_1(0) = \sum_{1\le i<j\le 3} (-1)^{i+j-1} \int_{\mathbb{R}_+} (\log(L_{i2}) - \log(L_{j2}))\, d\log(m_{ij}(y_2))$$

and

$$J_1(1) = \sum_{1\le i<j\le 3} (-1)^{i+j-1} \int_{\mathbb{R}_+} \left(\log(y_2 + \frac{L_{i3}}{L_{i2}}) - \log(y_2 + \frac{L_{j3}}{L_{j2}})\right) d\log(m_{ij}(y_2)).$$

By the result of ①, we have

Formula 5.5.11

$$J_1(0) = \sum_{1\le i<j\le 3} (-1)^{i+j-1} \left(\log(L_{i2}) - \log(L_{j2})\right)(\log(R_{ij}) - \log(S_{ij}))\,.$$

Now in order to get the precise expression of $J_1(0)$, it remains to calculate

$$\int_{\mathbb{R}_+} \left(\log(y_2 + \frac{L_{i3}}{L_{i2}}) - \log(y_2 + \frac{L_{j3}}{L_{j2}})\right) d\log(m_{ij}(y_2)) = \int_{\mathbb{R}_+} \left(\log(y_2 + \frac{L_{i3}}{L_{i2}}) - \log(y_2 + \frac{L_{j3}}{L_{j2}})\right) d\log(y_2 + \frac{S_{ij}}{R_{ij}}). \tag{5.6}$$

Observe that

$$\frac{R_{ij}}{S_{ij}} - \frac{L_{i3}}{L_{i2}} = \frac{L_{i1}}{L_{i2}} \cdot \frac{\begin{vmatrix} L_{i2} & L_{i3} \\ L_{j2} & L_{j3} \end{vmatrix}}{R_{ij}}; \quad \frac{R_{ij}}{S_{ij}} - \frac{L_{j3}}{L_{j2}} = \frac{L_{j1}}{L_{j2}} \cdot \frac{\begin{vmatrix} L_{i2} & L_{i3} \\ L_{j2} & L_{j3} \end{vmatrix}}{R_{ij}}.$$

Therefore we will have

$$0 < \frac{R_{ij}}{S_{ij}} < \frac{L_{i3}}{L_{i2}}, \frac{L_{j3}}{L_{j2}}$$

or

$$\frac{R_{ij}}{S_{ij}} > \frac{L_{i3}}{L_{i2}}, \frac{L_{j3}}{L_{j2}} > 0$$

In order to calculate (5.6), we will give a general setting.

Theorem 5.5.12 *Given $\alpha_1, \alpha_2, \beta \in \mathbb{R}_+$. Define the integral*

$$I(\alpha_1, \alpha_2; \beta)(T) = \int_0^T \log\left(\frac{y+\alpha_1}{y+\alpha_2}\right) d\log(y+\beta).$$

(1) If $0 < \beta < \alpha_1, \alpha_2$, then

$$I(\alpha_1, \alpha_2; \beta)(T) = \left(Li_2\left(\frac{\beta}{\alpha_2}\right) - Li_2\left(\frac{\beta}{\alpha_1}\right)\right) + \log(\alpha_2-\beta)\log\left(\frac{\beta}{\alpha_2}\right) - \log(\alpha_1-\beta)\log\left(\frac{\beta}{\alpha_1}\right)$$

$$+\frac{(\log(\alpha_2))^2 - (\log(\alpha_1))^2}{2} + O(\frac{1}{T}).$$

(2) If $0 < \alpha_1, \alpha_2 < \beta$, then

$$I(\alpha_1, \alpha_2; \beta)(T) = Li_2\left(\frac{\beta-\alpha_1}{T+\beta}\right) - Li_2\left(\frac{\beta-\alpha_2}{T+\beta}\right) + Li_2\left(\frac{\beta-\alpha_2}{\beta}\right) - Li_2\left(\frac{\beta-\alpha_1}{\beta}\right).$$

Proof 5.5.3.1 *(1)First of all, we define*

$$I(t) = \int_0^t \log(y+1)\frac{dy}{y}, \quad t > 0$$

take $y + 1 = \frac{1}{z}$ then

$$I(t) = -\int_{\frac{1}{1+t}}^1 \frac{\log(z)}{z(1-z)}dz = -\int_{\frac{1}{1+t}}^1 \frac{\log(z)}{z}dz - \int_{\frac{1}{1+t}}^1 \frac{\log(z)}{1-z}dz = -\left[\frac{(\log(z))^2}{2}\right]_{\frac{1}{1+t}}^1 - \int_{\frac{1}{1+t}}^1 \frac{\log(z)}{1-z}dz.$$

Take $x = 1 - z$, then

$$\int_{\frac{1}{1+t}}^1 \frac{\log(z)}{1-z}dz = -\int_0^{\frac{t}{1+t}} \frac{\log(1-x)}{x}d(-x) = -Li_2\left(\frac{t}{1+t}\right).$$

So we obtain

$$I(t) = \frac{(\log(1+t))^2}{2} + Li_2\left(\frac{t}{1+t}\right).$$

If $0 < \beta < \alpha_1, \alpha_2$. Taking

$$y + \beta = (\alpha_1 - \beta)z, \quad y + \alpha_1 = (\alpha_1 - \beta)(z+1),$$

$$\int_0^T \log(y+\alpha_1)d\log(y+\beta) = \int_{\frac{\beta}{\alpha_1-\beta}}^{\frac{T+\beta}{\alpha_1-\beta}} (\log(\alpha_1-\beta) + \log(z+1))\,\frac{dz}{z}$$

$$= \log(\alpha_1-\beta)\log(\frac{T+\beta}{\beta}) + I(\frac{T+\beta}{\alpha_1-\beta}) - I(\frac{\beta}{\alpha_1-\beta}).$$

In the same way we will have

$$\int_0^T \log(y+\alpha_2)d\log(y+\beta) = \int_{\frac{\beta}{\alpha_2-\beta}}^{\frac{T+\beta}{\alpha_2-\beta}} (\log(\alpha_2-\beta) + \log(z+1))\,\frac{dz}{z}$$

$$= \log(\alpha_2-\beta)\log(\frac{T+\beta}{\beta}) + I(\frac{T+\beta}{\alpha_2-\beta}) - I(\frac{\beta}{\alpha_2-\beta}).$$

Thus

$$I(\alpha_1,\alpha_2;\beta)(T)=\log\left(\frac{\alpha_1-\beta}{\alpha_2-\beta}\right)\log\left(\frac{T+\beta}{\beta}\right)+I(\frac{T+\beta}{\alpha_1-\beta})-I(\frac{T+\beta}{\alpha_2-\beta})+\int_{\frac{\beta}{\alpha_1-\beta}}^{\frac{\beta}{\alpha_2-\beta}}\log(z+1)\frac{dz}{z}.$$

However we observe that

$$I(\frac{T+\beta}{\alpha_1-\beta})-I(\frac{T+\beta}{\alpha_2-\beta})=\int_{\frac{T+\beta}{\alpha_2-\beta}}^{\frac{T+\beta}{\alpha_1-\beta}}\log(z+1)\frac{dz}{z}=\int_{\frac{T+\beta}{\alpha_2-\beta}}^{\frac{T+\beta}{\alpha_1-\beta}}\left(\log(z)+\log(1+1/z)\right)\frac{dz}{z},$$

so when T is sufficiently big, we will have

$$I(\frac{T+\beta}{\alpha_1-\beta})-I(\frac{T+\beta}{\alpha_2-\beta})=\left[\frac{(\log(z))^2}{2}\right]_{\frac{T+\beta}{\alpha_2-\beta}}^{\frac{T+\beta}{\alpha_1-\beta}}+O(\frac{1}{T})$$

$$=-\log(T+\beta)\log(\alpha_1-\beta)+\log(T+\beta)\log(\alpha_2-\beta)+\frac{(\log(\alpha_1-\beta))^2-(\log(\alpha_2-\beta))^2}{2}+O(\frac{1}{T}).$$

Meanwhile,

$$\int_{\frac{\beta}{\alpha_1-\beta}}^{\frac{\beta}{\alpha_2-\beta}}\log(z+1)\frac{dz}{z}=I(\frac{\beta}{\alpha_2-\beta})-I(\frac{\beta}{\alpha_1-\beta})$$

$$=\frac{\left(\log(\frac{\alpha_2}{\alpha_2-\beta})\right)^2}{2}-\frac{\left(\log(\frac{\alpha_1}{\alpha_1-\beta})\right)^2}{2}+\left(Li_2\left(\frac{\beta}{\alpha_2}\right)-Li_2\left(\frac{\beta}{\alpha_1}\right)\right).$$

In conclusion,

$$I(\alpha_1,\alpha_2;\beta)(T)=\left(Li_2\left(\frac{\beta}{\alpha_2}\right)-Li_2\left(\frac{\beta}{\alpha_1}\right)\right)+\log(\alpha_2-\beta)\log\left(\frac{\beta}{\alpha_2}\right)-\log(\alpha_1-\beta)\log\left(\frac{\beta}{\alpha_1}\right)$$

$$+\frac{(\log(\alpha_2))^2-(\log(\alpha_1))^2}{2}+O(\frac{1}{T}).$$

(2) Define

$$J(t;t_0)=\int_{t_0}^{t}\log(z-1)\frac{dz}{z},$$

where $1<t_0<t$. By taking $s=\frac{1}{z}$, we obtain

$$J(t;t_0)=-\int_{1/t}^{1/t_0}\frac{\log(1/s-1)}{1/s}(-\frac{1}{s^2})ds$$

$$=\int_{1/t}^{1/t_0}\frac{\log(1-s)}{s}ds-\int_{1/t}^{1/t_0}\frac{\log(s)}{s}ds$$

$$J(t;t_0)=Li_2(1/t)-Li_2(1/t_0)+\frac{(\log(1/t))^2}{2}-\frac{(\log(1/t_0))^2}{2}.$$

Now $0<\alpha_1,\alpha_2<\beta$,

$$y+\beta=(\beta-\alpha_1)z,\quad y+\alpha_1=(\beta-\alpha_1)(z-1),$$

$$\int_0^T \log(y+\alpha_1)d\log(y+\beta) = \int_{\frac{\beta}{\beta-\alpha_1}}^{\frac{T+\beta}{\beta-\alpha_1}} (\log(\beta-\alpha_1)+\log(z-1))\,\frac{dz}{z}$$

$$= \log(\beta-\alpha_1)\log(\frac{T+\beta}{\beta}) + J(\frac{T+\beta}{\beta-\alpha_1};\frac{\beta}{\beta-\alpha_1}).$$

In the same way we obtain

$$\int_0^T \log(y+\alpha_2)d\log(y+\beta) = \log(\beta-\alpha_2)\log(\frac{T+\beta}{\beta}) + J(\frac{T+\beta}{\beta-\alpha_2};\frac{\beta}{\beta-\alpha_2}).$$

Therefore

$$I(\alpha_1,\alpha_2;\beta)(T) = \log\left(\frac{\beta-\alpha_1}{\beta-\alpha_2}\right)\log\left(\frac{T+\beta}{\beta}\right) + J(\frac{T+\beta}{\beta-\alpha_1};\frac{\beta}{\beta-\alpha_1}) - J(\frac{T+\beta}{\beta-\alpha_2};\frac{\beta}{\beta-\alpha_2})$$

$$= Li_2\left(\frac{\beta-\alpha_1}{T+\beta}\right) - Li_2\left(\frac{\beta-\alpha_2}{T+\beta}\right) + Li_2\left(\frac{\beta-\alpha_2}{\beta}\right) - Li_2\left(\frac{\beta-\alpha_1}{\beta}\right)$$

$$+\log\left(\frac{\beta-\alpha_1}{\beta-\alpha_2}\right)\log\left(\frac{T+\beta}{\beta}\right) + \frac{\left(\log(\frac{\beta-\alpha_1}{T+\beta})\right)^2}{2} - \frac{\left(\log(\frac{\beta-\alpha_1}{\beta})\right)^2}{2} - \left(\frac{\left(\log(\frac{\beta-\alpha_2}{T+\beta})\right)^2}{2} - \frac{\left(\log(\frac{\beta-\alpha_2}{\beta})\right)^2}{2}\right).$$

It is not difficult to observe that

$$\log\left(\frac{\beta-\alpha_1}{\beta-\alpha_2}\right)\log\left(\frac{T+\beta}{\beta}\right) + \frac{\left(\log(\frac{\beta-\alpha_1}{T+\beta})\right)^2}{2} - \frac{\left(\log(\frac{\beta-\alpha_1}{\beta})\right)^2}{2} - \left(\frac{\left(\log(\frac{\beta-\alpha_2}{T+\beta})\right)^2}{2} - \frac{\left(\log(\frac{\beta-\alpha_2}{\beta})\right)^2}{2}\right) = 0.$$

Finally we get

$$I(\alpha_1,\alpha_2;\beta)(T) = Li_2\left(\frac{\beta-\alpha_1}{T+\beta}\right) - Li_2\left(\frac{\beta-\alpha_2}{T+\beta}\right) + Li_2\left(\frac{\beta-\alpha_2}{\beta}\right) - Li_2\left(\frac{\beta-\alpha_1}{\beta}\right).$$

Corollary 5.5.13 *Given $\alpha_1,\alpha_2,\beta \in \mathbb{R}_+$. Define*

$$I(\alpha_1,\alpha_2;\beta) = \lim_{T\to+\infty} I(\alpha_1,\alpha_2;\beta)(T)$$

then

$$I(\alpha_1,\alpha_2;\beta) = \int_{\mathbb{R}_+} \log\left(\frac{y+\alpha_1}{y+\alpha_2}\right) d\log(y+\beta).$$

(1) If $0<\beta<\alpha_1,\alpha_2$, then

$$I(\alpha_1,\alpha_2;\beta) = \left(Li_2\left(\frac{\beta}{\alpha_2}\right) - Li_2\left(\frac{\beta}{\alpha_1}\right)\right) + \log(\alpha_2-\beta)\log\left(\frac{\beta}{\alpha_2}\right) - \log(\alpha_1-\beta)\log\left(\frac{\beta}{\alpha_1}\right)$$

$$+\frac{(\log(\alpha_2))^2-(\log(\alpha_1))^2}{2}.$$

(2) If $0<\alpha_1,\alpha_2<\beta$, then

$$I(\alpha_1,\alpha_2;\beta) = Li_2\left(\frac{\beta-\alpha_2}{\beta}\right) - Li_2\left(\frac{\beta-\alpha_1}{\beta}\right).$$

