Chapter 1

Demailly's holomorphic Morse inequalities

The first aim of this Chapter is to provide the background material on differential geometry for the whole book. Then, in the last two sections, we present a heat kernel proof of Demailly's holomorphic Morse inequalities, Theorem 1.7.1.

This Chapter is organized as follows. In Section 1.1, we review connections on vector bundles. In Section 1.2, we explain different connections on the tangent bundle and their relations. In Section 1.3, we define the modified Dirac operator for an almost complex manifold and prove the related Lichnerowicz formula. We explain also the Atiyah-Singer index theorem for the modified Dirac operator. In Section 1.4, we show that the operator $\overline{\partial}^E + \overline{\partial}^{E,*}$ is a modified Dirac operator, and we establish the Lichnerowicz and Bochner-Kodaira-Nakano formulas for the Kodaira Laplacian. In Section 1.5, we deal with vanishing theorems for positive line bundles and the spectral gap property for the modified Dirac operator and the Kodaira Laplacian. In Section 1.6, we establish the asymptotic of the heat kernel which is the analytic core result of this Chapter. Finally, in Section 1.7, we prove Demailly's holomorphic Morse inequalities.

1.1 Connections on vector bundles

In this section, we review the definition on connections and the associated curvatures. Section 1.1.1 reviews some general facts on connections on vector bundles, and we specify them to the holomorphic case in Section 1.1.2.

1.1.1 Connection

Let E be a complex vector bundle over a smooth manifold X. Let TX be the tangent bundle and T^*X be the cotangent bundle. Let $\mathscr{C}^\infty(X, E)$ be the space of smooth sections of E on X. Let $\Omega^r(X,E)$ be the spaces of smooth r-forms on X with values in E, and set $\mathscr{C}^\infty(X) := \mathscr{C}^\infty(X,\mathbb{C})$, $\Omega^\bullet(X) := \Omega^\bullet(X,\mathbb{C})$.

Let $d : \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ be the exterior differential. It is characterized by a). $d^2 = 0$; b). for $\varphi \in \mathscr{C}^\infty(X)$, $d\varphi$ is the one form such that $(d\varphi)(U) = U(\varphi)$ for a vector field U; c). (Leibniz rule) for any $\alpha \in \Omega^q(X), \beta \in \Omega(X)$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^q \alpha \wedge d\beta. \tag{1.1.1}$$

Then we verify that for any 1-form α, vector fields U,V on X, we have

$$d\alpha(U,V) = U(\alpha(V)) - V(\alpha(U)) - \alpha([U,V]), \tag{1.1.2}$$

here $[U,V]$ is the Lie bracket of U and V.

A linear map $\nabla^E : \mathscr{C}^\infty(X,E) \to \mathscr{C}^\infty(X, T^*X \otimes E)$ is called a *connection* on E if for any $\varphi \in \mathscr{C}^\infty(X)$, $s \in \mathscr{C}^\infty(X,E)$ and $U \in TX$, we have

$$\nabla^E_U(\varphi s) = U(\varphi)\, s + \varphi \nabla^E_U\, s\,. \tag{1.1.3}$$

Connections on E always exist. Indeed, let $\{V_k\}_k$ an open covering of X such that $E|_{V_k}$ is trivial. If $\{\eta_{kl}\}_l$ is a local frame of $E|_{V_k}$, any section $s \in \mathscr{C}^\infty(V_k,E)$ has the form $s = \sum_l s_l \eta_{kl}$ with uniquely determined $s_l \in \mathscr{C}^\infty(V_k)$. We define a connection on $E|_{V_k}$ by $\nabla^E_k s := \sum_l ds_l \otimes \eta_{kl}$. Consider now a partition of unity $\{\psi_k\}_k$ subordinated to $\{V_k\}_k$. Then $\nabla^E s := \sum_k \nabla^E_k(\psi_k s)$, $s \in \mathscr{C}^\infty(X,E)$, defines a connection on E.

If ∇^E_1 is another connection on E, then by (1.1.3), $\nabla^E_1 - \nabla^E \in \Omega^1(X, \mathrm{End}(E))$.

If ∇^E is a connection on E, then there exists a unique extension $\nabla^E : \Omega^\bullet(X,E) \to \Omega^{\bullet+1}(X,E)$ verifying the Leibniz rule: for $\alpha \in \Omega^q(X)$, $s \in \Omega^r(X,E)$, we have

$$\nabla^E(\alpha \wedge s) = d\alpha \wedge s + (-1)^q \alpha \wedge \nabla^E s\,. \tag{1.1.4}$$

From (1.1.2), for $s \in \mathscr{C}^\infty(X,E)$ and vector fields U,V on X, we have

$$(\nabla^E)^2(U,V)s = \nabla^E_U \nabla^E_V s - \nabla^E_V \nabla^E_U s - \nabla^E_{[U,V]} s\,. \tag{1.1.5}$$

Then $(\nabla^E)^2(U,V)(\varphi s) = (\nabla^E)^2(U,\varphi V)s = (\nabla^E)^2(\varphi U,V)s = \varphi(\nabla^E)^2(U,V)s$ for any $\varphi \in \mathscr{C}^\infty(X)$. We deduce that:

Theorem and Definition 1.1.1. *The operator* $(\nabla^E)^2$ *defines a bundle morphism* $(\nabla^E)^2 : E \to \Lambda^2(T^*X) \otimes E$, *called the* curvature operator. *Therefore, there exists* $R^E \in \Omega^2(X, \mathrm{End}(E))$, *called the* curvature *of* ∇^E, *such that* $(\nabla^E)^2$ *is given by multiplication with* R^E, *i.e.,* $(\nabla^E)^2 s = R^E s \in \Omega^2(X,E)$ *for* $s \in \mathscr{C}^\infty(X,E)$.

Let h^E be a *Hermitian metric* on E, i.e., a smooth family $\{h^E_x\}_{x\in X}$ of sesquilinear maps $h^E_x : E_x \times E_x \to \mathbb{C}$ such that $h^E_x(\xi,\xi) > 0$ for any $\xi \in E_x \smallsetminus \{0\}$. We call (E,h^E) a Hermitian vector bundle on X. There always exist Hermitian metrics on E by using the partition of unity argument as above.

Definition 1.1.2. A connection ∇^E is said to be a *Hermitian connection* on (E, h^E) if for any $s_1, s_2 \in \mathscr{C}^\infty(X, E)$,

$$d\langle s_1, s_2\rangle_{h^E} = \langle \nabla^E s_1, s_2\rangle_{h^E} + \langle s_1, \nabla^E s_2\rangle_{h^E}. \tag{1.1.6}$$

There always exist Hermitian connections. In fact, let ∇_0^E be a connection on E, then $\langle \nabla_1^E s_1, s_2\rangle_{h^E} = d\langle s_1, s_2\rangle_{h^E} - \langle s_1, \nabla_0^E s_2\rangle_{h^E}$ defines a connection ∇_1^E on E. Now $\nabla^E = \frac{1}{2}(\nabla_0^E + \nabla_1^E)$ is a Hermitian connection on (E, h^E).

Let $\{\xi_l\}_{l=1}^m$ be a local frame of E. Denote by $h = (h_{lk} = \langle \xi_k, \xi_l\rangle_{h^E})$ the matrix of h^E with respect to $\{\xi_l\}_{l=1}^m$. The *connection form* $\theta = (\theta_k^l)$ of ∇^E with respect to $\{\xi_l\}_{l=1}^m$ is defined by, with local 1-forms θ_k^l,

$$\nabla^E \xi_k = \theta_k^l \xi_l\,. \tag{1.1.7}$$

Remark 1.1.3. If E is a real vector bundle on X, certainly, everything still holds, especially, a connection ∇^E is said to be an Euclidean connection on (E, h^E) if it preserves the Euclidean metric h^E.

