2.2 Actions of Lie groups on manifolds

In this section, we study the local structure of group actions on manifolds, in particular, we introduce the important notation: principal G-bundle.

2.2.1 Induced vector fields by group actions

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let M be a smooth manifold. We assume that G acts on the left on M as in (2.1.58).

Definition 2.2.1. For any $\xi \in \mathfrak{g}$, the vector field ξ^M on M induced by ξ is defined by: for any $x \in M$,

$$\xi_x^M = \frac{d}{dt}\Big|_{t=0} \exp(t\xi) \cdot x. \tag{2.2.1}$$

Let $e^{t\xi^M}$ ($t \in \mathbb{R}$) be the flow associated with $\xi^M \in \mathscr{C}^\infty(M, TM)$.

Lemma 2.2.2. *For any $g \in G, \xi \in \mathfrak{g}$, $x \in M$ and $t \in \mathbb{R}$, we have*

$$\begin{aligned} \exp(t\xi) \cdot x &= e^{t\xi^M}(x), \\ (\mathrm{Ad}_g \xi)^M &= g_* \xi^M. \end{aligned} \tag{2.2.2}$$

For the action G on $\mathfrak{g}^*$ by Ad_G^*, $\eta \in \mathfrak{g}$ and $\beta \in \mathfrak{g}^*$, we have

$$\eta_\beta^{\mathfrak{g}^*} = \mathrm{ad}_\eta^* \beta. \tag{2.2.3}$$

Proof. For any $t \in \mathbb{R}$, (2.1.11) and (2.1.14), we get

$$\frac{d}{dt}\exp(t\xi)\cdot x = \frac{d}{ds}\Big|_{s=0} \exp(s\xi)\exp(t\xi)\cdot x = \xi^M_{\exp(t\xi)\cdot x}, \quad \frac{d}{dt} e^{t\xi^M}(x) = \xi^M(e^{t\xi^M}(x)). \tag{2.2.4}$$

Then by $\exp(t\xi)\cdot x|_{t=0} = e^{t\xi^M}(x)\Big|_{t=0} = x$, and the uniqueness of the solutions of ordinary differential equations, we get the first equation of (2.2.2).

By $\exp(\mathrm{Ad}_g \xi) = g \cdot \exp \xi \cdot g^{-1}$ and the first equation of (2.2.2), we have

$$\begin{aligned} (\mathrm{Ad}_g \xi)_x^M &= \frac{d}{dt}\Big|_{t=0} \exp(t\mathrm{Ad}_g \xi)\cdot x = \frac{d}{dt}\Big|_{t=0} g \cdot \exp(t\xi)(g^{-1}x) \\ &= \frac{d}{dt}\Big|_{t=0} g \cdot e^{t\xi^M}(g^{-1}x) = (g_* \xi^M)(x) \ . \end{aligned} \tag{2.2.5}$$

Finally for any $X \in \mathfrak{g}$, $M = \mathfrak{g}^*$,

$$\begin{aligned} (\eta_\beta^M, X) &= \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}^*_{\exp(t\eta)}\beta, X) = \frac{d}{dt}\Big|_{t=0} (\beta, \mathrm{Ad}_{\exp(-t\eta)}X) \\ &= -(\beta, \mathrm{ad}_\eta X) = (\mathrm{ad}_\eta^* \beta, X). \end{aligned} \tag{2.2.6}$$

The proof of Lemma 2.2.2 is completed. □

Proposition 2.2.3. *The map*

$$(\mathfrak{g}, [\,,\,]) \to (\mathscr{C}^\infty(M, TM), [\,,\,]), \quad \xi \mapsto -\xi^M \tag{2.2.7}$$

is a morphism of Lie algebras, i.e., it is linear and for any $\xi, \eta \in \mathfrak{g}$

$$[\xi, \eta]^M = -[\xi^M, \eta^M]. \tag{2.2.8}$$

Proof. Fix $x \in M$, the orbit map $R_x : G \to M$, $g \mapsto g \cdot x$ is smooth and

$$\xi_x^M = \frac{d}{dt}\Big|_{t=0} \exp(t\xi) \cdot x = (dR_x)\xi. \tag{2.2.9}$$

Thus the map $\xi \in \mathfrak{g} \mapsto -\xi^M \in \mathscr{C}^\infty(M, TM)$ is linear. By Lemma 2.2.2, we have

$$\begin{aligned}[\xi, \eta]^M = (\mathrm{ad}_\xi \eta)^M = \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}_{\exp(t\xi)}\eta)^M = \frac{d}{dt}\Big|_{t=0} \exp(t\xi)_* \eta^M \\ = \frac{d}{dt}\Big|_{t=0} e_*^{t\xi^M} \eta^M = -[\xi^M, \eta^M].\end{aligned} \tag{2.2.10}$$

The proof of Proposition 2.2.3 is completed. □

2.2.2 The slice theorem

Let G be a compact Lie group. Let M be a smooth manifold with a left G-action. If $x \in M$, its orbit will be denoted by $G \cdot x$,

$$G \cdot x = \{y \in M : \ y = g \cdot x \text{ for some } g \in G\} \tag{2.2.11}$$

and its stabilizer G_x is defined by

$$G_x = \{g \in G : \ g \cdot x = x\}. \tag{2.2.12}$$

The G-action is called free if $G_x = \{e\}$ for any $x \in M$; locally free if G_x is discrete for any $x \in M$; effective if $\cap_{x\in M} G_x = \{e\}$.

Obviously the stabilizers of points are closed subgroups of G. Thus G_x is a Lie subgroup of G and G/G_x is a smooth manifold.

Proposition 2.2.4. *If G is a compact Lie group, its orbits are submanifolds of M.*

Proof. For $x \in M$, let $R_x : G \to M$, $g \mapsto g \cdot x$, be the orbit map. We evaluate the kernel of the differential of it, $T_g R_x : T_g G \to T_{gx} M$. By invariance, it is sufficient to study the case where $g = e$,

$$\ker(T_e R_x) = \{X \in \mathfrak{g} |\ X_x^M = 0\} \tag{2.2.13}$$

is the Lie algebra of G_x. So

$$R_x : G/G_x \to M \tag{2.2.14}$$

is an injective immersion. As G is compact, R_x is proper. So R_x is an embedding and $G{\cdot}x = \mathrm{Im} R_x$ is a submanifold of M. □

Now we consider the orbit space which is the quotient space $G\backslash M := \{Gx : x \in M\}$ endowed with the quotient topology. In general, the topology of the orbit space can be very bad, even not Hausdorff.