By this corollary, we return to the calculation of I_3 and will obtain

Formula 5.5.14

$$J_1(1) = \sum_{1\le i<j\le 3} (-1)^{i+j-1} I(\frac{L_{i3}}{L_{i2}}, \frac{L_{j3}}{L_{j2}}; \frac{S_{ij}}{R_{ij}}).$$

Now combining the results of Formulae 5.5.10, 5.5.11 and 5.5.14, we obtain the final result of

$$3\Delta \cdot I_3 = J_0(1) - (J_1(0) + J_1(1)),$$

that is

$$3\Delta \cdot I_3 = J_0^{(3)} - J_1^{(3)},$$

where

$$J_0^{(3)} = J_0(1) - J_1(0) = \sum_{1\le i<j\le 3} (-1)^{i+j-1} \left(\log\left(\frac{L_{i1}}{L_{i2}}\right) - \log\left(\frac{L_{j1}}{L_{j2}}\right)\right)\left(\log\left(\frac{R_{ij}}{S_{ij}}\right)\right);$$

$$J_1^{(3)} = J_1(1) = \sum_{1\le i<j\le 3} (-1)^{i+j-1} I(\frac{L_{i3}}{L_{i2}}, \frac{L_{j3}}{L_{j2}}; \frac{S_{ij}}{R_{ij}}).$$

5.5.4 The case $r = 4$.

$$I_4 = I_4(L_i) = \int_{U_{\mathbb{R}}} \frac{1}{L_1(u)L_2(u)L_3(u)L_4(u)} \mathrm{d}^{\times}u = \frac{1}{4}\int_{\mathbb{R}_+^3} \frac{dy_1\, dy_2\, dy_3}{\prod_{i=1}^4 (L_{i1}y_1 + L_{i2}y_2 + L_{i3}y_3 + L_{i4})}.$$

We will first give an explicit expression of I_4. Let us introduce some new notations.

Definition 5.5.15

$$R_{ij}^{(4)} = \begin{vmatrix} L_{i1} & L_{i2} \\ L_{j1} & L_{j2} \end{vmatrix}; \quad S_{ij}^{(4)} = \begin{vmatrix} L_{i1} & L_{i3} \\ L_{j1} & L_{j3} \end{vmatrix}; \quad U_{ij}^{(4)} = \begin{vmatrix} L_{i1} & L_{i4} \\ L_{j1} & L_{j4} \end{vmatrix}$$

$$R_{ijk}^{(4)} = \begin{vmatrix} L_{i1} & L_{i2} & L_{i3} \\ L_{j1} & L_{j2} & L_{j3} \\ L_{k1} & L_{k2} & L_{k3} \end{vmatrix}; \quad S_{ijk}^{(4)} = \begin{vmatrix} L_{i1} & L_{i2} & L_{i4} \\ L_{j1} & L_{j2} & L_{j4} \\ L_{k1} & L_{k2} & L_{k4} \end{vmatrix}; \quad V_{ijk}^{(4)} = \begin{vmatrix} L_{i1} & L_{i3} & L_{i4} \\ L_{j1} & L_{j3} & L_{j4} \\ L_{k1} & L_{k3} & L_{k4} \end{vmatrix}.$$

Theorem 5.5.16 (The case *r* = 4) *We fix* $\Delta_4 = det(L_1, L_2, L_3, L_4)$. *Let us define*

$$C_1(i) = \log(L_{i1}) - \log(L_{i2});$$

$$C_2(i,j) = (C_1(i) - C_1(j))\left(\log(R_{ij}^{(4)}) - \log(S_{ij}^{(4)})\right);$$

and

$$I(\alpha_1, \alpha_2; \beta) = \int_{\mathbb{R}_+} \log\left(\frac{y+\alpha_1}{y+\alpha_2}\right) d\log(y+\beta).$$

Then

$$4\Delta_4 \cdot I_4 = J_0^{(4)} - J_1^{(4)} - J_2^{(4)},$$

where

$$J_0^{(4)} = \sum_{1\le i_1<i_2<i_3\le 4} (-1)^{i_1+i_2+i_3} \left(C_2(i_1,i_2) - C_2(i_1,i_3) + C_2(i_2,i_3)\right) \left(\log(R^{(4)}_{i_1i_2i_3}) - \log(S^{(4)}_{i_1i_2i_3})\right);$$

$$J_1^{(4)} = \sum_{1\le i_1<i_2<i_3\le 4} (-1)^{i_1+i_2+i_3} J_1^{(4)}(i_1,i_2,i_3),$$

$$J_1^{(4)}(i_1,i_2,i_3) =$$

$$C_1(i_1) I\left(\frac{U^{(4)}_{i_1i_2}}{S^{(4)}_{i_1i_2}}, \frac{U^{(4)}_{i_1i_3}}{S^{(4)}_{i_1i_3}}; \frac{S^{(4)}_{i_1i_2i_3}}{R^{(4)}_{i_1i_2i_3}}\right) - C_1(i_2) I\left(\frac{U^{(4)}_{i_1i_2}}{S^{(4)}_{i_1i_2}}, \frac{U^{(4)}_{i_2i_3}}{S^{(4)}_{i_2i_3}}; \frac{S^{(4)}_{i_1i_2i_3}}{R^{(4)}_{i_1i_2i_3}}\right) + C_1(i_3) I\left(\frac{U^{(4)}_{i_1i_3}}{S^{(4)}_{i_1i_3}}, \frac{U^{(4)}_{i_2i_3}}{S^{(4)}_{i_2i_3}}; \frac{S^{(4)}_{i_1i_2i_3}}{R^{(4)}_{i_1i_2i_3}}\right);$$

$$J_2^{(4)} = \sum_{1\le i_1<i_2<i_3\le 4} (-1)^{i_1+i_2+i_3} J_2^{(4)}(i_1,i_2,i_3),$$

$$J_2^{(4)}(i_1,i_2,i_3) = \int_0^\infty I^{(4)}(i_1,i_2,i_3)(y_3)\, d\left(\log\left(y_3 + \frac{S^{(4)}_{i_1i_2i_3}}{R^{(4)}_{i_1i_2i_3}}\right)\right),$$

$$I^{(4)}(i_1,i_2,i_3)(y_3) =$$

$$I\left(\frac{L_{i_13}}{L_{i_12}}, \frac{L_{i_23}}{L_{i_22}}; \frac{S^{(4)}_{i_1i_2}y_3 + U^{(4)}_{i_1i_2}}{R^{(4)}_{i_1i_2}}\right) - I\left(\frac{L_{i_13}}{L_{i_12}}, \frac{L_{i_33}}{L_{i_32}}; \frac{S^{(4)}_{i_1i_3}y_3 + U^{(4)}_{i_1i_3}}{R^{(4)}_{i_1i_3}}\right) + I\left(\frac{L_{i_23}}{L_{i_22}}, \frac{L_{i_33}}{L_{i_32}}; \frac{S^{(4)}_{i_2i_3}y_3 + U^{(4)}_{i_2i_3}}{R^{(4)}_{i_2i_3}}\right).$$

Proof 5.5.4.1 (Proof of Theorem 5.5.16) *We begin again with the Eisenstein trick,*

$$\frac{\Delta_4}{\prod_{i=1}^4 (L_{i1}y_1 + L_{i2}y_2 + L_{i3}y_3 + L_{i4})} = \sum_{1\le i<j<k\le 4} (-1)^{i+j+k} \frac{R^{(4)}_{ijk}}{l_il_jl_k},$$

where

$$l_i = l_i(y_1,y_2,y_3) = L_{i1}y_1 + L_{i2}y_2 + L_{i3}y_3 + L_{i4}.$$

Define

$$m_{ijk}(y_3) = \begin{vmatrix} L_{i1} & L_{i2} & L_{i3}y_3 + L_{i4} \\ L_{j1} & L_{j2} & L_{j3}y_3 + L_{i4} \\ L_{k1} & L_{k2} & L_{k3}y_3 + L_{i4} \end{vmatrix} = R^{(4)}_{ijk}y_3 + S^{(4)}_{ijk}.$$

Then

$$\Delta_4 \int_{\mathbb{R}_+^3} \frac{dy_1dy_2dy_3}{\prod_{i=1}^4 (L_{i1}y_1 + L_{i2}y_2 + L_{i3}y_3 + L_{i4})} = \sum_{1\le i<j<k\le 4} (-1)^{i+j+k} \int_0^\infty \left(\int_{\mathbb{R}_+^2} \frac{m_{ijk}(y_3)}{l_il_jl_k} dy_1dy_2\right) \frac{R^{(4)}_{ijk}}{m_{ijk}(y_3)}\, dy_3.$$

If we define

$$3\Delta_3^{(ijk)}(y_3) I_3^{(ijk)}(y_3) = \int_{\mathbb{R}_+^2} \frac{m_{ijk}(y_3)}{l_il_jl_k} dy_1dy_2,$$

then replacing L_{i3} by $L_{i3}y_3 + L_{i4}$ and according to the result of Theorem 5.5.9, we obtain that

$$3\Delta_3^{(ijk)}(y_3) I_3^{(ijk)}(y_3) = J_0^{(3),(ijk)}(y_3) - J_1^{(3),(ijk)}(y_3),$$

where

$$J_0^{(3),(ijk)}(y_3) = \left(\log\left(\frac{L_{i1}}{L_{i2}}\right) - \log\left(\frac{L_{j1}}{L_{j2}}\right)\right)\left(\log\left(\frac{R_{ij}^{(4)}}{S_{ij}^{(4)}y_3 + U_{ij}^{(4)}}\right)\right)$$

$$-\left(\log\left(\frac{L_{i1}}{L_{i2}}\right) - \log\left(\frac{L_{k1}}{L_{k2}}\right)\right)\left(\log\left(\frac{R_{ik}^{(4)}}{S_{ik}^{(4)}y_3 + U_{ik}^{(4)}}\right)\right)$$

$$+\left(\log\left(\frac{L_{j1}}{L_{j2}}\right) - \log\left(\frac{L_{k1}}{L_{k2}}\right)\right)\left(\log\left(\frac{R_{jk}^{(4)}}{S_{jk}^{(4)}y_3 + U_{jk}^{(4)}}\right)\right);$$

$$J_1^{(3),(ijk)}(y_3) = I(\frac{L_{i3}}{L_{i2}}, \frac{L_{j3}}{L_{j2}}; \frac{S_{ij}^{(4)}y_3 + U_{ij}^{(4)}}{R_{ij}^{(4)}}) - I(\frac{L_{i3}}{L_{i2}}, \frac{L_{k3}}{L_{k2}}; \frac{S_{ik}^{(4)}y_3 + U_{ik}^{(4)}}{R_{ik}^{4}}) + I(\frac{L_{j3}}{L_{j2}}, \frac{L_{k3}}{L_{k2}}; \frac{S_{jk}^{(4)}y_3 + U_{jk}^{(4)}}{R_{jk}^{(4)}}),$$

and

$$\frac{R_{ijk}^{(4)}}{m_{ijk}(y_3)}\, dy_3 = d\log\left(R_{ijk}^{(4)}y_3 + S_{ijk}^{(4)}\right),$$

then

$$4\Delta_4 I_4 = \sum_{1\leq i<j<k\leq 4} (-1)^{i+j+k} \int_0^\infty \left(J_0^{(3),(ijk)}(y_3) - J_1^{(3),(ijk)}(y_3)\right) d\log\left(R_{ijk}^{(4)}y_3 + S_{ijk}^{(4)}\right).$$

Then define

$$J_{0,0}^{(3),(ijk)}(y_3) = \left(\log\left(\frac{L_{i1}}{L_{i2}}\right) - \log\left(\frac{L_{j1}}{L_{j2}}\right)\right)\left(\log\left(R_{ij}^{(4)}\right) - \log\left(S_{ij}^{(4)}\right)\right)$$

$$-\left(\log\left(\frac{L_{i1}}{L_{i2}}\right) - \log\left(\frac{L_{k1}}{L_{k2}}\right)\right)\left(\log\left(R_{ik}^{(4)}\right) - \log\left(S_{ik}^{(4)}\right)\right)$$

$$+\left(\log\left(\frac{L_{j1}}{L_{j2}}\right) - \log\left(\frac{L_{k1}}{L_{k2}}\right)\right)\left(\log\left(R_{jk}^{(4)}\right) - \log\left(S_{jk}^{(4)}\right)\right);$$

and

$$J_{0,1}^{(3),(ijk)}(y_3) = \left(\log\left(\frac{L_{i1}}{L_{i2}}\right) - \log\left(\frac{L_{j1}}{L_{j2}}\right)\right)\left(\log\left(y_3 + \frac{U_{ij}^{(4)}}{S_{ij}^{(4)}}\right)\right)$$

$$-\left(\log\left(\frac{L_{i1}}{L_{i2}}\right) - \log\left(\frac{L_{k1}}{L_{k2}}\right)\right)\left(\log\left(y_3 + \frac{U_{ik}^{(4)}}{S_{ik}^{(4)}}\right)\right)$$

$$+\left(\log\left(\frac{L_{j1}}{L_{j2}}\right)-\log\left(\frac{L_{k1}}{L_{k2}}\right)\right)\left(\log\left(y_3+\frac{U_{jk}^{(4)}}{S_{jk}^{(4)}}\right)\right);$$

It is easy to see that

$$J_0^{(3),(ijk)}(y_3)=J_{0,0}^{(3),(ijk)}(y_3)-J_{0,1}^{(3),(ijk)}(y_3).$$

Define

$$J_0^{(4)}=\sum_{1\le i<j<k\le 4}(-1)^{i+j+k}\int_0^\infty\left(J_{0,0}^{(3),(ijk)}(y_3)\right)d\log\left(R_{ijk}^{(4)}y_3+S_{ijk}^{(4)}\right);$$

and

$$J_1^{(4)}=\sum_{1\le i<j<k\le 4}(-1)^{i+j+k}\int_0^\infty\left(J_{0,1}^{(3),(ijk)}(y_3)\right)d\log\left(R_{ijk}^{(4)}y_3+S_{ijk}^{(4)}\right);$$

and

$$J_2^{(4)}=\sum_{1\le i<j<k\le 4}(-1)^{i+j+k}\int_0^\infty\left(J_1^{(3),(ijk)}(y_3)\right)d\log\left(R_{ijk}^{(4)}y_3+S_{ijk}^{(4)}\right).$$

Then

$$4\Delta_4 I_4=J_0^{(4)}-J_1^{(4)}-J_2^{(4)}.$$

For proving Theorem 5.5.16, we only need to calculate the three quantities above. This is done in the following three lemmas. We will give the proof in detail only for the first lemma, the other two proofs are quite similar and straightforward.