1.1.2 Chern connection

Let E be a holomorphic vector bundle over a complex manifold X. Let h^E be a Hermitian metric on E. We call (E, h^E) a holomorphic Hermitian vector bundle.

The almost complex structure J induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Let $T^{*(1,0)}X$ and $T^{*(0,1)}X$ be the corresponding dual bundles. Let

$$\Omega^{r,q}(X, E) := \mathscr{C}^\infty(X, \Lambda^r(T^{*(1,0)}X) \otimes \Lambda^q(T^{*(0,1)}X) \otimes E)$$

be the spaces of smooth (r, q)-forms on X with values in E.

The operator $\overline{\partial}^E : \mathscr{C}^\infty(X, E) \to \Omega^{0,1}(X, E)$ is well defined. Any section $s \in \mathscr{C}^\infty(X, E)$ has the local form $s = \sum_l \varphi_l \xi_l$ where $\{\xi_l\}_{l=1}^m$ is a local holomorphic frame of E and φ_l are smooth functions. We set $\overline{\partial}^E s = \sum_l (\overline{\partial}\varphi_l)\,\xi_l$, here $\overline{\partial}\varphi_l = \sum_j d\overline{z}_j \frac{\partial}{\partial \overline{z}_j}\varphi_l$ in holomorphic coordinates $(z_1, \cdots, z_n)$.

Definition 1.1.4. A connection ∇^E on E is said to be a *holomorphic connection* if $\nabla_U^E s = i_U(\overline{\partial}^E s)$ for any $U \in T^{(0,1)}X$ and $s \in \mathscr{C}^\infty(X, E)$.

Theorem 1.1.5. *There exists a unique* holomorphic Hermitian connection ∇^E *on* (E, h^E)*, called the* Chern connection*. With respect to a local holomorphic frame, the connection matrix is given by* $\theta = h^{-1} \cdot \partial h$.

Proof. By Definition 1.1.4, we have to define ∇_U^E just for $U \in T^{(1,0)}X$. Relation (1.1.6) implies for $U \in T^{(1,0)}X$, $s_1, s_2 \in \mathscr{C}^\infty(X, E)$,

$$U\langle s_1, s_2\rangle_{h^E} = \langle \nabla_U^E s_1, s_2\rangle_{h^E} + \langle s_1, \nabla_{\overline{U}}^E s_2\rangle_{h^E}\,. \tag{1.1.8}$$

Since $\nabla^E_{\overline{U}} s_2 = i_{\overline{U}}(\overline{\partial}^E s_2)$, the above equation defines ∇^E_U uniquely. Moreover, if $\{\xi_l\}_{l=1}^m$ is a local holomorphic frame, from (1.1.6) we deduce that $\theta = h^{-1} \cdot \partial h$. $\square$

Since E is holomorphic, similar to (1.1.4), the operator $\overline{\partial}^E$ extends naturally to $\overline{\partial}^E : \Omega^{\bullet,\bullet}(X,E) \longrightarrow \Omega^{\bullet,\bullet+1}(X,E)$ and $(\overline{\partial}^E)^2 = 0$.

Let ∇^E be the holomorphic Hermitian connection on (E, h^E). Then we have a decomposition of ∇^E after bidegree

$$\begin{aligned} &\nabla^E = (\nabla^E)^{1,0} + (\nabla^E)^{0,1}, \quad (\nabla^E)^{0,1} = \overline{\partial}^E, \\ &(\nabla^E)^{1,0} : \Omega^{\bullet,\bullet}(X,E) \longrightarrow \Omega^{\bullet+1,\bullet}(X,E). \end{aligned} \tag{1.1.9}$$

By (1.1.8), (1.1.9) and $(\overline{\partial}^E)^2 = 0$ we have

$$(\overline{\partial}^E)^2 = \big((\nabla^E)^{1,0}\big)^2 = 0, \quad (\nabla^E)^2 = \overline{\partial}^E (\nabla^E)^{1,0} + (\nabla^E)^{1,0}\overline{\partial}^E. \tag{1.1.10}$$

Thus the curvature $R^E \in \Omega^{1,1}(X, \mathrm{End}(E))$. If $\mathrm{rk}(E) = 1$, $\mathrm{End}(E)$ is trivial and R^E is canonically identified to a $(1,1)$-form on X, such that $\sqrt{-1}R^E$ is real.

In general, let us introduce an auxiliary Riemannian g^{TX} metric on X, compatible with the complex structure J (i.e., $g^{TX}(\cdot,\cdot) = g^{TX}(J\cdot, J\cdot)$). Then R^E induces a Hermitian matrix $\dot{R}^E \in \mathrm{End}(T^{(1,0)}X \otimes E)$ such that for $u, v \in T^{(1,0)}_x X$, $\xi, \eta \in E_x$, and $x \in X$,

$$\langle R^E(u,\overline{v})\xi, \eta\rangle_{h^E} = \langle \dot{R}^E(u\otimes\xi), v\otimes\eta\rangle. \tag{1.1.11}$$

Definition 1.1.6. We say that (E, h^E) is *Nakano positive* (*resp. semi-positive*) if $\dot{R}^E \in \mathrm{End}(T^{(1,0)}X \otimes E)$ is positive definite (resp. semi-definite), and *Griffiths positive* (*resp. semi-positive*) if $\langle R^E(v,\overline{v})\xi,\xi\rangle_{h^E} = \langle \dot{R}^E(v\otimes\xi), v\otimes\xi\rangle > 0$ (resp. $\geqslant 0$) for all non-zero $v \in T^{(1,0)}_x X$ and all non-zero $\xi \in E_x$. Certainly, these definitions do not depend on the choice of g^{TX}.

1.2 Connections on the tangent bundle

On the tangent bundle of a complex manifold, we can define several connections: the Levi-Civita connection, the holomorphic Hermitian (i.e. Chern) connection and Bismut connection. In this Section, we explain the relation between them. We shall see that these three connections coincide, if X is a Kähler manifold.

We start by recalling in Section 1.2.1 some facts about the Levi-Civita connection. In Section 1.3.1, we study in detail the holomorphic Hermitian connection on the tangent bundle. In Section 1.3.2, we define the Bismut connection.

Let (X, J) be a complex manifold with complex structure J and $\dim_{\mathbb{C}} X = n$. Let $T_h X$ be the holomorphic tangent bundle on X, and let TX be the corresponding real tangent bundle. Let g^{TX} be any Riemannian metric on TX compatible with J, i.e. $g^{TX}(Ju, Jv) = g^{TX}(u,v)$ for any $u, v \in T_x X$, $x \in X$. We will shortly express this relation by $g^{TX}(J\cdot, J\cdot) = g^{TX}(\cdot,\cdot)$.

