2.3.2 Examples of moment maps

Example 1: (Symplectic vector space) Let (V, ω) be a symplectic vector space of dimension $2n$. Then the symplectic group $\mathrm{Sp}(V) = \{g \in \mathrm{GL}(V) : g^*\omega = \omega\}$ with Lie algebra $\mathfrak{sp}(V)$, acts naturally on V. The map $\mu : V \to \mathfrak{sp}(V)^*$ defined by: for $v \in V, \xi \in \mathfrak{sp}(V) \subset \mathrm{End}(V)$,

$$(\mu(v), \xi) = \frac{1}{2}\omega(\xi v, v) \tag{2.3.30}$$

is a moment map for this $\mathrm{Sp}(V)$-action on V.

In fact, for $\xi \in \mathfrak{sp}(V)$, $u, v \in V$,

$$\xi_v^M = \frac{d}{dt}\bigg|_{t=0} \exp(t\xi)v = \xi v \quad \text{and} \quad \omega(\xi u, v) + \omega(u, \xi v) = 0. \tag{2.3.31}$$

So $\omega(\xi u, v) = \omega(\xi v, u)$, and by (2.3.30), for $g \in \mathrm{Sp}(V)$,

$$u(\mu(v), \xi) = \frac{1}{2}u(\omega(\xi v, v)) = \frac{1}{2}\omega(\xi u, v) + \frac{1}{2}\omega(\xi v, u) = \omega(\xi v, u), \tag{2.3.32}$$

and

$$\begin{aligned}(\mu(gv), \xi) &= \frac{1}{2}\omega(\xi g v, g v) = \frac{1}{2}\omega(g^{-1}\xi g v, v)\\ &= (\mu(v), \mathrm{Ad}_{g^{-1}}\xi) = (\mathrm{Ad}_g^*\mu(v), \xi).\end{aligned} \tag{2.3.33}$$

Thus $\mu : V \to \mathfrak{sp}(V)^*$ in (2.3.30) is a moment map.

If $V = \mathbb{R}^{2n}$ with the canonical symplectic form $\omega_{st} = \sum_{i=1}^n dx_i \wedge dy_i = \langle J_0\cdot, \cdot\rangle$, where $v = \begin{pmatrix} x \\ y \end{pmatrix}$, $x, y \in \mathbb{R}^n$ and $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the canonical complex structure on $\mathbb{R}^{2n}$ and $\langle\, ,\, \rangle$ is the canonical Euclidean metric on $\mathbb{R}^{2n}$ (cf. (1.1.31)). Then by (2.3.30),

$$(\mu(v), \xi) = \frac{1}{2}\omega(\xi v, v) = \frac{1}{2}\langle J_0\xi v, v\rangle = \frac{1}{2}v^t J_0 \xi v = \frac{1}{2}\operatorname{Tr}[vv^t J_0\xi]. \tag{2.3.34}$$

Let $B_{\mathfrak{sp}(2n)}$ be the bilinear form on $\mathfrak{sp}(2n)$ defined by: for $\xi, \eta \in \mathfrak{sp}(2n)$,

$$B_{\mathfrak{sp}(2n)}(\xi, \eta) = -\frac{1}{2}\operatorname{Tr}|_{\mathbb{R}^{2n}}[\xi\, \eta]. \tag{2.3.35}$$

It is Ad-invariant, as $\operatorname{Tr}|_{\mathbb{R}^{2n}}[\mathrm{Ad}_g\xi\, \mathrm{Ad}_g\eta] = \operatorname{Tr}|_{\mathbb{R}^{2n}}[\xi\, \eta]$ for any $g \in \mathrm{Sp}(2n)$. By (2.1.41) and (2.3.35), we verify that

$$B_{\mathfrak{sp}(2n)}|_{\mathfrak{u}} > 0, \quad B_{\mathfrak{sp}(2n)}|_{\mathfrak{p}} < 0 \quad \text{and} \quad B_{\mathfrak{sp}(2n)}(\mathfrak{u}, \mathfrak{p}) = 0. \tag{2.3.36}$$

Thus $B_{\mathfrak{sp}(2n)}$ is nondegenerate. The bilinear form $B_{\mathfrak{sp}(2n)}$ is called a Killing form on $\mathfrak{sp}(2n)$. From the general theory of Lie algebra (cf. Proposition 2.1.28), the nondegenerateness of the Killing form implies that $\mathfrak{sp}(2n)$ is semisimple.

