Proof. Let $s_1, s_2 \in \Omega^{0,\bullet}(X, E)$ with compact support and let α be the 1–form on X given by $\alpha(Y) = \langle c(Y)s_1, s_2\rangle_{\Lambda^{0,\bullet}\otimes E}$, for any vector field Y on X. Proposition 1.3.1 and (1.3.10) imply that for $x \in X$,

$$\langle s_1, D^c s_2\rangle_{\Lambda^{0,\bullet}\otimes E,x} = \langle D^c s_1, s_2\rangle_{\Lambda^{0,\bullet}\otimes E,x} - \mathrm{Tr}(\nabla\alpha)_x. \tag{1.3.17}$$

The integral over X of the last term vanishes by Proposition 1.2.1. Thus D^c is formally self–adjoint.

For $\zeta \in T^*X$, let $\zeta^* \in TX$ be the metric dual of ζ. The principal symbol $\sigma(D^c)$ of D^c is

$$\sigma(D^c)(\zeta) = \sqrt{-1}c(\zeta^*). \tag{1.3.18}$$

By (1.3.2), $(\sigma(D^c)(\zeta))^2 = |\zeta|^2$, which means, that $\sigma(D^c)(\zeta)$ is invertible for any $\zeta \neq 0$. Thus D^c is a first order elliptic differential operator. □

Let (F, h^F) be a Hermitian vector bundle on X and let ∇^F be a Hermitian connection on F. Then the usual **Bochner Laplacians** Δ^F, Δ are defined by

$$\Delta^F := -\sum_{i=1}^{2n}\Big((\nabla^F_{e_i})^2 - \nabla^F_{\nabla^{TX}_{e_i}e_i}\Big), \quad \Delta = \Delta^{\mathbb{C}}. \tag{1.3.19}$$

Let $s_1, s_2 \in \mathscr{C}^\infty(X, F)$, with compact support and let α be the 1-form on X given by $\alpha(Y)(x) = \langle\nabla^F_Y s_1, s_2\rangle(x)$, for any $Y \in T_xX$. Then by (1.2.6), (1.2.7), we get the following useful equation :

$$\begin{aligned}\int_X \langle\Delta^F s_1, s_2\rangle dv_X &= \int_X \langle\nabla^F s_1, \nabla^F s_2\rangle dv_X - \int_X \mathrm{Tr}(\nabla\alpha)dv_X\\ &= \int_X \langle\nabla^F s_1, \nabla^F s_2\rangle dv_X.\end{aligned} \tag{1.3.20}$$

We denote by Δ^{Cl} the Bochner Laplacian on $\Lambda(T^{*(0,1)}X)\otimes E$ associated to ∇^{Cl} as in (1.3.19). Now we prove the Lichnerowicz formula for D^c.

Theorem 1.3.5.

$$(D^c)^2 = \Delta^{Cl} + \frac{r^X}{4} + \frac{1}{2}\Big(R^E + \frac{1}{2}R^{\det}\Big)(e_i, e_j)c(e_i)c(e_j). \tag{1.3.21}$$

Proof. By (1.3.2), (1.3.6) and (1.3.15),

$$
\begin{aligned}
(D^c)^2 &= \frac{1}{2}\sum_{ij}\Big\{c(e_i)\nabla^{\mathrm{Cl}}_{e_i}c(e_j)\nabla^{\mathrm{Cl}}_{e_j} + c(e_j)\nabla^{\mathrm{Cl}}_{e_j}c(e_i)\nabla^{\mathrm{Cl}}_{e_i}\Big\}\\
&= \frac{1}{2}\sum_{ij}\Big\{(c(e_i)c(e_j)+c(e_j)c(e_i))\nabla^{\mathrm{Cl}}_{e_i}\nabla^{\mathrm{Cl}}_{e_j} + c(e_i)\left[\nabla^{\mathrm{Cl}}_{e_i}, c(e_j)\right]\nabla^{\mathrm{Cl}}_{e_j}\\
&\qquad + c(e_j)[\nabla^{\mathrm{Cl}}_{e_j}, c(e_i)]\nabla^{\mathrm{Cl}}_{e_i} + c(e_j)c(e_i)\left[\nabla^{\mathrm{Cl}}_{e_j},\nabla^{\mathrm{Cl}}_{e_i}\right]\Big\}\\
&= -\sum_i(\nabla^{\mathrm{Cl}}_{e_i})^2 + \sum_{ijk}\big\langle\nabla^{TX}_{e_i}e_j, e_k\big\rangle c(e_i)c(e_k)\nabla^{\mathrm{Cl}}_{e_j}\\
&\qquad + \frac{1}{2}\sum_{ij}c(e_j)c(e_i)\left[\nabla^{\mathrm{Cl}}_{e_j},\nabla^{\mathrm{Cl}}_{e_i}\right].
\end{aligned}
\tag{1.3.22}
$$

But we have

$$
\big\langle\nabla^{TX}_{e_i}e_j, e_k\big\rangle = -\big\langle e_j, \nabla^{TX}_{e_i}e_k\big\rangle. \tag{1.3.23}
$$

In view of (1.3.2), (1.3.23), we obtain

$$
\begin{aligned}
\big\langle\nabla^{TX}_{e_i}e_j, e_k\big\rangle c(e_i)c(e_k)\nabla^{\mathrm{Cl}}_{e_j} &= -c(e_i)c(e_k)\nabla^{\mathrm{Cl}}_{\nabla^{TX}_{e_i}e_k}\\
&= \nabla^{\mathrm{Cl}}_{\nabla^{TX}_{e_i}e_i} - \frac{1}{2}\sum_{i\neq k}c(e_i)c(e_k)\Big(\nabla^{\mathrm{Cl}}_{\nabla^{TX}_{e_i}e_k} - \nabla^{\mathrm{Cl}}_{\nabla^{TX}_{e_k}e_i}\Big)\\
&= \nabla^{\mathrm{Cl}}_{\nabla^{TX}_{e_i}e_i} - \frac{1}{2}c(e_i)c(e_k)\nabla^{\mathrm{Cl}}_{[e_i,e_k]}.
\end{aligned}
\tag{1.3.24}
$$

Comparing to (1.3.13), we have here

$$
(R^{\mathrm{Cl}} + R^E)(e_l, e_m) = \nabla^{\mathrm{Cl}}_{e_l}\nabla^{\mathrm{Cl}}_{e_m} - \nabla^{\mathrm{Cl}}_{e_m}\nabla^{\mathrm{Cl}}_{e_l} - \nabla^{\mathrm{Cl}}_{[e_l,e_m]}. \tag{1.3.25}
$$

(1.3.22)–(1.3.25) yield

$$
(D^c)^2 = -\sum_i\Big((\nabla^{\mathrm{Cl}}_{e_i})^2 - \nabla^{\mathrm{Cl}}_{\nabla^{TX}_{e_i}e_i}\Big) + \frac{1}{2}c(e_j)c(e_i)(R^{\mathrm{Cl}}+R^E)(e_j,e_i). \tag{1.3.26}
$$

To simplify the notation, set

$$
R_{ijkl} := \langle R^{TX}(e_j,e_i)e_k, e_l\rangle. \tag{1.3.27}
$$

By Proposition 1.3.2, we get

$$
\begin{aligned}
c(e_j)c(e_i)R^{\mathrm{Cl}}(e_j,e_i) = &-\frac{1}{4}R_{ijkl}c(e_i)c(e_j)c(e_k)c(e_l)\\
&+\frac{1}{2}c(e_i)c(e_j)R^{\det}(e_i,e_j).
\end{aligned}
\tag{1.3.28}
$$

By the second equation of (1.2.4) and (1.3.2),

$$\sum_{i\neq k\neq j} R_{ijkl}c(e_i)c(e_j)c(e_k) = 2\sum_{i<j<k}(R_{ijkl}+R_{jkil}+R_{kijl})c(e_i)c(e_j)c(e_k) = 0.$$

