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An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg

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Dedicated to Professor Shiing-shen Chern on his 86th birthday

Abstract. We present a direct analytic approach to the Guillemin-Sternberg conjecture [GS] that 'geometric quantization commutes with symplectic reduction', which was proved recently by Meinrenken [M1], [M2] and Vergne [V1], [V2] et al. Besides providing a new proof of this conjecture, our methods also lead immediately to further extensions in various contexts.

0. Introduction

Let (M, ω) be a closed symplectic manifold such that there is a Hermitian line bundle L over M admitting a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}(\nabla^L)^2 = \omega$. Let J be an almost complex structure on TM so that $g^{TM}(u, v) = \omega(u, Jv)$ defines a Riemannian metric on TM. Then one can construct canonically a Spin^c-Dirac operator

$$D^L : \Omega^{0,*}(M, L) \to \Omega^{0,*}(M, L) \ , \tag{0.1}$$

which gives the finite dimensional virtual vector space

$$Q(M, L) = \Omega^{0,\mathrm{even}}(M, L) \cap \ker D^L - \Omega^{0,\mathrm{odd}}(M, L) \cap \ker D^L \ . \tag{0.2}$$

Now suppose that (M, ω) admits a Hamiltonian action of a compact connected Lie group G with Lie algebra $\mathbf{g}$. Let $\mu : M \to \mathbf{g}^*$ be the corresponding moment map. We assume $0 \in \mathbf{g}^*$ is a regular value of μ, and for simplicity, we also assume G acts on $\mu^{-1}(0)$ freely. Then $M_G = \mu^{-1}(0)/G$ is

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smooth. Furthermore, ω descends to a symplectic form ω_G on M_G. Thus one gets the Marsden-Weinstein symplectic reduction (M_G, ω_G). On the other hand, a formula due to Kostant [Ko] (cf. (1.13) in Section 1) induces a natural $\mathbf{g}$ action on L. We make the assumption that this $\mathbf{g}$ action can be lifted to a G action on L. Then this G action preserves ∇^L. One can also assume, after an integration over G if necessary, that G preserves the Hermitian metric on L, as well as the almost complex structure J and thus also the metric g^{TM}. Furthermore, the pair (L, ∇^L) descends to a pair (L_G, ∇^{L_G}) over M_G so that the corresponding curvature condition $\frac{\sqrt{-1}}{2\pi}(\nabla^{L_G})^2 = \omega_G$ holds (cf. [GS]). The G-invariant almost complex structure J also descends to an almost complex structure on TM_G. Thus one can construct the corresponding Spin^c-Dirac operator on M_G, as well as the virtual vector space $Q(M_G, L_G)$.

Since G preserves everything, it commutes with D^L. Thus $Q(M, L)$ becomes a virtual representation of G. Denote by $Q(M, L)^G$ its G-trivial component.

We can now state the geometric quantization conjecture of Guillemin-Sternberg [GS] as follows.

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Theorem 0.1. *The following identity holds,*

$$\dim Q(M, L)^G = \dim Q(M_G, L_G) \ . \tag{0.3}$$

Theorem 0.1 was first proved by Guillemin-Sternberg [GS] in the holomorphic situation when (M, ω) is Kähler. They raised the conjecture for general symplectic manifolds. When G is abelian, this conjecture was first proved by Guillemin [G] in a special case, and later in general by Meinrenken [M1] and Vergne [V1], [V2] independently. The remaining nonabelian case was proved by Meinrenken [M2] using the symplectic cut techniques of Lerman [Le]. There are also closely related papers by Duistermaat-Guillemin-Meinrenken-Wu [DGMW], where the symplectic cut techniques were applied to the circle action case, and by Jeffrey-Kirwan [JK1], where the authors prove (0.3) under some extra conditions by using the nonabelian localization formulas of Witten [W1] and Jeffrey-Kirwan [JK2]. See also the survey paper by Sjamaar [S]. In all these works, the equivariant index theorem of Atiyah-Segal-Singer [AS], which expresses the analytic equivariant index through topological data on the fixed point sets, plays essential roles.

The purpose of this paper is, among other things, to give a direct analytic proof of Theorem 0.1.

The basic point is trying to consider the problem in the framework of Morse theory. Recall that $\mathscr{H} = |\mu|^2$, the norm square of the moment map, was taken by Kirwan [K1] as a kind of Morse function in her study of the cohomology of M_G. This function and in particular its associated Hamiltonian vector field $X^{\mathscr{H}}$ were also used in an essential way by Witten in his paper [W1] on nonabelian localizations. They further appeared in the papers of Jeffrey-Kirwan [JK2], Liu [Liu], Vergne [V3] and Wu [Wu] on this subject. Here for our purpose, we deform the Spin^c-Dirac operator D^L by using the Clifford action of $X^{\mathscr{H}}$. To be more precise, we deform D^L to $D^L_T = D^L + \frac{\sqrt{-1}T}{2} c(X^{\mathscr{H}})$ and study its properties as $T \to +\infty$.

Our first main result, proved in Theorem 2.1, is that under this deformation, the proof of Theorem 0.1 can be 'localized' to arbitrary small neighborhoods of $\mu^{-1}(0)$. Now as $0 \in \mathbf{g}^*$ is a regular value of μ, one finds that $\mu^{-1}(0)$ is a nondegenerate critical submanifold of $\mathscr{H}$ in the sense of Bott. This allows one to apply almost directly the techniques of Bismut-Lebeau [BL, Sects. 8, 9] near $\mu^{-1}(0)$ to complete the proof. It is remarkable that although the techniques in [BL] do not apply to D^L_T directly, they work perfectly when restricted to G-invariant subspaces. Consequently, the influence of [BL] on the present paper can be felt in many places in the text.

If G does not act on $\mu^{-1}(0)$ freely, then M_G is an orbifold. Our methods can also be applied to get the corresponding orbifold version of Theorem 0.1, which was also proved by Meinrenken [M2] in the nonabelian case. For simplicity, we only discuss in detail the case where G acts on $\mu^{-1}(0)$ freely.

Our methods also lead immediately to further generalizations of Theorem 0.1 in various contexts. Here we mention three of them.

For the first one, let E be a Hermitian G-equivariant vector bundle over M, admitting a Hermitian G-equivariant connection. Then E descends to a Hermitian vector bundle E_G over M_G. And we can define the corresponding virtual vector spaces $Q(M, E \otimes L)$ and $Q(M_G, E_G \otimes L_G)$ respectively.

Theorem 0.2. *There exists $m_0 > 0$ such that for any integer $m \geq m_0$,*

$$\dim Q(M, E \otimes L^m)^G = \dim Q(M_G, E_G \otimes L^m_G) \ . \tag{0.4}$$

The second result may be seen as an invariance property under reduction. It holds without the existence assumption of L.

Theorem 0.3. *If $\mu^{-1}(0)$ is not empty, then we have the equality of Todd genus,* $\langle \mathrm{Td}(TM), [M] \rangle = \langle \mathrm{Td}(TM_G), [M_G] \rangle$.

For the third one, we restrict ourselves to the holomorphic situation where (M, ω) is Kähler and L is holomorphic over M. In this case, we find that the above deformation corresponds exactly to the deformation of the Dolbeault operator by certain exponentials of $\mathscr{H}$. If we denote by $H^{0,p}(M, L)$ the pth

Dolbeault cohomology and by $H^{0,p}(M,L)^G$ its G-invariant part, then what we obtain is the following refined version of Theorem 0.1.

Theorem 0.4. *The following Morse type inequality holds for any integer p,*

$$\begin{aligned}&\dim H^{0,p}(M,L)^G - \dim H^{0,p-1}(M,L)^G + \cdots + (-1)^p \dim H^{0,0}(M,L)^G \\ &\leq \dim H^{0,p}(M_G,L_G) - \dim H^{0,p-1}(M_G,L_G) + \cdots + (-1)^p \dim H^{0,0}(M_G,L_G)\ .\end{aligned} \tag{0.5}$$

Theorem 0.3 has also been obtained independently by Meinrenken and Sjamaar [MS].

This paper is organized as follows. In Section 1, we introduce the basic analytic deformation of the Spin^c-Dirac operator and prove an essential Bochner type formula for the Laplacian of the deformed operator. In Section 2, we prove the basic result which allows us to localize the proof of Theorem 0.1 to sufficiently small neighborhoods of $\mu^{-1}(0)$. Section 3 is devoted to the proof of Theorem 0.1 by using the techniques of [BL]. The final Section 4 contains various extensions of Theorem 0.1 which can be proved with little modifications from our proof of Theorem 0.1. They include all of the three theorems mentioned above as well as some further extensions.

The results presented in this paper, which were contained in the preprint [TZ2], were announced in [TZ1]. The present paper is a revision of [TZ2], following various suggestions mainly due to J.-M. Bismut and one of the referees.

1. Hamiltonian actions and a deformation of Dirac operators

In this section, we recall the basic geometric setup, construct the corresponding Spin^c-Dirac operator and introduce a deformation which is the key to our proof of Theorem 0.1. In particular, a basic Bochner type formula for the Laplacian of the deformed operator is proved.

This section is organized as follows. In a), we review the construction of Spin^c-Dirac operators on symplectic manifolds. In b), we present the basic geometric setup about Hamiltonian actions on symplectic manifolds. In c), we introduce a deformation of Spin^c-Dirac operators and prove a formula for the Laplacian of the deformed operator.

a) Spin^c-Dirac operators on symplectic manifolds

Let (M,ω) be a compact symplectic manifold. Let J be an almost complex structure on TM such that

$$g^{TM}(v,w) = \omega(v,Jw) \tag{1.1}$$

defines a Riemannian metric on TM. It is well-known that such a J always exists, and that it is unique up to homotopy (cf. [McS, Prop. 4.1]). Let

$TM_{\mathbf{C}} = TM \otimes \mathbf{C}$ denote the complexification of the tangent bundle TM. Then one has the canonical splittings

$$TM_{\mathbf{C}} = T^{(1,0)}M \oplus T^{(0,1)}M,$$
$$\wedge^{*,*}(T^*M) = \bigoplus_{i,j=0}^{\dim_{\mathbf{C}} M} \wedge^{i,j}(T^*M) \ , \tag{1.2}$$

where

$$T^{(1,0)}M = \{z \in TM_{\mathbf{C}}; Jz = \sqrt{-1}z\},$$
$$T^{(0,1)}M = \{z \in TM_{\mathbf{C}}; Jz = -\sqrt{-1}z\},$$
$$\wedge^{i,j}(T^*M) = \wedge^i(T^{(1,0)*}M) \otimes \wedge^j(T^{(0,1)*}M) \ , \tag{1.3}$$

and $\dim_{\mathbf{C}} M = \frac{1}{2}\dim M$ is the complex dimension of M.

The almost complex structure J determines a canonical Spin^c-structure on TM (cf. [LM, Appendix D]). Furthermore, with g^{TM}, the fundamental $\mathbf{Z}_2$ Spin^c-bundle is given by

$$\wedge^{0,*}(T^*M) = \wedge^{0,\text{even}}(T^*M) \oplus \wedge^{0,\text{odd}}(T^*M) \ . \tag{1.4}$$

For any $X \in TM$ whose complexification has the decomposition $X = X_1 + X_2 \in T^{(1,0)}M \oplus T^{(0,1)}M$, let $\bar{X}_1{}^* \in T^{(0,1)*}M$ be the metric dual of X_1 (cf. [BL, Sect. 5]). Then $c(X) = \sqrt{2}\bar{X}_1{}^* \wedge -\sqrt{2}i_{X_2}$ defines the canonical Clifford action of X on $\wedge^{0,*}(T^*M)$ (cf [LM, Appendix D]). It interchanges $\wedge^{0,\text{even}}(T^*M)$ and $\wedge^{0,\text{odd}}(T^*M)$.

