Definition 1.2.6. A Euclidean metric g^V on V is $g^V \in \mathscr{C}^\infty(M, V^* \otimes V^*)$ and for any $x \in M$, g_x^V is a Euclidean metric on V_x. We will denote also $g_x^V(u,v)$ by $\langle u, v\rangle_x$ for $u, v \in V_x$.

Let $\{U_i\}_{i\in I}$ be an open covering of M such that $\Phi_i : V|_{U_i} \to U_i \times \mathbb{R}^r$ gives a trivialization of $V|_{U_i}$. Let $\{\varphi_i\}$ be a partition of unity associated with $\{U_i\}$, i.e., $\varphi_i : U_i \to [0,1]$ is $\mathscr{C}^\infty$ function with compact support, and for any compact set K of M, $\{i \in I : \varphi_i \neq 0 \text{ on } K\}$ is finite, and $\sum_i \varphi_i \equiv 1$ on M.

On U_i, we define g_i^V the Euclidean metric induced by the canonical metric on $\mathbb{R}^r$, by $g_i^V(u,v)_x = \langle \psi_i(u), \psi_i(v)\rangle$. Then $g(\cdot,\cdot) = \sum_i (\varphi_i g_i^V)(\cdot,\cdot)$ defines a Euclidean metric on V. Thus there always exists a Euclidean metric on V. Certainly, the space of Euclidean metrics on V is contractible.

Definition 1.2.7. We say (V,ω) is a symplectic vector bundle on M if $\omega \in \mathscr{C}^\infty(M, \Lambda^2 V^*)$ and for any $x \in M$, (V_x, ω_x) is a symplectic vector space.

Definition 1.2.8. Let (V_1, ω_1), (V_2, ω_2) be symplectic vector bundles on M, $\psi \in \mathscr{C}^\infty(M, \operatorname{Hom}(V_1, V_2))$. If for any $x \in M$, $\psi_x : (V_{1,x}, \omega_{1,x}) \to (V_{2,x}, \omega_{2,x})$ is a symplectic morphism, then we call ψ is a symplectic morphism of symplectic vector bundles. If moreover ψ_x is an isomorphism for any $x \in M$, then we call ψ is a symplectic isomorphism of symplectic vector bundles.

Definition 1.2.9. If $J \in \mathscr{C}^\infty(M, \operatorname{End}(V))$ such that for any $x \in M$, $J_x^2 = -\operatorname{Id}_{V_x}$, we call J a complex structure on V. Moreover, if (V,ω) is a symplectic vector bundle on M, and for any $x \in M$, J_x is a compatible complex structure on (V_x, ω_x), we call J a compatible complex structure on (V,ω).

Let (V,ω) be a symplectic vector bundle on M. For g^V a Euclidean metric on V, we define for $x \in M$ $A_{g,x} \in \operatorname{GL}(V_x)$ by (1.1.39), then as ω, g^V are $\mathscr{C}^\infty$, we get $A_g \in \mathscr{C}^\infty(M, \operatorname{End}(V))$. By the argument after (1.1.33) and (1.1.40), we know $J_g = (-A_g^2)^{-1/2} A_g$ is $\mathscr{C}^\infty$ on M and it defines a compatible complex structure on (V,ω). Thus there always exists a compatible complex structure on any symplectic vector bundle (V,ω). Moreover by the argument in Proposition 1.1.13, the set of compatible complex structures on a symplectic vector bundle (V,ω) is contractible.

Proposition 1.2.10. *Let W be a subbundle of a symplectic vector bundle (V,ω). We suppose that $N = W \cap W^{\perp_\omega}$ is of constant rank. Then we have the symplectic isomorphism of symplectic vector bundles on M*

$$(V,\omega) \simeq (W/N, \omega) \oplus (W^{\perp_\omega}/N, \omega) \oplus (N \oplus N^*, \omega_{st}). \tag{1.2.37}$$

Proof. Let g^V be a metric on V. Let W_1 (resp. W_2) be the orthogonal complement of the vector subbundle N in W (resp. $W^{\perp_\omega}$), then $(W_1, \omega|_{W_1}), (W_2, \omega|_{W_2})$ are symplectic vector bundles on M and $W_1 \perp_\omega W_2$. Thus $(W_3 = (W_1 \oplus W_2)^{\perp_\omega}, \omega|_{W_3})$ is also a symplectic vector bundle on M.

Now by the argument after (1.1.55), for any $x \in M$, N_x is a Lagrangian subspace of $(W_{3,x}, \omega|_{W_{3,x}})$. Let $J_3 \in \mathscr{C}^\infty(M, \operatorname{End}(W_3))$ be a compatible complex structure on $(W_{3,x}, \omega|_{W_{3,x}})$. Now the map ψ_x in (1.1.56) is $\mathscr{C}^\infty$ on M, thus ψ is a symplectic isomorphism from $(W_3, \omega|_{W_3})$ to $(N \oplus N^*, \omega_{st})$. In particular, we get a natural symplectic isomorphism of symplectic vector bundles on M,

$$(V,\omega) \simeq (W_1, \omega|_{W_1}) \oplus (W_2, \omega|_{W_2}) \oplus (N \oplus N^*, \omega_{st}). \tag{1.2.38}$$

The proof of Proposition 1.2.10 is completed. □

Proposition 1.2.11. *Let (V, ω) be a symplectic vector bundle over M. For $x \in M$, there is a open neighborhood U_x of x, and $\{e_j, f_j\}_j$ a frame of V over U_x such that*

$$\omega(e_i, f_j) = \delta_{ij}, \qquad \omega(f_i, f_j) = 0, \qquad \omega(e_i, e_j) = 0. \tag{1.2.39}$$

Such a frame is called symplectic frame.

Proof. The proof is similar to the proof of Proposition 1.1.7. We shall prove it by induction on $\mathrm{rk}(V)/2$. If $\mathrm{rk}(V) = 0$, nothing to prove. We suppose $\mathrm{rk}(V) > 0$ and the proposition is true for vector bundles whose rank is smaller than $\mathrm{rk}(V) - 2$.

Let $e_1 \in \mathscr{C}^\infty(U_x, V)$ be a nonvanishing section over U_x. As ω_x is nondegenerate, there is $f \in \mathscr{C}^\infty(U_x, V)$ such that $\omega_x(e_1, f) \neq 0$. By shrinking U_x, we can suppose for $y \in U_x$, we have $\omega_y(e_1, f) \neq 0$. Then, on U_x, we have

$$\omega(e_1, f_1) = 1, \text{ with } f_1 = \frac{f}{\omega_y(e_1, f)}. \tag{1.2.40}$$

Let W be the subbundle on U_x generated by e_1, f_1. Then W is symplectic. Thus we have $V|_{U_x} = W \oplus W^{\perp_\omega}$. As $\mathrm{rk}(W^{\perp_\omega}) = \mathrm{rk}(V) - 2$, by the induction hypotheses, by shrinking U_x, there is a symplectic frame $e_2, f_2, \ldots, e_k, f_k$ on U_x. Then $e_1, f_1, \ldots, e_k, f_k$ is a symplectic frame of V on U_x. □

1.2.3 Symplectic manifolds

Definition 1.2.12. For a manifold M, if $J \in \mathscr{C}^\infty(M, \mathrm{End}(TM))$ and for any $x \in M$, $J_x^2 = -\mathrm{Id}_{T_xM}$, we call that J an almost complex structure on TM and (M, J) is an almost complex manifold.

Definition 1.2.13. A 2-form ω on a manifold M is called a symplectic form on M, if ω is real and closed, and if for any $x \in M$, $\omega_x \in \Lambda^2(T_x^*M)$ is nondegenerate. In this case, (M, ω) is called a symplectic manifold.

For a submanifold W of a symplectic manifold (M, ω), we call W is a symplectic (resp. isotropic, coisotropic, Lagrangian) submanifold if for any $x \in W$, T_xW is a symplectic (resp. isotropic, coisotropic, Lagrangian) subspace of (T_xM, ω_x).

A diffeomorphism $\psi : M \to N$ is called a symplectic diffeomorphism (or symplectomorphism) for two symplectic manifolds (M, ω), (N, ω_N), if $\psi^*\omega_N = \omega$.

Let $J \in \mathscr{C}^\infty(M, \mathrm{End}(TM))$ be an almost complex structure on a symplectic manifold (M, ω), then we say J is a compatible almost complex structure if $\omega(\cdot, J\cdot)$ defines a J-invariant Riemannian metric on TM.

