Chapter 1

Basic symplectic geometry

This chapter is an introduction on symplectic geometry. Symplectic geometry has its origin in classical mechanics. Many important geometry problems can be naturally formulated in the context of symplectic geometry, thus it is also a widely useful language in mathematic physics, representation theory etc. Since 1970's, after Kostant and Souriau introduced the geometric quantization, symplectic geometry became an independent mathematic subject which is an extension of complex geometry. Complex geometry is a classical and still very active area, and Kähler manifolds in complex geometry are naturally symplectic manifolds which belong to a large class of manifolds: Poisson manifolds. These three classes of manifolds are basic objects of this chapter.

We start in Section 1.1 the definition on the symplectic vector spaces and show the space of its compatible complex structures is contractible. More precisely, we construct a smooth surjective map from the space of metrics on the vector space to the space of its compatible complex structures, this allows us to extend it easily to the symplectic vector bundles case. In Section 1.2, after recall basic facts on differential manifolds, we explain the Moser's trick which is very useful to treat the problems on the existence of certain diffeomorphisms and as applications, we establish the Darboux theorem which explain locally, any symplectic manifold is same as a symplectic vector space, thus any possible symplectic invariant should be of a global nature. In Section 1.3, we explain the Poisson structure on a symplectic manifold and give a brief introduction on Poisson manifolds. In Section 1.4, we recall the definition of a Kähler manifold.

1.1 Linear symplectic geometry

This section s a continuation of linear algebra. We explain basic facts on symplectic vector spaces, compatible complex structures and symplectic groups.

1.1.1 Symplectic vector spaces

Let V be a real vector space of dimension m. We will denote by V^* its dual space, and for $k \in \mathbb{N}$, let $\Lambda^k V^*$ be the space of antisymmetric (i.e., alternating) multilinear mappings from $\underbrace{V \times \cdots \times V}_{k \text{ times}}$ to $\mathbb{R}$. Certainly, for $k > m$, we have

$$\Lambda^k V^* = 0. \tag{1.1.1}$$

We get easily that

$$\Lambda^0 V^* = \mathbb{R}, \qquad \Lambda^1 V^* = V^*, \qquad \dim \Lambda^m V^* = 1. \tag{1.1.2}$$

A nonvanishing element of $\Lambda^m V^*$ defines an orientation of V. The antisymmetric multiplication for $\alpha \in \Lambda^k V^*$, $\beta \in \Lambda^r V^*$ is defined by: for $v_1, \ldots, v_{k+r} \in V$,

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+r}) := \frac{1}{k!r!} \sum_{\sigma \in S_{k+r}} (-1)^{|\sigma|} \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+r)}), \tag{1.1.3}$$

and $|\sigma|$ is the sign of $\sigma \in S_{k+r}$, the $(k+r)$-th permutation group. Then $\Lambda^\bullet V^* = \oplus_{k=0}^m \Lambda^k V^*$ becomes an algebra with its $\mathbb{Z}$-grading induced by its degree, and $\Lambda^\bullet V^*$ is called the exterior algebra of V^*. Any basis $\{e^j\}_{j=1}^m$ of V^* induces a following basis of $\Lambda^k V^*$:

$$e^I := e^{i_1} \wedge \cdots \wedge e^{i_k} \text{ for } I = \{1 \leq i_1 < \cdots < i_k \leq m\}. \tag{1.1.4}$$

We call a bilinear form $\theta : V \times V \to \mathbb{R}$ is nondegenerate, if for $v \in V$, $\theta(v, \cdot) = 0 \in V^*$ implies $v = 0$.

We call a bilinear form $g : V \times V \to \mathbb{R}$ is a scalar product (or Euclidean metric) on V if g is symmetric and positive, i.e., for any $u, v \in V$,

$$\begin{aligned} &\text{symmetric :} \quad g(u,v) = g(v,u), \\ &\text{positive :} \quad g(u,u) > 0 \quad \text{if } u \neq 0. \end{aligned} \tag{1.1.5}$$

Definition 1.1.1. We say (V, ω) is a symplectic vector space if V is a finite dimensional real vector space, and $\omega : V \times V \to \mathbb{R}$ is a nondegenerate antisymmetric bilinear form. In this case, we call ω a symplectic form on V.

Definition 1.1.2. Let (V_1, ω_1), (V_2, ω_2) be two symplectic vector spaces. A linear map $\phi : V_1 \to V_2$ is called symplectic, if

$$\omega_1 = \phi^* \omega_2 := \omega_2(\phi \cdot, \phi \cdot). \tag{1.1.6}$$

If the linear map $\phi : V_1 \to V_2$ is symplectic, then as ω_1 is nondegenerate, ϕ is injective. If ϕ is also an isomorphism, we call that ϕ is a symplectic isomorphism.

Proposition 1.1.3. *If (V, ω) is a symplectic vector space of dimension m, then m is even and $\omega^{m/2} \in \Lambda^m V^*$ is nonvanishing which defines an orientation of V. Moreover, the map*

$$v \in V \to \omega(v, \cdot) \in V^* \tag{1.1.7}$$

is an isomorphism.

Proof. Let $\langle\cdot,\cdot\rangle$ be a scalar product on V. Then there exists an antisymmetric invertible endomorphism $A \in \mathrm{End}(V)$ such that

$$\omega(\cdot,\cdot) = \langle\cdot, A\cdot\rangle. \tag{1.1.8}$$

As

$$\det A = \det(A^t) = (-1)^m \det A, \tag{1.1.9}$$

thus m is even.

If $\langle\cdot,\cdot\rangle'$ is another scalar product on V, and A' is the corresponding antisymmetric invertible endomorphism. Then there is $P \in \mathrm{GL}(V)$ such that $PAP^t = A'$. Thus $\det A$ and $\det A'$ have the same signature. This means V has a canonical orientation. In fact, this is equivalent to $\omega^{m/2} \in \Lambda^m V^*$ and $\omega^{m/2} \neq 0$.

As ω is nondegenerate, the map $v \in V \to \omega(v,\cdot) \in V^*$ is injective. As $\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} V^*$, (1.1.7) is an isomorphism. The proof of Proposition 1.1.3 is completed. □

The basic example is the following. In fact, as we will see in Theorem 1.1.15, it is the only symplectic vector space.

Example 1.1.4. Let L be a vector space. Then $L \oplus L^*$ is a symplectic vector space with a symplectic form $\omega^{L\oplus L^*}$ defined by: for (l_1, l_1^*), $(l_2, l_2^*) \in L \oplus L^*$,

$$\omega^{L\oplus L^*}((l_1, l_1^*), (l_2, l_2^*)) = (l_1, l_2^*) - (l_2, l_1^*), \tag{1.1.10}$$

here we denote by $(l_1, l_2^*) := l_2^*(l_1)$. In particular, if we identify $\mathbb{R}^n$ with $\mathbb{R}^{n*}$ by the canonical scalar product of $\mathbb{R}^n$ defined by: for $x = (x_1, \ldots, x_n)^t$, $y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$,

$$\langle x, y\rangle = \sum_{i=1}^n x_i y_i. \tag{1.1.11}$$

We call $\left(\mathbb{R}^{2n}, \omega_0\right) := (\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega^{\mathbb{R}^n\oplus\mathbb{R}^{n*}})$ the standard symplectic space. Sometimes, we also denote by ω_{st} the canonical symplectic form ω_0.

From now on, let (V, ω) be a symplectic vector space. For $W \subset V$ a linear subspace, let

$$W^{\perp_\omega} = \{v \in V : \omega(v, w) = 0, \text{ for all } w \in W\}, \tag{1.1.12}$$

be the ω-orthogonal complement of W. We denote by $v \perp_\omega u$ for $u, v \in V$ if $\omega(u, v) = 0$. In the same way, $u \perp_\omega W$ for $W \subset V$ if $\omega(u, v) = 0$ for any $v \in W$.