We identify $\mathfrak{sp}(2n)^*$ with $\mathfrak{sp}(2n)$ by $B_{\mathfrak{sp}(2n)}$. From (2.3.34), we have

$$\mu(v) = -vv^t J_0 = \begin{pmatrix} -xy^t & xx^t \\ -yy^t & yx^t \end{pmatrix} \in \mathfrak{sp}(2n) \simeq \mathfrak{sp}(2n)^*. \tag{2.3.37}$$

We can check directly $\mu(v) \in \mathfrak{sp}(2n)$, as $\mu(v)^t J_0 + J_0\mu(v) = -J_0^t vv^t J_0 - J_0 vv^t J_0 = 0$.

The unitary group $\mathrm{U}(n)$ is identified as a Lie subgroup of $\mathrm{Sp}(2n)$ via τ in (1.1.59). The embedding $i : \mathrm{U}(n) \to \mathrm{Sp}(2n)$ induces an injective morphism of Lie algebras $i : \mathfrak{u}(n) \to \mathfrak{sp}(2n)$ by $i = \tau|_{\mathfrak{u}(n)}$ which identifies $\mathfrak{u}(n)$ as $\mathfrak{u}$. The map i via (2.1.41) induces $i^* : \mathfrak{sp}(2n) \simeq \mathfrak{sp}(2n)^* = \mathfrak{p}^* \oplus \mathfrak{u}^* \to \mathfrak{u}^* = \tau(\mathfrak{u}(n))^* \simeq \mathfrak{u}(n)$. Note that the decomposition $\mathfrak{sp}(2n) = \mathfrak{p} \oplus \mathfrak{u}$ is simply the decomposition of a matrix as a sum of symmetric and antisymmetric matrices, and (2.3.36) implies the identification $\mathfrak{sp}(2n) \simeq \mathfrak{sp}(2n)^*$ identifies $\mathfrak{p}$ to $\mathfrak{p}^*$ and $\mathfrak{u}$ to $\mathfrak{u}^*$ via $B_{\mathfrak{sp}(2n)}$, thus i^* is the projection of a matrix to its antisymmetric part, i.e.,

$$i^* \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \alpha - \alpha^t & \beta - \gamma \\ \gamma - \beta & \alpha - \alpha^t \end{pmatrix}. \tag{2.3.38}$$

By Proposition 2.3.7, (2.3.37) and (2.3.38), we get

$$\begin{aligned} \tau(\mu_{\mathrm{U}(n)})(v) = i^* \circ \mu(v) &= \frac{1}{2} \begin{pmatrix} yx^t - xy^t & xx^t + yy^t \\ -xx^t - yy^t & yx^t - xy^t \end{pmatrix} \\ &= \frac{1}{2}\tau(yx^t - xy^t + \sqrt{-1}(-xx^t - yy^t)) \\ &= -\tau\Big(\frac{\sqrt{-1}}{2}(x + \sqrt{-1}y)(x^t - \sqrt{-1}y^t)\Big) = \tau\Big(-\frac{\sqrt{-1}}{2} zz^*\Big) \in \tau(\mathfrak{u}(n)), \end{aligned} \tag{2.3.39}$$

where $z = x + \sqrt{-1}y \in \mathbb{C}^n$. Thus the moment map of the $\mathrm{U}(n)$-action on $(\mathbb{C}^n, \omega_{\mathrm{st}})$ with $\omega_{\mathrm{st}} = \frac{\sqrt{-1}}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$, is

$$\mu_{\mathrm{U}(n)}(z) = -\frac{\sqrt{-1}}{2} zz^* \in \mathfrak{u}(n). \tag{2.3.40}$$

Note that by Exercise 2.3.4, the identification of $\mathfrak{u}(n)$ and $\mathfrak{u}(n)^*$ via $B_{\mathfrak{sp}(2n)}$ is exactly the identification via the Euclidean metric $B_{\mathfrak{u}(n)}(\xi, \eta) = -\operatorname{Tr}_{\mathbb{C}^n}[\xi\, \eta]$ on $\mathfrak{u}(n)$.

Example 2: (Cotangent bundle) Let $\pi : T^*M \to M$ be the cotangent bundle of a manifold M. By Example 1.2.14, we can define a canonical 1-form $\lambda_{st} = \sum p_i dx_i$ on T^*M which is independent of coordinates x_i on M and dual coordinates p_i on T^*M. Intrinsically,

$$(\lambda_{st})_{(x,p)}(v) = (p, d\pi(v))_x \quad \text{for } v \in T_{(x,p)}(T^*M). \tag{2.3.41}$$

The 2-form $\omega_{st} := -d\lambda_{st}$ is closed and nondegenerate, it defines the canonical symplectic form on T^*M.