Let (N, ω) be a symplectic manifold. By Proposition 1.1.3, N is even dimension. Let $\dim N = 2n$. Then $\omega^n \in \Omega^{2n}(N)$ induces a canonical orientation of N.

By the argument after Definition 1.2.9, there always exists a compatible almost complex structure on (N, ω).

If N is compact, we have

$$\int_N \omega^m > 0. \tag{1.2.41}$$

From (1.2.41), for any $0 \leqslant i \leqslant n$, $[w]^i \in H^{2i}(N, \mathbb{R})$ is non zero. In particular, $H^{2i}(N, \mathbb{R}) \neq 0$.

Example 1.2.14. Let M be a manifold of dimension m, and $\pi : T^*M \to M$ be the natural projection. The Liouville form $\lambda \in \Omega^1(T^*M)$ is defined by: for $x \in M, p \in T_x^*M$, $X \in T_{(x,p)}T^*M$,

$$(\lambda, X)_{(x,p)} := (p, d\pi_{(x,p)}X)_x. \tag{1.2.42}$$

Set

$$\omega^{T^*M} = -d\lambda. \tag{1.2.43}$$

Then ω^{T^*M} is a closed real 2-form on T^*M.

Let $\psi : U \subset M \to V \subset \mathbb{R}^m, q \to (x_1 = \psi_1(q), \ldots, x_m = \psi_m(q))$ be a local coordinate, then $\{\frac{\partial}{\partial x_j}\}$ is a local frame of TM, and $\{dx_j\}$ is a local frame of T^*M which gives the trivialization of TM, T^*M on U. Thus

$$T^*M \to V \times \mathbb{R}^m, \quad \Big(q, \sum_i p_i \psi^*(dx_i)\Big) \to (x_1, \ldots, x_m, p_1, \ldots, p_m)$$

is the induced local coordinate of $T^*M|_U$, and $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial p_j}\}$ is a local frame of $T(T^*M)$.

For $X = \sum_i X_i \frac{\partial}{\partial x_i} + P_i \frac{\partial}{\partial p_i}$, we have

$$(\lambda, X)_{(x,p)} = \sum_{i=1}^m p_i X_i = \Big(\sum_{i=1}^m p_i dx_i, X\Big). \tag{1.2.44}$$

From (1.2.43) and (1.2.44), we get

$$\lambda = \sum_{i=1}^m p_i dx_i, \qquad \omega^{T^*M} = \sum_{i=1}^m dx_i \wedge dp_i. \tag{1.2.45}$$

Hence, ω^{T^*M} is nondegenerate, and $\big(T^*M, \omega^{T^*M}\big)$ is a symplectic manifold.

Example 1.2.15. a) For $n > 1$, since $H^2(\mathbb{S}^{2n}, \mathbb{R}) = 0$, we conclude that $\mathbb{S}^{2n}$ is not symplectic.

b) Every orientable surface Σ with its volume form is symplectic.

In differential geometry, there are many problems related to the existence of diffeomorphisms. Moser proposed to transfer this problem as a problem on the existence of time dependent vector fields which is much easier to attack. Moser's trick has many applications and will be used repeatedly in our lecture. The following problem is a typical example for Moser's trick: Let M be a manifold. For $\alpha_0, \alpha_1 \in \Omega^\bullet(M)$, is there a diffeomorphism $\phi \in \mathrm{Diff}(M)$ such that $\phi^*\alpha_1 = \alpha_0$?

Moser proposed a solution by finding a family $\phi_t \in \mathrm{Diff}(M), \alpha_t \in \Omega^\bullet(M)$ $(t \in \mathbb{R})$, such that for $t \in \mathbb{R}$, we have

$$\phi_0 = \mathrm{Id}, \qquad \phi_t^*\alpha_t = \alpha. \tag{1.2.46}$$

In this case, we can take $\phi = \phi_1$.

Let X_t be the time dependent vector field on M defined by: for $x \in M, t \in \mathbb{R}$,

$$X_t\left(\phi_t(x)\right) = \frac{\partial}{\partial t}\phi_t(x). \tag{1.2.47}$$

Then by differential $\phi_t^*\alpha_t$ on t, we get (cf. Exercise 1.2.3)

$$\frac{d}{dt}\phi_t^*\alpha_t = \phi_t^*\left(L_{X_t}\alpha_t + \dot{\alpha_t}\right). \tag{1.2.48}$$

By (1.2.46), (1.2.48), we have

$$L_{X_t}\alpha_t + \dot{\alpha}_t = 0. \tag{1.2.49}$$

Inversely, if we find a time dependent vector field X_t such that (1.2.49) holds on $[0,1]$, and if the flow associated with $X.$ is defined for $t \in [0,1]$, we can take ϕ_t as the flow associated with $X.$. Note that if the support of X_t ($t \in [0,1]$) is in a compact set which independent of t, the associated flow is always defined for $t \in [0,1]$.

Theorem 1.2.16 (Moser's stability theorem)**.** *Let M be a compact manifold. Let ω_t ($t \in [0,1]$) be a smooth family of symplectic forms on M such that the class $[\omega_t] \in H^2(M,\mathbb{R})$ is independent of t. Then there exists $\phi \in \mathrm{Diff}(M)$ such that $\phi^*\omega_1 = \omega_0$.*

Proof. By Moser's trick, we need to find a time dependent vector field X_t such that

$$L_{X_t}\omega_t + \dot{\omega}_t = 0. \tag{1.2.50}$$

As $[\omega_t] \in H^2(M,\mathbb{R})$ is independent of t, then $\omega_t - \omega_0 \in \mathrm{Im}(d)$.

Let g^{TM} be a Riemannian metric on M. By the Hodge theorem (Theorem 1.2.4),

$$\gamma_t = d^*\Delta^{-1}(\omega_t - \omega_0) \tag{1.2.51}$$

is a smooth family on $t \in [0,1]$ of differential forms on M, such that

$$\omega_t - \omega_0 = d\gamma_t. \tag{1.2.52}$$

Thus

$$\dot{\omega}_t = d\beta_t \quad \text{with } \beta_t = \dot{\gamma}_t. \tag{1.2.53}$$

As ω_t is nondegenerate for $t \in [0,1]$, there exists $X_t \in \mathscr{C}^\infty(M,TM)$ smooth on t such that

$$i_{X_t}\omega_t + \beta_t = 0. \tag{1.2.54}$$

By Cartan's formula (1.2.20) and by the closedness of ω_t, this implies (1.2.50). As M is compact, the flow $\phi_t^{X.}$ associated with $X.$ is defined for $t \in [0,1]$. The proof of Theorem 1.2.16 is completed by taking $\phi = \phi_1^{X.}$. □

1.2.4 Darboux theorem

Proposition 1.2.17. *(Darboux lemma) Let W be a compact submanifold of a manifold M. If ω_0, ω_1 are two symplectic forms on M, such that $\omega_0|_W = \omega_1|_W \in \mathscr{C}^\infty(W,\Lambda^2(T^*M))$. Then there is a diffeomorphism $\phi : U_0 \to U_1$ between two neighborhoods of W, such that*

$$\phi|_W = \mathrm{Id}_W, \qquad \phi^*\omega_1 = \omega_0. \tag{1.2.55}$$

Proof. Set $\omega_t = (1-t)\omega_0 + t\omega_1$. As $\omega_0|_W = \omega_1|_W$, ω_t is a symplectic form on some compact neighborhood of W. By Moser's trick, we need to find X_t a smooth family on $t \in [0,1]$ of vector fields on a neighborhood of W, such that X_t is vanishing on W and

$$L_{X_t}\omega_t + \omega_1 - \omega_0 = 0. \tag{1.2.56}$$

If there is a form β such that

$$d\beta = \omega_1 - \omega_0, \qquad \beta|_W = 0, \tag{1.2.57}$$

as ω_t is nondegenerate on a neighborhood of W, for $t \in [0,1]$, there exists X_t such that

$$i_{X_t}\omega_t + \beta = 0. \tag{1.2.58}$$

By Cartan's formula (1.2.20), (1.2.58) implies (1.2.56). As W is compact, the flow $\phi_t^{X.}$ associated with $X.$ exists for $t \in [0,1]$. Take $\phi = \phi_1^{X.}$. Thus the second equation of (1.2.55) follows. Since $\beta|_W = 0$, by (1.2.58), $X_t|_W = 0$. Then the first equation of (1.2.55) follows.

For the existence of such form β, we need the following Lemmas.

