Chapter 2

Moment maps and symplectic reductions

In this chapter, we explain two main concepts of this book: moment map and symplectic reduction. The symplectic reduction has its original in classical Hamiltonian mechanic and was used by physicists since more that one century, it was introduced by Mardsen-Weinstein for mathematicians in 1970's, and it gives a systematical way to construct new symplectic manifolds from known symplectic manifolds. When we apply it formally to infinite dimensional spaces it gives an efficacy way to construct symplectic forms on the moduli spaces of certain geometric objects.

In Section 2.1, we give a short introduction to Lie groups, Lie algebras and Lie algebra cohomology. In particular, we establish Cartan-Chevalley-Eilenberg theorem: de Rham cohomology of a compact connected Lie group coincides with the cohomology of its Lie algebra, which is in fact at the origin of the notion of Lie algebra cohomology. In Section 2.2, we study the local structure of a group action on a manifold, so-called the slice theorem. In Section 2.3, we introduce the moment map and the symplectic reduction. The Cartan-Chevalley-Eilenberg theorem helps us to understand when a symplectic action is Hamiltonian. And the slice theorem shows the quotient space in the definition of symplectic reduction has a smooth structure. The most important point is that the symplectic form on the original manifold induces naturally a symplectic form on the reduced space. Section 2.4 is a brief introduction on symplectic cuts.

2.1 Introduction to Lie groups and Lie algebras

The aim of this section is to give a short introduction to Lie groups, Lie algebras, and Lie algebra cohomology.

2.1.1 Lie groups and Lie algebras

Set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1. Let $\mathfrak{g}$ be a $\mathbb{K}$-vector space. We call that an antisymmetric $\mathbb{K}$-bilinear form $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a Lie bracket, if it satisfies the Jacobi identity, i.e.,

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \quad \text{for any } X,Y,Z \in \mathfrak{g}. \tag{2.1.1}$$

In this case, we will call that $(\mathfrak{g}, [,])$ is a $\mathbb{K}$-Lie algebra. If $\mathfrak{k}$ is a $\mathbb{K}$-vector subspace of $\mathfrak{g}$ and $[u,v] \in \mathfrak{k}$ for any $u,v \in \mathfrak{k}$, then we call $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$.

A $\mathbb{K}$-linear map $\psi : \mathfrak{g} \to \mathfrak{h}$ of two $\mathbb{K}$-Lie algebras is called a morphism of Lie algebras if $[\psi(u), \psi(v)] = \psi([u,v])$ for any $u,v \in \mathfrak{g}$.

Here are some examples. We define the map $[\,,\,] : M_n(\mathbb{K}) \times M_n(\mathbb{K}) \to M_n(\mathbb{K})$ by

$$[A,B] = AB - BA \text{ for } A,B \in M_n(\mathbb{K}). \tag{2.1.2}$$

Then $M_n(\mathbb{K})$ is a $\mathbb{K}$-Lie algebra with Lie bracket $[\,,\,]$.

If M is a manifold, by (1.2.10), and by Exercise 1.2.2, $[,]$ is a Lie bracket on $\mathscr{C}^\infty(M,TM)$. If ω is a symplectic form on M, by Proposition 1.3.3, the Poisson algebra $(\mathscr{C}^\infty(M), \{\,,\,\})$ is a Lie algebra, and we verify that $\mathfrak{sympl}(M,\omega)$ and $\mathfrak{ham}(M,\omega)$ are also Lie algebras. Moreover, by (1.3.8), we have a morphism of Lie algebras

$$\mathscr{C}^\infty(M) \ni f \to -X_f \in \mathscr{C}^\infty(M,TM). \tag{2.1.3}$$

Definition 2.1.2. A group G is called a Lie group if it's a manifold, and the multiplication

$$(g,h) \in G \times G \to gh \in G \tag{2.1.4}$$

is a smooth map. We denote by $e \in G$ the identity element in G. If in addition that the multiplication commutes, we say G is an abelian Lie group.

A morphism $\varphi : G \to G'$ of two Lie groups G, G' is a $\mathscr{C}^\infty$ map and preserves the operations on groups, i.e., $\varphi(gh) = \varphi(g)\varphi(h)$ for any $g,h \in G$.

A Lie subgroup H of a Lie group G is an injective homomorphism of Lie groups $\varphi : H \to G$.

A torus means a Lie group isomorphic to the quotient group $\mathbb{R}^k/\mathbb{Z}^k$ for some k.

For a $\mathbb{K}$-vector space E, and a Lie group G, we call (E,ρ) a $\mathbb{K}$-representation of G, if $\rho : G \to \mathrm{GL}(E)$ is a morphism of Lie groups. The representation $\rho\colon G \to \mathrm{GL}(E)$ is said to be irreducible if any subspace of E which is stable by G is either $\{0\}$ or E.

A morphism $f : E \to F$ between representations is a linear map which is equivariant, i.e., $f(gv) = gf(v)$ for any $g \in G$ and $v \in E$.

Let G be a Lie group. For $h \in G$, the left multiplication L_h and the right multiplication R_h are defined by

$$L_h : g \in G \to hg \in G, \qquad R_h : g \in G \to gh \in G. \tag{2.1.5}$$

A vector field $X \in \mathscr{C}^\infty(G,TG)$ on G is called left (resp. right) invariant if for any $h \in G$, we have

$$L_{h*}X = X, \qquad (\text{resp. } R_{h*}X = X). \tag{2.1.6}$$

Let $\mathscr{C}^\infty(G,TG)^L$ be the space of left invariant vector fields on G. By Exercise 1.2.2, if X, $Y \in \mathscr{C}^\infty(G,TG)^L$, then $[X,Y] \in \mathscr{C}^\infty(G,TG)^L$. Thus $\mathscr{C}^\infty(G,TG)^L$ is a Lie algebra.

We denote by $\mathfrak{g} = T_eG$ the tangent space of G at e.

Proposition 2.1.3. *The linear maps*

$$\psi : X \in \mathfrak{g} \to X_L = ((dL_h)_e X)_{h\in G} \in \mathscr{C}^\infty(G, TG)^L \tag{2.1.7}$$

is well-defined, and is a bijection. In particular, for $X, Y \in \mathfrak{g}$, we define

$$[X, Y] = \psi^{-1}[X_L, Y_L], \tag{2.1.8}$$

then (2.1.8) *defines a Lie algebra structure on $\mathfrak{g}$, and we call $\mathfrak{g}$ the Lie algebra of G.*

Proof. For $W \in \mathscr{C}^\infty(G, TG)$, then $L_{g*}W = W$ for any $g \in G$ is equivalent to

$$(dL_g)(W(g^{-1}h)) = W(h) \quad \text{for } g, h \in G, \quad \text{i.e., } (dL_g)(W(e)) = W(g) \quad \text{for } g \in G. \tag{2.1.9}$$

Thus $h \to (dL_h)_e X$ is a left invariant vector on G and

$$Y \in \mathscr{C}^\infty(G, TG)^L \to Y_e \in \mathfrak{g} \tag{2.1.10}$$

is the inverse of ψ. Finally, the Jacobi identity for $[\,,\,]$ on $\mathfrak{g}$ is from the Jacobi identity for the vector fields and (2.1.8). The proof of Proposition 2.1.3 is completed. □

In the sequel, we do not distinguish $\mathfrak{g}$ and $\mathscr{C}^\infty(G, TG)^L$. If $X \in \mathfrak{g}$, let $\phi_t^X : G \to G$ be the flow associated with the left invariant vector field X. By the uniqueness of the solution of ordinary differential equations, we verify that

$$\phi_t^X\left(\phi_s^X(g)\right) = \phi_{t+s}^X(g), \qquad \phi_t^X(e) = \phi_1^{tX}(e). \tag{2.1.11}$$

Lemma 2.1.4. *For $X \in \mathfrak{g}$, ϕ_t^X is well-defined for all $t \in \mathbb{R}$, and we have*

$$\phi_t^X(g) = g\phi_t^X(e). \tag{2.1.12}$$

Proof. By the existence of the solution of ordinary differential equations, there is $\varepsilon > 0$ such that the flow $\phi_t^X(e)$ is well-defined for $|t| < \varepsilon$. For $|t| < \varepsilon, g \in G$, we have

$$\frac{\partial}{\partial t}\left(g\phi_t^X(e)\right) = (dL_g)\left(X_L\left(\phi_t^X(e)\right)\right) = X_L\left(g\phi_t^X(e)\right). \tag{2.1.13}$$