Thus

$$\begin{aligned} R_{ijkl}c(e_i)c(e_j)c(e_k)c(e_l) &= -R_{ijjl}c(e_i)c(e_l) + R_{ijil}c(e_j)c(e_l) \\ &= 2c(e_j)c(e_l)R_{ijil} = -2R_{ijij}\,. \end{aligned} \tag{1.3.29}$$

In the last equation of (1.3.29), we use that R_{ijil} is symmetric in j,l (which follows by the first equation of (1.2.4)). By (1.2.5) and (1.3.27), we get the right hand side of the (1.3.29) equals $-2r^X$. Hence (1.3.26)-(1.3.29) imply (1.3.21). □

1.3.3 Modified Dirac operator

For any $\mathbb{Z}_2$-graded vector space $V = V^+ \oplus V^-$, the natural $\mathbb{Z}_2$-grading on $\mathrm{End}(V)$ is defined by

$$\mathrm{End}(V)^+ = \mathrm{End}(V^+)\oplus\mathrm{End}(V^-), \quad \mathrm{End}(V)^- = \mathrm{Hom}(V^+,V^-)\oplus\mathrm{Hom}(V^-,V^+),$$

and we define $\deg B = 0$ for $B\in\mathrm{End}(V)^+$, and $\deg B = 1$ for $B\in\mathrm{End}(V)^-$. For $B, C\in\mathrm{End}(V)$, we define their supercommutator (or graded Lie bracket) by

$$[B,C] = BC - (-1)^{\deg B\cdot\deg C}CB. \tag{1.3.30}$$

For $B, B', C\in\mathrm{End}(V)$, the **Jacobi identity** holds:

$$\begin{aligned}(-1)^{\deg C\cdot\deg B'}\big[B',[B,C]\big] + (-1)^{\deg B'\cdot\deg B}\big[B,[C,B']\big] \\ + (-1)^{\deg B\cdot\deg C}\big[C,[B',B]\big] = 0.\end{aligned} \tag{1.3.31}$$

We will apply the above notation for spaces $\Lambda(T^{*(0,1)}X)$ and $\Omega^{0,\bullet}(X,E)$ with natural $\mathbb{Z}_2$-grading induced by the parity of the degree.
For $i_1 < \cdots < i_j$, we define

$$^c(e^{i_1}\wedge\cdots\wedge e^{i_j}) = c(e_{i_1})\cdots c(e_{i_j}). \tag{1.3.32}$$

Then by extending $\mathbb{C}$-linearly, ^cB is defined for any $B\in\Lambda(T^*X\otimes_{\mathbb{R}}\mathbb{C})$.
For $A\in\Lambda^3(T^*X)$, set $|A|^2 = \sum_{i<j<k}|A(e_i,e_j,e_k)|^2$. Now let A be a smooth section of $\Lambda^3(T^*X)$. Let

$$\nabla^A_U = \nabla^{\mathrm{Cl}}_U +^c(i_UA) \quad \text{for } U\in TX \tag{1.3.33}$$

be the Hermitian connection on $\Lambda(T^{*(0,1)}X)\otimes E$ induced by ∇^{Cl} and A. Let Δ^A be the Bochner Laplacian defined by ∇^A as in (1.3.19).

Definition 1.3.6. The **modified Dirac operators** $D^{c,A}$, $D^{c,A}_{\pm}$ are defined by

$$D^{c,A} := D^c + {}^cA, \quad D^{c,A}_{\pm} := D^c_{\pm} + {}^cA. \tag{1.3.34}$$

Theorem 1.3.7. *The modified Dirac operator $D^{c,A}$ is formally self–adjoint and*

$$(D^{c,A})^2 = \Delta^A + \frac{r^X}{4} + {}^c(R^E + \frac{1}{2}R^{\det}) + {}^c(dA) - 2|A|^2. \tag{1.3.35}$$

Proof. By Lemma 1.3.4 and (1.3.10), the operator $D^c + {}^cA$ is formally self–adjoint. By (1.3.6), $\nabla^{\mathrm{Cl}}_{e_i} {}^cA = {}^c(\nabla^{\mathrm{Cl}}_{e_i} A)$. From (1.2.44) and (1.3.2) and since A is odd degree, we have

$$\begin{aligned} &[c(e_i), {}^cA] = -2\,{}^c(i_{e_i}A), \\ &c(e_i)(\nabla^{\mathrm{Cl}}_{e_i} {}^cA) - (\nabla^{\mathrm{Cl}}_{e_i} {}^cA)c(e_i) = 2\,{}^c(e^i \wedge \nabla^{TX}_{e_i} A) = 2\,{}^c(dA). \end{aligned} \tag{1.3.36}$$

By (1.3.19), (1.3.33) and the first equation of (1.3.36),

$$\begin{aligned} \Delta^A = \Delta^{\mathrm{Cl}} &+ \frac{1}{2}\Big(\nabla^{\mathrm{Cl}}_{e_i}[c(e_i), {}^cA] + [c(e_i), {}^cA]\nabla^{\mathrm{Cl}}_{e_i}\Big) \\ &- \frac{1}{2}[c(\nabla^{TX}_{e_i} e_i), {}^cA] - \frac{1}{4}\sum_i [c(e_i), {}^cA]^2 \\ = \Delta^{\mathrm{Cl}} &- 2^c(i_{e_i}A)\nabla^{\mathrm{Cl}}_{e_i} + \frac{1}{2}[c(e_i), \nabla^{\mathrm{Cl}}_{e_i} {}^cA] - \sum_i {}^c(i_{e_i}A)^2. \end{aligned} \tag{1.3.37}$$

Then Theorem 1.3.5, (1.3.33), (1.3.36) and (1.3.37) imply

$$\begin{aligned} (D^c + {}^cA)^2 =& (D^c)^2 + [c(e_i), {}^cA]\nabla^{\mathrm{Cl}}_{e_i} + c(e_i)(\nabla^{\mathrm{Cl}}_{e_i} {}^cA) + ({}^cA)^2 \\ =& \Delta^A + ({}^cA)^2 + \sum_i {}^c(i_{e_i}A)^2 + c(e_i)(\nabla^{\mathrm{Cl}}_{e_i} {}^cA) \\ &- \frac{1}{2}[c(e_i), (\nabla^{\mathrm{Cl}}_{e_i} {}^cA)] + \frac{r^X}{4} + {}^c(R^E + \frac{1}{2}R^{\det}). \end{aligned} \tag{1.3.38}$$

Relations (1.3.36) and (1.3.38) yield

$$(D^c + {}^cA)^2 = \Delta^A + ({}^cA)^2 + \sum_i {}^c(i_{e_i}A)^2 + {}^c(dA) + \frac{r^X}{4} + {}^c(R^E + \frac{1}{2}R^{\det}). \tag{1.3.39}$$

Let $I = \{i_1, \cdots, i_m\}$ be an ordered subset of $\{1, \cdots, 2n\}$, and assume that all $i_j \in I$ are distinct. Let $|I|$ be the cardinal of I. Set ${}^ce_I = c(e_{i_1}) \cdots c(e_{i_m})$. Take $k \leqslant 2n$, and let I, J be two ordered subsets of $\{k+1, \cdots, 2n\}$ such that $I \cap J = \emptyset$. Then

$${}^ce_{1\cdots k}\,{}^ce_I\,{}^ce_{1\cdots k}\,{}^ce_J = (-1)^{k|I|}({}^ce_{1\cdots k})^2\,{}^ce_I\,{}^ce_J = (-1)^{k|I| + \frac{k(k+1)}{2}}\,{}^ce_I\,{}^ce_J. \tag{1.3.40}$$