Lemma 1.2.18. *Let M and N be two smooth manifolds and $\dim M = \dim N$, and let $Y \subset N$ be a compact submanifold. Let $\varphi : N \to M$ be a smooth map such that $\varphi|_Y$ is injective and $d\varphi_y : T_yN \to T_{\varphi(y)}M$ is bijective for any $y \in Y$. Then φ is a diffeomorphism on a neighborhood of Y.*

Proof. As Y is compact and $d\varphi|_Y$ is bijective, there exist open sets $U_1 \subset V_1, \dots, U_k \subset V_k$ such that $Y \subset \bigcup_{j=1}^k U_j$, $\overline{U}_j \subset V_j$ and $\varphi|_{V_j} : V_j \to \varphi(V_j)$ is a diffeomorphism for any $1 \le j \le k$. Then we strink successively U_j to keep $U_j \cap Y$ and such that $\varphi|_{\overline{\cup_{i=1}^j U_i}}$ is a diffeomorphism for each $1 \le j \le k$. To get the step j from the step $j-1$, as $\varphi(\cup_{i=1}^{j-1} U_i \setminus V_j) \cap \varphi(\overline{U_j} \cap Y) = \emptyset$, there exists $\overline{U_j} \cap Y \subset V_j' \subset V_j$ open such that $\varphi(\cup_{i=1}^{j-1} U_i \setminus V_j) \cap \varphi(V_j') = \emptyset$, then we replace U_j by $U_j \cap V_j'$. □

Lemma 1.2.19. *(Poincaré Lemma) Let $i : W \hookrightarrow M$ be a compact submanifold of a manifold M. If $\alpha \in \Omega^k(M)$ such that $i^*\alpha = 0$ and $d\alpha = 0$, then there is a $(k-1)$-form β defined on some neighborhood of W such that*

$$d\beta = \alpha, \qquad \beta|_W = 0 \in \mathscr{C}^\infty(W, \Lambda^{k-1}(T^*M)). \tag{1.2.59}$$

If $W = \{\mathrm{pt}\}$, $k \ge 1$, $i^*\alpha = 0$ always holds, and Lemma 1.2.19 reduces to the usual Poincaré Lemma.

Proof. Let g^{TM} be a metric on TM. Let $N = TM/TW$ be the norm bundle of W, and we identify N with $(TW)^\perp$ the orthogonal complement of TW in TM. By Lemma 1.2.18, for $\varepsilon > 0$ small enough, the exponential map

$$\exp^M : \mathcal{U}_\varepsilon = \{(y,Z) \in N_y : y \in W, |Z| < \varepsilon\} \ni (y,Z) \to \exp_y^M(Z) \in M. \tag{1.2.60}$$

defines a diffeomorphism of $\mathcal{U}_\varepsilon$ onto a neighborhood of W in M. Hence we identify $\mathcal{U}_\varepsilon$ as a neighborhood of W in M via exponential map (1.2.60).

For $s \in [0,1]$, set

$$\phi_s : (y,Z) \in \mathcal{U}_\varepsilon \to (y, sZ) \in \mathcal{U}_\varepsilon, \tag{1.2.61}$$

and let Y_s be the vector field on $\mathcal{U}_{s\varepsilon}$ defined by

$$Y_s(\phi_s(x)) = \frac{\partial}{\partial s}\phi_s(x). \tag{1.2.62}$$

Then Y_s is smooth on $s \in [0,1]$. Set

$$K(\alpha) = \int_0^1 \phi_s^* i_{Y_s}\alpha \, ds \in \Omega^{k-1}(\mathcal{U}_\varepsilon). \tag{1.2.63}$$

Then by Cartan's formula (1.2.20) and Exercise 1.2.3, and $\phi_0^*\alpha = i^*\alpha = 0$, we get

$$\alpha \;=\; \phi_1^*\alpha - \phi_0^*\alpha \;=\; \int_0^1 \phi_s^* L_{Y_s}\alpha\, ds \;=\; d\int_0^1 \phi_s^* i_{Y_s}\alpha\, ds + \int_0^1 \phi_s^* i_{Y_s} d\alpha\, ds \;=\; dK(\alpha). \quad (1.2.64)$$

Moreover, as $\phi_s|_W = \mathrm{Id}_W$, $Y_s = 0$ on W, thus $K(\alpha)|_W = 0$. The proof of Lemma 1.2.19 is completed. ☐

Now we return to the proof of Proposition 1.2.17. By Lemma 1.2.19, there exists β which satisfies (1.2.57). The proof of Proposition 1.2.17 is completed. ☐

Theorem 1.2.20. *Let $i : W \hookrightarrow M$ be a compact submanifold of a manifold M. If ω_0, ω_1 are two symplectic forms on M, such that*

a) $i^*\omega_0 = i^*\omega_1 \in \Omega^2(W)$,

b) $N = TW \cap (TW)^{\perp_{\omega_0}}$ is of constant rank on W.

c) There is a symplectic morphism $(TW^{\perp_{\omega_0}}/N, \omega_0) \simeq (TW^{\perp_{\omega_1}}/N, \omega_1)$.

Then there is a diffeomorphism $\phi : U_0 \to U_1$ between two neighborhoods of W, such that

$$\phi|_W = \mathrm{Id}_W, \qquad\qquad \phi^*\omega_1 = \omega_0. \quad (1.2.65)$$

Proof. We note that, as $i^*\omega_0 = i^*\omega_1$, we have

$$N = TW \cap (TW)^{\perp_{\omega_0}} = TW \cap (TW)^{\perp_{\omega_1}}. \quad (1.2.66)$$

By a), b), and Proposition 1.2.10, there exist symplectic subbundles $(V, \omega_0) = (V, \omega_1)$, (V_0, ω_0) and (V_1, ω_1) of $TM|_W$ on W such that

$$TW = V \oplus N, \qquad (TW)^{\perp_{\omega_0}} = V_0 \oplus N, \qquad (TW)^{\perp_{\omega_1}} = V_1 \oplus N, \quad (1.2.67)$$

and the symplectic direct decompositions

$$(TM|_W, \omega_0) = V \oplus V_0 \oplus (V \oplus V_0)^{\perp_{\omega_0}}, \qquad (TM|_W, \omega_1) = V \oplus V_1 \oplus (V \oplus V_1)^{\perp_{\omega_1}}. \quad (1.2.68)$$

Moreover by the argument after (1.1.55), N is a Lagrangian subbundle in $(V \oplus V_0)^{\perp_{\omega_0}}$ (resp. $(V \oplus V_1)^{\perp_{\omega_1}}$). Let J_0 (resp. J_1) be a compatible complex structure on $(V \oplus V_0)^{\perp_{\omega_0}}$ (resp. $(V \oplus V_1)^{\perp_{\omega_1}}$). By (1.1.56),

$$(V \oplus V_0)^{\perp_{\omega_0}} = N \oplus J_0 N \simeq N \oplus N^*, \qquad (V \oplus V_0)^{\perp_{\omega_1}} = N \oplus J_1 N \simeq N \oplus N^*. \quad (1.2.69)$$

This means that we have a symplectic isomorphism

$$\psi_1 : \big((V \oplus V_0)^{\perp_{\omega_0}}, \omega_0\big) \to \big((V \oplus V_1)^{\perp_{\omega_1}}, \omega_1\big) \quad \text{such that} \quad \psi_1|_N = \mathrm{Id}_N\,. \quad (1.2.70)$$

On the other hand, c) implies (V_0, ω_0) and (V_1, ω_1) is symplectic isomorphic.

Hence, by (1.2.68), there is a symplectic bundle map

$$\Phi : (TM|_W, \omega_0) \simeq (TM|_W, \omega_1) \quad \text{such that} \quad \Phi|_{TW} = \mathrm{Id}_{TW}\,. \quad (1.2.71)$$

Then Φ induces an automorphism of the normal bundle of W in M. Let φ be a diffeomorphism of two local coordinates $\mathcal{U}_\varepsilon$ and $\mathcal{U}_{\varepsilon'}$ as in (1.2.60) of W defined by

$$\varphi : (x, Z) \in \mathcal{U}_\varepsilon \to (x, \Phi(Z)) \in \mathcal{U}_{\varepsilon'}. \quad (1.2.72)$$

Then

$$\varphi^*\omega_1|_W = \omega_0|_W \in \mathscr{C}^\infty(W, \Lambda^2(T^*M)) \quad \text{and} \quad \varphi|_W = \mathrm{Id}_W\,. \quad (1.2.73)$$

From Proposition 1.2.17, Theorem 1.2.20 follows. ☐

Corollary 1.2.21 (Darboux theorem)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Then for any $y_0 \in M$, there is a local coordinate (x_i, p_j) near y_0 such that,*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dp_i. \tag{1.2.74}$$

Such local coordinate is called Darboux's coordinate. Corollary 1.2.21 means that locally, any symplectic manifolds are symplectically diffeomorphic, thus a possible symplectic invariant should be a global invariant of manifolds.