1.2.1 Levi-Civita connection

The results of the section apply for any Riemannian manifold (X, g^{TX}). We denote by $\langle \cdot, \cdot \rangle$ the $\mathbb{C}$-bilinear form on $TX \otimes_{\mathbb{R}} \mathbb{C}$ induced by the metric g^{TX}. Let ∇^{TX} be the Levi–Civita connection on (TX, g^{TX}). By the explicit equation for $\left\langle \nabla^{TX} \cdot, \cdot \right\rangle$, for U, V, W, Y vector fields on X,

$$\begin{aligned} 2\left\langle \nabla^{TX}_U V, W \right\rangle = U \langle V, W\rangle + V \langle U, W\rangle - W \langle U, V\rangle \\ - \langle U, [V, W]\rangle - \langle V, [U, W]\rangle + \langle W, [U, V]\rangle . \end{aligned} \quad (1.2.1)$$

∇^{TX} is the unique connection on TX which preserves the metric (satisfies (1.1.6)) and is torsion free, i.e.,

$$\nabla^{TX}_U V - \nabla^{TX}_V U = [U, V]. \quad (1.2.2)$$

The curvature $R^{TX} \in \Lambda^2(T^*X) \otimes \mathrm{End}(TX)$ of ∇^{TX} is defined by

$$R^{TX}(U, V) = \nabla^{TX}_U \nabla^{TX}_V - \nabla^{TX}_V \nabla^{TX}_U - \nabla^{TX}_{[U,V]}. \quad (1.2.3)$$

Then we have the following well know facts

$$\begin{aligned} &R^{TX}(U, V)W + R^{TX}(V, W)U + R^{TX}(W, U)V = 0, \\ &\langle R^{TX}(U, V)W, Y\rangle = \langle R^{TX}(W, Y)U, V\rangle. \end{aligned} \quad (1.2.4)$$

Let $\{e_i\}_{i=1}^{2n}$ be an orthonormal frame of TX and $\{e^i\}_{i=1}^{2n}$ its dual basis in T^*X. The **Ricci curvature** Ric and **scalar curvature** r^X of (TX, g^{TX}) are defined by

$$\mathrm{Ric} = -\sum_j \left\langle R^{TX}(\cdot, e_j)\cdot, e_j \right\rangle, \quad r^X = -\sum_{ij} \left\langle R^{TX}(e_i, e_j)e_i, e_j \right\rangle. \quad (1.2.5)$$

The Riemannian volume form dv_X of (TX, g^{TX}) has the form $dv_X = e^1 \wedge \cdots \wedge e^{2n}$ if the orthonormal frame $\{e_i\}$ is oriented.

If α is a 1-form on X, the function $\mathrm{Tr}(\nabla\alpha)$ is given by the formula

$$\mathrm{Tr}(\nabla\alpha) = \textstyle\sum_i e_i(\alpha(e_i)) - \alpha(\nabla^{TX}_{e_i} e_i). \quad (1.2.6)$$

The following formula is quite useful.

Proposition 1.2.1. *For any $\mathscr{C}^1$ 1-form α with compact support, we have*

$$\int_X \mathrm{Tr}(\nabla\alpha) dv_X = 0. \quad (1.2.7)$$

Proof. Let W be the vector field on X corresponding to α under the Riemannian metric g^{TX}, so that $\langle W, Y\rangle = (\alpha, Y)$ for any $Y \in TX$.

We denote by L_W the Lie derivative of the vector field W. Recall that for any vector field Y on X,

$$L_W Y = [W,Y] = \nabla^{TX}_W Y - \nabla^{TX}_Y W. \tag{1.2.8}$$

Thus by (1.2.8) and $\langle \nabla^{TX}_W e_j, e_j\rangle = 0$, we get

$$\begin{aligned} L_W dv_X &= \langle L_W e^j, e_j\rangle dv_X = -\langle e_j, L_W e_j\rangle dv_X \\ &= \left\langle \nabla^{TX}_{e_j} W, e_j\right\rangle dv_X = \left(e_j\langle W, e_j\rangle - \left\langle W, \nabla^{TX}_{e_j} e_j\right\rangle\right) dv_X \\ &= \mathrm{Tr}(\nabla\alpha) dv_X. \end{aligned} \tag{1.2.9}$$

We will denote by $\wedge$ and i the exterior and interior product respectively. E. Cartan's homotopy formula tells us that on the bundle of exterior differentials $\Lambda(T^*X)$,

$$L_W = d\cdot i_W + i_W \cdot d. \tag{1.2.10}$$

From (1.2.9) and (1.2.10), we get

$$0 = \int_X L_W dv_X = \int_X \mathrm{Tr}(\nabla\alpha) dv_X. \tag{1.2.11}$$

The proof of Proposition 1.2.1 is complete. □

For $x_0 \in X$, $W \in T_{x_0}X$, let $\mathbb{R} \ni u \to x_u = \exp^X_{x_0}(uW)$ be the geodesic in X such that $x_u|_{u=0} = x$, $\frac{dx_u}{du}|_{u=0} = W$. For $\varepsilon > 0$, we denote by $B^X(x_0,\varepsilon)$ and $B^{T_{x_0}X}(0,\varepsilon)$ the open balls in X and $T_{x_0}X$ with center x_0 and radius ε, respectively. Then the map $T_{x_0}X \ni Z \to \exp^X_{x_0}(Z) \in X$ is a diffeomorphism from $B^{T_{x_0}X}(0,\varepsilon)$ onto $B^X(x_0,\varepsilon)$ for ε small enough; by identifying $Z = \sum Z_i e_i \in T_x X$ with $(Z_1, \dots Z_{2n}) \in \mathbb{R}^{2n}$, it yields a local chart for X around x_0, called normal coordinate system at x_0. We will identify $B^{T_{x_0}X}(0,\varepsilon)$ with $B^X(x_0,\varepsilon)$ by this map.

Let $\{e_i\}_i$ be an oriented orthonormal basis of $T_{x_0}X$. We also denote by $\{e^i\}_i$ the dual basis of $\{e_i\}$. Let $\widetilde{e}_i(Z)$ be the parallel transport of e_i with respect to ∇^{TX} along the curve $[0,1] \ni u \to uZ$. Then $e_j = \frac{\partial}{\partial Z_j}$.

The **radial vector field** $\mathcal{R}$ is the vector field defined by $\mathcal{R} = \sum_i Z_i e_i$ with $(Z_1, \cdots, Z_{2n})$ the coordinate functions.

Proposition 1.2.2. *The following identities hold:*

$$\begin{aligned} &\mathcal{R} = \sum_j Z_j e_j = \sum_j Z_j \widetilde{e}_j(Z), \\ &\langle \mathcal{R}, e_j\rangle = Z_j. \end{aligned} \tag{1.2.12}$$

Proof. Note that $x_u : [0,1] \ni u \to uZ$ is a geodesic, and $\mathcal{R}(x_u) = u\frac{dx_u}{du}$, thus by the geodesic equation $\nabla^{TX}_{\frac{dx_u}{du}} \frac{dx_u}{du} = 0$, we get

$$\nabla^{TX}_{\mathcal{R}} \mathcal{R} = u \nabla^{TX}_{\frac{dx_u}{du}} (u\frac{dx_u}{du}) = u\frac{dx_u}{du} = \mathcal{R}. \tag{1.2.13}$$

Thus we have

$$\mathcal{R}\langle \mathcal{R}, \widetilde{e}_j\rangle = \langle \nabla^{TX}_{\mathcal{R}}\mathcal{R}, \widetilde{e}_j\rangle + \langle \mathcal{R}, \nabla^{TX}_{\mathcal{R}}\widetilde{e}_j\rangle = \langle \mathcal{R}, \widetilde{e}_j\rangle. \tag{1.2.14}$$

This means that $\langle \mathcal{R}, \widetilde{e}_j\rangle$ is homogeneous of order one. But

$$\langle \mathcal{R}, \widetilde{e}_j\rangle = \sum_k Z_k \langle e_k, \widetilde{e}_j\rangle = Z_j + \mathscr{O}(|Z|^2). \tag{1.2.15}$$

Thus from (1.2.14) and (1.2.15), we infer the first equation of (1.2.12).