Proposition 2.2.5. *If G is compact, then $G\backslash M$ is a Hausdorff space.*

Proof. Let $G \cdot x$ and $G \cdot y$ be two distinct orbits. They are compact as images of G. As M is Hausdorff, there is an open neighborhood U of x such that $\overline{U} \cap G \cdot y = \emptyset$. Now, for the quotient map $\pi : M \to G\backslash M$, $\pi(U)$ and the complement of $\pi(\overline{U})$ are both open. They don't intersect and the former contains $\pi(x)$ and the latter $\pi(y)$. □

For $x \in M$, then the tangent space of the orbit $G \cdot x$ is given by

$$T_x(G \cdot x) = \{\xi_x^M : \ \xi \in \mathfrak{g}\} \subset T_x M. \tag{2.2.15}$$

For any $h \in G_x$, $\xi \in \mathfrak{g}$, by (2.2.5), $dh(\xi_x^M) = (\mathrm{Ad}_h \xi)_x^M$, thus $T_x(G \cdot x)$ is preserved by G_x-action, and

$$N_x = T_x M / T_x(G \cdot x) \tag{2.2.16}$$

is a linear representation of G_x. We define

$$G \times_{G_x} N_x = G \times N_x / \sim \tag{2.2.17}$$

and $(g, v) \sim (f, w)$ if and only if there exists $h \in G_x$ such that

$$f = gh, \quad w = dh^{-1}(v). \tag{2.2.18}$$

We verify easily that $\sim$ is an equivalent relation on $G \times N_x$. And the zero section is $G \times_{G_x} \{0\} = G/G_x$.

Theorem 2.2.6 (Slice theorem)**.** *There exists an equivariant diffeomorphism from an equivariant open neighborhood of the zero section in $G \times_{G_x} N_x$ to an open neighborhood of $G \cdot x$ in M, which sends the zero section G/G_x onto the orbit $G \cdot x$ by the map R_x in (2.2.14).*

Proof. Let g_0^{TM} be a Riemannian metric on M. For any $g \in G$,

$$(g \cdot g_0^{TM})_x(u, v) = (g^{-1})^* g_0^{TM}(u, v)_x := g_0^{TM}(dg^{-1}(u), dg^{-1}(v))_{g^{-1}x} \tag{2.2.19}$$

for $u, v \in T_x M$, $x \in M$ defines a metric on TM. We say a metric g_1^{TM} on TM is G-invariant if $g \cdot g_1^{TM} = g_1^{TM}$ for any $g \in G$.

Let $d\mu$ be a Haar measure on G. Then

$$g^{TM} = \int_G (g \cdot g_0^{TM}) d\mu(g) \tag{2.2.20}$$

is a G-invariant metric on TM. In fact, for any $g_1 \in G$, as $L_{g_1^{-1}}^* d\mu = d\mu$, by (1.2.2) and Exercise 2.2.1, we get

$$\begin{aligned} g_1 \cdot g^{TM} = \int_G ((g_1 g) \cdot g_0^{TM}) d\mu(g) &= \int_G L_{g_1}^* \big(g \cdot g_0^{TM}(L_{g_1^{-1}}^* d\mu)(g)\big) \\ &= \int_G g \cdot g_0^{TM}(L_{g_1^{-1}}^* d\mu)(g) = g^{TM}. \end{aligned} \tag{2.2.21}$$

Let $B^{T_x M}(0, \delta)$ be the ball in $T_x M$ with the center 0 and the radius $\delta > 0$. Then for δ small, the exponential map with respect to the metric g^{TM},

$$\exp_x : T_x M \supset B^{T_x M}(0, \delta) \to M \tag{2.2.22}$$

is a diffeomorphism.

Note that for any $g \in G$, g is an isometry of (M, g^{TM}). So the image of a geodesic by g is a geodesic. This and the uniqueness of a geodesic for fixed starting point and its derivative imply that

$$g \cdot \exp_x v = \exp_{g \cdot x} dg(v), \qquad \text{for } x \in M, v \in T_x M, g \in G. \tag{2.2.23}$$

We identify N_x by $T_x(G\cdot x)^{\perp}$, the orthogonal complement of $T_x(G\cdot x)$ in (T_xM, g^{TM}). Let $B^{N_x}(0,r)$ be the ball in N_x with the center 0 and the radius r. For r small, we define a map

$$\phi : G\times N_x \supset G\times B^{N_x}(0,r)\to M, \quad (g,v)\mapsto g\cdot \exp_x v. \tag{2.2.24}$$

Then for any $h\in G_x$, by (2.2.23),

$$\phi((gh, dh^{-1}(v))) = gh\cdot \exp_x(dh^{-1}(v)) = gh\cdot h^{-1}\exp_x v = g\cdot \exp_x v = \phi((g,v)).$$

Since g^{TM} is a G-invariant metric, the G_x action preserves the ball $B^{N_x}(0,r)$. So the map

$$\varphi : G\times_{G_x} N_x \supset G\times_{G_x} B^{N_x}(0,r)\to M, \quad [g,v]\mapsto g\cdot \exp_x v \tag{2.2.25}$$

is well-defined. The map φ is G-equivariant, in fact for any $g'\in G$,

$$\varphi([g'g,v]) = g'g\cdot \exp_x v = g'\varphi([g,v]). \tag{2.2.26}$$

Now we need to prove that for r sufficient small, $\varphi : G\times_{G_x} B^{N_x}(0,r)\to \varphi(G\times_{G_x} B^{N_x}(0,r))$ is a diffeomorphism (we will explain the differential structure on $G\times_{G_x} B^{N_x}(0,r)$ in Corollary 2.2.7 if $G_x\neq\{e\}$). By Proposition 2.2.4, we know that $\varphi|_{G/G_x} : G\times_{G_x}\{0\}\to M$ is an embedding. By Lemma 1.2.18, we only need to prove that for any $g\in G$,

$$d\varphi_{[g,0]} : T_{[g,0]}(G\times_{G_x} N_x) = \mathfrak{g}\times_{G_x} N_x \to T_{gx}M \tag{2.2.27}$$

is bijective, with the equivalence relation $(\xi, v)\sim (\mathrm{Ad}_{h^{-1}}\xi, dh^{-1}(v))$ for any $\xi\in\mathfrak{g}$, $h\in G_x$ and $v\in N_x$. As φ is G-equivariant, we only need to establish (2.2.27) for $g=e$. By (2.2.25),

$$d\varphi_{[e,0]}[\xi,v] = \xi^M_x + v. \tag{2.2.28}$$

So

$$d\varphi_{[e,0]} : T_{[e,0]}(G\times_{G_x} N_x)\to T_xM = T_x(G\cdot x)\oplus N_x \tag{2.2.29}$$

is surjective. This implies that $d\varphi_{[e,0]}$ is bijective because

$$\dim T_{[e,0]}(G\times_{G_x} N_x) = \dim G + \dim N_x - \dim G_x = \dim M. \tag{2.2.30}$$