Definition 1.1.5. For W a linear subspace of a symplectic vector space (V, ω), we call

1. W is symplectic if $W \cap W^{\perp_\omega} = 0$;
2. W is isotropic if $W \subset W^{\perp_\omega}$;
3. W is coisotropic if $W^{\perp_\omega} \subset W$;
4. W is Lagrangian if $W = W^{\perp_\omega}$.

Proposition 1.1.6. *For W a linear subspace of (V, ω), we have*

$$\dim W + \dim W^{\perp_\omega} = \dim V, \qquad \left(W^{\perp_\omega}\right)^{\perp_\omega} = W. \tag{1.1.13}$$

If W is symplectic, then $W^{\perp_\omega}$ is also symplectic and we have the direct decomposition of symplectic vector spaces

$$(V, \omega) = (W, \omega|_W) \oplus (W^{\perp_\omega}, \omega|_{W^{\perp_\omega}}). \tag{1.1.14}$$

Proof. Let $\langle\, ,\, \rangle$ be a scalar product on V. Let $A \in \mathrm{End}(V)$ as in (1.1.8). Then $W^{\perp_\omega} = (AW)^\perp$. Hence,

$$\dim W^{\perp_\omega} = \dim(AW)^\perp = \dim V - \dim(AW). \tag{1.1.15}$$

As A is invertible, by (1.1.15), we get the first equation of (1.1.13), in particular, we have $\dim W = \dim \left(W^{\perp_\omega}\right)^{\perp_\omega}$. But by (1.1.12), we have $W \subset \left(W^{\perp_\omega}\right)^{\perp_\omega}$. This means the second equation of (1.1.13) holds.

If W is symplectic, then $W^{\perp_\omega} \cap \left(W^{\perp_\omega}\right)^{\perp_\omega} = W^{\perp_\omega} \cap W = \{0\}$, thus $W^{\perp_\omega}$ is symplectic. Now we get (1.1.14) by the first equation of (1.1.13).

The proof of Proposition 1.1.6 is completed. □

Proposition 1.1.7. *If (V, ω) is a linear symplectic space of dimension $2n$. Then there exists $e_1, f_1, \ldots, e_n, f_n$ a basis of V, such that, for $1 \leqslant i, j \leqslant n$,*

$$\omega(e_i, f_j) = \delta_{ij}, \qquad \omega(e_i, e_j) = 0, \qquad \omega(f_i, f_j) = 0. \tag{1.1.16}$$

This basis will be called a symplectic basis of V.

Proof. We shall prove this proposition by induction on $\dim V/2$. If $\dim V = 0$, certainly it holds. We suppose $\dim V \geqslant 2$ and the proposition is true for the symplectic vector space of dimension smaller than $\dim V - 2$.

Let $e_1 \in V\backslash\{0\}$. As ω is nondegenerate, there is $f_1 \in V$ such that

$$\omega(e_1, f_1) = 1. \tag{1.1.17}$$

Set $W = \mathbb{R}e_1 \oplus \mathbb{R}f_1$. By Proposition 1.1.6, we have a ω-orthogonal decomposition

$$V = W \oplus W^{\perp_\omega}. \tag{1.1.18}$$

By the induction hypotheses, we have a symplectic basis $e_2, f_2, \ldots, e_n, f_n$ on $W^{\perp_\omega}$. Hence, $\{e_1, f_1, \ldots, e_n, f_n\}$ is a symplectic basis of V.

The proof of Proposition 1.1.7 is completed. □

We give two applications of the symplectic basis.

Corollary 1.1.8. *Let ω, ω' be two symplectic forms on V. Then there exists $A \in \mathrm{GL}(V)$ such that*

$$\omega'(A\cdot, A\cdot) = \omega(\cdot, \cdot). \tag{1.1.19}$$

Proof. Let $\{e_i, f_j\}$ (resp. $\{e_i', f_j'\}$) be a symplectic basis of (V, ω) (resp. (V, ω')). Let $A \in \mathrm{End}(V)$ be defined by

$$Ae_i = e_i', \qquad Af_j = f_j'. \tag{1.1.20}$$

As $(e_i, f_j), (e_i', f_j')$ are bases of V, A is invertible. Moreover,

$$\begin{aligned} &\omega'(Ae_i, Ae_j) = \omega'(e_i', e_j') = 0, \qquad \omega'(Af_i, Af_j) = \omega'(f_i', f_j') = 0, \\ &\omega'(Ae_i, Af_j) = \omega'(e_i', f_j') = \delta_{ij}. \end{aligned} \tag{1.1.21}$$

This means (1.1.19) holds. The proof of Corollary 1.1.8 is completed. □

Corollary 1.1.9. *Let $\{e_i, f_i\}_{i=1}^n$ be a symplectic basis of (V,ω). Then*

$$L = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n \tag{1.1.22}$$

is a Lagrangian subspace of (V,ω). In particular, for any symplectic vector space, there always exists a Lagrangian subspace.

Proof. By (1.1.16) and (1.1.22), we have

$$L \subset L^{\perp_\omega}. \tag{1.1.23}$$

By (1.1.13) and (1.1.22), we have

$$\dim L^{\perp_\omega} = \dim V - \dim L = \dim L. \tag{1.1.24}$$

By (1.1.23) and (1.1.24), we get $L = L^{\perp_\omega}$, thus L is Lagrangian by Definition 1.1.5. The proof of Corollary 1.1.9 is completed. □

1.1.2 Compatible complex structures

Definition 1.1.10. Let V be a real vector space. If $J \in \mathrm{End}(V)$ such that $J^2 = -\mathrm{Id}_V$, we call J a complex structure on V. Moreover, if ω is a symplectic form on V, such that

$$g(\cdot,\cdot) = \omega(\cdot, J\cdot) \tag{1.1.25}$$

defines a scalar product on V, we call J a compatible complex structure on (V,ω). We denote by $\mathscr{J}(V,\omega)$ the space of compatible complex structures on (V,ω).

Proposition 1.1.11. *If J is a compatible complex structure on a symplectic vector space (V,ω), then ω is J-invariant, i.e.,*

$$\omega(J\cdot, J\cdot) = \omega(\cdot,\cdot). \tag{1.1.26}$$

Proof. By (1.1.25), we have

$$\begin{aligned} &\omega(\cdot,\cdot) = g(\cdot, -J\cdot), \\ &\omega(J\cdot, J\cdot) = g(J\cdot,\cdot) = g(\cdot, J^t\cdot). \end{aligned} \tag{1.1.27}$$

As ω is antisymmetric, J is antisymmetric with respect to g. Then

$$g(\cdot, J^t\cdot) = -g(\cdot, J\cdot) = -\omega(\cdot, J^2\cdot) = \omega(\cdot,\cdot). \tag{1.1.28}$$

From (1.1.27) and (1.1.28), we get (1.1.26). □

Example 1.1.12. **a)** Let $\{e_i, f_i\}_{i=1}^n$ be a symplectic basis of a symplectic vector space (V,ω). Set

$$Je_i = f_i, \qquad\qquad Jf_j = -e_j. \tag{1.1.29}$$

Then J is a compatible complex structure. In particular, $\mathscr{J}(V,\omega)$ is non-empty.

b) Let $J_0 \in \mathrm{End}(\mathbb{R}^{2n})$ be the standard complex structure of $\mathbb{R}^{2n}$ defined by

$$J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{1.1.30}$$

For $z = \begin{pmatrix} x \\ y \end{pmatrix}, z' = \begin{pmatrix} x' \\ y' \end{pmatrix} \in \mathbb{R}^{2n}$, we have

$$\omega_0(z, J_0 z') = \sum_{i=1}^{n} (x_i x_i' + y_i y_i'). \tag{1.1.31}$$

Hence J_0 is a compatible complex structure on the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ and $\omega_0(\cdot, J_0 \cdot)$ defines the canonical scalar product $\langle \, , \, \rangle$ on $\mathbb{R}^{2n}$.