Proof. For $y_0 \in M$, take $W = \{y_0\}$. Let $\phi : U_0 \subset M \to V_0 \subset \mathbb{R}^{2n}$ be a local chart such that $y_0 \in U_0$, $0 \in V_0$ and $\phi(y_0) = 0$. Then $(\phi^{-1*}\omega)_0 \in \Lambda^2(\mathbb{R}^{2n*})$ is a symplectic form on $\mathbb{R}^{2n}$. Thus by Corollary 1.1.8, there exists $\phi_1 \in \mathrm{GL}(2n, \mathbb{R})$ such that

$$\phi_1^*\omega_{st} = (\phi^{-1*}\omega)_0. \tag{1.2.75}$$

Now $\omega_1 := \phi^*\phi_1^*\omega_{st}$ is a symplectic form on U_0, and

$$\omega_{1,y_0} = \omega_{y_0} \in \Lambda^2(T^*_{y_0}M). \tag{1.2.76}$$

By Proposition 1.2.17, there are $U_1, U_2 \subset U_0$ neighborhoods of y_0, and a diffeomorphism $\phi_2 : U_1 \to U_2$ such that

$$\phi_2^*\omega_1 = \omega, \quad \phi_2(y_0) = y_0. \tag{1.2.77}$$

Take $\varphi = \phi_1 \circ \phi \circ \phi_2 : U_1 \to \phi_1 \circ \phi(U_2) \subset \mathbb{R}^{2n}$, then

$$\varphi^*\omega_{st} = \omega. \tag{1.2.78}$$

The proof of Corollary 1.2.21 is completed. □

Corollary 1.2.22. *If L is a compact Lagrangian submanifold of a symplectic manifold (M, ω), then there exist a neighborhood U of L in M, a neighborhood V of zero section of the symplectic manifold (T^*L, ω^{T^*L}) and a diffeomorphism $\phi : U \to V$ such that $\phi^*\omega^{T^*L} = \omega$.*

Proof. Take $W = L$. By Proposition 1.2.10 and (1.2.45), we have symplectic isomorphisms

$$(TM|_L, \omega) \simeq (TL \oplus T^*L, \omega^{TL \oplus T^*L}) \simeq (TT^*L|_L, \omega^{T^*L}). \tag{1.2.79}$$

We identify $T^*L|_L$ with the normal bundle of L in M. Then ω^{T^*L} induces a symplectic form in the neighborhood of L in M. By Proposition 1.2.17 and by (1.2.79), Corollary 1.2.22 follows. □

1.2.5 $\mathrm{Sympl}(M, \omega)$ and $\mathrm{Ham}(M, \omega)$

Let (M, ω) be a symplectic manifold. For $H \in \mathscr{C}^\infty([0,1] \times M, \mathbb{R})$, the time dependent Hamiltonian vector field X_{H_t}, which is smooth on $t \in [0, 1]$, is defined by

$$i_{X_{H_t}}\omega = dH_t. \tag{1.2.80}$$

Let

$$\phi_t^{H_\cdot} := \phi_t^{X_{H_\cdot}} \tag{1.2.81}$$

be the flow associated with X_{H_t}. In particular, if $H \in \mathscr{C}^\infty(M)$, we get a Hamiltonian vector field X_H associated with H. We suppose always that $\phi_t^{H_\cdot}$ is defined for $t \in \mathbb{R}$. As we mentioned before, this is the case if the support of H_t is in a compact set which independent of t.

Definition 1.2.23. We define $\mathrm{Sympl}(M,\omega)$ and $\mathrm{Ham}(M,\omega)$ the symplectic diffeomorphism group and the Hamiltonian diffeomorphism group by,

$$\begin{aligned}\mathrm{Sympl}(M,\omega) &= \{\phi \in \mathrm{Diff}(M) : \phi^*\omega = \omega\}, \\ \mathrm{Ham}(M,\omega) &= \{\phi \in \mathrm{Diff}(M) : \text{There exists } H \in \mathscr{C}^\infty([0,1]\times M) \text{ such that } \phi = \phi_1^{H_\cdot}\}.\end{aligned} \tag{1.2.82}$$

Certainly, $\mathrm{Sympl}(M,\omega)$ is a group. We will see in Proposition 1.2.25 that $\mathrm{Ham}(M,\omega)$ is also a group.

The following lemma is very useful.

Lemma 1.2.24. *For any* $\varphi \in \mathrm{Sympl}(M,\omega)$, $H \in \mathscr{C}^\infty(M)$, *we have*

$$X_{\varphi^* H} = \varphi_*^{-1} X_H. \tag{1.2.83}$$

Proof. By (1.2.80) and (1.2.82), for $y \in M$, we have

$$\begin{aligned}(i_{X_{\varphi^*H}}\omega)_y = d\,(\varphi^*H)_y = (\varphi^* dH)_y = (\varphi^*\omega(X_H,\cdot))_y \\ = \omega_{\varphi(y)}(X_H, \varphi_*\cdot)) = \omega_y(\varphi_*^{-1}X_H,\cdot).\end{aligned} \tag{1.2.84}$$

Thus (1.2.83) holds. □

Proposition 1.2.25. *The set* $\mathrm{Ham}(M,\omega)$ *is a connected normal subgroup of* $\mathrm{Sympl}(M,\omega)$.

Proof. By Cartan formula (1.2.20), and by (1.2.94), for $H_\cdot \in \mathscr{C}^\infty([0,1]\times M)$, we have

$$\frac{d}{dt}\phi_t^{H_\cdot,*}\omega = \phi_t^{H_\cdot,*} L_{X_{H_t}}\omega = \phi_t^{H_\cdot,*} d\, i_{X_{H_t}}\omega = \phi_t^{H,*} d\, dH_t = 0. \tag{1.2.85}$$

Thus, $\mathrm{Ham}(M,\omega)$ is a subset of $\mathrm{Sympl}(M,\omega)$.

For $s \in [0,1]$, $\phi_1^{sH_\cdot}$ is a continuous path in $\mathrm{Ham}(M,\omega)$ which connects Id_M and $\phi_1^{H_\cdot}$. Thus $\mathrm{Ham}(M,\omega)$ is connected.

Now, we start to prove that $\mathrm{Ham}(M,\omega)$ is a group. For $F, H \in \mathscr{C}^\infty([0,1]\times M,\mathbb{R})$, by Lemmas 1.2.1, 1.2.24 and (1.2.80), we have

$$\begin{aligned}\phi_t^{H_\cdot} \circ \phi_t^{F_\cdot} &= \phi_t^{X_{H_\cdot}} \circ \phi_t^{X_{F_\cdot}} = \phi_t^{X_{H_\cdot} + \phi_{\cdot *}^{X_{H_\cdot}} X_{F_\cdot}} = \phi_t^{H_\cdot + (\phi_\cdot^{X_{H_\cdot}})^{-1*} F_\cdot}, \\ \left(\phi_t^{H_\cdot}\right)^{-1} &= \left(\phi_t^{X_{H_\cdot}}\right)^{-1} = \phi_t^{-\left(\phi_\cdot^{X_{H_\cdot}}\right)_*^{-1} X_{H_\cdot}} = \phi_t^{-\left(\phi_\cdot^{X_{H_\cdot}}\right)^* H_\cdot}.\end{aligned} \tag{1.2.86}$$

Thus, $\mathrm{Ham}(M,\omega)$ is stable under composition and inverse. This means that it is a group.