By Lemma 1.2.18, the proof of Theorem 2.2.6 is completed. □

Corollary 2.2.7. *If for any $x\in M$, $G_x=\{e\}$, in other words, G is a free action on M, then there exists a differential structure on $G\backslash M$ with the quotient topology.*

Proof. By Proposition 2.2.5, $G\backslash M$ is Hausdorff. By Theorem 2.2.6, we have the G-equivariant diffeomorphism for any $x\in M$

$$\varphi_x : G\times B^{N_x}(0,r)\to \mathcal{V}_x\subset M, \quad (g,v)\to g\cdot \exp_x(v). \tag{2.2.31}$$

Locally, we have the homomorphism

$$\tilde{\varphi}_x : \mathcal{U}_x = B^{N_x}(0,r)\to W_x = G\backslash\mathcal{V}_x\subset G\backslash M, \quad v\to [\exp_x(v)]. \tag{2.2.32}$$

The subset $G\backslash\mathcal{V}_x$ is open in $G\backslash M$ since $\mathcal{V}_x$ is open in M. If $W_x\cap W_y\neq\emptyset$ for $x,y\in M$, we need to prove that

$$\tilde{\varphi}_y^{-1}\circ\tilde{\varphi}_x : \mathcal{U}_x\supset \tilde{\varphi}_x^{-1}(W_x\cap W_y)\to \tilde{\varphi}_y^{-1}(W_x\cap W_y)\subset\mathcal{U}_y \tag{2.2.33}$$

is $\mathscr{C}^\infty$. For $v\in\tilde{\varphi}_x^{-1}(W_x\cap W_y)$, then $\varphi_x([e,v])\in\mathcal{V}_y$, and thus

$$\varphi_y^{-1}\circ(\varphi_x([e,v])) = (g(v), \tilde{\varphi}_y^{-1}\circ\tilde{\varphi}_x(v))\in G\times B^{N_y}(0,r) \tag{2.2.34}$$

is $\mathscr{C}^\infty$ on $\mathcal{V}$. In particular, $\tilde{\varphi}_y^{-1}\circ\tilde{\varphi}_x$ is $\mathscr{C}^\infty$. Thus $\{\tilde{\varphi}_x\}$ define local coordinates of $G\backslash M$, and $G\backslash M$ is a smooth manifold. □

Definition 2.2.8. Let $\pi : P \to M$ be a smooth map of two manifolds. We say that P is a principal bundle over M with (right action) structure group G (or a principal G-bundle over M) if there is a covering of M by open sets $\{U_i\}$ and diffeomorphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times G$ such that for $x \in U_i \cap U_j$,

$$\varphi_i \circ \varphi_j^{-1}(x, g) = (x, \varphi_{ij}(x)g), \tag{2.2.35}$$

with $\varphi_{ij}(x) \in \mathscr{C}^\infty(U_i \cap U_j, G)$. Then G acts on the right on P by $q \cdot g =:= \varphi_i^{-1}(\varphi_i(q) \cdot g)$ for $q \in \pi^{-1}(U_i)$ with right G-action on $U_i \times G$ and by (2.2.35), $q \cdot g$ does not depend on the choice of U_i.

By Corollary 2.2.7, if G is compact and the action of G on a manifold P is free, then P/G is a smooth manifold. Moreover, $\pi : P \to P/G$ is a principal G-bundle. In fact, if $W_x \cap W_y \neq \emptyset$, then for $(h, v) \in G \times B^{N_x}(0, r)$,

$$\varphi_y^{-1} \circ \varphi_x(h, v) = (h \cdot g(v), \tilde{\varphi}_y^{-1} \circ \tilde{\varphi}_x v) \in G \times B^{N_y}(0, r). \tag{2.2.36}$$

Note that in the proof of Corollary 2.2.7, we do not use Theorem 2.1.13, thus it gives also a proof of Theorem 2.1.13 when H is compact.

Definition 2.2.9. Let $\pi : P \to M$ be a principal G-bundle. For $\rho : G \to \mathrm{GL}(E)$ a linear representation of G (i.e., ρ is a homomorphism of groups), where E is a finite-dimensional vector space, the vector bundle induced by ρ and π is defined by

$$P \times_G E = \{(p, v) : p \in P, v \in E\}/_\sim, \quad (p, v) \sim (pg, \rho(g)^{-1}(v)), \text{ for any } g \in G. \tag{2.2.37}$$

In fact, on U_i, we have a trivialization of $P \times_G E$ induced by $\varphi_i : \pi^{-1}(U_i) \to U_i \times G$,

$$\begin{array}{ccc} \pi^{-1}(U_i) \times_G E & \xrightarrow{\varphi_{U_i}} U_i \times G \times_G E & \xrightarrow{\rho_{U_i}} U_i \times E, \\ [p, v] & [(x, g), v] & (x, \rho(g)v), \end{array} \tag{2.2.38}$$

By (2.2.37), the last map in (2.2.38) is a canonical isomorphism and $\rho_{U_i}^{-1}(x, v) = [(x, e), v]$, thus

$$\rho_{U_i} \circ \varphi_{U_i} \circ (\rho_{U_j} \circ \varphi_{U_j})^{-1}(x, v) = \rho_{U_i}([(x, \varphi_{ij}(x)), v]) = (x, \rho(\varphi_{ij}(x))v), \tag{2.2.39}$$

and $\rho(\varphi_{ij}(x))$ is smooth. Thus $P \times_G E$ is a vector bundle on M.

Remark 2.2.10. If for any $x \in M$, $|G_x| < \infty$, then locally,

$$G\backslash\mathcal{V}_x = G\backslash G \times_{G_x} \mathcal{U}_x = G_x\backslash\mathcal{U}_x, \tag{2.2.40}$$

where the finite group G_x acts on $\mathcal{U}_x$ linearly. In this case, $G\backslash M$ is not a manifold. It is an orbifold.