Let λ be the complex line bundle

$$\lambda = \det(T^{(1,0)}M) \ . \tag{1.5}$$

We now temporarily assume that M is spin. In this case one can construct a square root $\lambda^{1/2}$ of λ, which together with the canonically induced Spin^c-structure on TM determine a Spin structure on TM. Let $S(TM) = S_+(TM) \oplus S_-(TM)$ be the corresponding $\mathbf{Z}_2$-graded bundle of spinors associated to (M, g^{TM}). Then, one has the following canonical identifications of Clifford modules (cf. [LM, Appendix D]),

$$S_+(TM) \otimes \lambda^{1/2} = \wedge^{0,\text{even}}(T^*M),$$
$$S_-(TM) \otimes \lambda^{1/2} = \wedge^{0,\text{odd}}(T^*M),$$
$$S(TM) \otimes \lambda^{1/2} = \wedge^{0,*}(T^*M) \ . \tag{1.6}$$

Let ∇^{TM} be the Levi-Civita connection of g^{TM}. Then ∇^{TM} together with the almost complex structure J induce via projection a canonical Hermitian connection $\nabla^{T^{(1,0)}M}$ on $T^{(1,0)}M$. This, in turn, induces a Hermitian connection ∇^{λ} on λ and thus a Hermitian connection $\nabla^{\lambda^{1/2}}$ on $\lambda^{1/2}$.

Also, ∇^{TM} lifts to a Hermitian connection $\nabla^{S(TM)}$ on $S(TM)$ preserving $S_{\pm}(TM)$. Let $\nabla^{S(TM)\otimes\lambda^{1/2}}$ be the tensor product connection on $S(TM) \otimes \lambda^{1/2}$ defined by

$$\nabla^{S(TM)\otimes\lambda^{1/2}} = \nabla^{S(TM)} \otimes \mathrm{Id}_{\lambda^{1/2}} + \mathrm{Id}_{S(TM)} \otimes \nabla^{\lambda^{1/2}} \quad . \tag{1.7}$$

Then $\nabla^{S(TM)\otimes\lambda^{1/2}}$ is a well-defined Hermitian connection on $\wedge^{0,*}(T^*M) = S(TM) \otimes \lambda^{1/2}$ and preserves the $\mathbf{Z}_2$-grading. We will also denote this connection by $\nabla^{\wedge^{0,*}(T^*M)}$.

Now for the general case without the assumption that M is spin, it is well-known that although $\lambda^{1/2}$ and $S(TM)$ might not exist, one can still construct their product which does exist (cf. [LM, Appendix D]). Furthermore, one can still construct the tensor product connection as above locally and get in fact a globally well-defined connection $\nabla^{\wedge^{0,*}(T^*M)}$ on $\wedge^{0,*}(T^*M)$. In particular, when doing local computations, one can use the above identifications just as in the spin case. From now on, we will drop the spin condition on M and adopt the above convention.

Now let E be a Hermitian vector bundle over M with a Hermitian connection ∇^E. Then the tensor product connection

$$\nabla^{\wedge^{0,*}(T^*M)\otimes E} = \nabla^{\wedge^{0,*}(T^*M)} \otimes \mathrm{Id}_E + \mathrm{Id}_{\wedge^{0,*}(T^*M)} \otimes \nabla^E \tag{1.8}$$

defines a Hermitian connection on $\wedge^{0,*}(T^*M) \otimes E$.

Denote by $\Omega^{0,*}(M,E)$ the set of smooth sections of $\wedge^{0,*}(T^*M) \otimes E$.

Let $\mathrm{e}_1, \ldots, \mathrm{e}_{\dim M}$ be an orthonormal base of TM.

Definition 1.1. *The Spin^c-Dirac operator D^E is defined by*

$$D^E = \sum_{j=1}^{\dim M} c(e_j)\nabla_{e_j}^{\wedge^{0,*}(T^*M)\otimes E} : \Omega^{0,*}(M,E) \to \Omega^{0,*}(M,E) \quad . \tag{1.9}$$

Clearly, D^E is a formally self-adjoint first order elliptic differential operator on $\Omega^{0,*}(M,E)$. Let $Q(M,E)$ be the virtual vector space defined by

$$Q(M,E) = \Omega^{0,\mathrm{even}}(M,E) \cap \ker D^E - \Omega^{0,\mathrm{odd}}(M,E) \cap \ker D^E \quad . \tag{1.10}$$

b) Hamiltonian group actions and the norm square of the moment map

From now on through the end of Section 3, we assume that there is a Hermitian line bundle L over M with a Hermitian connection ∇^L such that

$$\frac{\sqrt{-1}}{2\pi}(\nabla^L)^2 = \omega \quad . \tag{1.11}$$

Now suppose that (M,ω) admits a Hamiltonian action of a compact connected Lie group G with Lie algebra $\mathbf{g}$. Let $\mu : M \to \mathbf{g}^*$ be the corresponding moment map. Then for any $V \in \mathbf{g}^*$, one has by definition[1]

$$i_V\omega = d\langle\mu, V\rangle \quad . \tag{1.12}$$

[1] In this paper we will use the same notation for V and its induced vector field on M.

The following Kostant formula [Ko] defines an action of **g** on sections of L,

$$L_V s = \nabla_V^L s - 2\pi\sqrt{-1}\langle \mu, V\rangle s, \quad s \in \Gamma(L), \quad V \in \mathbf{g} \ . \tag{1.13}$$

We make the basic assumption that this **g** action can be lifted to a G action on L. From (1.11)–(1.13), one sees easily that this G action preserves ∇^L. After an integration over G when necessary, we can also assume that G preserves the Hermitian metric on L, the almost complex structure J and thus also the Riemannian metric g^{TM}.

Let **g** (and thus $\mathbf{g}^*$ also) be equipped with an AdG-invariant metric. Let $\mathscr{H} = |\mu|^2$ be the norm square of the moment map μ. Then $\mathscr{H}$ is a G-invariant function on M. In particular, its Hamiltonian vector field, denoted by $X^{\mathscr{H}}$, is G-invariant. The following formula for $X^{\mathscr{H}}$ is clear,

$$X^{\mathscr{H}} = -J(d\mathscr{H})^* \ . \tag{1.14}$$

Let $h_1, \ldots, h_{\dim G}$ be an orthonormal base of $\mathbf{g}^*$. Then μ has the expression

$$\mu = \sum_{i=1}^{\dim G} \mu_i h_i \ , \tag{1.15}$$

where each μ_i is a real valued function on M. Denote by V_i the Killing vector field on M induced by the dual of h_i.

By (1.15), one has

$$\mathscr{H} = |\mu|^2 = \sum_{i=1}^{\dim G} \mu_i^2 \ . \tag{1.16}$$

Also by (1.12) one has for each $1 \le i \le \dim G$ that,

$$i_{V_i}\omega = d\mu_i \ . \tag{1.17}$$

From (1.17) and (1.1), one finds that

$$J(d\mu_i)^* = -V_i \ . \tag{1.18}$$

From (1.14), (1.16) and (1.18) one obtains that

$$X^{\mathscr{H}} = -2J\sum_{i=1}^{\dim G} \mu_i (d\mu_i)^* = 2\sum_{i=1}^{\dim G} \mu_i V_i \ . \tag{1.19}$$

By (1.19) and (1.15), it is clear that $\mu^{-1}(0)$ is contained in the set of critical points of $\mathscr{H}$. An important fact is that there might be other critical points of $\mathscr{H}$ as well. This is the first difficulty which would make a proof of Theorem 0.1 non-trivial. Another important observation, which is essential for the analysis near $\mu^{-1}(0)$, is that by (1.19), $X^{\mathscr{H}}$ lies in the space of vector fields generated by **g** over M.

c) A deformation of Dirac operators and a Bochner type formula

For an integer m, let L^m be the m^{th} tensor power of the line bundle L.

In using the same notation and assumptions as in a) and b), we now introduce a deformation which is fundamental to this paper.

Definition 1.2. *For any $T \in \mathbf{R}$, let $D_T^{L^m}$ be the operator defined by*

$$D_T^{L^m} = D^{L^m} + \frac{\sqrt{-1}T}{2} c(X^{\mathscr{H}}) : \Omega^{0,*}(M, L^m) \to \Omega^{0,*}(M, L^m) \ . \qquad (1.20)$$

Clearly, $D_T^{L^m}$ is a formally self-adjoint first order elliptic differential operator. Furthermore, since G preserves everything and $X^{\mathscr{H}}$ is G-invariant, one sees that $D_T^{L^m}$ is G-equivariant.

Remark 1.3. The Hamiltonian vector field $X^{\mathscr{H}}$, or its dual one form, has played an important role in Witten's paper on nonabelian localizations [W1]. In some sense the above deformation might be viewed as a quantized version of the deformation used in [W1]. A similar deformation has also been used by Vergne [V2] on the symbol level.

Remark 1.4. When J is integrable so that (M, ω) is Kähler, one verifies that

$$D_T^{L^m} = \sqrt{2}\left(\mathrm{e}^{-T|\mu|^2/2}\bar{\partial}^{L^m}\mathrm{e}^{T|\mu|^2/2} + \mathrm{e}^{T|\mu|^2/2}\left(\bar{\partial}^{L^m}\right)^*\mathrm{e}^{-T|\mu|^2/2}\right) \ . \qquad (1.21)$$

In view of Kirwan [K1] and Witten [W2] (see also Mathai-Wu [MW]), one may regard $D_T^{L^m}$ as a Morse theoretic deformation of D^{L^m}.

Now for any $V \in \mathbf{g}$, denote by L_V the infinitesimal action induced by V on the corresponding vector bundles. Also, we will omit the superscripts L^m, TM, etc., from the context, when there will be no confusion.