Lemma 5.5.17

$$J_0^{(4)}=\sum_{1\le i<j<k\le 4}(-1)^{i+j+k}\left(C_2(i,j)-C_2(i,k)+C_2(j,k)\right)\left(\log(R_{ijk}^{(4)})-\log(S_{ijk}^{(4)})\right).$$

Proof 5.5.4.2 (Proof:) *We begin with the following integral*

$$\int_0^\infty\left(\log\left(\frac{L_{i1}}{L_{i2}}\right)-\log\left(\frac{L_{j1}}{L_{j2}}\right)\right)\left(\log\left(R_{ij}^{(4)}\right)-\log\left(S_{ij}^{(4)}\right)\right)d\log\left(R_{ijk}^{(4)}y_3+S_{ijk}^{(4)}\right)$$

$$=\lim_{T\to\infty}\left(\log\left(\frac{L_{i1}}{L_{i2}}\right)-\log\left(\frac{L_{j1}}{L_{j2}}\right)\right)\left(\log\left(R_{ij}^{(4)}\right)-\log\left(S_{ij}^{(4)}\right)\right)\left(\log\left(R_{ijk}^{(4)}T+S_{ijk}^{(4)}\right)-\log\left(S_{ijk}^{(4)}\right)\right)$$

$$=\lim_{T\to\infty}C_2(i,j)\left(\log\left(R_{ijk}^{(4)}T+S_{ijk}^{(4)}\right)-\log\left(S_{ijk}^{(4)}\right)\right).$$

Then

$$\int_0^\infty\left(J_{0,0}^{(3),(ijk)}(y_3)\right)d\log\left(R_{ijk}^{(4)}y_3+S_{ijk}^{(4)}\right)$$
$$=\lim_{T\to\infty}\left(C_2(i,j)-C_2(i,k)+C_2(j,k)\right)\left(\log\left(R_{ijk}^{(4)}T+S_{ijk}^{(4)}\right)-\log\left(S_{ijk}^{(4)}\right)\right).$$

Therefore the cocycle relation $\sum(-1)^{i+j+k}$*, acting on the integral above, yields a cancelation of the terms involving* $\log(T)$ *and gives us*

$$\sum_{1\le i<j<k\le 4}(-1)^{i+j+k}\left(J_{0,0}^{(3),(ijk)}(y_3)\right)d\log\left(R_{ijk}^{(4)}y_3+S_{ijk}^{(4)}\right)$$

$$=\sum_{1\le i<j<k\le 4}(-1)^{i+j+k}\left(C_2(i,j)-C_2(i,k)+C_2(j,k)\right)\left(\log\left(R_{ijk}^{(4)}\right)-\log\left(S_{ijk}^{(4)}\right)\right).$$

Lemma 5.5.18

$$J_1^{(4)}=\sum_{1\le i<j<k\le 4}(-1)^{i+j+k}J_1^{(4)}(i,j,k),$$

where

$$J_1^{(4)}(i,j,k)=C_1(i)I\left(\frac{U_{ij}^{(4)}}{S_{ij}^{(4)}},\frac{U_{ik}^{(4)}}{S_{ik}^{(4)}};\frac{S_{ijk}^{(4)}}{R_{ijk}^{(4)}}\right)-C_1(j)I\left(\frac{U_{ij}^{(4)}}{S_{ij}^{(4)}},\frac{U_{jk}^{(4)}}{S_{jk}^{(4)}};\frac{S_{ijk}^{(4)}}{R_{ijk}^{(4)}}\right)+C_1(k)I\left(\frac{U_{ik}^{(4)}}{S_{ik}^{(4)}},\frac{U_{jk}^{(4)}}{S_{jk}^{(4)}};\frac{S_{ijk}^{(4)}}{R_{ijk}^{(4)}}\right).$$

Lemma 5.5.19

$$J_2^{(4)}=\sum_{1\le i<j<k\le 4}(-1)^{i+j+k}J_2^{(4)}(i,j,k),$$

where

$$J_2^{(4)}(i,j,k)=\int_0^\infty I^{(4)}(i,j,k)(y_3)d\left(\log\left(y_3+\frac{S_{ijk}^{(4)}}{R_{ijk}^{(4)}}\right)\right),$$

and

$$I^{(4)}(i,j,k)(y_3)=I\left(\frac{L_{i3}}{L_{i2}},\frac{L_{j3}}{L_{j2}};\frac{S_{ij}^{(4)}y_3+U_{ij}^{(4)}}{R_{ij}^{(4)}}\right)-I\left(\frac{L_{i3}}{L_{i2}},\frac{L_{k3}}{L_{k2}};\frac{S_{ik}^{(4)}y_3+U_{ik}^{(4)}}{R_{ik}^{(4)}}\right)+I\left(\frac{L_{j3}}{L_{j2}},\frac{L_{k3}}{L_{k2}};\frac{S_{jk}^{(4)}y_3+U_{jk}^{(4)}}{R_{jk}^{(4)}}\right).$$

Remark 5.5.20 *The passage from the result of case* $r=3$ *to the case of* $r=4$ *can be generalized to any* $r\ge 2$*. Such a recurrence relation inspires us to do a general calculation for any general* $r\ge 2$*. We will theorefore give a definition of so-called generalized* m*-polylogarithms over a totally real field* F*. However, the coefficients such as* $R_{ijk}^{(4)}$ *are very inconvenient for general* r*, thus we will give a general definition of these coefficients in a uniform way.*

5.6 Generalized multiple zeta values for general F (II): general case

In this section we assume that a totally real field F is of degree $r>2$. Let us begin the preparation for the general calculation and give the notations in a uniform way as follows.

5.6.1 Preliminaries

Definition 5.6.1 *Let us define the set*

$$N(r) = \{1, \dots, r\}.$$

Let X and Y be two subsets of $N(r)$ of the same cardinality. Given the linear forms as in Formula 5.5.2, we will define the matrix

$$A_{X,Y} = (L_{ij})_{i \in X, j \in Y},$$

where L_{ij} is always the coefficient of the linear forms in question. Then we define

$$R^{(r)}_{X,Y} = det(A_{X,Y}).$$

For example, if $r = 4$, if we take $X_1 = \{i, j, k\}$, $1 \leq i < j < k \leq 4$ and $Y_1 = \{1, 2, 3\}$, then

$$R^{(4)}_{X_1,Y_1} = R^{(4)}_{ijk}.$$

If we take $Y_2 = \{1, 2, 4\}$, then

$$R^{(4)}_{X_1,Y_2} = S^{(4)}_{ijk}.$$

Let $X_2 = \{i, j\}$, $1 \leq i < j \leq 4$ and $Y_3 = \{1, 2\}$ and $Y_4 = \{1, 3\}$ then

$$R^{(4)}_{X_2,Y_3} = R^{(4)}_{ij}, \quad R^{(4)}_{X_2,Y_4} = S^{(4)}_{ij}.$$

Now we will define a cocycle relation operator which acts on other sets, later on we will make such an operator act on the set of indices.

Definition 5.6.2 *Assume that $n \in \mathbb{N} \setminus \{0\}$.*

Let A be a finite totally ordered set. We define

$$T_n(A) = \left\{ \underline{a} = (a_1, \dots, a_n) \in A^n \middle| a_1 < a_2 < \dots < a_n \right\}.$$

It is easy to see that $|T_n(A)| = \binom{|A|}{n}$.

Definition 5.6.3 *The n-cocycle relation operator is defined as*

$$Cycl_n(1, \dots, n) = \sum_{i=0}^{n-1} (-1)^i (1, 2, \dots, n-(i+1), \widehat{n-i}, n-(i-1), \dots, n),$$

where $\widehat{(\cdot)}$ means omitting the term $(n-i)$.

Then the cocycle operator acts on $\underline{a} \in T_n(A)$,

$$Cycl_n(\underline{a}) = \sum_{i=0}^{n-1} (-1)^i (a_1, a_2, \dots, a_{n-(i+1)}, \widehat{a_{n-i}}, a_{n-(i-1)}, \dots, a_n).$$

If f is a function depending on the multiple index $\underline{j} \in T_{n-1}(A)$, we deduce an action of the cocycle operator on f,

$$Cycl_n(f) = \sum_{i=0}^{n-1} (-1)^i f_{\underline{a}(i)},$$

where

$$\underline{a}(i) = (a_1, a_2, \ldots, a_{n-(i+1)}, \widehat{a_{n-i}}, a_{n-(i-1)}, \ldots, a_n),$$

and

$$\underline{j} \in \{\underline{a}(0), \ldots, \underline{a}(n-1)\}.$$

If there is no ambiguity, we can simply write

$$Cycl_n(f) = Cycl_n(\underline{j})(f_{\underline{j}})$$

For example, if $l_i(y) = \sum_{j=1}^n L_{ij} y_j$, $0 \le i \le n$ and if

$$N(n) = \{0, 1, \ldots, n\},$$

$$N_i = \{0, 1, \ldots, \widehat{(n-i)}, \ldots, n\},$$

$$\underline{n_i} = (0, 1, \ldots, \widehat{(n-i)}, \ldots, n)$$

$$f_{\underline{n_i}}(y) = \frac{det((L_{kh})_{k \in N_i, 1 \le h \le n})}{\prod_{j \ne n-i} l_j(y)},$$

Denote

$$det(l_{i_1}, \ldots, l_{i_n}) = det((L_{kh})_{k \in \{i_1, \ldots, i_n\}, 1 \le h \le n}),$$

then the Eisenstein trick

$$\sum_{i=0}^{n} (-1)^i \, \frac{det(l_0, l_1, \ldots, \widehat{l_i}, \ldots, l_n)}{\prod_{j \ne i} l_j(y)} = 0,$$

after replacing i by $n-i$, can be rewritten as

$$Cycl_{n+1}(f)(y) = 0.$$

5.6.2 General case of r and generalized polylogarithms over F

The cocycle relation

$$\sum_{i=0}^{n} (-1)^i \, \frac{det(l_0, l_1, \ldots, \widehat{l_{n-i}}, \ldots, l_n)}{\prod_{j \ne n-i} l_j(y)} = 0$$

is equivalent to

$$\sum_{i=0}^{n-1} (-1)^i \, \frac{det(l_0, l_1, \ldots, \widehat{l_{n-i}}, \ldots, l_n)}{\prod_{j \ne n-i} l_j(y)} + (-1)^n \frac{det(l_1, \ldots, l_n)}{l_1 \cdots l_n} = 0,$$

then

$$\frac{det(l_1, \ldots, l_n)}{l_1 \cdots l_n} = \sum_{i=0}^{n-1} (-1)^{n+i-1} \, \frac{det(l_0, l_1, \ldots, \widehat{l_{n-i}}, \ldots, l_n)}{\prod_{j \ne n-i} l_j(y)}$$

As in the previous section, we fix $l_0 = y_n$ and take $y_n = 1$, then

$$det(l_0, l_1, \ldots, \widehat{l_{n-i}}, \ldots, l_n) = (-1)^{1+n}\widetilde{det}(l_1, \ldots, \widehat{l_{n-i}}, \ldots, l_n),$$

where

$$\widetilde{det}(l_{i_1}, \ldots, l_{i_{n-1}}) = det\left((L_{kh})_{k\in\{i_1,\ldots,i_{n-1}\},1\le h\le n-1}\right)$$

Therefore

$$\frac{det(l_1, \ldots, l_n)}{l_1 \cdots l_n} = \sum_{i=0}^{n-1}(-1)^i \frac{\widetilde{det}(l_1, \ldots, \widehat{l_{n-i}}, \ldots, l_n)}{l_1 \cdots \widehat{l_{n-i}} \cdots l_n}. \tag{5.7}$$

Now recall us

$$N(n) = \{1, \ldots, n\},$$

and

$$\underline{i} = (i_1, \ldots, i_{n-1}) \in T_{n-1}(N(n)).$$

then (5.7) can be written as

$$\frac{det(l_1, \ldots, l_n)}{l_1 \cdots l_n} = Cycl_n(f) = Cycl_n(\underline{i})(f_{\underline{i}}). \tag{5.8}$$

Results of general r. Now we will talk about the case of a general r. We fix a totally real field F of degree r. Let us recall the fundamental integral of the Hecke transform.

$$I_r = I_r(L_i) = \int_{U_\mathbb{R}} \prod_{i=1}^{r} \frac{1}{L_i(u)} \mathrm{d}^\times u = \frac{1}{r}\int_{\mathbb{R}_+^{r-1}} \prod_{i=1}^{r} \frac{dy}{\sum_{j=1}^{r-1}(L_{ij}y_j) + L_{ir}}.$$

Let us fix the notation

$$\Delta_r = det(L_1, \ldots, L_r).$$

then

$$r\Delta_r I_r = \int_{\mathbb{R}_+^{r-1}} \frac{det(L_1, \ldots, L_r)}{\prod_{i=1}^{r}(\sum_{j=1}^{r-1}(L_{ij}y_j) + L_{ir})}\, dy_1 \cdots dy_{r-1}.$$

Theorem 5.6.4 (General Relation of I_r) *Given a totally real field F of degree r.*

$$N(r) = \{1, \ldots, r\}; \quad \mathrm{I}_{(r)}(\underline{i}) = \{i_1, \ldots, i_{r-1}\},$$

and

$$\underline{i} = (i_1, \ldots, i_{r-1}) \in T_{r-1}(N(r)).$$

And according to Definition 5.6.1

$$R^{(r)}_{\underline{i},r-1} = R^{(r)}_{\mathrm{I}_{(r)}(\underline{i}),N(r)\setminus\{r-1\}};$$

$$R^{(r)}_{\underline{i},r} = R^{(r)}_{\mathrm{I}_{(r)}(\underline{i}),N(r)\setminus\{r\}}.$$

Then we can obtain the result of the fundamental integral I_r from the result of I_{r-1},

$$r\Delta_r I_r = Cycl_r(\underline{i})\left(\int_0^\infty (r-1)\Delta^{(\underline{i})}_{r-1}(y_{r-1}) I^{(\underline{i})}_{r-1}(y_{r-1})\, d\log\left(y_{r-1} + \frac{R^{(r)}_{\underline{i},r-1}}{R^{(r)}_{\underline{i},r}}\right)\right),$$

where

$$\Delta_{r-1}^{(\underline{i})}(y_{r-1}) = R_{\underline{i},r}^{(r)} y_{r-1} + R_{\underline{i},r-1}^{(r)}$$

and

$$(r-1)\Delta_{r-1}^{(\underline{i})}(y_{r-1}) I_{r-1}^{(\underline{i})}(y_{r-1}) = \int_{\mathbb{R}_+^{r-2}} \frac{\Delta_{r-1}^{(\underline{i})}(y_{r-1})}{\prod_{k=1}^{r-1}(\sum_{j=1}^{r-1}(L_{i_k j} y_j) + L_{i_k r})} \, dy_1 \cdots dy_{r-2}.$$

In fact, we have considered the r-dimensional linear forms

$$l_i(y) = \sum_{j=1}^{r-1}(L_{ij} y_j) + L_{ir} y_r, \quad y_r = 1, \ 1 \le i \le r$$

as $r-1$-dimensional linear forms with a parameter y_{r-1} by regarding $L_{i(r-1)} y_{r-1} + L_{ir}$ as the $(r-1)$-th coefficient.