2.2.3 Poisson structure on $\mathfrak{g}^*$

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then G acts on $\mathfrak{g}^*$ by Ad^*-action. For $\beta \in \mathfrak{g}^*$, the coadjoint orbit of β is defined by

$$\mathcal{O}_\beta := G \cdot \beta = \mathrm{Ad}_G^* \beta \subset \mathfrak{g}^*. \tag{2.2.41}$$

Let

$$G_\beta = \{g \in G : \mathrm{Ad}_g^* \beta = \beta\} \tag{2.2.42}$$

be the stabilizer of β, then G_β is a closed subgroup of G, thus G_β is a Lie subgroup of G, and by Theorem 2.1.13, $\mathcal{O}_\beta \simeq G/G_\beta$ is a manifold with quotient topology of G. Note that $[g] \in G/G_\beta \to \mathrm{Ad}_g^*\beta \in \mathfrak{g}^*$ is smooth and proper, thus $\mathcal{O}_\beta$ is a submanifold of $\mathfrak{g}^*$, and its tangent space at $\gamma \in \mathcal{O}_\beta$ is

$$T_\gamma \mathcal{O}_\beta = \{\mathrm{ad}_X^*\gamma \in T_\gamma\mathfrak{g}^* \simeq \mathfrak{g}^* : X \in \mathfrak{g}\}. \tag{2.2.43}$$

Definition 2.2.11. For any $F, G \in \mathscr{C}^\infty(\mathfrak{g}^*)$, we define $\{F, G\} \in \mathscr{C}^\infty(\mathfrak{g}^*)$ by: for any $\beta \in \mathfrak{g}^*$,

$$\{F, G\}(\beta) = (\beta, [dF_\beta, dG_\beta]), \tag{2.2.44}$$

where dF_β, the differential of F at β, lies in $T_\beta^*\mathfrak{g}^*$ which is naturally identified with $\mathfrak{g}$ and $[\ ,\ ]$ is the Lie bracket of $\mathfrak{g}$.

Theorem 2.2.12. *The bracket* $\{\ ,\ \}$ *on* $\mathscr{C}^\infty(\mathfrak{g}^*)$ *is a Poisson bracket, i.e.,* $(\mathfrak{g}^*, \{\ ,\ \})$ *is a Poisson manifold.*

Proof. By (2.2.44) and the Leibniz's rule, we get for any $F, G, H \in \mathscr{C}^\infty(\mathfrak{g}^*)$,

$$\{F, G\} = -\{G, F\}, \quad \{FH, G\} = H\{F, G\} + F\{H, G\}. \tag{2.2.45}$$

What we should prove is the Jacobi identity and we will show that it is a direct consequence of the Jacobi identity of the Lie bracket of $\mathfrak{g}$.

By (2.1.20) and (2.2.44), we have

$$\{\{F, G\}, H\}(\beta) = (\beta, [d\{F, G\}_\beta, dH_\beta]) = (\mathrm{ad}_{dH_\beta}^*\beta, d\{F, G\}_\beta). \tag{2.2.46}$$

We compute first $d\{F, G\}$. Let's choose a base $\{e_j\}$ of $\mathfrak{g}$ and its dual basis $\{e_i^*\}$ of $\mathfrak{g}^*$. Then at $\beta = \beta_i e_i^*$

$$\begin{aligned} dF_\beta &= \frac{\partial F}{\partial \beta_i}(\beta) e_i, \\ [dF_\beta, dG_\beta] &= \frac{\partial F}{\partial \beta_i}(\beta)\frac{\partial G}{\partial \beta_j}(\beta)[e_i, e_j]. \end{aligned} \tag{2.2.47}$$

By (2.1.20), (2.2.44) and (2.2.47), we have

$$\begin{aligned} d\{F,G\}_\beta &= d(\beta, [dF_\beta, dG_\beta]) \\ &= (e_k^*, [dF_\beta, dG_\beta])e_k + \frac{\partial^2 F}{\partial\beta_i\partial\beta_\ell}(\beta)\Big(\beta, [e_i, dG_\beta]\Big)e_\ell + \frac{\partial^2 G}{\partial\beta_j\partial\beta_\ell}(\beta)\Big(\beta, [dF_\beta, e_j]\Big)e_\ell \\ &= [dF_\beta, dG_\beta] + \frac{\partial^2 F}{\partial\beta_i\partial\beta_\ell}(\beta)(\beta, -\mathrm{ad}_{dG_\beta} e_i)e_\ell + \frac{\partial^2 G}{\partial\beta_j\partial\beta_\ell}(\beta)(\beta, \mathrm{ad}_{dF_\beta} e_j)e_\ell \\ &= [dF_\beta, dG_\beta] + \frac{\partial^2 F}{\partial\beta_i\partial\beta_\ell}(\beta)(\mathrm{ad}_{dG_\beta}^*\beta, e_i)e_\ell + \frac{\partial^2 G}{\partial\beta_j\partial\beta_\ell}(\beta)(-\mathrm{ad}_{dF_\beta}^*\beta, e_j)e_\ell \\ &= [dF_\beta, dG_\beta] + D^2F(\mathrm{ad}_{dG_\beta}^*\beta, \cdot) - D^2G(\mathrm{ad}_{dF_\beta}^*\beta, \cdot), \end{aligned} \tag{2.2.48}$$

where D^2F is the Hessian of the function F. Therefore, by (2.1.20), (2.2.46) and (2.2.48), we get

$$\begin{aligned} \{\{F, G\}, H\}_\beta = {} & (\mathrm{ad}_{dH}^*\beta, [dF, dG]) + (\mathrm{ad}_{dH}^*\beta, D^2F(\mathrm{ad}_{dG}^*\beta, \cdot)) \\ & - (\mathrm{ad}_{dH}^*\beta, D^2G(\mathrm{ad}_{dF}^*\beta, \cdot)) \\ = {} & (\beta, [[dF, dG], dH]) + D^2F(\mathrm{ad}_{dG}^*\beta, \mathrm{ad}_{dH}^*\beta) - D^2G(\mathrm{ad}_{dF}^*\beta, \mathrm{ad}_{dH}^*\beta). \end{aligned} \tag{2.2.49}$$

By the Jacobi identity (2.1.1) of the Lie bracket of $\mathfrak{g}$ and (2.2.49), we get the Jacobi identity of this Poisson bracket. ∎