Now we recall some results from linear algebra. Let

$$\begin{aligned} \mathbf{P}_m =& \{A \in M_m(\mathbb{R}) : A \text{ is a symmetric positive definite matrix}\}, \\ \mathbf{p}_m =& \{A \in M_m(\mathbb{R}) : A \text{ is a symmetric matrix}\}. \end{aligned} \tag{1.1.32}$$

Then $\mathbf{P}_m$ is an open subset of the $m(m+1)/2$-dimensional vector space $\mathbf{p}_m$.

For $A \in \mathbf{P}_m$, $s \in \mathbb{R}$, we can define the s-th power A^s by

$$A^s = \frac{1}{2\pi i} \int_{\lambda \in \Gamma} \frac{\lambda^s}{\lambda - A} d\lambda, \tag{1.1.33}$$

where Γ is the oriented contour indicated in the figure 1.1 such that Spec$A \subset]r_1, r_2[$. From

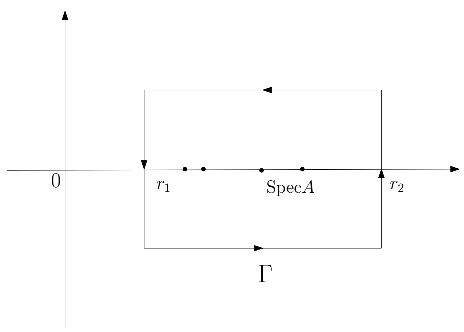

Figure 1.1: The contour Γ.

(1.1.33), we know the s-th power from $\mathbf{P}_m$ to $\mathbf{P}_m$ is smooth, and for $A \in \mathbf{P}_m, C \in M_m(\mathbb{R})$,

$$A^s C = C A^s \quad \text{ if } AC = CA. \tag{1.1.34}$$

Let $\{\lambda_j\} \subset \mathbb{R}_+^*$ be the eigenvalues of A and E_{λ_j} be the eigenspace associated with λ_j, i.e.,

$$A|_{E_{\lambda_j}} = \lambda_j \ \mathrm{Id}_{E_{\lambda_j}}, \quad \bigoplus_j E_{\lambda_j} = \mathbb{R}^m, \tag{1.1.35}$$

then by (1.1.33), A^s is defined by

$$A^s|_{E_{\lambda_j}} = \lambda_j^s \ \mathrm{Id}_{E_{\lambda_j}}. \tag{1.1.36}$$

We can also obtain A^s by first diagonalizing A. As A is a symmetric positive matrix, there exists $Q \in \mathrm{O}(m)$ and $\lambda_1 > 0, \dots, \lambda_m > 0$ such that,

$$A = Q \, \mathrm{diag}(\lambda_1, \dots, \lambda_m) \, Q^{-1}. \tag{1.1.37}$$

By (1.1.33), we have

$$A^s = Q\,\mathrm{diag}(\lambda_1^s, \ldots, \lambda_m^s)\,Q^{-1}. \tag{1.1.38}$$

Thus A^s is still a positive symmetric matrix.

Now for a real vector space V with a Euclidean metric $\langle\,,\,\rangle$, if $A \in \mathrm{GL}(V)$ and A is symmetric and positive with respect to $\langle\,,\,\rangle$, $A^s \in \mathrm{GL}(V)$ is well-defined by (1.1.33), and A^s is symmetric and positive.

Let (V, ω) be a symplectic vector space. Let $\mathscr{M}(V)$ be the space of Euclidean metrics on V. For $g \in \mathscr{M}(V)$, there is a unique $A_g \in \mathrm{GL}(V)$ such that

$$\omega(\cdot,\cdot) = g(A_g\cdot,\cdot). \tag{1.1.39}$$

Moreover, as ω is antisymmetric, A_g is antisymmetric with respect to g, thus $-A_g^2$ is symmetric and positive. Set

$$J_g = (-A_g^2)^{-1/2} A_g. \tag{1.1.40}$$

Then J_g is a compatible complex structure on (V, ω). In fact, for $u \neq 0 \in V$, we have

$$\begin{aligned} J_g^2 &= (-A_g^2)^{-1/2} A_g (-A_g^2)^{-1/2} A_g = -\,\mathrm{Id}_V, \\ \omega(u, J_g u) &= g(A_g u, (-A_g^2)^{-1/2} A_g u) = g(u, (-A_g^2)^{1/2} u) > 0. \end{aligned} \tag{1.1.41}$$

Proposition 1.1.13. *The injection*

$$i : J \in \mathscr{J}(V,\omega) \to g_J = \omega(\cdot, J\cdot) \in \mathscr{M}(V) \tag{1.1.42}$$

is a retract by deformation. In particular, $\mathscr{J}(V,\omega)$ is not empty and contractible.

Proof. Let $r : \mathscr{M}(V) \to \mathscr{J}(V,\omega)$ be a map defined by

$$r(g) = J_g. \tag{1.1.43}$$

By (1.1.40) and (1.1.43), we get

$$A_{g_J} = J, \quad r \circ i(J) = (-JJ)^{-1/2}\, J = J. \tag{1.1.44}$$

Thus r is a retract of i.

On the other hand, by (1.1.41),

$$i \circ r : g \in \mathscr{M}(V) \to \omega(\cdot, J_g\cdot) = g(\cdot, (-A_g^2)^{1/2}\cdot) \in \mathscr{M}(V). \tag{1.1.45}$$

For $s \in [0,1]$, set

$$H_s : g \in \mathscr{M}(V) \to g(\cdot, (-A_g^2)^{s/2}\cdot) \in \mathscr{M}(V). \tag{1.1.46}$$

Then $H_0 = \mathrm{Id}, H_1 = i \circ r$. This means $i \circ r$ is homotopy to the identity.

We have proved that $\mathscr{J}(V,\omega)$ has the same homotopy type of $\mathscr{M}(V)$. As $\mathscr{M}(V)$ is convex, thus contractible. Hence $\mathscr{J}(V,\omega)$ is contractible. We can also prove it directly: Fix $J_0 \in \mathscr{J}(V,\omega)$, we define the continuous map $\Phi : [0,1] \times \mathscr{J}(V,\omega) \to \mathscr{J}(V,\omega)$ by

$$\Phi(t, J) = r(t g_{J_0} + (1-t) g_J), \tag{1.1.47}$$

then

$$\Phi(0, J) = r(g_J) = J, \quad \Phi(1, J) = r(g_{J_0}) = J_0. \tag{1.1.48}$$

The proof of Proposition 1.1.13 is completed. □

We note that the maps i and r are all smooth.

Remark 1.1.14. Let J be a compatible complex structure on (V,ω). Then $g_J = \omega(\cdot, J\cdot)$ is a J-invariant scalar product on V and for any $u, v \in V$,

$$\omega(u,v) = g_J(Ju,v) =: \langle Ju, v\rangle_J. \tag{1.1.49}$$

If W is a Lagrangian subspace of (V,ω), then by (1.1.49), we have

$$W \cap JW = \{0\}, \qquad \text{and } W \perp_{g_J} JW. \tag{1.1.50}$$

We use the compatible complex structures to understand the symplectic basis now.

