A survey on “quantization commutes with reduction”

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- (L, h^L) a Hermitian line bundle over M carrying a Hermitian connection ∇^L such that

$$\frac{\sqrt{-1}}{2\pi} \left(\nabla^L\right)^2 = \omega.$$

L the pre-quantum line bundle on (M, ω).

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For $u\in T^{(1,0)}M$, set $c(u)=\sqrt{2}\overline{u}^*\wedge$, $c(\overline{u})=-\sqrt{2}i_{\overline{u}}$.
spin^c Dirac operator
$$D^L=\sum_j c(e_j)\nabla^{\mathrm{Cl}}_{e_j}:\Omega^{0,\frac{\text{even}}{\text{odd}}}(M,L)\to\Omega^{0,\frac{\text{odd}}{\text{even}}}(M,L)$$
Self-adjoint 1-order elliptic op. $D^L_\pm:=D^L|_{\Omega^{0,\frac{\text{even}}{\text{odd}}}(M,L)}$.

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- When (M,ω,J) is Kähler, and L holomorphic
$$D^L = \sqrt{2}\left(\overline{\partial}^L + \left(\overline{\partial}^L\right)^*\right).$$

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- Atiyah-Singer (1963) : $Q(L) = \int_M \mathrm{Td}(T^{(1,0)}M)\,\mathrm{ch}(L)$

$$= \int_M \det\left(\frac{e^{\sqrt{-1}R^{T^{(1,0)}M}/2\pi}}{1 - e^{-\sqrt{-1}R^{T^{(1,0)}M}/2\pi}}\right) e^{\omega}.$$

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- $Q(L)^\gamma$ the multiplicity of V^G_γ in $Q(L)$. How to compute $Q(L)^\gamma$?

Symplectic reduction

- Moment map $\mu : M \to \mathfrak{g}^*$ is defined by

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- For a regular value $\nu \in \mathfrak{g}^*$ of μ, symplectic reduction :

$$M_\nu = \mu^{-1}(G \cdot \nu)/G$$

  M_ν is a compact symplectic orbifold.
  $J,\ \omega,\ L \Longrightarrow J_\nu, \omega_\nu, L_\nu$ on M_ν.

Example : Borel-Weil-Bott theorem

- For $\gamma \in \Lambda^*_+$, $\mathcal{O}_\gamma = G \cdot \gamma$ the orbit of the co-adjoint action of G on $\mathfrak{g}^*$, and $\rho_\gamma : T \to \mathbb{C}$ the representation of the maximal torus T with weight γ.

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  moment map of G-action is the inclusion $\mathcal{O}_\gamma \hookrightarrow \mathfrak{g}^*$.
- $M_\gamma = \mathcal{O}_\gamma / G \simeq \mathrm{pt}$,

$$H^{0,0}(\mathcal{O}_\gamma, F) = V_\gamma^G, \quad H^{0,j}(\mathcal{O}_\gamma, F) = 0 \quad \text{for } j > 0.$$

Guillemin-Sternberg conjecture I

- $$\begin{array}{ccccccc}
 & & \mu^{-1}(G\cdot\nu) & \hookrightarrow & M & \dashleftarrow & \omega, J, D^{L} \\
 & & \downarrow{\scriptstyle G} & & & \nwarrow & \\
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- Guillemin-Sternberg conjecture (1982) : For any $\gamma \in \Lambda_+^*$,

$$Q(L)^\gamma = Q\left(L_\gamma\right).$$

Equivalently,

$$Q(L) := \mathrm{Ind}(D^L) = \bigoplus_{\gamma\in\Lambda_+^*} Q\left(L_\gamma\right)\cdot V_\gamma^G.$$

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- Other proofs : Duistermaart-Guillemin-Meinrenken-Wu (for circle actions) and Jeffrey-Kirwan (for non-abelian group actions with certain extra conditions)
  Paradan, using the transversal index theory, 2001.
  Etc ...

Guillemin-Sternberg conjecture III

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- Teleman, Braverman, Weiping Zhang, 2000 : for any j,

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- Suppose $\mu : M \to \mathfrak{g}^*$ is proper.

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- Schmid (1975) : $H^{0,k}_{(2)}(\mathcal{O}_\gamma, F) = \mathcal{H}_\gamma$ if $k = \dim(G/K)/2$; 0 other case.