Since the Levi-Civita connection ∇^{TX} is torsion free and $[\mathcal{R}, e_i] = -e_i$, we have

$$\langle \mathcal{R}, \nabla^{TX}_{\mathcal{R}} e_i\rangle = \langle \mathcal{R}, \nabla^{TX}_{e_i}\mathcal{R}\rangle + \langle \mathcal{R}, [\mathcal{R}, e_i]\rangle = \frac{1}{2} e_i \langle \mathcal{R}, \mathcal{R}\rangle - \langle \mathcal{R}, e_i\rangle. \tag{1.2.16}$$

From (1.2.13) and (1.2.16), we obtain

$$\mathcal{R}\langle \mathcal{R}, e_i\rangle = \langle \nabla^{TX}_{\mathcal{R}}\mathcal{R}, e_i\rangle + \langle \mathcal{R}, \nabla^{TX}_{\mathcal{R}} e_i\rangle = \frac{1}{2} e_i \langle \mathcal{R}, \mathcal{R}\rangle = Z_i. \tag{1.2.17}$$

But $\langle \mathcal{R}, e_i\rangle = \sum_j Z_j \langle e_j, e_i\rangle = Z_i + \mathscr{O}(|Z|^2)$. Thus we get the second equation of (1.2.12). $\square$

For $\alpha = (\alpha_1, \cdots, \alpha_{2n}) \in \mathbb{N}^{2n}$, set $Z^\alpha = Z_1^{\alpha_1} \cdots Z_{2n}^{\alpha_{2n}}$.

Lemma 1.2.3. *If $\widetilde{e}_i(Z)$ is written in the the basis $\{e_i\}$, its Taylor expansion up to order r is determined by the Taylor expansion up to order $r-2$ of $R_{mqkl} = \langle R^{TX}(e_q, e_m)e_k, e_l\rangle_Z$. Moreover we have*

$$\widetilde{e}_i(Z) = e_i - \frac{1}{6}\sum_j \left\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\right\rangle_{x_0} e_j + \sum_{|\alpha|\geqslant 3} \Big(\frac{\partial^\alpha}{\partial Z^\alpha}\widetilde{e}_i\Big)(0)\frac{Z^\alpha}{\alpha!}. \tag{1.2.18}$$

Thus the Taylor expansion up to order r of $g_{ij}(Z) = g^{TX}(e_i, e_j)(Z) = \langle e_i, e_j\rangle_Z$ is a polynomial of the Taylor expansion up to order $r-2$ of R_{mqkl}; moreover

$$g_{ij}(Z) = \delta_{ij} + \frac{1}{3}\left\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\right\rangle_{x_0} + \mathscr{O}(|Z|^3). \tag{1.2.19}$$

Proof. Let Γ^{TX} be the connection form of ∇^{TX} with respect to the frame $\{\widetilde{e}_i\}$ of TX, then $\nabla^{TX} = d + \Gamma^{TX}$. Let $\partial_i = \nabla_{e_i}$ be the partial derivatives along e_i. By the definition of our fixed frame, we have $i_{\mathcal{R}}\Gamma^{TX} = 0$. Thus

$$L_{\mathcal{R}}\Gamma^{TX} = [i_{\mathcal{R}}, d]\Gamma^{TX} = i_{\mathcal{R}}(d\Gamma^{TX} + \Gamma^{TX}\wedge\Gamma^{TX}) = i_{\mathcal{R}}R^{TX}. \tag{1.2.20}$$

Let $\widetilde{\theta}(Z) = (\theta^i_j(Z))^{2n}_{i,j=1}$ be the $2n \times 2n$-matrix such that

$$e_i = \sum_j \theta^j_i(Z)\widetilde{e}_j(Z), \quad \widetilde{e}_j(Z) = (\widetilde{\theta}(Z)^{-1})^k_j e_k. \tag{1.2.21}$$

Set $\theta^j(Z) = \sum_i \theta_i^j(Z) e^i$ and

$$\theta = \sum_j e^j \otimes e_j = \sum_j \theta^j \widetilde{e}_j \in T^*X \otimes TX. \tag{1.2.22}$$

As ∇^{TX} is torsion free, $\nabla^{TX}\theta = 0$, thus the $\mathbb{R}^{2n}$-valued 1-form $\theta = (\theta^j(Z))$ satisfies the structure equation,

$$d\theta + \Gamma^{TX} \wedge \theta = 0. \tag{1.2.23}$$

Observe first that under our trivialization by $\{\widetilde{e}_i\}$, by (1.2.12), for the $\mathbb{R}^{2n}$-valued function $i_{\mathcal{R}}\theta$,

$$i_{\mathcal{R}}\theta = \sum_j Z_j e_j = (Z_1, \cdots, Z_{2n}) =: Z. \tag{1.2.24}$$

Substituting (1.2.12), (1.2.24) and $(L_{\mathcal{R}} - 1)Z = 0$, into the identity $i_{\mathcal{R}}(d\theta + \Gamma^{TX} \wedge \theta) = 0$, from (1.2.20), we obtain

$$(L_{\mathcal{R}} - 1)L_{\mathcal{R}}\theta = (L_{\mathcal{R}} - 1)(dZ + \Gamma^{TX} Z) = (L_{\mathcal{R}}\Gamma^{TX})Z = (i_{\mathcal{R}} R^{TX})Z. \tag{1.2.25}$$

Where we consider the curvature R^{TX} as a matrix of two forms and θ is a $\mathbb{R}^{2n}$-valued one form. The i-th component of $R^{TX}Z, \theta$ is $\langle R^{TX}\mathcal{R}, \widetilde{e}_i\rangle$, θ^i, from (1.2.25), we get

$$i_{e_j}(L_{\mathcal{R}} - 1)L_{\mathcal{R}}\theta^i(Z) = \left\langle R^{TX}(\mathcal{R}, e_j)\mathcal{R}, \widetilde{e}_i \right\rangle(Z). \tag{1.2.26}$$

By (1.2.12), $L_{\mathcal{R}} e^j = e^j$. Thus from the Taylor expansion of $\theta_j^i(Z)$, we get

$$\sum_{|\alpha| \geqslant 1} (|\alpha|^2 + |\alpha|)(\partial^\alpha \theta_j^i)(0) \frac{Z^\alpha}{\alpha!} = \left\langle R^{TX}(\mathcal{R}, e_j)\mathcal{R}, \widetilde{e}_i \right\rangle(Z). \tag{1.2.27}$$

Now by (1.2.21) and $\theta_j^i(x_0) = \delta_{ij}$, (1.2.27) determines the Taylor expansion of $\theta_j^i(Z)$ up to order m in terms of the Taylor expansion of the coefficients of R^{TX} up to order $m - 2$. And

$$(\widetilde{\theta}^{-1})_j^i = \delta_{ij} - \frac{1}{6} \left\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j \right\rangle_{x_0} + \mathscr{O}(|Z|^3). \tag{1.2.28}$$

By (1.2.21), (1.2.27), we infer (1.2.18).

From (1.2.21),

$$g_{ij}(Z) = \theta_i^k(Z)\theta_j^k(Z). \tag{1.2.29}$$

Thus the rest of Lemma 1.2.3 follows from (1.2.28) and (1.2.29). The proof of Lemma 1.2.3 is complete. □

Let E be a complex vector bundle on X, and let ∇^E be a connection on E with curvature $R^E := (\nabla^E)^2$. Let $(\mathcal{U}, Z_1, \dots, Z_{2n})$ be a local chart of X such that $0 \in \mathcal{U}$ represents $x_0 \in X$. Set $\mathcal{R} = \sum_i Z_i \frac{\partial}{\partial Z_i}$. Now we identify E_Z to E_{x_0} by parallel transport with respect to the connection ∇^E along the curve $[0,1] \ni u \to uZ$; this gives a trivialization of E near 0. We denote by Γ^E the connection form with respect to this trivialization of E near 0. Then in the frame $e_j = \frac{\partial}{\partial Z_j}$, Γ^E becomes a function with values in $\mathbb{R}^{2n} \otimes \mathrm{End}(\mathbb{C}^{\mathrm{rk}(E)})$ and $\nabla^E = d + \Gamma^E$.