By the uniqueness of the solution of ordinary differential equations and by (2.1.13), we know that for $g \in G$, for $|t| < \varepsilon$, the flow $\phi_t^X(g)$ is well-defined, and (2.1.12) holds. Thus by the first equation of (2.1.11), we know that ϕ_t^X is defined for $t \in \mathbb{R}$. Using the same argument again, we know that (2.1.12) holds for $t \in \mathbb{R}$. □

The exponential map $\exp : \mathfrak{g} \to G$ is defined by: for $X \in \mathfrak{g}$,

$$\exp(X) := e^X := \phi_1^X(e). \tag{2.1.14}$$

By (2.1.11), (2.1.12), we have

$$\phi_t^X(g) = ge^{tX}. \tag{2.1.15}$$

Now we give another description of the Lie bracket on $\mathfrak{g}$. For $h \in G$, set

$$C_h = L_h R_{h^{-1}} : G \to G. \tag{2.1.16}$$

Then C_h is a homomorphism of G. By taking the derivation at e, we get

$$\mathrm{Ad}_h := (dC_h)_e : \mathfrak{g} \to \mathfrak{g}. \tag{2.1.17}$$

Since for $h_1, h_2 \in G$, we have $C_{h_1} C_{h_2} = C_{h_1 h_2}$, by (2.1.17), we get

$$\mathrm{Ad}_{h_1}\mathrm{Ad}_{h_2} = \mathrm{Ad}_{h_1 h_2} \in \mathrm{GL}(\mathfrak{g}), \qquad \mathrm{Ad}_h \mathrm{Ad}_{h^{-1}} = \mathrm{Id}_{\mathfrak{g}}\,. \tag{2.1.18}$$

Thus, we get a homomorphism of groups $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ which is called the adjoint representation of G. By taking the derivation of Ad_h on h at e, we get for $X \in \mathfrak{g}$,

$$\mathrm{ad}_X := (d\mathrm{Ad})_e\,(X) \in \mathrm{End}(\mathfrak{g}) =: \mathfrak{gl}(\mathfrak{g}). \tag{2.1.19}$$

Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. For $\alpha \in \mathfrak{g}^*$, $h \in G$, $X \in \mathfrak{g}$, we define $\mathrm{Ad}_h^*\alpha$, $\mathrm{ad}_X^*\alpha \in \mathfrak{g}^*$ by: for $Y \in \mathfrak{g}$,

$$(\mathrm{Ad}_h^*\alpha, Y) := (\alpha, \mathrm{Ad}_{h^{-1}}Y), \qquad (\mathrm{ad}_X^*\alpha, Y) := -(\alpha, \mathrm{ad}_X Y). \tag{2.1.20}$$

By (2.1.18) and (2.1.20), we get for $h_1, h_2 \in G$,

$$\mathrm{Ad}_{h_1}^*\mathrm{Ad}_{h_2}^* = \mathrm{Ad}_{h_1 h_2}^*. \tag{2.1.21}$$

Thus $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ is a homomorphism of groups which is called the coadjoint representation of G. By (2.1.19), we have

$$\mathrm{ad}_X^* = (d\mathrm{Ad}^*)_e\,(X). \tag{2.1.22}$$

Proposition 2.1.5. *If $X, Y \in \mathfrak{g}$, we have*

$$\mathrm{ad}_X Y = [X, Y], \tag{2.1.23}$$

and $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is a homomorphism of Lie algebras with Lie bracket* (2.1.2) *on* $\mathfrak{gl}(\mathfrak{g})$.

Proof. By Definition, we have

$$[X, Y] = \frac{\partial}{\partial t}\Big|_{t=0} \left(\phi_{-t,*}^X Y\right)_e, \qquad \left(\phi_{-t,*}^X Y\right)_e = \frac{\partial}{\partial s}\Big|_{s=0} \phi_{-t}^X\Big(\phi_s^Y\left(\phi_t^X(e)\right)\Big). \tag{2.1.24}$$

By (2.1.15) and (2.1.16), we get

$$\phi_{-t}^X\Big(\phi_s^Y\left(\phi_t^X(e)\right)\Big) = e^{tX}e^{sY}e^{-tX} = C_{e^{tX}}e^{sY}. \tag{2.1.25}$$

By (2.1.17), (2.1.19), (2.1.24) and (2.1.25), we have

$$[X, Y] = \frac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} C_{e^{tX}}e^{sY} = \frac{\partial}{\partial t}\Big|_{t=0} \mathrm{Ad}_{e^{tX}} Y = \mathrm{ad}_X Y. \tag{2.1.26}$$

By Jacobi identity and (2.1.26), we get for $X, Y, Z \in \mathfrak{g}$,

$$\begin{aligned} \mathrm{ad}_{[X,Y]} Z = [[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]] \\ = \mathrm{ad}_X \mathrm{ad}_Y Z - \mathrm{ad}_Y \mathrm{ad}_X Z = [\mathrm{ad}_X, \mathrm{ad}_Y] Z. \end{aligned} \tag{2.1.27}$$

The proof of Proposition 2.1.5 is completed. □

Example 2.1.6. For $m \in \mathbb{N}^*$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the group $\mathrm{GL}(m, \mathbb{K})$ is a Lie group, with its Lie algebra $\mathfrak{gl}(m, \mathbb{K}) = M_m(\mathbb{K})$, the space of $\mathbb{K}$-valued $m \times m$ matrices. For $A, B \in M_m(\mathbb{K})$, $g \in \mathrm{GL}(m, \mathbb{K})$, the exponential map is given by

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}, \tag{2.1.28}$$

and we have via the usual matrix operations

$$\mathrm{Ad}_g(A) = gAg^{-1}, \qquad \mathrm{ad}_A B = [A, B] = AB - BA. \tag{2.1.29}$$

Proof. For $x = (x_1, \dots, x_m)^t \in \mathbb{K}^m$, we define the norm $|x|$ of x by

$$|x|^2 = \sum_{i=1}^{m} |x_i|^2. \tag{2.1.30}$$

For $A \in M_m(\mathbb{K})$, we denote by $\|A\|$ the operator norm, that is

$$\|A\| = \sup_{x \in \mathbb{K}^m \setminus \{0\}} \frac{|Ax|}{|x|}. \tag{2.1.31}$$

Using $\|A^k\| \leqslant \|A\|^k$, we know that the sum of the right-hand side of (2.1.28) is absolutely and uniformly convergent, we denote it by $\phi(A)$. Moreover, by (2.1.28), for $t, s \in \mathbb{R}$,

$$\phi((t+s)A) = \phi(tA)\phi(sA). \tag{2.1.32}$$

By Definition and by (2.1.32), we get $\frac{\partial}{\partial t}\phi(tA) = \phi(tA)A$. Thus, the exponential map $\exp(A)$ is given by $\phi(A)$, the right-hand side of (2.1.28).

By Definition, for $h \in \mathrm{GL}(m, \mathbb{K})$, $A, B \in M_m(\mathbb{K})$, we have

$$\begin{aligned} \mathrm{Ad}_h A &= (dC_h)_e A = \frac{\partial}{\partial t}\Big|_{t=0} C_h(e^{tA}) = \frac{\partial}{\partial t}\Big|_{t=0} h e^{tA} h^{-1} = hAh^{-1}, \\ \mathrm{ad}_A B &= \frac{\partial}{\partial t}\Big|_{t=0} \mathrm{Ad}_{e^{tA}} B = \frac{\partial}{\partial t}\Big|_{t=0} e^{tA} B e^{-tA} = AB - BA. \end{aligned} \tag{2.1.33}$$

□

Remark 2.1.7. As $d\exp_e : \mathfrak{g} \to \mathfrak{g}$ is identity, we know $\exp : \mathfrak{g} \to G$ is a diffeomorphism from an open neighborhood $U \subset \mathfrak{g}$ of 0 to an open neighborhood $V \subset G$ of e. Thus if G is connected, for any $g \in G$, there exist $v_1, \dots, v_k \in \mathfrak{g}$ such that $g = \prod_j e^{v_j}$, but $\exp : \mathfrak{g} \to G$ need not be surjective. If G is compact and connected, then we can prove $\exp : \mathfrak{g} \to G$ is surjective. Here is an example: for any $A \in \mathfrak{sl}(2, \mathbb{R})$, the Lie algebra of $\mathrm{SL}(2, \mathbb{R})$, there exists $Q \in \mathrm{O}(2)$, $\lambda \in \mathbb{R}$ or $\sqrt{-1}\mathbb{R}$ such that $A = Q \begin{pmatrix} \lambda & * \\ 0 & -\lambda \end{pmatrix} Q^t$, thus $\mathrm{Tr}[e^A] \geq -2$. But $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ for any $x < -1$.