Since A is odd degree, (1.3.40) imply

$$
\begin{aligned}
{}^c(i_{e_i}A)^2 &= \sum_{k=0}^{2} \sum_{i_1<\cdots<i_k} (-1)^{\frac{k(k-1)}{2}}\, {}^c((i_{e_{i_1}}\cdots i_{e_{i_k}} i_{e_i}A)^2),\\
{}^c(A)^2 &= \sum_{k=0}^{3} \sum_{i_1<\cdots<i_k} (-1)^{\frac{k(k+1)}{2}}\, {}^c((i_{e_{i_1}}\cdots i_{e_{i_k}} A)^2).
\end{aligned} \tag{1.3.41}
$$

Observe that since $A \in \Lambda^3(T^*X)$, $A^2 = 0$ and $(i_{e_{i_1}} i_{e_{i_2}} A)^2 = 0$. Thus

$$
({}^cA)^2 + \sum_i {}^c(i_{e_i}A)^2 = -2 \sum_{i_1<i_2<i_3} (i_{e_{i_1}} i_{e_{i_2}} i_{e_{i_3}} A)^2 = -2|A|^2. \tag{1.3.42}
$$

From (1.3.39) and (1.3.42), we infer (1.3.35). □

1.3.4 Atiyah-Singer index theorem

Theorem 1.3.8. *If X is compact, the modified Dirac operator $D^{c,A}$ is an essentially self–adjoint Fredholm operator, thus its kernel* $\mathrm{Ker}(D^{c,A})$ *is a finite dimensional complex vector space.*

Proof. At first, if $s_k \in L^2_{0,\bullet}(X,E)$, $D^{c,A}s_k = 0$ and $\lim_{k\to\infty} s_k = s \in L^2_{0,\bullet}(X,E)$, then $D^{c,A}s = 0$ in the sense of distributions. By Theorem A.3.4, $s \in \Omega^{0,\bullet}(X,E)$ and $s \in \mathrm{Ker}(D^{c,A})$. Thus the space $\mathrm{Ker}(D^{c,A})$ is closed, so a Hilbert space. Since X is compact, Theorems A.3.1, A.3.2 and Lemma 1.3.4 imply that $D^{c,A}$ is essentially self–adjoint and the unit ball

$$
B = \{s \in L^2_{0,\bullet}(X,E) : \|s\|_{L^2} \leqslant 1,\ D^{c,A}s = 0\} \subset \mathrm{Ker}(D^{c,A}) \tag{1.3.43}
$$

is compact. Thus $\mathrm{Ker}(D^{c,A})$ is finite dimensional and $D^{c,A}$ is Fredhlom. □

When X is compact, we define the index $\mathrm{Ind}(D^{c,A}_+)$ of $D^{c,A}_+$ as

$$
\begin{aligned}
\mathrm{Ind}(D^{c,A}_+) &:= \dim \mathrm{Ker}(D^{c,A}_+) - \dim \mathrm{Coker}(D^{c,A}_+)\\
&= \dim \mathrm{Ker}(D^{c,A}_+) - \dim \mathrm{Ker}(D^{c,A}_-).
\end{aligned} \tag{1.3.44}
$$

For any Hermitian (complex) vector bundle (F, h^F) with Hermitian connection ∇^F and curvature R^F on X, set

$$
\begin{aligned}
\mathrm{ch}(F,\nabla^F) &:= \mathrm{Tr}\left[\exp\left(\frac{-R^F}{2\pi\sqrt{-1}}\right)\right],\\
c_1(F,\nabla^F) &:= \mathrm{Tr}\left[\frac{-R^F}{2\pi\sqrt{-1}}\right],\\
\mathrm{Td}(F,\nabla^F) &:= \det\left(\frac{R^F/(2\pi\sqrt{-1})}{\exp(R^F/(2\pi\sqrt{-1}))-1}\right).
\end{aligned} \tag{1.3.45}
$$

By Appendix B.5, these are closed real differential forms on X and their cohomology classes do not depend on the choice of the metric h^F and connection ∇^F. The corresponding cohomology classes are called the **Chern class** of F, the **first Chern class** of F, the **Todd class** of F, respectively, and we denote them by $\mathrm{ch}(F)$, $c_1(F)$, $\mathrm{Td}(F) \in H^*(X, \mathbb{R})$.

Theorem 1.3.9 (Atiyah-Singer index Theorem). *If X is compact,* $\mathrm{Ind}(D_+^{c,A})$ *is a topological invariant given by*

$$\mathrm{Ind}(D_+^{c,A}) = \int_X \mathrm{Td}(T^{(1,0)}X)\ \mathrm{ch}(E). \tag{1.3.46}$$

1.4 Lichnerowicz formula for $\square^E$

This Section is organized as follows. In Section 1.4.1, we exhibit the relation between the operator $\overline{\partial}^E + \overline{\partial}^{E,*}$ and the Dirac operator D^c. In Section 1.4.2, we prove Bismut's Lichnerowicz formula for the Kodaira Laplacian $\square^E$. In Section 1.4.3, we establish the Bochner-Kodaira-Nakano formula for $\square^E$. In Section 1.4.4, we prove the Bochner-Kodaira-Nakano formula with boundary term.

We will use the notation from Sections 1.2, 1.3.

1.4.1 The operator $\overline{\partial}^E + \overline{\partial}^{E,*}$

Let (X, J) be a complex manifold with complex structure J and $\dim_{\mathbb{C}} X = n$, and let g^{TX} be any Riemannian metric on TX compatible with J. We consider a holomorphic Hermitian vector bundle (E, h^E) on X. Let ∇^E be the holomorphic Hermitian (i.e. Chern) connection on (E, h^E) whose curvature is R^E.
Let $\overline{\partial}^E$ be the Dolbeault operator acting on $\Omega^{0,\bullet}(X, E) := \oplus_q \Omega^{0,q}(X, E)$. Then

$$(\overline{\partial}^E)^2 = 0. \tag{1.4.1}$$

The complex $(\Omega^{0,\bullet}(X, E), \overline{\partial}^E)$ is called the Dolbeault complex and its cohomology, called Dolbeault cohomology of X with values in E, is denoted by $H^{0,\bullet}(X, E)$.

By the Dolbeault isomorphism (Theorem B.4.4), $H^{0,\bullet}(X, E)$ is canonically isomorphic to the q-th cohomology group $H^q(X, \mathscr{O}_X(E))$ of the sheaf $\mathscr{O}_X(E)$ of holomorphic sections of E over X. We shortly denote $H^q(X, E) := H^q(X, \mathscr{O}_X(E))$. Especially for $q = 0$,

$$H^{0,0}(X, E) = H^0(X, \mathscr{O}_X(E)) = H^0(X, E). \tag{1.4.2}$$

Let $\overline{\partial}^{E,*}$ be the formal adjoint of $\overline{\partial}^E$ on the Dolbeault complex $\Omega^{0,\bullet}(X, E)$ with respect to the scalar product $\langle \cdot, \cdot \rangle$ in (1.3.14). Set

$$\begin{aligned} D &= \sqrt{2}\big(\overline{\partial}^E + \overline{\partial}^{E,*}\big)\,, \\ \square^E &= \overline{\partial}^E\, \overline{\partial}^{E,*} + \overline{\partial}^{E,*}\, \overline{\partial}^E. \end{aligned} \tag{1.4.3}$$

Then $\square^E$ is called the Kodaira Laplacian and

$$D^2 = 2\,\square^E. \tag{1.4.4}$$

Thus D^2 preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X,E)$. It is a fundamental result, that the elements of $\mathrm{Ker}(\square^E)$, called **harmonic forms**, represent the Dolbeault cohomology. The following Theorem follows from the more general Theorem 3.1.8 on non-compact manifolds (cf. Remark 3.1.10).