Theorem 1.1.15. *(Linear normal form) If (V,ω) is a symplectic vector space of dimension $2n$. Then*

1) there exists a symplectic base $\{e_j, f_j\}_j$ of (V,ω), i.e., for $1 \leqslant i,j \leqslant n$,

$$\omega(e_i,e_j) = \omega(f_i,f_j) = 0, \qquad\qquad \omega(e_i,f_j) = \delta_{ij}\,. \tag{1.1.51}$$

2) If $W \subset V$ is a subspace, then there exists a symplectic base $\{e_j, f_j\}_j$ of (V,ω) such that

$$\begin{aligned} &W = \mathrm{Span}\{e_1,\ldots,e_{k+l},f_1,\ldots,f_k\},\\ &W^{\perp_\omega} = \mathrm{Span}\{e_{k+1},\ldots,e_n,f_{k+l+1},\ldots,f_n\},\\ &N = W\cap W^{\perp_\omega} = \mathrm{Span}\{e_{k+1},\ldots,e_{k+l}\}.\end{aligned} \tag{1.1.52}$$

Thus we have a symplectic isomorphism of vector spaces

$$V \simeq W/N \oplus W^{\perp_\omega}/N \oplus (N\oplus N^*, \omega_{st}). \tag{1.1.53}$$

Proof. 1) is Proposition 1.1.7, here we reprove it by using complex structures. We fix a $J \in \mathscr{J}(V,\omega)$. Take $e_1 \in V$ such that $\langle e_1, e_1\rangle_J = 1$, then $W_1 = \mathrm{Span}\{e_1, Je_1\}$ is a symplectic subspace of (V,ω). Consider $W_1^{\perp_g} \subset V$ the orthogonal complement of W_1 in $(V, \langle\,,\,\rangle_J)$, by the recurrence on the dimension, we get an orthonormal basis $\{e_j, f_j = Je_j\}_j$ of $(V, \langle\,,\,\rangle_J)$. Then by (1.1.49), we get (1.1.51).

2) Now let $W_1 \subset W$, $W_2 \subset W^{\perp_\omega}$ be subspaces such that

$$W = W_1 \oplus N, \quad W^{\perp_\omega} = W_2 \oplus N. \tag{1.1.54}$$

Then $(W_1, \omega|_{W_1}), (W_2, \omega|_{W_2})$ are symplectic vector spaces, and we have the orthogonal decomposition of symplectic vector spaces,

$$(V,\omega) = (W_1,\omega|_{W_1}) \oplus (W_2,\omega|_{W_2}) \oplus (W_3,\omega|_{W_3}) \text{ with } W_3 = (W_1\oplus W_2)^{\perp_\omega}. \tag{1.1.55}$$

We claim that N is a Lagrangian subspace of $(W_3, \omega|_{W_3})$. In fact, as $N \subset W$, we get $N \perp_\omega W_1$, $N \perp_\omega W_2$, thus

$$N \subset W_3 = (W_1 \oplus W_2)^{\perp_\omega}.$$

As $N = W \cap W^{\perp_\omega}$, we get $\omega|_N = 0$, i.e., $N \subset N^{\perp_{\omega|_{W_3}}}$. But if $x \in N^{\perp_{\omega|_{W_3}}}$ as $x \perp_\omega W_1, x\perp_\omega W_2$, from (1.1.54), we get

$$x \in W^{\perp_\omega} \text{ and } x \in (W^{\perp_\omega})^{\perp_\omega} = W, \text{ thus } x \in N.$$

Take a $J_3 \in \mathscr{J}(W_3, \omega|_{W_3})$, then by (1.1.50), J_3N is orthogonal with N in $(W_3, \omega|_{W_3}(\cdot, J_3\cdot))$ and $W_3 = N \oplus J_3N$. We define the map $\psi : W_3 \to N \oplus N^*$ by taking the identity on N and

$$\psi(v) = \omega|_{W_3}(\cdot, v) \in N^* \quad \text{for } v \in J_3N. \tag{1.1.56}$$

Then ψ is a symplectic isomorphism from $(W_3, \omega|_{W_3})$ to $(N \oplus N^*, \omega_{st})$. Thus we get (1.1.53).

By taking a symplectic basis for $(W_1, \omega|_{W_1})$ and $(W_2, \omega|_{W_2})$, we get (1.1.52).

The proof of Theorem 1.1.15 is completed. □

Remark 1.1.16. In the notation of Theorem 1.1.15, W is symplectic if and only if $l = 0$, W is isotropic if and only if $k = 0$, W is coisotropic if and only if $k + l = n$, W is Lagrangian if and only if $k = 0, l = n$.

1.1.3 Symplectic groups

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by

$$\imath : z = (x_1 + iy_1, \ldots, x_n + iy_n)^t \in \mathbb{C}^n \to \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2n}, \tag{1.1.57}$$

with $x = (x_1, \ldots, x_n)^t, y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$. For $Z \in M_n(\mathbb{C})$, we define $\imath(Z) \in M_{2n}(\mathbb{R})$ by: for $z \in \mathbb{C}^n$,

$$\imath(Zz) = \imath(Z)\imath(z). \tag{1.1.58}$$

Then $\imath$ induces naturally an injection of matrix groups

$$\imath : Z = X + iY \in M_n(\mathbb{C}) \to \imath(Z) = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M_{2n}(\mathbb{R}). \tag{1.1.59}$$

From (1.1.58), $\imath$ identifies $\mathrm{GL}(n, \mathbb{C})$ as a subgroup of $\mathrm{GL}(2n, \mathbb{R})$, and

$$\imath(i) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = J_0 \tag{1.1.60}$$

is the canonical complex structure on $\mathbb{R}^{2n}$. Moreover, for $A \in \mathrm{GL}(n, \mathbb{C})$,

$$\imath(A^*) = \imath(A)^t. \tag{1.1.61}$$

Thus we get

$$\imath(\mathrm{U}(n)) \subset \mathrm{O}(2n). \tag{1.1.62}$$

We identify $\mathrm{U}(n)$ as a subgroup of $\mathrm{O}(2n)$.

Let (V, ω) be a symplectic vector space. Then the symplectic group $\mathrm{Sp}(V)$ is defined as

$$\mathrm{Sp}(V) = \{A \in \mathrm{GL}(V) : \omega(A\cdot, A\cdot) = \omega(\cdot, \cdot)\}. \tag{1.1.63}$$

Clearly, $\mathrm{Sp}(V)$ is a subgroup of $\mathrm{GL}(V)$. We denote also $\mathrm{Sp}(V)$ by $\mathrm{Sp}(V, \omega)$.

Let $\{e_j, f_j\}_j$ be a symplectic basis of (V, ω). For $x = (x_1, \ldots, x_n)^t$, $y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$, set

$$\phi : \sum_{i=1}^n x_i e_i + y_i f_i \in V \to \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2n}. \tag{1.1.64}$$

Then $\phi : (V, \omega) \to (\mathbb{R}^{2n}, \omega_0)$ is a symplectic isomorphism. We denote by $\mathrm{Sp}(2n)$ the symplectic group of $(\mathbb{R}^{2n}, \omega_0)$. Then by (1.1.31) and (1.1.63), we have

$$A \in \mathrm{Sp}(2n) \ \text{ if and only if } A^t J_0 A = J_0. \tag{1.1.65}$$

Proposition 1.1.17. *We have*

$$\mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n). \tag{1.1.66}$$

Proof. We check first

$$A \in M_n(\mathbb{C}) \Longleftrightarrow A \in M_{2n}(\mathbb{R}) \text{ and } AJ_0 = J_0 A. \tag{1.1.67}$$

In fact, the $\Rightarrow$ direction is trivial. For the $\Leftarrow$ direction, if $A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, then

$$AJ_0 = \begin{pmatrix} Y & -X \\ W & -Z \end{pmatrix}, \qquad J_0 A = \begin{pmatrix} -Z & -W \\ X & Y \end{pmatrix}. \tag{1.1.68}$$

Hence, $AJ_0 = J_0 A$ is equivalent to

$$X = W, \qquad Y = -Z. \tag{1.1.69}$$

By (1.1.59) and (1.1.69), (1.1.67) holds.

From (1.1.58) and (1.1.59), for $X, Y \in M_n(\mathbb{R})$,

$$\det \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \neq 0 \Longleftrightarrow \det(X + iY) \neq 0. \tag{1.1.70}$$

By (1.1.67) and (1.1.70), we have

$$\mathrm{GL}(n, \mathbb{C}) = \{A \in \mathrm{GL}(2n, \mathbb{R}) : J_0 A = AJ_0\}. \tag{1.1.71}$$

By (1.1.65) and (1.1.71), a matrix A belongs to two of the three groups $\mathrm{Sp}(2n), \mathrm{GL}(n, \mathbb{C})$ and $\mathrm{O}(2n)$ will be in the other group. Thus we get the first two equations of (1.1.66).