Lemma 1.2.4. *The Taylor coefficients of $\Gamma^E(e_j)(Z)$ at x_0 up to order r are determined by Taylor coefficients of R^E up to order $r-1$. More precisely,*

$$\sum_{|\alpha|=r} (\partial^\alpha \Gamma^E)_{x_0}(e_j)\frac{Z^\alpha}{\alpha!} = \frac{1}{r+1}\sum_{|\alpha|=r-1} (\partial^\alpha R^E)_{x_0}(\mathcal{R}, e_j)\frac{Z^\alpha}{\alpha!}. \tag{1.2.30}$$

Especially,

$$\Gamma^E_Z(e_j) = \frac{1}{2} R^E_{x_0}(\mathcal{R}, e_j) + \mathscr{O}(|Z|^2). \tag{1.2.31}$$

Proof. By the definition of our fixed frame, we have $R^E = d\Gamma^E + \Gamma^E \wedge \Gamma^E$ and

$$i_{\mathcal{R}}\Gamma^E = 0, \qquad L_{\mathcal{R}}\Gamma^E = [i_{\mathcal{R}}, d]\Gamma^E = i_{\mathcal{R}}(d\Gamma^E + \Gamma^E \wedge \Gamma^E) = i_{\mathcal{R}} R^E. \tag{1.2.32}$$

Using $L_{\mathcal{R}} dZ^j = dZ^j$ and expanding both sides of the second equation of (1.2.32) in Taylor's series of at $Z = 0$, we obtain

$$\sum_\alpha (|\alpha|+1)(\partial^\alpha \Gamma^E)_{x_0}(e_j)\frac{Z^\alpha}{\alpha!} = \sum_\alpha (\partial^\alpha R^E)_{x_0}(\mathcal{R}, e_j)\frac{Z^\alpha}{\alpha!}. \tag{1.2.33}$$

By equating coefficients of Z^α of both sides, we get Lemma 1.2.4. □

1.2.2 Chern connection

Recall that $T^{(1,0)}X$ is a holomorphic vector bundle with Hermitian metric $h^{T^{(1,0)}X}$ induced by g^{TX}. The map $T_hX \ni Y \to \frac{1}{2}(Y - \sqrt{-1}JY) \in T^{(1,0)}X$ induces the natural identification of T_hX and $T^{(1,0)}X$.

We will denote by $\langle \cdot, \cdot \rangle$ the $\mathbb{C}$-bilinear form on $TX \otimes_{\mathbb{R}} \mathbb{C}$ induced by g^{TX}. Note that $\langle \cdot, \cdot \rangle$ vanishes on $T^{(1,0)}X \times T^{(1,0)}X$ and on $T^{(0,1)}X \times T^{(0,1)}X$.

For $U \in TX \otimes_{\mathbb{R}} \mathbb{C}$, we will denote by $U^{(1,0)}, U^{(0,1)}$ its components in $T^{(1,0)}X$ and $T^{(0,1)}X$. Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}X$ with dual frame $\{w^j\}_{j=1}^n$. Then

$$e_{2j-1} = \tfrac{1}{\sqrt{2}}(w_j + \overline{w}_j) \quad \text{and} \quad e_{2j} = \tfrac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j), \quad j = 1, \dots, n, \tag{1.2.34}$$

form an orthonormal frame of TX. We fix this notation throughout the book and use it without further notice.

Let $\nabla^{T^{(1,0)}X}$ be the holomorphic Hermitian connection on $(T^{(1,0)}X, h^{T^{(1,0)}X})$ with curvature $R^{T^{(1,0)}X}$. For $v \in \mathscr{C}^\infty(X, T^{(0,1)}X)$, we define $\nabla^{T^{(0,1)}X} v := \overline{\nabla^{T^{(1,0)}X}\overline{v}}$. Then $\nabla^{T^{(0,1)}X}$ defines a connection on $T^{(0,1)}X$. Set

$$\widetilde{\nabla}^{TX} = \nabla^{T^{(1,0)}X} \oplus \nabla^{T^{(0,1)}X}. \tag{1.2.35}$$

Then $\widetilde{\nabla}^{TX}$ is a connection on $TX \otimes_{\mathbb{R}} \mathbb{C}$ and it preserves TX; we still denote by $\widetilde{\nabla}^{TX}$ the induced connection on TX. Then $\widetilde{\nabla}^{TX}$ preserves the metric g^{TX}.

Let T be the torsion of the connection $\widetilde{\nabla}^{TX}$. Then $T \in \Lambda^2(T^*X) \otimes TX$ is defined by

$$T(U,V) = \widetilde{\nabla}^{TX}_U V - \widetilde{\nabla}^{TX}_V U - [U,V], \tag{1.2.36}$$

for vector fields U and V on X. Hence

$$\begin{aligned}&T \text{ maps } T^{(1,0)}X \otimes T^{(1,0)}X \text{ (resp. } T^{(0,1)}X \otimes T^{(0,1)}X) \text{ into } T^{(1,0)}X\\ &\text{(resp. } T^{(0,1)}X) \text{ and vanishes on } T^{(1,0)}X \otimes T^{(0,1)}X.\end{aligned} \tag{1.2.37}$$

Set

$$S = \widetilde{\nabla}^{TX} - \nabla^{TX}, \quad \mathcal{S} = \sum_i S(e_i)e_i. \tag{1.2.38}$$

Then S is a real 1-form on X taking values in the skew-adjoint endomorphisms of TX. Since ∇^{TX} is torsion free, we have for $U, V \in TX$,

$$T(U,V) = S(U)V - S(V)U. \tag{1.2.39}$$

Moreover, from (1.2.1), (1.2.36) and (1.2.38) and ∇^{TX} preserves g^{TX}, we obtain directly

$$2\langle S(U)V, W\rangle - \langle T(U,V), W\rangle - \langle T(W,U), V\rangle + \langle T(V,W), U\rangle = 0. \tag{1.2.40}$$

By (1.2.37), (1.2.39) and (1.2.40), we get

$$\begin{aligned}&\langle S(w_i)w_k, w_j\rangle = 0,\\ &2\langle S(w_i)\overline{w}_k, w_j\rangle = 2\langle S(\overline{w}_k)w_i, w_j\rangle = -\langle T(w_i,w_j), \overline{w}_k\rangle.\end{aligned} \tag{1.2.41}$$

Since $T(w_i, \overline{w}_j) = 0$, $S(\overline{w}_j)w_i = S(w_i)\overline{w}_j$, and so

$$\begin{aligned}\mathcal{S} = 2S(w_j)\overline{w}_j &= \langle T(w_i,w_j), \overline{w}_j\rangle\, \overline{w}_i + \langle T(\overline{w}_i,\overline{w}_j), w_j\rangle\, w_i\\ &= \langle T(e_i,e_j), e_j\rangle\, e_i,\\ 2\langle S(\cdot)w_j, \overline{w}_j\rangle &= \langle T(w_i,w_j), \overline{w}_j\rangle\, w^i - \langle T(\overline{w}_i,\overline{w}_j), w_j\rangle\, \overline{w}^i.\end{aligned} \tag{1.2.42}$$

The connection $\widetilde{\nabla}^{TX}$ on TX induces naturally a covariant derivative on the exterior bundle $\Lambda(T^*X)$ and we still denote it by $\widetilde{\nabla}^{TX}$. For any differential forms α, β and vector field Y, it satisfies

$$\widetilde{\nabla}^{TX}_Y(\alpha \wedge \beta) = (\widetilde{\nabla}^{TX}_Y \alpha) \wedge \beta + \alpha \wedge \widetilde{\nabla}^{TX}_Y \beta. \tag{1.2.43}$$

For a 1-form α and vector fields U, V, we have $(\widetilde{\nabla}^{TX}_U \alpha, V) = U(\alpha, V) - (\alpha, \widetilde{\nabla}^{TX}_U V)$. Likewise, ∇^{TX} induces naturally a connection ∇^{TX} on $\Lambda(T^*X)$.