Let us compute the Hamiltonian vector field of this Poisson structure. For $F, H \in \mathscr{C}^\infty(\mathfrak{g}^*)$, let X_H be the Hamiltonian vector field associated with H. Then by (1.3.12), for $\beta \in \mathfrak{g}^*$,

$$(X_H \cdot F)(\beta) = \{F, H\}(\beta) = (\beta, [dF_\beta, dH_\beta]) = (\mathrm{ad}^*_{dH_\beta}\beta, dF_\beta). \tag{2.2.50}$$

Thus

$$X_H(\beta) = \mathrm{ad}^*_{dH_\beta}\beta. \tag{2.2.51}$$

By (2.2.43), the Hamiltonian vector fields span precisely the tangent space of the coadjoint orbit $\mathcal{O}_\beta$, thus the symplectic foliation $\mathcal{F}$ in Section 1.3.3 at $\beta \in \mathfrak{g}^*$ is given by

$$\mathcal{F}_\beta = T_\beta \mathcal{O}_\beta. \tag{2.2.52}$$

Proposition 2.2.13. *The coadjoint orbits in $\mathfrak{g}^*$ are the symplectic leaves of the Poisson manifold $(\mathfrak{g}^*, \{\ ,\ \})$, with the symplectic form*

$$\omega_\beta(\lambda_\beta^{\mathfrak{g}^*}, \mu_\beta^{\mathfrak{g}^*}) = (\beta, [\lambda, \mu]), \quad \lambda, \mu \in \mathfrak{g}. \tag{2.2.53}$$

Proof. By (2.2.52), $\mathcal{O}_\beta$ are the symplectic leaves of $(\mathfrak{g}^*, \{\ ,\ \})$.

For $\lambda, \mu \in \mathfrak{g}$, we define the functions $F, G \in \mathscr{C}^\infty(\mathfrak{g}^*)$ by

$$F(\beta) = (\beta, \lambda), \quad G(\beta) = (\beta, \mu). \tag{2.2.54}$$

Then we have $dF_\beta = \lambda$ and $dG_\beta = \mu$. Thus by (1.3.13), (2.1.20) and (2.2.44),

$$\begin{aligned} B_\beta(\lambda, \mu) &= B_\beta(dF_\beta, dG_\beta) = \{F, G\}(\beta) \\ &= (\beta, [\lambda, \mu]) = -(\mathrm{ad}^*_\lambda \beta, \mu). \end{aligned} \tag{2.2.55}$$

By (2.2.3) and (2.2.55), we have

$$i_\lambda B_\beta = -\mathrm{ad}^*_\lambda \beta = -\lambda_\beta^{\mathfrak{g}^*}. \tag{2.2.56}$$

Thus by (1.3.14), (1.3.21) and (2.2.55), we have

$$\omega_\beta(\lambda_\beta^{\mathfrak{g}^*}, \mu_\beta^{\mathfrak{g}^*}) = \omega_\beta(-i_\lambda B_\beta, -i_\mu B_\beta) = B_\beta(\lambda, \mu) = (\beta, [\lambda, \mu]). \tag{2.2.57}$$

The proof of Proposition 2.2.13 is completed. □

Exercise 2.2.1. Verify in (2.2.19), for any $g_1, g_2 \in G$, $g_1 \cdot (g_2 \cdot g_0^{TM}) = (g_1 g_2) \cdot g_0^{TM}$.

Exercise 2.2.2. (Coadjoint orbits in $\mathfrak{so}(3)^*$) Let $\{e_i\}$ be the canonical orthonormal basis of $\mathbb{R}^3$. For $X = (x_1, x_2, x_3)^t, Y = (y_1, y_2, y_3)^t \in \mathbb{R}^3$, the inner product $\langle X, Y\rangle$ is given by $X \cdot Y = \sum_{i=1}^3 x_i y_i$, and the cross product (or vector product) of X and Y is defined by

$$\begin{aligned} X \times Y = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \\ =(x_2y_3 - x_3y_2)e_1 - (x_1y_3 - x_3y_1)e_2 + (x_1y_2 - x_2y_1)e_3. \end{aligned} \tag{2.2.58}$$

1. Verify that the following map is an isomorphism of vector spaces from $\mathbb{R}^3$ to $\mathfrak{so}(3)$, the Lie algebra of SO(3):

$$\mathbb{R}^3 \ni \begin{pmatrix} x \\ y \\ z \end{pmatrix} = X \mapsto \widehat{X} = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \in \mathfrak{so}(3). \tag{2.2.59}$$

2. For $X, Y, Z \in \mathbb{R}^3$, verify that

$$\begin{aligned} \langle X \times Y, Z \rangle &= (X \times Y) \cdot Z = \det(X, Y, Z), \\ X \times (Y \times Z) &= (X \cdot Z)Y - (X \cdot Y)Z. \end{aligned} \tag{2.2.60}$$

   In particular $X \perp X \times Y, Y \perp X \times Y$, and $\langle X \times Y, Z \rangle$ is antisymmetric on X, Y, Z.

3. $[\widehat{X}, \widehat{Y}] = \widehat{X \times Y} = \widehat{\widehat{X} \cdot Y}$. Conclude that the Killing form B on $\mathfrak{so}(3)$ in (2.1.109) is given by $B(\widehat{X}, \widehat{Y}) = \mathrm{Tr}[\widehat{X}\widehat{Y}]$.

4. $-\frac{1}{2}\,\mathrm{Tr}[\widehat{X}\widehat{Y}] = X \cdot Y$, thus the map (2.2.59) is an isometry from $(\mathbb{R}^3, \langle\ \ \rangle)$ onto $(\mathfrak{so}(3), -\frac{1}{2}B)$.

5. $\mathrm{Ad}_A \widehat{X} = \widehat{AX}$, for $A \in \mathrm{SO}(3)$. Thus (2.2.59) is a morphism of SO(3)-representations for the adjoint action of SO(3) on $\mathfrak{so}(3)$.

6. We identify $\mathfrak{so}(3)$ with $\mathfrak{so}(3)^*$ via the metric $-\frac{1}{2}B$, then for $A \in \mathrm{SO}(3)$, $\widehat{\xi} \in \mathfrak{so}(3)^*$, we have

$$\mathrm{Ad}^*_A \widehat{\xi} = \widehat{A\,\xi}, \quad \mathrm{ad}^*_{\widehat{X}} \widehat{\xi} = \widehat{X \times \xi} = \widehat{\widehat{X}\,\xi}. \tag{2.2.61}$$

   Thus the induces map from $\mathbb{R}^3$ onto $\mathfrak{so}(3)^*$ is a morphism of SO(3)-representations for the coadjoint action of SO(3) on $\mathfrak{so}(3)^*$.