It remains to show it is a normal subgroup of $\mathrm{Sympl}(M,\omega)$. For any $\phi \in \mathrm{Sympl}(M,\omega)$, by Lemma 1.2.1 and by (1.2.83), we have

$$\phi^{-1} \circ \phi_t^{H_\cdot} \circ \phi = \phi^{-1} \circ \phi_t^{X_{H_\cdot}} \circ \phi = \phi_t^{\phi_*^{-1} X_{H_\cdot}} = \phi_t^{X_{\phi^* H_\cdot}}. \tag{1.2.87}$$

This means $\mathrm{Ham}(M,\omega)$ is normal. The proof of Proposition 1.2.25 is completed. □

Let $\mathfrak{sympl}(M,\omega)$ and $\mathfrak{ham}(M,\omega)$ be formal Lie algebras of the Lie groups $\mathrm{Sympl}(M,\omega)$ and $\mathrm{Ham}(M,\omega)$, i.e., their tangent spaces at the identity element cf. §2.1.1. Then by (1.2.82),

$$\begin{aligned}\mathfrak{sympl}(M,\omega) &= \{X \in \mathscr{C}^\infty(M,TM) : L_X\omega = 0\}, \\ \mathfrak{ham}(M,\omega) &= \{X \in \mathscr{C}^\infty(M,TM) : \text{there is } H \in \mathscr{C}^\infty(M) \text{ such that } i_X\omega = dH\}.\end{aligned} \tag{1.2.88}$$

Exercise 1.2.1. Let $X \in \mathscr{C}^\infty(M, TM)$. On a local chart $x = (x_1, \ldots, x_n)$, for $|t|$ small enough, we write $X = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}$, and the associated flow $\phi_t^X(x) = (\phi_1(t,x), \ldots, \phi_n(t,x))$. Verify that (1.2.5) in this local coordinate is the ordinary differential equation : for $j \in \{1, \ldots, n\}$,

$$\frac{\partial}{\partial t}\phi_j(t,x) = X_j(\phi_t^X(x)), \quad \phi_j(0,x) = x_j. \tag{1.2.89}$$

Exercise 1.2.2. Let $\phi : M \to N$ be a diffeomorphism. For $\alpha \in \Omega^\bullet(N)$, $X, Y, Z \in \mathscr{C}^\infty(M, TM)$, we have:

a) For $t \in \mathbb{R}$, we have $e^{t\phi_* X} = \phi \circ e^{tX} \circ \phi^{-1}$, conclude that (1.2.11) holds.

b) $[\phi_* X, \phi_* Y] = \phi_*[X, Y]$.

c) By taking the differential of the above identity with $\phi_t = e^{tZ}$, we deduce the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0. \tag{1.2.90}$$

d) For $\psi : M \to N$ a smooth map of manifolds, we have $d^M(\psi^*\alpha) = \psi^*(d^N\alpha)$.

Exercise 1.2.3. For $X. \in \mathscr{C}^\infty(\mathbb{R} \times M, TM)$, let $\phi_t^{X.} : M \to M$ be the flow associated with $X.$.

a) For $Y \in \mathscr{C}^\infty(M, TM)$, $f \in \mathscr{C}^\infty(M)$, verify that $((\phi_{-t}^{X.})_* Y) f(y) = (Y(f \circ \phi_{-t}^{X.}))(\phi_t^{X.} y)$ for any $y \in M$, conclude that

$$[X_0, Y] = \frac{\partial}{\partial t}\Big|_{t=0} \left(\phi_{-t}^{X.}\right)_* Y. \tag{1.2.91}$$

b) For $\alpha \in \Omega^\bullet(M)$, we have

$$L_{X_0}\alpha = \frac{\partial}{\partial t}\Big|_{t=0} \left(\phi_t^{X.}\right)^* \alpha. \tag{1.2.92}$$

c) We fix $t \in \mathbb{R}$. For $s \in \mathbb{R}$, set $\varphi_s = \phi_{t+s}^{X.} \circ \left(\phi_t^{X.}\right)^{-1}$. Prove that for $x \in M$,

$$\frac{\partial}{\partial s}\varphi_s(x) = X_{t+s}\left(\varphi_s(x)\right). \tag{1.2.93}$$

Thus, φ_s is the flow associated with the time dependent vector field $Y_s = X_{t+s}$.

d) Verifies first (1.2.3). For $\alpha \in \Omega^\bullet(M)$, prove that we have

$$\frac{\partial}{\partial t}\left(\phi_t^{X.}\right)^* \alpha = \left(\phi_t^{X.}\right)^* L_{X_t}\alpha. \tag{1.2.94}$$

Conclude that

$$\frac{\partial}{\partial t}\left((\phi_t^{X.})^{-1}\right)^* \alpha = -L_{X_t}\left((\phi_t^{X.})^{-1}\right)^* \alpha. \tag{1.2.95}$$

Exercise 1.2.4. Verify directly from (1.2.88) that

a) $\mathfrak{ham}(M,\omega) \subset \mathfrak{sympl}(M,\omega)$.

b) if $X, Y \in \mathfrak{sympl}(M,\omega)$ (resp. $(\mathfrak{ham}(M,\omega))$, then $[X,Y] \in \mathfrak{sympl}(M,\omega)$ (resp. $(\mathfrak{ham}(M,\omega))$.

1.3 Poisson manifolds

We show every symplectic manifold has a natural Poisson structure, and many results on Hamiltonian vector fields still hold for Poisson manifolds.

1.3.1 Poisson structure on symplectic manifolds

Let (M, ω) be a symplectic manifold of dimension $2n$.

Definition 1.3.1. The Poisson bracket $\{,\} : \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ is defined by, for $f, g \in \mathscr{C}^\infty(M)$,

$$\{f, g\} := \omega(X_f, X_g). \tag{1.3.1}$$

By (1.2.80) and (1.3.1), we have

$$\{f, g\} = \omega(X_f, X_g) = -X_f\, g = X_g\, f. \tag{1.3.2}$$

In a Darboux coordinate $(x_1, \dots, x_n, p_1, \dots, p_n)$, we have $\omega = \sum_i dx_i \wedge dp_i$, thus

$$X_f = \sum_{i=1}^n \frac{\partial f}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial p_i}, \quad \{f, g\} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i}. \tag{1.3.3}$$

Proposition 1.3.2. *The Poisson bracket is invariant under symplectic morphisms, i.e., for any $\phi \in \mathrm{Sympl}(M, \omega)$, $f, g \in \mathscr{C}^\infty(M)$, we have*

$$\{\phi^* f, \phi^* g\} = \phi^*\{f, g\}. \tag{1.3.4}$$

Proof. For any $\phi \in \mathrm{Sympl}(M, \omega)$, by (1.2.83) and (1.3.1), we have

$$\begin{aligned}\{\phi^* f, \phi^* g\}_x = \omega(X_{\phi^* f}, X_{\phi^* g})_x = \omega(\phi_*^{-1} X_f, \phi_*^{-1} X_g)_x = \left(\left(\phi^{-1}\right)^* \omega\right)(X_f, X_g)_{\phi(x)} \\ = \omega(X_f, X_g)_{\phi(x)} = \{f, g\}_{\phi(x)} = \left(\phi^*\{f, g\}\right)_x.\end{aligned} \tag{1.3.5}$$

The proof of Proposition 1.3.2 is completed. □

Proposition 1.3.3. *For any $f, g, h \in \mathscr{C}^\infty(M)$, we have*

a) $\{f, g\} = -\{g, f\}$,

b) $\{fg, h\} = f\{g, h\} + g\{f, h\}$,

c) (Jacobi identity) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.

Proof. a), b) is evident from (1.3.1) or from (1.3.3). By Proposition 1.2.25, for $h \in \mathscr{C}^\infty(M)$, $\phi_t^{X_h} = e^{tX_h} \in \mathrm{Sympl}(M, \omega)$, by (1.3.5), we get

$$\phi_t^{X_h *}\{f, g\} = \{\phi_t^{X_h *} f, \phi_t^{X_h *} g\}. \tag{1.3.6}$$

By differential (1.3.6) at $t = 0$, we have

$$L_{X_h}\{f, g\} = \{L_{X_h} f, g\} + \{f, L_{X_h} g\}, \tag{1.3.7}$$

which is equivalent to c). The proof of Proposition 1.3.3 is completed. □

Proposition 1.3.4. *For $f, g \in \mathscr{C}^\infty(M)$, we have*

$$X_{\{f,g\}} = -[X_f, X_g]. \tag{1.3.8}$$

Proof. For $h \in \mathscr{C}^\infty(M)$, by Proposition 1.3.3 and (1.3.2), we have

$$\begin{aligned} X_{\{f,g\}}h = -\{\{f,g\},h\} &= \{\{g,h\},f\} + \{\{h,f\},g\} \\ &= -X_f X_g h + X_g X_f h = -\left[X_f, X_g\right] h. \end{aligned} \tag{1.3.9}$$

The proof of Proposition 1.3.4 is completed. □

1.3.2 Poisson manifolds

Let M be a manifold.

Definition 1.3.5. We say that $(M, \{,\})$ is a Poisson manifold, if $\{,\} : \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ is a $\mathbb{R}$-bilinear map such that for any $f, g, h \in \mathscr{C}^\infty(M)$, we have

a) $\{f,g\} = -\{g,f\}$,

b) $\{fg,h\} = f\{g,h\} + g\{f,h\}$,

c) (Jacobi identity) $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$.