Theorem 1.4.1 (Hodge theory)**.** *If X is a compact complex manifold, then for any $q \in \mathbb{N}$, we have the following direct sum decomposition*

$$\begin{aligned} \Omega^{0,q}(X,E) &= \mathrm{Ker}(D|_{\Omega^{0,q}}) \oplus \mathrm{Im}(\square^E|_{\Omega^{0,q}}) \\ &= \mathrm{Ker}(D|_{\Omega^{0,q}}) \oplus \mathrm{Im}(\overline{\partial}^E|_{\Omega^{0,q-1}}) \oplus \mathrm{Im}(\overline{\partial}^{E,*}|_{\Omega^{0,q+1}}). \end{aligned} \tag{1.4.5}$$

Thus for any $q \in \mathbb{N}$, we have the canonical isomorphism,

$$\mathrm{Ker}(D|_{\Omega^{0,q}}) = \mathrm{Ker}(D^2|_{\Omega^{0,q}}) \simeq H^{0,q}(X,E). \tag{1.4.6}$$

Especially, $H^q(X,E) \simeq H^{0,q}(X,E)$ is finite dimensional.

Definition 1.4.2. The **Bergman kernel** of E is $P(x,x')$, $(x,x' \in X)$, the Schwartz kernel of P, the orthogonal projection from $(L^2(X,\Lambda(T^{*(0,1)}X)\otimes E), \langle\ \ \rangle)$ onto $\mathrm{Ker}(D)$, the kernel of D acting on $\Omega^{0,\bullet}(X,E) \cap L^2(X,\Lambda(T^{*(0,1)}X)\otimes E)$, with respect to the Riemannian volume form $dv_X(x')$. Especially,

$$P(x,x') \in (\Lambda(T^{*(0,1)}X)\otimes E)_x \otimes (\Lambda(T^{*(0,1)}X)\otimes E)^*_{x'}.$$

Remark 1.4.3. From Theorem 1.4.1, the Bergman kernel $P(x,x')$ is smooth on $x,x' \in X$ when X is compact. In general, by the ellipticity of D and Schwartz kernel Theorem, we know $P(x,x')$ is $\mathscr{C}^\infty$ (cf. Problem 1.5).

Recall that the tensors $S, T, \mathcal{S}, T_{as}$ were defined in (1.2.38) and (1.2.48).

Lemma 1.4.4. *For the operators $\overline{\partial}^E, (\nabla^E)^{1,0}$ acting on $\Omega^{\bullet,\bullet}(X,E)$ in (1.1.9), we have*

$$\begin{aligned} \overline{\partial}^E &= \overline{w}^j \wedge \widetilde{\nabla}^{TX}_{\overline{w}_j} + i_{T^{(0,1)}} \\ &= \overline{w}^j \wedge \widetilde{\nabla}^{TX}_{\overline{w}_j} + \frac{1}{2}\langle T(\overline{w}_j,\overline{w}_k), w_m\rangle \overline{w}^j \wedge \overline{w}^k \wedge i_{\overline{w}_m}\,, \end{aligned} \tag{1.4.7}$$

$$\begin{aligned} (\nabla^E)^{1,0} &= w^j \wedge \widetilde{\nabla}^{TX}_{w_j} + i_{T^{(1,0)}} \\ &= w^j \wedge \widetilde{\nabla}^{TX}_{w_j} + \frac{1}{2}\langle T(w_j,w_k), \overline{w}_m\rangle w^j \wedge w^k \wedge i_{w_m}\,. \end{aligned} \tag{1.4.8}$$

For the formal adjoints $\overline{\partial}^{E,*}$ *and* $(\nabla^E)^{1,0*}$ *of* $\overline{\partial}^E$ *and* $(\nabla^E)^{1,0}$ *with respect to* (1.3.14)*, we have*

$$\begin{aligned}\overline{\partial}^{E,*} &= -i_{\overline{w}_j}\widetilde{\nabla}^{TX}_{w_j} - \langle T(w_j,w_k),\overline{w}_k\rangle i_{\overline{w}_j}\\ &\quad + \frac{1}{2}\langle T(w_j,w_k),\overline{w}_m\rangle \overline{w}^m \wedge i_{\overline{w}_k}\wedge i_{\overline{w}_j}\,,\end{aligned}\tag{1.4.9}$$

$$\begin{aligned}(\nabla^E)^{1,0*} &= -i_{w_j}\widetilde{\nabla}^{TX}_{\overline{w}_j} - \langle T(\overline{w}_j,\overline{w}_k),w_k\rangle i_{w_j}\\ &\quad + \frac{1}{2}\langle T(\overline{w}_j,\overline{w}_k),w_m\rangle w^m \wedge i_{w_k} i_{w_j}.\end{aligned}\tag{1.4.10}$$

Proof. The operator $\overline{\partial}^E$ on E is given by

$$\overline{\partial}^E = \sum_{i=1}^n \overline{w}^i \wedge \nabla^E_{\overline{w}_i}.\tag{1.4.11}$$

We still denote by $\widetilde{\nabla}^{TX}$ the connection $\widetilde{\nabla}^{TX}\otimes 1 + 1\otimes\nabla^E$ and by i_T the operator $i_T\otimes 1$ on $\Lambda^{\bullet,\bullet}(T^*X)\otimes E$. From (1.2.44), we deduce

$$\nabla^E = \varepsilon\circ\widetilde{\nabla}^{TX} + i_T\,.\tag{1.4.12}$$

Relations (1.2.37) and (1.4.12) imply (1.4.7) and (1.4.8), by decomposition after bidegree and the definition of T. Observe that from (1.2.38), the $(0,1)$ and $(1,0)$-components of $\mathcal{S}$ are

$$\begin{aligned}\mathcal{S}^{(0,1)} &= \Big(\Big\langle \widetilde{\nabla}^{TX}_{w_i}\overline{w}_i, w_j\Big\rangle - \big\langle \nabla^{TX}_{e_k}e_k, w_j\big\rangle\Big)\overline{w}_j,\\ \mathcal{S}^{(1,0)} &= \Big(\Big\langle \widetilde{\nabla}^{TX}_{\overline{w}_i}w_i, \overline{w}_j\Big\rangle - \big\langle \nabla^{TX}_{e_k}e_k, \overline{w}_j\big\rangle\Big)w_j.\end{aligned}\tag{1.4.13}$$

Let $s_1,s_2\in\Omega_0^{\bullet,\bullet}(X,E)$ and let α be the $(0,1)$-form on X given for any vector field $U=U^{(1,0)}\oplus U^{(0,1)}\in T^{(1,0)}X\oplus T^{(0,1)}X$ on X, by $\alpha(U) = -\langle i_{U^{(0,1)}}s_1, s_2\rangle_{\Lambda^{\bullet,\bullet}\otimes E}$. Note that from (1.2.6),

$$\operatorname{Tr}(\nabla\alpha) = w_j\alpha(\overline{w}_j) + \overline{w}_j\alpha(w_j) - \alpha(\nabla^{TX}_{e_k}e_k).\tag{1.4.14}$$

Proceeding as in the proof of (1.3.17), (1.4.13) and (1.4.14) entail the following relation between pointwise scalar products:

$$\begin{aligned}\langle s_1, \overline{w}^i\widetilde{\nabla}^{TX}_{\overline{w}_i}s_2\rangle_{\Lambda^{\bullet,\bullet}\otimes E,x} &= -\langle i_{\overline{w}_i}\widetilde{\nabla}^{TX}_{w_i}s_1, s_2\rangle_{\Lambda^{\bullet,\bullet}\otimes E,x}\\ &\quad - \operatorname{Tr}(\nabla\alpha)_x + i_{\mathcal{S}^{(0,1)}}\alpha.\end{aligned}\tag{1.4.15}$$

The integral of the last term vanishes by Proposition 1.2.1, so integrating (1.4.15) and (1.2.42) over X, we infer (1.4.9).