7. The coadjoint orbits in $\mathfrak{so}(3)^*$ are $\mathcal{O}_r := \{\xi \in \mathbb{R}^3 : |\xi| = r\}$, the spheres of radius $r > 0$ with center 0, and the singular orbit $\{0\}$.

8. For any $v \in T_\xi \mathcal{O}_r \subset \mathbb{R}^3$, $\widehat{v} = \mathrm{ad}^*_{\widehat{\eta}} \widehat{\xi}$ with $\eta = -v \times \xi / r^2$.

9. The induced symplectic form on $\mathcal{O}_r$ is σ_0/r, where

$$\sigma_0(\xi) = i_{\xi/r} dv = \frac{1}{r}(x dy \wedge dz + y dz \wedge dx + z dx \wedge dy) \text{ for } \xi = (x, y, z)^t \in \mathcal{O}_r,$$

   is the volume form on $\mathcal{O}_r$ induced by the Euclidean volume form dv on $\mathbb{R}^3$.

2.3 Moment maps

We introduce the most geometric objects of this book: moment maps and associated symplectic reductions.

2.3.1 Basic properties of moment maps

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and (M, ω) be a symplectic manifold of dimension $2n$. We assume that G acts on the left on M. We define the G-action on the space of functions from M to $\mathfrak{g}^*$ by: for $g \in G$, $x \in M$, $\phi : M \to \mathfrak{g}^*$,

$$(g \cdot \phi)_x = \mathrm{Ad}_g^*(\phi(g^{-1}x)). \tag{2.3.1}$$

If $g \cdot \phi = \phi$ for any $g \in G$, then we say that $\phi : M \to \mathfrak{g}^*$ is G-equivariant.

We call that G acts symplectically on (M, ω) if its action preserves ω, i.e., for any $g \in G$, $g^*\omega = \omega$.

Definition 2.3.1. A symplectic action of a Lie group G on a symplectic manifold (M, ω) is called Hamiltonian if there exists a map $\mu : M \to \mathfrak{g}^*$ satisfying

$$d(\mu, \xi) = i_{\xi^M}\omega \quad \text{for } \xi \in \mathfrak{g}; \tag{2.3.2}$$

and μ is G-equivariant, i.e.,

$$\mu(gx) = \mathrm{Ad}_g^*\mu(x) \quad \text{for } g \in G,\ x \in M. \tag{2.3.3}$$

This map μ is called a moment map for the G-action on M.

For any $\xi \in \mathfrak{g}$, by (1.2.80) and (2.3.2), the Hamiltonian vector field $X_{(\mu,\xi)}$ of the function $(\mu, \xi) = \mu(\xi)$ is given by

$$X_{(\mu,\xi)} = X_{\mu(\xi)} = \xi^M. \tag{2.3.4}$$

Thus $\xi^M \in \mathfrak{ham}(M, \omega) \subset \mathfrak{sympl}(M, \omega)$. If G is connected, then Remark 2.1.7 and (2.3.2) imply that $g^*\omega = \omega$ for any $g \in G$, i.e., we can drop the condition that G acts symplectically on (M, ω) in Definition 2.3.1.

Lemma 2.3.2. *Let $\mu : M \to \mathfrak{g}^*$ be the moment map for a Hamiltonian G-action on (M, ω), then for $\eta, \xi \in \mathfrak{g}$,*

$$(\mu, [\eta, \xi]) = \omega(\eta^M, \xi^M), \tag{2.3.5}$$

and μ defines a morphism of Lie algebras

$$\mu : (\mathfrak{g}, [,]) \to (\mathscr{C}^\infty(M), \{,\}), \quad \xi \mapsto (\mu, \xi). \tag{2.3.6}$$

Proof. At first, $\mu : \mathfrak{g} \to \mathscr{C}^\infty(M)$ is linear. For $\eta, \xi \in \mathfrak{g}$, by (1.3.2), (2.3.2) and (2.3.4), we get for $x \in M$,

$$\{\mu(\eta), \mu(\xi)\}_x = \omega(X_{\mu(\eta)}, X_{\mu(\xi)})_x = \omega(\eta^M, \xi^M)_x = \xi^M(\mu, \eta)_x. \tag{2.3.7}$$

From (2.1.20) and (2.3.3), we get

$$\begin{aligned} \xi^M(\mu, \eta)_x &= \frac{d}{dt}\Big|_{t=0} (\mu(\exp(t\xi) \cdot x), \eta) = \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}^*_{\exp(t\xi)}\mu(x), \eta) \\ &= \frac{d}{dt}\Big|_{t=0} (\mu(x), \mathrm{Ad}_{\exp(-t\xi)}\eta) = (\mu(x), -[\xi, \eta]) \\ &= (\mu, [\eta, \xi])_x. \end{aligned} \tag{2.3.8}$$

From (2.3.7) and (2.3.8), the proof of Lemma 2.3.2 is completed. □

Remark 2.3.3. If G is connected, then we can replace (2.3.3) by that (2.3.6) defines a morphism of Lie algebras.

In fact, if $\mu : (\mathfrak{g}, [,]) \to (\mathscr{C}^\infty(M), \{,\})$ is a morphism of Lie algebras, then for $\eta, \xi \in \mathfrak{g}$, by (1.3.1), (2.3.2) and (2.3.7), we have

$$(\xi^M \cdot \mu, \eta) = \{\mu(\eta), \mu(\xi)\} = (\mu, [\eta, \xi]) = (\mathrm{ad}^*_\xi \mu, \eta). \tag{2.3.9}$$

Thus $\frac{\partial}{\partial t}\Big(\mu(e^{t\xi} \cdot x), \mathrm{Ad}_{e^{t\xi}}\eta\Big) = \Big((\xi^M \cdot \mu)(e^{t\xi} \cdot x), \mathrm{Ad}_{e^{t\xi}}\eta\Big) + \Big(\mu(e^{t\xi} \cdot x), [\xi, \mathrm{Ad}_{e^{t\xi}}\eta]\Big) = 0$ for any $t \in \mathbb{R}$. This implies that (2.3.3) holds for $g = e^{t\xi}$. If G is connected, Remark 2.1.7 and (2.3.3) for $e^{t\xi}$ imply that $\mu : M \to \mathfrak{g}^*$ is G-equivariant.