The operator $\{,\}$ is called a Poisson bracket on M.

By Proposition 1.3.3, every symplectic manifold is a Poisson manifold.
Let $(M, \{,\})$ be a Poisson manifold.

Lemma 1.3.6. *For $f, g \in \mathscr{C}^\infty(M)$, $x_0 \in M$, we have*

$$\begin{aligned} &\{1, f\} = 0, \\ &\{f,g\}_{x_0} = 0, \ \textit{if} \ df_{x_0} = 0. \end{aligned} \tag{1.3.10}$$

Proof. By Definition 1.3.5, $\{1,f\} = 1\cdot\{1,f\}+1\cdot\{1,f\}$, thus we get the first equation of (1.3.10).

At first, if $f|_U = 0$ for a neighborhood U of x_0, then $\{f,g\}_{x_0} = 0$. In fact, let $\varphi \in \mathscr{C}_c^\infty(U)$, $\varphi = 1$ near x_0, then $(1-\varphi)f = f$, thus $\{f,g\}_{x_0} = \{(1-\varphi)f, g\}_{x_0} = (1-\varphi)(x_0)\{f,g\}_{x_0} + f(x_0)\{1-\varphi, g\}_{x_0} = 0$.

Now on a local chart U_0 near x_0, $\psi \in \mathscr{C}_c^\infty(U)$, $\psi = 1$ near x_0, then near x_0,

$$f(x) = f(x_0) + \sum_j (\psi x_j)(h_j\psi) \text{ with } h_j(x) = \int_0^1 \frac{\partial f}{\partial x_j}(tx)dt, \tag{1.3.11}$$

and $\psi x_j, h_j\psi$ are $\mathscr{C}^\infty$ functions on M. Thus

$$\{f,g\}_{x_0} = \{f(x_0), g\}_0 + \{(x_j\psi)(h_j\psi), g\}_0 = h_j(0)\{x_j\psi, g\}_0 \quad \text{with } h_j(0) = \frac{\partial f}{\partial x_j}(0).$$

The proof of Lemma 1.3.6 is completed. □

By (1.3.10), $\{\cdot, g\}$ defines a vector field X_g on M for $g \in \mathscr{C}^\infty(M)$ by

$$X_g f = \{f,g\} \quad \text{for } f \in \mathscr{C}^\infty(M). \tag{1.3.12}$$

We call X_g the Hamiltonian vector field of g.

Definition 1.3.7. We define $B \in \mathscr{C}^\infty(M, \Lambda^2(TM))$ by: for $f, g \in \mathscr{C}^\infty(M)$,

$$B_x(df_x, dg_x) = \{f, g\}_x. \tag{1.3.13}$$

By (1.3.10), for $x \in M$, $\{f, g\}_x$ only depends on $df_x, dg_x \in T_x^*M$, thus (1.3.13) is well-defined. By (1.3.12) and (1.3.13), we have

$$X_f = -i_{df} B. \tag{1.3.14}$$

In particular, B defines a linear map form T^*M to TM. But in general, B can be degenerate. As we will see in Proposition 1.3.9, if B is nondegenerate, then M is a symplectic manifold.

Proposition 1.3.8. *Let $(M, \{,\})$ be a Poisson manifold. Then for $f \in \mathscr{C}^\infty(M)$,*

a) *Let $H. \in \mathscr{C}^\infty(\mathbb{R} \times M, \mathbb{R})$ and $\phi_t^{X_{H.}}$ be the flow associated with the Hamiltonian vector field X_{H_t} of $H.$, then $\phi_t^{X_{H.}}$ preserves the Poisson bracket $\{\,,\,\}$. We also have $L_{X_f} B = 0$.*

b) *If $\phi \in \mathrm{Diff}(M)$ preserves $\{,\}$, then $\phi_*^{-1} X_f = X_{\phi^* f}$.*

c) *We have a morphism of Lie algebras*

$$f \in (\mathscr{C}^\infty(M), \{,\}) \to -X_f \in \mathscr{C}^\infty(M, TM), \tag{1.3.15}$$

i.e., the map is linear and for $f, g \in \mathscr{C}^\infty(M)$, we have

$$X_{\{f,g\}} = -[X_f, X_g]. \tag{1.3.16}$$

Proof. By Definition 1.3.5 c), we know

$$L_{X_{H_t}}\{f, g\} = \{L_{X_{H_t}} f, g\} + \{f, L_{X_{H_t}} g\}. \tag{1.3.17}$$

By Lemma 1.2.1 b) or Exercise 1.2.3 d), we have

$$\frac{\partial}{\partial t}(((\phi_t^{X_{H.}})^{-1})^* f)(y) = -d(\phi_t^{X_{H.}})^{-1} X_{H_t}(y) \cdot f = -(L_{X_{H_t}}((\phi_t^{X_{H.}})^{-1})^* f)(y). \tag{1.3.18}$$

From (1.3.17) and (1.3.18), we get $\frac{\partial}{\partial t}(\phi_t^{X_{H.}})^* \Big\{ ((\phi_t^{X_{H.}})^{-1})^* f, ((\phi_t^{X_{H.}})^{-1})^* g \Big\} = 0$. Thus (1.3.4) holds for $\phi_t^{X_{H.}}$, i.e., $\phi_t^{X_{H.}}$ preserves the Poisson bracket $\{\,,\,\}$.

For any $f, g, h \in \mathscr{C}^\infty(M)$, by (1.3.12) and (1.3.13), we have

$$\begin{aligned}(L_{X_h} B)(df, dg) =& L_{X_h}(B(df, dg)) - B(L_{X_h} df, dg) - B(df, L_{X_h} dg) \\ =& \{\{f, g\}, h\} - \{\{f, h\}, g\} - \{f, \{g, h\}\} = 0.\end{aligned} \tag{1.3.19}$$

Thus a) holds.

If ϕ preserves $\{,\}$, by (1.3.12), we get

$$\begin{aligned}\left(\phi_*^{-1} X_h\right) g_x = dg \left(\phi_*^{-1} X_h\right)_x = d(g \circ \phi^{-1})(X_h)_{\phi(x)} = \{g \circ \phi^{-1}, h\}_{\phi(x)} \\ = \{g, \phi^* h\}_x = \left(X_{\phi^* h} g\right)_x.\end{aligned} \tag{1.3.20}$$

Thus b) holds. From the same argument of (1.3.9), we get c). □

1.3.3 Symplectic foliation

Let $(M, \{,\})$ be a Poisson manifold of dimension m.

Proposition 1.3.9. *If B is nondegenerate, then for any $f, g \in \mathscr{C}^\infty(M)$,*

$$\omega(X_f, X_g) := \{f, g\} \tag{1.3.21}$$

defines a symplectic form on M.

Proof. As B is nondegenerate, by (1.3.14), we have

$$\{X_h(x) : h \in \mathscr{C}^\infty(M)\} = T_xM. \tag{1.3.22}$$

Thus ω is a well-defined 2-form on M. As B is nondegenerate, it implies ω is nondegenerate. By Exercise 1.3.1, Jacobi identity implies ω is closed. Hence, ω is a symplectic form on M. □

Recall that a distribution of TM is a subset $\mathscr{F}$ of TM such that for any $x \in M$, $\mathscr{F} \cap T_xM$ is a vector space. The leaves of $\mathscr{F}$ is defined as follows: Two points are on the same leaf if they can be connected by a piecewise smooth path tangent to $\mathscr{F}$. A distribution $\mathscr{F}$ is integrable if for any $x_0 \in M$, there exist a submanifold $L \subset M$ such that $x_0 \in L$ and for any $x \in L$, $\mathscr{F}_x = T_xL$.