For $\varphi \in \mathrm{Diff}(M)$, the group of diffeomorphisms of M, we denote by $d\varphi : T_xM \to T_{\varphi(x)}M$ its differential, and $(d\varphi)^* : T^*_{\varphi(x)}M \to T^*_xM$ its dual. Then φ induces a diffeomorphism $\tilde{\varphi}$ of $Q := T^*M$ defined by

$$\tilde{\varphi} : T^*_xM \to T^*_{\varphi(x)}M, \quad \tilde{\varphi}(x, p) = (\varphi(x), (d\varphi^{-1})^*(p)). \tag{2.3.42}$$

Certainly, $\mathrm{Diff}(M) \ni \varphi \to \tilde{\varphi} \in \mathrm{Diff}(T^*M)$ identifies $\mathrm{Diff}(M)$ as a subgroup of $\mathrm{Diff}(T^*M)$. Thus we will not distinct φ and $\tilde{\varphi}$, and $\mathrm{Diff}(M)$ acts naturally on T^*M.

The formal Lie algebra of $\mathrm{Diff}(M)$ is $\mathscr{C}^\infty(M, TM)$, the space of vector fields on M. We apply formally the Ad, ad actions in (2.1.18)–(2.1.20) to the group $\mathrm{Diff}(M)$ and its Lie algebra $\mathscr{C}^\infty(M, TM)$, then for $\varphi \in \mathrm{Diff}(M)$, $X \in \mathscr{C}^\infty(M, TM)$, we get by (1.2.4),

$$(\mathrm{Ad}_\varphi X)_x = \frac{\partial}{\partial t}|_{t=0} \varphi \circ e^{tX} \circ \varphi^{-1}(x) = (d\varphi)(X(\varphi^{-1}(x))) = (\varphi_* X)_x. \tag{2.3.43}$$

For $X \in \mathscr{C}^\infty(M, TM)$, (2.3.42) induces an infinitesimal action $X^Q \in \mathscr{C}^\infty(Q, TQ)$ on Q by: for $(x,p) \in T_x^*M$

$$X^Q(x,p) = \frac{d}{dt}\Big|_{t=0} e^{tX} \cdot (x,p). \tag{2.3.44}$$

On local coordinates (cf. Example 1.2.14),

$$X^Q(x,p) = (X_x, -L_X p), \tag{2.3.45}$$

here p is understood as a constant section of T^*M, and $L_X p$ defined by (1.2.9).

Lemma 2.3.8. *The forms λ_{st}, ω_{st} are $\mathrm{Diff}(M)$-invariant, i.e., for any $\varphi \in \mathrm{Diff}(M)$, $\tilde{\varphi}^*\lambda_{st} = \lambda_{st}$. The map*

$$\mu : T^*M \to \mathscr{C}^\infty(M, TM)^*, \quad (\mu, X)_{(x,p)} = i_{X^Q}(\lambda_{st})_{(x,p)} = p(X_x) \in \mathscr{C}^\infty(T^*M). \tag{2.3.46}$$

is the moment map on T^*M with respect to the $\mathrm{Diff}(M)$-action. In other words, $\mu_{(x,p)} = p \circ \delta_x$, where $\delta_x(X) = X_x$.

Proof. By (2.3.41) and $\pi \circ \tilde{\varphi} = \varphi \circ \pi$, we have

$$\begin{aligned}(\tilde{\varphi}^*\lambda_{st})_{(x,p)}(v) &= (\lambda_{st})_{(\varphi(x),(d\varphi^{-1})^*(p))}(d\tilde{\varphi}(v)) = ((d\varphi^{-1})^*p, d\pi \circ d\tilde{\varphi}(v)) \\ &= ((d\varphi^{-1})^*p, d\varphi \circ d\pi(v)) = p(d\pi(v)) = (\lambda_{st})_{(x,p)}(v).\end{aligned} \tag{2.3.47}$$

Thus $\tilde{\varphi}^*\omega_{st} = -d\tilde{\varphi}^*\lambda_{st} = \omega_{st}$, i.e., ω_{st} is also $\mathrm{Diff}(M)$-invariant.