Proof 5.6.2.1 *We always have the same notations.*

$$l_i(y) = \sum_{j=1}^{r-1}(L_{ij} y_j) + L_{ir}, \quad 1 \le i \le r.$$

Our starting point of the proof is the cocycle relations (5.7) and (5.8).

$$\frac{det(l_1, \ldots, l_n)}{l_1 \cdots l_n} = \sum_{i=0}^{n-1} (-1)^i \frac{\widetilde{det}(l_1, \ldots, \widehat{l_{n-i}}, \ldots, l_n)}{l_1 \cdots \widehat{l_{n-i}} \cdots l_n}.$$

In the notation of Definition 5.6.1, we define

$$\widetilde{det}(l_{i_1}, \ldots, l_{i_r - 1}) = R_{\underline{i},r}^{(r)},$$

then

$$\frac{\widetilde{det}(l_{i_1}, \ldots, l_{i_r - 1})}{l_{i_1} \cdots l_{i_{r-1}}} = \frac{R_{\underline{i},r}^{(r)} y_{r-1} + R_{\underline{i},r-1}^{(r)}}{l_{i_1}, \ldots, l_{i_r - 1}} \times \frac{R_{\underline{i},r}^{(r)}}{R_{\underline{i},r}^{(r)} y_{r-1} + R_{\underline{i},r-1}^{(r)}}$$

and

$$\frac{R_{\underline{i},r}^{(r)}}{R_{\underline{i},r}^{(r)} y_{r-1} + R_{\underline{i},r-1}^{(r)}} \, dy_{r-1} = d \log \left(y_{r-1} + \frac{R_{\underline{i},r-1}^{(r)}}{R_{\underline{i},r}^{(r)}} \right).$$

This finishes the proof of the theorem.

Theorem 5.6.5 ***(General expression of the basic integral of the Hecke transform)*** *The fundamental integral of the Hecke transform can be written as a sum of of $r-1$ terms*

$$r\Delta_r I_r = J_0^{(r)} - J_1^{(r)} - \ldots - J_{r-2}^{(r)},$$

where the term $J_0^{(r)}$ is a sum of logarithms, and $J_1^{(r)}$ involves dilogarithms and logarithms, $J_j^{(r)}$ $(1 \le j \le r-2)$ are sums of iterated integrals of degree $j+1$ of differential forms $\frac{dy}{y+\beta_j}$, which we call the generalized $(j+1)$-logarithms, where β_j is a rational functions of the coefficients L_{ij}.

Proof 5.6.2.2 *We will prove this theorem by mathematical induction.*

The first step. *Theorem 5.5.4, Theorem 5.5.9 and Theorem 5.5.16 imply this theorem when $r = 2, 3, 4$, respectively.*

The inductive step. *We suppose that the theorem holds when $r \leq k-1$, now we will prove when $r = k$ $(k \geq 3)$. From Theorem 5.6.4 and the inductive hypothesis, we can deduce that*

$$Cycl_r(\underline{i})\left(\int_0^\infty J_0^{(r-1),(\underline{i})}(y_{r-1})\; d\log\left(y_{r-1} + \frac{R^{(r)}_{\underline{i},r-1}}{R^{(r)}_{\underline{i},r}}\right)\right) = J_0^{(r)} - J_1^{(r)};$$

$$Cycl_r(\underline{i})\left(\int_0^\infty J_j^{(r-1),(\underline{i})}(y_{r-1})\; d\log\left(y_{r-1} + \frac{R^{(r)}_{\underline{i},r-1}}{R^{(r)}_{\underline{i},r}}\right)\right) = J_{j+1}^{(r)}, \quad 1 \leq j \leq r-3.$$

The theorem is deduced from the two relations above. Moreover we can give a more precise expression of the first term. To do that, we will define a family of functions by induction. Recall that $\underline{i}$ is a $(r-1)$-tuple, we will define a filtration between the tuples. Let

$${}^{(h)}\underline{i} = ({}^{(h)}i_1, \ldots, {}^{(h)}i_h),\ 1 \leq h \leq r-1$$

be a h-tuple, where the upper left symbol ${}^{(h)}$ signifies the size h of this tuple.

$${}^{(r-1)}\underline{i} = \underline{i} = (i_1, \ldots, i_{r-1}).$$

Let the set

$$\mathrm{I}_r({}^{(h)}\underline{i}) = \left\{{}^{(h)}i_1, \ldots, {}^{(h)}i_h\right\}.$$

Then we define the filtration relation as follows

$${}^{(h-1)}\underline{i} = ({}^{(h-1)}i_1, \ldots, {}^{(h-1)}i_{h-1}) \subset {}^{(h)}\underline{i}$$

if

$${}^{(h)}i_\alpha \in \left\{{}^{(h)}i_\beta,\ 1 \leq \beta \leq h\right\},\ 1 \leq \alpha \leq h-1.$$

In the notation of Definition 5.6.1, define

$$C_1(i) = \log(L_{i1}) - \log(L_{i2});$$

$$C_2(i,j) = (C_1(i) - C_1(j))\left(\log\left(R^{(r)}_{\{i,j\},\{1,2\}}\right) - \log\left(R^{(r)}_{\{i,j\},\{1,3\}}\right)\right);$$

then

$$C_h({}^{(h)}\underline{i}) = \left(Cycl_h({}^{(h)}\underline{i})\left(C_{h-1}({}^{(h-1)}\underline{i})\right)\right)\left(\log\left(R^{(r)}_{\mathrm{I}_r({}^{(h)}\underline{i}),\{1,\ldots,h\}}\right) - \log\left(R^{(r)}_{\mathrm{I}_r({}^{(h)}\underline{i}),\{1,\ldots,h-2,h+1\}}\right)\right),$$

where $2 \leq h \leq r-1$.

$$C_{r-1}(\underline{i}) = \left(Cycl_{r-1}(\underline{i})\left(C_{r-2}({}^{(r-2)}\underline{i})\right)\right)\left(\log\left(R^{(r)}_{\mathrm{I}_r(\underline{i}),N(r)\smallsetminus\{r\}}\right) - \log\left(R^{(r)}_{\mathrm{I}_r(\underline{i}),N(r)\smallsetminus\{r-1\}}\right)\right).$$

Using the above formulas, one obtains by direct calculation that

$$J_0^{(r)} = Cycl_r(\underline{i})(C_{r-1}(\underline{i})).$$

Chapter 6

The values $Z_{I,F}(\Gamma, S)$ for imaginary quadratic fields

Francis Brown formulated a program about the modularity of mixed motives [4], with the purpose of generalizing multiple zeta values and special values of L-functions of modular forms at all integers. In Brown's observation, there exsits a modular phenomena in the ring of multiple zeta values relating a mysterious depth filtration. He proposed that a geometric understanding of these phenomena should put multiple zeta values and modular form for $SL_2(\mathbb{Z})$ in the same framework.

$$\begin{tikzcd}
\mathbb{P}^1 \smallsetminus \{0,1,\infty\} \arrow[r, Rightarrow, "replaced", "with"'] & \Gamma\backslash\mathcal{H} \\
\pi_1^{un} \arrow[r, Rightarrow, "replaced", "with"'] & G_\Gamma \\
MZVs \arrow[r, Rightarrow, "replaced", "with"'] & MMVs \\
\mathscr{M}\mathscr{T}(\mathbb{Z}) \arrow[r, Rightarrow, "replaced", "with"'] & \mathscr{M}\mathscr{M}\mathscr{M}_\Gamma,
\end{tikzcd}$$

where Γ is a subgroup of $SL_2(\mathbb{Z})$ with finite index, and G_Γ is relative completion of SL_2 respect to Γ. $MMVs$ signifies multiple modular values, which are regularized iterated integrals of modular forms on an orbifold $\Gamma\backslash\backslash\mathcal{H}$ and are also periods of the hyperthetical Tannakian category $\mathscr{M}\mathscr{M}\mathscr{M}_\Gamma$ of mixed modular motives.

$\mathscr{M}\mathscr{T}(\mathbb{Z})$ is the mixed Tate motives over $\mathbb{Z}$, and the category $\mathscr{M}\mathscr{M}\mathscr{M}_\Gamma$ should be generated by iterated extension of motives of modular forms.

Remark 6.0.6 *We want to give some remarks on Brown's MMVs and Goncharov's Hodge correlators. If g is a cusp form for $SL_2(\mathbb{Z})$, then the special value $L(g,n) \times (2\pi i)^?$ is a multiple modular value for $SL_2(\mathbb{Z})$, and is also a period in the sense of Kontsevich- Zagier. Goncharov's Hodge correlators contain the special values $L(f,n)$ of modular forms for $GL_2(\mathbb{Q})$ of weight $k \geq 2$, outside of critical strip. In fact the simplest Hodge correlators in this case coincide with the Rankin-Selberg integrals for $L(f,n)$.*

We could ask if our generalized multiple zeta values can be immerged into such a program.

We begin with the classical problem, studied by Gangl-Kaneko-Zagier and Goncharov.

Recall the work of Gangl-Kaneko-Zagier and Goncharov's results. For this chapter we suppose that the field K is an imaginary quadratic field. We will show that our generalized multiple zeta value, for an imaginary quadratic field, could also be a good candidate as being multiple Eisenstein series, with the hope to be related to some non-holomorphic new modular forms. We recall the theorem 6 in [5].

Theorem 6.0.7 (Gangl, Kaneko and Zagier) *[5] The Fourier expansion of*

$$G_{r,s}(\tau) = \sum_{\substack{\mathbf{m},\mathbf{n} \in \mathbb{Z}\tau+\mathbb{Z} \\ m>n>0}} \frac{1}{\mathbf{m}^r \mathbf{n}^s}$$

for $r \geq 3$ and $s \geq 2$ and $\tau \in \mathcal{H}$ the upper half plane is given by

$$(2\pi i)^{-r-s} G_{r,s} = (2\pi i)^{-r-s}\zeta(r,s) + \sum_{\substack{h+p=r+s \\ h,p>1}} (2\pi i)^{-p} C_{r,s}^{p} g_h(q)\zeta(p) + g_{r,s}(q)$$

with $q = e^{2\pi i \tau}$ and

$$C_{r,s}^{p} = \delta_{s,p} + (-1)^s C_{p-1}^{s-1} + (-1)^{p-r} C_{p-1}^{r-1} \in \mathbb{Z}$$

and

$$g_{r,s}(q) = \frac{(-1)^{r+s}}{(r-1)!(s-1)!} \sum_{\substack{m>n>0 \\ u>0,v>0}} u^{r-1} v^{s-1} q^{um+vn} \in \mathbb{Q}[[q]],$$

$$g_k(q) = \frac{(-1)^k}{(k-1)!} \sum_{u,n>0} u^{k-1} q^{un}.$$

New result over an imaginary quadratic field. We can also get a similar result for our generalized multiple zeta value $Z_{I,K}(\Gamma, S)$ if the field $K = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic field. In fact, it is also the first example of a non-totally real field case, where the Hecke transform is trivial.

Notation 6.0.8 *We will note $\tau = \sqrt{-D}$ if $-D \equiv 2, 3(4)$ and $\tau = \frac{1+\sqrt{-D}}{2}$ if $-D \equiv 1(4)$. Let $N(\alpha) = \alpha \cdot \overline{\alpha}$ be the standard norm of F.*

For an imaginary quadratic field, our definition about higher plectic Green functions still works, however there is no $u \in U$ the subgroup of $U \subset O_{K,+}^{\times}$.