Let β be the $(1,0)$ form on X given by $\beta(U) = -\big\langle i_{U^{(1,0)}} s_1, s_2 \big\rangle_{\Lambda^{\bullet,\bullet}\otimes E}$. Then as in (1.4.15),

$$\langle s_1, w^j \widetilde{\nabla}^{TX}_{w_j} s_2 \rangle_{\Lambda^{\bullet,\bullet}\otimes E,x} = -\langle i_{w_j} \widetilde{\nabla}^{TX}_{\overline{w}_j} s_1, s_2 \rangle_{\Lambda^{\bullet,\bullet}\otimes E,x} - \operatorname{Tr}(\nabla\beta)_x + i_{\mathcal{S}^{(1,0)}}\beta. \quad (1.4.16)$$

Integration of (1.4.16) and (1.2.42) gives (1.4.10). $\square$

In this Section, in the definition (1.3.15) of the spin^c Dirac operator D^c, we choose $\nabla^{\det}$ to be the holomorphic Hermitian connection on $\det(T^{(1,0)}X)$. Consequently D is a modified Dirac operator.

Theorem 1.4.5. *We have the following identity*

$$D = D^c - \frac{1}{4}{}^c(T_{as}). \quad (1.4.17)$$

Proof. In view of (1.3.1), (1.4.7) and (1.4.9), we have

$$\begin{aligned} \sqrt{2}\,\overline{\partial}^E &= c(w_i)\widetilde{\nabla}^{TX}_{\overline{w}_i} - \frac{1}{4} c(w_i)c(w_j)c(T(\overline{w}_i,\overline{w}_j)),\\ \sqrt{2}\,\overline{\partial}^{E,*} &= c(\overline{w}_i)\widetilde{\nabla}^{TX}_{w_i} + \frac{\sqrt{2}}{2}\langle T(w_i,w_j),\overline{w}_k\rangle i_{\overline{w}_j} i_{\overline{w}_i} \wedge \overline{w}^k\\ &= c(\overline{w}_i)\widetilde{\nabla}^{TX}_{w_i} + \frac{1}{4} c(\overline{w}_j)c(\overline{w}_i)c(T(w_i,w_j)). \end{aligned} \quad (1.4.18)$$

Taking into account (1.4.3) and (1.4.18), we get

$$\begin{aligned} D =& c(w_i)\widetilde{\nabla}^{TX}_{\overline{w}_i} + c(\overline{w}_i)\widetilde{\nabla}^{TX}_{w_i}\\ &- \frac{1}{4} c(w_i)c(w_j)c(T(\overline{w}_i,\overline{w}_j)) - \frac{1}{4} c(\overline{w}_i)c(\overline{w}_j)c(T(w_i,w_j)). \end{aligned} \quad (1.4.19)$$

Let $\Gamma^{T^{(1,0)}X} \in T^*X \otimes \operatorname{End}(T^{(1,0)}X)$ be the connection form of $\nabla^{T^{(1,0)}X}$ associated to the frames $\{w_j\}$. Note that for the frame $\{\overline{w}^{j_1}\wedge\cdots\wedge\overline{w}^{j_k}, 1 \leqslant j_1 < \cdots < j_k \leqslant n\}$,

$$\begin{aligned} \widetilde{\nabla}^{TX} &= d + \big\langle \Gamma^{T^{(1,0)}X} w_l, \overline{w}_m \big\rangle\, \overline{w}^m \wedge i_{\overline{w}_l},\\ \Gamma^{\det} &= \operatorname{Tr}[\Gamma^{T^{(1,0)}X}]. \end{aligned} \quad (1.4.20)$$

Comparing with (1.2.38), (1.3.3), (1.3.5), we obtain

$$\widetilde{\nabla}^{TX} = \nabla^{\mathrm{Cl}} + \frac{1}{4}\sum_{ij} \big\langle S(\cdot)e_i, e_j \big\rangle\, c(e_i)c(e_j). \quad (1.4.21)$$

Clearly, by (1.2.38),

$$\frac{1}{4}\Big(\big\langle S(e_i)e_i, e_j\big\rangle\,(c(e_i))^2 c(e_j) + \big\langle S(e_i)e_j, e_i\big\rangle\, c(e_i)c(e_j)c(e_i)\Big) = -\frac{1}{2} c(\mathcal{S}). \quad (1.4.22)$$

Thus (1.2.39), (1.4.21), (1.4.22) imply

$$\begin{aligned} & c(w_i)\widetilde{\nabla}^{TX}_{\overline{w}_i} + c(\overline{w}_i)\widetilde{\nabla}^{TX}_{w_i} \\ & \quad = D^c - \frac{1}{2}c(\mathcal{S}) + \frac{1}{4}\sum_{j\neq i\neq k}\big\langle S(e_i)e_j, e_k\big\rangle\, c(e_i)c(e_j)c(e_k) \\ & \quad = D^c - \frac{1}{2}c(\mathcal{S}) + \frac{1}{4}{}^c(T_{as}). \end{aligned} \tag{1.4.23}$$

Using (1.2.42), we get

$$\begin{aligned} & \frac{1}{4}c(\overline{w}_i)c(\overline{w}_j)c(T(w_i,w_j)) + \frac{1}{4}c(w_i)c(w_j)c(T(\overline{w}_i,\overline{w}_j)) \\ & \quad = \frac{1}{4}\big\langle T(e_i,e_j),e_k\big\rangle c(e_i)c(e_j)c(e_k) = \frac{1}{2}{}^c(T_{as}) - \frac{1}{2}c(\mathcal{S}). \end{aligned} \tag{1.4.24}$$

Finally (1.4.19), (1.4.23) and (1.4.24) imply (1.4.17). □

When X is compact, the Euler number $\chi(X,E)$ of the holomorphic vector bundle E is defined by

$$\chi(X,E) = \sum_{q=0}^{n}(-1)^q \dim H^q(X,E). \tag{1.4.25}$$

From Theorems 1.3.9, 1.4.1, 1.4.5, we obtain:

Theorem 1.4.6 (Riemann-Roch-Hirzebruch Theorem). *If X is compact, then*

$$\chi(X,E) = \int_X \mathrm{Td}(T_hX)\ \mathrm{ch}(E). \tag{1.4.26}$$

1.4.2 Bismut's Lichnerowicz formula for $\square^E$

Recall that the Bismut connection ∇^B preserves the complex structure on TX by Lemma 1.2.10, thus, as in (1.2.43), it induces a natural connection ∇^B on $\Lambda(T^{*(0,1)}X)$ which preserves its $\mathbb{Z}$-grading. Let $\nabla^{B,\Lambda^{0,\bullet}}$, $\nabla^{B,\Lambda^{0,\bullet}\otimes E}$ be the connections on $\Lambda(T^{*(0,1)}X)$, $\Lambda(T^{*(0,1)}X)\otimes E$defined by

$$\begin{aligned} \nabla^{B,\Lambda^{0,\bullet}} &= \nabla^B + \big\langle S(\cdot)w_j,\overline{w}_j\big\rangle, \\ \nabla^{B,\Lambda^{0,\bullet}\otimes E} &= \nabla^{B,\Lambda^{0,\bullet}}\otimes 1 + 1\otimes\nabla^E. \end{aligned} \tag{1.4.27}$$

By (1.2.42), $\big\langle S(\cdot)w_j,\overline{w}_j\big\rangle$ is a purely imaginary form, thus $\nabla^{B,\Lambda^{0,\bullet}\otimes E}$ is a Hermitian connection on $\Lambda(T^{*(0,1)}X)\otimes E$ which preserves its $\mathbb{Z}$-grading. We denote by $R^{B,\Lambda^{0,\bullet}}$ the curvature of $\nabla^{B,\Lambda^{0,\bullet}}$.
By (1.2.60), (1.3.3) and (1.3.8), as in (1.4.21), we get for $U\in TX$,

$$\nabla_U^{B,\Lambda^{0,\bullet}\otimes E} = \nabla_U^{\mathrm{Cl}} + \frac{1}{2}{}^c(S^B(U)) = \nabla_U^{\mathrm{Cl}} - \frac{1}{4}{}^c(i_U T_{as}). \tag{1.4.28}$$

As in (1.3.19), we denote by $\Delta^{B,\Lambda^{0,\bullet}\otimes E}$ the Bochner Laplacian defined by $\nabla^{B,\Lambda^{0,\bullet}\otimes E}$.