For any $X \in \mathscr{C}^\infty(M, TM)$, by (2.3.47), we have

$$L_{X^Q}\lambda_{st} = 0. \tag{2.3.48}$$

Then by Cartan formula (1.2.20) and (2.3.48), we get

$$di_{X^Q}\lambda_{st} = -i_{X^Q}d\lambda_{st} = i_{X^Q}\omega_{st}. \tag{2.3.49}$$

From (2.3.49), what we need to prove now is that μ is $\mathrm{Diff}(M)$-equivariant. In fact, for $\varphi \in \mathrm{Diff}(M)$,

$$\begin{aligned}(\mu_{\varphi(x,p)}, X) &= (\mu_{(\varphi(x),(d\varphi^{-1})^*(p))}, X) = ((d\varphi^{-1})^*p)(X_{\varphi(x)}) = p((d\varphi^{-1})X_{\varphi(x)}) \\ &= p(\mathrm{Ad}_{\varphi^{-1}}X)_x = (\mu_{(x,p)}, \mathrm{Ad}_{\varphi^{-1}}X) = ((\mathrm{Ad}^*_\varphi \mu)_{(x,p)}, X).\end{aligned} \tag{2.3.50}$$

The proof of Lemma 2.3.8 is completed. □

Proposition 2.3.9. *(Coadjoint orbit) Let G be a Lie group. For $\beta \in \mathfrak{g}^*$, the Ad^*-action of G on its coadjoint orbit $M = \mathcal{O}_\beta = G \cdot \beta = \mathrm{Ad}_G \cdot \beta \subset \mathfrak{g}^*$ is Hamiltonian and the natural injection $\mu : \mathcal{O}_\beta \to \mathfrak{g}^*$ is a moment map.*

Proof. By (2.2.43), $T_\alpha\mathcal{O}_\beta = \{\eta^M_\alpha : \eta \in \mathfrak{g}\} = \{\mathrm{ad}^*_\eta\alpha : \eta \in \mathfrak{g}\}$. The form ω is defined by: for $\xi, \eta \in \mathfrak{g}$,

$$\omega(\xi^M, \eta^M)_\alpha = (\alpha, [\xi, \eta]) = -(\mathrm{ad}^*_\xi\alpha, \eta). \tag{2.3.51}$$

Here we prove directly that ω is a symplectic form on $\mathcal{O}_\beta$. At first, for $g \in G$, by (1.2.4) and (2.2.2),

$$\begin{aligned}(g^*\omega)(\xi^M, \eta^M)_\alpha &= \omega((\mathrm{Ad}_g\xi)^M, (\mathrm{Ad}_g\eta)^M)_{\mathrm{Ad}^*_g\alpha} \\ &= (\mathrm{Ad}^*_g\alpha, [\mathrm{Ad}_g\xi, \mathrm{Ad}_g\eta]) = (\mathrm{Ad}^*_g\alpha, \mathrm{Ad}_g[\xi,\eta]) = (\alpha, [\xi,\eta]).\end{aligned} \tag{2.3.52}$$

Thus ω is G-invariant. If $\omega(\xi^M, \eta^M)_\alpha = 0$ for any $\eta \in \mathfrak{g}$, then by (2.3.51), $\mathrm{ad}_\xi^* \alpha = 0$, but $\xi_\alpha^M = \mathrm{ad}_\xi^* \alpha$. This concludes that ω is nondegenerate.

Now we check that μ is G-equivalent. In fact, for $\alpha \in \mathcal{O}_\beta$, $g \in G$, we have

$$\mu_{g\cdot\alpha} = \mu_{\mathrm{Ad}_g^*\alpha} = \mathrm{Ad}_g^*\alpha = \mathrm{Ad}_g^*\mu_\alpha. \tag{2.3.53}$$

Thus from (2.2.53), for $\xi, \eta \in \mathfrak{g}$, we get

$$\begin{aligned} d(\mu, \xi)_\alpha(\eta_\alpha^M) &= \left.\frac{d}{dt}\right|_{t=0} (\mathrm{Ad}_{e^{t\eta}}^*\alpha, \xi) = (\mathrm{ad}_\eta^*\alpha, \xi) \\ &= (\alpha, [\xi, \eta]) = \omega_\alpha(\xi_\alpha^M, \eta_\alpha^M). \end{aligned} \tag{2.3.54}$$

Since the vector fields η^M span the tangent space, (2.3.54) means that $d\mu(\xi) = i_{\xi^M}\omega$. Finally, by Cartan formula (1.2.20) and ω is G-invariant, we get for $\eta \in \mathfrak{g}$,

$$0 = d^2\mu(\eta) = di_{\eta^M}\omega = L_{\eta^M}\omega - i_{\eta^M}d\omega = -i_{\eta^M}d\omega. \tag{2.3.55}$$

By using again that the vector fields η^M span the tangent space, we see $d\omega = 0$. Thus $(\mathcal{O}_\beta, \omega)$ is a symplectic manifold and μ is the moment map. The proof of Proposition 2.3.9 is completed. □