We denote also by ε the exterior product $T^*X \otimes \Lambda^{\bullet}(T^*X) \to \Lambda^{\bullet+1}(T^*X)$.

Lemma 1.2.5. *For the exterior differentiation operator d acting on smooth sections of $\Lambda(T^*X)$, we have*

$$d = \varepsilon \circ \widetilde{\nabla}^{TX} + i_T, \qquad d = \varepsilon \circ \nabla^{TX}. \tag{1.2.44}$$

Proof. We denote by $\mathbf{d} := \varepsilon \circ \widetilde{\nabla}^{TX} + i_T$. Then by using (1.2.43), we know that for any homogeneous differential forms α, β, we have

$$\mathbf{d}(\alpha \wedge \beta) = (\mathbf{d}\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathbf{d}\beta. \tag{1.2.45}$$

From Leibniz's rule (1.2.45), it suffices to show that $\mathbf{d}$ agrees with d on functions (which is clear) and 1–forms. Now, for any smooth function f on X, we have

$$\begin{aligned}
\varepsilon \circ \widetilde{\nabla}^{TX} df &= e^i \wedge e^j \langle \widetilde{\nabla}^{TX}_{e_i} df, e_j \rangle \\
&= e^i \wedge e^j \Big(e_i(e_j(f)) - \langle df, \widetilde{\nabla}^{TX}_{e_i} e_j \rangle \Big) \\
&= \frac{1}{2} e^i \wedge e^j \Big(e_i(e_j(f)) - \langle df, \widetilde{\nabla}^{TX}_{e_i} e_j \rangle - \Big(e_j(e_i(f)) - \langle df, \widetilde{\nabla}^{TX}_{e_j} e_i \rangle \Big) \Big) \\
&= -\frac{1}{2} e^i \wedge e^j \langle df, T(e_i, e_j) \rangle = -i_T df.
\end{aligned} \tag{1.2.46}$$

Thus $\mathbf{d}$ coincides also d on 1–forms. Thus we get the first equation of (1.2.44). As ∇^{TX} is torsion free, from the above argument, we obtain the second equation of (1.2.44). ◻

If $B \in \Lambda^2(T^*X) \otimes TX$ we will denote by B_{as} the anti-symmetrization of the tensor $V, W, Y \to \langle B(V, W), Y \rangle$. Then

$$B_{as}(V, W, Y) = \langle B(V, W), Y \rangle - \langle B(V, Y), W \rangle - \langle B(Y, W), V \rangle. \tag{1.2.47}$$

Especially from (1.2.37), we infer

$$\begin{aligned}
T_{as} &= \frac{1}{2} \langle T(e_i, e_j), e_k \rangle e^i \wedge e^j \wedge e^k \\
&= \frac{1}{2} \langle T(w_i, w_j), \overline{w}_k \rangle w^i \wedge w^j \wedge \overline{w}^k + \frac{1}{2} \langle T(\overline{w}_i, \overline{w}_j), w_k \rangle \overline{w}^i \wedge \overline{w}^j \wedge w^k \\
&=: T^{(1,0)}_{as} + T^{(0,1)}_{as}.
\end{aligned} \tag{1.2.48}$$

Here $T^{(1,0)}_{as}$, $T^{(0,1)}_{as}$ are the anti-symmetrizations of the component $T^{(1,0)}$, $T^{(0,1)}$ of T in $T^{(1,0)}X$ and $T^{(0,1)}X$.

Let Θ be the real $(1,1)$-form defined by

$$\Theta(X, Y) = g^{TX}(JX, Y). \tag{1.2.49}$$

Note that the exterior differentiation operator d acting on smooth sections of $\Lambda(T^*X)$ has the decomposition

$$d = \partial + \overline{\partial}. \tag{1.2.50}$$

Proposition 1.2.6. *We have the identity of* 3*–forms on* X,

$$T_{as} = -\sqrt{-1}(\partial - \overline{\partial})\Theta. \tag{1.2.51}$$

Proof. By (1.2.34), we know that $\Theta = \sqrt{-1}\sum_i w^i \wedge \overline{w}^i$. Thus

$$\begin{aligned}\widetilde{\nabla}^{TX}\Theta &= \sqrt{-1}((\widetilde{\nabla}^{TX}w^i)\wedge \overline{w}^i + w^i \wedge \widetilde{\nabla}^{TX}\overline{w}^i)\\ &= \sqrt{-1}\left(-\langle \widetilde{\nabla}^{TX}w_i, \overline{w}_j\rangle - \langle w_i, \widetilde{\nabla}^{TX}\overline{w}_j\rangle\right) w^i\wedge \overline{w}^j = 0.\end{aligned} \tag{1.2.52}$$

From (1.2.44), (1.2.48) and (1.2.52) we have

$$d\Theta = i_T\Theta = \sqrt{-1}(T_{as}^{(1,0)} - T_{as}^{(0,1)}). \tag{1.2.53}$$

The relations (1.2.48) and (1.2.53) yield

$$\partial\Theta = \sqrt{-1}T_{as}^{(1,0)}, \qquad \overline{\partial}\Theta = -\sqrt{-1}T_{as}^{(0,1)}. \tag{1.2.54}$$

(1.2.54) imply (1.2.51). □

Definition 1.2.7. We call Θ as in (1.2.49) a Hermitian form on X and (X, Θ, g^{TX}) a complex Hermitian manifold. The metric $g^{TX} = \Theta(\cdot, J\cdot)$ on TX is called a **Kähler metric** if Θ is a closed form, i.e. $d\Theta = 0$. In this case, the form Θ is called a **Kähler form** on X, and the complex manifold (X, J) is called a **Kähler manifold**.

Let $\nabla^X J \in T^*X \otimes \mathrm{End}(TX)$ be the covariant derivative of J induced by the Levi-Civita connection ∇^{TX}.

Theorem 1.2.8. (X, J, Θ) *is Kähler if and only if the bundle* $T^{(1,0)}X$ *and* $T^{(0,1)}X$ *are preserved by the Levi-Civita connection* ∇^{TX}*, or in other words, if and only if* $\nabla^X J = 0$*. In this case,*

$$\nabla^{TX} = \widetilde{\nabla}^{TX}, \quad S = 0, \quad T = 0. \tag{1.2.55}$$

Proof. As Θ is a $(1,1)$-form, by (1.2.41), (1.2.48) and (1.2.51), $d\Theta = 0$ is equivalent to $T_{as} = 0$ and equivalent to $S(\overline{w}_k)w_i \in T^{(1,0)}X$ for any i, k. But this means that the bundles $T^{(1,0)}X$ and $T^{(0,1)}X$ are preserved by ∇^{TX}. Hence (1.2.55) is equivalent to (X, Θ) being Kähler.
Moreover, as J acts by multiplication with $\sqrt{-1}$ on $T^{(1,0)}X$, we get for $U \in TX$,

$$\begin{aligned}\langle S(U)w_i, w_j\rangle &= -\left\langle \nabla^{TX}_U w_i, w_j\right\rangle = -\frac{1}{2}\left\langle \nabla^{TX}_U (1-\sqrt{-1}J)w_i, w_j\right\rangle\\ &= \frac{1}{2}\sqrt{-1}\left\langle (\nabla^X_U J)w_i, w_j\right\rangle,\end{aligned} \tag{1.2.56}$$

by (1.2.38). Now, from $J^2 = -1$ we deduce

$$J(\nabla^X J) + (\nabla^X J)J = 0. \tag{1.2.57}$$

This means that $(\nabla^X J)$ exchanges $T^{(1,0)}X$ and $T^{(0,1)}X$. By (1.2.44) and (1.2.56), $\nabla^X J = 0$ is equivalent to $S(\overline{w}_k)w_i \in T^{(1,0)}X$ for any i, k. The proof of Theorem 1.2.8 is complete. □