Theorem 1.4.7.

$$D^2 = \Delta^{B,\Lambda^{0,\bullet}\otimes E} + \frac{r^X}{4} + {}^c\,(R^E + \frac{1}{2}\operatorname{Tr}[R^{T^{(1,0)}X}]) + \frac{\sqrt{-1}}{2}{}^c(\overline{\partial}\partial\Theta) - \frac{1}{8}|(\partial - \overline{\partial})\Theta|^2. \quad (1.4.29)$$

Proof. Let $R^{\det}$ be the curvature of the holomorphic Hermitian connection on $\det(T^{(1,0)}X)$. Then

$$R^{\det} = \operatorname{Tr}[R^{T^{(1,0)}X}]. \quad (1.4.30)$$

Theorem 1.3.7 and relations (1.2.51), (1.4.17) and (1.4.30) entail (1.4.29). □

Remark 1.4.8. If (X,Θ) is Kähler, then $\nabla^{B,E}$ coincides with $\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}$, the connection on $\Lambda(T^{*(0,1)}X)\otimes E$ induced by the holomorphic Hermitian connections $\nabla^{T^{(1,0)}X}$ and ∇^E. Moreover, $r^X = 2R^{\det}(w_i,\overline{w}_i)$. (1.4.29) reads

$$\begin{aligned} D^2 =& \Delta^{\Lambda(T^{*(0,1)}X)\otimes E} - R^E(w_j,\overline{w}_j) \\ &+ 2\Big(R^E + \frac{1}{2}\operatorname{Tr}[R^{T^{(1,0)}X}]\Big)(w_i,\overline{w}_j)\overline{w}^j \wedge i_{\overline{w}_i}. \end{aligned} \quad (1.4.31)$$

1.4.3 Bochner-Kodaira-Nakano formula

Let Θ be the real $(1,1)$-form associated to g^{TX} as in (1.2.49). We define the Lefschetz operator $L = (\Theta\wedge)\otimes 1$ on $\Lambda^{\bullet,\bullet}(T^*X)\otimes E$ and its adjoint $\Lambda = i(\Theta)$ with respect to the Hermitian product $\langle\cdot,\cdot\rangle_{\Lambda^{\bullet,\bullet}\otimes E}$ induced by g^{TX} and h^E. For $\{w_j\}_{j=1}^n$ a local orthonormal frame of $T^{(1,0)}X$, we have

$$L = \sqrt{-1}w^j \wedge \overline{w}^j\wedge, \quad \Lambda = -\sqrt{-1}i_{\overline{w}_j}i_{w_j}. \quad (1.4.32)$$

Let us define the formal adjoints $(\nabla^E)^{1,0*}$ of $(\nabla^E)^{1,0}$ and $(\nabla^E)^{0,1*} = \overline{\partial}^{E,*}$ of $(\nabla^E)^{0,1} = \overline{\partial}^E$ with respect to (1.3.14) as in Lemma 1.4.4. We use next the the supercommutator defined in (1.3.30), and we apply it on $\Omega^{\bullet,\bullet}(X,E)$ endowed with natural $\mathbb{Z}_2$-grading induced by the parity of degree.

Definition 1.4.9. The **holomorphic** and **anti-holomorphic Kodaira Laplacians** are defined by:

$$\begin{aligned} \overline{\square}^E =& \big[(\nabla^E)^{1,0}, (\nabla^E)^{1,0*}\big], \\ \square^E =& \big[\overline{\partial}^E, \overline{\partial}^{E,*}\big]. \end{aligned} \quad (1.4.33)$$

The **Hermitian torsion operator** is defined by

$$\mathcal{T} := [\Lambda, \partial\Theta] = [i(\Theta), \partial\Theta]. \quad (1.4.34)$$

Let us express now $\mathcal{T}$ in terms of the torsion T of the connection $\widetilde{\nabla}^{TX}$.

Lemma 1.4.10. *We have*

$$\mathcal{T} = \frac{1}{2}\,\langle T(w_j, w_k), \overline{w}_m\rangle \Big[2\,w^k \wedge \overline{w}^m \wedge i_{\overline{w}_j} - 2\,\delta_{jm} w^k - w^j \wedge w^k \wedge i_{w_m}\Big]. \quad (1.4.35)$$

Proof. From (1.2.48), (1.2.54) and (1.4.34), we obtain

$$\mathcal{T} = \frac{\sqrt{-1}}{2}\,\langle T(w_j, w_k), \overline{w}_m\rangle \Big\{[\Lambda, \omega^j] \wedge \omega^k \wedge \overline{w}^m + \omega^j \wedge [\Lambda, \omega^k] \wedge \overline{w}^m + \omega^j \wedge \omega^k \wedge [\Lambda, \overline{w}^m]\Big\}. \quad (1.4.36)$$

By the formula (1.4.32) for Λ, we easily get

$$[\Lambda, \omega^j] = -\sqrt{-1} i_{\overline{w}_j}\,, \quad [\Lambda, \overline{w}^m] = \sqrt{-1} i_{w_m}. \quad (1.4.37)$$

Now, (1.4.36), (1.4.37) together with $T(w_j, w_k) = -T(w_k, w_j)$ imply the desired relation (1.4.35). □

We have the following generalization of the usual Kähler identities in the presence of torsion.

Theorem 1.4.11 (generalized Kähler identities)**.**

$$\big[\,\overline{\partial}^{E,*}, L\big] = \sqrt{-1}\big((\nabla^E)^{1,0} + \mathcal{T}\big), \quad (1.4.38a)$$

$$\big[(\nabla^E)^{1,0*}, L\big] = -\sqrt{-1}\big(\overline{\partial}^E + \overline{\mathcal{T}}\big), \quad (1.4.38b)$$

$$\big[\Lambda, \overline{\partial}^E\big] = -\sqrt{-1}\big((\nabla^E)^{1,0*} + \mathcal{T}^*\big), \quad (1.4.38c)$$

$$\big[\Lambda, (\nabla^E)^{1,0}\big] = \sqrt{-1}\big(\overline{\partial}^{E,*} + \overline{\mathcal{T}}^*\big). \quad (1.4.38d)$$

Proof. Remark that the third and forth formulas are the adjoints of the first two. Thus it suffices to prove (1.4.38a), (1.4.38b). Using (1.4.9) we find

$$\big[\,\overline{\partial}^{E,*}, L\big] = [-i_{\overline{w}_i}\widetilde{\nabla}^{TX}_{w_i}, L] - \langle T(w_j, w_k), \overline{w}_k\rangle [i_{\overline{w}_j}, L] + \frac{1}{2}\langle T(w_j, w_k), \overline{w}_m\rangle [\overline{w}^m \wedge i_{\overline{w}_k} i_{\overline{w}_j}, L]\,. \quad (1.4.39)$$

By (1.4.32),

$$\big[i_{\overline{w}_j}, L\big] = -\sqrt{-1}\,w^j \wedge, \quad [i_{w_j}, L] = \sqrt{-1}\,\overline{w}^j \wedge\,. \quad (1.4.40)$$