Proposition 2.1.8. *A compact connected abelian Lie group G is a torus.*

Proof. As G is abelian, the map $\psi : G \times G \to G$, $\psi(x, y) = xy$ is a homomorphism of Lie groups. As $d\psi_{(e,e)}(u, v) = u + v$ for any $u, v \in \mathfrak{g}$, Exercise 2.1.2 a) implies that

$$\exp(u)\exp(v) = \exp(u + v). \tag{2.1.34}$$

Thus $\exp : \mathfrak{g} \to G$ is a morphism of Lie groups in view $\mathfrak{g}$ as an additive Lie group. As G is connected, the argument in Remark 2.1.7 and (2.1.34) imply that $\exp : \mathfrak{g} \to G$ is surjective and $\Gamma := \ker(\exp)$ is a discrete subgroup of $\mathfrak{g}$. Thus $G = \mathfrak{g}/\Gamma$. As G is compact, we conclude from Exercise 2.1.3 that there exists $e_1, \ldots, e_{\dim G} \in \Gamma$ such that

$$\Gamma = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{\dim G} \quad \text{and } G = \oplus_j \mathbb{R}/\mathbb{Z}e_j. \tag{2.1.35}$$

The proof of Proposition 2.1.8 is completed. □

Lemma 2.1.9. *Let H be an abelian compact Lie group and let $\rho\colon H \to \mathrm{GL}(V)$ be a finite-dimensional complex irreducible representation of H. Then* $\dim V = 1$.

Proof. Let g^V be a H-invariant metric on V. If $h \in H$, since h acts by isometry, we can diagonalize $\rho(h)$:

$$V = \bigoplus_{\lambda \in \mathrm{Spec}(\rho(h))} V_\lambda \quad \text{with } \rho(h)|_{V_\lambda} = \lambda \,\mathrm{Id}_{V_\lambda}\,. \tag{2.1.36}$$

Now for any $h' \in H$, since $hh' = h'h$, h' preserve the V_λ's, so the V_λ's are stable by H, hence there is a $\lambda_0(h)$ such that $\rho(h) = \lambda_0(h)\,\mathrm{Id}_V$. Thus, $\rho(H) \subset \mathbb{C}\,\mathrm{Id}_V$ and if $\dim V > 1$, (V, ρ) cannot be irreducible. □

Example 2.1.10. Let $\mathfrak{u}(n)$, $\mathfrak{sp}(2n)$ be the Lie algebras of $\mathrm{U}(n)$, $\mathrm{Sp}(2n)$. Then we have

$$\begin{aligned}\mathfrak{sp}(2n) &= \{A \in M_{2n}(\mathbb{R}) : A^t J_0 = -J_0 A\} \\ &= \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^t \end{pmatrix} : \beta^t = \beta, \gamma^t = \gamma, \alpha, \beta, \gamma \in M_n(\mathbb{R}) \right\}.\end{aligned} \tag{2.1.37}$$

In fact, by (1.1.76),

$$A \in \mathrm{Sp}(2n) \quad \text{if and only if } J_0 = A^t J_0 A. \tag{2.1.38}$$

Thus $\mathfrak{sp}(2n)$ is a subset of the right-hand side of (2.1.37). Now if $A \in M_{2n}(\mathbb{R})$, $A = J_0^{-1}(-A^t)J_0$, we know that $e^{sA} = J_0^{-1} e^{-sA^t} J_0 = J_0^{-1}(e^{-sA})^t J_0$ for $s \in \mathbb{R}$. This implies $e^{-sA} \in \mathrm{Sp}(2n)$ thus $A = \frac{\partial}{\partial s}|_{s=0} e^{sA} \in \mathfrak{sp}(2n)$.

The Cartan involution $\Theta : \mathrm{Sp}(2n) \to \mathrm{Sp}(2n)$ of $\mathrm{Sp}(2n)$ is defined by $\Theta(g) = (g^t)^{-1}$. Let $\vartheta : \mathfrak{sp}(2n) \to \mathfrak{sp}(2n)$ be the differential of Θ. Then

$$\Theta^2 = \mathrm{Id}, \quad \vartheta(A) = -A^t. \tag{2.1.39}$$

By (1.1.66) and (2.1.39), we get

$$\{g \in \mathrm{Sp}(2n) : \Theta(g) = g\} = \mathrm{U}(n)(= \tau(\mathrm{U}(n))), \quad \ker(\vartheta - \mathrm{Id}) = \mathfrak{u}(n). \tag{2.1.40}$$

Thus we have the Cartan decomposition of $\mathfrak{sp}(2n)$:

$$\mathfrak{sp}(2n) = \mathfrak{u} \oplus \mathfrak{p}, \tag{2.1.41}$$

with $\mathfrak{u}$ its anti-symmetric part and $\mathfrak{p}$ its symmetric part:

$$\begin{aligned}\mathfrak{u} &:= \ker(\vartheta - \mathrm{Id}) = \left\{ \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} : \alpha^t = -\alpha, \beta^t = \beta, \quad \alpha, \beta \in M_n(\mathbb{R}) \right\}, \\ \mathfrak{p} &:= \ker(\vartheta + \mathrm{Id}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} : \alpha^t = \alpha, \beta^t = \beta, \quad \alpha, \beta \in M_n(\mathbb{R}) \right\}.\end{aligned} \tag{2.1.42}$$

By using Θ is an automorphism of Lie groups or from the direct computation, we get

$$[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{u}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{u}, \quad [\mathfrak{u}, \mathfrak{p}] \subset \mathfrak{p}. \tag{2.1.43}$$

Theorem 2.1.11. *The map*

$$\exp : \mathfrak{p} \to \{B \in \mathrm{Sp}(2n) : B \text{ is symmetric and positive } \}, \quad A \to e^A, \tag{2.1.44}$$

is well-defined and bijective. In particular, for any $B \in \mathrm{Sp}(2n)$, there exists unique $A \in \mathfrak{p}$, $Q \in \mathrm{U}(n)$ such that $B = e^A Q$.

Proof. At first, if $A \in \mathfrak{p}$, then e^A is symmetric and positive, and $e^A \in \mathrm{Sp}(2n)$ by the argument after (2.1.38). Thus (2.1.44) is well-defined.

If $B \in \mathrm{Sp}(2n)$ is symmetric and positive, then there exists $Q \in \mathrm{O}(2n)$ such that

$$B = Q\,\mathrm{diag}(\lambda_1, \dots, \lambda_{2n})\,Q^t, \tag{2.1.45}$$

and $\lambda_j > 0$ for $1 \leq j \leq 2n$. Set

$$A = Q\,\mathrm{diag}(\log\lambda_1, \dots, \log\lambda_{2n})\,Q^t, \tag{2.1.46}$$

and from (1.1.38), $B^s = e^{sA}$ for any $s \in \mathbb{R}$. By Proposition 1.1.18, we know $e^{sA} \in \mathrm{Sp}(2n)$. Thus $A \in \mathfrak{sp}(2n)$, as A is symmetric, we get $A \in \mathfrak{p}$. Thus (2.1.44) is surjective.

If $B = e^A$ and $A \in \mathfrak{p}$. Then there exists $Q_1 \in \mathrm{O}(2n)$ such that $A = Q_1\mathrm{diag}(\mu_1, \dots, \mu_{2n})\,Q_1^t$, thus $B = Q_1\mathrm{diag}(e^{\mu_1}, \dots, e^{\mu_{2n}})\,Q_1^t$. By (1.1.38), we get for $s \in \mathbb{R}$,

$$B^s = Q_1\mathrm{diag}(e^{s\mu_1}, \dots, e^{s\mu_{2n}})\,Q_1^t = e^{sA}.$$

By taking the differential at $s = 0$, we get $A = \frac{\partial}{\partial s}|_{s=0} B^s$. Thus (2.1.44) is injective.

Now the last part is from Proposition 1.1.18. The proof of Theorem 2.1.11 is completed. □

Now we state some results of the theory of Lie groups which we will not prove here.

Theorem 2.1.12. *A closed subgroup of a Lie group is a Lie subgroup.*

Theorem 2.1.13. *If H is a closed Lie subgroup of G, then the homogeneous space G/H is a smooth manifold with quotient topology of G.*

In fact, Theorem 2.1.13 is a direct consequence of Corollary 2.2.7 when G is compact, whose proof does not use Theorem 2.1.13.

Definition 2.1.14. Let M be a connected topology space. We say that M is simply connected if for any continuous closed curve $\gamma : \mathbb{S}^1 \to M$, there exists a continuous map $\gamma_t : \mathbb{S}^1 \times [0, 1] \to M$ such that $\gamma_1(\mathbb{S}^1) = pt$ (i.e., constant map) and $\gamma_0(\mathbb{S}^1) = \gamma$. That is, the fundamental group $\pi_1(M)$ of M is zero.