For $x \in M$, let

$$\mathcal{F}_x = \mathrm{Span}\{X_f(x) : f \in \mathscr{C}^\infty(M)\}, \tag{1.3.23}$$

then the subset $\mathcal{F} = \cup_{x\in M}\mathcal{F}_x$ of TM defines a distribution on M. Note that the vector space $\mathcal{F}_x$ need not have constant rank. By Proposition 1.3.8 a), b), for any $H \in \mathscr{C}^\infty(M)$, the flow $\phi_t^{X_H}$ of X_H preserves $\mathscr{F}$, thus

$$d\phi_t^{X_H}(\mathcal{F}_x) = \mathcal{F}_{\phi_t^{X_H}(x)}. \tag{1.3.24}$$

Now for any $x_0 \in M$, there exist $h_1, \ldots, h_k \in \mathscr{C}^\infty(M)$ such that $X_{h_1}(x_0), \ldots, X_{h_k}(x_0)$ is a basis of $\mathcal{F}_{x_0}$, and there exists an open neighbourhood W of $0 \in \mathbb{R}^k$ such that the map

$$W \ni (t_1, \ldots, t_k) \to \phi_{t_k}^{X_{h_k}} \ldots \phi_{t_1}^{X_{h_1}}(x_0) \in M$$

is well-defined, and thus get an k-dimensional submanifold L of M through x_0. As the tangent spaces of $x \in L$ are given by the action of the flot $\phi_{t_j}^{X_{h_j}}$ on $\mathcal{F}_{x_0}$, by (1.3.24), it must be $\mathcal{F}_x$. Thus we verified that the distribution $\mathcal{F}$ is integrable. The distribution $\mathcal{F}$ is called a symplectic foliation of $(M, \{\ ,\ \})$.

If $\mathcal{F}$ is a subbundle of TM, and L is a leave of $\mathcal{F}$. Then by Proposition 1.3.9, L is a symplectic manifold.

Theorem 1.3.10. *If in some neighborhood U_0 of $m_0 \in M$, the symplectic foliation $\mathcal{F}$ is a subbundle of TM, then there is a neighborhood $\mathscr{U}$ of m_0 and a coordinate (x_i, p_j, z_k) on $\mathscr{U}$ such that*

$$\{x_i, p_j\} = \delta_{ij}, \tag{1.3.25}$$

and zero for the other Poisson brackets.

Such a coordinate is called a Poisson-Darboux coordinate.

Proof. Assume that $\mathcal{F}$ is a subbundle of TM near $m_0 \in M$. Note first that $\{\cdot,\cdot\}|_{\mathcal{F}}$ is nondegenerate: if $\{h,f\}_x = 0$ for any $h \in \mathscr{C}^\infty(M)$, then by (1.3.12), $X_h(f)_x = 0$, thus $df|_{\mathcal{F}_x} = 0$ and (1.3.21) defines $\omega \in \mathscr{C}^\infty(U_0, \Lambda^2\mathcal{F}^*)$ and ω is nondegenerate along $\mathcal{F}$, and along the leaf, ω is closed.

We assume that $\operatorname{rk}\mathcal{F} = 2l$. By Frobenius's theorem, these exist a neighborhood $\mathcal{U}$ (resp. U, resp. V) of y_0 (resp. 0) in M (resp. $\mathbb{R}^{2l}, \mathbb{R}^{m-2l}$), and a local coordinate

$$\varphi : m \in \mathcal{U} \to (y,z) \in U \times V \subset \mathbb{R}^{2l} \times \mathbb{R}^{m-2l}, \tag{1.3.26}$$

such that $\varphi(m_0) = 0$, and

$$\mathcal{F} = \operatorname{Span}\left\{\frac{\partial}{\partial y_i}\right\}_{i=1}^{2l}. \tag{1.3.27}$$

As the function z_j are constant along the leaf of $\mathcal{F}$, thus for any $h \in \mathscr{C}^\infty(\mathcal{U})$, as $X_h \in \mathscr{C}^\infty(\mathcal{U},\mathcal{F})$,

$$\{h, z_j\}_m = X_h z_j = 0. \tag{1.3.28}$$

We identify $\mathcal{U}$ with $U \times V$. For $z \in V$, let $e_i(z), f_j(z)$ be a symplectic frame on $\mathcal{F}|_{\{0\}\times V}$, and let $\{e_j, f_j\}_j$ be the canonical symplectic basis in $(\mathbb{R}^{2l}, \omega_{st})$. We define $\phi : U \times V \to \mathbb{R}^{2l} \times V$, which is linear on the first factor, by

$$\phi\left(\sum_{i=1}^{l} a_i e_i(z) + b_i f_i(z), z\right) = \left(\sum_{i=1}^{l} a_i e_i + b_i f_i, z\right). \tag{1.3.29}$$

Then,

$$\phi^*\omega_{st}|_{\{0\}\times V} = \omega|_{\{0\}\times V} \in \Lambda^2(T^*U). \tag{1.3.30}$$

Now we repeat the proof of Darboux's theorem. For $t \in [0,1]$, set

$$\omega_t = (1-t)\omega + t\omega_1, \qquad \text{with } \omega_1 = \phi^*\omega_{st}. \tag{1.3.31}$$

By the construction, we have $\omega, \omega_1 \in \mathscr{C}^\infty(\mathcal{U}, \Lambda^2(T^*U))$ and $d^U\omega = 0$. We would like to find $X_t \in \mathscr{C}^\infty(\mathcal{U}, TU)$ a smooth family of time dependent vector fields on $\mathcal{U}$ in the direction U, and $\beta_t \in \mathscr{C}^\infty(\mathcal{U}, T^*U)$ such that

$$\dot{\omega}_t = d^U\beta_t, \qquad i_{X_t}\omega_t + \beta_t = 0, \qquad X_t|_{\{0\}\times V} = 0, \tag{1.3.32}$$

with d^U the exterior differential along U.

For $s \in [0,1]$, let $\phi_s : U \times V \to U \times V$ be the smooth map defined by

$$\phi_s(y,z) = (sy, z), \tag{1.3.33}$$

and Y_s be the smooth family of time dependent vector fields on $\mathcal{U}$ in the direction U defined by

$$Y_s(\phi_s(y,z)) = \frac{\partial}{\partial s}\phi_s(y,z). \tag{1.3.34}$$

Let $\beta \in \mathscr{C}^\infty(\mathcal{U}, T^*U)$ be defined by

$$\beta = \int_0^1 \phi_s^* i_{Y_s}(\omega_1 - \omega)\, ds. \tag{1.3.35}$$

Thus, by (1.3.30) and (1.3.31), as in (1.2.64), we get $i^*(\omega_1 - \omega) = 0$ and

$$\dot{\omega}_t = \omega_1 - \omega = d^U \beta, \qquad \beta|_{\{0\}\times V} = 0. \tag{1.3.36}$$

As ω_t is nondegenerate in the direction U near y_0, there exists $X_t \in \mathscr{C}^\infty([0,1]\times\mathcal{U}, T^*U)$ such that (1.3.32) holds. By shrinking U and V, the flow $\phi_t^{X_\cdot}$ associated with X_t is defined on $[0,1]$. Take $\Phi = \phi_1^{X_\cdot}$. By (1.3.32), we have

$$\Phi^* \circ \phi^* \omega_{st} = \omega. \tag{1.3.37}$$

The proof of Theorem 1.3.10 is completed. □

Exercise 1.3.1. If $\omega \in \Omega^2(M)$ is only nondegenerate on M, for $f, g, h \in \mathscr{C}^\infty(M)$, we can still define $X_f \in \mathscr{C}^\infty(M, TM)$ by (1.2.80), and $\{f, g\}$ by (1.3.1). Verify that

1. $d\omega(X_f, X_g, X_h) = \sum_{(f,g,h)} X_f \omega(X_g, X_h) - \omega([X_f, X_g], X_h)$, where $\sum_{(f,g,h)}$ is the cyclic sum on f, g, h.
2. $X_f \omega(X_g, X_h) = \{\{g, h\}, f\} = -X_f X_g h$.
3. $X_f \omega(X_g, X_h) - \omega([X_f, X_g], X_h) = -\{\{h, f\}, g\}$.
4. Conclude that

$$-d\omega(X_f, X_g, X_h) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}. \tag{1.3.38}$$

1.4 Kähler manifolds

Let X be a complex manifold with complex structure J. The almost complex structure J induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Then we verify that $T^{(1,0)}X = \{v - \sqrt{-1}Jv : v \in TX\}$. Let $T^{*(1,0)}X$ and $T^{*(0,1)}X$ be the corresponding dual bundles. Let

$$\Omega^{r,q}(X) := \mathscr{C}^\infty(X, \Lambda^r(T^{*(1,0)}X) \otimes \Lambda^q(T^{*(0,1)}X))$$

be the spaces of smooth (r,q)-forms on X.