Let Γ_1 be the graph Figure 1.

$$G_{I,\mathbb{Q}(\sqrt{-D}),\Gamma_1,\partial\Gamma_1}(\{x_v\}_{v\in\partial\Gamma_1}, 1) = \sum_{\alpha+\beta+\gamma=0} \frac{e^{-2\pi i Tr(\alpha\cdot x_0+\beta\cdot x_1+\gamma\cdot x_2)}}{N(\alpha)^{\sigma_1} N(\beta)^{\sigma_2} N(\gamma)^{\sigma_3}}$$

$$Z_{I,\mathbb{Q}(\sqrt{-D})}(\Gamma_1, \partial\Gamma_1) = G_{I,\mathbb{Q}(\sqrt{-D}),\Gamma_1,\partial\Gamma_1}(\{0\}_{v\in\partial\Gamma_1}, 1) = \sum_{\alpha,\beta} \frac{1}{N(\alpha)^{\sigma_1} N(\beta)^{\sigma_2} N(\alpha+\beta)^{\sigma_3}}.$$

If $\alpha = a_1 + a_2\tau \in O_F, a_1, a_2 \in \mathbb{Z}$ and $\beta = b_1 + b_2\tau \in O_F, b_1, b_2 \in \mathbb{Z}$, then

$$Z_{I,\mathbb{Q}(\sqrt{-D})}(\Gamma_1, \partial\Gamma_1) =$$

$$\sum \frac{1}{(a_1+a_2\tau)^{\sigma_1}(a_1+a_2\bar{\tau})^{\sigma_1}(b_1+b_2\tau)^{\sigma_2}(b_1+b_2\bar{\tau})^{\sigma_2}(a_1+b_1+(a_2+b_2)\tau)^{\sigma_3}(a_1+b_1+(a_2+b_2)\bar{\tau})^{\sigma_3}}.$$

We can also add a new sign μ for each edge of the graph Γ_1, such that $\alpha > 0$ and $\beta > 0$, where $\alpha = a_1 + a_2\tau > 0$ means that if $a_2 > 0$ or $a_2 = 0$, $a_1 > 0$. Then

$$Z_{I,\mu,\mathbb{Q}(\sqrt{-D})}(\Gamma_1, \partial\Gamma_1) = \sum_{\alpha,\beta>0} \frac{1}{N(\alpha)^{\sigma_1}N(\beta)^{\sigma_2}N(\alpha+\beta)^{\sigma_3}}$$

$$= \sum_{\substack{a_1+a_2\tau>0\\ b_1+b_2>0}} \frac{1}{(a_1+a_2\tau)^{\sigma_1}(a_1+a_2\bar{\tau})^{\sigma_1}(b_1+b_2\tau)^{\sigma_2}(b_1+b_2\bar{\tau})^{\sigma_2}(a_1+b_1+(a_2+b_2)\tau)^{\sigma_3}(a_1+b_1+(a_2+b_2)\bar{\tau})^{\sigma_3}}$$

$$= \sum_{\substack{a_2=b_2=0\\ a_1>0,b_1>0}} + \sum_{\substack{a_2=0,b_2>0\\ b_1\in\mathbb{Z}}} + \sum_{\substack{a_2>0,b_2=0\\ a_1\in\mathbb{Z}}} + \sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}}.$$

The first sum $\sum_{\substack{a_2=b_2=0\\ a_1>0,b_1>0}}$.

$$\sum_{\substack{a_2=b_2=0\\ a_1>0,b_1>0}} = \sum_{a_1>0,b_1>0} \frac{1}{(a_1)^{2\sigma_1}(b_1)^{2\sigma_2}(a_1+b_1)^{2\sigma_3}} = Z_{I,\nu,\mathbb{Q}}(\Gamma_1, \partial\Gamma_1),$$

where

$$Z_{I,\nu,\mathbb{Q}}(\Gamma_1, \partial\Gamma_1) = \sum_{r+s=2\sigma_1+2\sigma_2} \left(C_{r-1}^{2\sigma_1-1} + C_{r-1}^{2\sigma_2-1}\right)\zeta(s, r+2\sigma_3).$$

The second sum $\sum_{\substack{a_2=0,b_2>0\\ a_1>0,b_1\in\mathbb{Z}}}$.

$$\sum_{\substack{a_2=0,b_2>0\\ b_1\in\mathbb{Z}}} = \sum_{\substack{a_1>0\\ b_2>0,b_1\in\mathbb{Z}}} \frac{1}{(a_1)^{2\sigma_1}(b_1+b_2\tau)^{\sigma_2}(b_1+b_2\bar{\tau})^{\sigma_2}(a_1+b_1+b_2\tau)^{\sigma_3}(a_1+b_1+b_2\bar{\tau})^{\sigma_3}}$$

Now we apply the Eisenstein's formula,

$$\frac{1}{(b_1+b_2\tau)^{\sigma_2}(a_1+b_1+b_2\tau)^{\sigma_3}} = \sum_{r_1+s_1=\sigma_2+\sigma_3} \left(\frac{(-1)^{\sigma_2}C_{r_1-1}^{\sigma_2-1}}{a_1^{r_1}(a_1+b_1+b_2\tau)^{s_1}} + \frac{(-1)^{\sigma_2+s_1}C_{r_1-1}^{\sigma_3-1}}{a_1^{r_1}(b_1+b_2\tau)^{s_1}}\right).$$

And

$$\frac{1}{(b_1+b_2\bar{\tau})^{\sigma_2}(a_1+b_1+b_2\bar{\tau})^{\sigma_3}} = \sum_{r_2+s_2=\sigma_2+\sigma_3} \left(\frac{(-1)^{\sigma_2}C_{r_2-1}^{\sigma_2-1}}{a_1^{r_2}(a_1+b_1+b_2\bar{\tau})^{s_2}} + \frac{(-1)^{\sigma_2+s_2}C_{r_2-1}^{\sigma_3-1}}{a_1^{r_2}(b_1+b_2\bar{\tau})^{s_2}}\right).$$

Then

$$\sum_{\substack{a_2=0,b_2>0\\ b_1\in\mathbb{Z}}} = A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = \sum_{\substack{a_1>0\\ b_2>0,b_1\in\mathbb{Z}}} \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3}} \frac{C_{r_1-1}^{\sigma_2-1}C_{r_2-1}^{\sigma_2-1}}{a_1^{r_1+r_2+\sigma_1}(a_1+b_1+b_2\tau)^{s_1}(a_1+b_1+b_2\bar{\tau})^{s_2}}$$

$$A_2 = \sum_{\substack{a_1>0\\ b_2>0,b_1\in\mathbb{Z}}} \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3}} \frac{(-1)^{s_2} C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_3-1}}{a_1^{r_1+r_2+\sigma_1}(a_1+b_1+b_2\tau)^{s_1}(b_1+b_2\bar{\tau})^{s_2}}$$

$$A_3 = \sum_{\substack{a_1>0\\ b_2>0,b_1\in\mathbb{Z}}} \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3}} \frac{(-1)^{s_1} C_{r_2-1}^{\sigma_2-1} C_{r_1-1}^{\sigma_3-1}}{a_1^{r_1+r_2+\sigma_1}(a_1+b_1+b_2\bar{\tau})^{s_1}(b_1+b_2\tau)^{s_2}}$$

$$A_4 = \sum_{\substack{a_1>0\\ b_2>0,b_1\in\mathbb{Z}}} \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3}} \frac{(-1)^{s_1+s_2} C_{r_2-1}^{\sigma_3-1} C_{r_1-1}^{\sigma_3-1}}{a_1^{r_1+r_2+\sigma_1}(b_1+b_2\tau)^{s_1}(b_1+b_2\bar{\tau})^{s_2}}.$$

(1) The first term can computed by the Lipschitz summation formula, due to H. Maass and John Hawkins.

$$A_1 = \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3}} \sum_{\substack{a_1>0,b_2>0\\ b_1'\in\mathbb{Z}}} \frac{C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_2-1}}{a_1^{r_1+r_2+\sigma_1}(b_1'+b_2\tau)^{s_1}(b_1'+b_2\bar{\tau})^{s_2}}$$

$$= \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3}} C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_2-1} \sum_{a_1>0} \frac{1}{a_1^{r_1+r_2+\sigma_1}} \sum_{b_2>0,b_1'\in\mathbb{Z}} \frac{1}{(b_1'+b_2\tau)^{s_1}(b_1'+b_2\bar{\tau})^{s_2}},$$

where we remplaced $a_1 + b_1$ by b_1'.

Theorem 6.0.9 (Lipschitz summation formula, Hans Maass 96') *[17]*

$$\sum_{m\in\mathbb{Z}} \frac{e^{-2\pi i\mu m}}{(\tau+m)^{\alpha_1}(\bar{\tau}+m)^{\alpha_2}} = \frac{(2\pi)^{\alpha_1+\alpha_2}(-1)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \sum_{n\in\mathbb{Z}} a_{n+\mu}(y,\alpha_1,\alpha_2)e^{2\pi i(n+\mu)x},$$

where $\tau = x + iy \in \mathcal{H} = \{z \in \mathbb{C} : Imz > 0\}$, $\mu \in \mathbb{R}$, $Re(\alpha_1+\alpha_2) > 1$ *and* $a_{n+\mu}(y,\alpha_1,\alpha_2)$

$$= \begin{cases} \Gamma(\alpha_1+\alpha_2-1)(4\pi y)^{1-\alpha_1-\alpha_2}, & \textit{if } n+\mu = 0 \\ (n+\mu)^{\alpha_1+\alpha_2-1}e^{-2\pi(n+\mu)y}\sigma(4\pi(n+\mu)y,\alpha_1,\alpha_2), & \textit{if } n+\mu > 0 \\ (-n-\mu)^{\alpha_1+\alpha_2-1}e^{2\pi(n+\mu)y}\sigma(-4\pi(n+\mu)y,\alpha_2,\alpha_1), & \textit{if } n+\mu < 0 \end{cases}$$

where $\sigma(\eta,\alpha,\beta)$ *denotes the special function which has the following integral representation:*

$$\int_0^\infty (u+1)^{\alpha-1}u^{\beta-1}e^{-\eta u}du, \alpha \in \mathbb{C}, Re(\beta) > 0, Re(\eta) > 0.$$

Remark 6.0.10 *The function* $\sigma(\eta,\alpha,\beta)$ *is a generalization of* $\Gamma(\beta)$. *In fact, the confluent hypergeometric function of the second kind*

$$\Psi(\beta,\alpha+\beta;\eta) = \frac{\sigma(\eta,\alpha,\beta)}{\Gamma(\beta)}$$

is an entire function in α *and* β.

Definition 6.0.11 *For the simplicity of use, we denote*

$$\phi_{\alpha_1,\alpha_2}(\tau)=\sum_{m\in\mathbb{Z}}\frac{1}{(\tau+m)^{\alpha_1}(\bar{\tau}+m)^{\alpha_2}}.$$

Then

$$\phi_{\alpha_1,\alpha_2}(\tau)=(2\pi i)^{\alpha_1+\alpha_2}(-1)^{\alpha_1-\alpha_2}\frac{\Gamma(\alpha_1+\alpha_2-1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}(4\pi y)^{1-\alpha_1-\alpha_2}$$

$$+\sum_{n\in\mathbb{N}_+}\frac{(2\pi)^{\alpha_1+\alpha_2}(-1)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1)}n^{\alpha_1+\alpha_2-1}\frac{\sigma(4\pi ny,\alpha_1,\alpha_2)}{\Gamma(\alpha_2)}e^{2\pi in\tau}$$

$$+\sum_{n\in\mathbb{N}_-}\frac{(2\pi)^{\alpha_1+\alpha_2}(-1)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_2)}(-n)^{\alpha_1+\alpha_2-1}\frac{\sigma(-4\pi ny,\alpha_2,\alpha_1)}{\Gamma(\alpha_1)}e^{2\pi in\bar{\tau}}.$$

Then we can get

$$A_1=\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3}}C_{r_1-1}^{\sigma_2-1}C_{r_2-1}^{\sigma_2-1}\zeta(r_1+r_2+\sigma_1)\sum_{b_2\in\mathbb{N}_+^*}\phi_{s_1,s_2}(b_2\tau).$$

(2)In the same way, we can get

$$A_4=\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3}}(-1)^{s_1+s_2}C_{r_1-1}^{\sigma_3-1}C_{r_2-1}^{\sigma_3-1}\zeta(r_1+r_2+\sigma_1)\sum_{b_2\in\mathbb{N}_+^*}\phi_{s_1,s_2}(b_2\tau).$$

(3)Now we turn to the second and the third term.

$$\frac{1}{a_1^{r_1+r_2+\sigma_1}(a_1+b_1+b_2\tau)^{s_1}}=\sum_{t_1+t_2=r_1+r_2+\sigma_1+s_1}\left(\frac{(-1)^{r_1+r_2+\sigma_1}C_{t_1-1}^{r_1+r_2+\sigma_1-1}}{(b_1+b_2\tau)^{t_1}(a_1+b_1+b_2\tau)^{t_2}}+\frac{(-1)^{r_1+r_2+\sigma_1+t_2}C_{t_1-1}^{s_1-1}}{(b_1+b_2\tau)^{t_1}a_1^{t_2}}\right).$$

Thus the second term can be written as

$$A_2=B_1+B_2,$$

where

$$B_1=\sum_{\substack{a_1>0\\ b_2>0,b_1\in\mathbb{Z}}}\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3\\ t_1+t_2=r_1+r_2+\sigma_1+s_1}}\frac{(-1)^{r_1+r_2+\sigma_1+s_2}C_{r_1-1}^{\sigma_2-1}C_{r_2-1}^{\sigma_3-1}C_{t_1-1}^{r_1+r_2+\sigma_1-1}}{(b_1+b_2\tau)^{t_1}(b_1+b_2\bar{\tau})^{s_2}(a_1+b_1+b_2\tau)^{t_2}},$$

and

$$B_2=\sum_{\substack{a_1>0\\ b_2>0,b_1\in\mathbb{Z}}}\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3\\ t_1+t_2=r_1+r_2+\sigma_1+s_1}}\frac{(-1)^{r_1+r_2+\sigma_1+s_2+t_2}C_{r_1-1}^{\sigma_2-1}C_{r_2-1}^{\sigma_3-1}C_{t_1-1}^{s_1-1}}{(b_1+b_2\bar{\tau})^{s_2}(b_1+b_2\tau)^{t_1}a_1^{t_2}}.$$

It's easy to see that we can again apply the theorem 6.0.9 to calculate the term B_2.

$$B_2=\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3\\ t_1+t_2=r_1+r_2+\sigma_1+s_1}}(-1)^{r_1+r_2+\sigma_1+s_2+t_2}C_{r_1-1}^{\sigma_2-1}C_{r_2-1}^{\sigma_3-1}C_{t_1-1}^{s_1-1}\sum_{a_1>0}\frac{1}{a_1^{t_2}}\sum_{\substack{b_2>0\\ b_1\in\mathbb{Z}}}\frac{1}{(b_1+b_2\bar{\tau})^{s_2}(b_1+b_2\tau)^{t_1}}$$

$$=\sum_{\substack{r_1+s_1=\sigma_2+\sigma_3\\ r_2+s_2=\sigma_2+\sigma_3\\ t_1+t_2=r_1+r_2+\sigma_1+s_1}}(-1)^{r_1+r_2+\sigma_1+s_2+t_2}C_{r_1-1}^{\sigma_2-1}C_{r_2-1}^{\sigma_3-1}C_{t_1-1}^{s_1-1}\zeta(t_2)\times\sum_{b_2\in\mathbb{N}_+^*}\phi_{t_1,s_2}(b_2\tau).$$

In order to calculate the term B_1, we will mention another theorem in [17].