It remains to show the last equation of (1.1.66). For $A \in \mathrm{GL}(n, \mathbb{C})$, by (1.1.61),

$$\imath(A) \in \mathrm{O}(2n) \Longleftrightarrow \imath(A)\imath(A)^t = I \Longleftrightarrow \imath(AA^*) = I \Longleftrightarrow AA^* = I. \tag{1.1.72}$$

Thus $\mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) \subset \mathrm{U}(n)$. By (1.1.62), this means the last equation of (1.1.66) holds. $\square$

For $A \in \mathrm{GL}(V)$, let A^t be the adjoint of A with respect to a scalar product $\langle\, , \rangle$ on V, set

$$|A| = (AA^t)^{1/2}, \qquad U = |A|^{-1} A. \tag{1.1.73}$$

Then $|A|$ is symmetric positive. As

$$UU^t = |A|^{-1} AA^t |A|^{-1} = |A|^{-1} |A|^2 |A|^{-1} = \mathrm{Id}_V, \tag{1.1.74}$$

we get $U \in \mathrm{O}(V)$ the orthogonal group of $(V, \langle\, , \rangle)$. The decomposition

$$A = |A| U \tag{1.1.75}$$

will be called the polar decomposition. Moreover, the polar decomposition is unique, i.e., if $A = BU'$ where B is symmetric positive and $U' \in \mathrm{O}(V)$, then $B = |A|, U' = U$.

By (1.1.33), (1.1.73), the map $\mathrm{GL}(V) \ni A \to |A|$ and $\mathrm{GL}(V) \ni A \to U$ are smooth.

Proposition 1.1.18. *Let (V, ω) be a symplectic vector space. Let J be a compatible complex structure on (V, ω). If $A \in \mathrm{Sp}(V)$, then the transpose $A^t \in \mathrm{Sp}(V)$. Moreover, if A is symmetric positive, then for $s \in \mathbb{R}$, $A^s \in \mathrm{Sp}(V)$. In particular, for $A \in \mathrm{Sp}(V)$, in the polar decomposition (1.1.75), $|A| \in \mathrm{Sp}(V)$ and $U \in \mathrm{U}(V)$ the unitary group of (V, J, ω), i.e., the space of complex automorphisms of (V, J) preserving the scalar product $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$.*

Proof. By (1.1.25) and (1.1.63), $A \in \mathrm{Sp}(V)$ is equivalent to

$$J = A^t J A. \tag{1.1.76}$$

Thus,

$$J^{-1} = A^{-1} J^{-1} (A^t)^{-1}. \tag{1.1.77}$$

As $J^{-1} = -J$, we have $(A^t)^{-1} \in \mathrm{Sp}(V)$. Thus $A^t \in \mathrm{Sp}(V)$.

If $A \in \mathrm{Sp}(V)$ is symmetric and positive, by (1.1.76), we have

$$A^{-1} = JAJ^{-1}. \tag{1.1.78}$$

This implies $(\lambda - A^{-1})^{-1} = J(\lambda - A)^{-1}J^{-1}$ for $\lambda \notin \mathrm{Spec}(A)$. Combining with (1.1.33), we get

$$A^{-s} = JA^s J^{-1} \quad \text{for } s \in \mathbb{R}, \tag{1.1.79}$$

which is equivalent to $A^s \in \mathrm{Sp}(V)$.

Thus in the decomposition (1.1.73), we have $|A| \in \mathrm{Sp}(V)$ and this implies $U \in \mathrm{Sp}(V)$. By Proposition 1.1.17, we get $U \in \mathrm{U}(V) = \{A \in \mathrm{O}(V) : JA = AJ\}$. □

Remark 1.1.19. From Proposition 1.1.18, we have

$$\mathrm{Sp}(2n)/\mathrm{U}(n) := \{g\,U(n) : g \in \mathrm{Sp}(2n)\} \simeq \{A \in \mathrm{Sp}(2n) : A \text{ is symmetric and positive}\}. \tag{1.1.80}$$

and the map

$$\psi : [0,1] \times \mathrm{Sp}(2n) \to \mathrm{Sp}(2n), \quad \psi(s, A) = P^s Q, \tag{1.1.81}$$

here $A = PQ$ is the polar decomposition of A, is a retract by deformation from $\mathrm{Sp}(2n)$ to $\mathrm{U}(n)$. In fact $\psi(1,\cdot) = \mathrm{Id}_{\mathrm{Sp}(2n)}$, $\psi(0,\cdot) : \mathrm{Sp}(2n) \to \mathrm{U}(n)$ and $\psi(0,\cdot)|_{\mathrm{U}(n)} = \mathrm{Id}_{\mathrm{U}(n)}$.

In particular we conclude that $\mathrm{Sp}(2n)$ is connected, as $\mathrm{U}(n)$ is connected.

Proposition 1.1.20. *Let (V, ω) be a symplectic vector space of dimension $2n$. Then we have*

$$\mathscr{J}(V, \omega) \simeq \mathrm{Sp}(2n)/\mathrm{U}(n). \tag{1.1.82}$$

Hence, $\mathscr{J}(V,\omega)$ is a noncompact symmetric space, and $\mathrm{Sp}(2n)/\mathrm{U}(n)$ is contractible.

Proof. By (1.1.64), $\phi : (V, \omega) \to (\mathbb{R}^{2n}, \omega_0)$ is a symplectic isomorphism. Thus we can work directly for $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$ and let $\{e_j, J_0 e_j\}_j$ be an orthonormal basis of $(\mathbb{R}^{2n}, \langle \cdot, \cdot \rangle = \omega_0(\cdot, J_0 \cdot))$.

For $(A, J) \in \mathrm{Sp}(2n) \times \mathscr{J}(\mathbb{R}^{2n}, \omega_0)$, by (1.1.25), for $u \neq 0 \in V$,

$$\left(AJA^{-1}\right)^2 = -\mathrm{Id}_V, \qquad \omega(u, AJA^{-1}u) = \omega(A^{-1}u, JA^{-1}u) > 0. \tag{1.1.83}$$

Thus we can define the $\mathrm{Sp}(2n)$-action on $\mathscr{J}(\mathbb{R}^{2n}, \omega_0)$ by

$$(A, J) \in \mathrm{Sp}(V, \omega) \times \mathscr{J}(V, \omega) \to AJA^{-1} \in \mathscr{J}(V, \omega). \tag{1.1.84}$$

By Proposition 1.1.17, the stabilizer at J_0 is $\mathrm{U}(n)$, thus the stabilizer at AJ_0A^{-1} is $A \cdot U(n)$.

It remains to show that this $\mathrm{Sp}(2n)$-action is transitive. Let $\{e'_j, Je'_j\}_j$ be an orthonormal basis of $(\mathbb{R}^{2n}, g(\cdot,\cdot) = \omega_0(\cdot, J\cdot))$. Let $A \in \mathrm{GL}(2n, \mathbb{R})$ be defined by

$$Ae_j = e'_j, \qquad AJ_0 e_j = Je'_j. \tag{1.1.85}$$

As A sends one symplectic basis to another, $A \in \mathrm{Sp}(2n)$. Moreover, by (1.1.85), we have

$$AJ_0A^{-1}e'_j = AJ_0e_j = Je'_j, \qquad AJ_0A^{-1}Je_j = -Ae_j = -e'_j = J^2e'_j. \tag{1.1.86}$$

Thus, $AJ_0A^{-1} = J$. This means $\mathrm{Sp}(2n)$ act transitively on $\mathscr{J}(\mathbb{R}^{2n}, \omega_0)$, thus (1.1.82) holds. From Proposition 1.1.13 and (1.1.82), $\mathrm{Sp}(2n)/\mathrm{U}(n)$ is contractible. The proof of Proposition 1.1.20 is completed. □

Exercise 1.1.1. In (1.1.39), verify that if J is a compatible complex structure on (V, ω) such that g is J-invariant (i.e., $g(J\cdot, J\cdot) = g(\cdot, \cdot)$), then J is given by (1.1.40).