We assume now that a connected Lie group G acts symplectically on a connected symplectic manifold (M, ω). Then by Lemma 2.3.2 and Remark 2.3.3, the existence and uniqueness of the moment map is equivalent to the existence and uniqueness of the lifting μ such that the following diagram of morphisms of Lie algebras is commutative:

$$\begin{array}{ccccc} & & & & (\mathfrak{g}, [\,,\,]) \\ & & & \overset{\mu}{\swarrow} & \big\downarrow \\ 0 \to \mathbb{R} & \longrightarrow & (\mathscr{C}^\infty(M), \{\,,\,\}) & \longrightarrow & (\mathfrak{sympl}(M, \omega), [\,,\,]) \end{array} \tag{2.3.10}$$

here the map $\mathfrak{g} \to \mathfrak{sympl}(M, \omega)$ is $\xi \to -\xi^M$, and then map $\mathscr{C}^\infty(M) \to \mathfrak{sympl}(M, \omega)$ is $f \to -X_f$.

Proposition 2.3.4. *If M is connected compact and $H^1(M, \mathbb{R}) = 0$, then every symplectic action of a connected Lie group G is Hamiltonian.*

Proof. Let $\{\xi_i\}_{i=1}^{\dim G}$ be a basis of $\mathfrak{g}$. As the G-action is symplectic, we have $L_{\xi_i^M}\omega = 0$, thus $di_{\xi_i^M}\omega = 0$. Since $H^1(M, \mathbb{R}) = 0$, there exists $\mu_i \in \mathscr{C}^\infty(M)$ such that

$$i_{\xi_i^M}\omega = d\mu_i. \tag{2.3.11}$$

As M is compact, we can fix the free constant by requiring $\int_M \mu_i \omega^n = 0$. We define $\mu : M \to \mathfrak{g}^*$ by

$$\mu\Big(\sum_i a_i \xi_i\Big) = \sum_i a_i \mu_i, \text{ where } a_i \in \mathbb{R}. \tag{2.3.12}$$

Then we have

$$i_{\xi^M}\omega = d\mu(\xi) \text{ and } \int_M \mu(\xi)\omega^n = 0 \text{ for any } \xi \in \mathfrak{g}. \tag{2.3.13}$$

For $\xi, \eta \in \mathfrak{g}$, set

$$C(\xi, \eta) := \mu([\xi, \eta]) - \{\mu(\xi), \mu(\eta)\}. \tag{2.3.14}$$

By (1.3.8), (2.2.8) and (2.3.4), we get

$$X_{\mu([\xi,\eta])} = [\xi, \eta]^M = -[\xi^M, \eta^M] = -[X_{\mu(\xi)}, X_{\mu(\eta)}] = X_{\{\mu(\xi),\mu(\eta)\}}. \tag{2.3.15}$$

By (1.2.80), (2.3.15) implies that $dC(\xi, \eta) = 0$, thus $C(\xi, \eta)$ is a constant function on M as M is connected. Now by (1.3.1), (2.3.13), as $L_{\eta^M}\omega = 0$, we have

$$\begin{aligned} \int_M \{\mu(\xi), \mu(\eta)\}\omega^n &= \int_M \omega(\xi^M, \eta^M)\omega^n = \int_M \eta^M(\mu(\xi))\omega^n \\ &= \int_M L_{\eta^M}(\mu(\xi)\omega^n) = \int_M d\, i_{\eta^M}(\mu(\xi)\omega^n) = 0. \end{aligned} \tag{2.3.16}$$

By (2.3.13) and (2.3.16), we get $C(\xi,\eta)=0$ for any $\xi,\eta\in\mathfrak{g}$. By Remark 2.3.3, we know that the symplectic G-action is Hamiltonian if G is connected. □

Proposition 2.3.5. *Assume a connected Lie group G acts symplectically on a connected symplectic manifold (M,ω). If $H^1(\mathfrak{g})=0$, then there exists at most a moment map.*

Assume one of the following conditions holds:
(a) *M is compact and $H^1(\mathfrak{g})=0$,*
(b) *$H^1(\mathfrak{g})=H^2(\mathfrak{g})=0$,*
then there exists a unique moment map for this G-action on M.

Proof. First, we claim that

$$[\mathfrak{sympl}(M,\omega),\mathfrak{sympl}(M,\omega)]\subset\mathfrak{ham}(M,\omega). \tag{2.3.17}$$

In fact, for $X,Y\in\mathfrak{sympl}(M,\omega)$, we have $L_X\omega=L_Y\omega=0$. Then by (1.2.20) and the formula $[L_X,i_Y]=i_{[X,Y]}$, we have

$$\begin{aligned} i_{[X,Y]}\omega &= L_Xi_Y\omega-i_YL_X\omega=L_Xi_Y\omega=i_Xdi_Y\omega+di_Xi_Y\omega\\ &=i_XL_Y\omega-i_Xi_Yd\omega+d(\omega(Y,X))=d(\omega(Y,X)). \end{aligned} \tag{2.3.18}$$

Thus $[X,Y]$ is the Hamiltonian vector field associated with $\omega(Y,X)$.

For the uniqueness of the moment map, we assume that there exist two moment maps μ_1 and μ_2 satisfying Definition 2.3.1. Then for ξ, $\eta\in\mathfrak{g}$, by Lemma 2.3.2 and (2.3.4),

$$\begin{aligned} \mu_1([\xi,\eta]) &= \{\mu_1(\xi),\mu_1(\eta)\}=\omega(X_{\mu_1(\xi)},X_{\mu_1(\eta)})=\omega(X_{\mu_2(\xi)},X_{\mu_2(\eta)})\\ &=\mu_2([\xi,\eta]). \end{aligned} \tag{2.3.19}$$

Therefore, the moment map is unique if $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$ which is equivalent to $H^1(\mathfrak{g})=0$ by Proposition 2.1.26.

Assume now that $H^1(\mathfrak{g})=0$, then by Proposition 2.1.26, $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$. Thus for any $\zeta\in\mathfrak{g}$, there exist $u_j,v_j\in\mathfrak{g}$, such that $\zeta=\sum_{j=1}^k[u_j,v_j]$. Set

$$\mu(\zeta):=\sum_j\omega(u_j^M,v_j^M). \tag{2.3.20}$$

Then by (2.2.8) and (2.3.18),

$$d(\mu(\zeta))=\sum_j d(\omega(u_j^M,v_j^M))=\sum_j i_{[v_j^M,u_j^M]}\omega=\sum_j i_{[u_j,v_j]^M}\omega=i_{\zeta^M}\omega. \tag{2.3.21}$$

By (2.3.21), $\mu(\zeta)$ is defined up to a constant.