Theorem 6.0.12 (Two-variable summation formula, Pasles and De Azevedo Pribitkin, 2001) *[17] If $\mu \in \mathbb{R}$, $Re(\alpha+\beta) > 1$, $Re(\gamma) > 0$, $\tau = x + iy \in \mathcal{H}$ and $z \in \mathcal{H}$, then*

$$\sum_{m \in \mathbb{Z}} \frac{e^{-2\pi i \mu m}}{(\tau+m)^{\alpha}(\bar{\tau}+m)^{\beta}(z+m)^{\gamma}} = \frac{(2\pi)^{\alpha+\beta+\gamma}(-i)^{\alpha-\beta+\gamma}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \sum_{\substack{n,l \in \mathbb{Z} \\ l+\mu>0}} \varphi_{n,l+\mu}(\tau, z, \alpha, \beta, \gamma) e^{2\pi i[nx+(l+\mu)z]},$$

where

$$\varphi_{n,l+\mu}(\tau, z, \alpha, \beta, \gamma) = \begin{cases} e^{-2\pi n y} \int_0^b (n+t)^{\alpha+\beta-1}(l+\mu-t)^{\gamma-1} e^{2\pi i(\tau-z)t} \sigma(4\pi(n+t)y, \alpha, \beta) dt, & \textit{if } n \geq 0 \\ e^{2\pi n y} \int_0^b (-n-t)^{\alpha+\beta-1}(l+\mu-t)^{\gamma-1} e^{2\pi i(\bar{\tau}-z)t} \sigma(-4\pi(n+t)y, \beta, \alpha) dt, & \textit{if } n < 0, \end{cases}$$

here

$$b = b(l+[\mu]) = \begin{cases} \{\mu\}, & \textit{if } l+[\mu] = 0 \\ 1, & \textit{if } l+[\mu] \geq 1. \end{cases}$$

$\{\mu\}$ denotes the fractional part and $[\mu]$ denotes the integer part of μ.

Definition 6.0.13

$$\psi_{\alpha,\beta,\gamma}(\tau, z) = \sum_{m \in \mathbb{Z}} \frac{1}{(\tau+m)^{\alpha}(\bar{\tau}+m)^{\beta}(z+m)^{\gamma}}$$

$$= \frac{(2\pi)^{\alpha+\beta+\gamma}(-i)^{\alpha-\beta+\gamma}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \sum_{\substack{n,l \in \mathbb{Z} \\ l>0}} \varphi_{n,l}(\tau, z, \alpha, \beta, \gamma) e^{2\pi i[nx+lz]}.$$

Then we can deduce that

$$B_1 = \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ t_1+t_2=r_1+r_2+\sigma_1+s_1}} (-1)^{r_1+r_2+\sigma_1+s_2} C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_3-1} C_{t_1-1}^{r_1+r_2+\sigma_1-1} \sum_{a_1>0} \sum_{b_2>0} \psi_{t_1,s_2,t_2}(b_2\tau, b_2\tau + a_1).$$

Finally, we get

$$A_2 = \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ t_1+t_2=r_1+r_2+\sigma_1+s_1}} (-1)^{r_1+r_2+\sigma_1+s_2+t_2} C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_3-1} C_{t_1-1}^{s_1-1} \zeta(t_2) \times \sum_{b_2 \in \mathbb{N}_+^*} \phi_{t_1,s_2}(b_2\tau)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ t_1+t_2=r_1+r_2+\sigma_1+s_1}} (-1)^{r_1+r_2+\sigma_1+s_2} C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_3-1} C_{t_1-1}^{r_1+r_2+\sigma_1-1} \sum_{a_1>0} \sum_{b_2>0} \psi_{t_1,s_2,t_2}(b_2\tau, b_2\tau + a_1).$$

(4) At last we turn to the third term A_3.

$$\frac{1}{a_1^{r_1+r_2+\sigma_1}(b_1+b_2\tau)^{s_2}} = \sum_{\substack{k_1+k_2= \\ r_1+r_2+\sigma_1+s_2}} \left(\frac{C_{k_1-1}^{r_1+r_2+\sigma_1-1}}{(a_1+b_1+b_2\tau)^{k_1}(b_1+b_2\tau)^{k2}} + \frac{C_{k_1-1}^{s_2-1}}{(a_1+b_1+b_2\tau)^{k_1} a_1^{k_2}} \right)$$

Then

$$A_3 = \sum_{\substack{a_1>0 \\ b_2>0, b_1 \in \mathbb{Z}}} \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ k_1+k_2=r_1+r_2+\sigma_1+s_2}} \frac{(-1)^{s_1} C_{r_2-1}^{\sigma_2-1} C_{r_1-1}^{\sigma_3-1} C_{k_1-1}^{r_1+r_2+\sigma_1-1}}{(a_1+b_1+b_2\tau)^{k_1}(a_1+b_1+b_2\bar{\tau})^{s_1}(b_1+b_2\tau)^{k2}}$$

$$+ \sum_{\substack{a_1>0 \\ b_2>0, b_1\in\mathbb{Z}}} \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ k_1+k_2=r_1+r_2+\sigma_1+s_2}} \frac{(-1)^{s_1} C_{r_2-1}^{\sigma_2-1} C_{r_1-1}^{\sigma_3-1} C_{k_1-1}^{s_2-1}}{(a_1+b_1+b_2\tau)^{k_1}(a_1+b_1+b_2\bar{\tau})^{s_1} a_1^{k_2}}.$$

By a similar method as for A_2, we finally get

$$A_3 = \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ k_1+k_2=r_1+r_2+\sigma_1+s_2}} (-1)^{s_1} C_{r_2-1}^{\sigma_2-1} C_{r_1-1}^{\sigma_3-1} C_{k_1-1}^{s_2-1} \zeta(k_2) \times \sum_{b_2\in\mathbb{N}_+^*} \phi_{k_1,s_1}(b_2\tau)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ k_1+k_2=r_1+r_2+\sigma_1+s_1}} (-1)^{s_1} C_{r_2-1}^{\sigma_2-1} C_{r_1-1}^{\sigma_3-1} C_{k_1-1}^{r_1+r_2+\sigma_1-1} \sum_{a_1>0} \sum_{b_2>0} \psi_{k_1,s_1,k_2}(b_2\tau, b_2\tau-a_1).$$

In concusion, we obtain the second sum

$$\sum_{\substack{a_2=0,b_2>0 \\ b_1\in\mathbb{Z}}} = A_1 + A_2 + A_3 + A_4 = \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3}} C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_2-1} \zeta(r_1+r_2+\sigma_1) \sum_{b_2\in\mathbb{N}_+^*} \phi_{s_1,s_2}(b_2\tau)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3}} (-1)^{s_1+s_2} C_{r_1-1}^{\sigma_3-1} C_{r_2-1}^{\sigma_3-1} \zeta(r_1+r_2+\sigma_1) \sum_{b_2\in\mathbb{N}_+^*} \phi_{s_1,s_2}(b_2\tau)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ t_1+t_2=r_1+r_2+\sigma_1+s_1}} (-1)^{r_1+r_2+\sigma_1+s_2+t_2} C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_3-1} C_{t_1-1}^{s_1-1} \zeta(t_2) \times \sum_{b_2\in\mathbb{N}_+^*} \phi_{t_1,s_2}(b_2\tau)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ t_1+t_2=r_1+r_2+\sigma_1+s_1}} (-1)^{r_1+r_2+\sigma_1+s_2} C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_3-1} C_{t_1-1}^{r_1+r_2+\sigma_1-1} \sum_{a_1>0} \sum_{b_2>0} \psi_{t_1,s_2,t_2}(b_2\tau, b_2\tau+a_1)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ k_1+k_2=r_1+r_2+\sigma_1+s_2}} (-1)^{s_1} C_{r_2-1}^{\sigma_2-1} C_{r_1-1}^{\sigma_3-1} C_{k_1-1}^{s_2-1} \zeta(k_2) \times \sum_{b_2\in\mathbb{N}_+^*} \phi_{k_1,s_1}(b_2\tau)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ k_1+k_2=r_1+r_2+\sigma_1+s_1}} (-1)^{s_1} C_{r_2-1}^{\sigma_2-1} C_{r_1-1}^{\sigma_3-1} C_{k_1-1}^{r_1+r_2+\sigma_1-1} \sum_{a_1>0} \sum_{b_2>0} \psi_{k_1,s_1,k_2}(b_2\tau, b_2\tau-a_1).$$

The third sum $\sum_{\substack{a_2>0,b_2=0 \\ a_1\in\mathbb{Z},b_1>0}}$.

By the symmetry between a_1 and b_1 and the symmetry between a_2 and b_2, we can deduce the third sum from the result of the second sum.

$$\sum_{\substack{b_2=0,b_1>0 \\ a_2>0,a_1\in\mathbb{Z}}} = \sum_{\substack{r_1'+s_1'=\sigma_1+\sigma_3 \\ r_2'+s_2'=\sigma_1+\sigma_3}} C_{r_1'-1}^{\sigma_1-1} C_{r_2'-1}^{\sigma_1-1} \zeta(r_1'+r_2'+\sigma_2) \sum_{a_2\in\mathbb{N}_+^*} \phi_{s_1',s_2'}(a_2\tau)$$

$$+ \sum_{\substack{r'_1+s'_1=\sigma_1+\sigma_3\\ r'_2+s'_2=\sigma_1+\sigma_3}} (-1)^{s'_1+s'_2} C^{\sigma_3-1}_{r'_1-1} C^{\sigma_3-1}_{r'_2-1} \zeta(r'_1+r'_2+\sigma_2) \sum_{a_2\in\mathbb{N}^*_+} \phi_{s'_1,s'_2}(a_2\tau)$$

$$+ \sum_{\substack{r'_1+s'_1=\sigma_1+\sigma_3\\ r'_2+s'_2=\sigma_1+\sigma_3\\ t'_1+t'_2=r'_1+r'_2+\sigma_2+s'_1}} (-1)^{r'_1+r'_2+\sigma_2+s'_2+t'_2} C^{\sigma_1-1}_{r'_1-1} C^{\sigma_3-1}_{r'_2-1} C^{s'_1-1}_{t'_1-1} \zeta(t'_2) \times \sum_{a_2\in\mathbb{N}^*_+} \phi_{t'_1,s'_2}(a_2\tau)$$

$$+ \sum_{\substack{r'_1+s'_1=\sigma_1+\sigma_3\\ r'_2+s'_2=\sigma_1+\sigma_3\\ t'_1+t'_2=r'_1+r'_2+\sigma_2+s'_1}} (-1)^{r'_1+r'_2+\sigma_2+s'_2} C^{\sigma_1-1}_{r'_1-1} C^{\sigma_3-1}_{r'_2-1} C^{r'_1+r'_2+\sigma_2-1}_{t'_1-1} \sum_{b_1>0} \sum_{a_2>0} \psi_{t'_1,s'_2,t'_2}(a_2\tau, a_2\tau+b_1)$$

$$+ \sum_{\substack{r'_1+s'_1=\sigma_1+\sigma_3\\ r'_2+s'_2=\sigma_1+\sigma_3\\ k'_1+k'_2=r'_1+r'_2+\sigma_2+s'_2}} (-1)^{s'_1} C^{\sigma_1-1}_{r'_2-1} C^{\sigma_3-1}_{r'_1-1} C^{s'_2-1}_{k'_1-1} \zeta(k'_2) \times \sum_{a_2\in\mathbb{N}^*_+} \phi_{s'_2,k'_1}(a_2\tau)$$

$$+ \sum_{\substack{r'_1+s'_1=\sigma_1+\sigma_3\\ r'_2+s'_2=\sigma_1+\sigma_3\\ k'_1+k'_2=r'_1+r'_2+\sigma_2+s'_1}} (-1)^{s'_1} C^{\sigma_1-1}_{r'_2-1} C^{\sigma_3-1}_{r'_1-1} C^{r'_1+r'_2+\sigma_2-1}_{k_1-1} \sum_{b_1>0} \sum_{a_2>0} \psi_{k'_1,s'_1,k'_2}(a_2\tau, a_2\tau-b_1).$$

The fourth sum $\sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}}$.