1.2.3 Bismut connection

Let S^B denote the 1-form with values in the antisymmetric elements of $\mathrm{End}(TX)$ which satisfies for $U, V, W \in TX$,

$$\langle S^B(U)V, W\rangle = \frac{\sqrt{-1}}{2}\Big((\partial - \overline{\partial})\Theta\Big)(U, V, W) = -\frac{1}{2}T_{as}(U, V, W). \tag{1.2.58}$$

By (1.2.40), (1.2.47), (1.2.58), we have for $U, V, W \in TX$,

$$\big\langle (S^B - S)(U)V, W\big\rangle = -\big\langle T(U, V), W\big\rangle + \big\langle T(U, W), V\big\rangle. \tag{1.2.59}$$

Relations (1.2.41), (1.2.48) and (1.2.58) yield

$$\begin{aligned}
\langle S^B(e_j)\omega_l, \overline{\omega}_m\rangle &= -\frac{1}{2}\langle T(e_j, \omega_l), \overline{\omega}_m\rangle + \frac{1}{2}\langle T(e_j, \overline{\omega}_m), \omega_l\rangle \\
&= -\langle S(e_j)\omega_l, \overline{\omega}_m\rangle, \\
\langle S^B(e_j)\omega_l, \omega_m\rangle &= -\frac{1}{2}\langle T(\omega_l, \omega_m), e_j\rangle = \langle S(e_j)\omega_l, \omega_m\rangle.
\end{aligned} \tag{1.2.60}$$

Definition 1.2.9. The **Bismut connection** ∇^B on TX is defined by

$$\nabla^B := \nabla^{TX} + S^B = \widetilde{\nabla}^{TX} + S^B - S. \tag{1.2.61}$$

In view of (1.2.58), the torsion of ∇^B is $2S^B$ which is a skew-symmetric tensor.

The connection ∇^B will be used in the Lichnerowicz formula (1.4.29).

Lemma 1.2.10. *The connection ∇^B preserves the complex structure of TX.*

Proof. Using (1.2.60), we find that for $V, W \in T^{(1,0)}X$, $\big\langle (S^B - S)(U)V, W\big\rangle = 0$, for any $U \in TX$. Equivalently, $(S^B - S)(U)$ is a complex endomorphism of TX. Using (1.2.61), we find that ∇^B preserves the complex structure of TX. □

1.3 Spin^c Dirac operator

This Section is organized as follows. In Section 1.3.1, we define the Clifford connection. In Section 1.3.2, we define the spin^c Dirac operator on a complex manifold and prove the related Lichnerowicz formula. In Section 1.3.3, we obtain the Lichnerowicz formula for the modified Dirac operator. In Section 1.3.4, we explain also the Atiyah-Singer index theorem for the modified Dirac operator.

In this Section, we work on a smooth manifold with an almost complex structure J.

1.3.1 Clifford connection

Let (X, J) be a smooth manifold with J an almost complex structure on TX. Let g^{TX} be any Riemannian metric on TX compatible with J. Let $h^{\Lambda^{0,\bullet}}$ be the Hermitian metric on $\Lambda(T^{*(0,1)}X)$ induced by g^{TX}.

The fundamental $\mathbb{Z}_2$ spinor bundle induced by J is given by $\Lambda(T^{*(0,1)}X)$, and its $\mathbb{Z}_2$-grading is defined by $\Lambda(T^{*(0,1)}X) = \Lambda^{\text{even}}(T^{*(0,1)}X) \oplus \Lambda^{\text{odd}}(T^{*(0,1)}X)$. For any $v \in TX$ with decomposition $v = v^{(1,0)} + v^{(0,1)} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $\overline{v}^{(1,0),*} \in T^{*(0,1)}X$ be the metric dual of $v^{(1,0)}$. Then

$$c(v) = \sqrt{2}(\overline{v}^{(1,0),*} \wedge - i_{v^{(0,1)}}) \tag{1.3.1}$$

defines the Clifford action of v on $\Lambda(T^{*(0,1)}X)$, where $\wedge$ and i denote the exterior and interior product, respectively. We verify easily that for $U, V \in TX$,

$$c(U)c(V) + c(V)c(U) = -2\langle U, V\rangle. \tag{1.3.2}$$

For a skew-adjoint endomorphism A of TX, from (1.3.1), using the notation of (1.2.34),

$$\begin{aligned}\frac{1}{4}\langle Ae_i, e_j\rangle c(e_i)c(e_j) = &-\frac{1}{2}\big\langle Aw_j, \overline{w}_j\big\rangle + \big\langle Aw_l, \overline{w}_m\big\rangle\, \overline{w}^m \wedge i_{\overline{w}_l}\\ &+ \frac{1}{2}\big\langle Aw_l, w_m\big\rangle\, i_{\overline{w}_l}\, i_{\overline{w}_m} + \frac{1}{2}\big\langle A\overline{w}_l, \overline{w}_m\big\rangle\, \overline{w}^l \wedge \overline{w}^m \wedge .\end{aligned} \tag{1.3.3}$$

Let $\nabla^{\det}$ be a Hermitian connection on $\det(T^{(1,0)}X)$ endowed with metric induced by g^{TX}. Let $R^{\det}$ be its curvature. Let $P^{T^{(1,0)}X}$ be the natural projection from $TX \otimes_{\mathbb{R}} \mathbb{C}$ onto $T^{(1,0)}X$. Then the connection $\nabla^{1,0} = P^{T^{(1,0)}X}\nabla^{TX}P^{T^{(1,0)}X}$ on $T^{(1,0)}X$ induces naturally a connection $\nabla^{\det_1}$ on $\det(T^{(1,0)}X)$.

Let $\Gamma^{TX} \in T^*X \otimes \operatorname{End}(TX)$, $\Gamma^{\det}$ be the connection forms of ∇^{TX}, $\nabla^{\det}$ associated to the frames $\{e_j\}$, $w_1 \wedge \cdots \wedge w_n$, i.e.

$$\begin{aligned}&\nabla^{TX}_{e_i} e_j = \Gamma^{TX}(e_i)e_j, \quad \nabla^{\det}(w_1 \wedge \cdots \wedge w_n) = \Gamma^{\det} w_1 \wedge \cdots \wedge w_n,\\ &\nabla^{\det_1}(w_1 \wedge \cdots \wedge w_n) = \Big(\textstyle\sum_j \langle \Gamma^{TX} w_j, \overline{w}_j\rangle\Big) w_1 \wedge \cdots \wedge w_n.\end{aligned} \tag{1.3.4}$$

The **Clifford connection** ∇^{Cl} on $\Lambda(T^{*(0,1)}X)$ is defined for the frame $\{\overline{w}^{j_1} \wedge \cdots \wedge \overline{w}^{j_k},\ 1 \leqslant j_1 < \cdots < j_k \leqslant n\}$ by the local formula

$$\nabla^{\text{Cl}} = d + \frac{1}{4}\big\langle \Gamma^{TX} e_i, e_j\big\rangle c(e_i)c(e_j) + \frac{1}{2}\Gamma^{\det}. \tag{1.3.5}$$

Proposition 1.3.1. *∇^{Cl} defines a Hermitian connection on $\Lambda(T^{*(0,1)}X)$ and preserves its $\mathbb{Z}_2$-grading. For any V, W vector fields of TX on X, we have*

$$\left[\nabla^{Cl}_V, c(W)\right] = c(\nabla^{TX}_V W). \tag{1.3.6}$$

Proof. At first, by (1.3.4) and (1.3.5), we have

$$\begin{aligned}\left[\nabla^{\mathrm{Cl}}_V, c(e_k)\right] &= \frac{1}{4}\left[\langle\Gamma^{TX}(V)e_i, e_j\rangle c(e_i)c(e_j), c(e_k)\right]\\ &= \langle\Gamma^{TX}(V)e_k, e_j\rangle c(e_j) = c(\nabla^{TX}_V e_k).\end{aligned} \tag{1.3.7}$$

Thus if ∇^{Cl} is well defined, we get (1.3.6) from (1.3.7).