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- Suppose temporary : $\{x \in M : \mu^M(x) = 0\}$ is compact.
- For $x \in M,\ v \in T_xM$,

$$\sigma^M_{L,\mu}(x,v) = \pi^*\left(\sqrt{-1}c(v+\mu^M)\otimes \mathrm{Id}_L\right)\Big|_{(x,v)}$$
$$: \pi^*(\Lambda^{\mathrm{even}}(T^{*(0,1)}M)\otimes L) \to \pi^*(\Lambda^{\mathrm{odd}}(T^{*(0,1)}M)\otimes L))$$

where $c(\cdot)$ is the Clifford action on $\Lambda(T^{*(0,1)}M)$.
Then $\sigma^M_{L,\mu}$ is a transversally elliptic symbol in the sense of Atiyah (1974), as $\{(x,v)\in T_GM; \sigma^M_{L,\mu}(x,v)=0\} = \{(x,0)\in T_GM; \mu^M(x)=0\}$ is compact.

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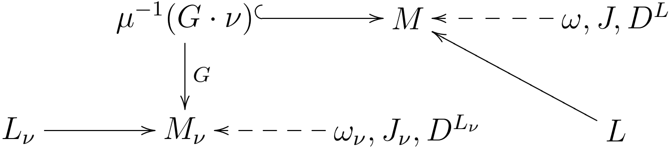
- In many cases,

$$\operatorname{Ind}\left(\sigma^M_{L,\mu}\right) = \operatorname{Ker}_{L^2}(D^L_+) - \operatorname{Ker}_{L^2}(D^L_-) \in R[G].$$

Recall : Symplectic reduction

- 

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- Paradan 2009 : New proof of Ma-Zhang's theorem : symplectic cuts and the wonderful compactifications of Concini-Procesi.

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- Theorem (Ma-Zhang) : $\forall\ \gamma\in\Lambda^*_+$, $\exists\ a_\gamma>0$ such that $\mathrm{Ind}_\gamma\left(\sigma^{M_a}_{L,\mu}\right)$ does not depend on any regular value $a>a_\gamma$ of $\mathcal{H}=|\mu|^2$. We denote it by $Q(L)^\gamma$.

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- Theorem (Ma-Zhang) : $\forall\ \gamma \in \Lambda_+^*$, $\exists\ a_\gamma > 0$ such that $\operatorname{Ind}_\gamma\left(\sigma_{L,\mu}^{M_a}\right)$ does not depend on any regular value $a > a_\gamma$ of $\mathcal{H} = |\mu|^2$. We denote it by $Q(L)^\gamma$.
- Theorem (Ma-Zhang) : For any $\gamma \in \Lambda_+^*$,

$$Q(L)^\gamma = Q(L_\gamma).$$

Quantization commutes with reduction

- $\sigma^{M_a}_{L,\mu}$ has an index :

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- Theorem (Ma-Zhang) : For any $\gamma \in \Lambda^*_+$,

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- If $\{x \in M : \mu^M(x) = 0\}$ is compact, then

$$Q(L)^\gamma = \mathrm{Ind}_\gamma\left(\sigma^{M}_{L,\mu}\right).$$

Thus the above result implies Vergne's conjecture.

Transversal index =APS index

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- Assume that $\{x \in M : \mu^M(x) = 0\}$ is compact. For $T > 0$, let D_T^L be the deformed Dirac operator introduced by Tian-Zhang :

$$D_T^L = D^L + \sqrt{-1}T\, c\left(\mu^M\right) : \Omega^{0,\bullet}(M, L) \to \Omega^{0,\bullet}(M, L) .$$

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- First step : For $\gamma \in \Lambda_+^*$, $\exists\, T_\gamma > 0$ s.t. $\forall\, T > T_\gamma$

$$\mathrm{Ind}_\gamma\left(\sigma_{L,\mu}^{M_a}\right) = \mathrm{Ind}_\gamma\left(D_{+,T,\mathrm{APS},M_a}^L\right) .$$

The proof of Vergne's conjecture for $\gamma = 0$ is then easy.

Product formula

- (N, ω^N, J^N) compact symplectic manifold with a prequantum line bundle (F, h^F, ∇^F). We suppose that G acts on N and the action lifts to F. For $\gamma \in \Lambda^*_+$, set

$$Q\,(F)^{-\gamma} = \dim \mathrm{Hom}_G((V^G_\gamma)^*, Q(F)),$$

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- Take $N = \mathcal{O}_\gamma$, then $Q(L \otimes F^*)^{\nu=0} = Q(L_\gamma)$, Borel-Weil-Bott theorem implies $Q(F^*)^{-\nu} = \delta_{\gamma,\nu}$, thus $Q(L_\gamma) = Q(L)^\gamma$. In particular, Vergne's conjecture holds.

Thank you!