Theorem 2.1.15. (Second theorem of Lie) *Let G and H be two connected Lie groups and $\pi_1(G) = 0$. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras of these two Lie groups, then there exists a unique homomorphism $\Phi : G \to H$ of Lie groups such that $d\Phi|_e = \phi$, for e the identity element of G.*

Let $\Omega_L^\bullet(G)$ be the space of left invariant differential forms on G. We define for $\alpha \in \Lambda^\bullet \mathfrak{g}^*$, $g \in G$,

$$\tau_L(\alpha)_g = (dL_g)^*\alpha \in \Lambda^\bullet(T_g^*G), \tag{2.1.47}$$

i.e., for $Y_1, \dots, Y_k \in T_gG$, we have

$$\tau_L(\alpha)_g(Y_1, \dots, Y_k) = \alpha(dL_{g^{-1}}Y_1, \dots, dL_{g^{-1}}Y_k). \tag{2.1.48}$$

Then we verify as in Proposition 2.1.3 that $\tau_L(\alpha) \in \Omega_L^\bullet(G)$, and $\tau_L : \Lambda^\bullet \mathfrak{g}^* \to \Omega_L^\bullet(G)$ is an isomorphism of algebras. In particular, G is orientable.

When G is compact, if $\alpha \in \Omega^{\dim G}(G)$ is left invariant, and

$$\int_G \alpha = 1, \tag{2.1.49}$$

with the orientation on G induced by α, then we call α a left Haar form on G.

Proposition 2.1.16. *If G is compact and connected, then a left Haar form is also right invariant.*

Proof. Let α be a left Haar form on G. For $g \in G$, let $\varrho_g \in \mathscr{C}^\infty(G, \mathbb{R}^\times)$ be the nonvanishing function on G defined by, for $h \in G$

$$\left(R_g^* \alpha\right)_h = \varrho_g(h)\, \alpha_h. \tag{2.1.50}$$

As α is left invariant, we have

$$L_h^* R_g^* \alpha = (L_h^* \varrho_g)\, \alpha. \tag{2.1.51}$$

By $L_h R_g = R_g L_h$, we get

$$L_h^* R_g^* \alpha = R_g^* L_h^* \alpha = R_g^* \alpha = \varrho_g \alpha. \tag{2.1.52}$$

Thus ϱ_g is a left invariant function. Hence, it is a constant.

On the other hand, by $R_h R_g = R_{gh}$, we have

$$R_g^* R_h^* \alpha = \varrho_h R_g^* \alpha = \varrho_h \varrho_g \alpha, \qquad R_g^* R_h^* \alpha = R_{gh}^* \alpha = \varrho_{gh} \alpha. \tag{2.1.53}$$

As R_g-action is smooth on g, thus by (2.1.50) and (2.1.53), the function

$$\varrho : g \in G \to \varrho_g \in \mathbb{R}^\times \tag{2.1.54}$$

is multiplicative and smooth. Moreover, $\rho_\cdot$ is positive near the identity element.

We claim that if G is compact, for all $g \in G$, $\varrho_g = \pm 1$. Otherwise, there is $g \in G$ such that $|\varrho_g| > 1$. Then $\lim_{k\to\infty} |\varrho_{g^k}| = \lim_{k\to\infty} |\varrho_g^k| = \infty$, which contradicts with the compactness of G.

If G is moreover connected, we get $\rho \equiv 1$. The proof of Proposition 2.1.16 is completed. □

Remark 2.1.17. For a compact Lie group G and α in (2.1.49), then $\varrho \equiv 1$ or -1 on each connected component of G, thus α need not be right invariant. We define the Haar measure $d\mu$ on G by $\int_G f d\mu := \int_G f\alpha$ for $f \in \mathscr{C}^\infty(G)$. It is a left and right invariant measure on G, i.e., for any $h \in G$,

$$\int_G (R_h^* f)\, d\mu = \int_G (L_h^* f)\, d\mu = \int_G f d\mu. \tag{2.1.55}$$

In fact R_h preserves (resp. inverses) the orientation induced by α if $\varrho_h = 1$ (resp. -1), thus

$$\int_G (R_h^* f)\, d\mu = \int_G R_h^*(f R_{h^{-1}}^* \alpha) = \frac{\varrho_h}{|\varrho_h|} \int_G f R_{h^{-1}}^* \alpha = \frac{\varrho_h}{|\varrho_h|} \varrho_{h^{-1}} \int_G f\alpha = \int_G f d\mu. \tag{2.1.56}$$

In the same way, L_h preserves the orientation and $\int_G (L_h^* f)\alpha = \int_G L_h^*(f L_{h^{-1}}^* \alpha) = \int_G f L_{h^{-1}}^* \alpha = \int_G f d\mu$. The proof of Proposition 2.1.16 implies also a left invariant positive measure on G is right invariant (as automatically $\rho > 0$) and the Haar measure is unique.

2.1.2 Cohomology groups of a Lie group and of a Lie algebra

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let M be a manifold of dimension m. We say that G acts (leftly) on M via a smooth map

$$\vartheta : (g,x) \in G \times M \to \vartheta(g,x) = g \cdot x \in M, \tag{2.1.57}$$

if for any $g, h \in G, x \in M$, we have

$$e \cdot x = x, \qquad (gh) \cdot x = g \cdot (h \cdot x). \tag{2.1.58}$$

If G acts on M, then we call M a G-manifold. For $g \in G$, we still denote by $g : M \to M$ the diffeomorphism $M \ni x \to g \cdot x \in M$.

Definition 2.1.18. Let $\Omega^\bullet(M)^G$ be the space of G-invariant differential forms on M, i.e.,

$$\Omega^\bullet(M)^G = \{\alpha \in \Omega^\bullet(M) : g^*\alpha = \alpha, \text{for any } g \in G\}. \tag{2.1.59}$$

As the exterior differential d commutes with g^*, $(\Omega^\bullet(M)^G, d)$ is a subcomplex of $(\Omega^\bullet(M), d)$, i.e., $d(\Omega^\bullet(M)^G) \subset \Omega^{\bullet+1}(M)^G$. Let $H^\bullet(M,\mathbb{R})^G$ be the cohomology group of $(\Omega^\bullet(M)^G, d)$, i.e., for $k \in \mathbb{N}$,

$$H^k(M,\mathbb{R})^G = \frac{\ker(d|_{\Omega^k(M)^G})}{\operatorname{Im}(d|_{\Omega^{k-1}(M)^G})}. \tag{2.1.60}$$

We introduce first some notations. Let π_G, π_M be the natural projections from $G \times M$ to G and M. The fiberwise integration $\int^G : \Omega^\bullet(G \times M) \to \Omega^{\bullet - \dim G}(M)$ is defined by: for $\alpha \in \Omega^\bullet(M), \beta \in \Omega^\bullet(G)$ and $f \in \mathscr{C}^\infty(G \times M)$,

$$\int^G f\pi_M^*(\alpha) \wedge \pi_G^*(\beta) = \alpha \int_G f\beta. \tag{2.1.61}$$

For a manifold X, we denote by d^X the exterior differential on X. We claim that for any $\psi \in \Omega^\bullet(G \times M)$,

$$d^M \int^G \psi = \int^G d^{G\times M}\psi. \tag{2.1.62}$$

At first, for $f \in \mathscr{C}^\infty(G \times M)$, $\beta \in \Omega^\bullet(G)$, we have

$$d^{G\times M}(f\pi_G^*\beta) = d^M f \wedge \pi_G^*\beta + d^G\left(f\pi_G^*(\beta)\right). \tag{2.1.63}$$

By Stokes formula and by (2.1.61), we get

$$\int^G d^{G\times M}\left(f\pi_G^*\beta\right) = \int^G d^M f \wedge \pi_G^*\beta = d^M\left(\int^G f\pi_G^*\beta\right). \tag{2.1.64}$$

Observe that by (2.1.61), for $\alpha \in \Omega^\bullet(M), \gamma \in \Omega^\bullet(G \times M)$, we have

$$\int^G \pi_M^*\alpha \wedge \gamma = \alpha \int^G \gamma. \tag{2.1.65}$$

Thus (2.1.64) implies (2.1.62).