Exercise 1.1.2. If $A \in \mathrm{Sp}(2n)$, then

1. $\det(A) = 1$.
2. If $\lambda \in \mathrm{Spec}(A)$, then $\lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1} \in \mathrm{Spec}(A)$.
3. If $\lambda, \mu \in \mathrm{Spec}(A)$, $\lambda\mu \neq 1$, then $E_\lambda \perp_\omega E_\mu$.

Exercise 1.1.3. Let $\{e_j, J_0e_j\}_j$ be the canonical basis of $(\mathbb{R}^{2n}, \omega_0)$. Set $L_0 = \mathrm{Span}\{e_1, \ldots, e_n\}$.

1. Verify that L_0 is Lagrangian in $(\mathbb{R}^{2n}, \omega_0)$.
2. If $L \subset (\mathbb{R}^{2n}, \omega_0)$ is Lagrangian, J is a compatible complex structure of $(\mathbb{R}^{2n}, \omega_0)$, verify that $L \perp JL$ in $(\mathbb{R}^{2n}, \omega_0(\cdot, J\cdot))$.
3. If $L \subset (\mathbb{R}^{2n}, \omega_0)$ is Lagrangian, $A \in U(n)$, then $AL := \imath(A)L$ is also Lagrangian.
4. If $L \subset (\mathbb{R}^{2n}, \omega_0)$ is Lagrangian, then there exists $A \in U(n)$ such that $AL_0 = L$.
5. Conclude that the set of Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega_0)$ is isomorphic to $\mathrm{U}(n)/\mathrm{O}(n) := \{A\mathrm{O}(n) : A \in \mathrm{U}(n)\}$, and we identify $\mathrm{O}(n)$ as a subgroup of $\mathrm{U}(n)$ by the natural injection.

Exercise 1.1.4. Show that $\mathrm{Sp}(2) = \mathrm{SL}(2, \mathbb{R})$ and $\mathrm{Sp}(2)/\mathrm{U}(1) = \mathbb{H}$, where

$$\mathbb{H} = \{x + iy \in \mathbb{C} : x \in \mathbb{R}, y > 0\}. \tag{1.1.87}$$

is the Poincaré upper half-plane.

Exercise 1.1.5. The aim of this exercise is to show that $\mathrm{Sp}(2n)/U(n)$ can be identified as the Siegel upper half-plane $\mathbb{H}_n$:

$$\mathbb{H}_n = \{X + iY \in \mathrm{GL}(n, \mathbb{C}) : X, Y \in \mathrm{GL}(n, \mathbb{R}) \text{ both symmetric, and } Y \text{ is positive}\}. \tag{1.1.88}$$

1. Verify that $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n)$ if and only if
$$A^tC = C^tA, \quad B^tD = D^tB, \quad A^tD - C^tB = I. \tag{1.1.89}$$
2. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n)$, $Z = X + iY \in \mathbb{H}_n$, verify that
$$(C\overline{Z} + D)^t(AZ + B) - (A\overline{Z} + B)^t(CZ + D) = 2iY. \tag{1.1.90}$$
Conclude that the matrix $CZ + D$ is invertible and $(AZ + B)(CZ + D)^{-1} \in \mathbb{H}_n$.

3. Show that $(g, Z) \in \mathrm{Sp}(2n) \times \mathbb{H}_n \to g \circ Z = (AZ+B)(CZ+D)^{-1} \in \mathbb{H}_n$ defines a $\mathrm{Sp}(2n)$-action on $\mathbb{H}_n$.

4. Show this action is transitive: for $Z = X + iY \in \mathbb{H}_n$, verify that $\begin{pmatrix} Y^{1/2} & XY^{-1/2} \\ 0 & Y^{-1/2} \end{pmatrix} \in \mathrm{Sp}(2n)$, and $Z = \begin{pmatrix} Y^{1/2} & XY^{-1/2} \\ 0 & Y^{-1/2} \end{pmatrix} \circ iI$.

5. Verify that the stabilizer of iI is $\mathrm{U}(n)$. Conclude that $\mathrm{Sp}(2n)/U(n) \simeq \mathbb{H}_n$.

1.2 Symplectic manifolds

In this section, we find the version on manifolds of many results on symplectic vector spaces, usually with extra efforts.

1.2.1 Basic calculus on manifolds

In this section, we recall the Cartan's formula and the Hodge decomposition theorem.

We say a manifold always means a $\mathscr{C}^\infty$ manifold without boundary. Let M be a manifold of dimension m. We denote by TM its tangent bundle, and by T^*M its cotangent bundle. Then the fiberwise exterior algebra of T^*M forms a vector bundle $\Lambda^\bullet(T^*M)$ on M, and for $k \in \mathbb{N}$, the space of k-th differential forms $\Omega^k(M)$ is the space of smooth sections of $\Lambda^k(T^*M)$ on M, i.e., $\Omega^k(M) = \mathscr{C}^\infty(M, \Lambda^k(T^*M))$, and $\Omega^\bullet(M) = \oplus_{k=0}^m \Omega^k(M)$. For $\alpha \in \Omega^k(M)$, we denote its degree by $\deg \alpha$, thus $\deg \alpha = k$. Let $\Omega_c^\bullet(M)$ be the space of elements of $\Omega^\bullet(M)$ with compact support.

Let $\phi : M \to N$ be a smooth map between two manifolds. We denote by

$$d_x\phi : T_xM \to T_{\phi(x)}N \tag{1.2.1}$$

the differential of ϕ at $x \in M$. The pull back of a differential form is defined by: for $\beta \in \Omega^k(N)$ and for $X_1, \dots, X_k \in T_xM$,

$$(\phi^*\beta)_x(X_1, \dots, X_k) = \beta_{\phi(x)}\left(d\phi_x(X_1), \dots, d\phi_x(X_k)\right). \tag{1.2.2}$$

Then $\phi^*\beta$ is a differential form on M. If $\psi : M' \to M$ is another smooth map between two manifolds, then by (1.2.2), we verify that

$$\psi^*(\phi^*\beta) = (\phi \circ \psi)^*\beta \quad \text{for } \beta \in \Omega^\bullet(N). \tag{1.2.3}$$

If ϕ is a diffeomorphism, the push forward of a vector field $X \in \mathscr{C}^\infty(M, TM)$ is defined by

$$(\phi_*X)_y = d\phi_{\phi^{-1}(y)}(X_{\phi^{-1}(y)}) \in T_yN. \tag{1.2.4}$$

Thus ϕ_*X is a vector field on N.

If $X_\cdot \in \mathscr{C}^\infty(\mathbb{R} \times M, TM)$ is a time dependent vector field on M, the flow $\phi_t^{X_\cdot} : M \to M$ associated with $X_\cdot$ is the solution of the ordinary differential equation on M,

$$\frac{\partial}{\partial t}\phi_t^{X_\cdot}(x) = X_t(\phi_t^{X_\cdot}(x)), \qquad \phi_0^{X_\cdot}(x) = x. \tag{1.2.5}$$

If $\phi_t^{X_\cdot}$ is defined, $\phi_t^{X_\cdot} : M \to M$ is a diffeomorphism. If X is time independent, then we will also denote by

$$e^{tX} := \phi_t^X : M \to M. \tag{1.2.6}$$

Lemma 1.2.1. *For $t \in \mathbb{R}$, let $\phi_t^{X_\cdot}, \phi_t^{Y_\cdot}$ be the flows associated with the time dependent vector fields $X_\cdot, Y_\cdot$ on M. Then*

a) $\phi_t^{X_\cdot} \circ \phi_t^{Y_\cdot} = \phi_t^{X_\cdot + \phi_*^{X_\cdot} Y_\cdot}$.

b) $\left(\phi_t^{X_\cdot}\right)^{-1} = \phi_t^{-((\phi_\cdot^{X_\cdot})^{-1})_* X_\cdot}$.

c) *For any $\phi \in \mathrm{Diff}(M)$, we have $\phi \circ \phi_t^{X_\cdot} \circ \phi^{-1} = \phi_t^{\phi_* X_\cdot}$.*

Proof. By (1.2.4) and (1.2.5), for any $x \in M$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\phi_t^{X\cdot} \circ \phi_t^{Y\cdot}(x) &= X_t\left(\phi_t^{X\cdot} \circ \phi_t^{Y\cdot}(x)\right) + d\phi_t^{X\cdot}\left(Y_t(\phi_t^{Y\cdot}(x))\right) \\ &= X_t\left(\phi_t^{X\cdot} \circ \phi_t^{Y\cdot}(x)\right) + \phi_{t*}^{X\cdot} Y_t\left(\phi_t^{X\cdot} \circ \phi_t^{Y\cdot}(x)\right). \quad (1.2.7)\end{aligned}$$

By the uniqueness of the solution of ordinary differential equations, we know a) holds.