Lemma 1.5. *The following formula for operators acting on $\Omega^{0,*}(M, L^m)$ holds,*

$$L_V = \nabla_V - 2m\pi\sqrt{-1}\langle\mu, V\rangle - \frac{1}{4}\sum_{j=1}^{\dim M} c(\mathrm{e}_j)c(\nabla_{e_j}V) - \frac{1}{2}\mathrm{Tr}\left[\nabla_{\cdot}^{T^{(1,0)}M}V|_{T^{(1,0)}M}\right] \ . \qquad (1.22)$$

Proof. Since the formula to be proved is purely local, we may well assume that both $S(TM)$ and $\lambda^{1/2}$ are well-defined. As G preserves everything, it is clear that the identifications of Clifford modules in (1.6) are all G-equivariant. In particular, G preserves $S(TM)$, $\lambda^{1/2}$ and the associated connections. On the other hand, recall that the Lie derivative L_V on TM is given by

$$L_V X = \nabla_V^{TM} X - \nabla_X^{TM} V, \quad X \in \Gamma(TM) \ . \qquad (1.23)$$

Thus the following formulas of infinitesimal actions are clear,

$$L_V|_{S(TM)} = \nabla_V^{S(TM)} - \sum_{j,k=1}^{\dim M} \frac{1}{4} \langle \nabla_{e_j}^{TM} V, e_k \rangle c(e_j) c(e_k),$$
$$L_V|_{\lambda^{1/2}} = \nabla_V^{\lambda^{1/2}} - \frac{1}{2} \mathrm{Tr}[\nabla_\cdot^{T^{(1,0)}M} V|_{T^{(1,0)}M}] \ . \tag{1.24}$$

(1.22) follows from (1.24) and the Kostant formula (1.13). □

Theorem 1.6. *The following Bochner type formula holds*,

$$D_T^{L^m,2} = D^{L^m,2} + \frac{\sqrt{-1}T}{4} \sum_{j=1}^{\dim M} c(e_j) c(\nabla_{e_j}^{TM} X^{\mathscr{H}}) - \frac{\sqrt{-1}T}{2} \mathrm{Tr}\left[\nabla_\cdot^{T^{(1,0)}M} X^{\mathscr{H}}|_{T^{(1,0)}M}\right]$$
$$+ \frac{T}{2} \sum_{i=1}^{\dim G} (\sqrt{-1} c(JV_i) c(V_i) + |V_i|^2) + 4m\pi T \mathscr{H}$$
$$- 2\sqrt{-1}T \sum_{i=1}^{\dim G} \mu_i L_{V_i} + \frac{T^2}{4} |X^{\mathscr{H}}|^2 \ . \tag{1.25}$$

Proof. From (1.20), one deduces easily that

$$D_T^2 = D^2 + \frac{\sqrt{-1}T}{2} \sum_{j=1}^{\dim M} (c(e_j) \nabla_{e_j} c(X^{\mathscr{H}}) + c(X^{\mathscr{H}}) c(e_j) \nabla_{e_j}) + \frac{T^2}{4} |X^{\mathscr{H}}|^2$$
$$= D^2 + \frac{\sqrt{-1}T}{2} \sum_{j=1}^{\dim M} c(e_j) c(\nabla_{e_j} X^{\mathscr{H}}) - \sqrt{-1} T \nabla_{X^{\mathscr{H}}} + \frac{T^2}{4} |X^{\mathscr{H}}|^2 \ . \tag{1.26}$$

Now from (1.16), (1.18), (1.19) and Lemma 1.5, one deduces that

$$\nabla_{X^{\mathscr{H}}} = 2 \sum_{i=1}^{\dim G} \mu_i L_{V_i} + 4m\pi\sqrt{-1}\mathscr{H} + \frac{1}{4} \sum_{j=1}^{\dim M} c(e_j) c(\nabla_{e_j} X^{\mathscr{H}})$$
$$- \frac{1}{2} \sum_{i=1}^{\dim G} c((d\mu_i)^*) c(V_i) + \frac{1}{2} \mathrm{Tr}[\nabla^{T^{(1,0)}M} X^{\mathscr{H}}]$$
$$- \sum_{i=1}^{\dim G} \sum_{j=1}^{\dim M} \left\langle \frac{1}{2} \left(1 + \frac{J}{\sqrt{-1}}\right) (i_{e_j} d\mu_i) V_i, e_j \right\rangle$$
$$= 2 \sum_{i=1}^{\dim G} \mu_i L_{V_i} + 4m\pi\sqrt{-1}\mathscr{H}$$
$$+ \frac{1}{4} \sum_{j=1}^{\dim M} c(e_j) c(\nabla_{e_j} X^{\mathscr{H}}) + \frac{1}{2} \mathrm{Tr}\left[\nabla^{T^{(1,0)}M} X^{\mathscr{H}}\right]$$

$$-\frac{1}{2}\sum_{i=1}^{\dim G} c(JV_i)c(V_i) + \frac{\sqrt{-1}}{2}\sum_{i=1}^{\dim G} |V_i|^2 \ . \tag{1.27}$$

(1.25) follows from (1.26) and (1.27). □

Corollary 1.7. *The following formula holds when restricted to the G-invariant part,* $\Omega_G^{0,*}(M, L^m)$, *of* $\Omega^{0,*}(M, L^m)$,

$$D_T^{L^m,2} = D^{L^m,2} + \frac{\sqrt{-1}T}{4}\sum_{j=1}^{\dim M} c(e_j)c(\nabla_{e_j}^{TM}X^{\mathscr{H}}) - \frac{\sqrt{-1}T}{2}\mathrm{Tr}\left[\nabla_{\cdot}^{T^{(1,0)}M}X^{\mathscr{H}}|_{T^{(1,0)}M}\right]$$
$$+\frac{T}{2}\sum_{i=1}^{\dim G}\left(\sqrt{-1}c(JV_i)c(V_i) + |V_i|^2\right) + 4m\pi T\mathscr{H} + \frac{T^2}{4}|X^{\mathscr{H}}|^2 \ . \tag{1.28}$$

Proof. (1.28) follows from Theorem 1.6 by noting that for any $s \in \Omega_G^{0,*}(M, L^m)$ and $1 \le i \le \dim G$,

$$L_{V_i}s = 0 \ . \tag{1.29}$$

□

Definition 1.8. *For any* $T \in \mathbf{R}$, *let* $F_T^{L^m}$ *be the differential operator acting on* $\Omega^{0,*}(M, L^m)$ *defined by*

$$F_T^{L^m} = D_T^{L^m,2} + 2\sqrt{-1}T\sum_{i=1}^{\dim G}\mu_i L_{V_i} \ . \tag{1.30}$$

Remark 1.9. Observe that the coefficient of T in $F_T^{L^m}$ is of order zero. This makes it possible to apply the methods and techniques in the paper of Bismut-Lebeau [BL] to our problem.

2. Localization to neighborhoods of $\mu^{-1}(0)$

In this section, we prove a key estimate which will allow us to localize our problem to sufficiently small neighborhoods of $\mu^{-1}(0)$. To do this, one must overcome some difficulties near the critical points of $\mathscr{H} = |\mu|^2$ outside of $\mu^{-1}(0)$. As we will show, the presence of the term $4m\pi T\mathscr{H}$ in (1.28) is exactly the key to resolve these difficulties.

This section is organized as follows. In a), we state the main result of this section. In b) and c), we prove the main result stated in a).

a) *An estimate outside of* $\mu^{-1}(0)$

The main result of this section can be stated as follows.

Theorem 2.1. *If $m > 0$, then for any open neighborhood U of $\mu^{-1}(0)$, there exist constants $C > 0,\ b > 0$ such that for any $T \geq 1$ and any $s \in \Omega_G^{0,*}(M, L^m)$ with* Supp $s \subset M \backslash U$, *one has the following estimate of Sobolev norms,*

$$\left\| D_T^{L^m} s \right\|_0^2 \ \geq \ C\Big(\|s\|_1^2 + (T-b)\|s\|_0^2 \Big) \ . \tag{2.1}$$

First of all note that if $m_0 > 0$ verifies the following inequality over $M \backslash U$,

$$4m_0\pi\mathscr{H} + \frac{\sqrt{-1}}{4}\sum_{j=1}^{\dim M} c(e_j)c(\nabla_{e_j}X^{\mathscr{H}}) - \frac{\sqrt{-1}}{2}\mathrm{Tr}\Big[\nabla_{\cdot}^{T^{(1,0)}M}X^{\mathscr{H}}|_{T^{(1,0)}M}\Big]$$

$$+\frac{1}{2}\sum_{i=1}^{\dim G}\Big(\sqrt{-1}c(JV_i)c(V_i) + |V_i|^2\Big) > 0 \ , \tag{2.2}$$

then Theorem 2.1 follows trivially for any $m \geq m_0$ from Corollary 1.7 and Definition 1.8. Thus what we need is a proof for those 'small' m's. Without loss of generality we now assume $m = 1$.

b) A local estimate around each point outside of $\mu^{-1}(0)$

The proof of Theorem 2.1 is divided into two steps. The first step is to prove the following result.

Proposition 2.2. *For any $x \in M \backslash \mu^{-1}(0)$, there exists an open neighborhood U_x of x such that there exist $C_x > 0,\ b_x > 0$ such that for any $T \geq 1$ and any $s \in \Omega^{0,*}(M, L)$ with* Supp $s \subset U_x$, *one has*

$$\langle F_T^L s, s\rangle \geq C_x(\|s\|_1^2 + (T - b_x)\|s\|_0^2) \ . \tag{2.3}$$

The rest of this subsection is devoted to a proof of Proposition 2.2, which is divided into two cases: x is or isn't a critical point of $\mathscr{H}$.

Case 1. x is not a critical point of $\mathscr{H}$. In this case, $X^{\mathscr{H}}(x) \neq 0$. (2.3) then follows trivially from (1.28) and (1.30).

Case 2. x is a critical point of $\mathscr{H}$. This is the more difficult case. Let $e_1, \ldots, e_{\dim M}$ be an orthonormal base of TM. Let

$$\Delta = \sum_{j=1}^{\dim M}\Big(\nabla_{e_j}^2 - \nabla_{\nabla_{e_j}^{TM}e_j}\Big) \tag{2.4}$$

be the Bochner Laplacian acting on $\Omega^{0,*}(M, L)$. From the Lichnerowicz formula for $D^{L,2}$ ([Li], [LM, Appendix D]), one finds

$$D^{L,2} = -\Delta + O(1) \ . \tag{2.5}$$

From (1.28), (1.30) and (2.5), one gets,

$$F_T^L = -\Delta + \frac{\sqrt{-1}T}{4}\sum_{j=1}^{\dim M} c(e_j)c(\nabla_{e_j}X^{\mathscr{H}}) - \frac{\sqrt{-1}T}{2}\mathrm{Tr}\Big[\nabla_{\cdot}^{T^{(1,0)}M}X^{\mathscr{H}}|_{T^{(1,0)}M}\Big]$$
$$+\frac{T}{2}\sum_{i=1}^{\dim G}\Big(\sqrt{-1}c(JV_i)c(V_i) + |V_i|^2\Big) + 4\pi T\mathscr{H} + \frac{T^2}{4}|X^{\mathscr{H}}|^2 + O(1)\ . \tag{2.6}$$

Now let $f_1, \cdots, f_{\dim M}$ be an orthonormal base of T_xM. Let $(y_1, \cdots, y_{\dim M})$ be the normal coordinate system with respect to $\{f_j\}_{j=1}^{\dim M}$ near x. Clearly, one can choose $f_1, \cdots, f_{\dim M}$ so that $\mathscr{H} = |\mu|^2$ has the following expression near x,

$$\mathscr{H}(y) = \mathscr{H}(x) + \sum_{j=1}^{\dim M} a_j y_j^2 + O(|y|^3)\ , \tag{2.7}$$

where the a_j's may possibly be zero.

Lemma 2.3. *The following inequality holds at the point x,*

$$\frac{\sqrt{-1}}{4}\sum_{j=1}^{\dim M} c(e_j)c(\nabla_{e_j}X^{\mathscr{H}}) - \frac{\sqrt{-1}}{2}\mathrm{Tr}\Big[\nabla_{\cdot}^{T^{(1,0)}M}X^{\mathscr{H}}|_{T^{(1,0)}M}\Big] \geq -\sum_{j=1}^{\dim M}|a_j|\ , \tag{2.8}$$

where the inequality is strict if at least one of the a_j's is negative.

Proof. For any $e \in TM$, write its complexification as $\mathrm{e} = \mathrm{e}^{1,0} + \mathrm{e}^{0,1}$ with $\mathrm{e}^{1,0} \in T^{(1,0)}M$, $\mathrm{e}^{0,1} \in T^{(0,1)}M$. Then one deduces easily that,

$$c(e)c(Je) = \sqrt{-1}\Big(|e|^2 - 4i_{e^{0,1}}\overline{e^{1,0}}^*\wedge\Big) = \sqrt{-1}(4\overline{e^{1,0}}^*\wedge i_{e^{0,1}} - |e|^2)\ , \tag{2.9}$$

where $\overline{e^{1,0}}^* \in T^{(0,1)*}M$ is the metric dual of $e^{1,0}$ as in Section 1a).