By (1.2.52), $\widetilde{\nabla}^{TX}_{w_i} L = L \widetilde{\nabla}^{TX}_{w_i}$ so from (1.4.40)

$$[-i_{\overline{w}_j}\widetilde{\nabla}^{TX}_{w_j}, L] = -[i_{\overline{w}_j}, L]\widetilde{\nabla}^{TX}_{w_j} = \sqrt{-1}\,w^j \wedge \widetilde{\nabla}^{TX}_{w_j}\,. \quad (1.4.41)$$

By (1.4.40), we infer

$$\begin{aligned}[\overline{w}^m \wedge i_{\overline{w}_k} i_{\overline{w}_j}, L] &= \overline{w}^m \wedge \big([i_{\overline{w}_k}, L] i_{\overline{w}_j} + i_{\overline{w}_k}[i_{\overline{w}_j}, L]\big) \\ &= -\sqrt{-1}\overline{w}^m \wedge (\omega^k \wedge i_{\overline{w}_j} + i_{\overline{w}_k}\omega^j).\end{aligned} \quad (1.4.42)$$

Relations (1.4.39)-(1.4.42) yield finally

$$\begin{aligned}\big[\,\overline{\partial}^{E,*}, L\big] = \sqrt{-1}\,w^j \wedge \widetilde{\nabla}^{TX}_{w_j} + \sqrt{-1}\,\langle T(w_j, w_k), \overline{w}_k\rangle w^j \\ + \sqrt{-1}\,\langle T(w_j, w_k), \overline{w}_m\rangle w^k \wedge \overline{w}^m \wedge i_{\overline{w}_j}.\end{aligned} \tag{1.4.43}$$

Adding (1.4.8) and (1.4.35) shows that $\sqrt{-1}\big((\nabla^E)^{1,0} + \mathcal{T}\big)$ equals the right hand side of (1.4.43), hence (1.4.38a) holds.

Formula (1.4.38b) can be proved along similar lines as (1.4.38a). Alternatively, as the computation is local, we can choose a local holomorphic frame of E and using (1.4.40), we reduce the proof to the case of a trivial line bundle E. But then (1.4.38b) follows from (1.4.38a) by conjugation. □

Theorem 1.4.12 (Bochner–Kodaira–Nakano formula)**.**

$$\square^E = \overline{\square}^E + [\sqrt{-1}R^E, \Lambda] + [(\nabla^E)^{1,0}, \mathcal{T}^*] - [(\nabla^E)^{0,1}, \overline{\mathcal{T}}^*]\,. \tag{1.4.44}$$

Proof. From (1.4.38d) we deduce that $\overline{\partial}^{E,*} = -\sqrt{-1}\,[\Lambda, (\nabla^E)^{1,0}] - \overline{\mathcal{T}}^*$. Thus

$$\square^E = [\,\overline{\partial}^E, \overline{\partial}^{E,*}] = -\sqrt{-1}\,\big[\,\overline{\partial}^E, [\Lambda, (\nabla^E)^{1,0}]\big] - [\,\overline{\partial}^E, \overline{\mathcal{T}}^*]\,. \tag{1.4.45}$$

The Jacobi identity (1.3.31) implies

$$\big[\,\overline{\partial}^E, [\Lambda, (\nabla^E)^{1,0}]\big] = \big[\Lambda, [(\nabla^E)^{1,0}, \overline{\partial}^E]\big] + \big[(\nabla^E)^{1,0}, [\,\overline{\partial}^E, \Lambda]\big]\,. \tag{1.4.46}$$

Since $(\overline{\partial}^E)^2 = 0$, $((\nabla^E)^{1,0})^2 = 0$, we have

$$R^E = (\nabla^E)^2 = [(\nabla^E)^{1,0}, \overline{\partial}^E]. \tag{1.4.47}$$

Using the expression of $[\,\overline{\partial}^E, \Lambda]$ given in (1.4.38c) we find

$$\big[(\nabla^E)^{1,0}, [\,\overline{\partial}^E, \Lambda]\big] = \sqrt{-1}\,\big[(\nabla^E)^{1,0}, (\nabla^E)^{1,0*}\big] + \sqrt{-1}\,\big[(\nabla^E)^{1,0}, \mathcal{T}^*\big]\,. \tag{1.4.48}$$

Taking into account the definition of $\overline{\square}^E$ (cf. (1.4.33)), we conclude (1.4.44) from (1.4.45)–(1.4.48). □

Corollary 1.4.13. *Assume that (X, g^{TX}) is Kähler. Then*

$$\square^E = \overline{\square}^E + [\sqrt{-1}R^E, \Lambda]\,, \tag{1.4.49a}$$

$$\Delta = 2\square = 2\overline{\square}\,. \tag{1.4.49b}$$

Here $\overline{\square} := \overline{\square}^{\mathbb{C}} = \partial\partial^* + \partial^*\partial$; $\square := \square^{\mathbb{C}}$ are usual ∂-Laplacian and $\overline{\partial}$-Laplacian, $\Delta = dd^* + d^*d$ is the Bochner Laplacian on $\Lambda(T^*X)$ and d^* is the adjoint of d.

Therefore, the Hodge decomposition holds for the de Rham cohomology group $H^\bullet(X, \mathbb{C})$:

(a) $H^j(X, \mathbb{C}) \cong \oplus_{p+q=j} H^q(X, \mathscr{O}^p_X) \cong \oplus_{p+q=j} H^{p,q}(X)$,

(b) $H^{p,q}(X) \cong \overline{H^{q,p}(X)}$.

We denote here by $H^{p,q}(X) := H^{p,q}(X, \mathbb{C})$ the Dolbeault cohomology groups.

Proof. Indeed, by Theorem 1.2.8, g^{TX} is Kähler if and only if $\mathcal{T}=0$, so (1.4.49a) follows trivially from (1.4.44). By taking $E=\mathbb{C}$ with a trivial metric, we obtain $\square=\overline{\square}$. Moreover

$$\Delta=[d,d^*]=[\partial+\overline{\partial},\partial^*+\overline{\partial}^*]=\square+\overline{\square}+[\partial,\overline{\partial}^*]+[\overline{\partial},\partial^*], \tag{1.4.50}$$

and the two latter brackets vanish (Problem 1.6). By the real analogue of Theorem 1.4.1 (Hodge theory), $H^\bullet(X,\mathbb{C})\simeq \operatorname{Ker}(\Delta)$. This completes the proof. □

Theorem 1.4.14 (Nakano's inequality)**.** *For any* $s\in\Omega_0^{\bullet,\bullet}(X,E)$,

$$\begin{aligned}\frac{3}{2}\langle\square^E s,s\rangle \geqslant \big\langle[\sqrt{-1}R^E,\Lambda]s,s\big\rangle \\ -\frac{1}{2}\big(\|\mathcal{T}s\|^2_{L^2}+\|\mathcal{T}^*s\|^2_{L^2}+\|\overline{\mathcal{T}}s\|^2_{L^2}+\|\overline{\mathcal{T}}^*s\|^2_{L^2}\big).\end{aligned} \tag{1.4.51}$$

If (X,g^{TX}) *is Kähler, then*

$$\langle\square^E s,s\rangle \geqslant \big\langle[\sqrt{-1}R^E,\Lambda]s,s\big\rangle. \tag{1.4.52}$$