Now we apply the Eisenstein's formula,

$$\frac{1}{(b_1+b_2\tau)^{\sigma_2}(a_1+b_1+(a_2+b_2)\tau)^{\sigma_3}}$$

$$= \sum_{l_1+h_1=\sigma_2+\sigma_3} \left(\frac{(-1)^{\sigma_2} C^{\sigma_2-1}_{l_1-1}}{(a_1+a_2\tau)^{l_1}(a_1+b_1+(a_2+b_2)\tau)^{h_1}} + \frac{(-1)^{\sigma_2+h_1} C^{\sigma_3-1}_{l_1-1}}{(a_1+a_2\tau)^{l_1}(b_1+b_2\tau)^{h_1}} \right).$$

And

$$\frac{1}{(b_1+b_2\bar{\tau})^{\sigma_2}(a_1+b_1+(a_2+b_2)\bar{\tau})^{\sigma_3}}$$

$$= \sum_{l_2+h_2=\sigma_2+\sigma_3} \left(\frac{(-1)^{\sigma_2} C^{\sigma_2-1}_{l_2-1}}{(a_1+a_2\bar{\tau})^{l_2}(a_1+b_1+(a_2+b_2)\bar{\tau})^{h_2}} + \frac{(-1)^{\sigma_2+h_2} C^{\sigma_3-1}_{l_2-1}}{(a_1+a_2\bar{\tau})^{l_2}(b_1+b_2\bar{\tau})^{h_2}} \right).$$

Then

$$\sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}} = T_1+T_2+T_3+T_4,$$

where

$$T_1 = \sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}} \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3}} \frac{C^{\sigma_2-1}_{l_1-1} C^{\sigma_2-1}_{l_2-1}}{(a_1+a_2\tau)^{l_1+\sigma_1}(a_1+a_2\bar{\tau})^{l_2+\sigma_1}(a_1+b_1+(a_2+b_2)\tau)^{h_1}(a_1+b_1+(a_2+b_2)\bar{\tau})^{h_2}},$$

$$T_2 = \sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}} \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3}} \frac{(-1)^{h_2} C^{\sigma_2-1}_{l_1-1} C^{\sigma_3-1}_{l_2-1}}{(a_1+a_2\tau)^{l_1+\sigma_1}(a_1+a_2\bar{\tau})^{l_2+\sigma_1}(a_1+b_1+(a_2+b_2)\tau)^{h_1}(b_1+b_2\bar{\tau})^{h_2}},$$

$$T_3 = \sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}} \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3}} \frac{(-1)^{s_1} C^{\sigma_2-1}_{l_2-1} C^{\sigma_3-1}_{l_1-1}}{(a_1+a_2\tau)^{l_1+\sigma_1}(a_1+a_2\bar{\tau})^{l_2+\sigma_1}(a_1+b_1+(a_2+b_2)\bar{\tau})^{h_1}(b_1+b_2\tau)^{h_2}},$$

$$T_4 = \sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}} \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3}} \frac{(-1)^{h_1+h_2} C^{\sigma_3-1}_{l_1-1} C^{\sigma_3-1}_{l_2-1}}{(a_1+a_2\tau)^{l_1+\sigma_1}(a_1+a_2\bar{\tau})^{l_2+\sigma_1}(b_1+b_2\tau)^{h_1}(b_1+b_2\bar{\tau})^{h_2}}.$$

In the following paragraph, we will calculate each T_i.

(1) T_1.

$$T_1 = \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3}} C^{\sigma_2-1}_{l_1-1} C^{\sigma_2-1}_{l_2-1} \sum_{a_2>0,b_2>0} \phi_{l_1+\sigma_1,l_2+\sigma_1}(a_2\tau)\phi_{h_1,h_2}((a_2+b_2)\tau)$$

$$= \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3}} C^{\sigma_2-1}_{l_1-1} C^{\sigma_2-1}_{l_2-1} \sum_{b_2'>a_2>0} \phi_{l_1+\sigma_1,l_2+\sigma_1}(a_2\tau)\phi_{h_1,h_2}(b_2'\tau).$$

(2) T_4.

$$T_4 = \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3}} (-1)^{h_1+h_2} C^{\sigma_3-1}_{l_1-1} C^{\sigma_3-1}_{l_2-1} \sum_{a_2>0,b_2>0} \phi_{l_1+\sigma_1,l_2+\sigma_1}(a_2\tau)\phi_{h_1,h_2}(b_2\tau).$$

(3) T_2.

Recall that the classical Lipschitz summation formula :

$$\sum_{m\in\mathbb{Z}} \frac{1}{(\tau+m)^s} = \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n\in\mathbb{N}_+^*} n^{s-1} e^{2\pi i n\tau}.$$

Then we will denote

$$\lambda_s(\tau) = \sum_{m\in\mathbb{Z}} \frac{1}{(\tau+m)^s}.$$

Then we get also get

$$\lambda_s(\bar{\tau}) = \sum_{m\in\mathbb{Z}} \frac{1}{(\bar{\tau}+m)^s} = \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n\in\mathbb{N}_-^*} n^{s-1} e^{2\pi i n\bar{\tau}}.$$

$$\frac{1}{(a_1+b_1+(a_2+b_2)\tau)^{h_1}(b_1+b_2\bar{\tau})^{h_2}} =$$

$$\sum_{m+p=h_1+h_2}\left(\frac{(-1)^{h_2}C_{m-1}^{h_1-1}}{(a_1+a_2\tau+b_2(\tau-\bar{\tau}))^m(b_1+b_2\bar{\tau})^p}+\frac{(-1)^{h_2+p}C_{m-1}^{h_2-1}}{(a_1+a_2\tau+b_2(\tau-\bar{\tau}))^m(a_1+b_1+(a_2+b_2)\tau)^p}\right).$$

$$T_2 = C_1 + C_2,$$

$$C_1 = \sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}}\sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ m+p=h_1+h_2}}\frac{C_{l_1-1}^{\sigma_2-1}C_{l_2-1}^{\sigma_3-1}C_{m-1}^{h_1-1}}{(a_1+a_2\tau)^{l_1+\sigma_1}(a_1+a_2\bar{\tau})^{l_2+\sigma_1}(a_1+a_2\tau+b_2(\tau-\bar{\tau}))^m(b_1+b_2\bar{\tau})^p};$$

$$C_2 = \sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}}\sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ m+p=h_1+h_2}}\frac{(-1)^pC_{l_1-1}^{\sigma_2-1}C_{l_2-1}^{\sigma_3-1}C_{m-1}^{h_2-1}}{(a_1+a_2\tau)^{l_1+\sigma_1}(a_1+a_2\bar{\tau})^{l_2+\sigma_1}(a_1+a_2\tau+b_2(\tau-\bar{\tau}))^m(a_1+b_1+(a_2+b_2)\tau)^p}.$$

We observe that $b_2(\tau-\bar{\tau})$ is an element in the upper half plan $\mathcal{H}$, thus we can apply the theorem of double-varaiable summation formula 6.0.12.

$$C_1 = \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ m+p=h_1+h_2}}\sum_{\substack{a_2>0\\ b_2>0}}\sum_{a_1\in\mathbb{Z}}\frac{C_{l_1-1}^{\sigma_2-1}C_{l_2-1}^{\sigma_3-1}C_{m-1}^{h_1-1}}{(a_1+a_2\tau)^{l_1+\sigma_1}(a_1+a_2\bar{\tau})^{l_2+\sigma_1}(a_1+a_2\tau+b_2(\tau-\bar{\tau}))^m}\sum_{b_1\in\mathbb{Z}}\frac{1}{(b_1+b_2\bar{\tau})^p}$$

$$= \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ m+p=h_1+h_2}}C_{l_1-1}^{\sigma_2-1}C_{l_2-1}^{\sigma_3-1}C_{m-1}^{h_1-1}\sum_{\substack{a_2>0\\ b_2>0}}\psi_{l_1+\sigma_1,l_2+\sigma_1,m}(a_2\tau,a_2\tau+b_2(\tau-\bar{\tau}))\lambda_p(b_2\bar{\tau}).$$

By the same argument, we get

$$C_2 = \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ m+p=h_1+h_2}}\sum_{\substack{a_2>0\\ b_2>0}}(-1)^pC_{l_1-1}^{\sigma_2-1}C_{l_2-1}^{\sigma_3-1}C_{m-1}^{h_2-1}\psi_{l_1+\sigma_1,l_2+\sigma_1,m}(a_2\tau,a_2\tau+b_2(\tau-\bar{\tau}))\lambda_p((a_2+b_2)\tau).$$

$$T_2 = \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ m+p=h_1+h_2}}C_{l_1-1}^{\sigma_2-1}C_{l_2-1}^{\sigma_3-1}C_{m-1}^{h_1-1}\sum_{\substack{a_2>0\\ b_2>0}}\psi_{l_1+\sigma_1,l_2+\sigma_1,m}(a_2\tau,a_2\tau+b_2(\tau-\bar{\tau}))\lambda_p(b_2\bar{\tau})$$

$$+\sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ m+p=h_1+h_2}}(-1)^pC_{l_1-1}^{\sigma_2-1}C_{l_2-1}^{\sigma_3-1}C_{m-1}^{h_2-1}\sum_{\substack{a_2>0\\ b_2>0}}\psi_{l_1+\sigma_1,l_2+\sigma_1,m}(a_2\tau,a_2\tau+b_2(\tau-\bar{\tau}))\lambda_p((a_2+b_2)\tau).$$

(4)T_3.

Using the similar method as we did for T_2, we can obtain an expanion for T_3. However, $b_2(\bar{\tau}-\tau)$ is no longer in the upper half plan. So we will try another way.

$$\frac{1}{(a_1+a_2\tau)^{l_1+\sigma_1}(b_1+b_2\tau)^{h_2}} = \sum_{\substack{i_1+j_1=\\ l_1+\sigma_1+h_2}}\left(\frac{C_{i_1-1}^{l_1+\sigma_1-1}}{(a_1+b_1+(a_2+b_2)\tau)^{i_1}(b_1+b_2\tau)^{j_1}}+\frac{C_{i_1-1}^{h_2-1}}{(a_1+b_1+(a_2+b_2)\tau)^{i_1}(a_1+a_2\tau\right.$$

$$T_3 = \sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}} \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ i_1+j_1=\\ l_1+\sigma_1+h_2}} \frac{(-1)^{s_1} C_{l_2-1}^{\sigma_2-1} C_{l_1-1}^{\sigma_3-1} C_{i_1-1}^{l_1+\sigma_1-1}}{(a_1+a_2\bar{\tau})^{l_2+\sigma_1}(a_1+b_1+(a_2+b_2)\tau)^{i_1}(a_1+b_1+(a_2+b_2)\bar{\tau})^{h_1}(b_1+b_2\tau)^{j_1}}$$

$$+ \sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}} \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ i_1+j_1=\\ l_1+\sigma_1+h_2}} \frac{(-1)^{s_1} C_{l_2-1}^{\sigma_2-1} C_{l_1-1}^{\sigma_3-1} C_{i_1-1}^{h_2-1}}{(a_1+a_2\bar{\tau})^{l_2+\sigma_1}(a_1+b_1+(a_2+b_2)\tau)^{i_1}(a_1+b_1+(a_2+b_2)\bar{\tau})^{h_1}(a_1+a_2\tau)^{j_1}}.$$

Since

$$\frac{1}{(a_1+a_2\bar{\tau})^{l_2+\sigma_1}(a_1+b_1+(a_2+b_2)\bar{\tau})^{h_1}} = \sum_{\substack{i_2+j_2=\\ l_2+\sigma_1+h_1}} \left(\frac{C_{i_2-1}^{l_2+\sigma_1-1}}{(b_1+b_2\bar{\tau})^{i_2}(a_1+b_1+(a_2+b_2)\bar{\tau})^{j_2}} + \frac{C_{i_2-1}^{h_1-1}}{(b_1+b_2\bar{\tau})(a_1+a_2\bar{\tau})} \right.$$

therefore

$$\frac{1}{(a_1+a_2\bar{\tau})^{l_2+\sigma_1}(a_1+b_1+(a_2+b_2)\bar{\tau})^{h_1}(a_1+b_1+(a_2+b_2)\tau)^{i_1}(b_1+b_2\tau)^{j_1}}$$

$$= \sum_{\substack{i_2+j_2=\\ l_2+\sigma_1+h_1}} \frac{C_{i_2-1}^{l_2+\sigma_1-1}}{(b_1+b_2\bar{\tau})^{i_2}(a_1+b_1+(a_2+b_2)\bar{\tau})^{j_2}(a_1+b_1+(a_2+b_2)\tau)^{i_1}(b_1+b_2\tau)^{j_1}}$$

$$+ \frac{C_{i_2-1}^{h_1-1}}{(b_1+b_2\bar{\tau})^{i_2}(a_1+a_2\bar{\tau})^{j_2}(a_1+b_1+(a_2+b_2)\tau)^{i_1}(b_1+b_2\tau)^{j_1}}.$$

So finally we obtain:

$$T_3 = \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ i_1+j_1=\\ l_1+\sigma_1+h_2}} (-1)^{s_1} C_{l_2-1}^{\sigma_2} C_{l_1-1}^{\sigma_3-1} C_{i_1-1}^{h_2-1} \sum_{b_2'>a_2>0} \phi_{j_1,l_2+\sigma_1}(a_2\tau)\phi_{i_1,h_1}(b_2'\tau)+$$

$$+ \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ i_1+j_1=\\ l_1+\sigma_1+h_2=\\ i_2+j_2\\ l_2+\sigma_1+h_1}} (-1)^{s_1} C_{l_2-1}^{\sigma_2} C_{l_1-1}^{\sigma_3-1} C_{i_1-1}^{l_1+\sigma_1-1} C_{i_2-1}^{l_2+\sigma_1-1} \sum_{a_2>b_2>0} \phi_{j_1,i_2}(b_2\tau)\phi_{i_1,j_2}(a_2\tau)$$

$$+ \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ i_1+j_1=\\ l_1+\sigma_1+h_2=\\ i_2+j_2\\ l_2+\sigma_1+h_1\\ t_1+t_2=i_1+j_2}} (-1)^{s_1+j_2} C_{l_2-1}^{\sigma_2} C_{l_1-1}^{\sigma_3-1} C_{i_1-1}^{l_1+\sigma_1-1} C_{i_2-1}^{h_1-1} C_{t_1-1}^{i_1-1} \sum_{a_2,b_2>0} \psi_{j_1,i_2,t_1}(b_2\tau, b_2\tau+a_2(\tau-\bar{\tau}))\lambda_{t_2}(a_2\bar{\tau})$$

$$+\sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ i_1+j_1=\\ l_1+\sigma_1+h_2=\\ i_2+j_2\\ l_2+\sigma_1+h_1\\ t_1+t_2=i_1+j_2}} (-1)^{s_1+j_2+t_2} C^{\sigma_2}_{l_2-1} C^{\sigma_3-1}_{l_1-1} C^{l_1+\sigma_1-1}_{i_1-1} C^{h_1-1}_{i_2-1} C^{j_2-1}_{t_1-1} \sum_{a_2,b_2>0} \psi_{j_1,i_2,t_1}(b_2\tau, b_2+a_2(\tau-\bar{\tau}))\lambda_{t_2}((a_2+b_2)\tau)$$