Now let $e_1, \ldots, \mathrm{e}_{\dim M}$ be an orthonormal base of TM near x so that $e_j = f_j,\ 1 \leq j \leq \dim M$, at x. From (2.7) and (1.14), one finds that

$$X^{\mathscr{H}} = -2\sum_{j=1}^{\dim M} h_j(y)Je_j \tag{2.10}$$

with each

$$h_j(y) = a_j y_j + O(|y|^2)\ . \tag{2.11}$$

From (2.9)–(2.11), one deduces that at the point x,

$$\frac{\sqrt{-1}}{4}\sum_{j=1}^{\dim M} c(e_j)c(\nabla_{e_j}X^{\mathscr{H}}) - \frac{\sqrt{-1}}{2}\mathrm{Tr}\Big[\nabla_{\cdot}^{T^{(1,0)}M}X^{\mathscr{H}}|_{T^{(1,0)}M}\Big]$$

$$= -\frac{\sqrt{-1}}{2}\sum_{j=1}^{\dim M} a_j c(f_j)c(Jf_j) - \sqrt{-1}\sum_{j=1}^{\dim M}\left\langle \frac{1}{2}\left(1+\frac{J}{\sqrt{-1}}\right)(-a_j Jf_j), f_j\right\rangle$$

$$= -2\sum_{j=1}^{\dim M} a_j i_{f_j^{0,1}} \overline{f_j^{1,0}}^{*} \wedge \geq -\sum_{j=1}^{\dim M} |a_j| \ , \tag{2.12}$$

where the last inequality is strict if at least one of the a_j's is negative. □

On the other hand, from (2.9) one finds that for any $1 \leq i \leq \dim G$,

$$\sqrt{-1}c(JV_i)c(V_i) + |V_i|^2 \geq 0 \ . \tag{2.13}$$

One also gets from (2.10) that

$$|X^{\mathscr{H}}(y)|^2 = 4\sum_{j=1}^{\dim M} h_j(y)^2 \ . \tag{2.14}$$

From (2.6), (2.8), (2.13) and (2.14), one finds that near x,

$$F_T^L \geq -\Delta - T\sum_{j=1}^{\dim M} |a_j| + 4\pi T\mathscr{H}(x) + T^2\sum_{j=1}^{\dim M} h_j^2 + O(1+T|y|) \ . \tag{2.15}$$

Now let $\alpha > 0$, which will be further fixed later, be sufficiently small so that the orthonormal base $\{e_j\}_{j=1}^{\dim M}$ is well defined over

$$B_\alpha(x) = \{y \in M;\ d(y,x) < \alpha\} \ . \tag{2.16}$$

For any $1 \leq j \leq \dim M$, let $(\nabla_{e_j})^*$ be the formal adjoint of ∇_{e_j} on $B_\alpha(x)$. Set

$$-\Delta_T = \sum_{j=1}^{\dim M} ((\nabla_{e_j})^* + T(\operatorname{sgn} a_j)h_j)(\nabla_{e_j} + T(\operatorname{sgn} a_j)h_j) \ . \tag{2.17}$$

Clearly, $-\Delta_T$ is nonnegative when acting on compactly supported sections over $B_\alpha(x)$. Furthermore, with (2.11) one verifies easily that

$$-\Delta_T = -\Delta - T\sum_{j=1}^{\dim M} |a_j| + T^2\sum_{j=1}^{\dim M} h_j^2 + O(\partial + 1 + T|y|) \ , \tag{2.18}$$

where by $O(\partial + 1 + T|y|)$ we mean a first order differential operator

$$\sum_{j=1}^{\dim M} b_j(y)\frac{\partial}{\partial y_j} + c(y) \tag{2.19}$$

with

$$b_j(y) = O(1), \quad c(y) = O(1 + T|y|) \ . \tag{2.20}$$

We will also use similar notation for other operators.

From (2.15), (2.17) and (2.18), one deduces that, when acting on sections with compact support in $B_\alpha(x)$,

$$\begin{aligned} F_T^L &\geq -\Delta_T + 4\pi T \mathscr{H}(x) + O(\partial + 1 + T|y|) \\ &\geq -\frac{1}{k}\Delta - \frac{T}{k}\sum_{j=1}^{\dim M} |a_j| + 4\pi T\mathscr{H}(x) + O(\partial + 1 + T|y|) \end{aligned} \tag{2.21}$$

for any $k \geq 1$.

Now for any $s \in \Omega^{0,*}(M, L)$ with Supp $s \subset B_\alpha(x)$, it is standard that

$$\langle -\Delta s, s\rangle \geq C_1 \|s\|_1^2 - C_2\|s\|_0^2 \tag{2.22}$$

for some constants $C_1 > 0,\ C_2 > 0$. Also, by Cauchy inequality, one has

$$\langle O(\partial)s, s\rangle \leq C_3\alpha\|s\|_1^2 + \frac{C_4}{\alpha}\|s\|_0^2 \tag{2.23}$$

for some constants $C_3 > 0,\ C_4 > 0$. Finally, one verifies the obvious estimate

$$\langle O(1 + T|y|)s, s\rangle \leq C_5(1 + T\alpha)\|s\|_0^2 \tag{2.24}$$

for some constant $C_5 > 0$. From (2.21) through (2.24) one gets

$$\begin{aligned} \langle F_T^L s, s\rangle \geq& \left(\frac{C_6}{k} - C_7\alpha\right)\|s\|_1^2 + T\left(4\pi\mathscr{H}(x) - C_8\alpha - \frac{C_9}{k}\right)\|s\|_0^2 \\ &- \left(\frac{C_{10}}{\alpha} + C_{11}\right)\|s\|_0^2 \end{aligned} \tag{2.25}$$

for some constants $C_i > 0,\ i = 6, \ldots, 11$.

Now since $\mu(x) \neq 0$, we first choose k large enough so that

$$4\pi\mathscr{H}(x) - \frac{C_9}{k} > \frac{C_9}{k} \ . \tag{2.26}$$

Then choose α small enough so that

$$\frac{C_6}{k} - C_7\alpha > 0, \quad \frac{C_9}{k} - C_8\alpha > 0 \ . \tag{2.27}$$

With this choice of α, by (2.25)–(2.27), we get (2.3) at x.

The proof of Proposition 2.2 is completed. □

c) The global estimate outside of $\mu^{-1}(0)$

In this subsection we prove Theorem 2.1 by gluing together the estimate (2.3) outside of $\mu^{-1}(0)$. This follows for example from an easy trick used in [BL, pp. 115–117]. To be more precise, since U is open in M, $M\backslash U$ is compact. So by the finite covering principle and by Proposition 2.2, there is a finite number of open subsets $\{U_i\}_{i=1}^{l}$ so that $\cup_i U_i \supset M\backslash U$ and that (2.3) holds on each U_i, $i = 1, \ldots, l$. The next step, as in [BL], is to construct a family of smooth functions $\{\phi_i\}$ so that $\{\phi_i^2\}$ forms a partition of unity subordinating to the family of open subsets $\{U_i\}$. Then by proceeding exactly the same as in [BL, pp. 115–117], one sees that (2.3) actually holds for any section with compact support in $M\backslash U$.

Now for any $s \in \Omega_G^{0,*}(M, L)$ with $\mathrm{Supp}\, s \subset M\backslash U$, one has by (1.29) and (1.30) that,

$$\left\langle D_T^{L,2} s, s \right\rangle = \left\langle F_T^L s, s \right\rangle \ . \tag{2.28}$$

Theorem 2.1 follows from (2.28) and the discussions above. □

Remark 2.4. As one might have noticed, one of the key points to the proof of Theorem 2.1 is the inequality (2.26) at each critical point $x \in M\backslash\mu^{-1}(0)$ of $\mathscr{H}$. If at each such x, at least one of the a_j's is negative, then by Lemma 2.3 we have a strict inequality in (2.8). This allows us to replace (2.26) by a similar inequality without appealing to $\mathscr{H}$ and get a proof of Theorem 2.1 for the case of $m = 0$. In particular, the existence of L is no longer necessary.

3. A proof of Theorem 0.1

In this section, we prove Theorem 0.1. As will be seen, Theorem 2.1 enables us to localize the problem to arbitrary small neighborhoods of $\mu^{-1}(0)$. Now since $0 \in \mathbf{g}^*$ is a regular value of the moment map μ, $\mu^{-1}(0)$ is a nondegenerate critical submanifold, in the sense of Bott, of $\mathscr{H} = |\mu|^2$. Thus near $\mu^{-1}(0)$, we have a situation closely related to what is considered in the paper of Bismut and Lebeau [BL, Sects. 8, 9]. It turns out that the methods and techniques in [BL] can be applied here directly, with little modifications, to complete the proof of Theorem 0.1.

A surprising feature is that the resulting Dirac type operator on $\mu^{-1}(0)/G$ through this procedure is not identical to D^{L_G}.

The present version of this section differs from that in [TZ2]. Here we follow a suggestion, first proposed to us by Bismut, to consider the canonical G-principal fibration on any sufficiently small G-invariant neighborhood of $\mu^{-1}(0)$ and then apply [BL] to the base manifold to get the result. Similar suggestion was also proposed to us by one of the referees.

This section is organized as follows. In a), we review Theorem 0.1 for convenience. In b), we examine the canonical G-principal fibration structure near $\mu^{-1}(0)$. In c), we identify the restriction of D_T^L on G-invariant sections to an operator on the base manifold. In the self-contained subsection d), we prove a basic result concerning the Spin^c-Dirac operator and the harmonic oscillator on a flat space. In e), we apply [BL] to our situation to give a proof of Theorem 0.1. Finally in f), we present an explicit formula for the Dirac type operator, obtained through this procedure, on $\mu^{-1}(0)/G$.

a) Symplectic reduction and the quantization formula

From now on, we assume that $0 \in \mathbf{g}^*$ is a regular value of the moment map μ. Then $\mu^{-1}(0)$ is a closed submanifold of M, and is invariant under the action of G. For simplicity, we assume that G acts on $\mu^{-1}(0)$ freely so that we have a canonical principal fibration with smooth base $M_G = \mu^{-1}(0)/G$,

$$G \to \mu^{-1}(0) \xrightarrow{\pi} M_G = \mu^{-1}(0)/G \ . \tag{3.1}$$

Denote by $i : \mu^{-1}(0) \hookrightarrow M$ the canonical embedding. Then $i^*\omega$ is a G-invariant closed two form on $\mu^{-1}(0)$. It descends to a symplectic form ω_G on M_G such that

$$\pi^*\omega_G = i^*\omega \ . \tag{3.2}$$

On the other hand, L descends through i^*L to a Hermitian line bundle L_G over M_G. Moreover, by the Kostant formula (1.13), ∇^L induces a Hermitian connection ∇^{L_G} on L_G such that

$$\frac{\sqrt{-1}}{2\pi}(\nabla^{L_G})^2 = \omega_G \ , \tag{3.3}$$

(cf. [GS]). Let D^{L_G} be the Spin^c-Dirac operator acting on $\Omega^{0,*}(M_G, L_G)$ and $Q(M_G, L_G)$ the associated virtual vector space defined by (1.10). Now since D^L is G-equivariant, $Q(M, L)$ is a virtual representation of G. Denote by $Q(M, L)^G$ its G-invariant part.

The purpose of this section is to give an analytic proof of Theorem 0.1, which we recall for convenience as follows.