Let $\{\xi_i\}_{i=1}^{\dim G}$ be a basis of $\mathfrak{g}$, and we fix $\mu_i=\mu(\xi_i)$ and extend linearly on $\mathfrak{g}$ by (2.3.12), then by (2.3.15), $C(\xi,\eta)$ in (2.3.14) is a constant function on M as M is connected.

• If M is compact, we can normalize μ by requiring $\int_M\mu(\xi_i)\omega^n=0$ for $1\le i\le\dim G$. Then $C(\xi,\eta)\equiv0$ as in the proof of Proposition 2.3.4.

• If M may be noncompact, the equation (2.3.14) defines an element $C\in\Lambda^2\mathfrak{g}^*$. For $\xi,\eta,\zeta\in\mathfrak{g}$, by (2.1.87) and (2.3.14),

$$\begin{aligned} (\delta C)(\xi,\eta,\zeta) &= -C([\xi,\eta],\zeta)+C([\xi,\zeta],\eta)-C([\eta,\zeta],\xi)\\ &=-\sum_{(\xi,\eta,\zeta)}\Big(\mu([[\xi,\eta],\zeta])-\{\mu([\xi,\eta]),\mu(\zeta)\}\Big), \end{aligned} \tag{2.3.22}$$

here $\sum_{(\xi,\eta,\zeta)}$ is the cyclic sum of ξ,η,ζ.

By the Jacobi identity $\sum_{(\xi,\eta,\zeta)}[[\xi,\eta],\zeta]=0$, (2.3.14) and (2.3.22), we get

$$(\delta C)(\xi,\eta,\zeta)=\sum_{(\xi,\eta,\zeta)}\Big(\{\{\mu(\xi),\mu(\eta)\},\mu(\zeta)\}+\{C(\xi,\eta),\mu(\zeta)\}\Big)=0, \tag{2.3.23}$$

in the last equality, we use the Jacobi identity for $\{\,,\,\}$ and $C(\xi,\eta)$ is a constant function on M.

By the condition $H^2(\mathfrak{g})=0$ and (2.3.23), there exists $\alpha\in\mathfrak{g}^*$ such that $C=\delta\alpha$. That is, by (2.1.87), $C(\xi,\eta)=-\alpha([\xi,\eta])$.

We define $\tilde{\mu}(\xi)=\mu(\xi)+(\alpha,\xi)$. Then

$$\begin{aligned}\tilde{\mu}([\xi,\eta])-\{\tilde{\mu}(\xi),\tilde{\mu}(\eta)\}&=\mu([\xi,\eta])+(\alpha,[\xi,\eta])-\{\mu(\xi)+(\alpha,\xi),\mu(\eta)+(\alpha,\eta)\}\\&=C(\xi,\eta)+(\alpha,[\xi,\eta])=0.\end{aligned} \tag{2.3.24}$$

Then by Remark 2.3.3, $\tilde{\mu}$ is a moment map and the G-action is Hamiltonian. □

From Propositions 2.1.29 and 2.3.5, we get

Corollary 2.3.6. *If a connected semisimple Lie group G acts symplectically on a connected symplectic manifold (M,ω), then the G-action is Hamiltonian and there is a unique moment map assciated with this G-action.*

We collect now some properties of the moment map.

Proposition 2.3.7. *Let $\mu: M\to\mathfrak{g}^*$ be a moment map. Then*
1) For $x\in M$, let $\mathfrak{g}_x$ be the Lie algebra of the stabilizer $G_x=\{g\in G: g\cdot x=x\}$ of x, then

$$\mathfrak{g}_x=(\mathrm{Im}\ d\mu_x)^{\perp}\subset\mathfrak{g}. \tag{2.3.25}$$

2) The map $\mu: M\to(\mathfrak{g}^*,\{,\})$ is a homomorphism of Poisson manifolds. That is, for $F,G\in\mathscr{C}^\infty(\mathfrak{g}^*)$, we have

$$\mu^*\{F,G\}=\{\mu^*F,\mu^*G\}. \tag{2.3.26}$$

3) (Functional property) If $H\subset G$ is a Lie subgroup of G, we have an embedding of their Lie algebras $i:\mathfrak{h}\to\mathfrak{g}$ and the dual map $i^*:\mathfrak{g}^*\to\mathfrak{h}^*$. Then by the definition, $\mu_H:=i^*\circ\mu: M\to\mathfrak{h}^*$ is a moment map for the H-action on M.

Proof. In fact, by Lemma 2.2.2, for $\xi\in\mathfrak{g}$,

$$\xi\in\mathfrak{g}_x\Leftrightarrow\xi^M_x=0\Leftrightarrow(d\mu_x,\xi)=i_{\xi^M_x}\omega=0. \tag{2.3.27}$$

Thus 1) holds.

For $F,G\in\mathscr{C}^\infty(\mathfrak{g}^*)$, $x\in M$, set $F_1(\gamma)=(dF_{\mu(x)},\gamma)$, $G_1(\gamma)=(dG_{\mu(x)},\gamma)$ two linear functions on $\mathfrak{g}^*$, then $F_1=dF_1=dF_{\mu(x)}$, $G_1=dG_1=dG_{\mu(x)}\in\mathfrak{g}$ and by Lemma 2.3.2, (1.3.1), (2.2.44) and (2.3.4) ,

$$\begin{aligned}\mu^*\{F,G\}_x=\{F,G\}_{\mu(x)}&=(\mu(x),[dF_{\mu(x)},dG_{\mu(x)}])\\&=(\mu(x),[F_1,G_1])=\{(\mu,F_1),(\mu,G_1)\}_x.\end{aligned} \tag{2.3.28}$$

Recall that the value of $\{\mu^*F,\mu^*G\}$ at $x\in M$ depends only on the differentials of μ^*F and μ^*G at x and

$$d(\mu^*F)_x=d(F\circ\mu)_x=dF_{\mu(x)}(d\mu_x)=(d\mu_x,F_1)=d(\mu,F_1)_x. \tag{2.3.29}$$

From (2.3.28), (2.3.29), we get (2.3.26).

The part 3) is trivial. The proof of Proposition 2.3.7 is completed. □