Proof. Let $s\in\Omega_0^{\bullet,\bullet}(X,E)$. Since

$$\begin{aligned}\langle\square^E s,s\rangle &= \|\overline{\partial}^E s\|^2_{L^2}+\|\overline{\partial}^{E,*}s\|^2_{L^2},\\ \langle\overline{\square}^E s,s\rangle &= \|(\nabla^E)^{1,0}s\|^2_{L^2}+\|(\nabla^E)^{1,0*}s\|^2_{L^2},\end{aligned} \tag{1.4.53}$$

we deduce from (1.4.44) that

$$\begin{aligned}\|\overline{\partial}^E s\|^2_{L^2}+\|\overline{\partial}^{E,*}s\|^2_{L^2} =&\|(\nabla^E)^{1,0}s\|^2_{L^2}+\|(\nabla^E)^{1,0*}s\|^2_{L^2}\\ &+\big\langle[\sqrt{-1}R^E,\Lambda]s,s\big\rangle+\big\langle[(\nabla^E)^{1,0},\mathcal{T}^*]s,s\big\rangle-\big\langle[\overline{\partial}^E,\overline{\mathcal{T}}^*]s,s\big\rangle.\end{aligned} \tag{1.4.54}$$

By the Cauchy–Schwarz inequality, we find

$$\begin{aligned}\Big|\big\langle[(\nabla^E)^{1,0},\mathcal{T}^*]s,s\big\rangle\Big| &\leqslant \frac{1}{2}\Big(\|(\nabla^E)^{1,0}s\|^2_{L^2}+\|(\nabla^E)^{1,0*}s\|^2_{L^2}+\|\mathcal{T}s\|^2_{L^2}+\|\mathcal{T}^*s\|^2_{L^2}\Big),\\ \big|\big\langle[\overline{\partial}^E,\overline{\mathcal{T}}^*]s,s\big\rangle\big| &\leqslant \frac{1}{2}\big(\|\overline{\partial}^E s\|^2_{L^2}+\|\overline{\partial}^{E,*}s\|^2_{L^2}+\|\overline{\mathcal{T}}s\|^2_{L^2}+\|\overline{\mathcal{T}}^*s\|^2_{L^2}\big).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{3}{2}\big(\|\overline{\partial}^E s\|^2_{L^2}+\|\overline{\partial}^{E,*}s\|^2_{L^2}\big) \geqslant \frac{1}{2}\big(\|(\nabla^E)^{1,0}s\|^2_{L^2}+\|(\nabla^E)^{1,0*}s\|^2_{L^2}\big)\\ +\langle[\sqrt{-1}R^E,\Lambda]s,s\rangle-\frac{1}{2}\big(\|\mathcal{T}s\|^2_{L^2}+\|\mathcal{T}^*s\|^2_{L^2}+\|\overline{\mathcal{T}}s\|^2_{L^2}+\|\overline{\mathcal{T}}^*s\|^2_{L^2}\big),\end{aligned} \tag{1.4.55}$$

whereby the conclusion. □

For the purpose of proving vanishing theorems and the spectral gap for forms of bidegree $(0,q)$ with values in a positive bundle (especially on non–compact manifolds or with boundary), we derive sometimes another form of the Bochner-Kodaira-Nakano formula. Set $\widetilde{E} = E \otimes K_X^*$ where $K_X^* = \Lambda^n(T^{(1,0)}X) = \det(T^{(1,0)}X)$. Since $K_X \otimes K_X^* \cong \mathbb{C}$, there exists a natural isometry

$$\begin{aligned} &\Psi = \sim : \Lambda^{0,q}(T^*X) \otimes E \longrightarrow \Lambda^{n,q}(T^*X) \otimes \widetilde{E}, \\ &\Psi\, s = \widetilde{s} = (w^1 \wedge \ldots \wedge w^n \wedge s) \otimes (w_1 \wedge \ldots \wedge w_n), \end{aligned} \tag{1.4.56}$$

where $\{w_j\}_{j=1}^n$ a local orthonormal frame of $T^{(1,0)}X$.

Theorem 1.4.15. *For any* $s \in \Omega^{0,\bullet}(X,E)$*, we have*

$$\begin{aligned} \square^E s = \Psi^{-1}\overline{\square}^{\widetilde{E}}\Psi s + R^{E\otimes K_X^*}(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s \\ +\Psi^{-1}(\nabla^{\widetilde{E}})^{1,0}\mathcal{T}^*\Psi s - \big[\,\overline{\partial}^E, \Psi^{-1}\overline{\mathcal{T}}^*\Psi\big]s. \end{aligned} \tag{1.4.57}$$

Proof. We apply (1.4.44) for $\widetilde{s}$:

$$\square^{\widetilde{E}}\widetilde{s} = \overline{\square}^{\widetilde{E}}\widetilde{s} + \big[\sqrt{-1}R^{\widetilde{E}}, \Lambda\big]\widetilde{s} + \big[(\nabla^{\widetilde{E}})^{1,0}, \mathcal{T}^*\big]\widetilde{s} - \big[\,\overline{\partial}^{\widetilde{E}}, \overline{\mathcal{T}}^*\big]\widetilde{s}. \tag{1.4.58}$$

Since K_X^* is a holomorphic bundle,

$$\overline{\partial}^{\widetilde{E}}\widetilde{s} = (\overline{\partial}^E s)^{\sim}\,, \quad \overline{\partial}^{\widetilde{E},*}\widetilde{s} = (\overline{\partial}^{E,*} s)^{\sim}\,, \quad \square^{\widetilde{E}}\widetilde{s} = (\square^E s)^{\sim}\,. \tag{1.4.59}$$

Hence $\Psi^{-1}\square^{\widetilde{E}}\widetilde{s} = \square^E s$. Likewise

$$\begin{aligned} &\Psi^{-1}\big[\,\overline{\partial}^{\widetilde{E}}, \overline{\mathcal{T}}^*\big]\widetilde{s} = \big[\,\overline{\partial}^E, \Psi^{-1}\overline{\mathcal{T}}^*\Psi\big]s, \\ &\Psi^{-1}\big[(\nabla^{\widetilde{E}})^{1,0}, \mathcal{T}^*\big]\widetilde{s} = \Psi^{-1}(\nabla^{\widetilde{E}})^{1,0}\mathcal{T}^*\widetilde{s}, \\ &\Psi^{-1}\overline{\square}^{\widetilde{E}}\Psi s = \Psi^{-1}(\nabla^{\widetilde{E}})^{1,0}(\nabla^{\widetilde{E}})^{1,0*}\Psi s\,. \end{aligned} \tag{1.4.60}$$

By (1.4.37) we have

$$[\sqrt{-1}R^{\widetilde{E}}, \Lambda] = R^{\widetilde{E}}(w_j, \overline{w}_k)\big(w^j \wedge i_{w_k} - i_{\overline{w}_j}\overline{w}^k \wedge\big), \tag{1.4.61}$$

thus

$$\Psi^{-1}\big[\sqrt{-1}R^{\widetilde{E}}, \Lambda\big]\widetilde{s} = R^{E\otimes K_X^*}(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s. \tag{1.4.62}$$

From(1.4.59), (1.4.60) and (1.4.62), we obtain (1.4.57). □

Remark 1.4.16. Assume now that g^{TX} is Kähler. Then $\mathcal{T} = 0$, and $\widetilde{\nabla}^{TX}$ on $\Lambda(T^{*(0,1)}X) \otimes E$ is induced by the holomorphic Hermitian connections $\nabla^{T^{(1,0)}X}$, ∇^E. On $\Omega^{0,\bullet}(X,E)$, set $\Delta^{0,\bullet} = -\sum_i(\widetilde{\nabla}^{TX}_{w_i}\widetilde{\nabla}^{TX}_{\overline{w}_i} - \widetilde{\nabla}^{TX}_{\nabla^{TX}_{w_i}\overline{w}_i})$. From (1.4.8) and (1.4.10), for $s \in \Omega^{0,\bullet}(X,E)$, we obtain $\Psi^{-1}\overline{\square}^{\widetilde{E}}\Psi s = \Delta^{0,\bullet}s$. We infer from (1.4.57):

$$\square^E s = \Delta^{0,\bullet}s + R^{E\otimes K_X^*}(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s \quad \text{for } s \in \Omega^{0,\bullet}(X,E)\,. \tag{1.4.63}$$