Theorem 3.1. *The following identity holds,*

$$\dim Q(M, L)^G = \dim Q(M_G, L_G) \ . \tag{3.4}$$

Our proof of Theorem 3.1 is based on the following simple observation. Since D_T^L is G-equivariant, for any $T \in \mathbf{R}$, one has

$$\dim Q(M, L)^G = \dim(\ker D_T^L) \cap \Omega_G^{0,\mathrm{even}}(M, L) - \dim(\ker D_T^L) \cap \Omega_G^{0,\mathrm{odd}}(M, L). \tag{3.5}$$

One then takes $T \to +\infty$ and examines the behavior of the eigenvalues of $D_T^L|_{\Omega_G^{0,*}(M,L)}$ under this limit. The result in Section 2 allows us to reduce the problem to sufficiently small neighborhoods of $\mu^{-1}(0)$, on which the problem will be worked out in this section.

b) A principal fibration near $\mu^{-1}(0)$

Since G acts on $\mu^{-1}(0)$ freely, it is clear that there exists a sufficiently small G-invariant open neighborhood U of $\mu^{-1}(0)$ such that G acts on U freely. Thus U/G is a smooth open manifold.

Now consider the canonical G-principal fibration

$$G \to U \xrightarrow{\pi} U/G. \tag{3.6}$$

Let E (resp. $T^H U$) be the vertical (resp. horizontal) tangent bundle of the fibration (3.6), determined by the Levi-Civita connection $\nabla^{TU} = \nabla^{TM}|_U$ of the Riemannian metric $g^{TU} = g^{TM}|_U$ on U. Then one has the following orthogonal splittings of the tangent bundle TU and the corresponding metric,

$$TU = E \oplus T^H U, \quad g^{TU} = g^E \oplus g^{T^H U} \ . \tag{3.7}$$

It is clear that the above constructions are G-equivariant. This in particular determines a Riemannian metric $g^{T(U/G)}$ on U/G so that one has the following identifications,

$$\pi^* T(U/G) = T^H U, \quad \pi^* g^{T(U/G)} = g^{T^H U} \ . \tag{3.8}$$

c) The restriction of D_T^L *to the G-invariant part and the induced Dirac operator on* U/G

In this subsection, we identify the restriction of D_T^L on $\Omega_G^{0,*}(U,L)$, the G-invariant part of $\Omega^{0,*}(U,L)$, with a Dirac type operator on U/G.

If F is a G-equivariant Hermitian vector bundle over U, then it induces canonically a Hermitian vector bundle $F_{U/G}$ over U/G such that $\pi^* F_{U/G} = F$. We will denote by $\pi_G^F : \Gamma_G(F) \to \Gamma(F_{U/G})$ the canonical isomorphism which maps a G-invariant section of F to the corresponding section of $F_{U/G}$. We will usually omit the superscript F from π_G^F when there will be no confusion in the context. Also, if ∇^F is a G-equivariant Hermitian connection on F, then one verifies easily that

$$\nabla_e^{F_{U/G}} s := \pi_G \nabla_{\pi^* e}^F \pi_G^{-1} s, \quad e \in \Gamma(T(U/G)), \quad s \in \Gamma(F_{U/G}) \ , \tag{3.9}$$

where $\pi^* e \in T^H U$ is determined by the identification (3.8), gives rise to a Hermitian connection $\nabla^{F_{U/G}}$ on $F_{U/G}$.

Now let h be the smooth positive function on U/G defined by

$$h(x) = \sqrt{\mathrm{vol}(\pi^{-1}(x))}, \quad x \in U/G \ , \tag{3.10}$$

the square root of the volume of the fibered group over x.

Definition 3.2. *Let R be the bounded operator defined by*

$$R = h\pi_G : \Omega_G^{0,*}(U, L) \to \Gamma((\wedge^{0,*}(T^*U) \otimes L)_{U/G}) \ . \tag{3.11}$$

Clearly, the map R is an isometry. We now try to write $R D_T^L R^{-1}$ explicitly as a Dirac type operator on U/G. We start with the following easy but essential observation.

Lemma 3.3. *If $e_1, \ldots, e_{\dim G}$ is an orthonormal base of the vertical tangent bundle E, then $\tilde{B} := \sum_{i=1}^{\dim G} c(e_i) \nabla_{e_i}^{\wedge^{0,*}(T^*U)\otimes L}$ is a bounded operator, when acting on $\Omega_G^{0,*}(U, L)$.*

Proof. We need only to show that each $\nabla_{e_i}^{\wedge^{0,*}(T^*U)\otimes L}$ is bounded. Clearly, the Killing vector fields V_i, $1 \le i \le \dim G$, are linearly independent over U and span the vertical tangent bundle E. Thus each e_i, $1 \le i \le \dim G$, can be expressed through a linear combination of the V_i's with smooth functions as coefficients. Lemma 3.3 then follows from (1.29) and the trivial fact that each $L_{V_i} - \nabla_{V_i}^{\wedge^{0,*}(T^*U)\otimes L}$ is a bounded operator on $\Omega^{0,*}(U, L)$. □

If S is an endomorphism of $\wedge^{0,*}(T^*U) \otimes L$, we will denote by

$$S_{U/G} = \pi_G S \pi_G^{-1} \tag{3.12}$$

its induced endomorphism over U/G.

Let $f_1, \ldots, f_{\dim U/G}$ be an orthonormal base of $T(U/G)$. For simplicity, we use the following notation for Clifford actions on the corresponding vector bundles over U/G,

$$c(f_j) = \pi_G c(\pi^* f_j) \pi_G^{-1}, \quad 1 \le j \le \dim U/G \ . \tag{3.13}$$

Definition 3.4. *Let $D_{U/G}^L$ be the differential operator acting on $\pi_G \Omega_G^{0,*}(U, L) = \Gamma((\wedge^{0,*}(T^*U) \otimes L)_{U/G})$ defined by*

$$D_{U/G}^L = \sum_{j=1}^{\dim U/G} c(f_j) \nabla_{f_j}^{(\wedge^{0,*}(T^*U)\otimes L)_{U/G}} \ . \tag{3.14}$$

Proposition 3.5. *The following identity holds for operators acting on $\pi_G \Omega_G^{0,*}(U, L)$,*

$$R D^L R^{-1} = D_{U/G}^L - \frac{1}{h} c((dh)^*) + \tilde{B}_{U/G} \ . \tag{3.15}$$

Proof. (3.15) follows from Definition 1.1, Lemma 3.3, (3.10), (3.14) and a direct verification. □

On the other hand, one knows from (1.19) that the G-invariant Hamiltonian vector field $X^{\mathscr{H}}$ lies in $\Gamma(E)$ over U. The induced Clifford action $\pi_G c(X^{\mathscr{H}})\pi_G^{-1}$ acts on $\pi_G\Omega_G^{0,*}(U,L)$. We denote this action by $c(\pi_G X^{\mathscr{H}})$. From Proposition 3.5, Definition 1.2 and the fact that R is an isometry, one gets

Corollary 3.6. *The following identity holds for the formally self-adjoint operators acting on* $\pi_G\Omega_G^{0,*}(U,L)$,

$$RD_T^L R^{-1} = D_{U/G}^L - \frac{1}{h}c((dh)^*) + \tilde{B}_{U/G} + \frac{\sqrt{-1}T}{2}c(\pi_G X^{\mathscr{H}}) \ . \tag{3.16}$$

Remark 3.7. As a simple but important observation, one sees from (1.19) that $c(\pi_G X^{\mathscr{H}})$ anticommutes with every $c(f_j)$, $1 \leq j \leq \dim U/G$.

Remark 3.8. As another important observation from (1.19), one sees that if U is small enough, then there exists a constant $C > 0$ such that $|X^{\mathscr{H}}|^2 \geq C\mathscr{H}$ on U. When pushing everything down to U/G, one then gets on U/G that

$$-c(\pi_G X^{\mathscr{H}})^2 \geq C\mathscr{H}_{U/G} \ . \tag{3.17}$$

(3.17) is a natural analogue of [BL, Prop. 8.14]. By this and by Corollary 3.6 and Remark 3.7, one sees that the operator $RD_T^L R^{-1}$ on U/G is of the same nature as the operator $D^X + TV$ studied in [BL]. Thus one is able to apply the results in [BL] almost directly to our situation.

d) Spin^c-Dirac operators and harmonic oscillators

The result in this subsection will play an important role in our proof of Theorem 3.1, in the same way as [BL, Theorem 7.4] plays in [BL, Sect. 9]. This subsection is otherwise self-contained.

Let $W = V \oplus V'$ be a $\mathbf{Z}_2$-vector space such that $\dim V = \dim V'$. We assume V and V' be Euclidean with the Euclidean metrics g^V, $g^{V'}$ respectively. Then W carries the orthogonal direct sum metric. Let $J \in \mathrm{End}^{\mathrm{odd}}(W)$ be an isometry so that $J^2 = -\mathrm{Id}$. We assume the existence of J. Then J defines a complex structure on W. Let $f_1, \ldots, f_{\dim V}$ be an orthonormal base of V, then $Jf_1, \ldots, Jf_{\dim V}$ is an orthonormal base of V'. They together form an orthonormal base of W.

We consider $\wedge^{0,*}(T^*W)$ as a vector bundle over V. Let ∇ be the flat covariant derivative acting on $\Gamma(\wedge^{0,*}(T^*W))$. Set

$$D^V = \sum_{i=1}^{\dim V} c(f_i)\nabla_{f_i} \ : \ \Gamma(\wedge^{0,*}(T^*W)) \to \Gamma(\wedge^{0,*}(T^*W)) \ . \tag{3.18}$$

Let $a_i > 0,\ 1 \le i \le \dim V$ be positive constants. Let $(y_1, \ldots, y_{\dim V})$ be the coordinate system associated to $f_1, \ldots, f_{\dim V}$. For any $T \in \mathbf{R}$, set

$$D_T^V = D^V - \sqrt{-1}T \sum_{i=1}^{\dim V} a_i y_i c(Jf_i) \ . \tag{3.19}$$

Let Δ be the Laplacian

$$\Delta = \sum_{i=1}^{\dim V} (\nabla_{f_i})^2 \ . \tag{3.20}$$

Proposition 3.9. *The following identity holds,*

$$(D_T^V)^2 = -\Delta - \sqrt{-1}T \sum_{i=1}^{\dim V} a_i c(f_i) c(Jf_i) + T^2 \sum_{i=1}^{\dim V} a_i^2 y_i^2 \ . \tag{3.21}$$

Proof. (3.21) follows from (3.19), (3.20) and a direct verification. □

Theorem 3.10. (i) *If* $T > 0$, *then the kernel of* $(D_T^V)^2$ *is one dimensional and is generated by* $\exp(-\frac{T}{2}\sum_{i=1}^{\dim V} a_i y_i^2)$. *Furthermore, there exists a positive number* $C > 0$ *such that all the nonzero eigenvalues of* $(D_T^V)^2$ *are larger than* CT; (ii) *If* $T < 0$, *then the kernel of* $(D_T^V)^2$ *is one dimensional and is generated by* $\exp(\frac{T}{2}\sum_{i=1}^{\dim V} a_i y_i^2)\ \overline{f_1^{1,0}}^* \wedge \cdots \wedge \overline{f_{\dim V}^{1,0}}^*$. *Furthermore, there exists a positive number* $C' > 0$ *such that all the nonzero eigenvalues of* $(D_T^V)^2$ *are larger than* $-C'T$.