In conclusion, we get

$$\sum_{\substack{a_2>0,b_2>0\\ a_1,b_1\in\mathbb{Z}}} = \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3}} C^{\sigma_2-1}_{l_1-1} C^{\sigma_2-1}_{l_2-1} \sum_{b_2'>a_2>0} \phi_{l_1+\sigma_1,l_2+\sigma_1}(a_2\tau)\phi_{h_1,h_2}(b_2'\tau)$$

$$+\sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3}} (-1)^{h_1+h_2} C^{\sigma_3-1}_{l_1-1} C^{\sigma_3-1}_{l_2-1} \sum_{a_2>0,b_2>0} \phi_{l_1+\sigma_1,l_2+\sigma_1}(a_2\tau)\phi_{h_1,h_2}(b_2\tau)$$

$$+\sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ m+p=h_1+h_2}} C^{\sigma_2-1}_{l_1-1} C^{\sigma_3-1}_{l_2-1} C^{h_1-1}_{m-1} \sum_{\substack{a_2>0\\ b_2>0}} \psi_{l_1+\sigma_1,l_2+\sigma_1,m}(a_2\tau, a_2\tau+b_2(\tau-\bar{\tau}))\lambda_p(b_2\bar{\tau})$$

$$+\sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ m+p=h_1+h_2}} (-1)^p C^{\sigma_2-1}_{l_1-1} C^{\sigma_3-1}_{l_2-1} C^{h_2-1}_{m-1} \sum_{\substack{a_2>0\\ b_2>0}} \psi_{l_1+\sigma_1,l_2+\sigma_1,m}(a_2\tau, a_2\tau+b_2(\tau-\bar{\tau}))\lambda_p((a_2+b_2)\tau)$$

$$+\sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ i_1+j_1=\\ l_1+\sigma_1+h_2}} (-1)^{s_1} C^{\sigma_2}_{l_2-1} C^{\sigma_3-1}_{l_1-1} C^{h_2-1}_{i_1-1} \sum_{b_2'>a_2>0} \phi_{j_1,l_2+\sigma_1}(a_2\tau)\phi_{i_1,h_1}(b_2'\tau)+$$

$$+\sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ i_1+j_1=\\ l_1+\sigma_1+h_2=\\ i_2+j_2\\ l_2+\sigma_1+h_1}} (-1)^{s_1} C^{\sigma_2}_{l_2-1} C^{\sigma_3-1}_{l_1-1} C^{l_1+\sigma_1-1}_{i_1-1} C^{l_2+\sigma_1-1}_{i_2-1} \sum_{a_2>b_2>0} \phi_{j_1,i_2}(b_2\tau)\phi_{i_1,j_2}(a_2\tau)$$

$$+\sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ i_1+j_1=\\ l_1+\sigma_1+h_2=\\ i_2+j_2\\ l_2+\sigma_1+h_1\\ t_1+t_2=i_1+j_2}} (-1)^{s_1+j_2} C^{\sigma_2}_{l_2-1} C^{\sigma_3-1}_{l_1-1} C^{l_1+\sigma_1-1}_{i_1-1} C^{h_1-1}_{i_2-1} C^{i_1-1}_{t_1-1} \sum_{a_2,b_2>0} \psi_{j_1,i_2,t_1}(b_2\tau, b_2\tau+a_2(\tau-\bar{\tau}))\lambda_{t_2}(a_2\bar{\tau})$$

$$+\sum_{\substack{l_1+h_1=\sigma_2+\sigma_3\\ l_2+h_2=\sigma_2+\sigma_3\\ i_1+j_1=\\ l_1+\sigma_1+h_2=\\ i_2+j_2\\ l_2+\sigma_1+h_1\\ t_1+t_2=i_1+j_2}} (-1)^{s_1+j_2+t_2} C^{\sigma_2}_{l_2-1} C^{\sigma_3-1}_{l_1-1} C^{l_1+\sigma_1-1}_{i_1-1} C^{h_1-1}_{i_2-1} C^{j_2-1}_{t_1-1} \sum_{a_2,b_2>0} \psi_{j_1,i_2,t_1}(b_2\tau, b_2+a_2(\tau-\bar{\tau}))\lambda_{t_2}((a_2+b_2)\tau)$$

Theorem 6.0.14 (The Fourier expansion of $Z_{I,\mu,\mathbb{Q}(\sqrt{-\mathbb{D}})}(\Gamma_1, \partial\Gamma_1)$)

$$Z_{I,\mu,\mathbb{Q}(\sqrt{-\mathbb{D}})}(\Gamma_1, \partial\Gamma_1) = Z_{I,\mu,\mathbb{Q}}(\Gamma_1, \partial\Gamma_1)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3}} C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_2-1} \zeta(r_1 + r_2 + \sigma_1) \sum_{b_2 \in \mathbb{N}_+^*} \phi_{s_1,s_2}(b_2\tau)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3}} (-1)^{s_1+s_2} C_{r_1-1}^{\sigma_3-1} C_{r_2-1}^{\sigma_3-1} \zeta(r_1 + r_2 + \sigma_1) \sum_{b_2 \in \mathbb{N}_+^*} \phi_{s_1,s_2}(b_2\tau)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ t_1+t_2=r_1+r_2+\sigma_1+s_1}} (-1)^{r_1+r_2+\sigma_1+s_2+t_2} C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_3-1} C_{t_1-1}^{s_1-1} \zeta(t_2) \times \sum_{b_2 \in \mathbb{N}_+^*} \phi_{t_1,s_2}(b_2\tau)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ k_1+k_2=r_1+r_2+\sigma_1+s_2}} (-1)^{s_1} C_{r_2-1}^{\sigma_2-1} C_{r_1-1}^{\sigma_3-1} C_{k_1-1}^{s_2-1} \zeta(k_2) \times \sum_{b_2 \in \mathbb{N}_+^*} \phi_{k_1,s_1}(b_2\tau)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ t_1+t_2=r_1+r_2+\sigma_1+s_1}} (-1)^{r_1+r_2+\sigma_1+s_2} C_{r_1-1}^{\sigma_2-1} C_{r_2-1}^{\sigma_3-1} C_{t_1-1}^{r_1+r_2+\sigma_1-1} \sum_{a_1>0} \sum_{b_2>0} \psi_{t_1,s_2,t_2}(b_2\tau, b_2\tau + a_1)$$

$$+ \sum_{\substack{r_1+s_1=\sigma_2+\sigma_3 \\ r_2+s_2=\sigma_2+\sigma_3 \\ k_1+k_2=r_1+r_2+\sigma_1+s_1}} (-1)^{s_1} C_{r_2-1}^{\sigma_2-1} C_{r_1-1}^{\sigma_3-1} C_{k_1-1}^{r_1+r_2+\sigma_1-1} \sum_{a_1>0} \sum_{b_2>0} \psi_{k_1,s_1,k_2}(b_2\tau, b_2\tau - a_1)$$

$$+ \sum_{\substack{r'_1+s'_1=\sigma_1+\sigma_3 \\ r'_2+s'_2=\sigma_1+\sigma_3}} C_{r'_1-1}^{\sigma_1-1} C_{r'_2-1}^{\sigma_1-1} \zeta(r'_1 + r'_2 + \sigma_2) \sum_{a_2 \in \mathbb{N}_+^*} \phi_{s'_1,s'_2}(a_2\tau)$$

$$+ \sum_{\substack{r'_1+s'_1=\sigma_1+\sigma_3 \\ r'_2+s'_2=\sigma_1+\sigma_3}} (-1)^{s'_1+s'_2} C_{r'_1-1}^{\sigma_3-1} C_{r'_2-1}^{\sigma_3-1} \zeta(r'_1 + r'_2 + \sigma_2) \sum_{a_2 \in \mathbb{N}_+^*} \phi_{s'_1,s'_2}(a_2\tau)$$

$$+ \sum_{\substack{r'_1+s'_1=\sigma_1+\sigma_3 \\ r'_2+s'_2=\sigma_1+\sigma_3 \\ t'_1+t'_2=r'_1+r'_2+\sigma_2+s'_1}} (-1)^{r'_1+r'_2+\sigma_2+s'_2+t'_2} C_{r'_1-1}^{\sigma_1-1} C_{r'_2-1}^{\sigma_3-1} C_{t'_1-1}^{s'_1-1} \zeta(t'_2) \times \sum_{a_2 \in \mathbb{N}_+^*} \phi_{t'_1,s'_2}(a_2\tau)$$

$$+ \sum_{\substack{r'_1+s'_1=\sigma_1+\sigma_3 \\ r'_2+s'_2=\sigma_1+\sigma_3 \\ k'_1+k'_2=r'_1+r'_2+\sigma_2+s'_2}} (-1)^{s'_1} C_{r'_2-1}^{\sigma_1-1} C_{r'_1-1}^{\sigma_3-1} C_{k'_1-1}^{s'_2-1} \zeta(k'_2) \times \sum_{a_2 \in \mathbb{N}_+^*} \phi_{s'_2,k'_1}(a_2\tau)$$

$$+ \sum_{\substack{r'_1+s'_1=\sigma_1+\sigma_3 \\ r'_2+s'_2=\sigma_1+\sigma_3 \\ t'_1+t'_2=r'_1+r'_2+\sigma_2+s'_1}} (-1)^{r'_1+r'_2+\sigma_2+s'_2} C^{\sigma_1-1}_{r'_1-1} C^{\sigma_3-1}_{r'_2-1} C^{r'_1+r'_2+\sigma_2-1}_{t'_1-1} \sum_{b_1>0} \sum_{a_2>0} \psi_{t'_1,s'_2,t'_2}(a_2\tau, a_2\tau + b_1)$$

$$+ \sum_{\substack{r'_1+s'_1=\sigma_1+\sigma_3 \\ r'_2+s'_2=\sigma_1+\sigma_3 \\ k'_1+k'_2=r'_1+r'_2+\sigma_2+s'_1}} (-1)^{s'_1} C^{\sigma_1-1}_{r'_2-1} C^{\sigma_3-1}_{r'_1-1} C^{r'_1+r'_2+\sigma_2-1}_{k_1-1} \sum_{b_1>0} \sum_{a_2>0} \psi_{k'_1,s'_1,k'_2}(a_2\tau, a_2\tau - b_1).$$

$$+ \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3 \\ l_2+h_2=\sigma_2+\sigma_3}} C^{\sigma_2-1}_{l_1-1} C^{\sigma_2-1}_{l_2-1} \sum_{b'_2>a_2>0} \phi_{l_1+\sigma_1,l_2+\sigma_1}(a_2\tau)\phi_{h_1,h_2}(b'_2\tau)$$

$$+ \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3 \\ l_2+h_2=\sigma_2+\sigma_3}} (-1)^{h_1+h_2} C^{\sigma_3-1}_{l_1-1} C^{\sigma_3-1}_{l_2-1} \sum_{a_2>0,b_2>0} \phi_{l_1+\sigma_1,l_2+\sigma_1}(a_2\tau)\phi_{h_1,h_2}(b_2\tau)$$

$$+ \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3 \\ l_2+h_2=\sigma_2+\sigma_3 \\ m+p=h_1+h_2}} C^{\sigma_2-1}_{l_1-1} C^{\sigma_3-1}_{l_2-1} C^{h_1-1}_{m-1} \sum_{\substack{a_2>0 \\ b_2>0}} \psi_{l_1+\sigma_1,l_2+\sigma_1,m}(a_2\tau, a_2\tau + b_2(\tau - \bar{\tau}))\lambda_p(b_2\bar{\tau})$$

$$+ \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3 \\ l_2+h_2=\sigma_2+\sigma_3 \\ m+p=h_1+h_2}} (-1)^p C^{\sigma_2-1}_{l_1-1} C^{\sigma_3-1}_{l_2-1} C^{h_2-1}_{m-1} \sum_{\substack{a_2>0 \\ b_2>0}} \psi_{l_1+\sigma_1,l_2+\sigma_1,m}(a_2\tau, a_2\tau + b_2(\tau - \bar{\tau}))\lambda_p((a_2 + b_2)\tau)$$

$$+ \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3 \\ l_2+h_2=\sigma_2+\sigma_3 \\ i_1+j_1= \\ l_1+\sigma_1+h_2}} (-1)^{s_1} C^{\sigma_2}_{l_2-1} C^{\sigma_3-1}_{l_1-1} C^{h_2-1}_{i_1-1} \sum_{b'_2>a_2>0} \phi_{j_1,l_2+\sigma_1}(a_2\tau)\phi_{i_1,h_1}(b'_2\tau)+$$

$$+ \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3 \\ l_2+h_2=\sigma_2+\sigma_3 \\ i_1+j_1= \\ l_1+\sigma_1+h_2= \\ i_2+j_2 \\ l_2+\sigma_1+h_1}} (-1)^{s_1} C^{\sigma_2}_{l_2-1} C^{\sigma_3-1}_{l_1-1} C^{l_1+\sigma_1-1}_{i_1-1} C^{l_2+\sigma_1-1}_{i_2-1} \sum_{a_2>b_2>0} \phi_{j_1,i_2}(b_2\tau)\phi_{i_1,j_2}(a_2\tau)$$

$$+ \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3 \\ l_2+h_2=\sigma_2+\sigma_3 \\ i_1+j_1= \\ l_1+\sigma_1+h_2= \\ i_2+j_2 \\ l_2+\sigma_1+h_1 \\ t_1+t_2=i_1+j_2}} (-1)^{s_1+j_2} C^{\sigma_2}_{l_2-1} C^{\sigma_3-1}_{l_1-1} C^{l_1+\sigma_1-1}_{i_1-1} C^{h_1-1}_{i_2-1} C^{i_1-1}_{t_1-1} \sum_{a_2,b_2>0} \psi_{j_1,i_2,t_1}(b_2\tau, b_2\tau{+}a_2(\tau{-}\bar{\tau}))\lambda_{t_2}(a_2\bar{\tau})$$

$$+ \sum_{\substack{l_1+h_1=\sigma_2+\sigma_3 \\ l_2+h_2=\sigma_2+\sigma_3 \\ i_1+j_1= \\ l_1+\sigma_1+h_2= \\ i_2+j_2 \\ l_2+\sigma_1+h_1 \\ t_1+t_2=i_1+j_2}} (-1)^{s_1+j_2+t_2} C^{\sigma_2}_{l_2-1} C^{\sigma_3-1}_{l_1-1} C^{l_1+\sigma_1-1}_{i_1-1} C^{h_1-1}_{i_2-1} C^{j_2-1}_{t_1-1} \sum_{a_2,b_2>0} \psi_{j_1,i_2,t_1}(b_2\tau, b_2{+}a_2(\tau{-}\bar{\tau}))\lambda_{t_2}((a_2{+}b_2)\tau)$$

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