Theorem 2.1.19. *If G is compact and connected, then the inclusion $i : \Omega^\bullet(M)^G \to \Omega^\bullet(M)$ is a quasi-isomorphism of complexes, i.e., i induces an isomorphism $H^\bullet(M,\mathbb{R})^G \simeq H^\bullet(M,\mathbb{R})$.*

Proof. Let $\theta \in \Omega^{\dim G}(G)$ be the left Haar form on G as in (2.1.49). Let $r : \Omega^\bullet(M) \to \Omega^\bullet(M)$ be the linear map defined by: for $\alpha \in \Omega^\bullet(M)$, $x \in M$,

$$r(\alpha)_x = \int_{g\in G} (g^*\alpha)_x \, \theta(g). \tag{2.1.66}$$

For $\alpha \in \Omega^\bullet(M)$, $x \in M$, we define $\alpha^x(g) = (g^*\alpha)_x \in \Lambda^\bullet(T_x^*M)$ as a function on G. For any $h \in G$, we get by Remark 2.1.17,

$$\begin{aligned}(h^*r(\alpha))_x = \int_{g\in G} (h^*g^*\alpha)_x \, \theta(g) = \int_{g\in G} ((gh)^*\alpha)_x \, \theta(g) \\ = \int_{g\in G} (R_h^*\alpha^x)(g)\theta(g) = r(\alpha)_x.\end{aligned} \tag{2.1.67}$$

Thus, $r(\alpha) \in \Omega^\bullet(M)^G$. Clearly, we have

$$r \circ i = \mathrm{Id}_{\Omega^\bullet(M)^G} . \tag{2.1.68}$$

For $\alpha \in \Omega^\bullet(M)$, as $\theta(g)$ has maximal degree on G, from (2.1.62), we get

$$\begin{aligned}r(d^M\alpha)_x = \int_{g\in G} \left(g^*d^M\alpha\right)_x \theta(g) = \int_{g\in G} \left(d^M g^*\alpha\right)_x \theta(g) \\ = \Big(\int^G d^{G\times M}(g^*\alpha \wedge \theta(g))\Big)_x = d^M r(\alpha)_x.\end{aligned} \tag{2.1.69}$$

This means that $r : (\Omega^\bullet(M), d) \to (\Omega^\bullet(M)^G, d)$ is a morphism of complexes. Let

$$i_* : H^\bullet(M,\mathbb{R})^G \to H^\bullet(M,\mathbb{R}), \qquad r_* : H^\bullet(M,\mathbb{R}) \to H^\bullet(M,\mathbb{R})^G, \tag{2.1.70}$$

be the maps induced by i, r. Then (2.1.68) implies

$$r_* \circ i_* = \mathrm{Id}_{H^\bullet(M,\mathbb{R})^G} . \tag{2.1.71}$$

We will show that $i \circ r$ is chain homotopy to $\mathrm{Id}_{\Omega^\bullet(M)}$.

As $\theta(g)$ has maximal degree, by (2.1.61), we have

$$i \circ r(\alpha) = \int_G g^*\alpha \, \theta(g) = \int^G \vartheta^*(\alpha) \wedge \pi_G^*(\theta). \tag{2.1.72}$$

Let $\varphi : U \subset G \to V \subset \mathbb{R}^{\dim G}$ be a local coordinate near $e \in G$ such that $\varphi(e) = 0$ and V convex. Let σ be a form on G with degree $\dim G$ such that

$$\mathrm{Supp}(\sigma) \subset U \qquad \text{and} \qquad \int_G \sigma = 1. \tag{2.1.73}$$

As G is connected, we get $H^0(G,\mathbb{R}) = \mathbb{R}$. Moreover as G is compact and orientable, by Poincaré duality,

$$[\alpha] \in H^{\dim G}(G,\mathbb{R}) \to \int_G \alpha \in \mathbb{R}, \tag{2.1.74}$$

is an isomorphism. Thus $[\sigma]$ and $[\theta]$ are in the same cohomology class. This means there is $\eta \in \Omega^{\dim G-1}(G)$ such that

$$d^G\eta = \theta - \sigma. \tag{2.1.75}$$

By (2.1.72) and (2.1.75), we get

$$i \circ r(\alpha) = \int^G \vartheta^*(\alpha) \wedge \pi_G^*(\sigma) + \int^G \vartheta^*(\alpha) \wedge \pi_G^*(d^G\eta). \tag{2.1.76}$$

Let $H_1 : \Omega^k(M) \to \Omega^{k-1}(M)$ be the linear map defined by: for $\alpha \in \Omega^k(M)$,

$$H_1(\alpha) = (-1)^k \int^G \vartheta^*(\alpha) \wedge \pi_G^*(\eta). \tag{2.1.77}$$

For $\alpha \in \Omega^k(M)$, by (2.1.62) and $\vartheta^*(d^M\alpha) = d^{G\times M}\vartheta^*(\alpha)$, we get

$$\begin{aligned}(d^M H_1 + H_1 d^M)\alpha = (-1)^k d^M \int^G \vartheta^*(\alpha) \wedge \pi_G^*(\eta) + (-1)^{k+1} \int^G \vartheta^*(d^M\alpha) \wedge \pi_G^*(\eta) \\ = \int^G \vartheta^*(\alpha) \wedge \pi_G^*(d^G\eta).\end{aligned} \tag{2.1.78}$$

We identify $U \subset G$ and $V \subset \mathbb{R}^{\dim G}$ via φ. For $s \in [0,1]$, set $\phi_s : (u,x) \in U \times M \to (su,x) \in U \times M$, and let X_s be the vector field on $sU \times M$ defined as in (1.2.47). For $\beta \in \Omega^\bullet(U \times M)$, set

$$K(\beta) = \int_0^1 \phi_s^* \left(i_{X_s}\beta\right) ds. \tag{2.1.79}$$

We know from (1.2.64) that

$$\left(d^{G\times M}K + Kd^{G\times M}\right)\beta = \beta - \phi_0^*\beta. \tag{2.1.80}$$

Let $H_2 : \Omega^k(M) \to \Omega^{k-1}(M)$ be a linear map defined by: for $\alpha \in \Omega^k(M)$,

$$H_2(\alpha) = \int^G K(\vartheta^*\alpha) \wedge \pi_G^*(\sigma). \tag{2.1.81}$$

By (2.1.62), (2.1.73), σ has maximal degree on G, and $d^{G\times M}(\vartheta^*\alpha) = \vartheta^* d^M\alpha$, we get

$$\begin{aligned}(d^M H_2 + H_2 d^M)\alpha &= d^M \int^G K(\vartheta^*\alpha) \wedge \pi_G^*(\sigma) + \int^G K\left(\vartheta^*(d^M\alpha)\right) \wedge \pi_G^*(\sigma) \\ &= \int^G \left(d^{G\times M}K(\vartheta^*\alpha) + K(d^{G\times M}(\vartheta^*\alpha))\right) \wedge \pi_G^*(\sigma) = \int^G (\vartheta^*\alpha - \phi_0^*\vartheta^*\alpha) \wedge \pi_G^*(\sigma) \\ &= \int^G \vartheta^*\alpha \wedge \pi_G^*(\sigma) - \alpha.\end{aligned} \tag{2.1.82}$$

Take $H = H_1 + H_2$. By (2.1.76), (2.1.78) and (2.1.82), we get

$$i \circ r - \mathrm{Id}_{\Omega^\bullet(M)} = dH + Hd. \tag{2.1.83}$$

This means $i \circ r$ is chain homotopy to identity. Thus

$$i_* \circ r_* = \mathrm{Id}_{H^\bullet(M,\mathbb{R})}. \tag{2.1.84}$$

The proof of Theorem 2.1.19 is completed. □

Remark 2.1.20. As G is connected, for any $g \in G$, there exists a smooth path $[0,1] \ni t \to g_t \in G$ from e to g. Let X_t be the vector field on M associated with the flow g_t. Then for any closed $\alpha \in \Omega^\bullet(M)$, by (1.2.20) and (1.2.94), we get

$$g^*\alpha - \alpha = \int_0^1 \frac{\partial}{\partial t} g_t^*\alpha = \int_0^1 g_t^* L_{X_t}\alpha = d\int_0^1 g_t^* i_{X_t}\alpha. \tag{2.1.85}$$

Thus G acts on $H^\bullet(M,\mathbb{R})$ as identity map, i.e., trivially on $H^\bullet(M,\mathbb{R})$, which is a direct consequence of Theorem 2.1.19. However, this fact does not imply directly Theorem 2.1.19, which means there exists a closed $\beta \in \Omega^\bullet(M)^G$, such that $\beta - \alpha$ is exact.

For $j \in \mathbb{N}$, let $H^j(M,\mathbb{Z})$ be the j-th cohomology group of M with integer coefficients, then

$$H^j(M,\mathbb{R}) = H^j(M,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.1.86}$$

As G is connected and $H^j(M,\mathbb{Z})$ is discrete, the G-action on $H^j(M,\mathbb{Z})$ must be trivial. Thus we get a topological proof of the fact that G acts trivially on $H^\bullet(M,\mathbb{R})$.