Proof. Assume first $T > 0$. Consider the orthogonal splitting,

$$\wedge^{0,*}(T^*W) = \mathbf{C} \oplus \bigoplus_{i=1}^{\dim V} \wedge^{0,i}(T^*W) \ . \tag{3.22}$$

From Proposition 3.9 and (2.9), one sees easily that when restricted to $\mathbf{C} = \wedge^{0,0}(T^*W)$,

$$(D_T^V)^2 = -\Delta - T \sum_{i=1}^{\dim V} a_i + T^2 \sum_{i=1}^{\dim V} a_i^2 y_i^2 \ , \tag{3.23}$$

while when restricted to $\oplus_{i=1}^{\dim V} \wedge^{0,*} (T^*W)$,

$$(D_T^V)^2 \ge -\Delta - T \sum_{i=1}^{\dim V} a_i + T^2 \sum_{i=1}^{\dim V} a_i^2 y_i^2 + 2T \min\{a_i; 1 \le i \le \dim V\} \ . \tag{3.24}$$

Now by the standard properties of harmonic oscillators, $-\Delta - T\sum_{i=1}^{\dim V} a_i + T^2 \sum_{i=1}^{\dim V} a_i^2 y_i^2$ is a nonnegative operator with a one dimensional kernel generated by $\exp\left(-\frac{T}{2}\sum_{i=1}^{\dim V} a_i y_i^2\right)$ and all the nonzero eigenvalues are larger than CT for some $C > 0$. Combining this fact with (3.23), (3.24), we complete the proof of the part (i). The proof of the part (ii) is entirely similar. $\square$

e) A proof of Theorem 3.1

Let N_G be the normal bundle to $M_G = \mu^{-1}(0)/G$ in U/G. Let $i_G : M_G \hookrightarrow U/G$ denote the canonical isometric embedding. Then one has the obvious orthogonal splittings

$$\begin{aligned} T(U/G)|_{M_G} &= N_G \oplus TM_G, \\ g^{T(U/G)}|_{M_G} &= g^{N_G} \oplus g^{TM_G} \ . \end{aligned} \tag{3.25}$$

On the other hand, if we denote by J_G the canonically induced almost complex structure on $(TU)_{U/G}|_{M_G} = (i^* TU)_{U/G}$,

$$J_G = \pi_G i^* J = i_G^* J_{U/G} \ , \tag{3.26}$$

then from (1.1) and (1.12) one verifies easily that

$$J_G N = (i^* E)_{U/G} \ . \tag{3.27}$$

From (3.25)–(3.27) one gets easily the orthogonal splittings

$$\begin{aligned} (TU)_{U/G}|_{M_G} &= N_G \oplus J_G N_G \oplus TM_G, \\ g^{(TU)_{U/G}}|_{M_G} &= g^{N_G} \oplus g^{J_G N_G} \oplus g^{TM_G} \ , \end{aligned} \tag{3.28}$$

with J_G preserves $N_{G,J} = N_G \oplus J_G N_G$ and TM_G. Thus one can construct canonically the Hermitian vector bundles $N_{G,J}^{(1,0)}$ and $T^{(1,0)}M_G$ etc, which further give the canonical identification of Hermitian vector bundles,

$$\wedge^{0,*}(T^*U)_{U/G}|_{M_G} = \wedge^{0,*}(N_{G,J}^*) \hat{\otimes} \wedge^{0,*} (T^*M_G) \ . \tag{3.29}$$

Let $L_G = L_{U/G}|_{M_G}$ be the Hermitian vector bundle over M_G with the Hermitian connection $\nabla^{L_G} = i_G^* \nabla^{L_{U/G}}$. Then one verifies easily that (cf. [GS])

$$\frac{\sqrt{-1}}{2\pi} (\nabla^{L_G})^2 = \omega_G \ . \tag{3.30}$$

Definition 3.11. *Let D^{L_G} be the canonical Spin^c Dirac operator acting on $\Omega^{0,*}(M_G, L_G)$ defined in the same way as in Definition* 1.1 *from $J_G|_{TM_G}$, g^{TM_G} and the pair (L_G, ∇^{L_G}).*

On the other hand, from (3.14), (3.29) one can define a differential operator D^H acting on $\Gamma(\wedge^{0,*}(N^*_{G,J})\hat{\otimes} \wedge^{0,*} (T^*M_G) \otimes L_G)$ defined by

$$D^H = \sum_{j=1}^{\dim U/G} c(f_j)(i_G^* \nabla^{(\wedge^{0,*}(T^*U)\otimes L)_{U/G}})(f_j) \ , \tag{3.31}$$

where we use the same notation for the f_j's and their restriction on M_G. Clearly, if we assume that when restricted to M_G, $f_1, \dots, f_{\dim M_G}$ is an orthonormal base of TM_G, then one has

$$D^H = \sum_{j=1}^{\dim M_G} c(f_j)(i_G^* \nabla^{(\wedge^{0,*}(T^*U)\otimes L)_{U/G}})(f_j) \ . \tag{3.32}$$

Let p be the canonical orthogonal projection

$$p : \wedge^{0,*}(N^*_{G,J})\hat{\otimes} \wedge^{0,*} (T^*M_G) \otimes L_G \to \wedge^{0,*}(T^*M_G) \otimes L_G \ , \tag{3.33}$$

which acts as identity on $\wedge^{0,0}(N^*_{G,J})\otimes \wedge^{0,*} (T^*M_G) \otimes L_G \simeq \wedge^{0,*}(T^*M_G) \otimes L_G$ and maps each $\wedge^{0,i}(N^*_{G,J}) \otimes \wedge^{0,*}(T^*M_G) \otimes L_G$, $i \geq 1$, to zero.

Let $\tilde{M} \in \mathrm{End}(\wedge^{0,*}(T^*M_G) \otimes L_G)$ be defined by

$$\tilde{M} = p((\tilde{B}_{U/G} - \frac{1}{h} c((dh)^*))|_{M_G})p \ . \tag{3.34}$$

Definition 3.12. *The differential operator $D_Q^{L_G}$ acting on $\Omega^{0,*}(M_G, L_G)$ is defined by*

$$D_Q^{L_G} = pD^H p + \tilde{M} \ . \tag{3.35}$$

Clearly, $D_Q^{L_G}$ is a first order elliptic differential operator.

We can now apply the techniques and results of Bismut-Lebeau [BL, Sects. 8, 9] to complete the proof of Theorem 3.1.

In fact, Theorem 2.1, which is the analogue of [BL, Prop. 9.13], allows us to localize the problem to an arbitrary small neighborhood U of $\mu^{-1}(0)$. By (3.6) and Corollary 3.6, one can further reduce the problem to the study of the operator $RD_T^L R^{-1}$ on U/G, to which the arguments in [BL] can be applied directly (see Remarks 3.7 and 3.8). In particular, the part (i) of Theorem 3.10 plays exactly the same role as [BL, Theorem 7.14] plays in [BL], while (3.31)–(3.35), Definitions 3.4, 3.11, Corollary 3.6 and again the part (i) of Theorem 3.10 give rise to the limiting operator $D_Q^{L_G}$ on M_G (compare with [BL, (8.94) and (9.20)]).

In doing all these, note that here we are no longer in the holomorphic situation as in [BL], where many estimates have actually taken the advantage of the holomorphic properties there. However, it is easy to see that this does not cause any trouble in modifying the estimates in [BL] so that all the arguments can actually go through here. As a typical example, we do not have direct analogues of [BL, (9.68), (9.69)] with zero on the right hand side, but with $O(|Z|)$ and $O(|Z|\partial + 1)$ respectively instead. And one sees easily how to modify the estimates in [BL] to work for our situation.

To summarize, by proceeding in exactly the same way as in [BL, Sects. 8, 9], we get the following analogue of [BL, (9.156)].

Theorem 3.13. (i) *The operator $D_Q^{L_G}$ is a formally self-adjoint Dirac type operator which has the same principal symbol as that of D^{L_G}*; (ii) *There exist $C > 0$, $T_0 > 0$ such that there are no nonzero eigenvalues of $D_Q^{L_G,2}$ in $[0, C]$, and that for any $T \geq T_0$, the number of eigenvalues of $D_T^{L,2}|_{\Omega_G^{0,*}(M,L)}$ in $[0, C]$ is equal to the dimension of the kernel of $D_Q^{L_G}$.*

Now it is a simple fact that the positive and negative eigenvalues of $D_T^L|_{\Omega_G^{0,*}(M,L)}$ are in 1–1 correspondence to each other. Thus one gets from Theorem 3.13 that

$$\begin{aligned}\dim(\ker D_T^L) \cap \Omega_G^{0,\mathrm{even}}(M,L) - \dim(\ker D_T^L) \cap \Omega_G^{0,\mathrm{odd}}(M,L)& \\ = \mathrm{ind}\, D_Q^{L_G}|_{\Omega^{0,\mathrm{even}}(M_G,L_G)}\ .& \end{aligned} \tag{3.36}$$

(3.4) then follows from (3.5), (3.36) and the trivial identity

$$\mathrm{ind}\, D^{L_G}|_{\Omega^{0,\mathrm{even}}(M_G,L_G)} = \mathrm{ind}\, D_Q^{L_G}|_{\Omega^{0,\mathrm{even}}(M_G,L_G)}\ . \tag{3.37}$$

The proof of Theorem 3.1 is completed. □

Remark 3.14. In [BL], one can go further to obtain an equality between the kernels of Dirac type operators (cf. [BL, (9.161)]). Since here we have no analogue of [BL, Theorem 1.7], the non-zero small eigenvalues appearing in Theorem 3.13 may well exist.

f) An explicit formula for $D_Q^{L_G}$

We present a more explicit formula for $D_Q^{L_G}$ by evaluating the terms pD^Hp and $\tilde{M}$ in (3.35) separately.

Proposition 3.15. *The following identity holds for operators acting on $\Omega^{0,*}(M_G, L_G)$,*

$$pD^Hp = D^{L_G}\ . \tag{3.38}$$

Proof. Let us go back to the embedding $i : \mu^{-1}(0) \hookrightarrow U$. Let N be the normal bundle to $\mu^{-1}(0)$ in U. Then by (1.1) and (1.12) one finds that $E|_{\mu^{-1}(0)} = JN$. Thus one has the canonical orthogonal splittings

$$\begin{aligned} TU|_{\mu^{-1}(0)} &= N \oplus JN \oplus \pi^*(TM_G), \\ g^{TU}|_{\mu^{-1}(0)} &= g^N \oplus g^{JN} \oplus \pi^* g^{TM_G}\ , \end{aligned} \tag{3.39}$$

with J preserves $N_J = N \oplus JN$ and $W_J = \pi^*(TM_G)$. Thus one can construct the Hermitian vector bundles $N_J^{(1,0)}$, $W_J^{(1,0)}$ which in turn give the canonical identification of Hermitian vector bundles

$$\wedge^{0,*}(T^*U)|_{\mu^{-1}(0)} = \wedge^{0,*}(N_J^*) \hat{\otimes} \wedge^{0,*}(W_J^*) \ . \tag{3.40}$$

Let P (resp. $P^\perp$) be the orthogonal projection from $TU|_{\mu^{-1}(0)}$ to N_J (resp. W_J). Set

$$\nabla^{N_J} = Pi^*\nabla^{TU}P,$$
$$\nabla^{W_J} = P^\perp i^*\nabla^{TU}P^\perp \ , \tag{3.41}$$

and

$$A = i^*\nabla^{TU} - \nabla^{N_J} - \nabla^{W_J} \ . \tag{3.42}$$

Then ∇^{N_J} (resp. ∇^{W_J}) is a Euclidean connection on N_J (resp. W_J), while A is a one form taking values in skew-endomorphisms on $TU|_{\mu^{-1}(0)}$ which exchange N_J and W_J.