Now we observe that $c(w_{j_1})\cdots c(w_{j_k})\,1$, $(1 \leqslant j_1 < \cdots < j_k \leqslant n)$ generate a frame of $\Lambda(T^{*(0,1)}X)$. Taking into account (1.3.7), to verify that ∇^{Cl} does not depend on the choice of our frame $\{w_j\}_{j=1}^n$, we only need to verify that $\nabla^{\mathrm{Cl}}1$ is well defined.

Relations (1.2.38), (1.3.3), (1.3.4) and (1.3.5) entail

$$\begin{aligned}\nabla^{\mathrm{Cl}} = d &+ \frac{1}{2}(\nabla^{\det} - \nabla^{\det_1}) + \left\langle\Gamma^{TX} w_l, \overline{w}_m\right\rangle \overline{w}^m \wedge i_{\overline{w}_l}\\ &- \frac{1}{2}\left\langle S w_l, w_m\right\rangle i_{\overline{w}_l}\, i_{\overline{w}_m} - \frac{1}{2}\left\langle S\overline{w}_l, \overline{w}_m\right\rangle \overline{w}^l \wedge \overline{w}^m \wedge .\end{aligned} \tag{1.3.8}$$

From (1.3.8), we know

$$\nabla^{\mathrm{Cl}}1 = \frac{1}{2}(\nabla^{\det} - \nabla^{\det_1}) - \frac{1}{2}\sum_{lm}\left\langle S\overline{w}_l, \overline{w}_m\right\rangle \overline{w}^l \wedge \overline{w}^m. \tag{1.3.9}$$

Clearly, $\nabla^{\det} - \nabla^{\det_1}$ is a 1–form on X, and the right hand side of (1.3.9) does not depend on the choice of the frame w_j. Thus $\nabla^{\mathrm{Cl}}1$ is well defined.

Let $c(e_i)^*$ be the adjoint of $c(e_i)$ with respect to the Hermitian product on $\Lambda(T^{*(0,1)}X)$. By (1.3.1), we have

$$c(e_i)^* = -c(e_i). \tag{1.3.10}$$

Using (1.3.5), (1.3.10) and the anti–symmetry of $\left\langle\Gamma^{TX}e_i, e_j\right\rangle$ in i, j, we see that ∇^{Cl} preserves the Hermitian metric on $\Lambda(T^{*(0,1)}X)$.

Finally, from (1.3.5), ∇^{Cl} preserves the $\mathbb{Z}_2$-grading on $\Lambda(T^{*(0,1)}X)$. The proof of Proposition 1.3.1 is complete. □

Let R^{Cl} be the curvature of ∇^{Cl}.

Proposition 1.3.2. *We have the following identity :*

$$R^{Cl} = \frac{1}{4}\langle R^{TX}e_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}R^{\det}. \tag{1.3.11}$$

Proof. At first, observe that if i,j,k,l are different, then $[c(e_i)c(e_j),c(e_k)c(e_l)]=0$. Thus from (1.3.2),

$$\begin{aligned}
&\big[\big\langle\Gamma^{TX}(W)e_i,e_j\big\rangle c(e_i)c(e_j),\big\langle\Gamma^{TX}(V)e_k,e_l\big\rangle c(e_k)c(e_l)\big]\\
&\quad=4\sum_{i\neq j\neq k}\big\langle\Gamma^{TX}(W)e_i,e_j\big\rangle\big\langle\Gamma^{TX}(V)e_k,e_j\big\rangle\,[c(e_i)c(e_j),c(e_k)c(e_j)]\\
&\qquad=4\big\langle\Gamma^{TX}(W)e_i,\Gamma^{TX}(V)e_k\big\rangle(c(e_i)c(e_k)-c(e_k)c(e_i))\\
&\qquad\qquad=4\big\langle(\Gamma^{TX}\wedge\Gamma^{TX})(W,V)e_i,e_k\big\rangle c(e_i)c(e_k).
\end{aligned}\tag{1.3.12}$$

Moreover, we have

$$\begin{aligned}
&R^{TX}=d\Gamma^{TX}+\Gamma^{TX}\wedge\Gamma^{TX},\\
&R^{\mathrm{Cl}}(e_l,e_m)=\nabla^{\mathrm{Cl}}_{e_l}\nabla^{\mathrm{Cl}}_{e_m}-\nabla^{\mathrm{Cl}}_{e_m}\nabla^{\mathrm{Cl}}_{e_l}-\nabla^{\mathrm{Cl}}_{[e_l,e_m]}.
\end{aligned}\tag{1.3.13}$$

Finally, (1.3.5), (1.3.12) and (1.3.13) yield (1.3.11). □

1.3.2 Dirac operator and Lichnerowicz formula

Let (E,h^E) be a Hermitian vector bundle on X. Let ∇^E be a Hermitian connection on (E,h^E) with curvature R^E.

Set $\mathbf{E}^q=\Lambda^q(T^{*(0,1)}X)\otimes E$, $\mathbf{E}=\oplus_{q=0}^n E^q$. We still denote by ∇^{Cl} the connection on $\Lambda(T^{*(0,1)}X)\otimes E$ induced by ∇^{Cl} and ∇^E. Let $\Omega^{0,q}(X,E):=\mathscr{C}^\infty(X,\mathbf{E}^q)$ be the set of smooth sections of $\mathbf{E}^q$ on X.

Along the fibers of $\Lambda(T^{*(0,1)}X)\otimes E$, we consider the pointwise Hermitian product $\langle\cdot,\cdot\rangle_{\Lambda^{0,\bullet}\otimes E}$ induced by g^{TX} and h^E. The L^2–scalar product on $\Omega^{0,\bullet}(X,E)$ is given by

$$\langle s_1,s_2\rangle=\int_X\langle s_1(x),s_2(x)\rangle_{\Lambda^{0,\bullet}\otimes E}\,dv_X(x)\,.\tag{1.3.14}$$

We denote the corresponding norm with $\|\cdot\|_{L^2}$, and by $L^2(X,\Lambda(T^{*(0,1)}X)\otimes E)$ or $L^2_{0,\bullet}(X,E)$, the L^2 completion of $\Omega_0^{0,\bullet}(X,E)$, the subspace of $\Omega^{0,\bullet}(X,E)$ with compact support.

Definition 1.3.3. The **spin^c Dirac operator** D^c is defined by

$$D^c=\sum_{j=1}^{2n}c(e_j)\nabla^{\mathrm{Cl}}_{e_j}:\Omega^{0,\bullet}(X,E)\longrightarrow\Omega^{0,\bullet}(X,E)\,.\tag{1.3.15}$$

By Proposition 1.3.1 and (1.3.1), D^c interchanges $\Omega^{0,\mathrm{even}}(X,E)$ and $\Omega^{0,\mathrm{odd}}(X,E)$. We denote by

$$D^c_+=D^c|_{\Omega^{0,\mathrm{even}}(X,E)},\quad D^c_-=D^c|_{\Omega^{0,\mathrm{odd}}(X,E)}.\tag{1.3.16}$$

Lemma 1.3.4. *D^c is a formally self–adjoint, first order elliptic differential operator on $\Omega^{0,\bullet}(X,E)$.*