By applying a) for $Y_t = -\left((\phi_t^{X\cdot})^{-1}\right)_* X_t$, we get b).

By Definition, for any $x \in M$, we have

$$\frac{\partial}{\partial t}\phi \circ \phi_t^{X\cdot} \circ \phi^{-1}(x) = d\phi\left(X_t\left(\phi_t^{X\cdot} \circ \phi^{-1}(x)\right)\right) = (\phi_* X_t)\left(\phi \circ \phi_t^{X\cdot} \circ \phi^{-1}(x)\right). \quad (1.2.8)$$

By the uniqueness of the solution of ordinary differential equations, this means c) holds. The proof of Lemma 1.2.1 is completed. □

For $X, Y \in \mathscr{C}^\infty(M, TM)$, $\alpha \in \Omega^\bullet(M)$, the Lie derivation of Y and α in the direction X is defined by

$$L_X Y = \frac{\partial}{\partial t}\Big|_{t=0} \left(\phi_{-t}^X\right)_* Y, \qquad L_X \alpha = \frac{\partial}{\partial t}\Big|_{t=0} \left(\phi_t^X\right)^* \alpha. \quad (1.2.9)$$

For $f \in \mathscr{C}^\infty(M) = \Omega^0(M)$, by (1.2.9), we have $L_X f = Xf$. We verify that

$$[X, Y]f := XYf - YXf \quad (1.2.10)$$

defines a vector field $[X, Y]$ on M which is called the Lie bracket of vector fields X and Y. Classically, we have

$$L_X Y = [X, Y]. \quad (1.2.11)$$

Let $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ be the exterior differential on M. When we like to precise the manifold M, we denote also d by d^M. By Definition, for $\alpha \in \Omega^k(M)$, $X_0, \dots, X_k$ vector fields on M, we have

$$\begin{aligned}d\alpha(X_0, \dots, X_k) &= \sum_{i=0}^k (-1)^i X_i \alpha(X_0, \dots, \widehat{X_i}, \dots, X_k) \\ &\quad + \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k), \quad (1.2.12)\end{aligned}$$

where $\widehat{\ }$ means we omit the term.

If $x = (x_1, \dots, x_m)$ is a local coordinate system on $U \subset M$, if

$$\alpha(x) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m} \alpha_{i_1, \dots, i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad (1.2.13)$$

then, from (1.2.12), we get

$$d\alpha(x) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m, 1 \leqslant j \leqslant m} \frac{\partial \alpha_{i_1, \dots, i_k}}{\partial x_j}(x) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \quad (1.2.14)$$

By (1.2.14), we verify directly

$$d^2 = 0. \tag{1.2.15}$$

For $\alpha \in \Omega^\bullet(M)$, we say α is closed if $d\alpha = 0$; and we say α is exact if there exists β such that $\alpha = d\beta$.

By (1.2.15), $\mathrm{Im}(d|_{\Omega^{i-1}(M)}) \subset \ker(d|_{\Omega^i(M)})$. We define the i-th de Rham cohomology group of M by

$$H^i(M, \mathbb{R}) = \frac{\ker(d|_{\Omega^i(M)})}{\mathrm{Im}(d|_{\Omega^{i-1}(M)})}. \tag{1.2.16}$$

For $X \in \mathscr{C}^\infty(M, TM)$, we denote by $i_X : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ the contraction by X. On a local coordinate U, if α is given by (1.2.13), if

$$X = \sum_i X_i \frac{\partial}{\partial x_i}, \tag{1.2.17}$$

then,

$$i_X \alpha = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m, 1 \leqslant j \leqslant k} (-1)^{j-1} X_{i_j}\, \alpha_{i_1, \ldots, i_k}\, dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_k}. \tag{1.2.18}$$

By (1.2.14), (1.2.18), we get the following Leibniz rule: for $\alpha \in \Omega^k(M), \beta \in \Omega^\bullet(M)$, $X \in \mathscr{C}^\infty(M, TM)$,

$$\begin{aligned} d(\alpha \wedge \beta) &= (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta), \\ i_X(\alpha \wedge \beta) &= (i_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_X\beta). \end{aligned} \tag{1.2.19}$$

The following relation on the contraction, Lie derivative and exterior differential is very useful.

Theorem 1.2.2 (Cartan's formula)**.** *For $\alpha \in \Omega^\bullet(M)$, $X \in \mathscr{C}^\infty(M, TM)$, we have*

$$L_X \alpha = (d i_X + i_X d)\alpha. \tag{1.2.20}$$

Proof. For $\mathcal{L}_X = L_X$ or $di_X + i_X d$, by (1.2.9) and (1.2.19), for any $\alpha, \beta \in \Omega^\bullet(M)$, we get

$$\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X\beta). \tag{1.2.21}$$

Thus we only need to verify (1.2.20) for any $f \in \mathscr{C}^\infty(M)$ and $\alpha \in \Omega^1(M)$. For $Y \in \mathscr{C}^\infty(M, TM)$, we have

$$\begin{aligned} L_X f = Xf &= (df, X) = i_X df\,; \\ (di_X\alpha + i_X d\alpha)(Y) &= Y(\alpha(X)) + d\alpha(X, Y) = X(\alpha(Y)) - \alpha([X, Y]) \\ &= L_X(\alpha(Y)) - \alpha(L_X Y) = (L_X\alpha)(Y). \end{aligned} \tag{1.2.22}$$

Thus (1.2.20) holds. $\square$

The manifold M is orientable if there exists a nowhere vanishing m-form on M, in this case, M is oriented means we fix a nowhere vanishing m-form on M. If M is oriented, then for any $\beta \in \Omega_c^\bullet(M)$, we can define $\int_M \beta \in \mathbb{R}$, the integral of β on M. Note that $\int_M \beta = 0$ if $\deg \beta < m$. Moreover,

Theorem 1.2.3 (Stokes theorem)**.** *Assume M is oriented, then for any $\alpha \in \Omega^\bullet_c(M)$, we have*

$$\int_M d\alpha = 0. \tag{1.2.23}$$

Recall that any Euclidean metric on TM is called a Riemannian metric on M (or TM).

Let g^{TM} be a Riemannian metric on TM. For $x \in M, u, v \in T_xM$, we denote also $\langle u, v\rangle = g^{TM}(u,v)_x$ and $|v| = \sqrt{\langle v, v\rangle}$.