The connections ∇^{N_J}, ∇^{W_J} and the almost complex structure J induce canonically the Hermitian connections $\nabla^{\det(N_J^{(1,0)})}$, $\nabla^{\det(W_J^{(1,0)})}$ on $\det(N_J^{(1,0)})$, $\det(W_J^{(1,0)})$ respectively. Also, recall that the Hermitian line bundle λ has been defined in (1.5). The following identification of Hermitian line bundles is clear,

$$i^*\lambda = \det(N_J^{(1,0)}) \otimes \det(W_J^{(1,0)}) \ . \tag{3.43}$$

By (3.41) and (3.42), one further has the corresponding identification of Hermitian connections

$$\nabla^{i^*\lambda} = i^*\nabla^\lambda = \nabla^{\det(N_J^{(1,0)})} \otimes \mathrm{Id}_{\det(W_J^{(1,0)})} + \mathrm{Id}_{\det(N_J^{(1,0)})} \otimes \nabla^{\det(W_J^{(1,0)})} \ . \tag{3.44}$$

Now the almost complex structure J induces canonical Spin^c structures on N_J, W_J respectively (cf. [LM, Appendix D]). Thus one has the following identifications of Clifford modules analogous to those in (1.6) (cf. [LM, Appendix D]),

$$\wedge^{0,*}(N_J^*) = S(N_J) \otimes \det{}^{1/2}(N_J^{(1,0)}) \ ,$$
$$\wedge^{0,*}(W_J^*) = S(W_J) \otimes \det{}^{1/2}(W_J^{(1,0)}) \ . \tag{3.45}$$

Furthermore, by proceeding similarly as in Section 1a) and [LM, Appendix D], one constructs from ∇^{N_J}, $\nabla^{\det(N_J^{(1,0)})}$ (resp. ∇^{W_J}, $\nabla^{\det(W_J^{(1,0)})}$) the canonical Hermitian connection $\nabla^{\wedge^{0,*}(N_J^*)}$ (resp. $\nabla^{\wedge^{0,*}(W_J^*)}$) on $\wedge^{0,*}(N_J^*)$ (resp. $\wedge^{0,*}(W_J^*)$). We denote by ${}^0\nabla^{(\wedge^{0,*}(T^*U)\otimes L)|_{\mu^{-1}(0)}}$ the tensor product connection of $\nabla^{\wedge^{0,*}(N_J^*)}$, $\nabla^{\wedge^{0,*}(W_J^*)}$ and $i^*\nabla^L$.

From the definitions of these connections and from (3.41), (3.42) and (3.44), one verifies easily that if $e_1, \ldots, e_{\dim \mu^{-1}(0)}$ is an orthonormal base of

$T\mu^{-1}(0)$ such that $e_j \in W_J$ for $1 \leq j \leq \dim M_G$, then one has for any $1 \leq i \leq \dim \mu^{-1}(0)$,

$$\nabla_{e_i}^{\wedge^{0,*}(T^*U)\otimes L}\big|_{\mu^{-1}(0)} = {}^0\nabla_{e_i}^{(\wedge^{0,*}(T^*U)\otimes L)|_{\mu^{-1}(0)}} + \frac{1}{2}\sum_{s=1}^{\dim M_G}\sum_{t=\dim M_G+1}^{\dim \mu^{-1}(0)} \langle A(e_i)e_s, e_t\rangle c(e_s)c(e_t) + \frac{1}{2}\sum_{s=1}^{\dim M_G}\sum_{t=\dim M_G+1}^{\dim \mu^{-1}(0)} \langle A(e_i)e_s, Je_t\rangle c(e_s)c(Je_t) \ . \tag{3.46}$$

Now one verifies directly that

$$p\pi_G \sum_{i=1}^{\dim M_G} c(e_i)\,{}^0\nabla_{e_i}^{(\wedge^{0,*}(T^*U)\otimes L)|_{\mu^{-1}(0)}} \pi_G^{-1} p = D^{L_G} \ , \tag{3.47}$$

and that

$$p\pi_G c(e_i)c(e_s)c(e_t)\pi_G^{-1}p = p\pi_G c(e_i)c(e_s)c(Je_t)\pi_G^{-1}p = 0 \tag{3.48}$$

for $1 \leq i \leq \dim M_G$, $1 \leq s \leq \dim M_G$ and $\dim M_G + 1 \leq t \leq \dim \mu^{-1}(0)$.

(3.38) follows from (3.32), (3.33) and (3.46)–(3.48). □

We now compute $pi_G^*(\tilde{B}_{U/G})p$.

Set $\tilde{p} = \pi_G^{-1}p\pi_G$. From Lemma 3.3, (3.40), (3.46), (2.9) and the definition of the connection ${}^0\nabla^{(\wedge^{0,*}(T^*U)\otimes L)|_{\mu^{-1}(0)}}$, one verifies directly that

$$\begin{aligned}
\pi_G^{-1}pi_G^*(\tilde{B}_{U/G})p\pi_G &= \tilde{p}\tilde{B}\tilde{p} \\
&= \frac{1}{2}\sum_{i=\dim M_G+1}^{\dim \mu^{-1}(0)}\sum_{s=1}^{\dim M_G}\sum_{t=\dim M_G+1}^{\dim \mu^{-1}(0)} \langle A(e_i)e_s, e_t\rangle \tilde{p}c(e_i)c(e_s)c(e_t)\tilde{p} \\
&\quad + \frac{1}{2}\sum_{i=\dim M_G+1}^{\dim \mu^{-1}(0)}\sum_{s=1}^{\dim M_G}\sum_{t=\dim M_G+1}^{\dim \mu^{-1}(0)} \langle A(e_i)e_s, Je_t\rangle \tilde{p}c(e_i)c(e_s)c(Je_t)\tilde{p} \\
&= \frac{1}{2}\sum_{i=\dim M_G+1}^{\dim \mu^{-1}(0)}\sum_{s=1}^{\dim M_G} \langle A(e_i)Je_i, e_s\rangle \tilde{p}c(e_s)c(e_i)c(Je_i)\tilde{p} \\
&\quad - \frac{1}{2}\sum_{i=\dim M_G+1}^{\dim \mu^{-1}(0)} c(A(e_i)e_i) \\
&= -\frac{\sqrt{-1}}{2}\sum_{i=\dim M_G+1}^{\dim \mu^{-1}(0)} c(A(e_i)Je_i) \\
&\quad - \frac{1}{2}\sum_{i=\dim M_G+1}^{\dim \mu^{-1}(0)} c(A(e_i)e_i) \ .
\end{aligned} \tag{3.49}$$

Now set $\tilde{h} = h|_{M_G}$. One verifies directly (cf. [BGV, Ch. 10]) that

$$-\frac{1}{2}\tilde{h}\sum_{i=\dim M_G+1}^{\dim \mu^{-1}(0)} \pi_G(A(e_i)e_i) = (d\tilde{h})^* \ . \tag{3.50}$$

Combining (3.34), (3.38), (3.49), (3.50) and Definition 3.12, one gets

Theorem 3.16. *If* $b_1, \ldots, b_{\dim G}$ *is an orthonormal base of the vertical tangent bundle* $E|_{\mu^{-1}(0)}$ *of the fibration* (3.1), *then one has,*

$$D_Q^{L_G} = D^{L_G} - \frac{\sqrt{-1}}{2}\sum_{i=1}^{\dim G} c(\pi_G(A(b_i)Jb_i)) \ . \tag{3.51}$$

Remark 3.17. The fact that $D_Q^{L_G}$ and D^{L_G} might not be identical was first suggested to us by Bismut.

Now we assume that the almost complex structure J is integrable so that (M, ω) is Kähler. Then A commutes with J. By (3.50), one gets

$$\frac{\sqrt{-1}}{2}\sum_{i=1}^{\dim G} \pi_G(A(b_i)Jb_i) = \frac{\sqrt{-1}}{2}\sum_{i=1}^{\dim G} J_G\pi_G(A(b_i)b_i) = -\frac{\sqrt{-1}}{\tilde{h}}J_G(d\tilde{h})^* \ . \tag{3.52}$$

Also one verifies in this case that (M_G, ω_G) is Kähler (cf. [GS]). Furthermore, by (1.11) and (3.3) one finds that both L and L_G admit unique holomorphic structures with the Hermitian holomorphic connections ∇^L, ∇^{L_G} respectively. Thus one has the standard formula (cf. [BGV])

$$D^{L_G} = \sqrt{2}\big(\bar{\partial}^{L_G} + (\bar{\partial}^{L_G})^*\big) \ . \tag{3.53}$$

From (3.51)–(3.53), we obtain the important formula

$$D_Q^{L_G} = \sqrt{2}\big(\tilde{h}\bar{\partial}^{L_G}\tilde{h}^{-1} + \tilde{h}^{-1}(\bar{\partial}^{L_G})^*\tilde{h}\big) \ , \tag{3.54}$$

which is essential to the proof of Theorem 0.4 in Section 4d).

4. Generalizations and further comments

In this section, we present various generalizations of Theorem 0.1. All the proofs are very short and are readily given using the arguments in previous sections with little modifications.

This section is organized as follows. In a), we state and prove a generalized version of Theorem 0.1 which includes Theorem 0.2 as a special case.

We also give a proof of Theorem 0.3 in this subsection. In b), we prove a dual version of Theorem 0.1. In c), we prove the negative quantization formula for the case where G is abelian. Finally in d), we restrict ourselves to the holomorphic situation and prove certain Morse type inequalities which include Theorem 0.4 as a special case.

a) A generalized quantization formula

From now on, we no longer assume the existence of the line bundle L unless stated otherwise. The other assumptions and notation are the same as in the previous sections.

Let E be a G-equivariant Hermitian vector bundle over M admitting a G-equivariant Hermitian connection ∇^E. Then E induces canonically a Hermitian vector bundle E_G over M_G with a canonically induced Hermitian connection.

For any $V \in \mathbf{g}$, set

$$r_V^E = L_V|_E - \nabla_V^E \ . \tag{4.1}$$

By proceeding exactly the same as in Sections 1–3, one can prove the following result.

Theorem 4.1. *If at each critical point $x \in M\backslash\mu^{-1}(0)$ of $\mathscr{H}$, one has*

$$\sqrt{-1}\sum_{i=1}^{\dim G}\mu_i(x)r_{V_i}^E(x) > 0 \ , \tag{4.2}$$

then the following identity holds,

$$\dim Q(M,E)^G = \dim Q(M_G,E_G) \ . \tag{4.3}$$

Proof. The condition (4.2) simply replaces the condition $\mathscr{H}(x) > 0$ in (2.26) to make the arguments in Section 2 work in the present situation. Furthermore, the arguments in Section 3 work for general E. □

When the line bundle L exists, Theorem 0.1 follows from Theorem 4.1 by taking $E = L$. Similarly, Theorem 0.2 also follows as a special case of Theorem 4.1.

In view of Remark 2.4, if at each critical point $x \in M\backslash\mu^{-1}(0)$ of $\mathscr{H}$, at least one of the a_j's defined in (2.7) is negative, then the arguments in Section 2 still go through even if the inequality in (4.2) is replaced by the equality. A result of Kirwan (cf. [K1, 4.18, 4.20] and [K2, pp. 549]) implies that this actually happens when $\mu^{-1}(0)$ is nonempty. Thus one gets the following improved version of Theorem 4.1.