We will first apply Theorem 2.1.19 to the case $M = G$ with the left G action where G is a connected compact Lie group.

Recall that $\Omega_L^\bullet(G)$ is the space of left invariant differential forms on a Lie group G. Then as for any $g \in G$, $L_g^* \cdot d = d \cdot L_g^*$ on $\Omega^\bullet(G)$, thus $d(\Omega_L^\bullet(G)) \subset \Omega_L^{\bullet+1}(G)$, we know $(\Omega_L^\bullet(G), d)$ is a subcomplex of $(\Omega^\bullet(G), d)$. We denote by δ the differential on $\Lambda^\bullet\mathfrak{g}^*$ induced by the exterior differential d on $\Omega_L^\bullet(G)$ under the isomorphism $\tau_L : \Lambda^\bullet\mathfrak{g}^* \to \Omega_L^\bullet(G)$ in (2.1.47).

Proposition 2.1.21. *For $\alpha \in \Lambda^k\mathfrak{g}^*, X_0, \dots, X_k \in \mathfrak{g}$, we have*

$$\delta\alpha(X_0,\dots,X_k) = \sum_{0\leqslant i<j\leqslant k} (-1)^{i+j}\alpha([X_i,X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k). \tag{2.1.87}$$

Proof. As before we identify $X_0, \dots, X_k \in \mathfrak{g}$ with left invariant vector fields on G. Then for $0 \leqslant i \leqslant k$, $(\tau_L\alpha)(X_0, \dots, \widehat{X}_i, \dots, X_k)$ is a constant function on G and is equal to $\alpha(X_0, \dots, \widehat{X}_i, \dots, X_k)$. As $d\tau_L\alpha = \tau_L\delta\alpha$, by (1.2.12), we get

$$\begin{aligned}(\tau_L\delta\alpha)(X_0,\dots,X_k) &= (d\tau_L\alpha)(X_0,\dots,X_k)\\ &= \sum_{0\leqslant i<j\leqslant k} (-1)^{i+j}(\tau_L\alpha)([X_i,X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k).\end{aligned} \tag{2.1.88}$$

As $[X_i, X_j]$ is also a left invariant vector field on G, each term at the right hand side of (2.1.88) is constant on G and when we evaluate it at $e \in G$, it is the corresponding term in (2.1.87). The proof of Proposition 2.1.21 is completed. □

As $d^2 = 0$ on $\Omega_L^\bullet(G)$, by construction,

$$\delta^2 = (\tau_L^{-1} d\tau_L)^2 = 0 : \Lambda^\bullet\mathfrak{g}^* \to \Lambda^\bullet\mathfrak{g}^*. \tag{2.1.89}$$

We check directly that (2.1.89) still holds for any Lie algebra (not necessary for Lie algebras of compact Lie groups). This motives the following definition.

Definition 2.1.22. Let $\mathfrak{g}$ be a Lie algebra, and δ is given by (2.1.87). The cohomology group of Lie algebra $\mathfrak{g}$ is defined by: for $j \in \mathbb{N}$,

$$H^j(\mathfrak{g}) := H^j(\Lambda^\bullet\mathfrak{g}^*, \delta) := \frac{\ker(\delta|_{\Lambda^j\mathfrak{g}^*})}{\operatorname{Im}(\delta|_{\Lambda^{j-1}\mathfrak{g}^*})}. \tag{2.1.90}$$

Corollary 2.1.23. *If G is a connected compact Lie group with Lie algebra $\mathfrak{g}$, then*

$$H^\bullet(G,\mathbb{R}) = H^\bullet(\mathfrak{g}). \tag{2.1.91}$$

Proof. From Theorem 2.1.19 and Proposition 2.1.21, we get Corollary 2.1.23. □

The Ad^*-action of G (resp. ad^*-action of $\mathfrak{g}$) on $\mathfrak{g}^*$ induces corresponding action on $\Lambda^\bullet\mathfrak{g}^*$ by: for any $\alpha,\beta\in\Lambda^\bullet\mathfrak{g}^*$, $g\in G$, $X\in\mathfrak{g}$,

$$\mathrm{Ad}_g^*(\alpha\wedge\beta) = \mathrm{Ad}_g^*\alpha\wedge\mathrm{Ad}_g^*\beta, \quad \mathrm{ad}_X^*(\alpha\wedge\beta) = \mathrm{ad}_X^*\alpha\wedge\beta + \alpha\wedge\mathrm{ad}_X^*\beta. \tag{2.1.92}$$

We denote by $(\Lambda^\bullet\mathfrak{g}^*)^{\mathrm{Ad}}$ the subspace of $\Lambda^\bullet\mathfrak{g}^*$ which is Ad^*-invariant, and by

$$(\Lambda^\bullet\mathfrak{g}^*)^{\mathrm{ad}} := \{\alpha\in\Lambda^\bullet\mathfrak{g}^* : \mathrm{ad}_X^*\alpha = 0 \ \text{ for } X\in\mathfrak{g}\}, \tag{2.1.93}$$

the subspace of $\Lambda^\bullet\mathfrak{g}^*$ on which ad^*-action of $\mathfrak{g}$ is zero.

Next we consider G with a $G\times G$ action defined by : for $g_1,g_2,h\in G$,

$$I_{g_1,g_2}\cdot h = g_1 h g_2^{-1} = R_{g_2^{-1}}L_{g_1}h. \tag{2.1.94}$$

We denote by $C_I^\bullet(G)$ the space of $G\times G$-invariant differential forms on G. Then again $(C_I^\bullet(G),d)$ is a subcomplex of $(\Omega^\bullet(G),d)$.

Proposition 2.1.24. *We have*

$$\tau_L^{-1}\left(C_I^\bullet(G)\right) = (\Lambda^\bullet\mathfrak{g}^*)^{\mathrm{Ad}}, \qquad d|_{(\Lambda^\bullet\mathfrak{g}^*)^{\mathrm{Ad}}} = 0. \tag{2.1.95}$$

Moreover if G is connected, then

$$\tau_L^{-1}\left(C_I^\bullet(G)\right) = (\Lambda^\bullet\mathfrak{g}^*)^{\mathrm{ad}}. \tag{2.1.96}$$

Proof. By Definition, we have

$$\tau_L^{-1}(C_I^\bullet(G)) = \{\alpha\in\Lambda^\bullet\mathfrak{g}^* : R_g^*\tau_L\alpha = \tau_L\alpha, \text{ for all } g\in G\}. \tag{2.1.97}$$

For any $X_1,\ldots,X_k\in\mathfrak{g}$. If $\alpha\in\tau_L^{-1}(C_I^k(G))$, then for $g\in G$, by (2.1.20), (2.1.48), (2.1.92) and (2.1.97), as $\alpha = (\tau_L\alpha)_e$, we get

$$\begin{aligned}\alpha(X_1,\ldots,X_k) &= \left(R_g^*\tau_L\alpha\right)_e(X_1,\ldots,X_k) = (\tau_L\alpha)_{eg}(dR_gX_1,\ldots,dR_gX_k)\\ &= \alpha(\mathrm{Ad}_{g^{-1}}X_1,\ldots,\mathrm{Ad}_{g^{-1}}X_k) = (\mathrm{Ad}_g^*\alpha)(X_1,\ldots,X_k).\end{aligned} \tag{2.1.98}$$

Thus $\tau_L^{-1}(C_I^\bullet(G))\subset(\Lambda^\bullet\mathfrak{g}^*)^{\mathrm{Ad}}$. Inversely, if $\alpha = \mathrm{Ad}_g^*\alpha$ for any $g\in G$, then for $h,g\in G$,

$$\begin{aligned}\left(R_g^*\tau_L\alpha\right)_h(dL_hX_1,\ldots,dL_hX_k) &= (\tau_L\alpha)_{hg}(dR_gdL_hX_1,\ldots,dR_gdL_hX_k)\\ &= \alpha(\mathrm{Ad}_{g^{-1}}X_1,\ldots,\mathrm{Ad}_{g^{-1}}X_k) = (\mathrm{Ad}_g^*\alpha)(X_1,\ldots,X_k)\\ &= \alpha(X_1,\ldots,X_k) = (\tau_L\alpha)_h(dL_hX_1,\ldots,dL_hX_k).\end{aligned} \tag{2.1.99}$$

Thus $\tau_L\alpha\in C_I^\bullet(G)$ and we get the first identity of (2.1.95).

If G is connected, G is generated by a neighborhood of e. Then (2.1.96) is a consequence of Remark 2.1.7 and the first identity of (2.1.95).