On local holomorphic coordinates $(z_1, \ldots, z_n)$ with $z_j = x_j + \sqrt{-1}y_j$, we denote

$$\begin{aligned} &\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right), \\ &dz_j = dx_j + \sqrt{-1}dy_j, \quad d\overline{z}_j = dx_j - \sqrt{-1}dy_j. \end{aligned} \tag{1.4.1}$$

Then on holomorphic coordinates $(z_1, \ldots, z_n)$, the ∂, $\bar{\partial}$-operators on functions are defined by

$$\partial f = \sum_j dz_j \frac{\partial}{\partial z_j} f, \quad \bar{\partial} f = \sum_j d\overline{z}_j \frac{\partial}{\partial \overline{z}_j} f \quad \text{for } f \in \mathscr{C}^\infty(X, \mathbb{C}). \tag{1.4.2}$$

They extend naturally to

$$\partial : \Omega^{\bullet,\bullet}(X) \to \Omega^{\bullet+1,\bullet}(X), \quad \bar{\partial} : \Omega^{\bullet,\bullet}(X) \to \Omega^{\bullet,\bullet+1}(X), \tag{1.4.3}$$

which verify the Leibniz rule (1.2.19) for ∂, $\bar{\partial}$ and $\partial^2 = \bar{\partial}^2 = 0$. Thus we have the decomposition

$$d = \partial + \bar{\partial}, \quad \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0. \tag{1.4.4}$$

The operator $\bar{\partial}$ is called the Dolbeault operator.

Definition 1.4.1. A J-invariant Riemannian metric g^{TX} on TX is called a Kähler metric if $\Theta = g^{TX}(J\cdot, \cdot)$ is a closed form, i.e., $d\Theta = 0$. In this case, the form Θ is called a Kähler form on X, and the complex manifold (X, J) is called a Kähler manifold.

Certainly any Kähler manifold is a symplectic manifold. Kähler manifolds are an important class of symplectic manifolds.

Example 1.4.2. (Projective space) For $x, y \in \mathbb{C}^{n+1}\backslash\{0\}$, we say $x \sim y$ if there is $\lambda \in \mathbb{C}^\times$ such that $x = \lambda y$. We verify that $\sim$ is an equivalent relation. Then the complex projective space $\mathbb{CP}^n$ is the quotient space $(\mathbb{C}^{n+1}\backslash\{0\})/\sim$. Let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$ be the natural projection. For $z \in \mathbb{C}^{n+1}\backslash\{0\}$, usually we note by $[z] = [z_0 : \cdots : z_n] = \pi(z)$ which called the homogenous coordinate on $\mathbb{CP}^n$. Let $U_i = \{[z] \in \mathbb{CP}^n : z_i \neq 0\}$, then

$$\psi_i : U_i \to \mathbb{C}^n, \quad [z] \to \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i}\right) \tag{1.4.5}$$

defines holomorphic local coordinates of $\mathbb{CP}^n$, where $\widehat{\ }$ means we omit the term. Thus $\mathbb{CP}^n$ is a complex manifold.

Let $\widetilde{\omega}_{FS}$ be the real 2-form on $\mathbb{C}^{n+1}\backslash\{0\}$ defined by

$$\begin{aligned} \widetilde{\omega}_{FS,z} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(|z|^2\right) &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(|z_0|^2 + \cdots + |z_n|^2\right) \\ &= \frac{\sqrt{-1}}{2\pi}\left(\frac{\sum_{j=0}^n dz_j \wedge d\overline{z}_j}{|z|^2} - \frac{\sum_{j=0}^n \overline{z}_j dz_j \wedge \sum_{k=0}^n z_k d\overline{z}_k}{|z|^4}\right), \end{aligned} \tag{1.4.6}$$

with $|z|^2 = \sum_{j=0}^n |z_j|^2$.

Let $U \subset \mathbb{CP}^n$ be an open set and $\varphi : U \to \mathbb{C}^{n+1}\backslash\{0\}$ a lifting of U, i.e., a holomorphic map with $\pi \circ \varphi = \mathrm{Id}_U$. Then $\varphi^*\widetilde{\omega}_{FS}$ does not depend on the lifting φ. In fact, if $\varphi_1 : U \to \mathbb{C}^{n+1}\backslash\{0\}$ is another lifting, then for $[z] \in U$, there exists a holomorphic function $f : U \to \mathbb{C}^\times$ such that $\varphi_1([z]) = f([z])\varphi([z])$. As $\partial, \bar{\partial}$ commute with φ_1, φ, we get

$$\varphi_1^*\widetilde{\omega}_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(|\varphi_1([z])|^2\right) = \varphi^*\widetilde{\omega}_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(|f|^2\right). \tag{1.4.7}$$

But $\partial\bar{\partial}\log\left(|f|^2\right) = \partial(\overline{f}^{-1}\bar{\partial}\overline{f}) = 0$ as f is a holomorphic function. Thus $\varphi^*\widetilde{\omega}_{FS}$ defines a global differential form ω_{FS} on $\mathbb{CP}^n$. As $\widetilde{\omega}_{FS}$ is closed, we know $d\omega_{FS} = \varphi^* d\widetilde{\omega}_{FS} = 0$, thus ω_{FS} is closed.

To verify that ω_{FS} is positive, by symmetric on the coordinates, we only need to verify it on $\psi_0 : U_0 \to \mathbb{C}^n$, $\psi_0([z]) = (w_1, \dots, w_n) = w$ with $w_i = \frac{z_i}{z_0}$. Now for the lifting $\varphi : U_0 \to \mathbb{C}^{n+1}\backslash\{0\}$, $\varphi(w) = (1, w)$, we get for the coordinate ψ_0,

$$\begin{aligned}\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(1+|w|^2\right) &= \frac{\sqrt{-1}}{2\pi}\partial\Big(\frac{\sum_j w_j d\overline{w}_j}{1+|w|^2}\Big)\\ &= \frac{\sqrt{-1}}{2\pi}\Big(\frac{\sum_j dw_j \wedge d\overline{w}_j}{1+|w|^2} - \frac{\sum_j \overline{w}_j dw_j \wedge \sum_k w_k d\overline{w}_k}{(1+|w|^2)^2}\Big).\end{aligned} \tag{1.4.8}$$

Using the Cauchy-Schwarz inequality $|w|^2|u|^2 \geq |\sum_{j=1}^n u_j\overline{w}_j|^2$, we know ω_{FS} is positive, i.e., $\omega_{FS}(u, \overline{u}) > 0$ for $0 \neq u \in T^{(1,0)}\mathbb{CP}^n$.

Thus $(\mathbb{CP}^n, \omega_{FS})$ is symplectic, and ω_{FS} is so-called Fubini-Study form on $\mathbb{CP}^n$.

Let

$$i : [z_0 : z_1] \in \mathbb{CP}^1 \to [z_0 : z_1 : 0 : \dots : 0] \in \mathbb{CP}^n \tag{1.4.9}$$

be an embedding of $\mathbb{CP}^1$ to $\mathbb{CP}^n$. Then

$$\begin{aligned}\int_{\mathbb{CP}^1} i^*\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\int_{\mathbb{CP}^1}\partial\bar{\partial}\log\left(|z_0|^2+|z_1|^2\right) &= \frac{\sqrt{-1}}{2\pi}\int_{\mathbb{C}}\partial\bar{\partial}\log\left(1+|z|^2\right)\\ &= \frac{\sqrt{-1}}{2\pi}\int_{\mathbb{C}}\frac{dzd\bar{z}}{\left(1+|z|^2\right)^2} = \frac{1}{\pi}\int_{\mathbb{R}^2}\frac{dxdy}{\left(1+x^2+y^2\right)^2} = 1.\end{aligned} \tag{1.4.10}$$

Moreover, by (1.4.8),

$$\int_{\mathbb{CP}^n}\omega_{FS}^n = n!\left(\frac{\sqrt{-1}}{2\pi}\right)^n\int_{\mathbb{C}^n}\frac{dw_1 d\overline{w}_1 \dots dw_n d\overline{w}_n}{(1+|w|^2)^{n+1}} = 1. \tag{1.4.11}$$

Remark 1.4.3. By the result from algebraic topology, $[\omega_{FS}]$ is a generator of $H^2(\mathbb{CP}^n, \mathbb{Z})$, the 2th cohomology group of $\mathbb{CP}^n$ with $\mathbb{Z}$-coefficient.

1.5 Bibliographic notes

For basic material concerning manifolds, vector bundles and Riemannian geometry we refer to [23], [54] and [40]. The material of this chapter is very standard. We recommend also two sueful references on symplectic geometry [18] and on Poisson manifolds [53].

A proof of the Hodge theory, Theorem 1.2.4 can be found in [6, Theorem 3.54], [43, Theorem 1.4.1], [54].

For equivalent conditions on Kähler manifold in Section 1.4, cf. [43, Theorem 1.2.8], [6, §3.6].