Theorem 4.2. *If* $\mu^{-1}(0)$ *is nonempty and if at each critical point* $x \in M \backslash \mu^{-1}(0)$ *of* $\mathscr{H}$,

$$\sqrt{-1} \sum_{i=1}^{\dim G} \mu_i(x) r_{V_i}^E(x) \geq 0 \ , \tag{4.4}$$

then (4.3) *holds.*

As a special case of this theorem, one can take $E = \mathbf{C}$, the trivial line bundle. The rigidity theorem for the canonical Spin^c-Dirac operator on an almost complex manifold (cf. [H]) states that

$$\dim Q(M, \mathbf{C}) = \dim Q(M, \mathbf{C})^G \ . \tag{4.5}$$

Combining these two results with the Atiyah-Singer index theorem [AS], one obtains Theorem 0.3. □

b) *A dual quantization formula*

Let E be as in a) and the line bundle λ be as defined in (1.5). Let $*_E$ be the Hodge star operator (cf. [GH, pp. 151]),

$$*_E : \wedge^{0,q}(T^*M) \otimes E \to \wedge^{0,\dim_{\mathbf{C}} M - q}(T^*M) \otimes \lambda^{-1} \otimes E^* \ . \tag{4.6}$$

Clearly $*_E$ is G-equivariant. Furthermore, one verifies directly that

$$*_E D^E *_E^{-1} : \Omega^{0,*}(M, \lambda^{-1} \otimes E^*) \to \Omega^{0,*}(M, \lambda^{-1} \otimes E^*) \tag{4.7}$$

is a G-equivariant Dirac type operator with the same principal symbol as that of $D^{\lambda^{-1} \otimes E^*}$.

From (4.6) and (4.7) one gets immediately that

$$\dim Q(M, E)^G = (-1)^{\dim_{\mathbf{C}} M} \dim Q(M, \lambda^{-1} \otimes E^*)^G,$$

$$\dim Q(M_G, E_G) = (-1)^{\dim_{\mathbf{C}} M_G} \dim Q(M_G, \lambda_G^{-1} \otimes E_G^*) \ . \tag{4.8}$$

Combining (4.8) with Theorem 4.1, one gets

Theorem 4.3. *If the condition in Theorem* 4.1 *holds*, *then*

$$\dim Q(M, \lambda^{-1} \otimes E^*)^G = (-1)^{\dim G} \dim Q(M_G, \lambda_G^{-1} \otimes E_G^*) \ . \tag{4.9}$$

As an immediate consequence, in the case that L exists, one can take $E = L^m \otimes \lambda^{-1}$ to get

Corollary 4.4. *There exists* $m_0 > 0$ *such that for any* $m \geq m_0$,

$$\dim Q(M, L^{-m})^G = (-1)^{\dim G} \dim Q(M_G, L_G^{-m}) \ . \tag{4.10}$$

Remark 4.5. Theorem 4.3 as well as Corollary 4.4 can also be proved directly using the same arguments as in Sections 2, 3 by taking $T \to -\infty$ instead of taking $T \to +\infty$. In doing so, one uses the part (ii) of Theorem 3.10 instead of the part (i) of Theorem 3.10, when proceeding the arguments in Section 3. In fact, this was the approach taken in [TZ2]. The approach we now adopt here was suggested by one of the referees.

c) *Negative quantization formula for abelian actions*

In this subsection we prove the following improved version of Corollary 4.4.

Theorem 4.6. *If G is abelian, then* (4.10) *holds for any* $m \geq 1$.

Proof. We need only to prove the case when $m = 1$. By (4.8) it is sufficient to prove (4.3) for $E = L \otimes \lambda^{-1}$. In this case, (4.2) does not hold in general. However, in the special case that $G = S^1$, one verifies that

$$\sqrt{-1}\mu r_V^E + \frac{\sqrt{-1}}{4} \sum_{j=1}^{\dim M} c(e_j) c(\nabla_{e_j} X^{\mathscr{H}}) - \frac{\sqrt{-1}}{2} \mathrm{Tr}\left[\nabla_{\cdot}^{T^{(1,0)}M} X^{\mathscr{H}}\right] > - \sum_{j=1}^{\dim M} |a_j| \ , \tag{4.11}$$

at each critical point $x \in M \backslash \mu^{-1}(0)$ of $\mathscr{H}$. This is sufficient for the arguments in Section 2 to work to give a proof of (4.9). The higher rank case then follows from a standard 'reduction in stage' procedure (cf. [V2]). □

Remark 4.7. Theorem 4.6 has also been proved in Jeffrey-Kirwan [JK1, Theorem 6.2]. Furthermore, an example due to Vergne (cf. [JK1, pp. 686]) shows that the abelian condition in Theorem 4.6 is necessary.

d) *Quantization on Kähler manifolds: holomorphic Morse inequalities*

In this subsection we restrict ourselves to the case where J is integrable. Then (M, ω) is a Kähler manifold and G acts holomorphically on M. One verifies easily (cf. [GS]) that the induced almost complex structure J_G on M_G is also integrable so that (M_G, ω_G) is Kähler.

Now let E be a G-equivariant holomorphic Hermitian vector bundle. Then its unique holomorphic Hermitian connection is also G-equivariant. These together induce canonically a holomorphic Hermitian vector bundle E_G over M_G. Thus one has formulas (1.21) and (3.54), with L^m, L_G there being replaced by E, E_G respectively.

Now note that in the holomorphic category all the arguments in Sections 1–3 preserve the **Z**-grading nature of the problem. Therefore, one actually gets the following refined version of Theorem 4.1, where one still uses the superscript G to denote the G-invariant part.

Theorem 4.8. *If* (4.2) *holds at each critical point* $x \in M \backslash \mu^{-1}(0)$ *of* $\mathscr{H}$, *then for any integer* p, *one has the following Morse type inequality for Dolbeault cohomologies,*

$$\dim H^{0,p}(M,E)^G - \dim H^{0,p-1}(M,E)^G + \cdots + (-1)^p \dim H^{0,0}(M,E)^G$$
$$\leq \dim H^{0,p}(M_G,E_G) - \dim H^{0,p-1}(M_G,E_G) + \cdots + (-1)^p \dim H^{0,0}(M_G,E_G) \ . \tag{4.12}$$

Remark 4.9. If the line bundle L exists, then one can take $E = L$ in Theorem 4.8 to get Theorem 0.4. In this case, Guillemin-Sternberg [GS] proved that there is actually an equality between the dimensions of spaces of holomorphic sections,

$$\dim H^{0,0}(M,L)^G = \dim H^{0,0}(M_G,L_G) \ . \tag{4.13}$$

Remark 4.10. As a holomorphic refinement of Theorem 0.3, it is proved in Wu-Zhang [WuZ] that if G is abelian and $\mu^{-1}(0)$ is nonempty, then one has the equality of Hodge numbers for any integer p,

$$\dim H^{0,p}(M,\mathbf{C}) = \dim H^{0,p}(M_G,\mathbf{C}_G) \ . \tag{4.14}$$

References

[AS] M.F. Atiyah, I.M. Singer: The index of elliptic operators III. Ann. of Math. **87**, 546–604 (1968)

[BGV] N. Berline, E. Getzler, M. Vergne: Heat Kernels and Dirac Operators. Grundl. der Math. Wiss. vol. 298, Springer, 1992

[BL] J.-M. Bismut, G. Lebeau: Complex immersions and Quillen metrics. Publ. Math. IHES. **74**, 1–297 (1991)

[DGMW] H. Duistermaat, V. Guillemin, E. Meinrenken, S. Wu: Symplectic reduction and Riemann-Roch for circle actions. Math. Res. Lett. **2**, 259–266 (1995)

[G] V. Guillemin: Reduced phase space and Riemann-Roch. in Lie Groups and Geometry, R. Brylinski et al. ed., Birkhäuser (Progress in Math., 123), 1995, pp. 305–334

[GH] P. Griffiths, J. Harris: Principles of Algebraic Geometry. John Wiley and Sons, 1978.

[GS] V. Guillemin, S. Sternberg: Geometric quantization and multiplicities of group representations. Invent. Math. **67**, 515–538 (1982)

[H] F. Hirzebruch: Elliptic genera of level N for complex manifolds. in Differential Geometric Methods in Theoretical Physics, Kluwer, Dordrecht, 1988, pp. 37–63

[JK1] L. C. Jeffrey, F. C. Kirwan: Localization and quantization conjecture. Topology **36**, 647–693 (1997)

[JK2] L. C. Jeffrey, F. C. Kirwan: Localization for nonabelian group actions. Topology **34**, 291–327 (1995)

[K1] F. C. Kirwan: Cohomology of Quotients in Symplectic and Algebraic Geometry. Princeton Univ. Press, Princeton, 1984

[K2] F. C. Kirwan: Convexity properties of the moment mapping, III. Invent. Math. **77**, 547–552 (1984)

[Ko] B. Kostant: Quantization and unitary representations. in Modern Analysis and Applications. Lecture Notes in Math. vol. 170, Springer-Verlag, (1970), pp. 87–207

[LM] H. B. Lawson, M.-L. Michelsohn: Spin Geometry. Princeton Univ. Press, 1989

[Le] E. Lerman: Symplectic cuts. Math. Res. Lett. **2**, 247–258 (1995).

[Li] A. Lichnerowicz, Spineurs harmoniques. C. R. Acad. Sci. Paris, Série I, **257**, 7–9 (1963)

[Liu] K. Liu: Remarks on nonabelian localization. Intern. Math. Res. Notices No. **13**, 683–691 (1995)

[M1] E. Meinrenken: On Riemann-Roch formulas for multiplicities. J. A. M. S. **9**, 373–389 (1996)

[M2] E. Meinrenken: Symplectic surgery and the Spin^c-Dirac operator. Adv. in Math. To appear

[MS] E. Meinrenken, R. Sjamaar: Singular reduction and quantization. Preprint, (dg-ga/9707023) 1996

[MW] V. Mathai, S. Wu: Equivariant holomorphic Morse inequalities I: a heat kernel proof. J. Diff. Geom. **46**, 78–98 (1997)

[McS] D. Mcduff, D. Salamon: Introduction to Symplectic Topology. Clarendon Press, Oxford, 1995

[S] R. Sjamaar: Symplectic reduction and Riemann-Roch formulas for multiplicities. Bull. A. M. S. **38**, 327–338 (1996)

[TZ1] Y. Tian, W. Zhang: Symplectic reduction and quantization. C. R. Acad. Sci. Paris, Série I, **324**, 433–438 (1997)

[TZ2] Y. Tian and W. Zhang: Symplectic reduction and analytic localization. Preprint, 1996

[V1] M. Vergne: Multiplicity formula for geometric quantization, Part I. Duke Math. J. **82**, 143–179 (1996)

[V2] M. Vergne: Multiplicity formula for geometric quantization, Part II. Duke Math. J. **82**, 181–194 (1996)

[V3] M. Vergne: A note on the Jeffrey-Kirwan-Witten localisation formula. Topology **35**, 243–266 (1996)

[W1] E. Witten: Two dimensional gauge theories revisited. J. Geom. Phys. **9**, 303–368 (1992)

[W2] E. Witten: Holomorphic Morse inequalities. in Algebraic and Differential Topology, Teubner-Texte Math. Vol. **70**, Ed. G. Rassias, Teubner (Leipzig, 1984), pp. 318–333

[Wu] S. Wu: An integration formula for the square of moment maps of circle actions. Lett. Math. Phys. **29**, 311–328 (1993)

[WuZ] S. Wu, W. Zhang: Equivariant holomorphic Morse inequalities III: non-isolated fixed points. Preprint, 1997 (dg-ga/9701009), To appear in GAFA