For a curve $\gamma :]a, b[\to M$ $(a, b \in \mathbb{R})$, the length of γ is $|\gamma| = \int_a^b |\dot\gamma(t)|dt$, here $\dot\gamma(t) := \frac{\partial \gamma}{\partial t}(t) \in T_{\gamma(t)}M$. We say γ is a geodesic of (M, g^{TM}) if locally γ attends the minimal length, i.e., for any $c_1 \geq a$, there exists $\varepsilon > 0$ such that for $c_1 < c_2 < c_1 + \varepsilon$, the length of $\gamma|_{[c_1,c_2]}$ is minimal for all curve from $\gamma(c_1)$ to $\gamma(c_2)$. For $v \in T_xM$, $[0,1] \ni t \to \exp_x(tv)$ is defined to be the unique geodesic with starting point x and its derivative at x is v, and we call $T_xM \ni v \to \exp_x(v)$ the exponential maps of M.

The metric g^{TM} induces a Euclidean metric $\langle\cdot,\cdot\rangle_{\Lambda^\cdot(T^*M)}$ on $\Lambda^\cdot(T^*M)$. If $\{e_j\}$ is an orthonormal basis of TM and $\{e^j\}$ its dual basis, then $\{e^{i_1} \wedge \cdots \wedge e^{i_k}\}_{1\leq i_1<\cdots<i_k\leq m}$ is an orthonormal basis of $\Lambda(T^*M)$.

Let dv_M be the Riemannian volume form of (M, g^{TM}). Then dv_M is a m-form with values in $o(TM)$, the orientable bundle of M which is a real line bundle on M. When M is orientable, dv_M is just a m-form and

$$dv_M = e^1 \wedge \cdots \wedge e^m \tag{1.2.24}$$

for any oriented orthonormal frame $\{e_j\}$ of TM. On a local coordinate U, we can write

$$\begin{aligned} g^{TM}(x) &= \sum_{i,j} g_{ij}(x) dx_i \otimes dx_j, \\ dv_M &= \sqrt{\det(g_{ij})} dx_1 \wedge \cdots \wedge dx_m, \quad \text{if } M \text{ is oriented}, \end{aligned} \tag{1.2.25}$$

where $(g_{ij}(x))$ is a symmetric positive matrix.

For $s_1, s_2 \in \Omega^\bullet_c(M)$, we define

$$\langle s_1, s_2\rangle = \int_M \langle s_1, s_2\rangle_{\Lambda^\cdot(T^*M)} dv_M. \tag{1.2.26}$$

Then $\langle\,,\,\rangle$ is a scalar product on $\Omega^\bullet_c(M)$. Let d^* be the formal adjoint of d with respect to $\langle\,,\,\rangle$, i.e., for $s_1, s_2 \in \Omega^\bullet_c(M)$,

$$\langle d^* s_1, s_2\rangle := \langle s_1, ds_2\rangle. \tag{1.2.27}$$

By (1.2.15), we know that

$$d^* : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M), \quad \text{and } (d^*)^2 = 0. \tag{1.2.28}$$

Let Δ be the Hodge Laplacian on $\Omega^\bullet(M)$ defined by

$$\Delta = (d + d^*)^2 = dd^* + d^*d : \Omega^\bullet(M) \to \Omega^\bullet(M). \tag{1.2.29}$$

Clearly, Δ preserves the $\mathbb{Z}$-grading on $\Omega^\bullet(M)$ which is defined by its degree.

Theorem 1.2.4 (Hodge)**.** *If M is compact, for any $k \in \mathbb{N}$ and $0 \leqslant k \leqslant m$, we have the orthogonal decompositions,*

$$\Omega^k(M) = \ker(\Delta|_{\Omega^k(M)}) \oplus \mathrm{Im}(\Delta|_{\Omega^k(M)}), \tag{1.2.30}$$

and

$$\begin{aligned}\ker(\Delta|_{\Omega^k(M)}) &= \ker(d|_{\Omega^k(M)}) \cap \ker(d^*|_{\Omega^k(M)}), \\ \mathrm{Im}(\Delta|_{\Omega^k(M)}) &= \mathrm{Im}(d|_{\Omega^{k-1}(M)}) \oplus \mathrm{Im}(d^*|_{\Omega^{k+1}(M)}).\end{aligned} \tag{1.2.31}$$

In particular,

$$H^\bullet(M, \mathbb{R}) \simeq \ker d \cap \ker d^* = \ker \Delta. \tag{1.2.32}$$

Note that Δ is a second order elliptic differential operator, thus $\dim \ker \Delta < +\infty$ if M is compact. Combining this with (1.2.32), we get that if M is compact,

$$\dim H^j(M, \mathbb{R}) < +\infty \quad \text{for any } j \geq 0. \tag{1.2.33}$$

Let $\Delta^{-1} : \mathrm{Im}(\Delta) \to \Omega^\bullet(M)$ be the inverse of Δ: for any $\beta \in \mathrm{Im}(\Delta)$, by (1.2.30), there exists a unique $\alpha \in \mathrm{Im}(\Delta)$, such that $\beta = \Delta\alpha$, and we define $\Delta^{-1}\beta = \alpha$.

Form the proof of the Hodge Theorem, Theorem 1.2.4, if $\alpha_t \in \mathrm{Im}(d|_{\Omega^{k-1}(M)})$ is a smooth family on $t \in \mathbb{R}$ of differential forms on M, then

$$\beta_t = d^* \Delta^{-1} \alpha_t \tag{1.2.34}$$

is a smooth family on $t \in \mathbb{R}$ of differential forms on M such that

$$d\beta_t = \alpha_t. \tag{1.2.35}$$

1.2.2 Symplectic vector bundles

Let M be a manifold, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.2.5. Let E be a $\mathscr{C}^\infty$ manifold, $\pi : E \to M$ be a smooth map. We call that E is a $\mathbb{K}$-vector bundle on M of rank r if there exists an open covering $\{U_i\}$ of M, diffeomorphisms

$$\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{K}^r, \quad \Phi_i(v) = (\pi(v), \psi_i(v)), \tag{1.2.36}$$

such that if $U_i \cap U_j \neq \emptyset$, then

$$\Phi_{ji} := \Phi_j \circ \Phi_i^{-1} : U_i \cap U_j \times \mathbb{K}^r \to U_i \cap U_j \times \mathbb{K}^r, \Phi_{ji}(x, w) = (x, \psi_{ji}(x, w)),$$

$\psi_{ji}(x, w)$ is $\mathbb{K}$-linear on $w \in \mathbb{K}^r$, i.e., $\psi_{ji}(x, w) = \psi_{ji}(x)w$, and $\psi_{ji}(\cdot) \in \mathscr{C}^\infty(U_i \cap U_j, \mathrm{GL}(r, \mathbb{K}))$. We denote $r =: \mathrm{rk}(E)$. If $r = 1$, we call that E is a $\mathbb{K}$-line bundle.

Let $\pi : E \to M$ be a $\mathbb{K}$-vector bundle on M, we will denote $E|_U := \pi^{-1}(U)$ the restriction of E on a subset $U \subset M$. For $x \in M$, $E_x := \pi^{-1}(x)$ is the fiber of E at x, by the compatibility condition, the $\mathbb{K}$-vector space structure on E_x induced by (1.2.36) does not depend on the trivialization (1.2.36).

If F is another $\mathbb{K}$-vector bundle on M, then we define the dual of F: $F^* = \cup_{x \in M}\{F_x^*\}$, the direct sum of E and F: $E \oplus F = \cup_{x \in M}\{E_x \oplus F_x\}$, the tensor product of E and F: $E \otimes F = \cup_{x \in M}\{E_x \otimes F_x\}$. We verify directly they inherit naturally smooth structures, and they are vector bundles on M. We denote also $\mathrm{Hom}(E, F) = F \otimes E^*$.

A $\mathscr{C}^\infty$-map $\psi : E \to F$ is a morphism of $\mathbb{K}$-vector bundles if for any $x \in M$, ψ is a $\mathbb{K}$-linear map from E_x to F_x, i.e., $\psi \in \mathscr{C}^\infty(M, \mathrm{Hom}(E, F))$. If for any $x \in M$, ψ_x is an isomorphism from E_x to F_x, then we say that ψ is an isomorphism of $\mathbb{K}$-vector bundles.

Let V be a real vector bundle on M.