Let $\sigma : g \in G \to g^{-1} \in G$ be the inverse map. For $X \in \mathfrak{g}, g \in G$, we have

$$\sigma\left(ge^{tX}\right) = e^{-tX}g^{-1}. \tag{2.1.100}$$

This means

$$d\sigma_g(dL_g(X)) = -dR_{g^{-1}}X, \quad \text{i.e., } d\sigma_g = -dR_{g^{-1}}dL_{g^{-1}}. \tag{2.1.101}$$

Thus for $\alpha \in \Omega^k(G)$, we have

$$(\sigma^*\alpha)_g = (-1)^k\left(R^*_{g^{-1}}L^*_{g^{-1}}\alpha\right)_g. \tag{2.1.102}$$

If $\alpha \in C^k_I(G)$, by (2.1.102), we have

$$\sigma^*\alpha = (-1)^k\alpha, \qquad \sigma^*d\alpha = (-1)^{k+1}d\alpha. \tag{2.1.103}$$

From (2.1.103) and $\sigma^*d\alpha = d\sigma^*\alpha$, we get $d\alpha = 0$. The proof of Proposition 2.1.24 is completed. □

Theorem 2.1.25 (Cartan 1929, Chevalley-Eilenberg 1948)**.** *If G is a connected compact Lie group with Lie algebra $\mathfrak{g}$, then*

$$H^\bullet(G,\mathbb{R}) = H^\bullet(\mathfrak{g}) = (\Lambda^\bullet\mathfrak{g}^*)^{\mathrm{Ad}} = (\Lambda^\bullet\mathfrak{g}^*)^{\mathrm{ad}}. \tag{2.1.104}$$

Proof. It is a consequence of Theorem 2.1.19, Proposition 2.1.24 and (2.1.91). □

Now we give some results on the cohomology groups of Lie algebras.

Proposition 2.1.26. *If $\mathfrak{g}$ is a Lie algebra, then*

$$H^0(\mathfrak{g}) = \mathbb{R}, \qquad H^1(\mathfrak{g}) = (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*. \tag{2.1.105}$$

Proof. By (2.1.87), we have

$$\delta|_{\Lambda^0\mathfrak{g}^*} = 0. \tag{2.1.106}$$

From (2.1.106), we get the first identity of (2.1.105). For $\alpha \in \mathfrak{g}^*$, by (2.1.87),

$$\delta\alpha = 0 \quad \text{if and only if} \quad 0 = \delta\alpha(X,Y) = -\alpha([X,Y]) \text{ for all } X,Y \in \mathfrak{g}. \tag{2.1.107}$$

By (2.1.106) and (2.1.107), we have

$$H^1(\mathfrak{g}^*) = \ker(\delta|_{\Lambda^1\mathfrak{g}^*}) = (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*. \tag{2.1.108}$$

The proof of Proposition 2.1.26 is completed. □

Proposition 2.1.26 motives the following definition.

Definition 2.1.27. Let $\mathfrak{h}$ be a linear subspace of a Lie algebra $\mathfrak{g}$. The $\mathfrak{h}$ is called a Lie subalgebra of $\mathfrak{g}$ if $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$. If $[\mathfrak{h},\mathfrak{g}] \subset \mathfrak{h}$, then $\mathfrak{h}$ is called an idea of $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is called abelian if $[\mathfrak{g},\mathfrak{g}] = 0$. A Lie algebra $\mathfrak{g}$ is called simple, if it is not abelian and contains no nonzero proper ideals. A Lie algebra $\mathfrak{g}$ is called semisimple, if it is a direct sum of simple Lie algebras.

A Lie group is called semisimple (resp. simple) if its Lie algebra is semisimple (resp. simple).

If a Lie algebra $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Proposition 2.1.28. *a) A compact connected Lie group H with Lie algebra $\mathfrak{h}$ is semisimple if and only if its center $Z(H) := \{g \in H : gh = hg \text{ for any } h \in H\}$ is finite if and only if $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$.*
b) A Lie algebra $\mathfrak{g}$ is semisimple if and only if the symmetric bilinear form (Killing form)

$$B(a, b) := \mathrm{Tr}\,|_{\mathfrak{g}}[\mathrm{ad}_a \mathrm{ad}_b] \quad \text{for } a, b \in \mathfrak{g} \tag{2.1.109}$$

is nondegenerate on $\mathfrak{g}$.

Proposition 2.1.29 (Whitehead's lemma)**.** *If $\mathfrak{g}$ is a semisimple Lie algebra, then*

$$H^0(\mathfrak{g}) = \mathbb{R}, \qquad H^1(\mathfrak{g}) = 0, \qquad H^2(\mathfrak{g}) = 0. \tag{2.1.110}$$

Proof. The first and second identities of (2.1.110) follow form Proposition 2.1.26 and Definition 2.1.27. We will proof the third identity of (2.1.110) only for the case where $\mathfrak{g}$ is a Lie algebra of a compact connected Lie group G. By Theorem 2.1.25, we need to show

$$\left(\Lambda^2 \mathfrak{g}^*\right)^{\mathrm{ad}} = 0. \tag{2.1.111}$$

If $\alpha \in \left(\Lambda^2 \mathfrak{g}^*\right)^{\mathrm{ad}}$, then for any $X, Y, Z \in \mathfrak{g}$, by Proposition 2.1.24 and (2.1.87), we have

$$\begin{aligned} 0 &= \delta\alpha(X, Y, Z) = -\alpha([X, Y], Z) - \alpha([Y, Z], X) - \alpha([Z, X], Y), \\ 0 &= (\mathrm{ad}^*_X \alpha)(Y, Z) = -\alpha([X, Y], Z) - \alpha(Y, [X, Z]). \end{aligned} \tag{2.1.112}$$

From (2.1.112), we get

$$\alpha([Y, Z], X) = 0. \tag{2.1.113}$$

As $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we get $\alpha = 0$, which finishes the proof in the case where $\mathfrak{g}$ is a Lie algebra of a compact connected Lie group G. □

Remark 2.1.30. To get the third equation of (2.1.110) in general case, we need a result, so called Weyl's trick, from the semisimple Lie algebra. In fact for any real semisimple Lie algebra $\mathfrak{g}$, there is a real semisimple Lie algebra $\mathfrak{u}$ such that $\mathfrak{u}$ is the Lie algebra of a compact connected Lie group and $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{u} \otimes_{\mathbb{R}} \mathbb{C}$. Then

$$H^2(\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C} = H^2(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}) = H^2(\mathfrak{u} \otimes_{\mathbb{R}} \mathbb{C}) = H^2(\mathfrak{u}) \otimes_{\mathbb{R}} \mathbb{C} = 0. \tag{2.1.114}$$

By Theorem 2.1.25, Proposition 2.1.28 and (2.1.110), we get for a compact connected Lie group G with finite center,

$$H^0(G, \mathbb{R}) = \mathbb{R}, \quad H^1(G, \mathbb{R}) = H^2(G, \mathbb{R}) = 0. \tag{2.1.115}$$

Recall that the k-th homotopy group of a manifold M is the set of homotopy classes of maps $f : \mathbb{S}^k \to M$, which forms a group.

Theorem 2.1.31 (Bott 1956)**.** *If G is a compact, connected and simply connected simple Lie group, then*

$$\pi_1(G) = 0, \qquad \pi_2(G) = 0, \qquad \pi_3(G) \simeq \mathbb{Z}. \tag{2.1.116}$$

The proof of this theorem is more difficult and we omit the proof. Combining Hurewicz's theorem, we get

$$H^1(G,\mathbb{Z}) = H^2(G,\mathbb{Z}) = 0, \qquad H^0(G,\mathbb{Z}) = H^3(G,\mathbb{Z}) = \mathbb{Z}. \tag{2.1.117}$$

By (2.1.86), (2.1.116) (or (2.1.117)) refines (2.1.115). Here we give an example to illustrate this point.

Example 2.1.32. If $G = \mathrm{SU}(2)$, then $G = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1, \text{ and } a, b \in \mathbb{C} \right\} = \mathbb{S}^3$ the 3-dimensional sphere (as $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2)$ if and only if $|a|^2 + |b|^2 = |c|^2 + |d|^2 = ad - bc = 1$, and $\overline{a}b + \overline{c}d = 0$). We know well that

$$\pi_1(\mathbb{S}^3) = 0, \qquad \pi_2(\mathbb{S}^3) = 0, \qquad \pi_3(\mathbb{S}^3) \simeq \mathbb{Z}. \tag{2.1.118}$$

Exercise 2.1.1. We use x to denote the coordinate for the real number $\mathbb{R}$.

1. Consider the additive group $(\mathbb{R}, +)$, verify first it is a Lie group. We identify $\mathbb{R}$ as its Lie algebra by $a \in \mathbb{R} \to a\frac{\partial}{\partial x} \in \mathrm{Lie}(\mathbb{R})$. Compute $e^{tX_v}(y)$ for $y \in \mathbb{R}$. Conclude that $\exp(v) = v$.

2. Consider the multiplicative group $(\mathbb{R}^\times_+, \times)$, verify first it is a Lie group. We identify $\mathbb{R}$ as its Lie algebra by $a \in \mathbb{R} \to a\frac{\partial}{\partial x} \in \mathrm{Lie}(\mathbb{R}^\times_+)$. Compute $e^{tX_v}(y)$ for $y \in \mathbb{R}$. Conclude that $\exp(v) = e^v$.

Exercise 2.1.2. Let $\varphi : G \to H$ be a homomorphism of Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$. Let $\varphi_* = (d\varphi)_e : \mathfrak{g} \to \mathfrak{h}$ be the differential of φ.

1. Verify that the inverse map $G \ni g \to g^{-1} \in G$ is smooth. (Hint: for the map $\psi : G \times G \ni (g, h) \to gh \in G$, deduce from the implicit function theorem that $\psi^{-1}(e)$ is a smooth submanifold of $G \times G$.)

2. For $g \in G$, $v \in \mathfrak{g}$, verify $\varphi(\exp(v)) = \exp(\varphi_* v)$ and $\varphi_*(\mathrm{Ad}_g v) = \mathrm{Ad}_{\varphi(g)} \varphi_* v$.

3. Deduce that for $u, v \in \mathfrak{g}$, $\varphi_*(\mathrm{ad}_u v) = \mathrm{ad}_{\varphi_* u} \varphi_* v$.

4. Conclude that for $g \in G, u, v \in \mathfrak{g}$, $\mathrm{Ad}_g[u, v] = [\mathrm{Ad}_g u, \mathrm{Ad}_g v]$.

Exercise 2.1.3. Let Γ be a discrete subgroup of $\mathbb{R}^n$. Verify that

1. Take $0 \neq e_1 \in \Gamma$ of smallest positive norm, show that $\Gamma/\mathbb{Z}e_1$ is a discrete subgroup in $\mathbb{R}^n/\mathbb{R}e_1$. (Hint: Otherwise, there exist $g_j \in \Gamma, a_j \in] - \frac{1}{2}, \frac{1}{2}]$ such that $g_j - a_j e_1 \to 0$.)

2. Conclude that there exists linearly independent vectors $e_1, \ldots, e_k \in \Gamma$ such that $\Gamma = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_k$.

Exercise 2.1.4. Let $G = \mathbb{S}^1 \times \mathbb{S}^1$ be the two dimensional torus. For any irrational number α, the map $\psi : \mathbb{R} \to G$, $t \to (e^{it}, e^{i\alpha t})$ identifies $\mathbb{R}$ as a Lie subgroup $H = \{(e^{it}, e^{i\alpha t}) : t \in \mathbb{R}\}$ of G. Verify that H is not closed in G and $\overline{H} = G$.

Exercise 2.1.5. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we will denote by $\mathfrak{gl}(m, \mathbb{K})$, $\mathfrak{sl}(m, \mathbb{K})$, $\mathfrak{o}(m)$ and $\mathfrak{so}(m)$ the Lie algebras of $\mathrm{GL}(m, \mathbb{K})$, $\mathrm{SL}(m, \mathbb{K})$, $\mathrm{O}(m)$ and $\mathrm{SO}(m)$.

1. Verify that

$$\begin{aligned}\mathfrak{gl}(m,\mathbb{K}) &= M_m(\mathbb{K}),\\ \mathfrak{sl}(m,\mathbb{K}) &= \{A\in M_m(\mathbb{K}) : \mathrm{Tr}[A]=0\},\\ \mathfrak{u}(m) &= \{A\in M_m(\mathbb{C}) : A+A^*=0\},\\ \mathfrak{o}(m) &= \{A\in M_m(\mathbb{R}) : A+A^t=0\},\\ \mathfrak{so}(m) &= \{A\in \mathfrak{o}(m) : \mathrm{Tr}[A]=0\}.\end{aligned} \tag{2.1.119}$$

2. Verify that $\dim_{\mathbb{R}}\mathfrak{u}(m)=m^2$, $\dim_{\mathbb{R}}\mathfrak{o}(m)=m(m-1)/2$.

Exercise 2.1.6. Let G be the special affine group of $\mathbb{R}^m$ which is the semi-direct product of $\mathbb{R}^m$ by $\mathrm{SL}(m,\mathbb{R})$, i.e., the multiplication on $G:=\mathbb{R}^m\rtimes\mathrm{SL}(m,\mathbb{R})$ is given by

$$(u,A)\cdot(v,B)=(u+Av,AB)\quad \text{for}\ \ (u,A),(v,B)\in\mathbb{R}^m\times\mathrm{SL}(m,\mathbb{R}). \tag{2.1.120}$$

1. Verify that G can be represented as a subgroup of $\mathrm{SL}(m+1,\mathbb{R})$ of the form $\begin{pmatrix} A & u\\ 0 & 1\end{pmatrix}$ where $A\in\mathrm{SL}(m,\mathbb{R}), u\in\mathbb{R}^m$.
2. Verify that the Lie algebra of G is $\mathfrak{g}=\mathbb{R}^m\times\mathfrak{sl}(m,\mathbb{R})$ with the Lie bracket

$$[(u,A),(v,B)]=(Av-Bu,AB-BA)\quad \text{for}\ \ (u,A),(v,B)\in\mathbb{R}^m\times\mathfrak{sl}(m,\mathbb{R}). \tag{2.1.121}$$

3. Verify that $\mathbb{R}^m\times\{0\}$ is an ideal of $\mathfrak{g}$.
4. Conclude that G is not semisimple, but $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$.

Exercise 2.1.7. Let M be an oriented connected compact manifold of dimension m.

1. Verify that $H^0(M,\mathbb{R})=\{f\in\mathscr{C}^\infty(M) : df=0\}$. Verify that for $f\in\mathscr{C}^\infty(M)$, $df=0$ if and only if f is constant on M. Conclude that

$$H^0(M,\mathbb{R})=\{f\in\mathscr{C}^\infty(M) : f\equiv C\}\simeq\mathbb{R}. \tag{2.1.122}$$

2. Let g^{TM} be a Riemannian metric on M. We define the Hodge star operator $*:\Lambda^j(T_x^*M)\to\Lambda^{m-j}(T_x^*M)$ by

$$\alpha\wedge *_x\beta=\langle\alpha,\beta\rangle_{\Lambda(T_x^*M)}\,dv_M(x). \tag{2.1.123}$$

Prove that

$$*\cdot*|_{\Lambda^j(T_x^*M)}=(-1)^{(m-j)j}. \tag{2.1.124}$$

3. Verify that

$$d^*|_{\Omega^j(M)}=(-1)^{mj+m+1}*d*,\quad *\Delta=\Delta*\ . \tag{2.1.125}$$

4. Prove that $\alpha\in\ker(\Delta|_{\Omega^m(M)})$ if and only if $d(*\alpha)=0$. Conclude that $dv_M\in\ker(\Delta|_{\Omega^m(M)})$ and $H^m(M,\mathbb{R})\simeq\mathbb{R}\{dv_M\}$. Thus $H^m(M,\mathbb{R})\ni\alpha\to\int_M\alpha\in\mathbb{R}$ is an isomorphism.

5. Prove that $\alpha \in \ker(\Delta|_{\Omega^j(M)})$ if and only if $*\alpha \in \ker(\Delta|_{\Omega^{m-j}(M)})$. Conclude that the Poincaré duality holds, i.e., the bilinear form

$$H^j(M,\mathbb{R}) \times H^{m-j}(M,\mathbb{R}) \to H^m(M,\mathbb{R}) \to \mathbb{R}, \quad (\alpha,\beta) \to \alpha\wedge\beta \to \int_M \alpha\wedge\beta. \tag{2.1.126}$$

is nondegenerate.

Exercise 2.1.8. We use the notation in Remark 2.1.7. Let G be a connected Lie group. For $g \in G$, let $\gamma : [0,1] \to G$ be a continuous path from e to g. Prove that there exists $0 \leq t_1 < \cdots < t_k \leq 1$ such that $\gamma([0,1)] \subset \cup_{i=1}^{k} \gamma(t_j) \cdot V$, and conclude that $t_j \in V \ldots V$, in particular, the first part of Remark 2.1.7 holds.